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# Locally and globally small Riemann sums and Henstock integral of fuzzy-number-valued functions

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**Abstract** In this paper, we first define and discuss the locally small Riemann sums ($LSRS$) for fuzzy-number-valued functions. In addition the necessary and sufficient conditions have been obtained for a fuzzy-number-valued function which has ($LSRS$), i.e., if a fuzzy-number-valued function is Henstock ($H$) integrable on $[a, b]$ then it has ($LSRS$) and the converse is always true. Secondly, the globally small Riemann sums ($GSRS$) for fuzzy-number-valued functions is defined and discussed, and the necessary and sufficient conditions have been given for a fuzzy-number-valued function which has ($GSRS$), i.e., if a fuzzy-number-valued function is ($H$) integrable on $[a, b]$ then it has ($GSRS$) and the converse is always true. Finally, by Egorov's Theorem, we obtain the dominated convergence theorem for globally small Riemann sums ($GSRS$) of fuzzy-number-valued functions.
**Keywords:** Fuzzy numbers; fuzzy integrals; ($H$) integral; ($LSRS$); ($GSRS$).

## 1 Introduction

Since the concept of fuzzy sets was firstly introduced by Zadeh in 1965 [22], it has been studied extensively from many different aspects of the theory and applications, such as fuzzy topology, fuzzy analysis, fuzzy decision making and fuzzy logic, information science and so on. fuzzy integrals of fuzzy-number-valued functions have been studied by many authors from different points of views, including Goetschel [9], Nanda [15], Kaleva [12], Wu [18, 19] and other authors [1, 3, 4, 5, 6, 8]. The locally and globally small Riemann sums have been introduced by many authors from different points of views. In 1986, Schurle characterized the Lebesgue integral in ($LSRS$) (locally small Riemann sums) property [16]. The ($LSRS$) property has been used to characterized the Perron ($P$) integral on $[a, b]$ [17]. By considering the equivalency between the ($P$) integral and the Henstock-Kurzweil ($HK$) integral, the ($LSRS$) property has been used to characterized the ($HK$) integral on $[a, b]$ [13].

The ($LSRS$) property brought a research to have global characterization on the Riemann sums of an ($HK$) integrable function on $[a, b]$. This research has been done by considering the following fact: Every ($HK$) integrable function on $[a, b]$ is measurable, however, there is no guarantee the boundedness of the function. A measurable function $f$ is ($HK$) integrable on $[a, b]$ depends on it behaves on the set of $x$ in which $|f(x)|$ is large, i.e. $|f(x)| \geq N$ for some $N$. This fact has been characterized in ($GSRS$) (globally small Riemann sums) property [13].

The ($GSRS$) property involves one characteristic of the primitive of an ($HK$) integrable function. That is the primitive of the ($HK$) integral on $[a, b]$ is $ACG^*$ (generalized strongly absolutely continuous) on $[a, b]$. This is not a simple concept.

In 2015, Indrati [11] introduced a countably Lipschitz condition of a function which is simpler than the $ACG^*$, and proved that the ($HK$) integrable function or it's primitive could be characterized in countably Lipschitz condition. Also, by considering the characterization of the ($HK$) integral in the ($GSRS$) property, it showed that the relationship between ($GSRS$) property and countably Lipschitz condition of an ($HK$) integrable function on $[a, b]$.

In this paper, we first define and discuss the locally small Riemann sums ($LSRS$) for fuzzy-number-valued functions. In addition the necessary and sufficient conditions have been obtained for a fuzzy-number-valued

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function which has $(LSRS)$, i.e., if a fuzzy-number-valued function is $(H)$ integrable on $[a,b]$ then it has $(LSRS)$ and the converse is always true. Secondly, the globally small Riemann sums $(GSRS)$ for fuzzy-number-valued functions is defined and discussed, and the necessary and sufficient conditions have been given for a fuzzy-number-valued function which has $(GSRS)$, i.e., if a fuzzy-number-valued function is $(H)$ integrable on $[a,b]$ then it has $(GSRS)$ and the converse is always true. Finally, by Egorov's Theorem, we obtain the dominated convergence theorem for globally small Riemann sums $(GSRS)$ of fuzzy-number-valued functions.

The paper is organized as follows, in Section 2 we shall review the relevant concepts and properties of fuzzy sets and the definition of $(H)$ integrals for fuzzy-number-valued functions. Section 3 is devoted to discussing the locally small Riemann sums $(LSRS)$ of fuzzy-number-valued functions. In section 4 we shall investigate the globally small Riemann sums $(GSRS)$ of fuzzy-number-valued functions by Egorov's Theorem, we obtain the dominated convergence theorem for globally small Riemann sums $(GSRS)$ of fuzzy-number-valued functions. The last section provides Conclusions.

# 2 Preliminaries

**Definition 2.1** [10, 13] Let $\delta : [a,b] \to \mathbb{R}^+$ be a positive real-valued function. $P = \{[x_{i-1}, x_i]; \xi_i\}$ is said to be a $\delta$-fine division, if the following conditions are satisfied:

(1) $a = x_0 < x_1 < x_2 < ... < x_n = b$;

(2) $\xi_i \in [x_{i-1}, x_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))(i = 1, 2, \cdots, n)$.

For brevity, we write $P = \{[u,v]; \xi\}$, where $[u,v]$ denotes a typical interval in $P$ and $\xi$ is the associated point of $[u,v]$.

**Definition 2.2** [10, 13] A real-valued function $f(x)$ is said to be $(H)$ integrable to $G$ on $[a,b]$ if for every $\varepsilon > 0$ there is a function $\delta(\xi) > 0$ such that for any $\delta$-fine division $P = \{[u,v]; \xi\}$ we have

$$\Big| \sum_{(P)} f(\xi)(v-u) - G \Big| < \varepsilon \tag{2.1}$$

As usual, we write $(RH)\int_a^b f(x)\mathrm{d}x = G$ and $f(x) \in RH[a,b]$.

For the results about fuzzy number space $E^1$. we recall that $E^1 = \{u : \ R \to [0,1] : u$ satisfies (1)-(4) below$\}$:

(1) $u$ is normal, i.e., there exists a $x_0 \in R$ such that $u(x_0) = 1$;

(2) $u$ is a convex fuzzy set, i.e., $u(rx + (1-r)y) \geqslant \min(u(x), u(y)),\ x, y \in R,\ r \in [0,1]$;

(3) $u$ is upper semi-continuous;

(4) $cl\{x \in R : u(x) > 0\}$ is compact, where $clA$ denotes the closure of $A$.

For $0 < r \leqslant 1$, denote $[u]^r = \{x : u(x) \geqslant r\}$. Then from (1)-(4), it follows that the $r-$level set $[u]^r$ is a close interval for all $r \in [0,1]$ (refer to [2, 7, 9, 12, 14, 20, 21]). We write $u^r = [u]^r = [u^r_-, u^r_+]$ or $[u_-(r), u_+(r)]$.

For $u, v \in E^1$, $k \in \mathbb{R}$, the addition and scalar multiplication are defined by the equations:

$$[u+v]^r = [u]^r + [v]^r, \text{ i.e., } u^r_- + v^r_- = [u+v]^r_- \text{ and } u^r_+ + v^r_+ = [u+v]^r_+;$$

$$[ku]^r = k[u]^r, \text{ i.e., } [ku]^r_- = \min\{ku^r_-, ku^r_+\} \text{ and } [ku]^r_+ = \max\{ku^r_-, ku^r_+\},$$

respectively.

Define $D(u,v) = \sup\limits_{r\in[0,1]} d([u]^r, [v]^r) = \sup\limits_{r\in[0,1]} \max\{|u^r_- - v^r_-|, |u^r_+ - v^r_+|\}$, where $d$ is Hausdorff metric. Furthermore, we write

$$\|\tilde{u}\|_{E^1} = D(\tilde{u}, \tilde{0}) = \sup_{\lambda\in[0,1]} \max\{|u^-_\lambda|, |u^+_\lambda|\}.$$

Notice that $\|\cdot\|_{E^1} = D(\cdot, \tilde{0})$ doesn't stands for the norm of $E^1$.

For $u, v \in E^1,\ u \leqslant v$ means $u^r_- \leqslant v^r_-, u^r_+ \leqslant v^r_+$ (see [2, 7, 9, 12, 14, 20, 21]).

Using the results of [2, 7, 9, 12, 14, 20, 21], we recall that:

(1) $(E^1, D)$ is a complete metric space,

(2) $D(u+w, v+w) = D(u,v)$,

(3) $D(u+v, w+e) \leqslant D(u,w) + D(v,e)$,

(4) $D(ku, kv) = |k|D(u,v), k \in R$,

(5) $D(u+v,\tilde{0}) \leqslant D(u,\tilde{0}) + D(v,\tilde{0})$,
(6) $D(u+v,w) \leqslant D(u,w) + D(v,\tilde{0})$, where $\tilde{0} = \chi_{\{0\}}$ and $u,v,w,e \in E^1$.

**Definition 2.3** [18] A fuzzy-number-valued function $\tilde{f}(x)$ is said to be $(H)$ integrable to $\tilde{A} \in E^1$ if for every $\varepsilon > 0$ there is a function $\delta(\xi) > 0$ such that for any $\delta$-fine division $P = \{[u,v];\xi\}$ of $[a,b]$, we have

$$D(\sum \tilde{f}(\xi)(v-u), \tilde{A}) < \varepsilon \tag{2.2}$$

$$(FH)\int_a^b \tilde{f}(x)\mathrm{d}x = \tilde{A} \quad \text{and} \quad \tilde{f}(x) \in FH[a,b]. \tag{2.3}$$

**Lemma 2.1** [18] Let $\tilde{f} : [a,b] \to E^1$ be a fuzzy-number-valued function. Then $\tilde{f} \in FH[a,b]$ iff $f_-^r(x), f_+^r(x) \in H[a,b]$ uniformly for any $r \in [0,1]$, i.e., $\delta(\xi)$ in Definition 2.2 is independent of $r \in [0,1]$.

# 3 Locally small Riemann sums and Henstock $(H)$ integral of fuzzy-number-valued functions

In this section, we shall define locally small Riemann sums or in short $(LSRS)$ and show that it's the necessary and sufficient condition for $\tilde{f}(x)$ to be Henstock $(H)$ integrable on $[a,b]$.

**Definition 3.1** A fuzzy-number-valued function $\tilde{f} : [a,b] \to E^1$ is said to be have locally small Riemann sums or $(LSRS)$ if for every $\varepsilon > 0$ there is a $\delta(\xi) > 0$ such that for every $t \in [a,b]$, we have

$$\big\| \sum \tilde{f}(\xi)(v-u) \big\|_{E^1} < \varepsilon, \tag{3.1}$$

whenever $P = \{[u,v];\xi\}$ is a $\delta$-fine division of an interval $[r,s] \subset (t-\delta(t), t+\delta(t))$, $t \in [r,s]$ and $\Sigma$ sums over $P$.

If there exists a $z \in E^1$ such that $x = y + z$, then we call $z$ the $H-$ difference of $x$ and $y$, denoted by $x - y$.

**Lemma 3.1** [18] Let $\tilde{f} \in FH[a,b]$ and $\tilde{F}$ be the primitive of $\tilde{f}(x)$ then $\tilde{F}$ satisfies the $H-$ difference.

**Lemma 3.2** (Henstock Lemma). If a fuzzy-number-valued function $\tilde{f} : [a,b] \to E^1$ is $(H)$ integrable on $[a,b]$ with primitive $\tilde{F}$, i.e., for every $\varepsilon > 0$ there is a positive function $\delta(\xi) > 0$ such that for any $\delta$-fine division $P = \{[u,v];\xi\}$ of $[a,b]$, we have

$$D(\sum \tilde{f}(\xi)(v-u), \sum \tilde{F}(u,v)) < \varepsilon. \tag{3.2}$$

Then for any sum of parts $\sum\limits_1$ from $\sum$, we have

$$D(\sum_1 \tilde{f}(\xi)(v-u), \sum_1 \tilde{F}(u,v)) < \varepsilon. \tag{3.3}$$

The proof is similar to the Theorem 3.7 [13].

**Theorem 3.1** If $\tilde{f}(x)$ is $(H)$ integrable on $[a,b]$ then it has $LSRS$.

**Proof** Let $\tilde{F}$ be the primitive of $\tilde{f}(x)$. Given $\varepsilon > 0$ there is a $\delta(\xi) > 0$ such that for any $\delta$-fine division $P = \{[u,v];\xi\}$ of $[a,b]$, we have

$$D\big(\sum \tilde{f}(\xi)(v-u), \sum \tilde{F}(u,v)\big) < \varepsilon. \tag{3.4}$$

Where $\tilde{F}(u,v) = \tilde{F}(v) - \tilde{F}(u)$. By the continuity of $\tilde{F}$ at $\xi$,

$$D\big(\tilde{F}(u), \tilde{F}(v)\big) < \varepsilon \quad \text{whenever} \quad [u,v] \subset (\xi - \delta(\xi), \xi + \delta(\xi)).$$

Therefore for $t \in [a,b]$ and any $\delta$-fine division $P = \{[u,v];\xi\}$ of $[r,s] \subset (t-\delta(t), t+\delta(t))$, we have

$$\begin{aligned} \big\| \sum \tilde{f}(\xi)(v-u) \big\|_{E^1} &\leq D\big(\sum \tilde{f}(\xi)(v-u), \sum \tilde{F}(u,v)\big) + D\big(\tilde{F}(r), \tilde{F}(s)\big) \\ &< 2\varepsilon. \end{aligned}$$

That is $\tilde{f}(x)$ has $LSRS$.
This completes the proof. □

**Lemma 3.3** [18] (Cauchy criterion). A fuzzy-number-valued function $\tilde{f} : [a,b] \to E^1$ is $(H)$ integrable on $[a,b]$ iff for every $\varepsilon > 0$ there is a positive function $\delta(\xi) > 0$ such that whenever $P_1 = \{[u_1, v_1]; \xi_1\}$, $P_2 = \{[u_2, v_2]; \xi_2\}$ are two $\delta$-fine divisions, we have

$$D\big(\sum_{(P_1)} \tilde{f}(\xi_1)(v_1 - u_1), \sum_{(P_2)} \tilde{f}(\xi_2)(v_2 - u_2)\big) < \varepsilon. \tag{3.5}$$

**Theorem 3.2** If a fuzzy-number-valued function $\tilde{f} : [a,b] \to E^1$ has $LSRS$ on $[a,b]$ then $\tilde{f}(x)$ is $(H)$ integrable on any closed sub-interval $C \subset (a,b)$. (Where $C = [r,s]$).

**Proof** A fuzzy-number-valued function $\tilde{f} : [a,b] \to E^1$ has $LSRS$ means that for every $\varepsilon > 0$ there is a $\delta(\xi) > 0$ such that for every $t \in [a,b]$, we have

$$\big\| \sum \tilde{f}(\xi)(v-u) \big\|_{E^1} < \varepsilon, \tag{3.6}$$

whenever $P = \{[u,v]; \xi\}$ is a $\delta$-fine division of an interval $C \subset (t - \delta(t), t + \delta(t))$, $t \in C$ and $\Sigma$ sums over $P$.

(i) If there $t \in [a,b]$ with $C \subset (t - \delta(t), t + \delta(t))$ we have the following discussion:

(1) If $t \in C$ then for every $\varepsilon > 0$ there is a two $\delta$-fine divisions $P_1 = \{[u_1, v_1]; \xi_1\}$, $P_2 = \{[u_2, v_2]; \xi_2\}$ on $C$, such that

$$D\big(\sum_{(P_1)} \tilde{f}(\xi_1)(v_1 - u_1), \sum_{(P_2)} \tilde{f}(\xi_2)(v_2 - u_2)\big) < \varepsilon. \tag{3.7}$$

According to the Cauchy criterion, then $\tilde{f}(x)$ is $(H)$ integrable on $C$.

(2) If $t \notin C$ then there is a closed interval $E \subset (t - \delta(t), t + \delta(t))$, with the result that $t \in E$ and $C \subset E$ (where $E = [g,h]$ ). As a result, for every $\varepsilon > 0$ there is a two $\delta$-fine divisions $P_1 = \{[u_1, v_1]; \xi_1\}$, $P_2 = \{[u_2, v_2]; \xi_2\}$ on $E$, such that

$$D\big(\sum_{(P_1)} \tilde{f}(\xi_1)(v_1 - u_1), \sum_{(P_2)} \tilde{f}(\xi_2)(v_2 - u_2)\big) < \varepsilon. \tag{3.8}$$

According to the Cauchy criterion, then $\tilde{f}(x)$ is $(H)$ integrable on $E$. Because $C \subset E$ and $\tilde{f}(x)$ is $(H)$ integrable on $E$ then $\tilde{f}(x)$ is $(H)$ integrable on $C$.

(ii) If $C \nsubseteq (t - \delta(t), t + \delta(t))$ then there is a positive function $\delta$ on $[a,b]$ which resulted in the presence that $P = \{(C_i, t_i) : i = 1, 2, \cdots, k\}$ is a $\delta$-fine division of the interval $C$. It follows that $\tilde{f}(x)$ is $(H)$ integrable on $C_i$ for $i = 1, 2, \cdots, k$.

Then $\tilde{f}(x)$ is $(H)$ integrable on $C$.

This completes the proof. □

**Corollary 3.1** If a fuzzy-number-valued function $\tilde{f} : [a,b] \to E^1$ has $LSRS$ on $[a,b]$ then $\tilde{f}(x)$ is $(H)$ integrable on $C$ for any simple set $C \subset (a,b)$.

Notice that a simple set $C$ means that there exists finite closed sub-interval $C_i$ which belongs to $(a,b)$ such that $C = \bigcup_{i=1}^{k} C_i$.

**Theorem 3.3** If a fuzzy-number-valued function $\tilde{f} : [a,b] \to E^1$ has $LSRS$ on $[a,b]$ then $\tilde{f}(x)$ is $(H)$ integrable on $[a,b]$.

**Proof** A fuzzy-number-valued function $\tilde{f} : [a,b] \to E^1$ has $LSRS$ then for every $\varepsilon > 0$ there is $\delta^*(\xi) > 0$ such that for every $t \in [a,b]$, we have

$$\big\| \sum \tilde{f}(\xi)(v-u) \big\|_{E^1} < \varepsilon, \tag{3.9}$$

whenever $P = \{[u,v]; \xi\}$ is a $\delta^*$-fine division of an interval $C \subset (t - \delta(t), t + \delta(t))$, $t \in C$ and $\Sigma$ sums over $P$. According to the Corollary 3.1, $\tilde{f}(x)$ is $(H)$ integrable on $C$ for any simple set $C \subset (a,b)$.

Rows set $\{E_i\}$, $E_i \bigcap E_j = \phi$, $\forall i \neq j$ with property $(a,b) = \bigcup E_i$, $E_i$ is a closed interval. Thus for above $\varepsilon > 0$, there is a positive numbers $n_0$ with property

$$\mu\{[a,b] - \bigcup_{i \leq n_0} E_i\} < \varepsilon, \tag{3.10}$$

where $\mu$ is Lebesgue measure.

For any $i$, there is a positive function $\delta_i$ such that for any $\delta_i$-fine division on $E_i$, we have

$$D(\sum \tilde{f}(\xi)(v-u), (H)\int_{E_i} \tilde{f}(x)\mathrm{d}x) < \varepsilon. \tag{3.11}$$

Define a positive function $\delta$ by the formula:

$$\delta(\xi) = \begin{cases} \min\{\delta^*(\xi), \frac{1}{2}d(\xi, \partial[a,b])\} & \text{if } \xi \in \bigcup\limits_{i>n_0} E_i, \\ \min\{\delta^*(\xi), \delta_i(\xi)\}, & \text{if } \xi \in \bigcup\limits_{i\le n_0} E_i. \end{cases}$$

For each $\mathcal{C} = \{C\} = \{C_1, C_2, \cdots, C_k\}$ with $C_j = E_i \bigcap Q$ (where $Q = [u,v]$), for one $i \le n_0$ and one $Q$ with $\{[u,v];\xi\}$ is a $\delta$-fine division and $\xi \in (a,b)$, we have

(i) If $C_j = E_i$ for $i \le n_0$. Because $\tilde{f}(x)$ is $(H)$ integrable on $E_i$ and $\tilde{f}(x)$ is $(H)$ integrable on $C_j$ consequently $\tilde{f}(x)$ is $(H)$ integrable on $\bigcup\limits_{j=1}^{k} C_j$. Selected a positive function $\delta_*$ with $\delta_*(\xi) = \min\{\delta_j(\xi) : j = 1, 2, \cdots, k\}$, then for each $\delta_*$-fine division $P = \{[u,v];\xi\}$ on $\bigcup\limits_{j=1}^{k} C_j$, we have

$$D\big((H)\int_{\bigcup\limits_{j=1}^{k} C_j} \tilde{f}(x)\mathrm{d}x, \sum \tilde{f}(\xi)(v-u)\big) < \varepsilon. \tag{3.12}$$

Thus obtained:

$$\begin{aligned} \big\|\mathcal{C}\sum(H)\int_C \tilde{f}(x)\mathrm{d}x\big\|_{E^1} &\le D\big((H)\int_{\bigcup\limits_{j=1}^{k} C_j} \tilde{f}(x)\mathrm{d}x, \sum \tilde{f}(\xi)(v-u)\big) + \sum_{j=1}^{k} \big\|\sum \tilde{f}(\xi)(v-u)\big\|_{E^1} \\ &< \varepsilon + k\varepsilon. \end{aligned}$$

According to the properties of Cauchy, $\tilde{f}(x)$ is $(H)$ integrable on $[a,b]$.

(ii) If $C_j = E_i \bigcap Q$, for $i \le n_0$ and one $\delta$-fine $Q$ with $\{[u,v];\xi\}$ and $\xi \in (a,b)$ then $C_j \subset (\xi - \delta(\xi), \xi + \delta(\xi))$. According to the Theorem 3.2, then $\tilde{f}(x)$ is $(H)$ integrable on $C_j$. As the result $\tilde{f}(x)$ is $(H)$ integrable on $\bigcup\limits_{j=1}^{k} C_j$. Selected a positive function $\delta_1$ with property $\delta_1(\xi) \le \delta(\xi)$ then for each $\delta_*$-fine division $P = \{[u,v];\xi\}$ on $\bigcup\limits_{j=1}^{k} C_j$, we have

$$D\big((H)\int_{\bigcup\limits_{j=1}^{k} C_j} \tilde{f}(x)\mathrm{d}x, \sum \tilde{f}(\xi)(v-u)\big) < \varepsilon. \tag{3.13}$$

Thus obtained:

$$\begin{aligned} \big\|\mathcal{C}\sum(H)\int_C \tilde{f}(x)\mathrm{d}x\big\|_{E^1} &\le D\big((H)\int_{\bigcup\limits_{j=1}^{k} C_j} \tilde{f}(x)\mathrm{d}x, \sum \tilde{f}(\xi)(v-u)\big) + \sum_{j=1}^{k} \big\|\sum \tilde{f}(\xi)(v-u)\big\|_{E^1} \\ &< \varepsilon + k\varepsilon. \end{aligned}$$

According to the properties of Cauchy, $\tilde{f}(x)$ is $(H)$ integrable on $[a,b]$.
This completes the proof. □

**Corollary 3.2** A fuzzy-number-valued function $\tilde{f} : [a,b] \to E^1$ has $LSRS$ on $[a,b]$ iff $\tilde{f}(x)$ is $(H)$ integrable on $[a,b]$.

# 4 Globally small Riemann sums and Henstock $(H)$ integral of fuzzy-number-valued functions

In this section, we shall define globally small Riemann sums or in short $GSRS$ and show that it's the necessary and sufficient condition for $\tilde{f}(x)$ to be Henstock $(H)$ integrable on $[a,b]$.

**Definition 4.1** A fuzzy-number-valued function $\tilde{f} : [a,b] \to E^1$ is said to be have globally small Riemann sums or $(GSRS)$ if for every $\varepsilon > 0$ there exists a positive integer $N$ such that for every $n \geqslant N$ there is a $\delta_n(\xi) > 0$ and for every $\delta_n$-fine division $P = \{[u,v];\xi\}$ of $[a,b]$, we have

$$\Big\| \sum_{\|\tilde{f}(\xi)\|_{E^1} > n} \tilde{f}(\xi)(v-u) \Big\|_{E^1} < \varepsilon, \tag{4.1}$$

where the $\sum$ is taken over $P$ and for which $\|\tilde{f}(\xi)\|_{E^1} > n$.

**Theorem 4.1** Let $\tilde{f}(x)$ be $(H)$ integrable to $\tilde{F}(a,b)$ on $FH[a,b]$ and $\tilde{F}_n(a,b)$ the integral of $\tilde{f}_n(x)$ on $FH[a,b]$, where $\tilde{f}_n(x) = \tilde{f}(x)$ when $||\tilde{f}(x)||_{E^1} \leqslant n$ and $\tilde{0}$ otherwise. If $\tilde{F}_n(a,b) \to \tilde{F}(a,b)$ as $n \to \infty$ then $\tilde{f}(x)$ has $GSRS$.

**Proof** Given $\varepsilon > 0$ there is a $\delta_n(\xi) > 0$ such that for every $\delta_n$-fine division $P = \{[u,v];\xi\}$ of $[a,b]$, we have

$$D(\sum \tilde{f}_n(\xi)(v-u), \tilde{F}_n(a,b)) < \varepsilon, \tag{4.2}$$

where $(FH)\int_a^b \tilde{f}_n(x)\mathrm{d}x = \tilde{F}_n(a,b)$.

$$D(\sum \tilde{f}(\xi)(v-u), \tilde{F}(a,b)) < \varepsilon, \tag{4.3}$$

where $(FH)\int_a^b \tilde{f}(x)\mathrm{d}x = \tilde{F}(a,b)$.

Choose $N$ so that whenever $n \geqslant N$

$$D(\tilde{F}_n(a,b), \tilde{F}(a,b)) < \varepsilon. \tag{4.4}$$

Therefore for $n \geqslant N$ and $\delta_n$-fine division $P = \{[u,v];\xi\}$ of $[a,b]$, we have

$$\begin{aligned} & \Big\| \sum_{\|\tilde{f}(\xi)\|_{E^1} > n} \tilde{f}(\xi)(v-u) \Big\|_{E^1} = D(\sum \tilde{f}_n(\xi)(v-u), \sum \tilde{f}(\xi)(v-u)) \\ \leqslant\ & D(\sum \tilde{f}_n(\xi)(v-u), \tilde{F}_n(a,b)) + D(\tilde{F}_n(a,b), \tilde{F}(a,b)) + D(\tilde{F}(a,b), \sum \tilde{f}(\xi)(v-u)) \\ <\ & 3\varepsilon. \end{aligned}$$

Hence $\tilde{f}(x)$ has $GSRS$.

This completes the proof. □

**Theorem 4.2** A fuzzy-number-valued function $\tilde{f}(x)$ has $GSRS$ iff $\tilde{f}(x)$ is $(H)$ integrable on $[a,b]$ and $\tilde{F}_n(a,b) \to \tilde{F}(a,b)$ as $n \to \infty$ where $\tilde{F}_n(a,b)$ and $\tilde{F}(a,b)$ are defined as in Theorem 4.1.

**Proof** Theorem 4.1 proves the sufficiency. We shall prove only the necessity. Suppose $\tilde{f}(x)$ has $GSRS$. Note that $\tilde{f}_n(x)$, as defined in Theorem 4.1, is $(FH)$ integrable on $[a,b]$ for all $n$. Then for $n, m \geqslant N$ and a suitably chosen $\delta$-fine division $P = \{[u,v];\xi\}$, we have

$$\begin{aligned} & D(\tilde{F}_n(a,b), \tilde{F}_m(a,b)) \\ \leqslant\ & D(\tilde{F}_n(a,b), \sum_{\|\tilde{f}(\xi)\|_{E^1} \leqslant n} \tilde{f}(\xi)(v-u)) + D(\sum_{\|\tilde{f}(\xi)\|_{E^1} \leqslant m} \tilde{f}(\xi)(v-u), \tilde{F}_m(a,b)) \\ +\ & \Big\| \sum_{\|\tilde{f}(\xi)\|_{E^1} > n} \tilde{f}(\xi)(v-u) \Big\|_{E^1} + \Big\| \sum_{\|\tilde{f}(\xi)\|_{E^1} > m} \tilde{f}(\xi)(v-u) \Big\|_{E^1} \\ <\ & 4\varepsilon. \end{aligned}$$

That is, $\tilde{F}_n(a,b)$ converge to a fuzzy number, say $\tilde{F}(a,b)$, as $n \to \infty$. Again, for suitably chosen $N$ and $\delta(\xi)$ and for every $\delta$-fine division $P = \{[u,v];\xi\}$, we have

$$\begin{aligned} & D(\sum \tilde{f}(\xi)(v-u), \tilde{F}(a,b)) \leqslant D(\tilde{F}(a,b), \tilde{F}_N(a,b)) \\ +\ & D(\tilde{F}_N(a,b), \sum_{\|\tilde{f}(\xi)\|_{E^1} \leqslant N} \tilde{f}(\xi)(v-u)) + \Big\| \sum_{\|\tilde{f}(\xi)\|_{E^1} > N} \tilde{f}(\xi)(v-u) \Big\|_{E^1} \\ <\ & 3\varepsilon. \end{aligned}$$

That is, $\tilde{f}(x)$ is $(FH)$ integrable on $[a,b]$.

This completes the proof. □

**Theorem 4.3** Let $\tilde{f}_n(x) \in FH[a,b],\ n = 1,2,3\cdots$ and satisfy:

(1) $\lim\limits_{n\to\infty} \tilde{f}_n(x) = \tilde{f}(x)$ almost everywhere in $[a,b]$;

(2) there exists a Lebesgue $(L)$ integrable ($H$ integrable) function $h(x)$ on $[a,b]$ such that

$$D(\tilde{f}_n(x), \tilde{f}_m(x)) < h(x). \tag{4.5}$$

Then, $\tilde{f}_n(x)$ has $GSRS$ on $[a,b]$ uniformly for any $n$. Naturally, $\tilde{f}$ is $(SFH)$ integrable on $[a,b]$. Furthermore,

$$\lim_{n\to\infty} (SFH)\int_a^b \tilde{f}_n(x)\mathrm{d}x = (SFH)\int_a^b \tilde{f}(x)\mathrm{d}x. \tag{4.6}$$

**Proof** Let $\varepsilon > 0$. Since $H(x) = (L)\int_a^x h(t)\mathrm{d}t$ is absolutely continuous on $[a,b]$, there exists a positive number $\eta > 0$ such that $\sum |H(b_i) - H(a_i)| < \varepsilon$ whenever $\{[a_i,b_i]\}$ is a finite collection of non-overlapping intervals in $[a,b]$ that satisfy $\sum(b_i - a_i) < \eta$. Since $\lim\limits_{n\to\infty} \tilde{f}_n(x) = \tilde{f}(x)$ almost everywhere in $[a,b]$, and

$$\begin{aligned} D(\tilde{f}_n, \tilde{f}) &= \sup_{r\in[0,1]} \max\{|(f_n(x))_-^r - (f(x))_-^r|, |(f_n(x))_+^r - (f(x))_+^r|\} \\ &= \sup_{r_k\in[0,1]} \max\{|(f_n(x))_-^{r_k} - (f(x))_-^{r_k}|, |(f_n(x))_+^{r_k} - (f(x))_+^{r_k}|\} \end{aligned}$$

is a sequence of Lebesgue $(L)$ measurable functions, where $r_k \in [0,1]$ is the set of rational numbers, by Egorov's Theorem, there exists an open set G with $L(G) < \eta$ such that $\lim\limits_{n\to\infty} \tilde{f}_n(x) = \tilde{f}(x)$ uniformly for $x \in [a,b]\setminus G$. Then, there is an natural number $N$, such that for any $n, m > N$, and for any $x \in [a,b]\setminus G$, we have $D(\tilde{f}_n(x), \tilde{f}_m(x)) < \varepsilon$. Since $h(x)$ is $(H)$ integrable on $[a,b]$, there is a $\delta_h(\xi) > 0$ such that for any $\delta_h$-fine division $P = \{[u,v];\xi\}$ of $[a,b]$, we have

$$\big|\sum h(\xi)(v-u) - (L)\int_a^b h(t)\mathrm{d}t\big| < \varepsilon. \tag{4.7}$$

Define

$$\delta(\xi) = \begin{cases} \delta_h(\xi), & \text{if } \xi \in [a,b]\setminus G, \\ \delta(\xi),\ \text{satisfying } (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i)) \subset G, & \text{if } \xi \in [a,b]. \end{cases}$$

Then, it follows that for a $\delta$-fine division $P_0 = \{[x_{i-1}, x_i];\xi_i\}$ of $[a,b]$,

$$\begin{aligned} & D(\sum \tilde{f}_n(\xi_i)(x_i - x_{i-1}), \sum \tilde{f}_m(\xi_i)(x_i - x_{i-1})) \\ \le\ & D(\sum_{\xi_i\in[a,b]\setminus G} \tilde{f}_n(\xi_i)(x_i - x_{i-1}), \sum_{\xi_i\in[a,b]\setminus G} \tilde{f}_m(\xi_i)(x_i - x_{i-1})) \\ +\ & D(\sum_{\xi_i\in G} \tilde{f}_n(\xi_i)(x_i - x_{i-1}), \sum_{\xi_i\in G} \tilde{f}_m(\xi_i)(x_i - x_{i-1})) \\ \le\ & \sum_{\xi_i\in[a,b]\setminus G} D(\tilde{f}_n(\xi_i), \tilde{f}_m(\xi_i))(x_i - x_{i-1}) + \sum_{\xi_i\in G} D(\tilde{f}_n(\xi_i), \tilde{f}_m(\xi_i))(x_i - x_{i-1}) \\ <\ & \varepsilon(b-a) + \big|\sum_{\xi_i\in G} h(\xi_i)(x_i - x_{i-1}) - \int_G h(t)\mathrm{d}t\big| + \big|\int_G h(t)\mathrm{d}t\big| \\ <\ & \varepsilon(b-a) + 3\varepsilon. \end{aligned}$$

Hence, there is an natural number $N$ such that for any $n, m > N$, we have

$$\begin{aligned} & D(\tilde{F}_n[a,b], \tilde{F}_m[a,b]) \\ \le\ & D(\tilde{F}_n[a,b], \sum \tilde{f}_n(\xi_i)(x_i - x_{i-1})) + D(\tilde{F}_m[a,b], \sum \tilde{f}_m(\xi_i)(x_i - x_{i-1})) \\ +\ & D(\sum \tilde{f}_n(\xi_i)(x_i - x_{i-1}), \sum \tilde{f}_m(\xi_i)(x_i - x_{i-1})) \\ <\ & 3\varepsilon. \end{aligned}$$

Thus, $\tilde{F}_n[a,b]$ is a Cauchy sequence, and there is an natural number $N_1$ such that for any $n > N_1$, we have $D(\tilde{F}_n[a,b], \tilde{A}) < \varepsilon$. According to the $(FH)$ integrability of $\tilde{f}_{N_1}(x)$, there is a $\delta_{N_1}(\xi) > 0$ such that for any $\delta_{N_1}$-fine division $P = \{[u,v];\xi\}$ of $[a,b]$, for any $n > N_{N_1}$, we have

$$\begin{aligned} & D(\sum \tilde{f}_n(\xi)(v-u), \tilde{F}_n[a,b]) \\ \le\ & D(\tilde{F}_n[a,b], \tilde{F}_{N_1}[a,b]) + D(\sum \tilde{f}_{N_1}(\xi)(v-u), \tilde{F}_{N_1}[a,b]) \\ +\ & D(\sum \tilde{f}_n(\xi)(v-u), \sum \tilde{f}_{N_1}(\xi)(v-u)) \\ <\ & 3\varepsilon. \end{aligned}$$

This completes the proof. □

# 5 conclusions

In this paper, we introduced locally and globally small Riemann sums for fuzzy-number-valued functions. We proved that a fuzzy-number-valued functions is $(H)$ integrable on $[a,b]$ iff it has $(LSRS)$. Also it is proved that a fuzzy-number-valued functions is $(H)$ integrable on $[a,b]$ iff it has $(GSRS)$. Finally, by Egorov's Theorem, we obtained the dominated convergence theorem for $(GSRS)$ of fuzzy-number-valued functions.

## References

[1] B. Bede, S.G. Gal, Quadrature rules for integrals of fuzzy-number-valued functions, Fuzzy Sets and Systems, 145, 359-380(2004).

[2] P. Diamond, P.E. Kloeden, Metric spaces of fuzzy sets: theory and applications. World Scientific Publishing, Singapore, 1994.

[3] Y.H. Feng, Fuzzy-valued mappings with finite variation, fuzzy-valued measures and fuzzy-valued Lebesgue-Stieltjes integrals, Fuzzy Sets and Systems, 121, 227-236(2001).

[4] Z.T. Gong, Y.B. Shao, The controlled convergence theorems for the strong Henstock integrals of fuzzy-number-valued functions, Fuzzy Sets and Systems, 160, 1528-1546(2009).

[5] Z.T. Gong, L. Wang, The numerical calculus of expectations of fuzzy random variables, Fuzzy Sets and Systems, 158, 722-738(2007).

[6] Z.T. Gong, L.L. Wang, The Henstock-Stieltjes integral for fuzzy-number-valued functions, Information Sciences, 188, 276-297(2012).

[7] Z.T. Gong, C.X. Wu, The Mcshane Integral of Fuzzy-Valued Functions, Southeast Asian Bulletin of Mathematics, 24(3), 365-373(2000).

[8] M.E. Hamid, Z.T. Gong, The characterizations of McShane integral and Henstock integrals for fuzzy-number-valued functions with a small Riemann sum on a small set, J. of Computational Analysis and Applications, 19(5), 830-836(2015).

[9] R. Goetschel Jr., W. Voxman, Elementary fuzzy calculus, Fuzzy Sets and Systems, 18, 31-43(1986).

[10] R. Henstock, Theory of Integration, Butterworth, London, 1963.

[11] C.R. Indrati, Some Characteristics of the Henstock-Kurzweil in Countably Lipschitz Condition, The 7th SEAMS-UGM Conference 2015.

[12] O. Kaleva, Fuzzy differential equations, Fuzzy Sets Syst, 24, 301-317(1987).

[13] P.Y. Lee, Lanzhou Lectures on Henstock Integration, World Scientific, Singapore, 1989.

[14] M. Ma, Embedding Problem of Fuzzy number Space, Part4 , Fuzzy Sets and Systems, 58, 185-193(1993).

[15] S. Nanda, On fuzzy integrals, Fuzzy Sets and Systems, 32, 95-101(1989).

[16] A.W. Schurle, A new property equivalent to Lebesgue integrability, Proceedings of the American Mathematical Society, 96(1), 103-106(1986) .

[17] A.W. Schurle, A function is Perron integrable if it has locally small Riemann sums, Journal of the Australian Mathematical Society (Series A), 41(2), 224-232(1986).

[18] C.X. Wu, Z.T. Gong, On Henstock integral of fuzzy-number-valued functions (I), Fuzzy Sets and Systems, 120, 523-532(2001).

[19] H.C. Wu, The fuzzy Riemann integral and its numerical integration, Fuzzy Sets and Systems, 110, 1-25(2000).

[20] C.X. Wu, M. Ma, Embedding Problem of Fuzzy number Space, Part(I), Fuzzy Sets and Systems, 44, 33-38(1991).

[21] C.X. Wu, M. Ma, Embedding Problem of Fuzzy number Space, Part(II), Fuzzy Sets and Systems, 45, 189-202(1992).

[22] L.A. Zadeh, Fuzzy sets, Information Control, 8, 338-353(1965).

# THE GENERAL ITERATIVE METHODS FOR SPLIT VARIATIONAL INCLUSION PROBLEM AND FIXED POINT PROBLEM IN HILBERT SPACES

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**Abstract.** In this paper, we consider and analyze a general iterative method to approximate a common solution of split variational inclusion problem and fixed point problem for a nonexpansive mapping, which is the unique solution for the variational inequality in real Hilbert spaces. Furthermore, under reasonable conditions, the sequence generated by the proposed iterative scheme converges strongly to a common solution of split variational inclusion problem and fixed point problem for a nonexpansive mapping, which is a solution of a certain optimization problem related to a strongly positive linear operator. The results presented in this paper improve and extend the corresponding results reported by some authors recently.

**Keywords**: Split variational inclusion problem, Nonexpansive mapping, Fixed point problem, Hilbert space.
**AMS Subject Classification**: 47H10, 47H09.

## 1. Introduction

Throught out the paper unless otherwise stated, let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let $C$ and $Q$ be nonempty closed convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. A mapping $S:\mathcal{H}_1\to\mathcal{H}_1$ is called contraction, if there exists a constant $\alpha\in(0,1)$ such that

$$\|Sx-Sy\|\leq\alpha\|x-y\|,\forall x,y\in\mathcal{H}_1.$$

If $\alpha=1,S$ is called nonexpansive, that is,

$$\|Sx-Sy\|\leq\|x-y\|,\forall x,y\in\mathcal{H}_1.$$

Further, we consider the the following fixed point problem (in short, FPP) for a nonexpansive mapping $S:\mathcal{H}_1\to\mathcal{H}_1$ : Find $x\in\mathcal{H}_1$ such that

$$Sx=x. \tag{1.1}$$

The solution set of FPP (1.1) is denoted by $\mathrm{Fix}(S)$. It is well known that if $\mathrm{Fix}(S)\neq\emptyset$, $\mathrm{Fix}(S)$ is closed and convex. Next, let $T:\mathcal{H}_1\to\mathcal{H}_1$ be a single-valued mapping. We recall the following definitions:

(i) $T$ is said to be monotone, if

$$\langle Tx-Ty,x-y\rangle\geq 0,\ \ \forall x,y\in\mathcal{H}_1.$$

(ii) $T$ is said to be $\alpha$-strongly monotone, if there exists a constant $\alpha>0$ such that

$$\langle Tx-ty,x-y\rangle\geq\alpha\|x-y\|^2,\ \ \forall x,y\in\mathcal{H}_1.$$

(iii) $T$ is said to be $\beta$-inverse strongly monotone(or, $\beta$-ism), if there exists a constant $\beta>0$ such that

$$\langle Tx-ty,x-y\rangle\geq\beta\|Tx-Ty\|^2,\ \ \forall x,y\in\mathcal{H}_1.$$

(iv) $T$ is said to be firmly nonexpansive, if

$$\langle Tx-ty,x-y\rangle\geq\|Tx-Ty\|^2,\ \ \forall x,y\in\mathcal{H}_1.$$

Next, let $M:\mathcal{H}_1\to 2^{\mathcal{H}_1}$ be a multi-valued mappings. We recall the following definitions:

- $M$ is called monotone if for all $x,y\in\mathcal{H}_1,u\in Mx$ and $v\in My$ such that $\langle x-y,u-v\rangle\geq 0$.

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- A monotone mappings $M : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ is maximal if the Graph$(M)$ is not properly contained in the graph of any other monotone mapping.

For more precisely, a monotone mappings $M$ is maximal if and only if for $(x, u) \in \mathcal{H}_1 \times \mathcal{H}_1, \langle x-y, u-v\rangle \geq 0$, for every $(y, v) \in$ Graph$(M)$ implies that $u \in Mx$, where Graph$(M) := \{(x, y) \in \mathcal{H}_1 \times \mathcal{H}_1 : y \in Mx\}$. Let $M : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ be a multi-valued mappings. Then, the resolvent mapping associated with $M$, is defined by

$$J_\lambda^M(x) := (I + \lambda M)^{-1}(x), \ \forall x \in \mathcal{H}_1$$

for some $\lambda > 0$, where $I$ stands identity operator on $\mathcal{H}_1$. We note that for all $\lambda > 0$ the resolvent operator $J_\lambda^M$ is single-valued, nonexpansive and firmly nonexpansive.

For a given single-valued operator $F : \mathcal{H}_1 \to \mathcal{H}_1$, Hartman and Stampacchia [12] introduced the variational inequality problem (in short, VIP) :

$$\text{(VIP)} \quad \begin{cases} \text{Find } x^* \in C \text{ such that} \\ \langle F(x^*), x - x^*\rangle \geq 0, \forall x \in C. \end{cases}$$

The VIP is a powerful tool to investigate and study a wide class of unrelated problems arising in industrial, regional, physical, pure and applied sciences in a unified and general framwork. Variational inequalities have been extended and generallized in several direction using novel and new techniques. The following existence result of solutions for VIP can be found in [12]. Let $\mathcal{H}_1$ be a real Hilbert space, $C$ a nonempty, compact and convex subset $\mathcal{H}_1$. Then, if $F : C \to \mathcal{H}_1$ is continuous, there exists $x^* \in C$ such that

$$\langle F(x^*), x - x^*\rangle \geq 0, \forall x \in C.$$

Recently, in 2011, Moudafi [24] introduced the following split monotone variational inclusion problem (in short, SMVIP):

$$\text{(SMVIP)} \quad \begin{cases} \text{Find } x^* \in \mathcal{H}_1 \text{ such that } 0 \in f_1(x^*) + B_1(x^*), \\ y^* = Ax^* \in \mathcal{H}_2 \text{ solves } 0 \in f_2(y^*) + B_2(y^*). \end{cases}$$

where $B_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ is a multi-valued mappings on a Hilbert space $\mathcal{H}_1$, $B_2 : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ is a multi-valued mappings on a Hilbert space $\mathcal{H}_2$, $A : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator, $f_1 : \mathcal{H}_1 \to \mathcal{H}_1$ and $f_2 : \mathcal{H}_2 \to \mathcal{H}_2$ are two given single-valued operators. If $f_1 \equiv 0$ and $f_2 \equiv 0$, then SMVIP reduces to the following split variational inclusion problem (in short, SVIP) : Find $x^* \in \mathcal{H}_1$ such that

$$0 \in B_1(x^*), \tag{1.2}$$

and

$$y^* = Ax^* \in \mathcal{H}_2 \text{ solves } 0 \in B_2(y^*). \tag{1.3}$$

When looked separately, (1.2) is the variational inclusion problem and we denoted its solution set by SOLVIP$(B_1)$. The SVIP (1.2)-(1.3) constitutes a pair of variational inclusion problems which have to be solved so that the image $y^* = Ax^*$ under a given bounded linear operator $A$, of the solution $x^*$ of SVIP (1.2) in $\mathcal{H}_1$ is the solution of another SVIP (1.3) in another space $\mathcal{H}_2$, we denote the solution set of SVIP (1.3) by SOLVIP$(B_2)$. The solution set of The SVIP (1.2)-(1.3) is denoted by

$$\Gamma : \{x^* \in \mathcal{H}_1 : x^* \in \text{SOLVIP}(B_1) \text{ and } Ax^* \in \text{SOLVIP}(B_2)\}.$$

l,Recently, Byrne et al. [3] studied the weak and strong convergence of the following iterative method for SVIP. For given $x_0 \in \mathcal{H}_1$, compute iterative sequence $\{x_n\}$ generated by the following scheme.

$$x_{n+1} = J_\lambda^{B_1}(x)\Big(x_n + \gamma A^*(J_\lambda^{B_1} - I)Ax_n\Big), \tag{1.4}$$

for $\lambda > 0$ and $A^*$ is the adjoint of $A, L = \|A^*A\|$ and $\gamma \in (0, \frac{2}{L})$. It is proved, in [3], that the sequence $\{x_n\}$ generated by (1.4) converges strongly to $x^*$ which is the solution of SVIP.

Very recently, Kazami and Rizvi [13] studied and analyzed the strong convergence of the iterative method for approximating a common solution of SVIP and FPP for a nonexpansive mapping in a real Hilbert space. Let $g : \mathcal{H}_1 \to \mathcal{H}_1$ be a contraction mapping with constant $\alpha \in (0, 1)$ and $S : \mathcal{H}_1 \to \mathcal{H}_1$ be a nonexpansive mapping. For a given $x_0 \in \mathcal{H}_1$ arbitrarily, let $\{u_n\}$ and $\{x_n\}$ be generated by

$$\begin{aligned} u_n &= J_\lambda^{B_1}\left(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n\right); \\ x_{n+1} &= \alpha_n g(x_n) + (1 - \alpha_n)Su_n, \end{aligned} \tag{1.5}$$

where $\lambda > 0$ and $\gamma \in (0, \frac{1}{L})$, $L$ is the spectral radius of the operator $A^*A$ and $A^*$ is the adjoint of $A$ and $\{\alpha_n\}$ is a sequence in $(0, 1)$. They proved that, under some certain conditions imposed on the parameters $\{\alpha_n\}$, the sequences $\{u_n\}$ and $\{x_n\}$ both converge strongly to $z \in \text{Fix}(S) \cap \Gamma$, where $z = P_{\text{Fix}(S)\cap\Gamma} g(z)$.

On the other hand, iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, e.g., [11, 27, 28, 29] and the references therein. Convex minimization problems have a great impact and influence in the development of almost all branches of pure and applied sciences. A typical problem is to minimize a quadratic function over the set of the fixed points a nonexpansive mapping on a real Hilbert space:

$$\theta(x) = \min_{x \in C} \frac{1}{2}\langle Gx, x\rangle - \langle x, b\rangle, \tag{1.6}$$

where $G$ is a linear bounded operator, $C$ is the fixed point set of a nonexpansive mapping $T$ and $b$ is a given point in $H$. Let $H$ be a real Hilbert space. Recall that a linear bounded operator $B$ is strongly positive if there is a constant $\bar{\gamma} > 0$ with property

$$\langle Gx, x\rangle \geq \bar{\gamma}\|x\|^2 \quad \text{for all } x \in H. \tag{1.7}$$

Marino and Xu [25] introduced the following general iterative scheme basing on the viscosity approximation method introduced by Moudafi [14]:

$$x_{n+1} = (I - \alpha_n G)Tx_n + \alpha_n \beta f(x_n), n \geq 0. \tag{1.8}$$

where $G$ is a strongly positive bounded linear operator on $H$. They proved that if the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.8) converges strongly to the unique solution of the variational inequality

$$\langle (G - \beta f)x^*, x - x^*\rangle \geq 0, x \in C \tag{1.9}$$

which is the optimality condition for the minimization problem

$$\min_{x \in C} \frac{1}{2}\langle Gx, x\rangle - h(x),$$

where $h$ is a potential function for $\beta f (i.e., h'(x) = \beta f(x)$ for $x \in H)$.

Motivated by the work of Kazmi and Rizvi [13] and Moudafi [24] and Marino and Xu [25] and by the ongoing research in this direction, we suggest and analyze a general iterative method for approximating a common solution of SVIP and FPP which solves the variational inequality (1.9). More precisely, let $g : \mathcal{H}_1 \to \mathcal{H}_1$ be a contraction mapping with constant $\alpha \in (0, 1)$, $S : \mathcal{H}_1 \to \mathcal{H}_1$ be a nonexpansive mapping and $G : \mathcal{H}_1 \to \mathcal{H}_1$ be a strongly positive, bounded linear operator with constant $\mu$ and $0 < \beta < \frac{\mu}{\alpha}$. For a given $x_0 \in \mathcal{H}_1$ arbitrarily, let $\{u_n\}$ and $\{x_n\}$ generated by

$$\begin{aligned} u_n &= J_\lambda^{B_1}\left(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n\right); \\ x_{n+1} &= \alpha_n \beta f(x_n) + (I - \alpha_n G)Su_n, \end{aligned} \tag{1.10}$$

where $\lambda > 0$ and $\gamma \in (0, \frac{1}{L})$, $L$ is the spectral radius of the operator $A^*A$ and $A^*$ is the adjoint of $A$ and $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $B_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$, $B_2 : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ two multi-valued mappings on $\mathcal{H}_1$, and $\mathcal{H}_2$, respectively. We prove that the iterative method (1.10) converges strongly to a common element of SVIP and FPP for a nonexpansive mapping, which is a solution of a certain optimization problem related to a strongly positive linear operator. The result presented in this paper generalize the corresponding results of Kazmi and Rizvi [13] and Moudafi [24], Marino and Xu [25] and many others.

## 2. Preliminaries

For a real Hilbert space $\mathcal{H}_1$ with the norm $\|\cdot\|$ and the inner product $\langle\cdot,\cdot\rangle$, it is well known that for any $\lambda \in (0, 1)$,

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2, \ \forall x, y \in \mathcal{H}_1. \tag{2.1}$$

Further, every nonexpansive operator $T : \mathcal{H}_1 \to \mathcal{H}_1$ satisfies, for all $(x, y) \in \mathcal{H}_1 \times \mathcal{H}_1$, the inequality

$$\langle (x - T(x)) - (y - T(y)), T(y) - T(x)\rangle \leq \frac{1}{2}\|(T(x) - x) - (T(y) - y)\|^2 \tag{2.2}$$

and therefore, we get, for all $(x, y) \in \mathcal{H}_1 \times \text{Fix}(T)$,

$$\langle (x - T(x), (y - T(x)\rangle \leq \frac{1}{2}\|T(x) - x\|^2. \tag{2.3}$$

A mapping $T : \mathcal{H}_1 \to \mathcal{H}_1$ is said to be averaged if and only if it can be written as the average of the identity mapping and a nonexpansive mapping, i.e.,

$$T := (1 - \alpha)I + \alpha S$$

where $\alpha \in (0, 1)$ and $S : \mathcal{H}_1 \to \mathcal{H}_1$ is nonexpansive and $I$ is the identity operator on $\mathcal{H}_1$

**Proposition 2.1.** *(i) If $T = (1 - \alpha)S + \alpha V$, where $S : \mathcal{H}_1 \to \mathcal{H}_1$ is averaged, $V : \mathcal{H}_1 \to \mathcal{H}_1$ is nonexpansive and $\alpha \in (0, 1)$, then $T$ is averaged.*
*(ii) The composite of finitely many averaged mapping is averaged.*
*(iii) If the mapping are averaged and have a nonempty common fixed point, then*

$$\bigcap_{i=1}^{N} Fix(T_i) = Fix(T_1, T_2, \ldots, T_N).$$

*(iv) If $T$ is $\tau - ism$, then for $\gamma > 0, \gamma T$ is $\frac{\tau}{\gamma} - ism$*

*(v) $T$ is averaged if and only if , its complement $I - T$ is $\tau - ism$ for some $\tau > \frac{1}{2}$*

For every point $x \in \mathcal{H}_1$, there exists a unique nearest point in $C$, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \ \ \forall y \in C.$$

$P_C$ is called the (nearest point or metric) projection of $\mathcal{H}_1$ onto $C$. In addition, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \tag{2.4}$$

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \ \ \forall x \in \mathcal{H}_1, y \in C. \tag{2.5}$$

Recall that a mapping $T : \mathcal{H}_1 \to \mathcal{H}_1$ is said to be firmly nonexpansive mapping if

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle, \ \ \forall x, y \in \mathcal{H}_1.$$

It is well known that $P_C$ is a firmly nonexpansive mapping of $\mathcal{H}_1$ onto $C$ and satisfies

$$\|P_C x - P_C y\|^2 \leq \langle x - y, P_C x - P_C y\rangle, \ \ \forall x, y \in \mathcal{H}_1. \tag{2.6}$$

If $G$ an $\alpha-$inverse-strongly monotone mapping of $C$ into $\mathcal{H}_1$, then it is obvious that $G$ is $\frac{1}{\alpha}-Lipschitz$ continuous. We also have that for all $x, y \in C$ and $\lambda > 0$,

$$\begin{aligned} \|(I - \lambda G)x - (I - \lambda G)y\|^2 &= \|x - y - \lambda(Gx - Gy)\|^2 \\ &= \|x - y\|^2 - 2\lambda\langle Gx - Gy, x - y\rangle + \lambda^2\|Gx - Gy\|^2 \\ &\leq \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Gx - Gy\|^2 \end{aligned} \tag{2.7}$$

So, if $\lambda \leq 2\alpha$, then $I - \lambda G$ is a nonexpansive mapping of $C$ into $\mathcal{H}_1$.

Next, we denote weak convergence and strong convergence by notations $\rightharpoonup$ and $\to$, respectively. A space $X$ is said to satisfy Opials condition [31] if for each sequence $\{x_n\}$ in $X$ which converges weakly to a point $x \in X$, we have

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \ \ \forall y \in X, y \neq x.$$

**Lemma 2.2.** [25] *Let $\mathcal{H}_1$ be a Hilbert space, $C$ be a nonempty closed convex subset of $H$, and $f : \mathcal{H}_1 \to \mathcal{H}_1$ be a contraction with coefficient $0 < \alpha < 1$, and $G$ be a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$. Then, for $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$,*

$$\langle x - y, (G - \gamma f)x - (G - \gamma f)y\rangle \geq (\bar{\gamma} - \gamma\alpha)\|x - y\|^2, \ \ x, y \in \mathcal{H}_1.$$

*That is, $G - \gamma f$ is strongly monotone with coefficient $\bar{\gamma} - \gamma\alpha$.*

**Lemma 2.3.** [25] *Assume $G$ is a strongly positive linear bounded operator on a Hilbert space $\mathcal{H}_1$ with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|G\|^{-1}$. Then $\|I - \rho G\| \leq 1 - \rho\bar{\gamma}$.*

**Lemma 2.4.** [23] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space $X$ and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

**Lemma 2.5.** [31] *Let $\mathcal{H}_1$ be a Hilbert space, $C$ a closed convex subset of $\mathcal{H}_1$, and $S : C \to C$ a nonexpansive mapping with $F(S) \neq \emptyset$. If $\{x_n\}$ is a sequence in $C$ weakly converging to $x \in C$ and if $\{(I-S)x_n\}$ converges strongly to $y$, then $(I-S)x = y$; in particular, if $y = 0$, then $x \in Fix(S)$.*

**Lemma 2.6.** [26] *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\sigma_n)a_n + \delta_n,$$

*where $\{\sigma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that*

(1) $\sum_{n=1}^{\infty} \sigma_n = \infty$;

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\sigma_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n \sigma_n| < \infty$.

*Then $\lim_{n\to\infty} \alpha_n = 0$.*

## 3. Main Results

In this section, we prove a strong convergence theorem for the general iterative methods for approximating the common element of SVIP and FPP which is the unique solution for the variational inequality (1.9). First, we have the following technical lemma, which is immediately consequence of the definition of resolvent mapping:

**Lemma 3.1.** *SVIP is equivalent to find $x^* \in \mathcal{H}_1$ such that $y^* = Ax^* \in \mathcal{H}_2$, $x^* = J_\lambda^{B_1}(x^*)$ and $y^* = J_\lambda^{B_2}(y^*)$ for some $\lambda > 0$.*

**Theorem 3.2.** *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces. Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. Assume that $B_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ and $B_2 : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ are maximal monotone mappings. Let $S : \mathcal{H}_1 \to \mathcal{H}_1$ be a nonexpansive mapping such that $Fix(S) \cap \Gamma \neq \emptyset$. Let $f : \mathcal{H}_1 \to \mathcal{H}_1$ be a contraction mapping with constant $\alpha \in (0,1)$ and $G : \mathcal{H}_1 \to \mathcal{H}_1$ a strongly positive, bounded linear operator with constant $\mu$ such that $\|G\| = 1$, and $0 < \beta < \frac{\mu}{\alpha}$. For given $\forall x_0 \in \mathcal{H}_1$, let the sequences $\{u_n\}$ and $\{x_n\}$ be generated by (1.10), where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying the following conditions :*

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and*

(iii) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$.

*Then the sequences $\{u_n\}$ and $\{x_n\}$ both converge strongly to $z \in Fix(S) \cap \Gamma$, where $z = P_{Fix(S)\cap\Gamma}(I - G + \beta f)(z)$. Moreover, $z$ is a unique solution of the variational inequality (1.9).*

**Proof** We observe that $P_{\mathrm{Fix}(S)\cap\Gamma}(I - G + \beta f)$ is a contraction. Indeed, applying Lemma 2.3 with $\|G\| = 1$, we have,

$$\begin{aligned}
\|P_{\mathrm{Fix}(S)\cap\Gamma}(I - G + \beta f)(x) - P_{\mathrm{Fix}(S)\cap\Gamma}(I - G + \beta f)(y)\| &\leq \|(\beta f + (I-G))(x) - (\beta f + (I-G))(y)\| \\
&\leq \beta\|f(x) - f(y)\| + \|I - G\|\|x - y\| \\
&\leq \gamma\alpha\|x - y\| + (1 - \overline{\gamma})\|x - y\| \\
&\leq (1 - (\overline{\gamma} - \alpha\beta))\,\|x - y\|,
\end{aligned}$$

for all $x, y \in \mathcal{H}_1$. Therefore, Banach's Contraction Mapping Principle guarantees that $P_{\mathrm{Fix}(S)\cap\Gamma}(I - G + \beta f)$ has a unique fixed point, say $z \in \mathcal{H}_1$. That is, $z = Q_F(\gamma f + (I-G))(z)$. Next, we devide the proof into five steps as follows.

**Step 1.** We first show that the sequences $\{x_n\}$ is bounded. Let $p \in Fix(S) \cap \Gamma$, then we have that $p = J_\lambda^{B_1} p$, $Ap = J_\lambda^{B_2}(Ap)$ and $Sp = p$. So, we have

$$\begin{aligned}
\|u_n - p\|^2 &= \left\| J_\lambda^{B_1}\left(x_n + \gamma A^*\left(J_\lambda^{B_2} - I\right)Ax_n\right) - p \right\|^2 \\
&= \left\| J_\lambda^{B_1}\left(x_n + \gamma A^*\left(J_\lambda^{B_2} - I\right)Ax_n\right) - J_\lambda^{B_1} p \right\|^2 \\
&\leq \left\| x_n + \gamma A^*\left(J_\lambda^{B_2} - I\right)Ax_n - p \right\|^2
\end{aligned}$$

$$\leq \|x_n - p\|^2 + \gamma^2 \Big\| A^* \Big( J_\lambda^{B_2} - I \Big) A x_n \Big\|^2 + 2\gamma \Big\langle x_n - p, A^* \Big( J_\lambda^{B_2} - I \Big) A x_n \Big\rangle. \quad (3.1)$$

It follow that

$$\begin{aligned} \|u_n - p\|^2 &\leq \|x_n - p\|^2 + \gamma^2 \Big\langle \Big( J_\lambda^{B_2} - I \Big) A x_n, A A^* \Big( J_\lambda^{B_2} - I \Big) A x_n \Big\rangle \\ &\quad + 2\gamma \Big\langle x_n - p, A^* \Big( J_\lambda^{B_2} - I \Big) A x_n \Big\rangle. \end{aligned} \quad (3.2)$$

Since

$$\begin{aligned} \gamma^2 \Big\langle \Big( J_\lambda^{B_2} - I \Big) A x_n, A A^* \Big( J_\lambda^{B_2} - I \Big) A x_n \Big\rangle &\leq L\gamma^2 \Big\langle \Big( J_\lambda^{B_2} - I \Big) A x_n, \Big( J_\lambda^{B_2} - I \Big) A x_n \Big\rangle \\ &= L\gamma^2 \Big\| \Big( J_\lambda^{B_2} - I \Big) A x_n \Big\|^2 \end{aligned} \quad (3.3)$$

and using (2.3), we have

$$\begin{aligned} 2\gamma \Big\langle x_n - p, A^* \Big( J_\lambda^{B_2} - I \Big) A x_n \Big\rangle &= 2\gamma \Big\langle A(x_n - p), \Big( J_\lambda^{B_2} - I \Big) A x_n \Big\rangle \\ &= 2\gamma \Big\langle A(x_n - p) + \Big( J_\lambda^{B_2} - I \Big) A x_n - \Big( J_\lambda^{B_2} - I \Big) A x_n, \Big( J_\lambda^{B_2} - I \Big) A x_n \Big\rangle \\ &= 2\gamma \left\{ \Big\langle J_\lambda^{B_2} A x_n - A p, \Big( J_\lambda^{B_2} - I \Big) A x_n \Big\rangle - \Big\| \Big( J_\lambda^{B_2} - I \Big) A x_n \Big\|^2 \right\} \\ &\leq 2\gamma \left\{ \frac{1}{2} \Big\| \Big( J_\lambda^{B_2} - I \Big) A x_n \Big\|^2 - \Big\| \Big( J_\lambda^{B_2} - I \Big) A x_n \Big\|^2 \right\} \\ &\leq -\gamma \Big\| \Big( J_\lambda^{B_2} - I \Big) A x_n \Big\|^2. \end{aligned} \quad (3.4)$$

From the inequalities (3.2), (3.3) and (3.4), we can conclude that

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 + \gamma(L\gamma - 1) \Big\| \Big( J_\lambda^{B_2} - I \Big) A x_n \Big\|^2. \quad (3.5)$$

Since $\gamma \in (0, \frac{1}{L})$, we obtain

$$\|u_n - p\|^2 \leq \|x_n - p\|^2,$$

which implies that

$$\|u_n - p\| \leq \|x_n - p\|. \quad (3.6)$$

Therefore

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n \beta f(x_n) + (I - \alpha_n G) S u_n - p\| \\ &\leq \beta\alpha_n \|f(x_n) - f(p)\| + \|(I - \alpha_n G)\| \|S u_n - p\| + \alpha_n \|\beta f(p) - G p\| \\ &\leq \beta\alpha_n \alpha \|x_n - p\| + (1 - \alpha_n \mu) \|u_n - p\| + \alpha_n \|\beta f(p) - G p\| \\ &\leq \beta\alpha_n \alpha \|x_n - p\| + (1 - \alpha_n \mu) \|x_n - p\| + \alpha_n \|\beta f(p) - G p\| \\ &= \big( \beta\alpha_n \alpha + (1 - \alpha_n \mu) \big) \|x_n - p\| + \alpha_n \|\beta f(p) - G p\| \\ &= \big( 1 - \alpha_n (\mu - \beta\alpha) \big) \|x_n - p\| + \alpha_n (\mu - \beta\alpha) \frac{\|\beta f(p) - G p\|}{(\mu - \beta\alpha)} \\ &\leq \max \left\{ \|x_n - p\|, \frac{\|\beta f(p) - G p\|}{\mu - \beta\alpha} \right\}. \end{aligned}$$

By induction, we have

$$\|x_n - p\| \leq \max \left\{ \|x_1 - p\|, \frac{\|\beta f(p) - G p\|}{\mu - \beta\alpha} \right\}, \ \forall n \geq 1. \quad (3.7)$$

Hence $\{x_n\}$ is bounded and consequently, we deduce that $\{u_n\}, \{f(x_n)\}$ and $\{S u_n\}$ are bounded.

**Step 2.** We show that the sequences $\{x_n\}$ is asymptotically regular, i.e.,

$$\|x_{n+1} - x_n\| \to 0 \text{ as } n \to \infty.$$

For each $n \in \mathbb{N}$, we notice that

$$\begin{aligned}
\|x_{n+1}-x_n\| &= \|\alpha_n\beta f(x_n)+(I-\alpha_n G)Su_n-\big(\alpha_{n-1}\beta f(x_{n-1})+(I-\alpha_{n-1}G)Su_{n-1}\big)\| \\
&\leq \|(I-\alpha_n G)(Su_n-Su_{n-1})-(\alpha_n-\alpha_{n-1})GSu_{n-1} \\
&\quad +\beta\alpha_n\big(f(x_n)-f(x_{n-1})\big)+\beta(\alpha_n-\alpha_{n-1})f(x_{n-1})\| \\
&\leq (1-\alpha_n\mu)\|Su_n-Su_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|GSu_{n-1}\| \\
&\quad +\beta\alpha_n\|(f(x_n)-f(x_{n-1})\|+\beta|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})\| \\
&\leq (1-\alpha_n\mu)\|u_n-u_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|GSu_{n-1}\| \\
&\quad +\beta\alpha\alpha_n\|x_n-x_{n-1}\|+\beta|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})\| \\
&= (1-\alpha_n\mu)\|u_n-u_{n-1}\|+\beta\alpha\alpha_n\|x_n-x_{n-1}\| \\
&\quad +|\alpha_n-\alpha_{n-1}|\big(\|Gu_{n-1}\|+\beta\|f(x_{n-1})\|\big) \\
&\leq (1-\alpha_n\mu)\|u_n-u_{n-1}\|+\beta\alpha\alpha_n\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|K \qquad (3.8)
\end{aligned}$$

where $K=\sup\big\{\|Gu_{n-1}\|+\beta\|f(x_{n-1})\| : n\in\mathbb{N}\big\}$. Since, for $\gamma\in(0,\frac{1}{L})$, the mapping $J_\lambda^{B_1}(I+\gamma A^*(J_\lambda^{B_2}-I)A)$ is averaged and hence nonexpansive, therefore

$$\begin{aligned}
\|u_n-u_{n-1}\| &= \Big\|J_\lambda^{B_1}\big(x_n+\gamma A^*\Big(J_\lambda^{B_2}-I\Big)Ax_n\big)-J_\lambda^{B_1}\big(x_{n-1}+\gamma A^*\Big(J_\lambda^{B_2}-I\Big)Ax_{n-1}\big)\Big\| \\
&\leq \Big\|J_\lambda^{B_1}\big(I+\gamma A^*\Big(J_\lambda^{B_2}-I\Big)A\big)x_n-J_\lambda^{B_1}\big(I+\gamma A^*\Big(J_\lambda^{B_2}-I\Big)A\big)x_{n-1}\Big\| \\
&\leq \|x_n-x_{n-1}\|. \qquad (3.9)
\end{aligned}$$

It follows from (3.8) and (3.9) that

$$\|x_{n+1}-x_n\|\leq\big(1-\alpha_n(\mu-\beta\alpha)\big)\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|K.$$

By applying Lemma 2.6 with $\beta_n=\alpha_n(\mu-\beta\alpha)$ and $\delta_n=|\alpha_n-\alpha_{n-1}|K$, we obtain

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \qquad (3.10)$$

**Step 3.** We show that

$$\|x_{n+1}-p\|\to 0 \text{ as } n\to\infty.$$

For each $n\in\mathbb{N}$,

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\alpha_n\beta f(x_n)+(I-\alpha_n G)Su_n-p\|^2 \\
&= \|\alpha_n(\beta f(x_n)-Gp)+(I-\alpha_n G)(Su_n-p)\|^2 \\
&\leq \|(I-\alpha_n G)(Su_n-p)\|^2+2\alpha_n\langle\beta f(x_n)-Gp,x_{n+1}-p\rangle \\
&\leq (1-\alpha_n\mu)^2\|Su_n-p\|^2+2\alpha_n\beta\langle f(x_n)-f(p),x_{n+1}-p\rangle \\
&\quad +2\alpha_n\langle\beta f(x_n)-Gp,x_{n+1}-p\rangle \\
&\leq (1-\alpha_n\mu)^2\|u_n-p\|^2+2\alpha_n\beta\langle f(x_n)-f(p),x_{n+1}-p\rangle \\
&\quad +2\alpha_n\langle\beta f(p)-Gp,x_{n+1}-p\rangle \\
&\leq (1-\alpha_n\mu)^2\|u_n-p\|^2+2\alpha_n\beta\alpha\|x_n-p\|\|x_{n+1}-p\| \\
&\quad +2\alpha_n\|\beta f(p)-Gp\|\|x_{n+1}-p\|. \qquad (3.11)
\end{aligned}$$

Thus, from (3.5), we obtain

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq (1-\alpha_n\mu)^2\left\{\|x_n-p\|^2+\gamma(L\gamma-1)\Big\|\Big(J_\lambda^{B_2}-I\Big)Ax_n\Big\|^2\right\} \\
&\quad +2\alpha_n\beta\alpha\|x_n-p\|\|x_{n+1}-p\| \\
&\quad +2\alpha_n\|\beta f(p)-Gp\|\|x_{n+1}-p\| \\
&= \big(1-2\alpha_n\mu+(\alpha_n\mu)^2\big)\|x_n-p\|^2+(1-\alpha_n\mu)^2\Big(\gamma(L\gamma-1)\Big\|\Big(J_\lambda^{B_2}-I\Big)Ax_n\Big\|^2\Big) \\
&\quad +2\alpha_n\beta\alpha\|x_n-p\|\|x_{n+1}-p\| \\
&\quad +2\alpha_n\|\beta f(p)-Gp\|\|x_{n+1}-p\| \\
&\leq \|x_n-p\|^2+\alpha_n\mu^2\|x_n-p\|^2-\Big(1-\alpha_n\mu)^2(\gamma(1-L\gamma)\Big\|\Big(J_\lambda^{B_2}-I\Big)Ax_n\Big\|^2\Big)
\end{aligned}$$

$$+2\alpha_n\beta\alpha\|x_n-p\|\|x_{n+1}-p\|$$
$$+2\alpha_n\|\beta f(p)-Gp\|\|x_{n+1}-p\|. \tag{3.12}$$

Therefore,

$$\begin{aligned}(1-\alpha_n\mu)^2(\gamma(1-L\gamma)\left\|\left(J_\lambda^{B_2}-I\right)Ax_n\right\|^2 &\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n\mu^2\|x_n-p\|^2\\ &\quad+2\alpha_n\beta\alpha\|x_n-p\|\|x_{n+1}-p\|\\ &\quad+2\alpha_n\|\beta f(p)-Gp\|\|x_{n+1}-p\|\\ &\le \|x_{n+1}-x_n\|\big(\|x_n-p\|+\|x_{n+1}-p\|\big)+\alpha_n\mu^2\|x_n-p\|^2\\ &\quad+2\alpha_n\beta\alpha\|x_n-p\|\|x_{n+1}-p\|\\ &\quad+2\alpha_n\|\beta f(p)-Gp\|\|x_{n+1}-p\|.\end{aligned}$$

Since $\gamma(1-L\gamma)>0$, and $\alpha_n\to 0$ and $\|x_{n+1}-x_n\|\to 0$ as $n\to\infty$, we have

$$\lim_{n\to\infty}\left\|\left(J_\lambda^{B_2}-I\right)Ax_n\right\|=0. \tag{3.13}$$

Furthermore, using (3.7), (3.11) and $\gamma\in(0,\frac{1}{L})$, we notice that

$$\begin{aligned}\|u_n-p\|^2&=\left\|J_\lambda^{B_1}\big(x_n+\gamma A^*\left(J_\lambda^{B_2}-I\right)Ax_n\big)-p\right\|^2\\ &=\left\|J_\lambda^{B_1}\big(x_n+\gamma A^*\left(J_\lambda^{B_2}-I\right)Ax_n\big)-J_\lambda^{B_1}p\right\|^2\\ &\le\left\langle u_n-p,x_n+\gamma A^*\left(J_\lambda^{B_2}-I\right)Ax_n\big)-p\right\rangle\\ &=\frac{1}{2}\Big\{\|u_n-p\|^2+\|x_n+\gamma A^*\left(J_\lambda^{B_2}-I\right)Ax_n-p\|^2\\ &\qquad-\|(u_n-p)-\left(x_n+\gamma A^*\left(J_\lambda^{B_2}-I\right)Ax_n-p\right)\Big\}\\ &=\frac{1}{2}\Big\{\|u_n-p\|^2+\|x_n-p\|^2+\gamma(L\gamma-1)\left\|\left(J_\lambda^{B_2}-I\right)Ax_n\right\|^2\\ &\qquad-\|(u_n-x_n)-\gamma A^*\left(J_\lambda^{B_2}-I\right)Ax_n\|^2\Big\}\\ &\le\frac{1}{2}\Big\{\|u_n-p\|^2+\|x_n-p\|^2-\Big[\|u_n-x_n\|^2\\ &\qquad+\gamma^2\|A^*\left(J_\lambda^{B_2}-I\right)Ax_n\|^2-2\gamma\langle u_n-x_n,A^*\left(J_\lambda^{B_2}-I\right)Ax_n\rangle\Big]\Big\}\\ &\le\frac{1}{2}\Big\{\|u_n-p\|^2+\|x_n-p\|^2-\|u_n-x_n\|^2\\ &\qquad+2\gamma\|A(u_n-x_n)\|\|(J_\lambda^{B_2}-I)Ax_n\|\Big\}.\end{aligned}$$

Thus, we obtain

$$\|u_n-p\|^2\le\|x_n-p\|^2-\|u_n-x_n\|^2+2\gamma\|A(u_n-x_n)\|\|(J_\lambda^{B_2}-I)Ax_n\|. \tag{3.14}$$

It follows from (3.11) and (3.14) that

$$\begin{aligned}\|x_{n+1}-p\|^2&\le(1-\alpha_n\mu)^2\Big[\|x_n-p\|^2-\|u_n-x_n\|^2+2\gamma\|A(u_n-x_n)\|\|(J_\lambda^{B_2}-I)Ax_n\|\Big]\\ &\quad+2\alpha_n\beta\alpha\|x_n-p\|\|x_{n+1}-p\|\\ &\quad+2\alpha_n\|\beta f(p)-Gp\|\|x_{n+1}-p\|\\ &=(1-2\alpha_n\mu+(\alpha_n\mu)^2\|x_n-p\|^2-(1-\alpha_n\mu)^2\|u_n-x_n\|^2\\ &\quad+(1-\alpha_n\mu)^2 2\gamma\|A(u_n-x_n)\|\|(J_\lambda^{B_2}-I)Ax_n\|\\ &\quad+2\alpha_n\beta\alpha\|x_n-p\|\|x_{n+1}-p\|\\ &\quad+2\alpha_n\|\beta f(p)-Gp\|\|x_{n+1}-p\|\\ &\le\|x_n-p\|^2+\alpha_n\mu^2\|x_n-p\|^2-(1-\alpha_n\mu)^2\|u_n-x_n\|^2\\ &\quad+(1-\alpha_n\mu)^2 2\gamma\|A(u_n-x_n)\|\|(J_\lambda^{B_2}-I)Ax_n\|\end{aligned}$$

$$
\begin{aligned}
&+ 2\alpha_n\beta\alpha\|x_n - p\|\|x_{n+1} - p\| \\
&+ 2\alpha_n\|\beta f(p) - Gp\|\|x_{n+1} - p\|.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
(1-\alpha_n\mu)^2\|u_n - x_n\|^2 &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n\mu^2\|x_n - p\|^2 \\
&\quad + (1-\alpha_n\mu)^2 2\gamma\|A(u_n - x_n)\|\|(J_\lambda^{B_2} - I)Ax_n\| \\
&\quad + 2\alpha_n\beta\alpha\|x_n - p\|\|x_{n+1} - p\| \\
&\quad + 2\alpha_n\|\beta f(p) - Gp\|\|x_{n+1} - p\| \\
&\leq \|x_{n+1} - x_n\|\big(\|x_n - p\| + \|x_{n+1} - p\|\big) + \alpha_n\mu^2\|x_n - p\|^2 \\
&\quad + (1-\alpha_n\mu)^2 2\gamma\|A(u_n - x_n)\|\|(J_\lambda^{B_2} - I)Ax_n\| \\
&\quad + 2\alpha_n\beta\alpha\|x_n - p\|\|x_{n+1} - p\| \\
&\quad + 2\alpha_n\|\beta f(p) - Gp\|\|x_{n+1} - p\|.
\end{aligned}
$$

Since $\alpha_n \to 0$ as $n \to \infty$, and from (3.10) and (3.13), we obtain

$$\lim_{n\to\infty}\|u_n - x_n\| = 0. \tag{3.15}$$

Since $\|Su_n - u_n\| \leq \|Su_n - x_n\| + \|x_n - u_n\|$, it follows that

$$\|Su_n - u_n\| \to 0 \text{ as } n \to \infty. \tag{3.16}$$

**Step 4.** We will show that

$$\limsup_{n\to\infty}\langle (G-\beta f)z, x_n - z\rangle \leq 0, \text{where } z = P_{\mathrm{Fix}(S)\cap\Gamma}(I - G + \beta f)(z).$$

Since $\{u_n\}$ is bounded, we consider a weak cluster point $w$ of $\{u_n\}$. Hence, there exists a subsequence $\{u_{n_i}\}$ of $\{u_n\}$, which converges weakly to $w$. Now, $S$ being nonexpansive, by (3.16) and Lemma 2.5, we obtain that $w \in \mathrm{Fix}(S)$. On the other hand, $u_{n_i} = J_\lambda^{B_1}\left(x_{n_i} + \gamma A^*(J_\lambda^{B_2} - I)Ax_{n_i}\right)$ can be rewritten as

$$\frac{(x_{n_i} - u_{n_i}) + A^*(J_\lambda^{B_2} - I)Ax_{n_i}}{\lambda} \in B_1 u_{n_i}. \tag{3.17}$$

By passing to limit $i \to \infty$ in (3.17) and by taking into account (3.13), (3.15) and the fact that the graph of a maximal monotone operator is weakly-trongly closed, we obtain $0 \in B_1(w)$. Furthermore, since $\{u_n\}$ and $\{x_n\}$ have the same asymptoical behavior, $\{Ax_{n_i}\}$ weakly converges to $Aw$. Again, by (3.13) and the fact that the resolvent $J_\lambda^{B_2}$ is nonexpansive and Lemma 3.1, we obtain that $Aw \in B_2(Aw)$. Thus $w \in \mathrm{Fix}(S) \cap \Gamma$. Since $z = P_{\mathrm{Fix}(S)\cap\Gamma}(I - G + \beta f)(z)$. Indeed, we have

$$
\begin{aligned}
\limsup_{n\to\infty}\langle (G-\beta f)z, z - x_n\rangle &= \lim_{i\to\infty}\langle (G-\beta f)z, z - x_{n_i}\rangle \\
&= \langle (G-\beta f)z, z - w\rangle \leq 0.
\end{aligned} \tag{3.18}
$$

**Step 5.** Finally, we will show that $x_n \to z$ as $n \to \infty$. We have

$$
\begin{aligned}
\|x_{n+1} - z\|^2 &= \|\alpha_n\beta f(x_n) + (I - \alpha_n G)Su_n - z\|^2 \\
&= \|\alpha_n(\beta f(x_n) - Gz) + (I - \alpha_n G)(Su_n - z)\|^2 \\
&\leq \|(I - \alpha_n G)(Su_n - z)\|^2 + 2\alpha_n\langle \beta f(x_n) - Gz, x_{n+1} - z\rangle \\
&\leq (1-\alpha_n\mu)^2\|u_n - z\|^2 + 2\alpha_n\beta\langle f(x_n) - f(z), x_{n+1} - z\rangle \\
&\quad + 2\alpha_n\langle \beta f(x_n) - Gz, x_{n+1} - z\rangle \\
&\leq (1-\alpha_n\mu)^2\|x_n - z\|^2 + 2\alpha_n\beta\langle f(x_n) - f(z), x_{n+1} - z\rangle \\
&\quad + 2\alpha_n\langle \beta f(z) - Gz, x_{n+1} - z\rangle \\
&\leq (1-\alpha_n\mu)^2\|x_n - z\|^2 + 2\alpha_n\beta\alpha\|x_n - z\|\|x_{n+1} - z\| \\
&\quad + 2\alpha_n\|\beta f(z) - Gz\|\|x_{n+1} - z\| \\
&\leq (1-\alpha_n\mu)^2\|x_n - z\|^2 + \alpha_n\beta\alpha\big[\|x_n - z\|^2 + \|x_{n+1} - z\|^2\big] \\
&\quad + 2\alpha_n\|\beta f(z) - Gz\|\|x_{n+1} - z\|
\end{aligned}
$$

$$\begin{aligned}
&\leq \big((1-\alpha_n\mu)^2+\alpha_n\beta\alpha\big)\|x_n-z\|^2+\alpha_n\beta\alpha\|x_n-z\|^2+\|x_{n+1}-z\|^2\\
&\quad+2\alpha_n\|\beta f(z)-Gz\|\|x_{n+1}-z\|,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|x_{n+1}-z\|^2 &\leq \frac{1-2\alpha_n\mu+(\alpha_n\mu)^2+\alpha_n\beta\alpha}{1-\alpha_n\beta\alpha}\|x_n-z\|^2\\
&\quad+\frac{2\alpha_n}{1-\alpha_n\beta\alpha}\langle\beta f(z)-Gz,x_{n+1}-z\rangle\\
&=\left[1-\frac{2(\mu-\beta\alpha)\alpha_n}{1-\alpha_n\beta\alpha}\right]\|x_n-z\|^2+\frac{(\alpha_n\mu)^2}{1-\alpha_n\beta\alpha}\|x_n-z\|^2\\
&\quad+\frac{2\alpha_n}{1-\alpha_n\beta\alpha}\langle\beta f(z)-Gz,x_{n+1}-z\rangle\\
&\leq\left[1-\frac{2(\mu-\beta\alpha)\alpha_n}{1-\alpha_n\beta\alpha}\right]\|x_n-z\|^2\\
&\quad+\frac{2(\mu-\beta\alpha)\alpha_n}{1-\alpha_n\beta\alpha}\left[\frac{(\alpha_n\mu^2)M}{2(\mu-\beta\alpha)}+\frac{1}{\mu-\beta\alpha}\langle\beta f(z)-Gz,x_{n+1}-z\rangle\right]\\
&=(1-\sigma_n)\|x_n-z\|^2+\sigma_n\delta_n,
\end{aligned}$$

where $M=\sup\{\|x_n-z\|^2:n\in\mathbb{N}\}, \sigma_n=\dfrac{2(\mu-\beta\alpha)\alpha_n}{1-\alpha_n\beta\alpha}$ and $\delta_n=\dfrac{(\alpha_n\mu^2)M}{2(\mu-\beta\alpha)}+\dfrac{1}{\mu-\beta\alpha}\langle\beta f(z)-Gz,x_{n+1}-z\rangle$. It is easily to see that $\sigma_n\to 0,\sum_{n=1}^{\infty}\sigma_n=\infty$ and $\limsup_{n\to\infty}\dfrac{\delta_n}{\sigma_n}\leq 0$ by (3.18). Thus, by Lemma 2.6, we deduce that $x_n\to z$ as $n\to\infty$. Further it follows from $\|u_n-x_n\|\to 0, u_n\rightharpoonup w\in \mathrm{Fix}(S)\cap\Gamma$ and $x_n\to z$ as $n\to\infty$, that is $z=w$. This completes the proof. □

**Remark 3.3.** In general case, if $G$ is any strongly positive bounded linear operator with coefficient $\bar{\gamma}$ and $0<\gamma<\dfrac{\mu}{\alpha}$. We define a bounded linear operator $\overline{G}$ on $E$ by

$$\overline{G}=\|G\|^{-1}G.$$

It is easy to see that $\overline{G}$ is a strongly positive with coefficient $\|G\|^{-1}\mu>0$ such that $\|\overline{G}\|=1$ and

$$0<\|G\|^{-1}\gamma<\frac{\|G\|^{-1}\mu}{\alpha}.$$

Let the sequence $\{x_n\}$ be defined by, for any $x_0\in E$,

$$\begin{aligned}
u_n &= J_{\lambda}^{B_1}\left(x_n+\gamma A^*(J_{\lambda}^{B_2}-I)Ax_n\right);\\
x_{n+1} &= \alpha_n\beta\|G\|^{-1}f(x_n)+(I-\alpha_n\bar{G})Su_n,
\end{aligned}\tag{3.19}$$

Replacing $G$ with $\overline{G}$ in Theorem 3.2, we obtain the following result.

**Theorem 3.4.** *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces. Let $A:\mathcal{H}_1\to\mathcal{H}_2$ be a bounded linear operator. Assume that $B_1:\mathcal{H}_1\to 2^{\mathcal{H}_1}$ and $B_2:\mathcal{H}_2\to 2^{\mathcal{H}_2}$ are maximal monotone mappings. Let $S:\mathcal{H}_1\to\mathcal{H}_1$ be a nonexpansive mapping such that $Fix(S)\cap\Gamma\neq\emptyset$. Let $f:\mathcal{H}_1\to\mathcal{H}_1$ be a contraction mapping with constant $\alpha\in(0,1)$ and $G:\mathcal{H}_1\to\mathcal{H}_1$ a strongly positive, bounded linear operator with constant $\mu$ and $0<\beta<\frac{\mu}{\alpha}$. For given $\forall x_0\in\mathcal{H}_1$, let the sequences $\{u_n\}$ and $\{x_n\}$ be generated by (3.19), where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying the following conditions :*

(i) $\lim_{n\to\infty}\alpha_n=0$;
(ii) $\sum_{n=1}^{\infty}\alpha_n=\infty$ *and*
(iii) $\sum_{n=1}^{\infty}|\alpha_n-\alpha_{n-1}|<\infty$.

*Then the sequences $\{u_n\}$ and $\{x_n\}$ both converge strongly to $z\in Fix(S)\cap\Gamma$, where*

$$z=P_{Fix(S)\cap\Gamma}\left((I-\|G\|^{-1}(G+\gamma f)\tilde{x})\right)z.$$

*Moreover, $z$ is also a unique solution of the variational inequality* (1.9).

*Proof.* From Theorem 3.2, we have that $\{x_n\}$ converges strongly, as $n \to \infty$, to a point $z$ satisfying

$$z = P_{\mathrm{Fix}(S)\cap\Gamma}\left(I - \|G\|^{-1}(G+\gamma f)\right)z,$$

which is a unique solution of the variational inequality:

$$\|G\|^{-1}\langle (G-\beta f)z, x-z\rangle \geq 0, x \in \mathcal{H}_1 \tag{3.20}$$

It is easy to see that (3.20) is equivalent to (1.9). Hence $z$ is a unique solution of the variational inequality (1.9). □

Putting $G = I$ and $\beta = 1$ in Theorem 3.2, we have the following results immediately.

**Theorem 3.5.** [13] *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces. Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. Assume that $B_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ and $B_2 : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ are maximal monotone mappings. Let $S : \mathcal{H}_1 \to \mathcal{H}_1$ be a nonexpansive mapping such that Fix(S) $\cap\, \Gamma \neq \emptyset$. Let $f : \mathcal{H}_1 \to \mathcal{H}_1$ be a contraction mapping with constant $\alpha \in (0,1)$. For any given $x_0 \in \mathcal{H}_1$, let the sequences $\{u_n\}$ and $\{x_n\}$ generated by*

$$\begin{aligned} u_n &= J_\lambda^{B_1}\left(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n\right); \\ x_{n+1} &= \alpha_n f(x_n) + (1-\alpha_n)Su_n, \end{aligned} \tag{3.21}$$

*where $\lambda > 0$ and $\gamma \in (0, \frac{1}{L})$, $L$ is the spectral radius of the operator $A^*A$ and $A^*$ is the adjoint of $A$ and $\{\alpha_n\}$ is a sequence in $(0,1)$ such that*

(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and*
(iii) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$.

*Then the sequences $\{u_n\}$ and $\{x_n\}$ both converge strongly to $z \in$ Fix(S) $\cap\, \Gamma$, where $z = P_{Fix(S)\cap\Gamma} f(z)$.*

Applying Theorem 3.2, we can establish the strong convergence for new iterative method as the following theorem.

**Theorem 3.6.** *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces. Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. Assume that $B_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ and $B_2 : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ are maximal monotone mappings. Let $S : \mathcal{H}_1 \to \mathcal{H}_1$ be a nonexpansive mapping such that Fix(S) $\cap\, \Gamma \neq \emptyset$. Let $f : \mathcal{H}_1 \to \mathcal{H}_1$ be a contraction mapping with constant $\alpha \in (0,1)$ and $G : \mathcal{H}_1 \to \mathcal{H}_1$ a strongly positive, bounded linear operator with constant $\mu$ and $0 < \beta < \frac{\mu}{\alpha}$. For any given $y_1 \in \mathcal{H}_1$, let the sequences $\{u_n'\}$ and $\{y_n\}$ generated by*

$$\begin{aligned} u_n' &= J_\lambda^{B_1}\left(y_n + \gamma A^*(J_\lambda^{B_2} - I)Ay_n\right); \\ y_{n+1} &= \alpha_n\beta f(Su_n') + (I - \alpha_n G)Su_n', n \geq 1, \end{aligned} \tag{3.22}$$

*where $\lambda > 0$ and $\gamma \in (0, \frac{1}{L})$, $L$ is the spectral radius of the operator $A^*A$ and $A^*$ is the adjoint of $A$ and $\{\alpha_n\}$ is a sequence in $(0,1)$ such that*

(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and*
(iii) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$.

*Then the sequences $\{u_n'\}$ and $\{y_n\}$ both converge strongly to $z$ obtained in Theorem 3.2.*

*Proof.* Let $\{x_n\}$ be the sequence given by $x_1 = y_1$ and

$$\begin{aligned} u_n &= J_\lambda^{B_1}\left(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n\right); \\ x_{n+1} &= \alpha_n\beta f(x_n) + (I - \alpha_n G)Su_n, n \geq 1. \end{aligned} \tag{3.23}$$

From Theorem 3.2, $x_n \to z$. Next, we claim that $y_n \to z$. Since $J_\lambda^{B_1}$ and $J_\lambda^{B_2}$ both are firmly nonexpansive, they are averaged. For $\gamma \in (0, \frac{1}{L})$, $L$, the mapping $(I + \gamma A^*(J_\lambda^{B_2} I)A)$ is averaged, see [15] . It follows from Proposition 2.1 (ii) that the mapping $J_\lambda^{B_1}(I + \gamma A^*(J_\lambda^{B_2} I)A)$ is averaged and hence nonexpansive. For each $n \geq 1$, we can estimate the following

$$\begin{aligned} \|x_{n+1} - y_{n+1}\| &= \|\alpha_n\beta f(x_n) + (I - \alpha_n G)Su_n - \alpha_n\beta f(Su_n') - (I - \alpha_n G)Su_n'\| \\ &\leq \|\alpha_n\beta f(x_n) - \alpha_n\beta f(Su_n')\| + \|(I - \alpha_n G)Su_n - (I - \alpha_n G)Su_n'\| \\ &\leq \alpha_n\beta\alpha\|Su_n' - x_n\| + (1 - \alpha_n\mu)\|u_n - u_n'\| \end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n\beta\alpha\|Su'_n - Sz\| + \alpha_n\beta\alpha\|Sz - x_n\| + (1-\alpha_n\mu)\|x_n - y_n\| \\
&\leq \alpha_n\beta\alpha\|u'_n - z\| + \alpha_n\beta\alpha\|z - x_n\| + (1-\alpha_n\mu)\|x_n - y_n\| \\
&\leq \alpha_n\beta\alpha\|y_n - z\| + \alpha_n\beta\alpha\|z - x_n\| + (1-\alpha_n\mu)\|x_n - y_n\| \\
&\leq \alpha_n\beta\alpha\|y_n - x_n\| + \alpha_n\beta\alpha\|x_n - z\| + \alpha_n\beta\alpha\|z - x_n\| + (1-\alpha_n\mu)\|x_n - y_n\| \\
&= (1-\alpha_n(\mu-\beta\alpha))\|x_n - y_n\| + \alpha_n(\mu-\beta\alpha)\frac{2\beta\alpha}{\mu-\beta\alpha}\|x_n - z\|.
\end{aligned}$$

It follows from $\sum_{n=1}^{\infty}\alpha_n = \infty, \lim_{n\to\infty}\|x_n - z\| = 0$ and Lemma 2.6 that $\|x_n - y_n\| \to 0$. Consequently, $y_n \to z$ as required. □

## References

1. F.E. Browder, Convergence of approximants to fixed points of nonexpansive nonlinear mappings in Banach spaces. Arch. Ration. Mech. Anal. **24**(1967), 82-90.
2. F.E. Browder, W.V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert space. J. Math. Anal. Appl. **20**(1967), 197-228.
3. C. Byrne, Iterative oblique projection onto convex subsets and the split feasibility problem. Inverse Probl. 18, (2002), 441-453.
4. Byrne, C., Censor, Y., Gibali, A., Reich, S.: The split common null point problem. J. Nonlinear Convex Anal. 13, 759775 (2012)
5. Y. Censor, A. Gibali, S. Reich, The split variational inequality problem. The Technion-Israel Institute of Technology, Haifa(2010). arXiv:1009.3708.
6. L.C. Ceng, Q.H. Ansari, J.C. Yao, Relaxed extragradient methods for finding minimum-norm solutions of the split feasibility problem, Nonlinear Analysis: Theory, Methods Applications,In press.
7. L.C. Ceng, J.C. Yao, Strong convergence theorems by a relaxed extrgradient method for a general system of variational inequalities, Math. Methods Oper.Res. **67** (2008), 375-390.
8. P.L. Combettes, S.A. Hirstoaga, Equilibrium programming in Hilbert space. J. Nonlinear Convex Anal. **6** (2005), 117-136.
9. F. Cianciaruso, G. Marino, L. Muglia, Y. Hong, A hybrid projection algorithm for finding solutions of mixed equilibrium problem and variational inequality problem, Fixed Point Theory and Applications, vol.10.1155, 383-745, 2010.
10. S.S. Chang, Y.J. Cho, J.K. Kim, L. Yang, Multiple-set split feasibility problems for asymptotically strict pseudocontractions, Vol. 2012, Article ID 491760, 12 pp, 2012.
11. F. Deutsch, I. Yamada, Minimizing certain convex functions over the intersection of the fixed point set of nonexpansive mappings, Numer. Funct. Anal. Optim. 19 (1998) 33-56.
12. Hartman, P. and G. Stampacchia, On some nonlinear elliptic dirential functional equations, Acta. Math. 115, 153-188.1966.
13. K. R. Kazmi, S.H.Rizvi, An iterative method for split variational inclusion problem and fixed point problem for nonexpansive mapping, Optim Lett, DOI 10.1007/s11590-013-0629-2, 2013.
14. A. Moudafi, Viscosity approximation methods for fixed-points problems, J. Math. Anal. Appl. **241** (2000) 46–55.
15. A. Moudafi, Split monotone variational inclusions. J. Optim. Theory Appl. 150, 275-283 (2011).
16. K. Goebel, W.A.Kirk, Topics on Metric Fixed Point Theory. Cambridge University Press, Cambridge, 1990.
17. B. Qu and N. Xiu, A note on the CQ algorithm for the split feasibility problem, Inverse Problems, 21,(2005), 55-65.
18. X. Qin, Y.J. Cho, S.M. Kang, Viscosity approximation methods for generalized equilibrium problems and fixed point problems with applications, Nonlinear Anal. **72**, (2010), 99-112.
19. S. Takahashi, W. Takahashi, Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mappings in a Hilbert spaces, Nonlinear Anal. **69** (2008), 1025-1033.
20. S. Plubtieng, R. Punpaeng, A new iterative method for equilibrium problems and fixed point problems of nonexpansive mappings and monotone mappings, Appl. Math. Comput. **197** (2008), 548-558.
21. S. Plubtieng, R. Punpaeng, A general iterative method for equilibrium problems and fixed point problems in Hilbert spaces, J. Math. Anal. Appl. **336** (2007), 455-469.
22. S. Takahashi, W. Takahashi, Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces, J. Math. Anal. Appl. **331** (2007), 506-515.
23. T. Suzuki, Strong convergence of krasnoselskii and manns type sequences for one-parameter nonexpansive semigroups without bochner integrals. J. Math. Anal. Appl. **305** (2005), 227-239.
24. A. Moudafi, Split monotone variational inclusions, Journal of Optimization Theory and Applications, **150** (2011), 275-283.
25. G. Marino, H.K. Xu, A general iterative method for nonexpansive mapping in Hilbert spaces, J. Math. Anal. Appl. **318** (2006), 43-52.
26. H.K. Xu, Viscosity approximation methods for nonexpansive mappings. J. Math. Anal. Appl. **298** (2004), 279-291.
27. H.K. Xu, Iterative algorithms for nonlinear operators, J. London Math. Soc. 66 (2002) 240-256.
28. H.K. Xu, An iterative approach to quadratic optimization, J. Optim. Theory Appl. 116 (2003) 659-678.
29. I. Yamada, The hybrid steepest descent method for the variational inequality problem of the intersection of fixed point sets of nonexpansive mappings, in: D. Butnariu, Y. Censor, S. Reich (Eds.), Inherently Parallel Algorithm for Feasibility and Optimization, Elsevier, 2001, pp. 473504.

30. K. Shimoji, W. Takahashi, Strong convergence to common fixed points of infinite nonexpansive mapping and applications. Taiwanese J. Math. **5**, (2001), 387-404.
31. Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, Bull. Amer. Math. Soc. **73** , 595-597, 1967.
32. F.Wang, H.K. Xu, Cyclic algorithms for split feasibility problems in Hilbert spaces. Nonlinear Anal. doi:10.1016/j.na.2011.03.044, 2011.
33. Y.H. Yao, R.D. Chen, G. Marino, and Y.C. Liou, Applications of Fixed-Point and Optimization Methods to the Multiple-Set Split Feasibility Problem, Journal of Applied Mathematics Volume 2012, Article ID 927530, (2012), 21 PP.

# A FIXED POINT ALTERNATIVE TO THE STABILITY OF AN ADDITIVE $\rho$-FUNCTIONAL INEQUALITIES IN FUZZY BANACH SPACES

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ABSTRACT. In this paper, we solve the following additive $\rho$-functional inequalities

$$N\left(f(x+y)-f(x)-f(y)-\rho\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right),t\right)\geq\frac{t}{t+\varphi(x,y)} \tag{0.1}$$

and

$$N\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)-\rho\left(f(x+y)-f(x)-f(y)\right),t\right)\geq\frac{t}{t+\varphi(x,y)} \tag{0.2}$$

in fuzzy normed spaces, where $\rho$ is a fixed real number with $\rho\neq 1$.

Using the fixed point method, we prove the Hyers-Ulam stability of the additive $\rho$-functional inequalities (0.1) and (0.2) in fuzzy Banach spaces.

## 1. Introduction and preliminaries

Katsaras [23] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [14, 27, 52]. In particular, Bag and Samanta [3], following Cheng and Mordeson [10], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [26]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [4].

We use the definition of fuzzy normed spaces given in [3, 31, 32] to investigate the Hyers-Ulam stability of additive $\rho$-functional inequalities in fuzzy Banach spaces.

**Definition 1.1.** [3, 31, 32, 33] Let $X$ be a real vector space. A function $N: X\times\mathbb{R}\to[0,1]$ is called a *fuzzy norm* on $X$ if for all $x,y\in X$ and all $s,t\in\mathbb{R}$,

($N_1$) $N(x,t)=0$ for $t\leq 0$;
($N_2$) $x=0$ if and only if $N(x,t)=1$ for all $t>0$;
($N_3$) $N(cx,t)=N(x,\frac{t}{|c|})$ if $c\neq 0$;
($N_4$) $N(x+y,s+t)\geq\min\{N(x,s),N(y,t)\}$;
($N_5$) $N(x,\cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty}N(x,t)=1$.
($N_6$) for $x\neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

The pair $(X,N)$ is called a *fuzzy normed vector space*.

The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [30, 31].

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**Definition 1.2.** [3, 31, 32, 33] Let $(X, N)$ be a fuzzy normed vector space. A sequence $\{x_n\}$ in $X$ is said to *be convergent* or *converge* if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, $x$ is called the *limit* of the sequence $\{x_n\}$ and we denote it by $N$-$\lim_{n\to\infty} x_n = x$.

**Definition 1.3.** [3, 31, 32, 33] Let $(X, N)$ be a fuzzy normed vector space. A sequence $\{x_n\}$ in $X$ is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces $X$ and $Y$ is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to $x_0$ in $X$, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on $X$ (see [4]).

The stability problem of functional equations originated from a question of Ulam [51] concerning the stability of group homomorphisms.

The functional equation $f(x+y) = f(x)+f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [19] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Th.M. Rassias [43] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [15] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach.

The functional equation $f\left(\frac{x+y}{2}\right) = \frac{1}{2}f(x)+\frac{1}{2}f(y)$ is called the *Jensen equation*. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [9, 20, 22, 24, 25, 28, 39, 40, 41, 45, 46, 47, 48, 49, 50]).

Gilányi [17] showed that if $f$ satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(x-y)\| \leq \|f(x+y)\| \tag{1.1}$$

then $f$ satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(x+y) + f(x-y).$$

See also [44]. Fechner [13] and Gilányi [18] proved the Hyers-Ulam stability of the functional inequality (1.1). Park, Cho and Han [38] investigated the Cauchy additive functional inequality

$$\|f(x) + f(y) + f(z)\| \leq \|f(x+y+z)\| \tag{1.2}$$

and the Cauchy-Jensen additive functional inequality

$$\|f(x) + f(y) + 2f(z)\| \leq \left\|2f\left(\frac{x+y}{2} + z\right)\right\| \tag{1.3}$$

and proved the Hyers-Ulam stability of the functional inequalities (1.2) and (1.3) in Banach spaces.

Park [36, 37] defined additive $\rho$-functional inequalities and proved the Hyers-Ulam stability of the additive $\rho$-functional inequalities in Banach spaces and non-Archimedean Banach spaces.

We recall a fundamental result in fixed point theory.

Let $X$ be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on $X$ if $d$ satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

**Theorem 1.4.** [6, 11] Let $(X, d)$ be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either

$$d(J^n x, J^{n+1}x) = \infty$$

for all nonnegative integers $n$ or there exists a positive integer $n_0$ such that

(1) $d(J^n x, J^{n+1}x) < \infty, \qquad \forall n \geq n_0$;
(2) the sequence $\{J^n x\}$ converges to a fixed point $y^*$ of $J$;
(3) $y^*$ is the unique fixed point of $J$ in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;
(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

In 1996, G. Isac and Th.M. Rassias [21] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [1, 5, 7, 8, 12, 16, 30, 34, 35, 41, 42]).

In Section 2, we solve the additive $\rho$-functional inequality (0.1) and prove the Hyers-Ulam stability of the additive $\rho$-functional inequality (0.1) in fuzzy Banach spaces by using the fixed point method.

In Section 3, we solve the additive $\rho$-functional inequality (0.2) and prove the Hyers-Ulam stability of the additive $\rho$-functional inequality (0.2) in fuzzy Banach spaces by using the fixed point method.

Throughout this paper, assume that $X$ is a real vector space and $(Y, N)$ is a fuzzy Banach space.

## 2. Additive $\rho$-functional inequality (0.1)

In this section, we prove the Hyers-Ulam stability of the additive $\rho$-functional inequality (0.1) in fuzzy Banach spaces. Let $\rho$ be a real number with $\rho \neq 1$. We need the following lemma to prove the main results.

**Lemma 2.1.** *Let $f : X \to Y$ be a mapping satisfying*

$$f(x + y) - f(x) - f(y) = \rho \left( 2f \left( \frac{x + y}{2} \right) - f(x) - f(y) \right) \tag{2.1}$$

*for all $x, y \in X$. Then $f : X \to Y$ is additive.*

*Proof.* Letting $x = y = 0$ in (2.1), we get $-f(0) = 0$ and so $f(0) = 0$.

Replacing $y$ by $x$ in (2.1), we get $f(2x) - 2f(x) = 0$ and so $f(2x) = 2f(x)$ for all $x \in X$. Thus

$$f(x + y) - f(x) - f(y) = \rho \left( 2f \left( \frac{x + y}{2} \right) - f(x) - f(y) \right) = \rho(f(x + y) - f(x) - f(y))$$

and so $f(x + y) = f(x) + f(y)$ for all $x, y \in X$. □

**Theorem 2.2.** *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \leq \frac{L}{2}\varphi(2x,2y)$$

*for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying*

$$N\left(f(x+y) - f(x) - f(y) - \rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right), t\right) \geq \frac{t}{t+\varphi(x,y)} \tag{2.2}$$

*for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$N\left(f(x) - A(x), t\right) \geq \frac{(2-2L)t}{(2-2L)t + L\varphi(x,x)} \tag{2.3}$$

*for all $x \in X$ and all $t > 0$.*

*Proof.* Letting $y = x$ in (2.2), we get

$$N\left(f(2x) - 2f(x), t\right) \geq \frac{t}{t+\varphi(x,x)} \tag{2.4}$$

for all $x \in X$.

Consider the set

$$S := \{g : X \to Y\}$$

and introduce the generalized metric on $S$:

$$d(g,h) = \inf\left\{\mu \in \mathbb{R}_+ : N(g(x) - h(x), \mu t) \geq \frac{t}{t+\varphi(x,x)},\ \forall x \in X, \forall t > 0\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that $(S,d)$ is complete (see [29, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t+\varphi(x,x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right), L\varepsilon t\right) \\ &= N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{2}\varepsilon t\right) \\ &\geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \varphi\left(\frac{x}{2}, \frac{x}{2}\right)} \geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \frac{L}{2}\varphi(x,x)} \\ &= \frac{t}{t+\varphi(x,x)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g,h)$$

for all $g, h \in S$.

It follows from (2.4) that

$$N\left(f(x)-2f\left(\frac{x}{2}\right),\frac{L}{2}t\right)\geq\frac{t}{t+\varphi(x,x)}$$

for all $x\in X$ and all $t>0$. So $d(f,Jf)\leq\frac{L}{2}$.

By Theorem 1.4, there exists a mapping $A:X\to Y$ satisfying the following:

(1) $A$ is a fixed point of $J$, i.e.,

$$A\left(\frac{x}{2}\right)=\frac{1}{2}A(x) \tag{2.5}$$

for all $x\in X$. Since $f:X\to Y$ is odd, $A:X\to Y$ is an odd mapping. The mapping $A$ is a unique fixed point of $J$ in the set

$$M=\{g\in S:d(f,g)<\infty\}.$$

This implies that $A$ is a unique mapping satisfying (2.5) such that there exists a $\mu\in(0,\infty)$ satisfying

$$N(f(x)-A(x),\mu t)\geq\frac{t}{t+\varphi(x,x)}$$

for all $x\in X$;

(2) $d(J^nf,A)\to 0$ as $n\to\infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty}2^nf\left(\frac{x}{2^n}\right)=A(x)$$

for all $x\in X$;

(3) $d(f,A)\leq\frac{1}{1-L}d(f,Jf)$, which implies the inequality

$$d(f,A)\leq\frac{L}{2-2L}.$$

This implies that the inequality (2.3) holds.

By (2.2),

$$N\left(2^n\left(f\left(\frac{x+y}{2^n}\right)-f\left(\frac{x}{2^n}\right)-f\left(\frac{y}{2^n}\right)\right)\right.$$
$$\left.-\rho\left(2^{n+1}f\left(\frac{x+y}{2^{n+1}}\right)-2^nf\left(\frac{x}{2^n}\right)-2^nf\left(\frac{y}{2^n}\right)\right),2^nt\right)\geq\frac{t}{t+\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)}$$

for all $x,y\in X$, all $t>0$ and all $n\in\mathbb{N}$. So

$$N\left(2^n\left(f\left(\frac{x+y}{2^n}\right)-f\left(\frac{x}{2^n}\right)-f\left(\frac{y}{2^n}\right)\right)\right.$$
$$\left.-\rho\left(2^{n+1}f\left(\frac{x+y}{2^{n+1}}\right)-2^nf\left(\frac{x}{2^n}\right)-2^nf\left(\frac{y}{2^n}\right)\right),t\right)\geq\frac{\frac{t}{2^n}}{\frac{t}{2^n}+\frac{L^n}{2^n}\varphi(x,y)}$$

for all $x,y\in X$, all $t>0$ and all $n\in\mathbb{N}$. Since $\lim_{n\to\infty}\frac{\frac{t}{2^n}}{\frac{t}{2^n}+\frac{L^n}{2^n}\varphi(x,y)}=1$ for all $x,y\in X$ and all $t>0$,

$$A(x+y)-A(x)-A(y)=\rho\left(2A\left(\frac{x+y}{2}\right)-A(x)-A(y)\right)$$

for all $x,y\in X$. By Lemma 2.1, the mapping $A:X\to Y$ is Cauchy additive, as desired. $\square$

**Corollary 2.3.** *Let $\theta \geq 0$ and let $p$ be a real number with $p > 1$. Let $X$ be a normed vector space with norm $\| \cdot \|$. Let $f : X \to Y$ be amapping satisfying*

$$N\left(f(x+y) - f(x) - f(y) - \rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right), t\right) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \tag{2.6}$$

*for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$N\left(f(x) - A(x), t\right) \geq \frac{(2^p - 2)t}{(2^p - 2)t + 2\theta\|x\|^p}$$

*for all $x \in X$ and all $t > 0$.*

*Proof.* The proof follows from Theorem 2.2 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{1-p}$, and we get the desired result. □

**Theorem 2.4.** *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

*for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying (2.2). Then $A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n} f\left(2^n x\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$N\left(f(x) - A(x), t\right) \geq \frac{(2 - 2L)t}{(2 - 2L)t + \varphi(x, x)} \tag{2.7}$$

*for all $x \in X$ and all $t > 0$.*

*Proof.* Let $(S, d)$ be the generalized metric space defined in the proof of Theorem 2.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{2} g\left(2x\right)$$

for all $x \in X$.

It follows from (2.4) that

$$N\left(f(x) - \frac{1}{2}f(2x), \frac{1}{2}t\right) \geq \frac{t}{t + \varphi(x, x)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq \frac{1}{2}$. Hence

$$d(f, A) \leq \frac{1}{2 - 2L},$$

which implies that the inequality (2.7) holds.

The rest of the proof is similar to the proof of Theorem 2.2. □

**Corollary 2.5.** *Let $\theta \geq 0$ and let $p$ be a real number with $0 < p < 1$. Let $X$ be a normed vector space with norm $\| \cdot \|$. Then $A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$N\left(f(x) - A(x), t\right) \geq \frac{(2 - 2^p)t}{(2 - 2^p)t + 2\theta\|x\|^p}$$

*for all $x \in X$ and all $t > 0$.*

*Proof.* The proof follows from Theorem 2.4 by taking $\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{p-1}$, and we get the desired result. □

## 3. Additive $\rho$-functional inequality (0.2)

In this section, we prove the Hyers-Ulam stability of the additive $\rho$-functional inequality (0.2) in fuzzy Banach spaces. Let $\rho$ be a fuzzy number with $\rho \neq 1$.

**Lemma 3.1.** *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$2f\left(\frac{x+y}{2}\right) - f(x) - f(y) = \rho\left(f(x+y) - f(x) - f(y)\right) \tag{3.1}$$

*for all $x, y \in X$. Then $f : X \to Y$ is additive.*

*Proof.* Letting $y = 0$ in (3.1), we get $2f\left(\frac{x}{2}\right) - f(x) = 0$ and so $f(2x) = 2f(x)$ for all $x \in X$. Thus

$$f(x+y) - f(x) - f(y) = 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) = \rho(f(x+y) - f(x) - f(y))$$

and so $f(x+y) = f(x) + f(y)$ for all $x, y \in X$. □

**Theorem 3.2.** *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \leq \frac{L}{2}\varphi\left(2x, 2y\right)$$

*for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$N\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y) - \rho\left(f(x+y) - f(x) - f(y)\right), t\right) \geq \frac{t}{t + \varphi(x,y)} \tag{3.2}$$

*for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$N\left(f(x) - A(x), t\right) \geq \frac{(1-L)t}{(1-L)t + \varphi(x,0)} \tag{3.3}$$

*for all $x \in X$ and all $t > 0$.*

*Proof.* Letting $y = 0$ in (3.2), we get

$$N\left(f(x) - 2f\left(\frac{x}{2}\right), t\right) = N\left(2f\left(\frac{x}{2}\right) - f(x), t\right) \geq \frac{t}{t + \varphi(x,0)} \tag{3.4}$$

for all $x \in X$.

Consider the set

$$S := \{g : X \to Y\}$$

and introduce the generalized metric on $S$:

$$d(g,h) = \inf\left\{\mu \in \mathbb{R}_+ : N(g(x) - h(x), \mu t) \geq \frac{t}{t + \varphi(x,0)},\ \forall x \in X, \forall t > 0\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that $(S, d)$ is complete (see [29, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t + \varphi(x,0)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right), L\varepsilon t\right) \\ &= N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{2}\varepsilon t\right) \\ &\geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \varphi\left(\frac{x}{2}, 0\right)} \geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \frac{L}{2}\varphi(x,0)} \\ &= \frac{t}{t + \varphi(x,0)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g,h)$$

for all $g, h \in S$.

It follows from (3.4) that

$$N\left(f(x) - 2f\left(\frac{x}{2}\right), t\right) \geq \frac{t}{t + \varphi(x,0)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq 1$.

By Theorem 1.4, there exists a mapping $A : X \to Y$ satisfying the following:

(1) $A$ is a fixed point of $J$, i.e.,

$$A\left(\frac{x}{2}\right) = \frac{1}{2}A(x) \tag{3.5}$$

for all $x \in X$. The mapping $A$ is a unique fixed point of $J$ in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that $A$ is a unique mapping satisfying (3.5) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(f(x) - A(x), \mu t) \geq \frac{t}{t + \varphi(x,0)}$$

for all $x \in X$;

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in X$;

(3) $d(f, A) \leq \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, A) \leq \frac{1}{1-L}.$$

This implies that the inequality (3.3) holds.

By (3.2),

$$N\left(2^{n+1}f\left(\frac{x+y}{2^{n+1}}\right)-2^nf\left(\frac{x}{2^n}\right)-2^nf\left(\frac{y}{2^n}\right)\right.$$
$$\left.-\rho\left(2^n\left(f\left(\frac{x+y}{2^n}\right)-f\left(\frac{x}{2^n}\right)-f\left(\frac{y}{2^n}\right)\right)\right),2^nt\right)\geq\frac{t}{t+\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)}$$

for all $x,y\in X$, all $t>0$ and all $n\in\mathbb{N}$. So

$$N\left(2^{n+1}f\left(\frac{x+y}{2^{n+1}}\right)-2^nf\left(\frac{x}{2^n}\right)-2^nf\left(\frac{y}{2^n}\right)\right.$$
$$\left.-\rho\left(2^n\left(f\left(\frac{x+y}{2^n}\right)-f\left(\frac{x}{2^n}\right)-f\left(\frac{y}{2^n}\right)\right)\right),t\right)\geq\frac{\frac{t}{2^n}}{\frac{t}{2^n}+\frac{L^n}{2^n}\varphi\left(x,y\right)}$$

for all $x,y\in X$, all $t>0$ and all $n\in\mathbb{N}$. Since $\lim_{n\to\infty}\frac{\frac{t}{2^n}}{\frac{t}{2^n}+\frac{L^n}{2^n}\varphi(x,y)}=1$ for all $x,y\in X$ and all $t>0$,

$$2A\left(\frac{x+y}{2}\right)-A(x)-A(y)=\rho\left(A(x+y)-A(x)-A(y)\right)$$

for all $x,y\in X$. By Lemma 3.1, the mapping $A:X\to Y$ is Cauchy additive, as desired. □

**Corollary 3.3.** *Let $\theta\geq 0$ and let $p$ be a real number with $p>1$. Let $X$ be a normed vector space with norm $\|\cdot\|$. Let $f:X\to Y$ be a mapping satisfying $f(0)=0$ and*

$$N\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)-\rho\left(f(x+y)-f(x)-f(y)\right),t\right)\geq\frac{t}{t+\theta(\|x\|^p+\|y\|^p)}\quad(3.6)$$

*for all $x,y\in X$ and all $t>0$. Then $A(x):=N\text{-}\lim_{n\to\infty}2^nf(\frac{x}{2^n})$ exists for each $x\in X$ and defines an additive mapping $A:X\to Y$ such that*

$$N\left(f(x)-A(x),t\right)\geq\frac{(2^p-2)t}{(2^p-2)t+2^p\theta\|x\|^p}$$

*for all $x\in X$ and all $t>0$.*

*Proof.* The proof follows from Theorem 3.2 by taking $\varphi(x,y):=\theta(\|x\|^p+\|y\|^p)$ for all $x,y\in X$. Then we can choose $L=2^{1-p}$, and we get the desired result. □

**Theorem 3.4.** *Let $\varphi:X^2\to[0,\infty)$ be a function such that there exists an $L<1$ with*

$$\varphi(x,y)\leq 2L\varphi\left(\frac{x}{2},\frac{y}{2}\right)$$

*for all $x,y\in X$. Let $f:X\to Y$ be a mapping satisfying $f(0)=0$ and (3.3). Then $A(x):=N\text{-}\lim_{n\to\infty}\frac{1}{2^n}f\left(2^nx\right)$ exists for each $x\in X$ and defines an additive mapping $A:X\to Y$ such that*

$$N\left(f(x)-A(x),t\right)\geq\frac{(1-L)t}{(1-L)t+L\varphi(x,0)}\quad(3.7)$$

*for all $x\in X$ and all $t>0$.*

*Proof.* Let $(S,d)$ be the generalized metric space defined in the proof of Theorem 3.2.

Now we consider the linear mapping $J:S\to S$ such that

$$Jg(x):=\frac{1}{2}g\left(2x\right)$$

for all $x \in X$.

It follows from (3.4) that

$$N\left(f(x) - \frac{1}{2}f(2x), Lt\right) \geq \frac{t}{t + \varphi(x, 0)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq L$. Hence

$$d(f, A) \leq \frac{L}{1 - L},$$

which implies that the inequality (3.7) holds.

The rest of the proof is similar to the proof of Theorem 3.2. □

**Corollary 3.5.** *Let $\theta \geq 0$ and let $p$ be a real number with $0 < p < 1$. Let $X$ be a normed vector space with the norm $\| \cdot \|$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.6). Then $A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$N\left(f(x) - A(x), t\right) \geq \frac{(2 - 2^p)t}{(2 - 2^p)t + 2^p\theta\|x\|^p}$$

*for all $x \in X$.*

*Proof.* The proof follows from Theorem 3.4 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{p-1}$, and we get the desired result. □

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## References

[1] L. Aiemsomboon and W. Sintunavaat, *On new stability results for generalized Cauchy functional equations on groups by using Brzdek's fixed point theorem*, J. Fixed Point Theory Appl. (in press).

[2] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.

[3] T. Bag and S.K. Samanta, *Finite dimensional fuzzy normed linear spaces*, J. Fuzzy Math. **11** (2003), 687–705.

[4] T. Bag and S.K. Samanta, *Fuzzy bounded linear operators*, Fuzzy Sets and Systems **151** (2005), 513–547.

[5] A. Bahyrycz and J. Brzdek, *On functions that are approximate fixed points almost everywhere and Ulam's type stability*, J. Fixed Point Theory Appl. (in press).

[6] L. Cădariu and V. Radu, *Fixed points and the stability of Jensen's functional equation*, J. Inequal. Pure Appl. Math. **4**, no. 1, Art. ID 4 (2003).

[7] L. Cădariu and V. Radu, *On the stability of the Cauchy functional equation: a fixed point approach*, Grazer Math. Ber. **346** (2004), 43–52.

[8] L. Cădariu and V. Radu, *Fixed point methods for the generalized stability of functional equations in a single variable*, Fixed Point Theory Appl. **2008**, Art. ID 749392 (2008).

[9] I. Chang and Y. Lee, *Additive and quadratic type functional equation and its fuzzy stability*, Results Math. **63** (2013), 717–730.

[10] S.C. Cheng and J.M. Mordeson, *Fuzzy linear operators and fuzzy normed linear spaces*, Bull. Calcutta Math. Soc. **86** (1994), 429–436.

[11] J. Diaz and B. Margolis, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc. **74** (1968), 305–309.

[12] M. Eshaghi Gordji and Th.M. Rassias, *Fixed points and generalized stability for quadratic and quartic mappings in $C^*$-algebras*, J. Fixed Point Theory Appl. (in press).

[13] W. Fechner, *Stability of a functional inequalities associated with the Jordan-von Neumann functional equation*, Aequationes Math. **71** (2006), 149–161.
[14] C. Felbin, *Finite dimensional fuzzy normed linear spaces*, Fuzzy Sets and Systems **48** (1992), 239–248.
[15] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.
[16] A. G. Ghazanfari and Z. Alizadeh, *On approximate ternary m-derivations and $\sigma$-homomorphisms*, J. Fixed Point Theory Appl. (in press).
[17] A. Gilányi, *Eine zur Parallelogrammgleichung äquivalente Ungleichung*, Aequationes Math. **62** (2001), 303–309.
[18] A. Gilányi, *On a problem by K. Nikodem*, Math. Inequal. Appl. **5** (2002), 707–710.
[19] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A. **27** (1941), 222–224.
[20] D.H. Hyers, G. Isac and Th.M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, 1998.
[21] G. Isac and Th.M. Rassias, *Stability of $\psi$-additive mappings: Appications to nonlinear analysis*, Internat. J. Math. Math. Sci. **19** (1996), 219–228.
[22] S. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press lnc., Palm Harbor, Florida, 2001.
[23] A.K. Katsaras, *Fuzzy topological vector spaces $II$*, Fuzzy Sets and Systems **12** (1984), 143–154.
[24] H. Kim, M. Eshaghi Gordji, A. Javadian and I. Chang, *Homomorphisms and derivations on unital $C^*$-algebras related to Cauchy-Jensen functional inequality,* J. Math. Inequal. **6** (2012), 557–565.
[25] H. Kim, J. Lee and E. Son, *Approximate functional inequalities by additive mappings,* J. Math. Inequal. **6** (2012), 461–471.
[26] I. Kramosil and J. Michalek, *Fuzzy metric and statistical metric spaces*, Kybernetica **11** (1975), 326–334.
[27] S.V. Krishna and K.K.M. Sarma, *Separation of fuzzy normed linear spaces*, Fuzzy Sets and Systems **63** (1994), 207–217.
[28] J. Lee, C. Park and D. Shin, *An AQCQ-functional equation in matrix normed spaces*, Results Math. **27** (2013), 305–318.
[29] D. Miheţ and V. Radu, *On the stability of the additive Cauchy functional equation in random normed spaces*, J. Math. Anal. Appl. **343** (2008), 567–572.
[30] M. Mirzavaziri and M.S. Moslehian, *A fixed point approach to stability of a quadratic equation*, Bull. Braz. Math. Soc. **37** (2006), 361–376.
[31] A.K. Mirmostafaee, M. Mirzavaziri and M.S. Moslehian, *Fuzzy stability of the Jensen functional equation*, Fuzzy Sets and Systems **159** (2008), 730–738.
[32] A.K. Mirmostafaee and M.S. Moslehian, *Fuzzy versions of Hyers-Ulam-Rassias theorem*, Fuzzy Sets and Systems **159** (2008), 720–729.
[33] A.K. Mirmostafaee and M.S. Moslehian, *Fuzzy approximately cubic mappings*, Inform. Sci. **178** (2008), 3791–3798.
[34] C. Park, *Fixed points and Hyers-Ulam-Rassias stability of Cauchy-Jensen functional equations in Banach algebras*, Fixed Point Theory Appl. **2007**, Art. ID 50175 (2007).
[35] C. Park, *Generalized Hyers-Ulam-Rassias stability of quadratic functional equations: a fixed point approach*, Fixed Point Theory Appl. **2008**, Art. ID 493751 (2008).
[36] C. Park, *Additive $\rho$-functional inequalities and equations,* J. Math. Inequal. **9** (2015), 17–26.
[37] C. Park, *Additive $\rho$-functional inequalities in non-Archimedean normed spaces,* J. Math. Inequal. **9** (2015), 397–407.
[38] C. Park, Y. Cho and M. Han, *Stability of functional inequalities associated with Jordan-von Neumann type additive functional equations*, J. Inequal. Appl. **2007**, Art. ID 41820 (2007).
[39] C. Park, K. Ghasemi, S. G. Ghaleh and S. Jang, *Approximate n-Jordan $*$-homomorphisms in $C^*$-algebras,* J. Comput. Anal. Appl. **15** (2013), 365-368.
[40] C. Park, A. Najati and S. Jang, *Fixed points and fuzzy stability of an additive-quadratic functional equation,* J. Comput. Anal. Appl. **15** (2013), 452–462.
[41] C. Park and Th.M. Rassias, *Fixed points and generalized Hyers-Ulam stability of quadratic functional equations,* J. Math. Inequal. **1** (2007), 515–528.
[42] V. Radu, *The fixed point alternative and the stability of functional equations*, Fixed Point Theory **4** (2003), 91–96.

[43] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.
[44] J. Rätz, *On inequalities associated with the Jordan-von Neumann functional equation*, Aequationes Math. **66** (2003), 191–200.
[45] L. Reich, J. Smítal and M. Štefánková, *Singular solutions of the generalized Dhombres functional equation*, Results Math. **65** (2014), 251–261.
[46] S. Schin, D. Ki, J. Chang and M. Kim, *Random stability of quadratic functional equations: a fixed point approach*, J. Nonlinear Sci. Appl. **4** (2011), 37–49.
[47] S. Shagholi, M. Bavand Savadkouhi and M. Eshaghi Gordji, *Nearly ternary cubic homomorphism in ternary Fréchet algebras,* J. Comput. Anal. Appl. **13** (2011), 1106–1114.
[48] S. Shagholi, M. Eshaghi Gordji and M. Bavand Savadkouhi, *Stability of ternary quadratic derivation on ternary Banach algebras,* J. Comput. Anal. Appl. **13** (2011), 1097–1105.
[49] D. Shin, C. Park and Sh. Farhadabadi, *On the superstability of ternary Jordan $C^*$-homomorphisms,* J. Comput. Anal. Appl. **16** (2014), 964–973.
[50] D. Shin, C. Park and Sh. Farhadabadi, *Stability and superstability of $J^*$-homomorphisms and $J^*$-derivations for a generalized Cauchy-Jensen equation,* J. Comput. Anal. Appl. **17** (2014), 125–134.
[51] S. M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ. New York, 1960.
[52] J.Z. Xiao and X.H. Zhu, *Fuzzy normed spaces of operators and its completeness*, Fuzzy Sets and Systems **133** (2003), 389–399.

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# FOURIER SERIES OF HIGHER-ORDER GENOCCHI FUNCTIONS AND THEIR APPLICATIONS

TAEKYUN KIM, DAE SAN KIM, LEE CHAE JANG, AND DMITRY V. DOLGY

Abstract. In this paper, we derive some identities for higher-order Genocchi functions arising from Fourier series for them . In addition, we give some application of these identities related to Bernoulli function.

## 1. Introduction

The numbers $G_k$, $(k \geq 0)$, in the Taylor expansion

$$\frac{2t}{e^t+1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}, \quad (\text{see } [2-11]), \tag{1.1}$$

are known as the Genocchi numbers. These numbers arise in the series expansion of trigonometric functions, and are extremely important in the number theory and analysis. The Genocchi polynomials $G_n(x)$, $(n \geq 0)$, are defined by the generating function

$$\left(\frac{2t}{e^t+1}\right) e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}, \quad (\text{see } [5, 12, 13]). \tag{1.2}$$

Note that $G_n(x) \in \mathbb{Z}[x]$ with $\deg G_n(x) = n-1$, for $n \geq 1$. Let $f(x)$ be a square integrable function defined on $[-p, p]$. Then the Fourier series of $f(x)$ is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right), \tag{1.3}$$

where

$$a_0 = \frac{1}{p} \int_{-p}^{p} f(x) dx, \quad a_n = \frac{1}{p} \int_{-p}^{p} f(x) \cos \frac{n\pi}{p} x \; dx, \tag{1.4}$$

and

$$b_n = \frac{1}{p} \int_{-p}^{p} f(x) \sin \frac{n\pi}{p} x \; dx, \quad (\text{see } [6, 7]). \tag{1.5}$$

The Fourier series in (1.3) can be alternatively given as follows:

$$\sum_{n=-\infty}^{\infty} C_n e^{\frac{n\pi i}{p} x}, \quad (i = \sqrt{-1}), \tag{1.6}$$

where

$$C_n = \frac{1}{2p} \int_{-p}^{p} f(x) e^{-\frac{n\pi i}{p} x} dx, , \quad (\text{see } [6, 7]). \tag{1.7}$$

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For $r \in \mathbb{N}$, the higher-order Genocchi polynomials are defined by the generating function to be

$$\left(\frac{2t}{e^t+1}\right)^r e^{xt} = \sum_{n=0}^{\infty} G_n^{(r)}(x0\frac{t^n}{n!}, \quad \text{(see [5])}. \tag{1.8}$$

When $x = 0$, $G_n^{(r)} = G_n^{(r)}(0)$ are called the higher-order Genocchi numbers.

For any real number $x$, we define

$$< x >= x - [x] \in [0, 1), \tag{1.9}$$

where $[\cdot]$ is the Gauss symbol. Note that $< x >$ is the fractional part of $x$. Thus, $G_m^{(r)}(< x >)$ are functions defined on $(-\infty, \infty)$ and periodic with period 1, which are called Genocchi functions of order $r$.

In this paper, we derive some identities of Genocchi functions of order $r$ arising from Fourier series for them. In addition, we give some application of these identities related to Bernoulli functions.

## 2. Fourier series of higher-order Genocchi functions and their applications

From (1.8), we note that

$$G_m^{(r)}(x) = 0, \text{ for } 0 \le m \le r-1, \text{ and } G_r^{(r)}(x) = r!. \tag{2.1}$$

Now, we assume that $m \ge r+1 \ge 2$. We first observe that

$$G_m^{(r)}(x+1) = 2mG_{m-1}^{r-1)}(x) - G_m^{(r)}(x), \ (m \ge 0). \tag{2.2}$$

Indeed,

$$\begin{aligned}
\sum_{m=0}^{\infty} G_m^{(r)}(x+1)\frac{t^m}{m!} &= \left(\frac{2t}{e^t+1}\right)^r e^{(x+1)t} = \left(\frac{2t}{e^t+1}\right)^r e^{xt}(e^t+1-1) \\
&= \left(\frac{2t}{e^t+1}\right)^{r-1} e^{xt}\left(\frac{2t}{e^t+1}\right)(e^t+1) - \left(\frac{2t}{e^t+1}\right)^r e^{xt} \\
&= 2t\sum_{m=0}^{\infty} G_m^{(r-1)}(x)\frac{t^m}{m!} - \sum_{m=0}^{\infty} G_m^{(r)}(x)\frac{t^m}{m!} \\
&= 2\sum_{m=0}^{\infty} mG_{m-1}^{(r-1)}(x)\frac{t^m}{m!} - \sum_{m=0}^{\infty} G_m^{(r)}(x)\frac{t^m}{m!} \\
&= \sum_{m=0}^{\infty}\left(2mG_{m-1}^{(r-1)}(x) - G_m^{(r)}(x)\right)\frac{t^m}{m!}.
\end{aligned} \tag{2.3}$$

For $x = 0$ in , we have

$$G_m^{(r)}(1) = 2mG_{m-1}^{(r-1)}(0) - G_m^{(r)}(0). \tag{2.4}$$

By (2.4), we get

$$G_m^{(r)}(1) = G_m^{(r)}(0) \Leftrightarrow G_m^{(r)}(0) = mG_{m-1}^{(r-1)}(0). \tag{2.5}$$

$G_m^{(r)}(< x >)$ is piecewise $C^{\infty}$. Moreover, $G_m^{(r)}(< x >)$ is continuous for those $(r, m)$ with $G_m^{(r)}(0) = mG_{m-1}^{(r-1)}(0)$, and discontinuous with jump discontinuities at integers for those $(r, m)$ with $G_m^{(r)}(0) \neq$

$mG_{m-1}^{(r-1)}(0)$. The Fourier series of $G_m^{(r)}(< x >)$ is

$$\sum_{n=-\infty}^{\infty} C_n^{(r,m)} e^{2\pi i n x}, \ (i = \sqrt{-1}), \tag{2.6}$$

where

$$\begin{aligned} C_n^{(r,m)} &= \int_0^1 G_m^{(r)}(< x >) e^{-2\pi i n x} dx \\ &= \int_0^1 G_m^{(r)}(x) e^{-2\pi i n x} dx. \end{aligned} \tag{2.7}$$

Now, we observe that

$$\begin{aligned} C_n^{(r,m)} &= \int_0^1 G_m^{(r)}(x) e^{-2\pi i n x} dx \\ &= \frac{1}{m+1} \left[ G_{m+1}^{(r)} e^{-2\pi i n x} \right]_0^1 + \frac{2\pi i n}{m+1} \int_0^1 G_{m+1}^{(r)}(x) e^{-2\pi i n x} dx \\ &= \frac{1}{m+1} \left( G_{m+1}^{(r)}(0) - G_{m+1}^{(r)}(0) \right) + \frac{2\pi i n}{m+1} C_n^{(r,m+1)} \\ &= \frac{2}{m+1} \left( (m+1) G_m^{(r-1)}(0) - G_{m+1}^{(r)}(0) \right) + \frac{2\pi i n}{m+1} C_n^{(r,m+1)}. \end{aligned} \tag{2.8}$$

Replacing $m$ by $m-1$ in (2.8), we have

$$\frac{2\pi i n}{m} C_n^{(r,m)} = C_n^{(r,m-1)} + \frac{2}{m} \left( G_m^{(r)}(0) - m G_{m-1}^{(r-1)}(0) \right). \tag{2.9}$$

Assume first that $n \neq 0$. Then we have

$$\begin{aligned} C_n^{(r,m)} &= \frac{m}{2\pi i n} C_n^{(r,m-1)} + \frac{1}{\pi i n} \left( G_m^{(r)}(0) - m G_{m-1}^{(r-1)}(0) \right) \\ &= \frac{m}{2\pi i n} \left( \frac{m-1}{2\pi i n} C_n^{(r,,m-2)} + \frac{1}{\pi i n} \left( G_{m-1}^{(r)}(0) - (m-1) G_{m-2}^{(r-1)}(0) \right) \right) \\ &\quad + \frac{1}{\pi i n} \left( G_m^{(r)}(0) - m G_{m-1}^{(r-1)}(0) \right) \\ &= \frac{m(m-1)}{(2\pi i n)^2} C_n^{(r,m-2)} + \frac{m}{2} \frac{1}{(\pi i n)^2} \left( G_{m-1}^{(r)}(0) - (m-1) G_{m-2}^{(r-1)}(0) \right) \\ &\quad + \frac{1}{\pi i n} \left( G_m^{(r)}(0) - m G_{m-1}^{(r-1)}(0) \right) \\ &= \frac{m(m-1)}{(2\pi i n)^2} \left\{ \frac{m-2}{2\pi i n} C_n^{(r,m-3)} + \frac{1}{\pi i n} \left( G_{m-2}^{(r)}(0) - (m-2) G_{m-3}^{(r-1)}(0) \right) \right\} \\ &\quad + \frac{m}{2} \frac{1}{(\pi i n)^2} \left( G_{m-1}^{(r)}(0) - (m-1) G_{m-2}^{(r-1)}(0) \right) + \frac{1}{\pi i n} \left( G_m^{(r)}(0) - m G_{m-1}^{(r-1)}(0) \right) \\ &= \frac{m(m-1)(m-2)}{(2\pi i n)^3} C_n^{(r,m-3)} + \frac{m(m-1)}{2^2} \frac{1}{(\pi i n)^3} \left( G_{m-2}^{(r)}(0) - (m-2) G_{m-3}^{(r-1)}(0) \right) \\ &\quad + \frac{m}{2} \frac{1}{(\pi i n)^2} \left( G_{m-1}^{(r)}(0) - (m-1) G_{m-2}^{(r-1)}(0) \right) + \frac{1}{\pi i n} \left( G_m^{(r)}(0) - m G_{m-1}^{(r-1)}(0) \right) \\ &= \cdots \\ &= \frac{m!}{(2\pi i n)^{m-1}} C_n^{(r,1)} + \sum_{k=1}^{m-1} \frac{(m)_{k-1}}{2^{k-1}} \frac{1}{(\pi i n)^k} \left( G_{m+1-k}^{(r)}(0) - (m+1-k) G_{m-k}^{(r-1)}(0) \right). \end{aligned} \tag{2.10}$$

Note that

$$C_n^{(r,1)} = \int_0^1 G_1^{(r)}(x)e^{-2\pi inx}dx = 0, \tag{2.11}$$

since $G_1^{(r)}(x) = 0$, for $r \geq 2$ and $G_1^{(1)}(x) = 1$. From (2.10) and (2.11), we have

$$\begin{aligned} C_n^{(r,m)} &= \sum_{k=1}^{m-1} \frac{(m)_{k-1}}{2^{k-1}} \frac{1}{(\pi in)^k} \left( G_{m-k+1}^{(r)}(0) - (m-k+1)G_{m-k}^{(r-1)}(0) \right) \\ &= \sum_{k=1}^{\min\{m+1-r,m-1\}} \frac{(m)_{k-1}}{2^{k-1}} \frac{1}{(\pi in)^k} \left( G_{m-k+1}^{(r)}(0) - (m-k+1)G_{m-k}^{(r-1)}(0) \right) \end{aligned} \tag{2.12}$$

Here we used the fact that $G_m^{(r)} = 0$, for $0 \leq m \leq r-1$. Assume next that $n = 0$. Then we have

$$\begin{aligned} C_0^{(r,m)} &= \int_0^1 G_m^{(r)}(x)dx = \frac{1}{m+1} \left[ G_{m+1}^{(r)}(x) \right]_0^1 = \frac{1}{m+1} \left( G_{m+1}^{(r)}(1) - G_{m+1}^{(r)}(0) \right) \\ &= \frac{2}{m+1} \left( (m+1)G_m^{(r-1)}(0) - G_{m+1}^{(r)}(0) \right) \end{aligned} \tag{2.13}$$

Before proceeding further, we recall the following facts about Bernoulli functions $B_n(< x >)$ :

$$B_m(< x >) = -m! \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^{2\pi inx}}{(2\pi in)^m}, \text{ for } m \geq 2, \quad \text{(see [1])}, \tag{2.14}$$

and

$$-\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^{2\pi inx}}{2\pi in} = \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z} \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases} \tag{2.15}$$

The series in (2.14) converges uniformly, but that in (2.15) converges only pointwise. Assume first that $G_m^{(r)}(0) = mG_{m-1}^{(r-1)}(0)$. Then $G_m^{(r)}(1) = G_m^{(r)}(0)$. $G_m^{(r)}(< x >)$ is piecewise $C^\infty$, and continuous. Hence the Fourier series of $G_m^{(r)}(< x >)$ converges uniformly to $G_m^{(r)}(< x >)$, and we have

$$\begin{aligned} G_m^{(r)}(< x >) &= \frac{2}{m+1} \left( (m+1)G_m^{(r-1)}(0) - G_{m+1}^{(r)}(0) \right) \\ &+ \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left( \sum_{k=1}^{\min\{m+1-r,m-1\}} \frac{(m)_{k-1}}{2^{k-1}} \frac{1}{(\pi in)^k} \left( G_{m-k+1}^{(r)}(0) - (m-k+1)G_{m-k}^{(r-1)}(0) \right) \right) e^{2\pi inx}, \end{aligned} \tag{2.16}$$

for all $x \in (-\infty, \infty)$. In addition, we can express this in terms of Bernoulli functions $B_m(< x >)$.

$$\begin{aligned} G_m^{(r)}(< x >) &= \frac{2}{m+1}\left((m+1)G_m^{(r-1)}(0) - G_{m+1}^{(r)}(0)\right) \\ &\quad - \sum_{k=1}^{\min\{m+1-r,m-1\}} \frac{2(m)_{k-1}}{k!}\left(G_{m-k+1}^{(r)}(0) - (m-k+1)G_{m-k}^{(r-1)}(0)\right)\left(-k!\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^k}\right) \\ &= \frac{2}{m+1}\left((m+1)G_m^{(r-1)}(0) - G_{m+1}^{(r)}(0)\right) \\ &\quad + \sum_{k=2}^{\min\{m+1-r,m-1\}} \frac{2(m)_{k-1}}{k!}\left((m+1-k)G_{m-k}^{(r-1)}(0) - G_{m+1-k}^{(r)}(0)\right)B_k(< x >) \\ &\quad + 2\left(mG_{m-1}^{(r-1)}(0) - G_m^{(r)}(0)\right) \times \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z} \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases} \end{aligned} \tag{2.17}$$

Therefore, we obtain the following theorem.

**Theorem 2.1.** *Let $m \geq r+1 \geq 2$. Assume that*

$$G_m^{(r)}(0) = mG_{m-1}^{(r-1)}(0).$$

*Then we have*

$$\begin{aligned} (a)\ G_m^{(r)}(< x >) &= \frac{2}{m+1}\left((m+1)G_m^{(r-1)}(0) - G_{m+1}^{(r)}(0)\right) \\ &\quad + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\left(\sum_{k=1}^{\min\{m+1-r,m-1\}} \frac{(m)_{k-1}}{2^{k-1}}\frac{1}{(\pi in)^k}\left(G_{m-k+1}^{(r)}(0) - (m-k+1)G_{m-k}^{(r-1)}(0)\right)\right)e^{2\pi inx}, \end{aligned}$$

*for all $x \in (-\infty, \infty)$, where the convergence is uniform.*

$$\begin{aligned} (b)\ G_m^{(r)}(< x >) &= \frac{2}{m+1}\left((m+1)G_m^{(r-1)}(0) - G_{m+1}^{(r)}(0)\right) \\ &\quad + \sum_{k=2}^{\min\{m+1-r,m-1\}} \frac{2(m)_{k-1}}{k!}\left((m+1-k)G_{m-k}^{(r-1)}(0) - G_{m+1-k}^{(r)}(0)\right)B_k(< x >), \end{aligned}$$

*for all $x \in (-\infty, \infty)$, where $B_k(< x >)$ is the Bernoulli function.*

Assume next that $G_m^{(r)}(0) \neq mG_m^{(r-1)}(0)$. Then $G_m^{(r)}(1) \neq G_m^{(r)}(0)$, and hence $G_m^{(r)}(< x >)$ is piecewise $C^\infty$ and discontinuous with jump discontinuities at integers. Thus, the Fourier series of $G_m^{(r)}(< x >)$ converges pointwise to $G_m^{(r)}(< x >)$, for $x \notin \mathbb{Z}$, and converges to $\frac{1}{2}\left(G_m^{(r)}(1) + G_m^{(r)}(0)\right) = mG_{m-1}^{(r-1)}(0)$, for $x \in \mathbb{Z}$. Therefore, we obtain the following theorem.

**Theorem 2.2.** *Let $m \geq r+1 \geq 2$. Assume that $G_m^{(r)}(0) \neq mG_{m-1}^{(r)}(0)$. Then we have*

$$(a)\ \frac{2}{m+1}\left((m+1)G_m^{(r-1)}(0)-G_{m+1}^{(r)}(0)\right)+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(\sum_{k=1}^{\min\{m+1-r,m-1\}}\frac{(m)_{k-1}}{2^{k-1}}\left(\frac{1}{\pi in}\right)^k\right.$$
$$\left.\times\left(G_{m+1-k}^{(r)}(0)-(m+1-k)G_{m-k}^{(r-1)}(0)\right)\right)e^{2\pi inx}=\begin{cases}G_m^{(r)}(<x>), & \text{for } x\notin\mathbb{Z},\\ mG_{m-1}^{(r-1)}(0), & \text{for } x\in\mathbb{Z}.\end{cases}$$

*Here the convergence is pointwise.*

$$(b)\ \frac{2}{m+1}\left((m+1)G_m^{(r-1)}(0)-G_{m+1}^{(r)}(0)\right)$$
$$+\sum_{k=1}^{\min\{m+1-r,m-1\}}\frac{2(m)_{k-1}}{k!}\left((m+1-k)G_{m-k}^{(r-1)}(0)-G_{m+1-k}^{(r)}(0)\right)B_k(<x>)$$
$$=G_m^{(r)}(<x>),\quad \text{for } x\notin\mathbb{Z},$$

*and*

$$\frac{2}{m+1}\left((m+1)G_m^{(r-1)}(0)-G_{m+1}^{(r)}(0)\right)$$
$$+\sum_{k=2}^{\min\{m+1-r,m-1\}}\frac{2(m)_{k-1}}{k!}\left((m+1-k)G_{m-k}^{(r-1)}(0)-G_{m+1-k}^{(r)}(0)\right)B_k(<x>)$$
$$=mG_{m-1}^{(r-1)}(0),\quad \text{for } x\in\mathbb{Z}.$$

*Here $B_k(<x>)$ is the Bernoulli function.*

## References


1. M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions,* Dover, New York, 1970.
2. L. Carlitz, *Some unusual congruences for the Bernoulli and Genocchi numbers,* Duke Math. J., **35** (1968), 563–566.
3. J. M. Gandhi, *Some integrals for Genocchi numbers,* Math. Mag., **33**(1959/1960), 21–23.
4. J. M. Gandhi, *Congruences for Genocchi numbers,* Duke Math. J., **31** (1964), 519–527.
5. D. S. Kim and T. Kim, *Some identities involving Genocchi polynomials and numbers,* Ars combin., **121** (2015), 403–412.
6. T. Kim, *Note on the Euler numbers and polynomials,* Adv. Stud. Contemp. Math. (Kyungshang), **17**(2008), no. 2, 131–136.
7. T. Kim, *Euler numbers and polynomials associated with zeta functions,* Abstr. Apol. Anal., **2008** Art. ID 581582. 11 pp.
8. T. Kim, S.-H. Rim, D. V. Dolgy and S.-H. Lee, *Some identities of Genocchi polynomials arising from Genocchi bases,* J. Inequal. Appl., **2013** 2013:43, 6 pp.
9. T. Kim, J. Choi and Y. H. Kim, *A note on the values of Euler zeta functions at positive integers,* Adv. Stud. Contemp. Math. (Kyungshang), **22**(2012), no. 1, 27–34.
10. T. Kim, *Some identities for the Bernoulli the Euler and the Genocchi numbers and polynomials,* Adv. Stud. Contemp. Math. (Kyungshang), **20**(2010), no. 1, 23–28.
11. T. Kim, *On the multiple q-Genocchi and Euler numbers,* Russ. J. Math. Phys. **15**(2008), no. 4, 481–486.
12. D. H. Lehmer, *A new approach to Bernoulli polynomials,* Amer. Math. Monthly., **95**(1988), 905–911.
13. J. Riordan and P. R. Stein, *Proof of a conjecture on Genocchi numbers,* Discrete Math, **5**(1973), 381–388.

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# $(g,\varphi_{h,m})-$ CONVEX AND $(g,\log\varphi)-$ CONVEX DOMINATED FUNCTIONS AND HADAMARD TYPE INEQUALITIES RELATED TO THEM

MUSTAFA GÜRBÜZ

ABSTRACT. In this paper, we present the notion of $(g,\varphi_{h,m})-$convex and $(g,\log\varphi)-$convex dominated function and present some properties of them. Besides, we attain some Hermite-Hadamard-type inequalities for $(g,\varphi_{h,m})-$convex and $(g,\log\varphi)-$ convex dominated functions. Our results generalize some findings about Hermite-Hadamard-type inequalities in the literature.

## 1. INTRODUCTION

The inequality

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)\,dx\leq\frac{f(a)+f(b)}{2} \tag{1.1}$$

which holds for every convex function $f$, from a closed set $[a,b]$ to $\mathbb{R}$, is known in the literature as Hermite-Hadamard's inequality (see [13]).

In [1], Dragomir and Ionescu introduced the following class of functions.

**Definition 1.** *Let $g:I\to\mathbb{R}$ be a convex function on the interval $I$. The function $f:I\to\mathbb{R}$ is called $g-$convex dominated on $I$ if the following condition is satisfied:*

$$\begin{aligned}&|\lambda f(x)+(1-\lambda)f(y)-f(\lambda x+(1-\lambda)y)|\\ &\leq\quad \lambda g(x)+(1-\lambda)g(y)-g(\lambda x+(1-\lambda)y)\end{aligned} \tag{1.2}$$

*for all $x,y\in I$ and $\lambda\in[0,1]$.*

In [1] and [2], the authors connect together some disparate threads through a Hermite-Hadamard motif. The first of these threads is the unifying concept of a $g-$convex-dominated function. In [3], Hwang *et al.* established some inequalities of Fejér type for $g-$convex-dominated functions. Finally, in [4], [5] and [6] authors introduced several new different kinds of convex -dominated functions and then gave Hermite-Hadamard-type inequalities for this classes of functions.

In [7], S. Varošanec introduced the following class of functions.

$I$ and $J$ are intervals in $\mathbb{R}$, $(0,1)\subseteq J$ and functions $h$ and $f$ are real non-negative functions defined on $J$ and $I$, respectively.

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**Definition 2.** *Let $h$ be a non-negative function from $J$ which is a subset of $\mathbb{R}$ to $\mathbb{R}$, $h \not\equiv 0$. $f : I \to \mathbb{R}$ is called an $h-$convex function, or that $f$ belongs to the class $SX(h, I)$, if $f$ is non-negative and for all $x, y \in I$ and $\alpha \in (0, 1]$, we get*

$$f(\alpha x + (1-\alpha) y) \leq h(\alpha) f(x) + h(1-\alpha) f(y). \tag{1.3}$$

If the inequality (1.3) is reversed, then $f$ is said to be $h-$concave, i.e. $f \in SV(h, I)$.

Youness have defined the $\varphi-$convex functions in [9]. A function $\varphi : [a, b] \to [c, d]$ where $[a, b] \subset \mathbb{R}$:

**Definition 3.** *A function $f : [a, b] \to \mathbb{R}$ is said to be $\varphi-$convex on $[a, b]$ if for every two points $x \in [a, b]$, $y \in [a, b]$ and $t \in [0, 1]$ the following inequality holds:*

$$f(t\varphi(x) + (1-t)\varphi(y)) \leq tf(\varphi(x)) + (1-t) f(\varphi(y)). \tag{1.4}$$

In [8], Sarıkaya defined a new kind of $\varphi-$convexity using $h-$convexity as following:

**Definition 4.** *Let $I$ be an interval in $\mathbb{R}$ and $h : (0, 1) \to (0, \infty)$ be a given function. We say that a function $f : I \to [0, \infty)$ is $\varphi_h-$convex if*

$$f(t\varphi(x) + (1-t)\varphi(y)) \leq h(t) f(\varphi(x)) + h(1-t) f(\varphi(y)) \tag{1.5}$$

*for all $x, y \in I$ and $t \in (0, 1)$.*

If inequality (1.5) is reversed, then $f$ is said to be $\varphi_h-$concave. In particular, if $f$ satisfies (1.5) with $h(t) = t$, $h(t) = t^s$ $(s \in (0, 1))$, $h(t) = \frac{1}{t}$, and $h(t) = 1$, then $f$ is said to be $\varphi-$convex, $\varphi_s-$convex, $\varphi-$Godunova-Levin function and $\varphi-P-$function, respectively.

In [10], Özdemir *et al.* defined $(h-m)-$convexity and obtained Hermite-Hadamard-type inequalities as following.

**Definition 5.** *Let $h : J \subset \mathbb{R} \to \mathbb{R}$ be a non-negative function. We say that $f : [0, b] \to \mathbb{R}$ is a $(h-m)-$convex function, if $f$ is non-negative and for all $x, y \in [0, b]$, $m \in [0, 1]$ and $\alpha \in (0, 1)$, we have*

$$f(\alpha x + m(1-\alpha) y) \leq h(\alpha) f(x) + mh(1-\alpha) f(y). \tag{1.6}$$

*If the inequality is reversed, then $f$ is said to be $(h-m)-$concave function on $[0, b]$.*

In [2], Dragomir *et al.* proved the following theorem for $g-$convex dominated functions related to (1.1):

**Definition 6.** *A function $f : I \to [0, \infty)$ is said to be $\log-$convex or multiplicatively convex if $\log t$ is convex, or, equivalently, if for all $x, y \in I$ and $t \in [0, 1]$ one has the inequality*

$$f(tx + (1-t) y) \leq [f(x)]^t [f(y)]^{1-t}. \tag{1.7}$$

We note that if $f$ and $g$ are convex and $g$ is increasing, then $g \circ f$ is convex; moreover, since $f = \exp(\log f)$, it follows that a $\log-$convex function is convex, but the converse may not necessarily be true [12]. This follows directly from (1.7) because, by the aritmetic-geometric mean inequality, we have

$$[f(x)]^t [f(y)]^{1-t} \leq tf(x) + (1-t) f(y)$$

for all $x, y \in I$ and $t \in [0, 1]$.

For some results related to this classical results, (see [13], [14], [15], [16], [17]) and the references therein.

In [17], Sarıkaya has defined the $\log-\varphi-$convex function as following.

**Definition 7.** *Let us consider a $\varphi:[a,b]\to[a,b]$ where $[a,b]\subset\mathbb{R}$ and $I$ stands for a convex subset of $\mathbb{R}$. We say that a function $f:I\to\mathbb{R}^{+}$ is a $\log-\varphi-$convex if*

$$f\left(t\varphi\left(x\right)+\left(1-t\right)\varphi\left(y\right)\right)\leq\left[f\left(\varphi\left(x\right)\right)\right]^{t}\left[f\left(\varphi\left(y\right)\right)\right]^{1-t} \tag{1.8}$$

*for all $x,y\in I$ and $t\in[0,1]$.*

**Theorem 1.** *Let $g:I\to\mathbb{R}$ be a convex function and $f:I\to\mathbb{R}$ a $g-$convex dominated mapping. Then for all $a,b\in I$ with $a<b$,*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_{a}^{b}f\left(x\right)dx\right|\leq\frac{1}{b-a}\int_{a}^{b}g\left(x\right)dx-g\left(\frac{a+b}{2}\right) \tag{1.9}$$

*and*

$$\left|\frac{f\left(a\right)+f\left(b\right)}{2}-\frac{1}{b-a}\int_{a}^{b}f\left(x\right)dx\right|\leq\frac{g\left(a\right)+g\left(b\right)}{2}-\frac{1}{b-a}\int_{a}^{b}g\left(x\right)dx. \tag{1.10}$$

In [4], Kavurmacı *et al.* proved the next theorem:

**Theorem 2.** *Let $h:J\to\mathbb{R}$ be a non-negative function, $h\neq0$, $g:I\to\mathbb{R}$ be an $h-$convex function and the real function $f:I\to\mathbb{R}$ be a $(g,h)-$convex dominated on $I$. Then one has the inequalities:*

$$\left|\frac{1}{b-a}\int_{a}^{b}f\left(x\right)dx-\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)\right|\leq\frac{1}{b-a}\int_{a}^{b}g\left(x\right)dx-\frac{1}{2h\left(\frac{1}{2}\right)}g\left(\frac{a+b}{2}\right) \tag{1.11}$$

*and*

$$\left|\left[f\left(a\right)+f\left(b\right)\right]\int_{0}^{1}h\left(\lambda\right)d\lambda-\frac{1}{b-a}\int_{a}^{b}f\left(x\right)dx\right|\leq\left[g\left(a\right)+g\left(b\right)\right]\int_{0}^{1}h\left(\lambda\right)d\lambda-\frac{1}{b-a}\int_{a}^{b}g\left(x\right)dx \tag{1.12}$$

*for all $x,y\in I$ and $\lambda\in[0,1]$.*

In [6], Özdemir *et al.* proved the following theorem:

**Theorem 3.** *Let a nonnegative function $g:I\subseteq\mathbb{R}\to\mathbb{R}$ belong to the class of $P\left(I\right)$. The real function $f:I\subseteq\mathbb{R}\to\mathbb{R}$ is $(g,P\left(I\right))-$convex dominated on $I$. If $a,b\in I$ with $a<b$ and $f,g\in L_{1}\left[a,b\right]$, then one has the inequalities:*

$$\left|\frac{2}{b-a}\int_{a}^{b}f\left(x\right)dx-f\left(\frac{a+b}{2}\right)\right|\leq\frac{2}{b-a}\int_{a}^{b}g\left(x\right)dx-g\left(\frac{a+b}{2}\right) \tag{1.13}$$

*and*

$$\left|\left[f\left(a\right)+f\left(b\right)\right]-\frac{1}{b-a}\int_{a}^{b}f\left(x\right)dx\right|\leq\left[g\left(a\right)+g\left(b\right)\right]-\frac{1}{b-a}\int_{a}^{b}g\left(x\right)dx \tag{1.14}$$

*for all $x,y\in I$.*

In [11], Özdemir *et al.* proved the following theorem:

**Theorem 4.** *Let $h:(0,1)\to(0,\infty)$ be a given function, $g:I\to[0,\infty)$ be a given $\varphi_h-$convex function. If $f:I\to[0,\infty)$ is Lebesgue integrable and $(g,\varphi_h)-$convex dominated on $I$ for linear continious, non-constant function $\varphi:[a,b]\to[a,b]$, then the following inequalities hold:*

$$\left|\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)}f(x)\,dx-\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right| \tag{1.15}$$

$$\leq \frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)}g(x)\,dx-\frac{1}{2h\left(\frac{1}{2}\right)}g\left(\frac{\varphi(a)+\varphi(b)}{2}\right)$$

*and*

$$\left|[f(\varphi(a))+f(\varphi(b))]\int_0^1 h(t)\,dt-\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)}f(x)\,dx\right| \tag{1.16}$$

$$\leq [g(\varphi(a))+g(\varphi(b))]\int_0^1 h(t)\,dt-\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)}g(x)\,dx$$

*for all $x,y,a\in[0,b]$, $t\in(0,1)$ and $m\in(0,1]$.*

In the following sections our main results are given: We introduce the notion of $\left(g,\varphi_{h,m}\right)-$convex and $(g,\log\varphi)-$ convex dominated function and present some properties of them. Besides, we present some Hermite-Hadamard-type inequalities for $\left(g,\varphi_{h,m}\right)-$convex and $(g,\log\varphi)-$ convex dominated functions. Our results generalize the Hermite-Hadamard-type inequalities in [2], [4], [6] and [11].

## 2. $\left(g,\varphi_{h,m}\right)-$CONVEX DOMINATED FUNCTIONS

**Definition 8.** *Let $h:(0,1)\to\mathbb{R}$ be a non-negative function, $h\neq 0$, $g:[0,b]\subseteq[0,\infty)\to\mathbb{R}^+$ be a given $\varphi_{h,m}-$convex function. The real function $f:[0,b]\to\mathbb{R}^+$ is called $\left(g,\varphi_{h,m}\right)-$convex dominated on $[0,b]$ if the following condition is satisfied*

$$|h(t)f(\varphi(x))+mh(1-t)f(\varphi(y))-f(t\varphi(x)+m(1-t)\varphi(y))| \tag{2.1}$$

$$\leq h(t)g(\varphi(x))+mh(1-t)g(\varphi(y))-g(t\varphi(x)+m(1-t)\varphi(y))$$

*for all $x,y\in[0,b]$, $t\in(0,1)$ and $m\in[0,1]$.*

In particular, if $f$ satisfies (2.1) with $m=1$, then $f$ is said to be $(g,\varphi_h)-$convex dominated function. If the inequality (2.1) is reversed, then $f$ is said to be $\varphi_{h,m}-$concave dominated function on $[0,b]$.

The next simple characterisation of $\left(g,\varphi_{h,m}\right)-$convex dominated functions holds.

**Lemma 1.** *Let $h:(0,1)\to(0,\infty)$ be a given function, $g:[0,b]\subseteq[0,\infty)\to\mathbb{R}^+$ be a given $\varphi_{h,m}-$convex function and $f:[0,b]\to\mathbb{R}^+$ be a real function. The following statements are equivalent:*

1. *$f$ is $\left(g,\varphi_{h,m}\right)-$convex dominated on $[0,b]$.*
2. *The mappings $g-f$ and $g+f$ are $\varphi_{h,m}-$ convex on $[0,b]$.*
3. *There exist two $\varphi_{h,m}-$convex mappings $l,k$ defined on $[0,b]$ such that*

$$f=\tfrac{1}{2}(l-k)\quad\text{and}\quad g=\tfrac{1}{2}(l+k)\ .$$

*Proof.* 1$\Longleftrightarrow$2 The condition (2.1) is equivalent to

$$\begin{aligned} & g\left(t\varphi\left(x\right)+m\left(1-t\right)\varphi\left(y\right)\right)-h\left(t\right)g(\varphi\left(x\right))-mh(1-t)g(\varphi\left(y\right)) \\ \leq\ & h\left(t\right)f(\varphi\left(x\right))+mh(1-t)f(\varphi\left(y\right))-f\left(t\varphi\left(x\right)+m\left(1-t\right)\varphi\left(y\right)\right) \\ \leq\ & h\left(t\right)g(\varphi\left(x\right))+mh(1-t)g(\varphi\left(y\right))-g\left(t\varphi\left(x\right)+m\left(1-t\right)\varphi\left(y\right)\right) \end{aligned}$$

for all $x,y\in[0,b]$ and $t\in(0,1)$. The two inequalities may be rearranged as

$$\begin{aligned} & \left(g+f\right)\left(t\varphi\left(x\right)+m\left(1-t\right)\varphi\left(y\right)\right) \\ \leq\ & h\left(t\right)\left(g+f\right)\left(\varphi\left(x\right)\right)+mh(1-t)\left(g+f\right)\left(\varphi\left(y\right)\right) \end{aligned}$$

and

$$\begin{aligned} & \left(g-f\right)\left(t\varphi\left(x\right)+m\left(1-t\right)\varphi\left(y\right)\right) \\ \leq\ & h\left(t\right)\left(g-f\right)\left(\varphi\left(x\right)\right)+mh(1-t)\left(g-f\right)\left(\varphi\left(y\right)\right) \end{aligned}$$

which are eqivalent to the $\varphi_{h,m}-$convexity of $g+f$ and $g-f$, respectively.

2$\Longleftrightarrow$3 Let we define the mappings $f$, $g$ as $f=\frac{1}{2}\left(l-k\right)$ and $g=\frac{1}{2}\left(l+k\right)$. Then if we sum and subtract $f$ and $g$, respectively, we have $g+f=l$ and $g-f=k$. By the condition 2 in Lemma 1, the mappings $g-f$ and $g+f$ are $\varphi_{h,m}-$convex on $[0,b]$, so $l,k$ are $\varphi_{h,m}-$convex mappings on $[0,b]$ too. □

**Theorem 5.** *Let $h:(0,1)\to(0,\infty)$ be a given function, $g:[0,b]\subseteq[0,\infty)\to\mathbb{R}^{+}$ be a given $\varphi_{h,m}-$convex function. If $f$ is defined from $[0,b]$ to $[0,\infty)$ and it is Lebesgue integrable with $\left(g,\varphi_{h,m}\right)-$convex dominated on $[0,b]$ for linear continuous, non-constant function $\varphi:[0,b]\to[0,b]$, then the following inequalities hold:*

$$\begin{aligned} & \left|\frac{1}{\varphi\left(b\right)-\varphi\left(a\right)}\int_{\varphi(a)}^{\varphi(b)}f\left(x\right)dx+\frac{m^{2}}{\varphi\left(b\right)-\varphi\left(a\right)}\int_{\frac{\varphi(a)}{m}}^{\frac{\varphi(b)}{m}}f\left(x\right)dx-\frac{1}{h\left(\frac{1}{2}\right)}f\left(\frac{\varphi\left(a\right)+\varphi\left(b\right)}{2}\right)\right| \\ \leq\ & \frac{1}{\varphi\left(b\right)-\varphi\left(a\right)}\int_{\varphi(a)}^{\varphi(b)}g\left(x\right)dx+\frac{m^{2}}{\varphi\left(b\right)-\varphi\left(a\right)}\int_{\frac{\varphi(a)}{m}}^{\frac{\varphi(b)}{m}}g\left(x\right)dx-\frac{1}{h\left(\frac{1}{2}\right)}g\left(\frac{\varphi\left(a\right)+\varphi\left(b\right)}{2}\right) \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} & \left|\begin{array}{l}\left[f\left(\varphi\left(a\right)\right)+m\left\{f\left(\varphi\left(\frac{a}{m}\right)\right)+f\left(\varphi\left(\frac{b}{m}\right)\right)+mf\left(\varphi\left(\frac{b}{m^{2}}\right)\right)\right\}\right]\int_{0}^{1}h\left(t\right)dt \\ -\left[\frac{1}{m\varphi\left(\frac{b}{m}\right)-\varphi(a)}\int_{\varphi(a)}^{m\varphi\left(\frac{b}{m}\right)}f\left(x\right)dx+\frac{m}{m\varphi\left(\frac{b}{m^{2}}\right)-\varphi\left(\frac{a}{m}\right)}\int_{\varphi\left(\frac{a}{m}\right)}^{m\varphi\left(\frac{b}{m^{2}}\right)}f\left(x\right)dx\right]\end{array}\right| \\ \leq\ & \left[g\left(\varphi\left(a\right)\right)+m\left\{g\left(\varphi\left(\frac{a}{m}\right)\right)+g\left(\varphi\left(\frac{b}{m}\right)\right)+mg\left(\varphi\left(\frac{b}{m^{2}}\right)\right)\right\}\right]\int_{0}^{1}h\left(t\right)dt \\ & -\left[\frac{1}{m\varphi\left(\frac{b}{m}\right)-\varphi\left(a\right)}\int_{\varphi(a)}^{m\varphi\left(\frac{b}{m}\right)}g\left(x\right)dx+\frac{m}{m\varphi\left(\frac{b}{m^{2}}\right)-\varphi\left(\frac{a}{m}\right)}\int_{\varphi\left(\frac{a}{m}\right)}^{m\varphi\left(\frac{b}{m^{2}}\right)}g\left(x\right)dx\right] \end{aligned} \tag{2.3}$$

for all $x,y,a\in[0,b]$, $t\in(0,1)$ and $m\in(0,1]$.

*Proof.* By the Definition 8 with $t=\frac{1}{2}$, $x=\lambda a+(1-\lambda)b$, $y=(1-\lambda)\frac{a}{m}+\lambda\frac{b}{m}$, $\lambda\in[0,1]$ and $m\in(0,1]$, as the mapping $f$ is $\left(g,\varphi_{h,m}\right)-$convex dominated function, we have that

$$\begin{aligned}&\left|\begin{array}{c} h\left(\frac{1}{2}\right)\left[f\left(\varphi\left(\lambda a+(1-\lambda)b\right)\right)+mf\left(\varphi\left((1-\lambda)\frac{a}{m}+\lambda\frac{b}{m}\right)\right)\right] \\ -f\left(\frac{\varphi(\lambda a+(1-\lambda)b)+m\varphi\left((1-\lambda)\frac{a}{m}+\lambda\frac{b}{m}\right)}{2}\right)\end{array}\right| \\ \leq & \\ &h\left(\frac{1}{2}\right)\left[g\left(\varphi\left(\lambda a+(1-\lambda)b\right)\right)+mg\left(\varphi\left((1-\lambda)\frac{a}{m}+\lambda\frac{b}{m}\right)\right)\right] \\ &-g\left(\frac{\varphi\left(\lambda a+(1-\lambda)b\right)+m\varphi\left((1-\lambda)\frac{a}{m}+\lambda\frac{b}{m}\right)}{2}\right).\end{aligned}$$

Then using the linearity of $\varphi-$function, we have

$$\begin{aligned}&\left|h\left(\frac{1}{2}\right)\left[f\left(\lambda\varphi\left(a\right)+(1-\lambda)\varphi\left(b\right)\right)+mf\left(\frac{1-\lambda}{m}\varphi\left(a\right)+\frac{\lambda}{m}\varphi\left(b\right)\right)\right]-f\left(\frac{\varphi\left(a\right)+\varphi\left(b\right)}{2}\right)\right| \\ \leq & \\ &h\left(\frac{1}{2}\right)\left[g\left(\lambda\varphi\left(a\right)+(1-\lambda)\varphi\left(b\right)\right)+mg\left(\frac{1-\lambda}{m}\varphi\left(a\right)+\frac{\lambda}{m}\varphi\left(b\right)\right)\right]-g\left(\frac{\varphi\left(a\right)+\varphi\left(b\right)}{2}\right).\end{aligned}$$

If we integrate the above inequality with respect to $\lambda$ over $[0,1]$, the inequality (2.2) is proved.

Since $f$ is $\left(g,\varphi_{h,m}\right)-$convex dominated on $[0,b]$, we have

$$\begin{aligned}&\left|h\left(t\right)f\left(\varphi\left(x\right)\right)+mh\left(1-t\right)f\left(\varphi\left(y\right)\right)-f\left(t\varphi\left(x\right)+m\left(1-t\right)\varphi\left(y\right)\right)\right| \\ \leq\ & h\left(t\right)g\left(\varphi\left(x\right)\right)+mh\left(1-t\right)g\left(\varphi\left(y\right)\right)-g\left(t\varphi\left(x\right)+m\left(1-t\right)\varphi\left(y\right)\right)\end{aligned}$$

for all $x,y>0$ which gives for $x=a$ and $y=b$

$$\begin{aligned}&\left|h\left(t\right)f\left(\varphi\left(a\right)\right)+mh\left(1-t\right)f\left(\varphi\left(\frac{b}{m}\right)\right)-f\left(t\varphi\left(a\right)+m\left(1-t\right)\varphi\left(\frac{b}{m}\right)\right)\right| \\ \leq & \\ &h\left(t\right)g\left(\varphi\left(a\right)\right)+mh\left(1-t\right)g\left(\varphi\left(\frac{b}{m}\right)\right)-g\left(t\varphi\left(a\right)+m\left(1-t\right)\varphi\left(\frac{b}{m}\right)\right)\end{aligned}$$

and for $x=\frac{a}{m}$, $y=\frac{b}{m^2}$, multiplying with $m$,

$$\begin{aligned}&\left|mh\left(t\right)f\left(\varphi\left(\frac{a}{m}\right)\right)+m^2h\left(1-t\right)f\left(\varphi\left(\frac{b}{m^2}\right)\right)-mf\left(t\varphi\left(\frac{a}{m}\right)+m\left(1-t\right)\varphi\left(\frac{b}{m^2}\right)\right)\right| \\ \leq & \\ &mh\left(t\right)g\left(\varphi\left(\frac{a}{m}\right)\right)+m^2h\left(1-t\right)g\left(\varphi\left(\frac{b}{m^2}\right)\right)-mg\left(t\varphi\left(\frac{a}{m}\right)+m\left(1-t\right)\varphi\left(\frac{b}{m^2}\right)\right)\end{aligned}$$

for all $t \in (0,1)$. By properties of modulus and adding the above inequalities side by side we get,

$$\begin{aligned}
&\left| \begin{array}{l} h(t)\left[f(\varphi(a)) + mf\left(\varphi\left(\frac{a}{m}\right)\right)\right] + mh(1-t)\left[f\left(\varphi\left(\frac{b}{m}\right)\right) + mf\left(\varphi\left(\frac{b}{m^2}\right)\right)\right] \\ \quad -\left[f\left(t\varphi(a) + m(1-t)\varphi\left(\frac{b}{m}\right)\right) + mf\left(t\varphi\left(\frac{a}{m}\right) + m(1-t)\varphi\left(\frac{b}{m^2}\right)\right)\right] \end{array} \right| \\
\leq & \\
&h(t)\left[g(\varphi(a)) + mg\left(\varphi\left(\frac{a}{m}\right)\right)\right] + mh(1-t)\left[g\left(\varphi\left(\frac{b}{m}\right)\right) + mg\left(\varphi\left(\frac{b}{m^2}\right)\right)\right] \\
&-\left[g\left(t\varphi(a) + m(1-t)\varphi\left(\frac{b}{m}\right)\right) + mg\left(t\varphi\left(\frac{a}{m}\right) + m(1-t)\varphi\left(\frac{b}{m^2}\right)\right)\right]
\end{aligned}$$

Thus, integrating over $t$ on $[0,1]$ we obtain the inequality (2.3). □

**Remark 1.** *Under the assupmtions of Theorem 5, if we choose $m = 1$, the inequalities (2.2) and (2.3) reduce to Hermite-Hadamard type inequalities for $(g, \varphi_h) -$convex dominated functions given as (1.15) and (1.16) in* [11].

**Remark 2.** *Under the assupmtions of Theorem 5, if we choose $m = 1$, $h(t) = t$, $t \in (0,1)$ and the function $\varphi$ is the identity, then the inequalities (2.2) and (2.3) reduce to Hermite-Hadamard type inequalities for convex-dominated functions given as (1.9) and (1.10) in* [2].

**Remark 3.** *Under the assupmtions of Theorem 5, if we choose $m = 1$, $h(t) = t^s$, $t, s \in (0,1)$ and the function $\varphi$ is the identity, then the inequalities (2.2) and (2.3) reduce to Hermite-Hadamard type inequalities for $(g, s) -$convex-dominated functions given as (1.9) and (1.10) in* [4].

**Remark 4.** *Under the assupmtions of Theorem 5, if we choose $m = 1$, $h(t) = \frac{1}{t}$, $t \in (0,1)$ and the function $\varphi$ is the identity, then the inequalities (2.2) and (2.3) reduce to Hermite-Hadamard type inequalities for $(g, Q(I)) -$convex-dominated functions given as (1.9) and (1.10) in* [6].

## 3. $(g, \log\varphi) -$CONVEX DOMINATED FUNCTIONS

**Definition 9.** *Let $g : [a,b] \subseteq \mathbb{R} \to (0,\infty)$ be a given $\log -\varphi -$convex mapping where $\varphi : [a,b] \to [a,b]$. The real function $f : [a,b] \to (0,\infty)$ is called $(g, \log -\varphi) -$convex dominated on $[a,b]$ if it holds*

$$\begin{aligned}
&\left|[f(\varphi(x))]^t [f(\varphi(y))]^{1-t} - f(t\varphi(x) + (1-t)\varphi(y))\right| \\
\leq\ &[g(\varphi(x))]^t [g(\varphi(y))]^{1-t} - g(t\varphi(x) + (1-t)\varphi(y))
\end{aligned} \tag{3.1}$$

*for all $x, y \in [a,b]$ and $t \in [0,1]$.*

**Proposition 1.** *Let $g : [a,b] \subseteq \mathbb{R} \to (0,\infty)$ be a given $\log -\varphi -$convex mapping where $\varphi : [a,b] \to [a,b]$ and $f : [a,b] \to (0,\infty)$ be a $(g, \log -\varphi) -$convex dominated function on $[a,b]$. Then the mapping $g + f$ is $\varphi -$convex on $[a,b]$.*

*Proof.* The condition (3.1) is equivalent to

$$\begin{aligned} & g\left(t\varphi\left(x\right)+\left(1-t\right)\varphi\left(y\right)\right)-\left[g\left(\varphi\left(x\right)\right)\right]^{t}\left[g\left(\varphi\left(y\right)\right)\right]^{1-t} \\ \leq\ & \left[f\left(\varphi\left(x\right)\right)\right]^{t}\left[f\left(\varphi\left(y\right)\right)\right]^{1-t}-f\left(t\varphi\left(x\right)+\left(1-t\right)\varphi\left(y\right)\right) \\ \leq\ & \left[g\left(\varphi\left(x\right)\right)\right]^{t}\left[g\left(\varphi\left(y\right)\right)\right]^{1-t}-g\left(t\varphi\left(x\right)+\left(1-t\right)\varphi\left(y\right)\right) \end{aligned}$$

for all $x,y\in\left[a,b\right]$ and $t\in\left[0,1\right]$. The left side of the inequality may be rearranged as

$$\begin{aligned} & \left(g+f\right)\left(t\varphi\left(x\right)+\left(1-t\right)\varphi\left(y\right)\right) \\ \leq\ & \left[f\left(\varphi\left(x\right)\right)\right]^{t}\left[f\left(\varphi\left(y\right)\right)\right]^{1-t}+\left[g\left(\varphi\left(x\right)\right)\right]^{t}\left[g\left(\varphi\left(y\right)\right)\right]^{1-t} \\ \leq\ & tf\left(\varphi\left(x\right)\right)+\left(1-t\right)f\left(\varphi\left(y\right)\right)+tg\left(\varphi\left(x\right)\right)+\left(1-t\right)g\left(\varphi\left(y\right)\right) \\ =\ & t\left(f+g\right)\left(\varphi\left(x\right)\right)+\left(1-t\right)\left(f+g\right)\left(\varphi\left(y\right)\right) \end{aligned}$$

which is eqivalent to the $\varphi-$convexity of $f+g$. □

**Theorem 6.** *Let $g:\left[a,b\right]\subseteq\mathbb{R}\rightarrow\left(0,\infty\right)$ be a given $\log-\varphi-$convex mapping and $f:\left[a,b\right]\rightarrow\left(0,\infty\right)$ is Lebesgue integrable and $\left(g,\log-\varphi\right)-$convex dominated function on $\left[a,b\right]$ for linear continuous function $\varphi:\left[a,b\right]\rightarrow\left[a,b\right]$, then the following inequalities hold:*

$$\left|\frac{1}{\varphi\left(b\right)-\varphi\left(a\right)}\int_{\varphi(a)}^{\varphi(b)}G\left(f\left(x\right),f\left(\varphi\left(a\right)+\varphi\left(b\right)-x\right)\right)dx-f\left(\frac{\varphi\left(a\right)+\varphi\left(b\right)}{2}\right)\right|$$

(3.2)

$$\leq\ \frac{1}{\varphi\left(b\right)-\varphi\left(a\right)}\int_{\varphi(a)}^{\varphi(b)}G\left(g\left(x\right),g\left(\varphi\left(a\right)+\varphi\left(b\right)-x\right)\right)dx-g\left(\frac{\varphi\left(a\right)+\varphi\left(b\right)}{2}\right)$$

(3.3)

$$\left|\frac{1}{\varphi\left(b\right)-\varphi\left(a\right)}\int_{\varphi(a)}^{\varphi(b)}G\left(f\left(x\right),f\left(\varphi\left(a\right)+\varphi\left(b\right)-x\right)\right)dx-f\left(\frac{\varphi\left(a\right)+\varphi\left(b\right)}{2}\right)\right|$$

$$\leq\ \frac{1}{\varphi\left(b\right)-\varphi\left(a\right)}\int_{\varphi(a)}^{\varphi(b)}g\left(x\right)dx-g\left(\frac{\varphi\left(a\right)+\varphi\left(b\right)}{2}\right)$$

and

$$\left|L\left(f\left(\varphi\left(b\right)\right),f\left(\varphi\left(a\right)\right)\right)-\frac{1}{\varphi\left(b\right)-\varphi\left(a\right)}\int_{\varphi(a)}^{\varphi(b)}f\left(x\right)dx\right|$$

(3.4)

$$\leq\ L\left(g\left(\varphi\left(b\right)\right),g\left(\varphi\left(a\right)\right)\right)-\frac{1}{\varphi\left(b\right)-\varphi\left(a\right)}\int_{\varphi(a)}^{\varphi(b)}g\left(x\right)dx$$

for all $x,y\in\left[a,b\right]$.

*Proof.* By the Definition 3.1 with $t=\frac{1}{2}$, $x=\lambda a+(1-\lambda)b$, $y=(1-\lambda)a+\lambda b$ and $\lambda\in[0,1]$, as the mapping $f$ is $(g,\log-\varphi)-$convex dominated function, we have that

$$\left|\left[f\left(\varphi\left(\lambda a+(1-\lambda)b\right)\right)\right]^{\frac{1}{2}}\left[f\left(\varphi\left(\left(1-\lambda\right)a+\lambda b\right)\right)\right]^{\frac{1}{2}}-f\left(\frac{\varphi\left(\lambda a+(1-\lambda)b\right)+\varphi\left(\left(1-\lambda\right)a+\lambda b\right)}{2}\right)\right|$$

$$\leq\left[g\left(\varphi\left(\lambda a+(1-\lambda)b\right)\right)\right]^{\frac{1}{2}}\left[g\left(\varphi\left(\left(1-\lambda\right)a+\lambda b\right)\right)\right]^{\frac{1}{2}}-g\left(\frac{\varphi\left(\lambda a+(1-\lambda)b\right)+\varphi\left(\left(1-\lambda\right)a+\lambda b\right)}{2}\right).$$

Then using the linearity of $\varphi-$function we have

$$\left|\left[f\left(\lambda\varphi\left(a\right)+(1-\lambda)\varphi\left(b\right)\right)\right]^{\frac{1}{2}}\left[f\left(\left(1-\lambda\right)\varphi\left(a\right)+\lambda\varphi\left(b\right)\right)\right]^{\frac{1}{2}}-f\left(\frac{\varphi\left(a\right)+\varphi\left(b\right)}{2}\right)\right| \tag{3.5}$$

$$\leq$$

$$\left[g\left(\lambda\varphi\left(a\right)+(1-\lambda)\varphi\left(b\right)\right)\right]^{\frac{1}{2}}\left[g\left(\left(1-\lambda\right)\varphi\left(a\right)+\lambda\varphi\left(b\right)\right)\right]^{\frac{1}{2}}-g\left(\frac{\varphi\left(a\right)+\varphi\left(b\right)}{2}\right).$$

If we integrate the above inequality with respect to $\lambda$ over $[0,1]$, the inequality in (3.2) is proved.

On the other hand, if we use the inequality $\sqrt{ab}\leq\frac{1}{2}\left(a+b\right)$ for $a,b>0$ on (3.5) we have

$$\left|\left[f\left(\lambda\varphi\left(a\right)+(1-\lambda)\varphi\left(b\right)\right)\right]^{\frac{1}{2}}\left[f\left(\left(1-\lambda\right)\varphi\left(a\right)+\lambda\varphi\left(b\right)\right)\right]^{\frac{1}{2}}-f\left(\frac{\varphi\left(a\right)+\varphi\left(b\right)}{2}\right)\right|$$

$$\leq$$

$$\frac{g\left(\lambda\varphi\left(a\right)+\left(1-\lambda\right)\varphi\left(b\right)\right)+g\left(\left(1-\lambda\right)\varphi\left(a\right)+\lambda\varphi\left(b\right)\right)}{2}-g\left(\frac{\varphi\left(a\right)+\varphi\left(b\right)}{2}\right).$$

If we integrate the above inequality with respect to $\lambda$ over $[0,1]$, the inequality in (3.3) is proved.

To prove the inequality in (3.4), firstly we use the Definition 3.1 for $x=a$ and $y=b$, we have

$$\left|\left[f\left(\varphi\left(a\right)\right)\right]^{t}\left[f\left(\varphi\left(b\right)\right)\right]^{1-t}-f\left(t\varphi\left(a\right)+\left(1-t\right)\varphi\left(b\right)\right)\right|$$

$$\leq\left[g\left(\varphi\left(a\right)\right)\right]^{t}\left[g\left(\varphi\left(b\right)\right)\right]^{1-t}-g\left(t\varphi\left(a\right)+\left(1-t\right)\varphi\left(b\right)\right)$$

Then, we integrate the above inequality with respect to $t$ over $[0,1]$, we get

$$\left|f\left(\varphi\left(b\right)\right)\int_{0}^{1}\left[\frac{f\left(\varphi\left(a\right)\right)}{f\left(\varphi\left(b\right)\right)}\right]^{t}dt-\int_{0}^{1}f\left(t\varphi\left(a\right)+\left(1-t\right)\varphi\left(b\right)\right)dt\right|$$

$$\leq g\left(\varphi\left(b\right)\right)\int_{0}^{1}\left[\frac{g\left(\varphi\left(a\right)\right)}{g\left(\varphi\left(b\right)\right)}\right]^{t}dt-\int_{0}^{1}g\left(t\varphi\left(a\right)+\left(1-t\right)\varphi\left(b\right)\right)dt$$

$$=g\left(\varphi\left(b\right)\right)\left(\frac{g\left(\varphi\left(a\right)\right)}{g\left(\varphi\left(b\right)\right)}-1\right)\frac{1}{\log g\left(\varphi\left(a\right)\right)-\log g\left(\varphi\left(b\right)\right)}-\frac{1}{\varphi\left(b\right)-\varphi\left(a\right)}\int_{\varphi(a)}^{\varphi(b)}g\left(x\right)dx$$

$$=\frac{g\left(\varphi\left(b\right)\right)-g\left(\varphi\left(a\right)\right)}{\log f\left(\varphi\left(b\right)\right)-\log f\left(\varphi\left(a\right)\right)}-\frac{1}{\varphi\left(b\right)-\varphi\left(a\right)}\int_{\varphi(a)}^{\varphi(b)}g\left(x\right)dx$$

$$=L\left(g\left(\varphi\left(b\right)\right),g\left(\varphi\left(a\right)\right)\right)-\frac{1}{\varphi\left(b\right)-\varphi\left(a\right)}\int_{\varphi(a)}^{\varphi(b)}g\left(x\right)dx.$$

If similar calculation is made for the function $f$, the proof is completed. □

**Corollary 1.** *If function $\varphi$ is the identity in (3.2), (3.3) and (3.4), then we have*

$$\left|\frac{1}{b-a}\int_a^b G\left(f\left(x\right),f\left(a+b-x\right)\right)dx - f\left(\frac{a+b}{2}\right)\right|$$

(3.6)

$$\leq \frac{1}{b-a}\int_a^b G\left(g\left(x\right),g\left(a+b-x\right)\right)dx - g\left(\frac{a+b}{2}\right)$$

(3.7)
$$\left|\frac{1}{b-a}\int_a^b G\left(f\left(x\right),f\left(a+b-x\right)\right)dx - f\left(\frac{a+b}{2}\right)\right|$$

$$\leq \frac{1}{b-a}\int_a^b g\left(x\right)dx - g\left(\frac{a+b}{2}\right)$$

*and*

$$\left|L\left(f\left(b\right),f\left(a\right)\right) - \frac{1}{b-a}\int_a^b f\left(x\right)dx\right|$$

(3.8)

$$\leq L\left(g\left(b\right),g\left(a\right)\right) - \frac{1}{b-a}\int_a^b g\left(x\right)dx$$

*for all $x,y\in[a,b]$.*

## References


[1] Dragomir S.S. and Ionescu N.M. On some inequalities for convex-dominated functions, *Anal. Num. Theor. Approx.*, 19 (1990), 21-28. MR 936: 26014 ZBL No.733: 26010.

[2] Dragomir S.S., Pearce C.E.M. and Pečarić J.E. Means, $g-$Convex Dominated & Hadamard-Type Inequalities, *Tamsui Oxford Journal of Mathematical Sciences* 18(2) 2002, 161-173.161-173.

[3] Hwang S. and Ho M. Inequalities of Fejér Type for $G-$convex Dominated Functions, *Tamsui Oxford Journal of Mathematical Sciences*, 25 (1) (2009) 55-69.

[4] Kavurmacı H., Özdemir M.E. and Sarıkaya M.Z. New Definitions and Theorems via Different Kinds of Convex Dominated Functions, *submitted.*

[5] Özdemir M.E., Kavurmacı H. and Tunç M. Hermite-Hadamard-type inequalities for new different kinds of convex dominated functions, *submitted.*

[6] Özdemir M.E.,Tunç M. and Kavurmacı H. Two new different kinds of convex dominated functions and inequalities via Hermite-Hadamard type, *submitted.*

[7] S. Varošanec, On $h-$convexity, J. Math. Anal. Appl. 326 (2007) 303-311.

[8] Sarıkaya M.Z. On Hermite-Hadamard-type inequalities for $\varphi_h-$convex functions, *RGMIA Ress. Rep. Coll., 15(37) 2012.*

[9] Youness E. A. E-Convex Sets, E-Convex Functions and E - Convex Programming, *Journal of Optimization Theory and Applications*, 102, 2(1999), 439-450.

[10] Özdemir M.E. Akdemir A.O. and Set E. On $(h-m)-$Convexity and Hadamard-Type Inequalities, *Transylvanian Jour. of Math. and Mechanics,* 8, 1(2016), 51-58.

[11] Özdemir M.E., Gürbüz M. and Kavurmacı H. Hermite-Hadamard-Type Inequalities for $(g-\varphi_h)-$Convex Dominated Functions, *Journal of Ineq. and Appl.*, 2013:184.

[12] Pearce C.E.M., Pečarić J.E. and Simic V. Stolarsky means and Hadamard's inequality, *J. Math. Anal. Appl.* 220 (1998) 99-109.

[13] Dragomir S.S. and Pearce C.E.M. Selected Topics on Hermite-Hadamard Inequalities and Applications, *RGMIA Monographs,* Victoria University, 2000.

[14] Dragomir S.S. and Agarwal R.P. Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, *Appl. Math. Lett.,* 11(5) (1998), 91-95.

[15] Set E., Özdemir M.E. and Dragomir S.S. On the Hermite-Hadamard inequality and other integral inequalities involving two functions, *Journal of Ineq. and Appl.*, Art. ID 148102, 9 pages, 2010.

[16] Set E., Özdemir M.E. and Dragomir S.S. On Hadamard-Type inequalities involving several kinds of convexity, *Journal of Ineq. and Appl.,* Art. ID 286845, 12 pagesi 2010.

[17] Sarıkaya M.Z. On Hermite-Hadamard Inequalities for Product of Two $\log-\varphi-$Convex Functions, arXiv:1203.5495v1 [math.FA] 25 Mar 2012.

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# FIXED POINT THEOREM AND A UNIQUENESS THEOREM CONCERNING THE STABILITY OF FUNCTIONAL QUATIONS IN MODULAR SPACES

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Abstract. In this paper, we investigate a fixed point theorem in modular spaces, whose induced modular is lower semi-continunous, for a mapping with some conditions in place of the condition of bounded orbit. Using this fixed point theorem, we prove the generalized Hyers-Ulam stability for the following additive-quadratic functional equation

$$f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-2f(2x)+4f(x)=0$$

in modular spaces.

## 1. Introduction and preliminaries

A problem that mathematicians has dealt with, for almost fifty years, is how to generalize the classical function space $L^p$". A first attempt was made by Birnhaum and Orlicz in 1931. This generalization found many applications in differential and integral equations with kernels of nonpower types. The more abstract generalization was given by Nakano [13] based on replacing the particular integral form of the functional by an abstract one that satisfes some good properties. This functional was called modular. This idea was refined and generalized by Musielak and Orlicz [11] in 1959. Modular spaces have been studied for almost forty years and there is a large set of known applications of them in various parts of analysis([6], [7], [9], [10], [12], [14], [17], [20]).

It is well known that fixed point theories are one of powerful tools in solving mathematical problems. Banach's contraction principle is one of the pivotal results in fixed point theories and they have a board set of applications. Khamsi, Kozowski and Reich [4] investigated the fixed point theorem in modular spaces. In [5], Khamsi proved a series of fixed point theorems in modular spaces, where the modulars do not satisfy $\triangle_2$-conditions.

**Lemma 1.1.** [5] *Let $X_\rho$ be a modular space whose induced modular is lower semi-continuous and let $C \subseteq X_\rho$ be a $\rho$-complete subset. If $T : C \longrightarrow C$ is a $\rho$-contraction, that is, there is a constant $L \in [0,1)$ such that*

$$\rho(Tx-Ty) \leq L\rho(x-y), \ \ \forall x,y \in C$$

*and $T$ has a bounded orbit at a point $x_0 \in C$, then the sequence $\{T^n x_0\}$ is $\rho$-convergent to a point $w \in C$.*

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The stability problem for functional equations first was planed in 1940 by Ulam [18].

"Let $G_1$ be a group and $G_2$ a metric group with the metric $d$. Given a constant $\delta > 0$, does there exist a constant $c > 0$ such that if a mapping $f : G_1 \longrightarrow G_2$ satisfies $d(f(xy), f(x)f(y)) < c$ for all $x, y \in G_1$, then there exists a unique homomorphism $h : G_1 \longrightarrow G_2$ with $d(f(x), h(x)) < \delta$ for all $x \in G_1$?"

In the next year, Hyers [3] gave the first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by Aoki [1] for additive mappings and by Rassias [15] for linear mappings by considering an unbounded Cauchy difference, the latter of which has influenced many developments in the stability theory. This area is then referred to as the generalized Hyers-Ulam stability. In 1994, P. Găvruta [2] generalized these theorems for approximate additive mappings controlled by the unbounded Cauchy difference with regular conditions.

Recently, Sadeghi [16] presented a fixed point method to prove the generalized Hyers-Ulam stability of functional equations in modular spaces with the $\triangle_2$-condition and using the fixed point theorem Lemma 1.1, Wongkum, Chaipunya, and Kumam [19] proved the generalized Hyers-Ulam stability for quadratic mappings in a modular space whose modular is convex, lower semi-continuous but do not satisfy the $\triangle_2$-condition.

Lee and Jung [8] proved a general uniqueness theorem that can be easily applied to the (generalized) Hyers-Ulam stability of the Cauchy additive functional equation, the quadratic functional equation, and the quadratic-additive type functional equations in Banach spaces.

In this paper, we investigate a fixed point theorem in modular spaces, whose induced modular is lower semi-continuous, for a mapping with some conditions in place of the condition of a bounded orbit. Using this fixed point theorem, we will prove a general uniqueness theorem that can be applied to the generalized Hyers-Ulam stability of additive-quadratic functional equations in modular spaces without $\triangle_2$-conditions.

## 2. Fixed point Theorems in modular spaces

In this section, we will prove a fixed point theorem in modular spaces, whose induced modular is (convex) lower semi-continuous, for a mapping with some conditions in place of the condition of a bounded orbit.

**Definition 2.1.** Let $X$ be a vector space over a field $\mathbb{K}(=\mathbb{R}$ or $\mathbb{C})$.
(1) A generalized functional $\rho : X \longrightarrow [0, \infty]$ is called *a modular* if
(M1) $\rho(x) = 0$ if and only if $x = 0$,
(M2) $\rho(\alpha x) = \rho(x)$ for every scalar $\alpha$ with $|\alpha| = 1$, and
(M3) $\rho(z) \leq \rho(x) + \rho(y)$ whenever $z$ is a convex combination of $x$ and $y$.
(2) If (M3) is replaced by
(M4) $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$
for all $x, y \in V$ and for all nonnegative real numbers $\alpha$, $\beta$ with $\alpha + \beta = 1$, then we say that $\rho$ is *convex*.

**Remark 2.2.** Let $\rho$ be a modualr on a vector space $X$. Then by (M1) and (M3), we can easily show that for any positive real number $\delta$ with $\delta < 1$,

$$\rho(\delta x) \leq \rho(x)$$

for all $x \in X$ and hence we have

$$\rho(x) \leq \rho(2x)$$

for all $x \in X$.

For any modular $\rho$ on $X$, the modular space $X_\rho$ is defined by

$$X_\rho := \{x \in X \mid \rho(\lambda x) \to 0 \ as \ \lambda \to 0\}.$$

Let $X_\rho$ be a modular space and let $\{x_n\}$ be a sequence in $X_\rho$. Then (i) $\{x_n\}$ is called *$\rho$-convergent* to a point $x \in X_\rho$, denoted by

$$\lim_{n\to\infty} x_n =_\rho x \ \ or \ \ x =_\rho \lim_{n\to\infty} x_n,$$

if $\rho(x_n - x) \to 0$ as $n \to \infty$, (ii) $\{x_n\}$ is called *$\rho$-Cauchy* if for any $\epsilon > 0$, one has $\rho(x_n - x_m) < \epsilon$ for sufficiently large $m, n \in \mathbb{N}$, and (iii) a subset $K$ of $X_\rho$ is called *$\rho$-complete* if each $\rho$-Cauchy sequence in $K$ is $\rho$-convergent to a ponit in $K$.

**Proposition 2.3.** *Let $\rho$ be a modualr on a vector space $X$ and $S : X \longrightarrow X$ an one-to-one linear map. Define a map $\rho_S : X \longrightarrow [0, \infty]$ by*

$$\rho_S(x) = \rho(S(x)), \ \forall x \in X.$$

*Then we have*

(1) *$\rho_S$ is a modular on $X$,*
(2) *if $\rho$ is a convex modular on $X$, then $\rho_S$ is a convex modular on $X$, and*
(3) *if $\rho$ is lower semi-continuous, then $\rho_S$ is lower semi-continuous.*

*Suppose that $S$ is an isomorphism. Then we have*

(4) *$S(X_{\rho_S}) = X_\rho$ and*
(5) *if $C$ is a $\rho$-complete subset of $X_\rho$ and $S(C) = C$, then $C$ is a $\rho_S$-complete subset of $X_{\rho_S}$.*

*Proof.* (1) Suppose that $\rho_S(x) = 0$. Then by (M1), $S(x) = 0$ and since $S$ is one-to-one, $x = 0$. If $x = 0$, then $\rho_S(0) = \rho(S(0)) = \rho(0) = 0$. Hence $\rho_S$ satisfies (M1). Since $S$ is a linear map, $\rho_S$ satisfies (M2). Let $z = \alpha x + \beta y$ for $x, y \in X$ and non-negative real numbers $\alpha, \beta$ with $\alpha + \beta = 1$. Since $S$ is a linear map and $\rho$ is a modular, by (M3), we have

$$\rho_S(\alpha x + \beta y) = \rho(\alpha S(x) + \beta S(y)) \leq \rho(S(x)) + \rho(S(y)) \leq \rho_S(x) + \rho_S(y)$$

and thus $\rho_S$ satisfies (M3).

(2) is trivial.

(3) Suppose that $\{x_n\}$ is a sequence in $X_{\rho_S}$ such that $\{x_n\}$ is $\rho_S$-convergent to $x$ in $X_{\rho_S}$. Then $\{S(x_n)\}$ is $\rho$-convergent to $S(x)$. Since $\rho$ is lower semi-continuous and $\{S(x_n)\}$ is $\rho$-convergent to $S(x)$,

$$\rho_S(x) = \rho(S(x)) \leq \liminf_{n\to\infty} \rho(S(x_n)) \leq \liminf_{n\to\infty} \rho_S(x_n). \tag{2.1}$$

and hence $\rho_S$ is lower semi-continuous.

(4) Let $x \in X_\rho$. Then

$$\lim_{\lambda\to 0} \rho(\lambda x) = \lim_{\lambda\to 0} \rho_S(\lambda S^{-1}(x)) = 0$$

and so $S^{-1}(x) \in X_{\rho_S}$. Hence $X_\rho \subseteq S(X_{\rho_S})$. For the converse, let $x \in X_{\rho_S}$. Then

$$\lim_{\lambda\to 0} \rho_S(\lambda x) = \lim_{\lambda\to 0} \rho(\lambda S(x)) = 0$$

and so $S(x) \in X_\rho$. Hence $S(X_{\rho_S}) \subseteq X_\rho$.

(5) Suppose that $C$ is a $\rho$-complete subset of $X_\rho$ with $S(C) = C$. By (4), $C \subseteq X_{\rho_S}$. Let $\{x_n\}$ be a $\rho_S$-Cauchy sequence in $C$. For any $n, m \in \mathbb{N}$,

$$\rho(S(x_n) - S(x_m)) = \rho_S(x_n - x_m),$$

$\{S(x_n)\}$ is a $\rho$-Cauchy sequence in $C$. Since $C$ is a $\rho$-complete subset of $X_\rho$, there is an $y \in C$ such that $\{S(x_n)\}$ is $\rho$-convergent to $y$. Then clearly, $\{x_n\}$ is $\rho_S$-convergent to $S^{-1}(y) \in C$ and so $C$ is a $\rho_S$-complete subset of $X_{\rho_S}$. □

A modular space $X_\rho$ is said to *satisfy the $\triangle_2$-condition* if there exists $k \geq 2$ such that $\rho(2x) \leq k\rho(x)$ for all $x \in X$.

Now, we will prove a fixed point theorem in modular spaces where the map $T$ do not assume to be the boundedness of an orbit. Our results exploit one unifying hypothesis in which some conditions are assumed.

**Lemma 2.4.** *Let $X_\rho$ be a modular space whose induced modular is lower semi-continuous and let $C \subseteq X_\rho$ be a $\rho$-complete subset. Let $S : X \longrightarrow X$ be an isomorphism and $T : C \longrightarrow C$ a mapping such that $S(C) = C$ and $STx = TSx$ for all $x \in C$. Suppose that there is a constant $L \in [0, 1)$ and $x_o \in C$ such that $\rho(Tx_o - x_o) < \infty$ and*

$$\rho(x + y) \leq \rho_S(x) + \rho_S(y), \;\; \rho_S(Tx - Ty) \leq L\rho(x - y), \;\; \forall x, y \in C. \tag{2.2}$$

*Then there is a unique fixed point $w \in C$ of $T$ such that*

$$\rho(S^{-2}x_0 - w) \leq \frac{2}{1 - L}\rho(Tx_0 - x_0). \tag{2.3}$$

*Further, we have*

$$\lim_{n\to\infty} \rho(T^n S^{-2}x_0) =_\rho w.$$

*Proof.* By Proposition 2.3, $\rho_S$ is a modular, $C$ is a $\rho_S$-complete subset of $X_{\rho_S}$, and $S(X_{\rho_S}) = X_\rho$. By (M1) and (2.2), we have

$$\rho(x) \leq \rho_S(x), \;\; \rho_S(Tx - Ty) \leq L\rho(x - y) \leq L\rho_S(x - y)$$

for all $x, y \in C$ and so $T$ is a $\rho_S$-contraction. By (M1) and (2.2), we have

$$\begin{aligned}
\rho_S(S^{-2}T^2x_0 - S^{-2}x_0) &= \rho(S^{-1}T^2x_0 - S^{-1}x_0) \\
&\leq \rho_S(S^{-1}T^2x_0 - S^{-1}Tx_0) + \rho_S(S^{-1}Tx_0 - S^{-1}x_0) \\
&\leq L\rho(S^{-1}Tx_0 - S^{-1}x_0) + \rho(Tx_0 - x_0) \\
&\leq (L + 1)\rho(Tx_0 - x_0).
\end{aligned}$$

and

$$\begin{aligned}
\rho_S(S^{-2}T^nx_0 - S^{-2}x_0) &\leq \rho_S(S^{-1}T^nx_0 - S^{-1}Tx_0) + \rho_S(S^{-1}Tx_0 - S^{-1}x_0) \\
&\leq L\rho(S^{-1}T^{n-1}x_0 - S^{-1}x_0) + \rho(Tx_0 - x_0) \\
&= L\rho_S(S^{-2}T^{n-1}x_0 - S^{-2}x_0) + \rho(Tx_0 - x_0)
\end{aligned}$$

for all $n \in \mathbb{N}$. By induction, we have

$$\rho_S(S^{-2}T^nx_0 - S^{-2}x_0) \leq \Sigma_{k=0}^{n-1} L^k \rho(Tx_0 - x_0) \leq \frac{1}{1 - L}\rho(Tx_0 - x_0)$$

for all $n \in \mathbb{N}$. For any non-negative integers $m, n$ with $m > n$,

(2.4)
$$\begin{aligned}\rho_S(S^{-3}T^n x_0 - S^{-3}T^m x_0) &= \rho(S^{-2}T^n x_0 - S^{-2}T^m x_0)\\ &\le \rho_S(S^{-2}T^n x_0 - S^{-2}x_0) + \rho_S(S^{-2}T^m x_0 - S^{-2}x_0)\\ &\le \frac{2}{1-L}\rho(Tx_0 - x_0).\end{aligned}$$

Since $STx = TSx$ for all $x \in C$, $T$ has a bounded orbit at a point $S^{-3}x_0$ in $C \subseteq X_{\rho_S}$ and thus by Lemma 1.1, $\{T^n S^{-3}x_0\}$ is $\rho_S$-convergent to a point $\omega_0 \in C$. Let $\omega = S\omega_0$. Then

$$\lim_{n\to\infty} \rho(T^n S^{-2}x_0) =_\rho w$$

and since $\rho_S$ is lower semi-continuous, by (2.4), we have (2.3).

Now, we claim that $w$ is a unique fixed point of $T$. Since $\rho_S$ is a lower semi-continuous, we have

$$\begin{aligned}\rho(w - Tw) = \rho_S(w_0 - Tw_0) &\le \liminf_{n\to\infty} \rho_S(T^{n+1}S^{-3}x_0 - Tw_0)\\ &\le \liminf_{n\to\infty} L\rho(T^n S^{-3}x_0 - w_0) = 0\end{aligned}$$

and hence $w$ is a fixed point of $T$. Suppose that $v$ is another fixed point of $T$. Since $STx = TSx$ for all $x \in C$, by (2.2) and (2.3), we have

$$\begin{aligned}\rho(S^{-1}w - S^{-1}v) &\le \rho_S(T^n w - T^n S^{-2}x_0) + \rho_S(T^n S^{-2}x_0 - T^n v)\\ &\le L^n \frac{4}{1-L}\rho(Tx_0 - x_0)\end{aligned}$$

for all $n \in \mathbb{N}$ and since $0 \le L < 1$, $w = v$. □

Replacing (2.2) by (2.5), we have the following lemma. The proof is similar to the proof of Lemma 2.4.

**Lemma 2.5.** *Let $X_\rho$ be a modular space whose induced modular is lower semi-continuous and let $C \subseteq X_\rho$ be a $\rho$-complete subset. Let $S : X \longrightarrow X$ be an isomorphism and $T : C \longrightarrow C$ a mapping such that $S(C) = C$ and $STx = TSx$ for all $x \in C$. Suppose that there are real numbers $r, L$ and $x_o \in C$ such that $0 < r < 1$, $L \in [0, \frac{1}{r})$, $\rho(Tx_o - x_o) < \infty$, and*

$$\rho(x+y) \le r\rho_S(x) + r\rho_S(y), \ \ \rho_S(Tx - Ty) \le L\rho(x-y), \ \ \forall x, y \in C. \tag{2.5}$$

*Then there is a unique fixed point $w \in C$ of $T$ such that*

$$\lim_{n\to\infty} \rho(T^n S^{-2}x_0) =_\rho w$$

*and*

$$\rho(S^{-2}x_0 - w) \le \frac{2r^2}{1-rL}\rho(Tx_0 - x_0). \tag{2.6}$$

*Proof.* By Proposition 2.3, $\rho_S$ is a modular, $C$ is a $\rho_S$-complete subset of $X_{\rho_S}$, and $S(X_{\rho_S}) = X_\rho$. By (M1) and (2.5), we have

$$\rho(x) \le \rho_S(x), \ \ \rho_S(Tx - Ty) \le L\rho(x-y) \le rL\rho_S(x-y)$$

for all $x, y \in C$ and so $T$ is a $\rho_S$-contraction. By (M1) and (2.5), we have

$$\begin{aligned}\rho_S(S^{-2}T^2x_0 - S^{-2}x_0) &= \rho_S(S^{-1}T^2x_0 - S^{-1}x_0)\\ &\le r\rho_S(S^{-1}T^2x_0 - S^{-1}Tx_0) + r\rho_S(S^{-1}Tx_0 - S^{-1}x_0)\\ &\le rL\rho(S^{-1}Tx_0 - S^{-1}x_0) + r\rho(Tx_0 - x_0)\\ &\le r(rL+1)\rho(Tx_0 - x_0).\end{aligned}$$

and

$$\begin{aligned}\rho_S(S^{-2}T^nx_0 - S^{-2}x_0) &\le r\rho_S(S^{-1}T^nx_0 - S^{-1}Tx_0) + r\rho_S(S^{-1}Tx_0 - S^{-1}x_0)\\ &\le rL\rho(S^{-1}T^{n-1}x_0 - S^{-1}x_0) + r\rho(Tx_0 - x_0)\\ &= rL\rho_S(S^{-2}T^{n-1}x_0 - S^{-2}x_0) + r\rho(Tx_0 - x_0)\end{aligned}$$

for all $n \in \mathbb{N}$. By induction, we have

$$\rho_S(S^{-2}T^nx_0 - S^{-2}x_0) \le \Sigma_{k=0}^{n-1} r^{k+1}L^k\rho(Tx_0 - x_0) \le \frac{r}{1-rL}\rho(Tx_0 - x_0)$$

for all $n \in \mathbb{N}$. For any non-negative integers $m, n$ with $m > n$,

$$\begin{aligned}\rho_S(S^{-3}T^nx_0 - S^{-3}T^mx_0) &= \rho(S^{-2}T^nx_0 - S^{-2}T^mx_0)\\ &\le r\rho_S(S^{-2}T^nx_0 - S^{-2}x_0) + r\rho_S(S^{-2}T^mx_0 - S^{-2}x_0)\\ &\le \frac{2r^2}{1-rL}\rho(Tx_0 - x_0).\end{aligned}$$

The rest of the proof is similar to the proof of Lemma 2.4 . □

## 3. Uniquness theorem for the stability of functional equations and its applications

Throughout this section, we assume that $V$ is a linear space and $X_\rho$ is a $\rho$-complete modular space whose induced modular is lower semi-continuous. In this section, we prove that, if for given map $f : V \longrightarrow X_\rho$, there is a mapping $F : V \longrightarrow X_\rho$, which is near $f$ in $X_\rho$, with some properties possessed by additive-quadratic mappings, then $F$ is uniquely determined.

Define a set $\mathbb{M}$ by

$$\mathbb{M} := \{g : V \longrightarrow X_\rho \mid g(0) = 0\}$$

and a generalized function $\widetilde{\rho}$ on $\mathbb{M}$ by for each $g \in \mathbb{M}$,

$$\widetilde{\rho}(g) := \inf\{c > 0 \mid \rho(g(x)) \le c\Psi(x),\ \forall x \in V\},$$

where $\Psi : V \longrightarrow [0, \infty)$ is a mapping.

Similar to Lemma 10 in [19], we have the following lemma :

**Lemma 3.1.** *We have the following :*
(1) $\mathbb{M}$ *is a linear space,*
(2) $\widetilde{\rho}$ *is a modular on* $\mathbb{M}$*, and*
(3) *if* $\rho$ *is convex, then* $\widetilde{\rho}$ *is convex,*
(4) $\mathbb{M}_{\widetilde{\rho}} = \mathbb{M}$ *and* $\mathbb{M}_{\widetilde{\rho}}$ *is* $\widetilde{\rho}$*-complete, and*
(5) $\widetilde{\rho}$ *is lower semi-continuous.*

*Proof.* (1), (2), and (3) are trivial.

(4) By the definition of $\mathbb{M}_{\widetilde{\rho}}$, $\mathbb{M}_{\widetilde{\rho}} = \mathbb{M}$. Take any $\widetilde{\rho}$-Cauchy sequence $\{g_n\}$ in $\mathbb{M}_{\widetilde{\rho}}$. Then $\{g_n(x)\}$ is a $\rho$-Cauchy sequence in $X_\rho$ for all $x \in X$. Since $X_\rho$ is $\rho$-complete, there is a mapping $g : V \longrightarrow X_\rho$ such that $\rho(g_n(x) - g(x)) \longrightarrow 0$ as $n \longrightarrow \infty$ for all $x \in X$. For any $\epsilon > 0$, there is an $m \in \mathbb{N}$ such that

$$\rho(g_m(0) - g(0)) = \rho(g(0)) \leq \epsilon$$

and hence $g \in \mathbb{M}_{\widetilde{\rho}} = \mathbb{M}$. Let $\delta > 0$ be given. Since $\{g_n\}$ is a $\widetilde{\rho}$-Cauchy sequence in $\mathbb{M}_{\widetilde{\rho}}$, there is a $k \in \mathbb{N}$ such that for any $n, m \in \mathbb{N}$ with $n, m \geq k$,

$$\rho(g_n(x) - g_m(x)) \leq \delta\Psi(x), \forall x \in V$$

and since $\rho$ is a lower semi-continuous, we have

$$\rho(g_n(x) - g(x)) \leq \liminf_{m\to\infty} \rho(g_n(x) - g_m(x)) \leq \delta\Psi(x)$$

for all $x \in X$. Hence $\{g_n\}$ is $\widetilde{\rho}$-convergent and thus $\mathbb{M}_{\widetilde{\rho}}$ is $\widetilde{\rho}$-complete.

(5) Suppose that $\{g_n\}$ is a sequence in $\mathbb{M}_{\widetilde{\rho}}$ which is $\widetilde{\rho}$-convergent to $g \in \mathbb{M}_{\widetilde{\rho}}$. Then $\{g_n(x)\}$ is $\rho$-convergent to $g(x)$ for all $x \in V$. Let $\epsilon > 0$ be given. Then for any $n \in \mathbb{N}$, there is a positive real number $c_n$ such that

$$\widetilde{\rho}(g_n) \leq c_n \leq \widetilde{\rho}(g_n) + \epsilon$$

and so

$$\rho(g(x)) \leq \liminf_{n\to\infty} \rho(g_n(x)) \leq \liminf_{n\to\infty} c_n\Psi(x) \leq \Big(\liminf_{n\to\infty} \widetilde{\rho}(g_n) + \epsilon\Big)\Psi(x)$$

for all $x \in X$. Hence $\widetilde{\rho}$ is lower semi-continuous. □

Now, with Lemma 2.4 and Lemma 3.1, we will show the following uniquness concerning the stability of additive-quadratic mappings and using these, we prove the generalized Hyers-Ulam stability for additive-quadratic mappings

**Theorem 3.2.** *Let $\Phi : V \longrightarrow [0, \infty)$ be a mapping and $L$ a positive real number sucht that $0 \leq L < \frac{1}{2}$ and*

$$\Phi(2x) \leq L\Phi(x) \tag{3.1}$$

*for all $x \in V$. Let $f, F : V \longrightarrow X_\rho$ be mappings such that*

$$\rho(f(x) - F(x)) \leq M[\Phi(x) + \Phi(-x)] \tag{3.2}$$

*for all $x \in V$ and some non-negative real number $M$ and*

$$F(2x) = 3F(x) + F(-x) \tag{3.3}$$

*for all $x \in X$. Then $F$ is determined by*

$$\frac{1}{8}F(x) =_\rho \lim_{n\to\infty} \Big[\Big(\frac{1}{2^{2n+4}} + \frac{1}{2^{n+4}}\Big)f(2^n x) + \Big(\frac{1}{2^{2n+4}} - \frac{1}{2^{n+4}}\Big)f(-2^n x)\Big] \tag{3.4}$$

*for all $x \in V$.*

*Proof.* Let $\Psi(x) = \Phi(x) + \Phi(-x)$ for all $x \in V$. By Lemma 3.1, $\mathbb{M}_{\widetilde{\rho}} = \mathbb{M}$ is $\widetilde{\rho}$-complete and $\widetilde{\rho}$ is lower semi-continuous. Define $T : \mathbb{M}_{\widetilde{\rho}} \longrightarrow \mathbb{M}_{\widetilde{\rho}}$ by $Tg(x) = \frac{3}{8}g(2x) - \frac{1}{8}g(-2x)$ for all $g \in \mathbb{M}_{\widetilde{\rho}}$ and all $x \in V$ and $S : \mathbb{M}_{\widetilde{\rho}} \longrightarrow \mathbb{M}_{\widetilde{\rho}}$ by $Sg = 2g$ for

all $g \in \mathbb{M}_{\widetilde{\rho}}$. Then $S$ is an isomorphism. Suppose that $g, h \in \mathbb{M}_{\widetilde{\rho}}$ and $\widetilde{\rho}(g-h) \leq c$ for some positive real number $c$. By (M3) and (3.1), we have

$$\begin{aligned}\rho_S(Tg(x)-Th(x)) &= \rho(\frac{3}{4}g(2x)-\frac{1}{4}g(-2x)-\frac{3}{4}h(2x)+\frac{1}{4}h(-2x)) \\ &\leq \rho(g(2x)-h(2x))+\rho(g(-2x)-h(-2x)) \\ &\leq c(\Psi(2x)+\Psi(-2x)) \\ &\leq 2cL\Psi(x)\end{aligned}$$

for all $x \in V$ and so

$$\widetilde{\rho}_S(Tg-Th) \leq 2L\widetilde{\rho}(g-h).$$

By (M3), we have

$$\widetilde{\rho}(g-h) \leq \widetilde{\rho}_S(g)+\widetilde{\rho}_S(h)$$

for all $g, h \in \mathbb{M}_{\widetilde{\rho}}$. By (3.3), $F$ is a fixed point of $T$ and since $\widetilde{\rho}_S = \widetilde{\rho_S}$, by (3.2), we get

$$\begin{aligned}\rho(S^{-1}f(x)-TS^{-1}f(x)) &\leq \rho(f(x)-F(x))+\rho(TF(x)-Tf(x)) \\ &\leq M\Psi(x)+\rho_S(TF(x)-Tf(x)) \\ &\leq M\Psi(x)+2L\rho(F(x)-f(x))\end{aligned}$$

for all $x \in V$ and thus

$$\widetilde{\rho}(S^{-1}f-TS^{-1}f) \leq (1+2L)M < \infty.$$

By Lemma 2.4, there is a unique fixed point $G \in \mathbb{M}_{\widetilde{\rho}}$ of $T$ such that

$$\lim_{n\to\infty}\widetilde{\rho}(T^nS^{-3}f-G)=0$$

and

$$\widetilde{\rho}(S^{-3}f-G) \leq \frac{2}{1-2L}\widetilde{\rho}(S^{-1}f-TS^{-1}f).$$

Since $F$ is a fixed point of $T$, $S^{-3}F$ is a fixed point of $T$ and by (3.2), we have

$$\widetilde{\rho}(S^{-3}f-S^{-3}F) \leq \widetilde{\rho}(S^{-1}f-S^{-1}F) \leq \frac{2}{1-2L}\widetilde{\rho}(S^{-1}f-TS^{-1}f)$$

because $1 < \frac{2}{1-2L}$. Hence by the uniquness of $G$ in Lemma 2.4, $S^{-3}F = G$ and

$$\lim_{n\to\infty}\widetilde{\rho}\Big(T^nS^{-3}f-S^{-3}F\Big)=0. \tag{3.5}$$

By induction, there are sequences $\{a_n\}$ and $\{b_n\}$ such that

$$T^nf(x) = a_nf(2^nx)+b_nf(-2^nx)$$

for all $x \in V$ and all $n \in \mathbb{N}$. By the definition of $T$,

$$T^{n+1}f(x) = \Big(\frac{3}{8}a_n-\frac{1}{8}b_n\Big)f(2^{n+1}x)+\Big(\frac{3}{8}b_n-\frac{1}{8}a_n\Big)f(-2^{n+1}x)$$

and so

$$\begin{cases}a_{n+1} = \frac{3}{8}a_n-\frac{1}{8}b_n \\ \\ b_{n+1} = \frac{3}{8}b_n-\frac{1}{8}a_n\end{cases}$$

for all $n \in \mathbb{N}$. Hence

$$\begin{cases} a_{n+1} + b_{n+1} = \frac{1}{4}(a_n + b_n) \\ \\ a_{n+1} - b_{n+1} = \frac{1}{2}(a_n - b_n) \end{cases}$$

for all $n \in \mathbb{N}$ and so we get

$$\begin{cases} a_n = \frac{1}{2^{2n+1}} + \frac{1}{2^{n+1}} \\ \\ b_n = \frac{1}{2^{2n+1}} - \frac{1}{2^{n+1}} \end{cases}$$

for all $n \in \mathbb{N}$. Thus

$$S^{-3}F(x) =_\rho \lim_{n\to\infty} \Big[\Big(\frac{1}{2^{2n+4}} + \frac{1}{2^{n+4}}\Big)f(2^n x) + \Big(\frac{1}{2^{2n+4}} - \frac{1}{2^{n+4}}\Big)f(-2^n x)\Big]$$

for all $x \in V$ and hence we have (3.4). □

For any mapping $f : V \longrightarrow X_\rho$, let

$$f_o(x) = \frac{f(x) - f(-x)}{2}, \quad f_e(x) = \frac{f(x) + f(-x)}{2}.$$

Then $f_o$ is odd and $f_e$ is even. By the fact that $f(x) = f_o(x) + f_o(x)$ for all $x \in V$, we can easily show the following corollary :

**Corollary 3.3.** *All conditions in Theorem 3.2 are assumed. Then $F$ is determined by*

$$\frac{1}{8}F_o(x) =_\rho \lim_{n\to\infty} \frac{1}{2^{n+3}} f_o(2^n x), \tag{3.6}$$

$$\frac{1}{8}F_e(x) =_\rho \lim_{n\to\infty} \frac{1}{2^{2n+3}} f_e(2^n x), \tag{3.7}$$

*and*

$$\frac{1}{16}F(x) =_\rho \lim_{n\to\infty} \Big(\frac{1}{2^{2n+4}} f_e(2^n x) + \frac{1}{2^{n+4}} f_o(2^n x)\Big) \tag{3.8}$$

*for all $x \in V$.*

*Proof.* Note that

$$\begin{aligned} &\rho\Big[\frac{1}{8}F_o(x) - \frac{1}{2^{n+3}} f_o(2^n x)\Big] \\ &= \rho\Big[\frac{1}{16}F(x) - \frac{1}{16}F(-x) - \frac{1}{2^{n+4}} f(2^n x) + \frac{1}{2^{n+4}} f(-2^n x)\Big] \\ &\le \rho\Big[\frac{1}{8}F(x) - \Big(\frac{1}{2^{2n+4}} + \frac{1}{2^{n+4}}\Big)f(2^n x) - \Big(\frac{1}{2^{2n+4}} - \frac{1}{2^{n+4}}\Big)f(-2^n x)\Big] \\ &+ \rho\Big[\frac{1}{8}F(-x) - \Big(\frac{1}{2^{2n+4}} + \frac{1}{2^{n+4}}\Big)f(-2^n x) - \Big(\frac{1}{2^{2n+4}} - \frac{1}{2^{n+4}}\Big)f(2^n x)\Big] \end{aligned}$$

for all $x \in V$ and for all $n \in \mathbb{N}$. By (3.4), we have (3.6) and similarly, we have (3.7). Thus we get (3.8). □

If $F$ is additive or quadratic or additive-quadratic, then $F$ satisfies (3.3) and hence we have the following corollary :

**Corollary 3.4.** *All conditions in Theorem 3.2 are assumed. If $F$ is addi-ive(quadratic, addiitive-quadratic, resp.) then $F$ is determind by*

$$\frac{1}{8}F(x) =_\rho \lim_{n\to\infty} \frac{1}{2^{n+3}} f_o(2^n x) \quad \Big(\frac{1}{8}F(x) =_\rho \lim_{n\to\infty} \frac{1}{2^{2n+3}} f_e(2^n x),$$
$$\frac{1}{16}F(x) =_\rho \lim_{n\to\infty} \Big(\frac{1}{2^{2n+4}} f_e(2^n x) + \frac{1}{2^{n+4}} f_o(2^n x)\Big), \quad resp.\Big)$$

*for all $x \in V$.*

Similar to the proof of Theorem 3.2, we can show the following theorem for modular spaces with convex modulars.

**Theorem 3.5.** *All conditions in Theorem 3.2 are assumed. Suppose that $\rho$ is convex and $L$ is a positive real number sucht that $0 \le L < 2$. Then $F$ is determined by*

$$\frac{1}{8}F(x) =_\rho \lim_{n\to\infty} \Big[\Big(\frac{1}{2^{2n+4}} + \frac{1}{2^{n+4}}\Big) f(2^n x) + \Big(\frac{1}{2^{2n+4}} - \frac{1}{2^{n+4}}\Big) f(-2^n x)\Big] \tag{3.9}$$

*for all $x \in V$.*

*Proof.* Let $\Psi(x) = \Phi(x) + \Phi(-x)$ for all $x \in V$. By Lemma 3.1, $\mathbb{M}_{\widetilde{\rho}} = \mathbb{M}$ is $\widetilde{\rho}$-complete and $\widetilde{\rho}$ is lower semi-continuous. Define $T : \mathbb{M}_{\widetilde{\rho}} \longrightarrow \mathbb{M}_{\widetilde{\rho}}$ by $Tg(x) = \frac{3}{8}g(2x) - \frac{1}{8}g(-2x)$ for all $g \in \mathbb{M}_{\widetilde{\rho}}$ and all $x \in V$ and $S : \mathbb{M}_{\widetilde{\rho}} \longrightarrow \mathbb{M}_{\widetilde{\rho}}$ by $Sg = 2g$ for all $g \in \mathbb{M}_{\widetilde{\rho}}$. Then $S$ is an isomorphism. Suppose that $g, h \in \mathbb{M}_{\widetilde{\rho}}$ and $\widetilde{\rho}(g-h) \le c$ for some positive real number $c$. By (M3) and (3.1), we have

$$\begin{aligned}\rho_S(Tg(x) - Th(x)) &= \rho(\frac{3}{4}g(2x) - \frac{1}{4}g(-2x) - \frac{3}{4}h(2x) + \frac{1}{4}h(-2x)) \\ &\le \frac{3}{4}\rho(g(2x) - h(2x)) + \frac{1}{4}\rho(g(2x) - h(-2x)) \\ &\le cL\Psi(x)\end{aligned}$$

for all $x \in V$ and so

$$\widetilde{\rho}_S(Tg - Th) \le L\widetilde{\rho}(g-h).$$

Further, clearly we have

$$\widetilde{\rho}(g-h) \le \frac{1}{2}\widetilde{\rho}_S(g) + \frac{1}{2}\widetilde{\rho}_S(h)$$

for all $g, h \in \mathbb{M}_{\widetilde{\rho}}$. By (3.3), $F$ is a fixed point of $T$ and by (3.2), we get

$$\begin{aligned}\rho(S^{-1}f(x) - TS^{-1}f(x)) &\le \frac{1}{2}\rho(f(x) - F(x)) + \frac{1}{2}\rho(TF(x) - Tf(x)) \\ &\le \frac{1}{2}M\Psi(x) + \frac{1}{2}\rho_S(TF(x) - Tf(x)) \\ &\le \frac{1}{2}(1+L)M\Psi(x)\end{aligned}$$

for all $x \in V$ and thus

$$\widetilde{\rho}(S^{-1}f - TS^{-1}f) \le \frac{1}{2}(1+L)M < \infty.$$

By Lemma 2.5, there is a unique fixed point $G \in \mathbb{M}_{\widetilde{\rho}}$ of $T$ such that

$$\widetilde{\rho}(S^{-3}f - G) \le \frac{1}{2-L}\widetilde{\rho}(S^{-1}f - TS^{-1}f).$$

and further, we have

$$\lim_{n\to\infty}\widetilde{\rho}(T^nS^{-3}f-G)=0.$$

Since $F$ is a fixed point of $T$, $S^{-3}F$ is a fixed point of $T$ and

$$\widetilde{\rho}(S^{-3}f-S^{-3}F)\leq\frac{1}{4}\widetilde{\rho}(S^{-1}f-S^{-1}F)\leq\frac{1}{2-L}\widetilde{\rho}(S^{-1}f-TS^{-1}f)$$

because $\frac{1}{2}\leq\frac{1}{2-L}$. Hence by the uniquness of $G$ in Lemma 2.4, $S^{-3}F=G$. The rest proof is similar to the proof of Theorem 3.2. □

Using Lemma 2.5 and Theorem 3.5, we can show the generalized Hyers-Ulam stability for aditive-quadratic mappings.

**Corollary 3.6.** *Let $V$ be a linear space and $X_\rho$ a $\rho$-complete modular space whose induced modular is convex lower semi-continuous. Suppose that $f:V\longrightarrow X_\rho$ is a mapping such that*

$$\rho(f(x+y)+f(x-y)-2f(x)-f(y)-f(-y))\leq\phi(x,y) \tag{3.10}$$

*for all $x,y\in V$ and let $\phi:V^2\longrightarrow[0,\infty)$ be a mapping satisfying*

$$\phi(2x,2y)\leq L\phi(x,y),\ \forall x,y\in V \tag{3.11}$$

*for some real number $L$ with $0\leq L<2$. Then there is a unique additive-quadratic mapping $G:V\longrightarrow X_\rho$ such that*

$$\rho(\frac{1}{4}f(x)-G(x))\leq\frac{3}{8(2-L)}[\phi(x,x)+\phi(-x,-x)] \tag{3.12}$$

*for all $x\in V$.*

*Proof.* Let $\Phi(x)=\phi(x,x)$ and $\Psi(x)=\Phi(x)+\Phi(-x)$ for all $x\in V$. By Lemma 3.1, $\mathbb{M}_{\widetilde{\rho}}=\mathbb{M}$ is $\widetilde{\rho}$-complete and $\widetilde{\rho}$ is lower semi-continuous. Define $T:\mathbb{M}_{\widetilde{\rho}}\longrightarrow\mathbb{M}_{\widetilde{\rho}}$ by $Tg(x)=\frac{3}{8}g(2x)-\frac{1}{8}g(-2x)$ for all $g\in\mathbb{M}_{\widetilde{\rho}}$ and all $x\in V$ and $S:\mathbb{M}_{\widetilde{\rho}}\longrightarrow\mathbb{M}_{\widetilde{\rho}}$ by $Sg=2g$ for all $g\in\mathbb{M}_{\widetilde{\rho}}$. Then $S$ is an isomorphism and (2.5) in Lemma 2.5 holds for $r=\frac{1}{2}$. Letting $y=x$ in (3.10), we get

$$\rho(f(2x)-3f(x)-f(-x))\leq\phi(x,x) \tag{3.13}$$

for all $x\in V$ and by (3.13), we have

$$\rho(Tf(x)-f(x))\leq\frac{3}{8}\phi(x,x)+\frac{1}{8}\phi(-x,-x)\leq\frac{3}{8}\Psi(x)$$

for all $x\in V$. Hence we get

$$\widetilde{\rho}(Tf-f)\leq\frac{3}{8} \tag{3.14}$$

and by Lemma 2.5, there is a unique fixed point $G\in\mathbb{M}_{\widetilde{\rho}}$ of $T$ such that

$$\widetilde{\rho}(S^{-2}f-G)\leq\frac{3}{8(2-L)}.$$

For any $n\in\mathbb{N}$, let

$$a_n=\frac{1}{2^{2n+4}}+\frac{1}{2^{n+4}},\ b_n=\frac{1}{2^{2n+4}}-\frac{1}{2^{n+4}}.$$

Since $G$ is a fixed point of $T$, $G$ satisfies (3.3) and by Theorem 3.5, we have

$$\frac{1}{8}G(x) =_\rho \lim_{n\to\infty}[a_n f(2^n x) + b_n f(-2^n x)]$$

for all $x \in V$. By (M3), we get

(3.15)

$$\begin{aligned}
&\rho\Big(\frac{1}{2^9}G(x+y) + \frac{1}{2^9}G(x-y) - \frac{1}{2^8}G(x) - \frac{1}{2^9}G(y) - \frac{1}{2^9}G(-y)\Big)\\
&\le \frac{1}{2^6}\rho\Big(\frac{1}{8}G(x+y) - a_n f(2^n(x+y)) - b_n f(-2^n(x+y))\Big)\\
&+ \frac{1}{2^6}\rho\Big(\frac{1}{8}G(x-y) - a_n f(2^n(x-y)) - b_n f(-2^n(x-y))\Big)\\
&+ \frac{1}{2^6}\rho\Big(\frac{1}{4}G(x) - 2a_n f(2^n x) - 2b_n f(-2^n x)\Big)\\
&+ \frac{1}{2^6}\rho\Big(\frac{1}{8}G(y) - a_n f(2^n y) - b_n f(-2^n y)\Big)\\
&+ \frac{1}{2^6}\rho\Big(\frac{1}{8}G(-y) - a_n f(2^n(-y)) - b_n f(-2^n(-y))\Big)\\
&+ \frac{a_n}{2^6}\rho\Big(f(2^n(x+y)) + f(2^n(x-y)) - 2f(2^n x) - f(2^n y) - f(2^n(-y))\Big)\\
&+ \frac{|b_n|}{2^6}\rho\Big(f(-2^n(x+y)) + f(-2^n(x-y)) - 2f(-2^n x) - f(-2^n y) - f(-2^n(-y))\Big)
\end{aligned}$$

and by (3.11), we have

(3.16)

$$\begin{aligned}
&\frac{a_n}{2^6}\rho\Big(f(2^n(x+y)) + f(2^n(x-y)) - 2f(2^n x) - f(2^n y) - f(2^n(-y))\Big)\\
&+ \frac{|b_n|}{2^6}\rho\Big(f(-2^n(x+y)) + f(-2^n(x-y)) - 2f(-2^n x) - f(-2^n y) - f(-2^n(-y))\Big)\\
&\le \frac{a_n}{2^6}\phi(2^n x, 2^n y) + \frac{|b_n|}{2^6}\phi(-2^n x, -2^n y)\\
&\le L^n\Big[\frac{a_n}{2^6}\phi(x,y) + \frac{|b_n|}{2^6}\phi(-x,-y)\Big]
\end{aligned}$$

for all $x, y \in V$ and for all $n \in \mathbb{N}$. Since $0 < a_n,\ |b_n| < 2^{-n}$, by (3.15) and (3.16), we can show that $G$ is an additive-quadratic mapping. Since every additive-quadratic mapping satisifes (3.3), $G$ is a unique additive-quadratic mapping with (3.12). □

## References


[1] T. Aoki, On the stability of the linear transformation in Banach spaces, J. Math. Soc. Japan **2**(1950), 64-66.

[2] P. Găvruta, A generalization of the Hyer-Ulam-Rassias stability of approximately additive mappings, J. Math. Anal. Appl. **184**(1994), 431-436.

[3] D. H. Hyers, On the stability of the linear functional equation, Proc. Nat. Acad. Sci. **27**(1941), 222-224.

[4] M. A. Khamsi, W. M. Kozowski and S. Reich, Fixed point theory in modular function spaces, Nonlinear Anal. **14**(1990), 935-953.

[5] M. A. Khamsi, Quasicontraction mappings in modular spaces without 2-condition, Fixed Point Theory and Applications, **2008**(2008), 1-6.

[6] S. Koshi and T. Shimogaki, On F-norms of quasi-modular spaces , J. Fac. Sci. Hokkaido Univ. Ser. **15**(1961), 202-218.

[7] M. Krbec, Modular interpolation spaces, Z. Anal. Anwendungen **1** (1982), 25-40.

[8] Y. H. Lee and S. M. Jung, General uniqueness theorem concerning the stability of additive and quadratic functional equations journal of function spaces **2015**(2015), 1-8.
[9] A. Luxemburg, Banach function spaces, Ph. D. thesis, Delft Univrsity of technology, Delft, The Netherlands, 1959.
[10] L. Maligranda, Orlicz Spaces and Interpolation, in: Seminars in Math., Vol. 5, Univ. of Campinas, Brazil, 1989.
[11] J. Musielak and W. Orlicz ,On modular spaces, Studia Math. **18** (1959), 49-65.
[12] J. Musielak, Orlicz Spaces and Modular Spaces, Lecture Notes in Math., Vol. 1034, Springer-verlag, Berlin, 1983.
[13] H. Nakano, Modular semi-ordered spaces, Tokyo, Japan, 1959.
[14] W. Orlicz, Collected Papers, Vols. I, II, PWN, Warszawa, 1988.
[15] Th. M. Rassias, On the stability of the linear mapping in Banach sapces, Proc. Amer. Math. Sco. **72**(1978), 297-300.
[16] G. Sadeghi, A fixed point approach to stability of functional equations in modular spaces, Bulletin of the Malaysian Mathematical Sciences Society. Second Series, **37**(2014), 333-344.
[17] Ph. Turpin, Fubini inequalities and bounded multiplier property in generalized modular spaces, Comment. Math., Tomus specialis in honorem Ladislai Orlicz I (1978), 331-353.
[18] S. M. Ulam, Problems in Modern Mathematics, Wiley, New York; 1964.
[19] K. Wongkum, P. Chaipunya, and P. Kumam, On the generalized Ulam-Hyers-Rassias stability of quadratic mappings in modular spaces without $\triangle_2$-conditions, Journal of function spaces **2015**(2015), 1-6.
[20] S. Yamamuro, On conjugate spaces of Nakano spaces, Trans. Amer. Math. Soc. **90**(1959), 291-311.

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# Rough fuzzy ideals in $BCK/BCI$-algebras

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**Abstract.** The notions of rough ideals and rough fuzzy ideals in $BCK/BCI$-algebras are introduced and some properties of such ideals are investigated. The relations between the upper(lower) rough ideals and the upper (lower) approximations of their homomorphic images are discussed.

## 1. Introduction

The notion of rough sets was introduced by Pawlark ([11]). The theory of rough sets has emerged as another major mathematical approach for managing uncertainty that arises from inexact, noisy, or incomplete information. It is turning out to be methodologically significant to the domains of artificial intelligence and cognitive sciences, especially in the representation of reasoning with vague and/or imprecise knowledge, data analysis, machine learning, and knowledge discovery ([11,12]). The algebraic approach to rough sets was studied in [8]. Biswas and Nanda ([1]) introduced the notion of rough subgrougs, and Kuroki and Morderson ([6]) discussed the structure of rough sets and rough groups. Kuroki and Wang ([7]) gave some properties of lower and upper approximations with respect to the normal subgroups and the fuzzy normal subgroups, and Kuroki ([5]) introduced the notion of rough ideals in semigroup, which is an extended notion of ideals in semigroups, and gave some properties of such ideals. Xiao and Zhang ([13]) established the notion of rough prime ideals and rough fuzzy prime ideals in a semigroup. Imai and Iséki ([2]) introduced two classes of abstract algebras : $BCK$-algebras and $BCI$-algebras. It is known that the class of $BCK$-algebras is a proper subclass of the class of $BCI$-algebras. C. R. Lim and H. S. Kim ([8]) introduced the notion of a rough set in $BCK/BCI$-algebras. By introducing the notion of a quick ideal in $BCK/BCI$-algebras, they obtained some relations between quick ideals and upper(lower) rough quick ideals in $BCK/BCI$-algebras.

In this paper, we introduce the notion of rough ideals and rough fuzzy ideals in $BCK/BCI$-algebras, and we give some properties of such ideals. Also, we discuss the relations between the upper(lower) rough ideals and the upper (lower) approximations of their homomorphic images.

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## 2. **Preliminaries**

A *BCI-algebra* ([9]) is a non-empty set $X$ with a constant 0 and a binary operation " $*$ " satisfying the axioms, for all $x, y, z \in X$:

(i) $((x * y) * (x * z)) * (z * y) = 0$,
(ii) $(x * (x * y)) * y = 0$,
(iii) $x * x = 0$,
(iv) $x * y = 0$ and $y * x = 0$ imply $x = y$.

A $BCK$-algebra is a $BCI$-algebra satisfying the axiom:

(v) $0 * x = 0$ for all $x \in X$.

We can define a partial ordering $\leq$ on $X$ by $x \leq y$ if and only $x * y = 0$. In any $BCI$-algebra $X$, the following hold: for any $x, y, z \in X$,

(1) $x * 0 = x$,
(2) $(x * y) * z = (x * z) * y$,
(3) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$,
(4) $(x * z) * (y * z) \leq (x * z)$.

Let $X$ be a $BCK/BCI$-algebra and let $0 \in I \subseteq X$. A set $I$ is called an *ideal* of $X$ if for all $x, y \in X$, $x * y \in I$ and $y \in I$ imply $x \in I$. An ideal $I$ is said to be *closed* if $0 * x \in I$ whenever $x \in I$. Let $S$ be a non-empty subset of $X$. Then $S$ is called a *subalgebra* of $X$ if, for any $x, y \in S$, $x * y \in S$. A closed ideal of a $BCK/BCI$-algebra $X$ is a subalgebra of $X$. An equivalence relation $\rho$ on $X$ is called a *congruence relation* on $X$ if $(x * u, y * v) \in \rho$ for any $(x, y), (u, v) \in \rho$. We denote by $[a]_\rho$ the $\rho$-congruence class containing the element $a \in X$. Let $X/\rho$ be the set of all $\rho$-equivalence classes on $X$, i.e., $X/\rho := \{[a]_\rho | a \in X\}$. For any $[x]_\rho, [y]_\rho \in X/\rho$, if we define

$$[x]_\rho * [y]_\rho := [x * y]_\rho = \{z \in X | (z, x * y) \in \rho\},$$

then it is well defined, since $\rho$ is a congruence relation. A congruence relation $\rho$ on a $BCK/BCI$-algebra $X$ is said to be *regular* if $[x]_\rho * [y]_\rho = [0]_\rho = [y]_\rho * [x]_\rho$ implies $[x]_\rho = [y]_\rho$ for any $[x]_\rho, [y]_\rho \in X/\rho$.

**Theorem 2.1.** ([9]) *Let $X$ be a $BCK$-algebra and let $\rho$ be a congruence relation on $X$. Then $\rho$ is regular if and only if $X/\rho$ is a $BCK$-algebra.*

Let $I$ be an ideal of $X$. We define a relation $\rho_I$ on $X$ as follows:

$$\rho_I := \{(x, y) | x * y, y * x \in I\}. \qquad (*)$$

Then $\rho_I$ is a regular congruence relation ([4]). Let $Con(X)$ be the set of all congruences on $X$. We define a subset $I_\rho$ of $X$ from $\rho \in Con(X)$ by $I_\rho := \{x * y | (x, y) \in \rho\}$.

**Proposition 2.2.** ([4]) *Let $A$ be an ideal. If $A$ is closed, then $A = I_{\rho_A}$.*

Let $X$ be a $BCK/BCI$-algebra and let $\rho$ be a congruence relation on $X$. Let $\mathcal{P}(X)$ denote the power set of $X$ and $\mathcal{P}^*(X) = \mathcal{P}(X) \setminus \{\emptyset\}$. Define the functions $\rho_, \rho^- : \mathcal{P}(X) \to \mathcal{P}(X)$ as follows: for any $\emptyset \neq A \in \mathcal{P}(X)$, $\rho_-(A) := \{x \in X | [x]_\rho \subseteq A\}$ and $\rho^-(A) := \{x \in X | [x]_\rho \cap A \neq \emptyset\}$. The set $\rho_-(A)$ is called the *$\rho$-lower approximation* of $A$, while $\rho^-(A)$ is called the *$\rho$-upper approximation* of $A$. For a non-empty subset $A$ of $X$, $\rho(A) = (\rho_-(A), \rho^-(A))$ is called a *rough set* with respect to $\rho$ of $\mathcal{P}(X) \times \mathcal{P}(X)$ if $\rho_-(A) \neq \rho^-(A)$. A subset $A$ of $X$ is said to be *definable* if $\rho_-(A) = \rho^-(A)$. The pair $(X, \rho)$ is called an *approximation space*. A congruence relation $\rho$ on a set $X$ is called *complete* if $[x]_\rho * [y]_\rho = [x * y]_\rho$ for any $x, y \in X$.

## 3. Rough ideals in $BCK/BCI$-algebras

Let $X$ be a $BCK/BCI$-algebra and let $\emptyset \neq A \subseteq X$. Let $\rho$ be a congruence relation on $X$. Then $A$ is called an *upper* (a *lower*, respectively) *rough ideal* of $X$ if $\rho^-(A)$ ($\rho_-(A)$, respectively) is an *ideal* of $X$.

**Theorem 3.1.** *Let $\rho_I$ be a congruences relation on a $BCK/BCI$-algebra $X$ as in $(*)$. If $A$ is a closed ideal of $X$, then it is an upper rough ideal of $X$.*

*Proof.* Since $A$ is an ideal of $X$, $0 \in A$. Hence $A \cap [0]_{\rho_I} \neq \emptyset$. Therefore $0 \in {\rho_I}^-(A)$.

Let $x, y \in X$ with $x*y, y \in {\rho_I}^-(A)$. Then $([x]_{\rho_I} * [y]_{\rho_I}) \cap A = ([x*y]_{\rho_I}) \cap A \neq \emptyset$ and $[y]_{\rho_I} \cap A \neq \emptyset$. Hence there exist $\alpha, \beta \in A$ such that $\alpha \in [x]_{\rho_I} * [y]_{\rho_I} = [x * y]_{\rho_I}$ and $\beta \in [y]_{\rho_I}$. Therefore $\alpha = p * q$ for some $p \in [x]_{\rho_I}, q \in [y]_{\rho_I}$. Since $\beta, q \in [y]_{\rho_I}$, we have $(\beta, y), (y, q) \in \rho_I$ and so $(\beta, q) \in \rho_I$. Hence $[\beta]_{\rho_I} = [q]_{\rho_I}$. Since $(q * \beta, q * q) = (q * \beta, 0) \in \rho_I$, we have $(q * \beta) * 0 = q * \beta \in A$ by Proposition 2.2. Using $\beta \in A$, we have $q \in A$. Since $p * q, q \in A$ and $A$ is an ideal of $X$, we obtain $p \in A$. Therefore $p \in [x]_{\rho_I} \cap A \neq \emptyset$. Thus $x \in {\rho_I}^-(A)$, completing the proof. □

Theorem 3.1 shows that the notion of an upper rough ideal is an extended notion of a closed ideal in $BCK/BCI$-algebras.

**Example 3.2.** Let $X := \{0, 1, 2, 3\}$ be a $BCK$-algebra with the following Cayley table:

| $*$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 |
| 2 | 2 | 1 | 0 | 1 |
| 3 | 3 | 3 | 3 | 0 |

If we take $A := \{0, 2\}$, then it is not an ideal of $X$, since $1 * 2 = 0 \in A$, but $1 \notin A$. On the while, let $\rho$ be a congruence relation on $X$ such that $\{0, 1, 2\}, \{3\}$ are all $\rho$-congruence classes of $X$. Then $\rho^-(A) = \{0, 1, 2\}$ is an ideal of $X$.

**Theorem 3.3.** *Let $X$ be a $BCK/BCI$-algebra and let $A$ be a closed ideal of $X$. Then $\rho_-(A)$, if it is non-empty, is an ideal of $X$.*

*Proof.* Since $A$ is a closed ideal of $X$, it is a subalgebra of $X$. Since $\rho_-(A) \neq \emptyset$, $\rho_-(A)$ is a subalgebra of $X$. Hence $0 \in \rho_-(A)$. Let $x, y \in X$ with $x * y, y \in \rho_-(A)$. Then $[x]_\rho * [y]_\rho = [x * y]_\rho \subseteq A$, $[y]_\rho \subseteq A$. If $\alpha \in [x]_\rho$, then $(\alpha, x) \in \rho$. Since $\rho$ is a congruence relation on $X$, we have $(\alpha * y, x * y) \in \rho$ and so $\alpha * y \in [x * y]_\rho \subseteq A$. Since $A$ is an ideal of $X$ and $y \in A$, we get $\alpha \in A$, i.e., $[x]_\rho \subseteq A$, proving that $x \in \rho_-(A)$. □

Let $\rho$ be a regular congruence relation on a $BCK$-algebra $X$ and let $\emptyset \neq A \subseteq X$. The lower and upper approximations can be presented in an equivalent form as shown below:

$$\rho_-(A)/\rho = \{[x]_\rho \in X/\rho \mid [x]_\rho \subseteq A\}$$
$$\rho^-(A)/\rho = \{[x]_\rho \in X/\rho \mid [x]_\rho \cap A \neq \emptyset\}.$$

**Proposition 3.4.** *Let $\rho$ be a regular congruence relation on a $BCK$-algebra $X$. If $A$ is a subalgebra of $X$, then $\rho^-(A)/\rho$ is a subalgebra of the quotient $BCK$-algebra $X/\rho$.*

*Proof.* Since $A$ is a subalgebra of $X$, there exists an element $x \in A$ such that $[x]_\rho \cap A \neq \emptyset$, i.e., $\rho^-(A)/\rho \neq \emptyset$. Let $[x]_\rho$ and $[y]_\rho$ be any elements of $\rho^-(A)/\rho$. Then $[x]_\rho \cap A \neq \emptyset$ and $[y]_\rho \cap A \neq \emptyset$. This means that there exist $a, b \in X$ such that $a \in [x]_\rho \cap A$ and $b \in [y]_\rho \cap A$. Then $a * b \in [x]_\rho * [y]_\rho$. Since $A$ is a subalgebra of $X$, $a * b \in A$. This means that $[x]_\rho * [y]_\rho \in \rho^-(A)/\rho$, completing the proof. □

**Proposition 3.5.** *Let $\rho$ be a regular congruence relation on a $BCK$-algebra $X$. If $A$ is a subalgebra of $X$, then $\rho_-(A)/\rho$ is, if it is non-empty, a subalgebra of the quotient $BCK$-algebra $X/\rho$.*

*Proof.* Straightforward. □

**Theorem 3.6.** *Let $\rho_I$ be a regular congruence relation on a $BCK$-algebra $X$ as in $(*)$. If $A$ is an ideal of $X$, then ${\rho_I}^-(A)/\rho_I$ is an ideal of the quotient $BCK$-algebra $X/\rho_I$.*

*Proof.* Since $0 \in {\rho_I}^-(A)$, we have $[0]_{\rho_I} \cap A \neq \emptyset$ and hence $[0]_{\rho_I} \in {\rho_I}^-(A)/\rho_I$. Let $[x]_{\rho_I} * [y]_{\rho_I}, [y]_{\rho_I} \in {\rho_I}^-(A)/\rho_I$. Then $([x]_{\rho_I} * [y]_{\rho_I}) \cap A = [x * y]_{\rho_I} \cap A \neq \emptyset$ and $[y]_{\rho_I} \cap A \neq \emptyset$. Hence there exist $\alpha \in A$ with $\alpha \in [x]_{\rho_I} * [y]_{\rho_I} = [x * y]_{\rho_I}$ and $\beta \in A$ for some $\beta \in [y]_{\rho_I}$. Therefore $\alpha = p * q$ for some $p \in [x]_{\rho_I}, q \in [y]_{\rho_I}$. Since $\beta, q \in [y]_{\rho_I}$, we have $(\beta, y), (y, q) \in \rho_I$ and so $(\beta, q) \in \rho_I$. Hence $[\beta]_{\rho_I} = [q]_{\rho_I}$. Since $(q * \beta, q * q) = (q * \beta, 0) \in \rho_I$, we have $(q * \beta) * 0 = q * \beta \in A$ by Proposition 2.2. Using $\beta \in A$, we have $q \in A$. Since $A$ is an ideal of $X$ and $q \in A$, we have $p \in A$. Thus $p \in [x]_{\rho_I} \cap A$, proving $[x]_{\rho_I} \in {\rho_I}^-(A)/\rho_I$. □

**Theorem 3.7.** *Let $\rho$ be a regular congruence relation on a $BCK$-algebra $X$. If $A$ is an ideal of $X$, then $\rho_-(A)/\rho$, if it is non-empty, an ideal of the quotient $BCK$-algebra $X/\rho$.*

*Proof.* Since $\rho_-(A)/\rho \neq \emptyset$, $\rho_-(A)/\rho$ is a subalgebra of $X/\rho$. Hence $[0]_\rho \in \rho_-(A)/\rho$. Let $[x]_\rho * [y]_\rho, [y]_\rho \in \rho_-(A)/\rho$ for some $[x]_\rho \in X/\rho$. Hence $[x*y]_\rho \subseteq A$ and $[y]_\rho \subseteq A$. Therefore $x*y \in \rho_-(A), y \in \rho_-(A)$. If $\alpha \in [x]_\rho$, then $(\alpha, x) \in \rho$. Since $\rho$ is a congruence relation on $X$, we have $(\alpha * y, x * y) \in \rho$. Hence $\alpha * y \in [x*y]_\rho \subseteq A$. Hence $\alpha \in A$, because $A$ is an ideal of $X$ and $y \in A$. Therefore $[x]_\rho \subseteq A$, proving that $[x]_\rho \in \rho_-(A)/\rho$. □

**Theorem 3.8.** *Let $\rho$ be a regular congruence relation on a $BCK$-algebra $X$. If $A$ is an upper rough ideal of $X$, then $\rho^-(A)/\rho$ is an ideal of the quotient algebra $X/\rho$.*

*Proof.* Since $0 \in \rho^-(A)$, we have $[0]_\rho \cap A \neq \emptyset$ and hence $[0]_\rho \in \rho^-(A)/\rho$. Let $[x]_\rho * [y]_\rho = [x*y]_\rho, [y]_\rho \in \rho^-(A)/\rho$ for some $[x]_\rho \in X/\rho$. Then $([x]_\rho * [y]_\rho) \cap A = [x*y]_\rho \cap A \neq \emptyset$ and $[y]_\rho \cap A \neq \emptyset$. Hence $x*y, y \in \rho^-(A)$. Since $\rho^-(A)$ is an ideal of $X$, we have $x \in A$. Thus $x \in [x]_\rho \cap A \neq \emptyset$, proving $[x]_\rho \in \rho^-(A)/\rho$. □

**Theorem 3.9.** *Let $\rho$ be a regular congruence relation on a $BCK$-algebra $X$. If $A$ is a lower rough ideal of $X$, then $\rho_-(A)/\rho$ is, if it is non-empty, an ideal of the quotient $BCK$-algebra $X/\rho$.*

*Proof.* Since $\rho_-(A)/\rho \neq \emptyset$, $\rho_-(A)/\rho$ is a subalgebra of $X/\rho$ and hence $[0]_\rho \in \rho_-(A)/\rho$. Let $[x]_\rho * [y]_\rho, [y]_\rho \in \rho_-(A)/\rho$ for some $[x]_\rho \in X/\rho$. Hence $[x*y]_\rho \subseteq A$ and $[y]_\rho \subseteq A$. Therefore $x*y \in \rho_-(A), y \in \rho_-(A)$. Since $\rho_-(A)$ is an ideal of $X$, we have $x \in \rho_-(A)$. Thus $[x]_\rho \subseteq A$. □

## 4. **Approximations of fuzzy sets**

Let $\mu$ and $\lambda$ be two fuzzy subsets of $X$. The inclusion $\lambda \subseteq \mu$ is denoted by $\lambda(x) \leq \mu(x)$ for all $x \in X$, and $\mu \cap \lambda$ is defined by $(\mu \cap \lambda)(x) = \mu(x) \wedge \lambda(x)$ for all $x \in X$.

**Definition 4.1.** Let $\rho$ be a congruence relation on a $BCK/BCI$-algebra $X$ and $\mu$ a fuzzy subset of $X$. Then we define the fuzzy sets $\rho_-(\mu)$ and $\rho^-(\mu)$ as follows:

$$\rho_-(\mu)(x) := \wedge_{a \in [x]_\rho} \mu(a) \text{ and } \rho^-(\mu)(x) := \vee_{a \in [x]_\rho} \mu(a).$$

The fuzzy sets $\rho_-(\mu)$ and $\rho^-(\mu)$ are called the *$\rho$-lower approximations* and *$\rho$-upper approximations* of the fuzzy set $\mu$, respectively. A set $\rho(\mu) = (\rho_-(\mu), \rho^-(\mu))$ is called a *rough fuzzy set* with respect to $\rho$ if $\rho_-(\mu) \neq \rho^-(\mu)$.

**Definition 4.2.** ([3]) A fuzzy set $\mu$ of a $BCK/BCI$-algebra $X$ is called a *fuzzy ideal* of $X$ if

$(F_1)$ $\mu(0) \geq \mu(x)$ for all $x \in X$,

$(F_2)$ $\mu(x) \geq \min\{\mu(x*y), \mu(y)\}$ for all $x, y \in X$.

Let $\mu$ and $\nu$ be fuzzy ideals of a $BCK/BCI$-algebra $X$. Then $\mu \cap \nu$ is also a fuzzy ideal of $X$.

A fuzzy subset $\mu$ of a $BCK/BCI$-algebra $X$ is called an *upper* (a *lower*, respectively) *rough fuzzy ideal* of $X$ if $\rho^-(\mu)$ ($\rho_-(\mu)$, respectively) is a fuzzy ideal of $X$.

**Theorem 4.3.** *Let $\rho$ be a congruence relation on a $BCK/BCI$-algebra $X$. If $\mu$ is a fuzzy ideal of $X$, then $\rho^-(\mu)$ is a fuzzy ideal of $X$.*

*Proof.* Since $\mu$ is a fuzzy ideal of $X$, $\mu(0) \geq \mu(x)$ for all $x \in X$. Hence we obtain

$$\rho^-(\mu)(0) = \vee_{z\in[0]_\rho}\mu(z) \geq \vee_{x'\in[x]_\rho}\mu(x') = \rho^-(\mu)(x).$$

For any $x, y \in X$, we have

$$\begin{aligned}\rho^-(\mu)(x) = \vee_{x'\in[x]_\rho}\mu(x') &\geq \vee_{x'*y'\in[x]_\rho*[y]_\rho, y'\in[y]_\rho}\{\min\{\mu(x'*y'), \mu(y')\}\}\\ &= \vee_{x'*y'\in[x*y]_\rho, y'\in[y]_\rho}\{\min\{\mu(x'*y'), \mu(y')\}\}\\ &\geq \min\{\vee_{x'*y'\in[x*y]_\rho}\mu(x'*y'), \vee_{y'\in[y]_\rho}\mu(y')\}\\ &= \min\{\rho^-(\mu)(x*y), \rho^-(\mu)(y)\}.\end{aligned}$$

Thus $\rho^-(\mu)$ is a fuzzy ideal of $X$. □

**Theorem 4.4.** *Let $\rho$ be a congruence relation on a $BCK/BCI$-algebra $X$. If $\mu$ is a fuzzy ideal of $X$, then $\rho_-(\mu)$ is, if it is non-empty, a fuzzy ideal of $X$.*

*Proof.* Since $\mu$ is a fuzzy ideal of $X$, $\mu(0) \geq \mu(x)$ for all $x \in X$. Hence for all $x \in X$, we have

$$\rho_-(\mu)(0) = \wedge_{z\in[0]_\rho}\mu(z) \geq \wedge_{z'\in[x]_\rho}\mu(z') = \rho_-(\mu)(x).$$

For any $x, y \in X$, we obtain

$$\begin{aligned}\rho_-(\mu)(x) = \wedge_{x'\in[x]_\rho}\mu(x') &\geq \wedge_{x'*y'\in[x]_\rho*[y]_\rho, y'\in[y]_\rho}\{\min\{\mu(x'*y'), \mu(y')\}\}\\ &= \wedge_{x'*y'\in[x*y]_\rho, y'\in[y]_\rho}\{\min\{\mu(x'*y'), \mu(y')\}\}\\ &= \min\{\wedge_{x'*y'\in[x*y]_\rho}\mu(x'*y'), \wedge_{y'\in[y]_\rho}\mu(y')\}\\ &= \min\{\rho_-(\mu)(x*y), \rho_-(\mu)(y)\}.\end{aligned}$$

Thus $\rho_-(\mu)$ is a fuzzy ideal of $X$. □

Let $\mu$ be a fuzzy subset of a $BCK/BCI$-algebra $X$ and let $(\rho_-(\mu), \rho^-(\mu))$ be a rough fuzzy set. If $\rho_-(\mu)$ and $\rho^-(\mu)$ are fuzzy ideals of a $BCK/BCI$-algebra $X$, then we call $(\rho_-(\mu), \rho^-(\mu))$ a *rough fuzzy ideal* of $X$. Therefore we have:

**Corollary 4.5.** *If $\mu$ is a fuzzy ideal of a $BCK/BCI$-algebra $X$, then $(\rho_-(\mu), \rho^-(\mu))$ is a rough fuzzy ideal of $X$. If $\mu$, $\lambda$ are fuzzy ideals of a $BCK/BCI$-algebra $X$, then $(\rho_-(\mu\cap\lambda), \rho^-(\mu\cap\lambda))$ is a rough fuzzy ideal of $X$.*

Let $\mu$ be a fuzzy subset of a $BCK/BCI$-algebra $X$. Then the sets

$$\mu_t := \{x \in X | \mu(x) \geq t\},\ \mu_t^X := \{x \in X | \mu(x) > t\},$$

where $t \in [0, 1]$, are called respectively, a *$t$-level subset* and a *$t$-strong level subset* of $\mu$.

**Theorem 4.6.** ([3]) *Let $\mu$ be a fuzzy subset of a $BCK/BCI$-algebra $X$. Then $\mu$ is a fuzzy ideal of $X$ if and only if $\mu_t$ and $\mu_t^X$ are, if they are non-empty, ideals of $X$ for every $t \in [0, 1]$.*

**Lemma 4.7.** *Let $\rho$ be a congruence relation on a $BCK/BCI$-algebra $X$. If $\mu$ is a fuzzy subset of $X$ and $t \in [0,1]$, then*

(1) $(\rho_-(\mu))_t = \rho_-(\mu_t)$,
(2) $(\rho^-(\mu))_t^X = \rho^-(\mu_t^X)$.

*Proof.* (1) We have

$$x \in (\rho_-(\mu))_t \Leftrightarrow \rho_-(\mu)(x) \geq t \Leftrightarrow \wedge_{a\in[x]_\rho}\mu(a) \geq t$$
$$\Leftrightarrow \forall a \in [x]_\rho, \mu(a) \geq t \Leftrightarrow [x]_\rho \subseteq \mu_t \Leftrightarrow x \in \rho_-(\mu_t).$$

(2) Also we have

$$x \in (\rho^-(\mu))_t^X \Leftrightarrow \rho^-(\mu)(x) > t \Leftrightarrow \vee_{a\in[x]_\rho}\mu(a) > t$$
$$\Leftrightarrow \exists a \in [x]_\rho, \mu(a) > t \Leftrightarrow [x]_\rho \cap \mu_t^X \neq \emptyset \Leftrightarrow x \in \rho^-(\mu_t^X).$$

□

**Theorem 4.8.** *Let $\rho$ be a congruence relation on a $BCK/BCI$-algebra $X$. Then $\mu$ is a lower (an upper) rough fuzzy ideal of $X$ if and only if $\mu_t, \mu_t^X$ are, if they are non-empty, lower (upper) rough ideals of $X$ for every $t \in [0,1]$.*

*Proof.* By Theorem 4.6 and Lemma 4.7, we can obtain the conclusion easily. □

## 5. **Problems of Homomorphism**

**Lemma 5.1.** *Let $f$ be a surjective homomorphism of a $BCK/BCI$-algebra $X$ to a $BCK/BCI$-algebra $Y$ and let $A$ be any subset of $X$. Let $\rho_2$ be a congruence relation on $Y$, and $\rho_1 := \{(x_1, x_2) \in X \times X | (f(x_1), f(x_2)) \in \rho_2\}$. Then*

(1) *$\rho_1$ is a congruence relation on $X$,*
(2) *If $\rho_2$ is complete and $f$ is single-valued, then $\rho_1$ is complete,*
(3) *$f(\rho_1^-(A)) = \rho_2^-(f(A))$,*
(4) *$f(\rho_{1-}(A)) \subseteq \rho_{2-}(f(A))$. If $f$ is single-valued, then $f(\rho_{1-}(A)) = \rho_{2-}(f(A))$.*

*Proof.* (1) It is clear that $\rho_1$ is a congruence relation on $X$.
(2) Let $x'$ be any element of $[x_1 * x_2]_{\rho_1}$. Since $\rho_2$ is complete, by the definition of $\rho_1$, we know that $f(x') \in [f(x_1 * x_2)]_{\rho_2} = [f(x_1)]_{\rho_2} * [f(x_2)]_{\rho_2}$. Since $f$ is surjective, there exist $x_1', x_2' \in X$ such that $f(x_1') \in [f(x_1)]_{\rho_2}, f(x_2') \in [f(x_2)]_{\rho_2}$, and $f(x') = f(x_1') * f(x_2') = f(x_1' * x_2')$. Since $f$ is single-valued, by the definition of $\rho_1$, we have $x_1' \in [x_1]_{\rho_1}, x_2' \in [x_2]_{\rho_1}$, such that $x' = x_1' * x_2'$. Thus $x' \in [x_1]_{\rho_1} * [x_2]_{\rho_1}$. This means that $[x_1 * x_2]_{\rho_1} \subseteq [x_1]_{\rho_1} * [x_2]_{\rho_1}$. On the other hand, we have $[x_1]_{\rho_1} * [x_2]_{\rho_1} \subseteq [x_1 * x_2]_{\rho_1}$. Therefore $\rho_1$ is complete.
(3) Let $y$ be any element of $f(\rho_1{}^-(A))$. Then there exists $x \in \rho_1{}^-(A)$ such that $f(x) = y$. Hence $[x]_{\rho_1} \cap A \neq \emptyset$. Then there exists $x' \in [x]_{\rho_1} \cap A$. Then $f(x') \in f(A)$ and by the definition of $\rho_1$,

we have $f(x') \in [f(x)]_{\rho_2}$. So $[f(x)]\rho_2 \cap f(A) \neq \emptyset$, which implies $y = f(x) \in {\rho_2}^-(f(A))$. Thus $f({\rho_1}^-(A)) \subseteq {\rho_2}^-(A))$.

Conversely, let $y \in {\rho_2}^-(f(A))$. Then there exists $x \in X$ such that $f(x) = y$. Hence $[f(x)]_{\rho_2} \cap f(A) \neq \emptyset$. So there exists $x' \in A$ such that $f(x') \in f(A)$ and $f(x') \in [f(x)]_{\rho_2}$. Then by the definition of $\rho_1$, we have $x' \in [x]_{\rho_1}$. Thus $[x]_{\rho_1} \cap A \neq \emptyset$ which implies $x \in {\rho_1}^-(A)$. So $y = f(x) \in f({\rho_1}^-(A))$. It means that ${\rho_2}^-(f(A)) \subseteq f({\rho_1}^-(A))$. From the above, we have $f({\rho_1}^-(A)) = {\rho_2}^-(f(A))$.

(4) Let $y$ be any element of $f({\rho_1}_-(A))$. Then there exists $x \in {\rho_1}_-(A)$ such that $f(x) = y$, so we have $[x]_{\rho_1} \subseteq A$. Let $y' \in [y]_{\rho_2}$. Then there exists $x' \in X$ such that $f(x') = y'$ and $f(x') \in [f(x)]_{\rho_2}$. Hence $x' \in [x]_{\rho_1} \subseteq A$, and so $y' = f(x') \in f(A)$. Thus $[y]_{\rho_2} \subseteq f(A)$ which yields that $y \in {\rho_2}_-(f(A))$. So we have $f({\rho_1}_-(A)) \subseteq {\rho_2}_-(f(A))$.

Assume that $f$ is single-valued and suppose $y \in {\rho_2}_-(f(A))$. Then there exist $x \in X$ such that $f(x) = y$ and $[f(x)]_{\rho_2} \subseteq f(A)$. Let $x' \in [x]_{\rho_1}$. Then $f(x') \in [f(x)]_{\rho_2} \subseteq f(A)$, and so $x' \in A$. Thus $[x]_{\rho_1} \subseteq A$ which yields $x \in {\rho_1}_-(A)$. Then $y = f(x) \in f(\rho_1(A))$, and so ${\rho_2}_-(f(A)) \subseteq f({\rho_1}_-(A))$. From the above, we have $f({\rho_1}_-(A)) = {\rho_2}_-(f(A))$. □

**Theorem 5.2.** *Let $f$ be a surjective homomorphism of a $BCK/BCI$-algebra $X$ to a $BCK/BCI$-algebra $Y$. Let $\rho_2$ be a congruence relation on $Y$ and $A$ be a subset of $X$. If $\rho_1 := \{(x_1, x_2) \in X \times X | (f(x_1), f(x_2)) \in \rho_2\}$, then ${\rho_1}^-(A)$ is an ideal of $X$ if and only if ${\rho_2}^-(f(A))$ is an ideal of $Y$.*

*Proof.* Assume that ${\rho_1}^-(A)$ is an ideal of $X$. Since $0 \in {\rho_1}^-(A)$, $[0]_{\rho_1} \cap A \neq \emptyset$. Hence there exists $x' \in [0]_{\rho_1} \cap A$. Then $f(x') \in f(A)$, and by the definition of $\rho_1$, we have $f(x') \in [f(0)]_{\rho_2}$. So $[f(0)]_{\rho_2} \cap f(A) \neq \emptyset$ which means $f(0) \in {\rho_2}^-(f(A))$. Let $x', y' \in Y$ with $x', y' * x' \in {\rho_2}^-(f(A))$. Then there exist $x, z \in A$ such that $f(x) = x'$ and $f(z) = y' * x'$. Hence $[f(x)]_{\rho_2} \cap f(A) \neq \emptyset$ and $[f(z)]_{\rho_2} \cap f(A) \neq \emptyset$. Therefore there exists $b \in A$ such that $f(b) \in [f(x)]_{\rho_2}$. By the definition of $\rho_1$, $b \in [x]_{\rho_1}$ and so $b \in [x]_{\rho_1} \cap A$. Hence $[x]_{\rho_1} \cap A \neq \emptyset$. Thus $x \in {\rho_1}^-(A)$. Since $f$ is surjective, there exists $y \in X$ such that $f(y) = y'$. Put $u := y * ((y * x) * z)$. Then $u \in X$. Since

$$\begin{aligned} f((y * x) * z) =& f(y * x) * f(z) \\ =& f(y * x) * y' * x' \ (\because f(z) = y' * x') \\ =& (f(y) * f(x)) * (y' * x') \\ =& (y' * x') * (y' * x') = 0', \end{aligned}$$

we have $f(u) = f(y * ((y * x) * z)) = f(y) * f((y * x) * z) = f(y) * 0' = f(y) = y'$. Since $[f(z)]_{\rho_2} \cap f(A) \neq \emptyset$, we obtain

$$\begin{aligned} [y' * x']_{\rho_2} \cap f(A) =& ([y']_{\rho_2} * [x']_{\rho_2}) \cap f(A) \\ =& ([f(u)]_{\rho_2} * [f(x)]_{\rho_2}) \cap f(A) \\ =& [f(u * x)]_{\rho_2} \cap f(A) \neq \emptyset. \end{aligned}$$

Then there exists $a \in A$ such that $f(a) \in f(A)$ and $f(a) \in [f(u*x)]_{\rho_2}$. By the definition of $\rho_1$, we have $a \in [u*x]_{\rho_1}$. Hence $[u*x]_{\rho_1} \cap A \neq \emptyset$ and so $u*x \in {\rho_1}^-(A)$. Since ${\rho_1}^-(A)$ is an ideal of $X$ and $x \in {\rho_1}^-(A)$, we get $u \in {\rho_1}^-(A)$. Therefore $f(u) = y' \in f({\rho_1}^-(A)) = {\rho_2}^-(f(A))$. Thus ${\rho_2}^-(f(A))$ is an ideal of $Y$.

Conversely, suppose that ${\rho_2}^-(f(A))$ is an ideal of $Y$. Since $f(0) = 0' \in {\rho_2}^-(f(A))$, $[f(0)]_{\rho_2} \cap f(A) \neq \emptyset$. Hence there exists $y' \in [f(0)]_{\rho_2} \cap f(A)$. Since $f$ is surjective, there exists $x' \in X$ such that $f(x') = y'$. Hence $f(x') \in [f(0)]_{\rho_2} \cap f(A)$. Therefore $f(x') \in f(A)$. By the definition of $\rho_1$, $x' \in [0]_{\rho_1}$ and $x' \in A$. Hence $[0]_{\rho_1} \cap A \neq \emptyset$, which means $0 \in {\rho_1}^-(A)$.

Let $x_1, x_2 \in X$ with $x_1 * x_2, x_2 \in {\rho_1}^-(A)$. By Lemma 5.1, we obtain that $f(x_1 * x_2) = f(x_1) * f(x_2), f(x_2) \in f({\rho_1}^-(A)) = {\rho_2}^-(f(A))$. Since ${\rho_2}^-(f(A))$ is an ideal of $Y$, we have $f(x_1) \in {\rho_2}^-(f(A))$. Hence $[f(x_1)]_{\rho_2} \cap f(A) \neq \emptyset$. Therefore $y' \in [f(x_1)]_{\rho_2} \cap f(A)$. Since $f$ is surjective, there exists $x' \in X$ such that $f(x') = y'$. Hence $f(x') = y' \in [f(x_1)]_{\rho_2} \cap f(A)$. Therefore $f(x') \in f(A)$. By the definition of $\rho_1$, there exists $x' \in [x_1]_{\rho_1}$ and $x' \in A$. Therefore $[x_1]_{\rho_1} \cap A \neq \emptyset$, which means $x_1 \in {\rho_1}^-(A)$. Thus ${\rho_1}^-(A)$ is an ideal of $X$. □

**Theorem 5.3.** *Let $f$ be an isomorphism of a $BCK/BCI$-algebra $X$ to a $BCK/BCI$-algebra $Y$. Let $\rho_2$ be a complete congruence relation on $Y$ and let $A$ be a subset of $X$. If $\rho_1 := \{(x_1, x_2) \in X \times X | (f(x_1), f(x_2)) \in \rho_2\}$, then ${\rho_1}_-(A)$ is an ideal of $X$ if and only if ${\rho_2}_-(f(A))$ is an ideal of $Y$.*

*Proof.* By Lemma 5.1, we have $f({\rho_1}_-(A)) = {\rho_2}_-(f(A))$. The proof is similar to the proof of Theorem 5.2. □

By Theorem 5.2 and Theorem 5.3, we can obtain the following conclusion easily in quotient $BCK/BCI$-algebras.

**Corollary 5.4.** *Let $f$ be an isomorphism of a $BCK/BCI$-algebra $X$ to a $BCK/BCI$-algebra $Y$. Let $\rho_2$ be a complete congruence relation on $Y$ and let $A$ be a subset of $X$. If $\rho_1 := \{(x_1, x_2) \in X \times X | (f(x_1), f(x_2)) \in \rho_2\}$, then ${\rho_1}_-(A)/\rho_1$( resp. ${\rho_1}^-(A)/\rho_1$) is an ideal of $X/\rho_1$ if and only if ${\rho_2}_-(f(A))/\rho_2$(resp. ${\rho_2}^-(f(A))/\rho_2$) is an ideal of $Y/\rho_2$.*

**Theorem 5.5.** *Let $f$ be a surjective homomorphism of a $BCK/BCI$-algebra $X$ to a $BCK/BCI$-algebra $Y$. Let $\rho_2$ be a complete congruence relation on $Y$ and let $A$ be a fuzzy subset of $X$. If $\rho_1 := \{(x_1, x_2) \in X \times X | (f(x_1), f(x_2)) \in \rho_2\}$, then*

(1) *${\rho_1}^-(A)$ is a fuzzy ideal of $X$ if and only if ${\rho_2}^-(f(A))$ is a fuzzy ideal of $Y$.*
(2) *If $f$ is single-valued, then ${\rho_1}_-(A)$ is a fuzzy ideal of $X$ if and only if ${\rho_2}_-(f(A))$ is a fuzzy ideal of $X$.*

*Proof.* (1) By Theorem 4.6, we obtain that ${\rho_1}^-(A)$ is a fuzzy ideal of $X$ if and only if $({\rho_1}^-(A))_t^X$ is, if it is non-empty, an ideal of $X$ for every $t \in [0, 1]$. By Lemma 4.7, we have $({\rho_1}^-(A))_t^X = {\rho_1}^-(A_t^X)$.

By Theorem 5.2, we obtain that ${\rho_1}^-(A_t^X)$ is an ideal of $X$ if and only if ${\rho_2}^-(f(A_t^X))$ is an ideal of $Y$. It is clear that $f(A_t^X) = (f(A))_t^X$. From this and Lemma 4.7, we have

$$ {\rho_2}^-(f(A_t^X)) = {\rho_2}^-(f(A)_t^X) = ({\rho_2}^-(f(A)))_t^X. $$

By Theorem 4.6, we obtain that $({\rho_2}^-(f(A)))_t^X$ is an ideal of $Y$ for every $t \in [0,1]$ if and only if ${\rho_2}^-(f(A))$ is a fuzzy ideal of $Y$. Thus the conclusion holds.
(2) Since $f$ is single valued, by Lemma 5.1, we have $f(\rho_{1-}(A)) = \rho_{2-}(f(A))$. The proof is similar to that of (1). □

**Corollary 5.6.** *Let $f$ be an isomorphism of a $BCK/BCI$-algebra $X$ to a $BCK/BCI$-algebra $Y$. Let $\rho_2$ be a complete congruence relation on $Y$ and $A$ a fuzzy subset of $X$. If $\rho_1 := \{(x_1, x_2) \in X \times X | (f(x_1), f(x_2)) \in \rho_2\}$, then $\rho_{1-}(A_t)/\rho_1$ (resp. ${\rho_1}^-(A_t^X)/\rho_1$) is an ideal of $X/\rho_1$ if and only if $\rho_{2-}(f(A_t))/\rho_2$ (resp. ${\rho_2}^-(f(A_t^X))/\rho_2$) is an ideal of $Y/\rho_2$.*

## References


[1] R. Biswas and S. Nanda, *Rough groups and rough subgroups*, Bull. Pol. Ac. Math. **42** (1994), 251-254.
[2] K. Iséki and S. Tanaka, *An introduction to the theory of BCK-algebras*, Math. Japo. **23** (1978), 1-26.
[3] Y. B. Jun, *Chracterization of fuzzy ideals by their level ideals in BCK(BCI)-algebras*, Math. Japop. **38** (1993), 67-71.
[4] M. Kondo, *Congruences and closed ideals in BCI-algebras*, Math. Japo. **48** (1997), 491-496.
[5] N. Kuroki, *Rough ideals in semigroups*, Inform. Sci., **100** (1995), 139-163.
[6] N. Kuroki and J. N. Mordeson, *Structure of rough sets and rough groups*, J. Fuzzy Math. **5** (1997), 183-191.
[7] N. Kuroki and P. P. Wang, *The lower and upper approximations in a fuzzy group*, Infrom. Sci. **90** (1996), 203-220.
[8] C. R. Lim and H. S. Kim, *Rough ideals in BCK/BCI-algebras*, Bull. Pol. Ac. Math. **51**(2003), 59-67.
[9] J. Meng and Y. B. Jun, *BCK*-algebras, Kyung Moon Sa, Seoul, 1994.
[10] J. N. Mordeson, *Rough set theory applied to (fuzzy) ideals theory*, Fuzzy Sets and Systems **121** (2001), 315-324.
[11] Z. Pawlak, *Rough sets*, Int. J. Inform. Comp. Sci. **11** (1982), 341-356.
[12] Z. Pawlak, *Rough sets and fuzzy sets*, Fuzzy Sets and Systems **17** (1985), 99-102.
[13] Q. M. Xiao and Z. L. Zhang, *Rough prime ideals and rough fuzzy prime ideals in semigroups*, Inform. Sci. **176** (2006), 725-733.

# A Lebesgue integrable space of Boehmians for a class of $\mathcal{D}_\kappa$ transformations

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**Abstract** Boehmians are objects obtained by an abstract algebraic construction similar to that of field of quotients and it in some cases just gives the field of quotients. As Boehmian spaces are represented by convolution quotients, integral transforms have a natural extension onto appropriately defined spaces of Boehmians. In this paper, we have defined convolution products and a class of delta sequences and have examined the axioms necessary for generating the $\mathcal{D}_\kappa$ spaces of Boehmians. The extended $\mathcal{D}_\kappa$ transformation has therefore been defined as a one-to-one onto mapping continuous with respect to $\Delta$ and $\delta$ convergences. Over and above, it has been asserted that the necessary and sufficient conditions for an integrable sequence to be in the range of the $\mathcal{D}_\kappa$ transformation is that the class of quotients belongs to the range of the representative. Further results related to the inverse problem are also discussed.

**keywords:** Integral transform; analogue system; generalized integral; discrete system; Boehmian.

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## 1 Introduction

As some physical situations were determined by initial value problems which are not smoothly enough but are generalized functions, numerous integral transforms were defined in a context of distributions, ultradistributions, tempered distributions, tempered ultradistributions and Boehmian spaces. The Laplace transform method of right-side distributions was treated in [17] and [18] to solve various types of ordinary differential equations. In [19] Loonker and Banerji have given a solution of Volterra-Abel integral equations by aid of a distributional wavelet transform. Indeed, if the differential equation $\acute{u} = w$ , $w$ being the heaviside step function, is considered then no classical conclusion can be drawn at this point. But, on generalized sense, if $S$ denotes a space of rapid descents (rapidly decreasing functions) and $S^{'}$ be its dual of slow growth, then for every

$v \in S$ we write

$$\begin{aligned}
\acute{u}(v) &= -u(\acute{v}) \\
&= -\int_{-\infty}^{\infty} u(x)\,\acute{v}(x)\,dx \\
&= -\int_{-\infty}^{0} \alpha\acute{v}(x)\,dx - \int_{0}^{\infty} (x+\alpha)\,\acute{v}(x)\,dx \\
&= -\alpha v(0) + \alpha v(0) - \int_{0}^{\infty} x\acute{v}(x)\,dx \\
&= \int_{0}^{\infty} v(x)\,dx \\
&= \int_{-\infty}^{\infty} w(x)\,v(x)\,dx \\
&= w(v)
\end{aligned}$$

where $\alpha$ is some suitable constant.

Let $\kappa$ be a sampling period and $v_\alpha$ be an analogue function. In some engineering applications, the classical $\mathcal{D}$ transform was presented as an equivalence between discrete and analogue systems as [8]

$$\mathcal{D}v_\alpha(r,\kappa) := \mathcal{D}v_\alpha(r) := \frac{1}{r!}\int_{R_+} v_\alpha(t)\,e^{-t\kappa^{-1}}\left(t\kappa^{-1}\right)^r dt \tag{1}$$

where $\mathcal{D}(v_\alpha * y_\alpha)(r,\kappa) = \sum_{0}^{r} \mathcal{D}y_\alpha(r-k,\kappa)\,\mathcal{D}v_\alpha(r,\kappa)$, $*$ being the Fourier convolution product defined by [7]

$$(v_\alpha * y_\alpha)(t) := \int_{R_+} v_\alpha(\tau)\,y_\alpha(t-\tau)\,dt. \tag{2}$$

Let $x_\alpha$ be an analogue function and $\kappa$ be a sampling period. Then, treating $r$ as a positive real number, say $\xi$, then the existed integral, denoted by $\mathcal{D}_\kappa$, is given as

$$\mathcal{D}_\kappa v_\alpha(\xi,\kappa) = \frac{1}{\xi!}\int_{R_+} v_\alpha(t)\,e^{-t\kappa^{-1}}\left(t\kappa^{-1}\right)^\xi dt, \tag{3}$$

where $\xi \in R_+$; $R_+ := (0,\infty)$.

In this paper, without reading the efficiency of this integral in discrete and analogue systems, we attempt to investigate the extension of this integral to a class of Boehmians, being recent in the space of generalized functions. We derive virtuous products, give definitions and derive some properties of the existence of the given integral in the class of generalized functions.

Boehmian spaces were inaugurated by the idea of regular operators which is a

subalgebra of Mikusiński operators. According to literature, we briefly recall the general construction of Boehmian spaces. Let $\mathcal{G}$ be a group and $\mathcal{S}$ be a subgroup of $\mathcal{G}$. We assume to each pair of elements $f \in \mathcal{G}$ and $\omega \in \mathcal{S}$, is assigned the product $f * g$ such that :

(1) $\omega, \psi \in \mathcal{S}$ implies $\omega * \psi \in \mathcal{S}$ and $\omega * \psi = \psi * \omega$.

(2) $f \in \mathcal{G}$ and $\omega, \psi \in \mathcal{S}$ implies $(f * \omega) * \psi = f * (\omega \star \psi)$.

(3) $f, g \in \mathcal{G}, \omega \in \mathcal{S}$ and $\lambda \in R$, implies $(f+g)*\omega = f*\omega + g*\omega, \lambda(f*\omega) = (\lambda f)*\omega$.Let $\Delta$ be a family of sequences from $\mathcal{S}$ such that:

(1) $f, g \in \mathcal{G}, (\delta_n) \in \Delta$ and $f * \delta_n = g * \delta_n \, (n = 1, 2, ...)$ implies $f = g, n \in N$.

(2) $(\omega_n), (\delta_n) \in \Delta$ implies $(\omega_n * \psi_n) \in \Delta$.

Members of $\Delta$ are called delta sequences.

Let $\mathcal{A}$ be a pair of sequences defined by $\mathcal{A} = \left\{((f_n), (\omega_n)) : (f_n) \subseteq \mathcal{G}^N, (\omega_n) \in \Delta\right\}$, where $n \in N$, then members of $((f_n), (\omega_n)) \in \mathcal{A}$ are called quotient of sequences, denoted by $[f_n/\omega_n]$, if $f_n * \omega_m = f_m * \omega_n, \forall n, m \in N$. Two quotients of sequences $f_n/\omega_n$ and $g_n\psi_n$ are equivalent, $f_n/\omega_n \sim g_n\psi_n$, if $f_n * \psi_m = g_m * \omega_n, \forall n, m \in N$.

The relation $\sim$ is an equivalent relation on $\mathcal{A}$. The equivalence class containing $f_n/\omega_n$ is denoted by $[f_n/\omega_n]$. These equivalence classes are called Boehmians. The space of all Boehmians is denoted by $\beta_1$. The sum of two Boehmians and multiplication by a scalar can be defined in a natural way $[f_n/\omega_n] + [g_n/\psi_n] = [(f_n * \psi_n + g_n * \omega_n) / (\omega_n * \psi_n)], \alpha [f_n/\omega_n] = [\alpha f_n/\omega_n], \alpha \in C$, space of complex numbers.

The operations $*$ and $\mathcal{D}^\alpha$ are given by $[f_n/\omega_n]*[g_n/\psi_n] = [(f_n * g_n) / (\omega_n * \psi_n)]$ and $\mathcal{D}^\alpha [f_n/\omega_n] = [\mathcal{D}^\alpha f_n/\omega_n]$ whereas, $*$ can be extended to $\beta \times \mathcal{S}$ in the form that If $[f_n/\omega_n] \in \beta_1$ and $\omega \in \mathcal{S}$, then $[f_n/\omega_n] * \omega = [f_n * \omega/\omega_n]$ .

However, soon after the topic has been initiated, numerous integral transforms were extended to Boehmian spaces by many authors in [1] , [2] , [6], [9 − 16] , [20 − 23] and many others.

**Definition 1** The Mellin type convolution product $\odot$ between two signals $x_\alpha$ and $y_\alpha$ is defined by the integral equation ( see [4])

$$(v_\alpha \odot y_\alpha)(x) = \int_{R_+} v_\alpha\left(y^{-1}x\right) y_\alpha(x)\, y^{-1} dy \tag{4}$$

when the integral exists.

The Lebesgue space of integrable functions defined on $R_+^2$ is denoted by $\mathbf{l}^1\left(R_+^2\right)$ and the set of smooth functions of bounded supports over $R_+$ is denoted by $\vartheta(R_+)$ (see [3] for definition, properties and convergence in $\vartheta(R_+)$).

## 2 Convolution products and Boehmians

In this section, we establish the prerequisite axioms of the Boehmian space $\mathcal{B}\left(\mathbf{l}^1\left(R_+^2\right), \vartheta, \odot, \bullet\right)$ with the operations $\odot$ and $\bullet$ where $\bullet$ is a convolution product defined as follows.

**Definition 2** Let the casual analogue signals $v_\alpha$, $y_\alpha \in \mathbf{l}^1\left(R_+^2\right)$ be given. Then, between $v_\alpha$ and $y_\alpha$, we define a product $\bullet$ given as

$$\left(v_\alpha \bullet y_\alpha\right)(\xi,\kappa) = \int_{R_+} v_\alpha\left(\xi, y^{-1}\kappa\right) y_\alpha(y)\, dy \tag{5}$$

provided the above integral exists.

Proving axioms of the space $\mathcal{B}\left(\mathbf{l}^1\left(R_+^2\right), \vartheta, \odot, \bullet\right)$ begins with the following theorem.

**Theorem 3** Given $v_\alpha \in \mathbf{l}^1\left(R_+^2\right)$ and $y_\alpha \in \vartheta(R_+)$. Then we get $v_\alpha \bullet y_\alpha \in \mathbf{l}^1\left(R_+^2\right)$.

**Proof** The hypothesis that $v_\alpha \in \mathbf{l}^1\left(R_+^2\right)$ implies $\int\limits_{R_+^2} \left|v_\alpha\left(\xi, y^{-1}\kappa\right)\right| d\xi d\kappa < M_1\,(y>0)$. Hence, with the aid of the Fubini's theorem together with the hypothesis that $y_\alpha \in \vartheta(R_+)$ we confirm

$$\begin{aligned}
\int\limits_{R_+^2} \left|\left(v_\alpha \bullet y_\alpha\right)(\xi,\kappa)\right| d\xi d\kappa &= \int\limits_{R_+^2} \left| \int\limits_{R_+} v_\alpha\left(\xi, y^{-1}\kappa\right) y_\alpha(y)\, dy \right| d\xi d\kappa \\
&\leq \int\limits_{R_+^2} \int\limits_{P} \left|v_\alpha\left(\xi, y^{-1}\kappa\right)\right| \left|y_\alpha(y)\right| dy d\xi d\kappa \\
&\leq M_1 \int\limits_{P} \left|y_\alpha(y)\right| dy < \infty
\end{aligned}$$

where $P$ is an interval in $R_+$ including the support of $y_\alpha$.

Hence the theorem is finished.

**Theorem 4** Let $v_\alpha \in \mathbf{l}^1\left(R_+^2\right)$ and that $y_\alpha, z_\alpha \in \vartheta(R_+)$ be analogue signals. Then

$$v_\alpha \bullet \left(y_\alpha \odot z_\alpha\right) = \left(v_\alpha \bullet y_\alpha\right) \bullet z_\alpha.$$

**Proof** On account of (4) and (5) we are permitted to write

$$\left(v_\alpha \bullet \left(y_\alpha \odot z_\alpha\right)\right)(\xi,\kappa) = \int_{R_+^2} v_\alpha\left(\xi, y^{-1}\kappa\right) y_\alpha\left(t^{-1}y\right) dy z_\alpha(t)\, t^{-1} dt. \tag{6}$$

The substitution $u = yt^{-1}$ implies $dy = tdu$. Therefore, (6) can be expressed as

$$\begin{aligned}
\left(v_\alpha \bullet \left(y_\alpha \odot z_\alpha\right)\right)(\xi,\kappa) &= \int_{R_+^2} v_\alpha\left(\xi, u^{-1}\left(t^{-1}\kappa\right)\right) y_\alpha(u)\, z_\alpha(t)\, du dt \\
&= \int_{R_+} \left(v_\alpha \bullet y_\alpha\right)\left(\xi, t^{-1}\kappa\right) z_\alpha(t)\, dt.
\end{aligned}$$

The proof is therefore finished.

**Theorem 5** Given $v_\alpha \in \mathbf{l}^1\left(R_+^2\right)$. For every $y_\alpha \in \vartheta\left(R_+\right)$, we get

$$\mathcal{D}_\kappa\left(v_\alpha \odot y_\alpha\right)(\xi,\kappa) = \left(\left(\mathcal{D}_\kappa v_\alpha\right) \bullet y_\alpha\right)(\xi,\kappa)$$

**Proof** Applying (3) to (4) gives

$$\begin{aligned}\mathcal{D}_\kappa\left(v_\alpha \bullet y_\alpha\right)(\xi,\kappa) &= \frac{1}{\xi!}\int\limits_{R_+^2} v_\alpha\left(y^{-1}x\right) y_\alpha(y)\, y^{-1} dy \left(e^{-x\kappa^{-1}}\right)\left(x\kappa^{-1}\right)^\xi dx \\ &= \frac{1}{\xi!}\int\limits_{R_+^2} v_\alpha\left(y^{-1}x\right)\left(e^{-x\kappa^{-1}}\right)\left(x\kappa^{-1}\right)^\xi y_\alpha(y)\, y^{-1} dxdy. \quad (7)\end{aligned}$$

Let $zy = x$, then $dx = ydz$. Therefore, on account of (7) we obtain that

$$\mathcal{D}_\kappa\left(v_\alpha \odot y_\alpha\right)(\xi,\kappa) = \frac{1}{\xi!}\int\limits_{R_+^2} v_\alpha(x)\left(e^{-yz\kappa^{-1}}\right)\left(x\kappa^{-1}\right)^\xi y_\alpha(y)\, dzdy. \quad (8)$$

Hence, by the Fubini's theorem, we finish the proof of the theorem.

**Theorem 6** Given $\breve{r} \in C$, $v_\alpha \in \mathbf{l}^1\left(R_+^2\right)$ and $y_\alpha \in \vartheta\left(R_+\right)$. We get

$$\left(\breve{r} v_\alpha\right) \bullet y_\alpha = \breve{r}\left(v_\alpha \bullet y_\alpha\right).$$

Proof of this theorem is straightforward follows from definitions. Hence it is omitted.

**Theorem 7** Given $v_\alpha, z_\alpha \in \mathbf{l}^1\left(R_+^2\right)$. For every $y_\alpha \in \vartheta\left(R_+\right)$, we get

$$\left(v_\alpha + z_\alpha\right) \bullet y_\alpha = v_\alpha \bullet y_\alpha + z_\alpha \bullet y_\alpha.$$

Proof of above theorem follows from simple integration. Details are therefore omitted.

**Theorem 8** Given $v_{\alpha,n} \to v_\alpha$ as $n \to \infty$ in $\mathbf{l}^1\left(R_+^2\right)$. For every $y_\alpha \in \vartheta\left(R_+\right)$, we get $v_{\alpha,n} \bullet y_\alpha \to v_\alpha \bullet y_\alpha$ as $n \to \infty$.

Proof of above theorem is a direct conclusion of Theorem 4. Hence it is avoided.

By $\Delta$ we mean the subset of $\vartheta\left(R_+\right)$ such that for every sequence $\left(\mu_{\alpha,n}\right)_0^\infty \in \vartheta\left(R_+\right)$, $n \in N$, we have.

$i^{'}: \int\limits_{R_+} \mu_{\alpha,n} dy = 1$; $i^{''}: \int\limits_{R_+} \left|\mu_{\alpha,n}\right| dy < \infty$; $i^{'''}: supp\mu_{\alpha,n} \subseteq (0, a_n)$, $\lim_{n\to\infty} a_n = 0$.

Elements of $\Delta$ are said to be delta sequences or approximating identities.

**Theorem 9** Given $v_\alpha \in \mathbf{l}^1\left(R_+^2\right)$. For every $\left(\mu_{\alpha,n}\right) \in \vartheta\left(R_+\right)$, we get $\lim_{n\to\infty} v_\alpha \bullet \mu_{\alpha,n} = v_\alpha$.

**Proof** Let $v_\alpha \in \mathbf{l}^1\left(R_+^2\right)$ and $\vartheta\left(R_+^2\right)$ be the set of smooth functions of bounded supports over $R^2$, then $\vartheta\left(R_+^2\right)$ is dense in $\mathbf{l}^1\left(R_+^2\right)$. Hence, for a given $\epsilon > 0$, we can find $\psi_\alpha \in \vartheta\left(R_+^2\right)$ such that

$$\|v_\alpha - \psi_\alpha\| < \epsilon. \tag{9}$$

Define $g_\alpha(y) = \psi_\alpha\left(\xi, y^{-1}\kappa\right)$, then $g_\alpha(y)$ is uniformly continuous mapping in $\vartheta(R_+)$ for every $\xi,\ \kappa > 0$. Therefore, for each $\epsilon > 0$ we find $\delta > 0$ so that $-\epsilon < g_\alpha(y_1) - g_\alpha(y_2) < \epsilon$ wherever $-\delta < y - y^{-1} < \delta$.
Since $y$ and $y^{-1}$ belong to $R_+$ and that $g_\alpha \in \vartheta(R_+)$, we get

$$-\epsilon < g_\alpha(y) - g_\alpha\left(y^{-1}\right) < \epsilon \tag{10}$$

when $-\delta < y - y^{-1} < \delta$.
Also, since $\psi_\alpha$ is of bounded support in $\vartheta\left(R_+^2\right)$ it follows $supp\psi_\alpha\left(\xi, y^{-1}\kappa\right) \subseteq [a_1, a_2] \times P$ for some compact subset $P$ of $R_+$. Hence

$$\psi_\alpha\left(\xi, y^{-1}\kappa\right) = 0,\ \left(\xi, y^{-1}\kappa\right) \notin [a_1 - \delta, a_2 + \delta] \times P. \tag{11}$$

The hypothesis that $supp\mu_{\alpha,n} \to 0,\ n \to \infty$ asserts that we can find $N \in N$ such that

$$supp\mu_{\alpha,n}(y) \subset [0, \delta] \tag{12}$$

for every $n \geq N$. By the property $\int\limits_{R_+} \mu_{\alpha,n} dy = 1$ we write

$$\begin{aligned}
\|(\psi_\alpha \bullet \mu_{\alpha,n}) - \psi_\alpha\| &= \int\limits_{R_+^2} \left( \left| \int\limits_{R_+} \left(\psi_\alpha\left(\xi, y^{-1}\kappa\right) - \psi_\alpha(\xi, \kappa)\,\mu_{\alpha,n}(y)\right) dy \right| \right) d\xi d\kappa \\
&\leq \int\limits_{R_+^3} \left( \left| g_\alpha(y) - g_\alpha\left(y^{-1}\right) \right| |\mu_{\alpha,n}(y)| \right) dy d\xi d\kappa \\
&= \int\limits_P \int\limits_{a_1}^{a_2+\delta} \int\limits_0^\delta \left( \left| g_\alpha(y) - g_\alpha\left(y^{-1}\right) \right| |\mu_{\alpha,n}(y)| \right) dy d\xi d\kappa.
\end{aligned}$$

By virtue of (10) and (12) and the assumption that $\int\limits_{R_+} |\mu_{\alpha,n}|\, dy < M$ we have

$$\begin{aligned}
\|\psi_\alpha \bullet \mu_{\alpha,n} - \psi_\alpha\| &< \epsilon\,(a_2 + \delta) \int\limits_{a_1}^{a_2+\delta} \left( \int\limits_0^\delta |\mu_{\alpha,n}(y)|\, dy \right) d\xi \\
&= \epsilon\,(a_2 + \delta)\, M\,(a_2 - a_1 + \delta)
\end{aligned} \tag{13}$$

Finally, we write

$$\|(v_\alpha \bullet \mu_{\alpha,n}) - v_\alpha\| \leq \|(v_\alpha - \psi_\alpha) \bullet \mu_{\alpha,n}\| + \|(\psi_\alpha \bullet \mu_{\alpha,n}) - \psi_\alpha\| + \|\psi_\alpha - v_\alpha\|. \tag{14}$$

Hence, on account of (9) , (10) and (13) , (14) yields

$$\|(v_\alpha \bullet \mu_n) - v_\alpha\| \leq \epsilon \left(M + M\left(a_2 - a_1 + 2\delta\right) + 1\right).$$

This finishes the proof of our result.

The Boehmian space $\mathcal{B}\left(\mathbf{l}^1\left(R_+^2\right), \vartheta, \odot, \bullet\right)$ is therefore entirely performed. Similarly, one can proceed to generate the space $\mathcal{B}\left(\mathbf{l}^1\left(R_+^2\right), \vartheta, \odot, \odot\right)$.

We introduce addition in $\mathcal{B}\left(\mathbf{l}^1\left(R_+^2\right), \vartheta, \odot, \bullet\right)$ as $[(u_n)/(\epsilon_n)] + [(v_n)/(\varepsilon_n)] = [((u_n \bullet \epsilon_n) + (v_n \bullet \epsilon_n))/(\epsilon_n \odot \varepsilon_n)]$. In $\mathcal{B}\left(\mathbf{l}^1\left(R_+^2\right), \vartheta, \odot, \bullet\right)$ we define scalar multiplication as $\Omega\,[(u_n)/(\epsilon_n)] = [\Omega\,(u_n)/(\epsilon_n)] = [(\Omega u_n)/(\epsilon_n)], \Omega \in C$. We define the convolution $\bullet$ in $\mathcal{B}\left(\mathbf{l}^1\left(R_+^2\right), \vartheta, \odot, \bullet\right)$ as $[(\varphi_n)/(\epsilon_n)] \bullet [(v_n)/(\varepsilon_n)] = [(\varphi_n \bullet v_n)/(\epsilon_n \odot \varepsilon_n)]$. Also, we define differentiation in $\mathcal{B}\left(\mathbf{l}^1\left(R_+^2\right), \vartheta, \odot, \bullet\right)$ as $\mathrm{D}^\alpha\,[(u_n)/(\epsilon_n)] = [(\mathcal{D}^\alpha u_n)/(\epsilon_n)]$, $\alpha$ is a real number. For $\mathcal{B}\left(\mathbf{l}^1\left(R_+^2\right), \vartheta, \odot, \bullet\right) \bullet \mathbf{l}^1\left(R_+^2\right)$ the product is given as $[(u_n)/(\epsilon_n)] \bullet \varphi = [((u_n) \bullet \varphi)/(\epsilon_n)]$ where $[(u_n)(\delta_n)] \in \mathcal{B}\left(\mathbf{l}^1\left(R_+^2\right), \vartheta, \odot, \bullet\right)$ and $\varphi \in \mathbf{l}^1\left(R_+^2\right)$.

$(\beta_n)$ of $\mathcal{B}\left(\mathbf{l}^1\left(R_+^2\right), \vartheta, \odot, \bullet\right)$ is $\delta$ convergent to $\beta$ in $\mathcal{B}\left(\mathbf{l}^1\left(R_+^2\right), \vartheta, \odot, \bullet\right)$; i.e.$(\beta_n \to \beta)$, if we can find $(\epsilon_n)$ in $\Delta$ such that $\beta_n \bullet \epsilon_k; \beta \bullet \epsilon_k \in \mathbf{l}^1\left(R_+^2\right)$ $(\forall k, n \in N)$, where $\lim_{n\to\infty} \beta_n \bullet \epsilon_k = \beta \bullet \epsilon_k$ in $\mathbf{l}^1\left(R_+^2\right)$ $(\forall k \in N)$.

$(\beta_n)$ of $\mathcal{B}\left(\mathbf{l}^1\left(R_+^2\right), \vartheta, \odot, \bullet\right)$ is $\Delta$-convergent to $\beta$ in $\mathcal{B}\left(\mathbf{l}^1\left(R_+^2\right), \vartheta, \odot, \bullet\right)$; i.e. $(\beta_n \to \beta)$, if $(\epsilon_n) \in \Delta$, with $(\beta_n - \beta) \bullet \epsilon_n \in \mathbf{l}^1\left(R_+^2\right)$ $(\forall n \in N)$, and $\lim_{n\to\infty} (\beta_n - \beta) \bullet \epsilon_n = 0$ in $\mathbf{l}^1\left(R_+^2\right)$.

## 3 $\mathcal{D}_\kappa$ transform of Boehmians

Let $\beta = [(v_{\alpha,n})/(\mu_{\alpha,n})] \in \mathcal{B}\left(\mathbf{l}^1\left(R_+^2\right), \vartheta, \odot, \odot\right)$. Then we present the generalized transform $\widehat{\mathcal{D}_\kappa}$ of $\beta$ as

$$\widehat{\mathcal{D}_\kappa}\beta = [(\mathcal{D}_\kappa v_{\alpha,n})/(\mu_{\alpha,n})] \tag{15}$$

in the space $\mathcal{B}\left(\mathbf{l}^1\left(R_+^2\right), \vartheta, \odot, \bullet\right)$.

**Theorem 10** The mapping $\widehat{\mathcal{D}_\kappa}\beta : \mathcal{B}\left(\mathbf{l}^1\left(R_+^2\right), \vartheta, \odot, \odot\right) \to \mathcal{B}\left(\mathbf{l}^1\left(R_+^2\right), \vartheta, \odot, \bullet\right)$ is well-defined and linear.

**Proof** Given $[(v_{\alpha,n})/(r_{\alpha,n})] = [(y_{\alpha,n})/(\psi_{\alpha,n})] \in \mathcal{B}\left(\mathbf{l}^1\left(R_+^2\right), \vartheta, \odot, \odot\right)$. Then it follows that $v_{\alpha,n} \odot y_{\alpha,m} = y_{\alpha,n} \odot r_{\alpha,m}$. Employing $\mathcal{D}_\kappa$ for the previous equation directly gives $\mathcal{D}_\kappa v_{\alpha,n} \bullet y_{\alpha,m} = \mathcal{D}_\kappa y_{\alpha,n} \bullet r_{\alpha,m}, (\forall n, m \in N)$. That is,

$$\mathcal{D}_\kappa v_{\alpha,n}/r_{\alpha,n} \sim \mathcal{D}_\kappa y_{\alpha,n}/\psi_{\alpha,n}. \tag{16}$$

Finally, let the Boehmians $[(v_{\alpha,n})/(r_{\alpha,n})]$ and $[(y_{\alpha,n})/(\psi_{\alpha,n})]$ be equivalent in the space $\mathcal{B}\left(\mathbf{l}^1\left(R_+^2\right), \vartheta, \odot, \odot\right)$. Then by definitions and Theorem 5 the proof of our result follows.

**Theorem 11** The extended $\widehat{\mathcal{D}_\kappa}$ integral is consistent with $\mathcal{D}_\kappa : \mathbf{l}^1\left(R_+^2\right) \to \mathbf{l}^1\left(R_+^2\right)$.

**Proof** For every $v_\alpha \in \mathbf{l}^1\left(R_+^2\right)$, let $\beta$ be its representative in $\mathcal{B}\left(\mathbf{l}^1\left(R_+^2\right), \vartheta, \odot, \odot\right)$. Then indeed $\beta = [(v_\alpha \odot (\varphi_{\alpha,n}))/(\varphi_{\alpha,n})]$ where $(\varphi_{\alpha,n}) \in \Delta\ (\forall n \in N)$. On the other hand, its clear that $(\varphi_{\alpha,n})$ and its representative are independent for $n \in N$. Therefore by Theorem 5 we deduce $\widehat{\mathcal{D}_\kappa}(\beta) = [(\mathcal{D}_\kappa(v_\alpha \odot (\varphi_{\alpha,n})))/(\varphi_{\alpha,n})] = [(\mathcal{D}_\kappa(v_\alpha \bullet (\varphi_{\alpha,n})))/(\varphi_{\alpha,n})]$; that is the representative of $\mathcal{D}_\kappa f$ in $\mathbf{l}^1\left(R_+^2\right)$.
Hence the proof is finished.

**Theorem 12** Given $[(g_{\alpha,n})/(\psi_{\alpha,n})] \in \mathcal{B}\left(\mathbf{l}^1\left(R_+^2\right), \vartheta, \odot, \bullet\right)$. Then $[(g_{\alpha,n})/(\psi_{\alpha,n})]$ is in the range of $\widehat{\mathcal{D}_\kappa}$ iff $g_{\alpha,n}$ is in the range of $\mathcal{D}_\kappa\ (\forall n \in N)$.

**Proof** Indeed, when $[(g_{\alpha,n})/(\psi_{\alpha,n})]$ is in the range of $\widehat{\mathcal{D}_\kappa}$; then it is obvious that $g_{\alpha,n}$ is in the range of $\mathcal{D}_\kappa\ (\forall n \in N)$.
On the other hand, if $g_{\alpha,n}$ is in the range of $\mathcal{D}_\kappa\ (\forall n \in N)$, then it can be obtained an element $v_{\alpha,n} \in \mathbf{l}^1\left(R_+^2\right)$ satisfying $\mathcal{D}_\kappa v_{\alpha,n} = g_{\alpha,n}\ (\forall n \in N)$. Due to the assumption that $[(g_{\alpha,n})/(\psi_{\alpha,n})] \in \mathcal{B}\left(\mathbf{l}^1\left(R_+^2\right), \vartheta, \odot, \bullet\right)$, we have $g_{\alpha,n} \bullet \psi_{\alpha,m} = g_{\alpha,m} \bullet \psi_{\alpha,n}\ (\forall m, n \in N)$. That is, $\mathcal{D}_\kappa(v_{\alpha,n} \odot \varphi_{\alpha,n}) = \mathcal{D}_\kappa(v_{\alpha,m} \odot \varphi_{\alpha,n})\ (m, n \in N)$, where $v_{\alpha,n} \in \mathbf{l}^1\left(R_+^2\right)$ and $\varphi_{\alpha,n} \in \Delta$, $\forall n \in N$.
Now, injectivity of $\mathcal{D}_\kappa$ leads to $\mathrm{v}_{\alpha,n} \odot \varphi_{\alpha,m} = v_{\alpha,m} \odot \varphi_{\alpha,n}\ (m, n \in N)$.
Hence, the quotient $v_{\alpha,n}/\varphi_{\alpha,n}$ defines a quotient in $\mathcal{B}\left(\mathbf{l}^1\left(R_+^2\right), \vartheta, \odot, \odot\right)$. Therefore, we write $[(v_{\alpha,n})/(\varphi_{\alpha,n})] \in \mathcal{B}\left(\mathbf{l}^1\left(R_+^2\right), \vartheta, \odot, \odot\right)$ and $\widehat{\mathcal{D}_\kappa}([(v_{\alpha,n})/(\varphi_{\alpha,n})]) = [(g_{\alpha,n})/(\psi_{\alpha,n})]$.
Hence the proof is finished.

**Theorem 13** $\widehat{\mathcal{D}_\kappa} : \mathcal{B}\left(\mathbf{l}^1\left(R_+^2\right), \vartheta, \odot, \odot\right) \to \mathcal{B}\left(\mathbf{l}^1\left(R_+^2\right), \vartheta, \odot, \bullet\right)$ is an isomorphism.

**Proof** Let $\widehat{\mathcal{D}_\kappa}[(v_{\alpha,n})/(\varphi_{\alpha,n})] \cong \widehat{\mathcal{D}_\kappa}[(g_{\alpha,n})/(\psi_{\alpha,n})]$ in of $\mathcal{B}\left(\mathbf{l}^1\left(R_+^2\right), \vartheta, \odot, \bullet\right)$. By aid of Theorem 5 and the idea involving quotients of $\mathcal{B}\left(\mathbf{l}^1\left(R_+^2\right), \vartheta, \odot, \bullet\right)$ we deduce that $\mathcal{D}_\kappa v_{\alpha,n} \bullet \psi_{\alpha,m} = \mathcal{D}_\kappa v_{\alpha,m} \bullet \psi_{\alpha,n}$. Theorem 5 reveals $\mathrm{D}_\kappa(v_{\alpha,n} \odot \psi_{\alpha,m}) = \mathcal{D}_\kappa(g_{\alpha,m} \odot \psi_{\alpha,n})$. Properties of $\mathcal{D}_\kappa$ implies $v_{\alpha,n} \odot \psi_{\alpha,m} = g_{\alpha,m} \odot \phi_{\alpha,n}$. Therefore, the idea of quotients of $\mathcal{B}\left(\mathbf{l}^1\left(R_+^2\right), \vartheta, \odot, \odot\right)$ leads to $[(v_{\alpha,n})/(\varphi_{\alpha,n})] = [(g_{\alpha,n})/(\psi_{\alpha,n})]$.Surjectivity of $\widehat{\mathcal{D}_\kappa}$ can be derived as in the following manner. Let $[(\mathcal{D}_\kappa v_{\alpha,n})/(\phi_{\alpha,n})] \in \mathcal{B}\left(\mathbf{l}^1\left(R_+^2\right), \vartheta, \odot, \bullet\right)$ be given arbitrary. Then $\mathcal{D}_\kappa v_{\alpha,n} \bullet \phi_{\alpha,m} = \mathcal{D}_\kappa v_{\alpha,m} \bullet \phi_{\alpha,n}$ for every $m, n \in N$.
Once again, Theorem 5 leads to $\mathcal{D}_\kappa(v_{\alpha,n} \odot \varphi_{\alpha,m}) = \mathcal{D}_\kappa(v_{\alpha,m} \odot \varphi_{\alpha,n})$. Hence the proof is finished.

**Theorem 14** Given $\delta_1, \delta_2 \in \mathcal{B}\left(\mathbf{l}^1\left(R_+^2\right), \vartheta, \odot, \odot\right)$. We get $\widehat{\mathcal{D}_\kappa}(\delta_1 \odot \delta_2) = \widehat{\mathcal{D}_\kappa}\delta_1 \bullet \delta_2$.

**Proof** Let $\delta_1 = [(v_{\alpha,n})/(\varphi_{\alpha,n})] \in \mathcal{B}\left(\mathbf{l}^1\left(R_+^2\right), \vartheta, \odot, \odot\right)$. By (15) and calculations, we get $\widehat{\mathcal{D}_\kappa}([(v_{\alpha,n})/(\varphi_{\alpha,n})] \odot [(\kappa_{\alpha,n})/(\phi_{\alpha,n})]) = \left[\left(\widehat{\mathcal{D}_\kappa}(v_{\alpha,n})\right)/(\varphi_{\alpha,n})\right] \bullet [(\kappa_{\alpha,n})/(\phi_{\alpha,n})]$.
This finishes the proof of the theorem.

**Definition 15** Given $\left[\left(\widehat{\mathcal{D}_\kappa}(v_{\alpha,n})\right)/(\varphi_{\alpha,n})\right] \in \mathcal{B}\left(\mathbf{l}^1\left(R_+^2\right), \vartheta, \odot, \bullet\right)$. We define $\widehat{\mathcal{D}_\kappa}^{-1}$ as the inverse integral of $\widehat{\mathcal{D}_\kappa}$ as

$$\widehat{\mathcal{D}_\kappa}^{-1}\left[\left(\widehat{\mathcal{D}_\kappa}(v_{\alpha,n})\right)/(\varphi_{\alpha,n})\right] = [(v_{\alpha,n})/(\varphi_{\alpha,n})], \tag{17}$$

$(\phi_{\alpha,n}) \in \Delta$.

**Theorem 16** Given $[(v_{\alpha,n})/(\varphi_{\alpha,n})] \in \mathcal{B}\left(\mathbf{l}^1\left(R_+^2\right), \vartheta, \odot, \bullet\right)$. For every $\phi_\alpha \in \vartheta(R_+)$ we get $\widehat{\mathcal{D}_\kappa}\left([(v_{\alpha,n})/(\varphi_{\alpha,n})] \odot \phi_\alpha\right) = \left[\left(\widehat{\mathcal{D}_\kappa}(v_{\alpha,n})\right)/(\varphi_{\alpha,n})\right] \bullet \phi_\alpha$.

**Proof** Let $[(v_{\alpha,n})/(\varphi_{\alpha,n})] \in \mathcal{B}\left(\mathbf{l}^1\left(R_+^2\right), \vartheta, \odot, \bullet\right)$ and $\phi_\alpha \in \vartheta(R_+)$. Applying (15) and Theorem 5 give $\widehat{\mathcal{D}_\kappa}\left([(v_{\alpha,n})/(\varphi_{\alpha,n})] \odot \phi_\alpha\right) = \widehat{\mathcal{D}_\kappa}\left([(v_{\alpha,n})/(\varphi_{\alpha,n})] \bullet \phi_\alpha\right)$. Hence the proof is finished.

**Theorem 17** $\widehat{\mathcal{D}_\kappa}$ and $\widehat{\mathcal{D}_\kappa}^{-1}$ are continuous in terms of convergence of $\delta$ and $\Delta$ types.

**Proof** We now confirm that $\widehat{\mathcal{D}_\kappa}$ and $\widehat{\mathcal{D}_\kappa}^{-1}$ are continuous in terms of $\delta$. Let $\beta_n \to \beta \in \mathcal{B}\left(\mathbf{l}^1\left(R_+^2\right), \vartheta, \odot, \odot\right)$ as $n \to \infty$. By virtue of Theorem 1 we can find $v_{\alpha,n,k}$ and $v_{\alpha,k} \in \mathbf{l}^1\left(R_+^2\right)$ such that $\beta_n = [(v_{\alpha,n,k})/(\phi_{\alpha,k})]$ and $\beta = [(v_{\alpha,k})/(\phi_{\alpha,k})]$ with $\lim_{n\to\infty} v_{\alpha,n,k} = v_{\alpha,k}$ $(\forall k \in N)$. Continuity of $\mathcal{D}_\kappa$ transform yields $\lim_{n\to\infty} \mathcal{D}_\kappa v_{\alpha,n,k} = \mathcal{D}_\kappa v_{\alpha,k}$ in $\mathbf{l}^1\left(R_+^2\right)$. Thus $[(\mathcal{D}_\kappa v_{\alpha,n,k})/(\phi_{\alpha,k})] \to [(\mathcal{D}_\kappa v_{\alpha,k})/(\phi_{\alpha,k})] \in \mathcal{B}\left(\mathbf{l}^1\left(R_+^2\right), \vartheta, \odot, \bullet\right)$ as $n \to \infty$. in the sense of $\delta$ .

On the other hand, we show continuity of the inverse integral with respect to $\delta$ convergence. Let $g_{a,n} \to g_a$ in $\mathcal{B}\left(\mathbf{l}^1\left(R_+^2\right), \vartheta, \odot, \bullet\right)$ as $n \to \infty$. Then, a parity of Theorem 1 implies that we can write $g_{a,n} = [(\mathcal{D}_\kappa v_{\alpha,n,k})/(\phi_{\alpha,k})]$ and that $g_a = [(\mathcal{D}_\kappa v_{\alpha,k})/(\phi_{\alpha,k})]$ with the property that $\mathcal{D}_\kappa v_{\alpha,n,k} \to \mathcal{D}_\kappa v_{\alpha,k}$ as $n \to \infty$. Hence $v_{\alpha,n,k} \to v_{\alpha,k}$ as $n \to \infty$. Therefore, $[(v_{\alpha,n,k})/(\phi_{\alpha,k})] \to [(v_{\alpha,k})/(\phi_{\alpha,k})]$ as $n \to \infty$. By using (17) we get $\widehat{\mathcal{D}_\kappa}^{-1}[(\mathcal{D}_\kappa v_{\alpha,n,k})/(\phi_{\alpha,k})] \to \widehat{\mathcal{D}_\kappa}^{-1}[(\mathcal{D}_\kappa v_{\alpha,k})/(\phi_{\alpha,k})]$ as $n \to \infty$.

Now, we establish continuity of $\widehat{\mathcal{D}_\kappa}$ and $\widehat{\mathcal{D}_\kappa}^{-1}$ with respect to $\Delta$ convergence. Let $\beta_n \to \beta \in \mathcal{B}\left(\mathbf{l}^1\left(R_+^2\right), \vartheta, \odot, \odot\right)$ as $n \to \infty$. Then there exist $v_{\alpha,n} \in \mathbf{l}^1\left(R_+^2\right)$ and $(\phi_{\alpha,n}) \in \Delta$ such that $(\beta_n - \beta) \odot \phi_{\alpha,n} = [((v_{\alpha,n}) \odot \phi_{\alpha,k})/(\phi_{\alpha,k})]$ and $v_{\alpha,n} \to 0$ as $n \to \infty$. Employing (16) reveals $\widehat{\mathcal{D}_\kappa}((\beta_n - \beta) \odot \phi_{\alpha,n}) = [(\mathcal{D}_\kappa((v_{\alpha,n}) \odot \phi_{\alpha,k}))/(\phi_{\alpha,k})]$. Hence, it follows

$$\widehat{\mathcal{D}_\kappa}((\beta_n - \beta) \odot \phi_{\alpha,n}) = [(\mathcal{D}_\kappa(v_{\alpha,n,k} \bullet \phi_{\alpha,k}))/(\phi_{\alpha,k})] = \mathcal{D}_\kappa(v_{\alpha,n}) \to 0$$

as $n \to \infty$ in $\mathbf{l}^1\left(R_+^2\right)$.

Therefore, $\widehat{\mathcal{D}_\kappa}((\beta_n - \beta) \odot \phi_{\alpha,n}) = \left(\widehat{\mathcal{D}_\kappa}\beta_n - \widehat{\mathcal{D}_\kappa}\beta\right) \bullet \phi_{\alpha,n}$ as $n \to \infty$. Hence, $\widehat{\mathcal{D}_\kappa}\beta_n \to \widehat{\mathcal{D}_\kappa}\beta$ as $n \to \infty$ in $\Delta$ convergence.

Finally, let $g_{\alpha,n} \to g_\alpha$ in $\mathcal{B}\left(\mathbf{l}^1\left(R_+^2\right), \vartheta, \odot, \bullet\right)$ as $n \to \infty$ then by Theorem 1 we find $\mathcal{D}_\kappa v_{\alpha,k} \in \mathbf{l}^1\left(R_+^2\right)$ such that $(g_{\alpha,n} - g_\alpha) \bullet \phi_{\alpha,n} = [(\mathcal{D}_\kappa v_{\alpha,k} \bullet \phi_{\alpha,k})/(\phi_{\alpha,k})]$ where $\mathcal{D}_\kappa v_{\alpha,k} \to 0$ as $n \to \infty$ for some $(\phi_{\alpha,n}) \in \Delta$.

Using (17), we obtain

$$\widehat{\mathcal{D}_\kappa}^{-1}((g_{\alpha,n} - g_\alpha) \bullet \phi_{\alpha,n}) = \left[\left(\widehat{\mathcal{D}_\kappa}^{-1}(\mathcal{D}_\kappa v_{\alpha,k} \bullet \phi_{\alpha,k})\right)/(\phi_{\alpha,k})\right].$$

Theorem 5 implies $\widehat{\mathcal{D}_\kappa}^{-1}((g_{\alpha,n} - g_\alpha) \bullet \phi_{\alpha,n}) = [((v_{\alpha,k}) \odot \phi_{\alpha,k})/(\phi_{\alpha,k})] \sim v_{\alpha,n} \to 0$ as $n \to \infty$. Thus, $\widehat{\mathcal{D}_\kappa}^{-1}((g_{\alpha,n} - g_\alpha) \bullet \phi_{\alpha,n}) = \left(\widehat{\mathcal{D}_\kappa}^{-1} g_{\alpha,n} - \widehat{\mathcal{D}_\kappa}^{-1} g_\alpha\right) \odot \phi_{\alpha,n}$

as $n \to \infty$. Hence, we get

$$\widehat{\mathcal{D}_\kappa}^{-1} g_{\alpha,n} \to g_\alpha \in \mathcal{B}\left(\mathbf{l}^1\left(R_+^2\right), \vartheta, \odot, \odot\right)$$

as $n \to \infty$. (18)

This finishes the proof of the theorem.

# References


[1] Al-Omari, S.K.Q.; Some characteristics of $S$ transforms in a class of rapidly decreasing Boehmians, J. Pseudo-Differ. Operar. Applic. ; 2014,5(4): 527-537. DOI:10.1007/s11868-014-0102-8.

[2] Al-Omari, S.K.Q. and Al-Omari, J. F.; Some extensions of a certain integral transform to a quotient space of generalized functions, Open Math.; 2015, 13, 816-825.

[3] Zemanian, A.H.; Distribution theory and transform analysis, Dover Publications, Inc., New York.

[4] Al-Omari, S.K.Q.; On a Widder potential transform and its extension to a space of locally integrable Boehmians, J. Assoc. Arab Univer. Basic and Appl. Sci.; 2015, 85(18): 94-98.

[5] Al-Omari, S. K. Q. and Baleanu, D.; On the generalized Stieltjes transform of Fox's kernel function and its properties in the space of generalized functions, Journal of Computational Analysis and Applications (JoCAAA); 2017, 23,1, 108-118.

[6] Mikusinski, P. and Morimoto, M.; Boehmians on the sphere and their spherical harmonic expansions, Fract. Calc. Appl. Anal.; 2001, 4, 25-35.

[7] Zemanian, A.H.; Generalized integral transformation, Dover Publications, Inc., New York.

[8] Hekrdla, J.; New integral transform between analogue and discrete systems. Electronic Letters; 1988, 24(10): 620-623.

[9] Zayed, A.I. and Mikusinski, P.; On the extension of the Zak transform, Methods Appl. Anal.; 1995, 2, 160-172.

[10] Ganesan, C.: Weighted ultra distributions and Boehmians, Int. Journal of Math. Analysis; 2010, 4 (15), 703-712.

[11] Karunakaran, V., Ganesan, C.: Fourier transform on integrable Boehmians,Integ. Transf. Spec. Funct.;2009, 20,937-941.

[12] Nemzer, D.: A note on multipliers for integrable Boehmians, Fract. Calc. Appl.Anal.;2009, 12 , 87-96.

[13] Roopkumar, R.: An extension of distributional wavelet transform, Colloquium Mathematicum; 2009, 115 , 195-206.

[14] Mikusinski,P. , Fourier transform for integrable Boehmians, Rock. Moun. J. Math.; 1987, 17(3), 577-582 .

[15] Mikusinski,P., Tempered Boehmians and ultradistributions, Proc. Amer. Math. Soc. 123(3), 813-817 (1995).

[16] Nemzer, D. , One-parameter groups of Boehmians, Bull. Korean Math. Soci.; 2007, 44 ,419-428.

[17] Zemanian, A. H., Distribution theory and transform analysis, Dover Publications, Inc.,New York. First Published by McGraw-Hill, Inc. New York, 1965.

[18] Kananthai, A., The distribution solutions of ordinary differential equation with polynomial coefficients, Southeast Asian Bulle. Math., 2001, 25, 129-134.

[19] Loonker, D., Banerji, P. K., Solution of integral equations by generalized wavelet transform, Bol. Soc. Paran. Mat.; 2015, 33(2), 89-94.

[20] Al-Omari, S.K.Q. and Dumitru Baleanu, On generalized space of quaternions and its application to a class of Mellin transforms,J. Nonlinear Sci. Appl.; 2016, 9, 38983908.

[21] Al-Omari, S.K.Q., Some definition of a New integral transform between analogue and discrete systems. J. Appl. Funct. Analy.; 2015,10, 1-2,19-30.

[22] Mikusinski,P., Convergence of Bohemians, Japan J. Math.;1983, 9, 159179.

[23] Mikusinski,P., On flexibility of Boehmians, Integral Transform. Spec. Funct.; 1996, 4, 141146.

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# STABILITY OF THE SINE-COSINE FUNCTIONAL EQUATION IN HYPERFUNCTIONS

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ABSTRACT. Let $\mathcal{D}'_{L^\infty}(\mathbb{R}^{2n})$ and $\mathcal{A}'_{L^\infty}(\mathbb{R}^{2n})$ be the spaces of bounded distributions and bounded hyperfunctions respectively. In this paper we consider the Ulam's stability of the sine-cosine functional equation

$$u \circ T - u \otimes v + v \otimes u \in \mathcal{D}'_{L^\infty}(\mathbb{R}^{2n}) \text{ [resp. } \mathcal{A}'_{L^\infty}(\mathbb{R}^{2n})],$$

where $u, v$ are Gelfand hyperfunctions, $T : \mathbb{R}^{2n} \to \mathbb{R}^n$ such that $T(x, y) = x - y$ for all $x, y \in \mathbb{R}^n$, and $\circ$, $\otimes$ denote pullback and tensor product of generalized functions respectively.

## 1. INTRODUCTION

A certain formula or equation is applicable to model a physical process if a small change in the formula or equation gives rise to a small change in the corresponding result. When this happens we say the formula or equation is stable. In an application, a functional equation like the additive Cauchy functional equation $f(x+y) - f(x) - f(y) = 0$ may not be true for all $x, y \in \mathbb{R}$ but it may be true approximately, that is

$$f(x+y) - f(x) - f(y) \approx 0$$

for all $x, y \in \mathbb{R}$. This can be stated mathematically as

$$|f(x+y) - f(x) - f(y)| \leq \varepsilon \tag{1.1}$$

for some small positive $\varepsilon$ and for all $x, y \in \mathbb{R}$. We would like to know when small changes in a particular equation like the additive Cauchy functional equation have only small effects on its solutions. This is the essence of stability theory. In 1940, S.M. Ulam asked the following question:

*Let $f$ be a mapping from a group $G_1$ to a metric group $G_2$ with metric $d(\cdot, \cdot)$ such that*

$$d(f(xy),\, f(x)f(y)) \leq \varepsilon.$$

*Then does there exist a group homomorphism $h$ and $\delta_\epsilon > 0$ such that*

$$d(f(x),\, h(x)) \leq \delta_\epsilon$$

*for all $x \in G_1$?*

This problem was solved affirmatively by D. H. Hyers under the assumption that $G_2$ is a Banach space (see Hyers [19], Hyers-Isac-Rassias [20]). Since then Ulam problems of many other functional

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equations have been investigated [13, 14, 15, 21, 23, 24, 25, 26, 27, 28]. Among the results, Székelyhidi has developed his idea of using invariant subspaces of functions defined on a group or semigroup he prove the Ulam-Hyers stability problem for functional equation

$$f(x+y) = f(x)g(y) + g(x)f(y), \quad x, y \in \mathbb{R}^n, \tag{1.2}$$

which arises from the sine addition formula [30, 31]. Using his elegant idea, Chung and Chang [7] prove the parallel Ulam-Hyers stability problem for functional equation

$$f(x-y) = f(x)g(y) - g(x)f(y), \quad x, y \in \mathbb{R}^n, \tag{1.3}$$

which arises from the sine subtraction formula. As a result it was proved that if $f, g : \mathbb{R}^n \to \mathbb{C}$ satisfy

$$|f(x-y) - f(x)g(y) + g(x)f(y)| \leq M, \quad x, y \in \mathbb{R}^n \tag{1.4}$$

for some $M > 0$, then either there exist $\lambda_1$, $\lambda_2 \in \mathbb{C}$, not both zero, and $L > 0$ such that

$$|\lambda_1 f(x) - \lambda_2 g(x)| \leq L \tag{1.5}$$

for all $x \in \mathbb{R}^n$, or else

$$f(x-y) = f(x)g(y) - g(x)f(y) \tag{1.6}$$

for all $x, y \in \mathbb{R}^n$. Also in the sequel, the functions $f$ and $g$ satisfying both (1.4) and (1.5) were investigated.

Schwartz introduced the theory of distributions in his monograph *Théorie des distributions* [29] in which Schwartz systematizes the theory of generalized functions, basing it on the theory of linear topological spaces, relates all the earlier approaches, and obtains many important results. After his elegant theory appeared, many important concepts and results on the classical spaces of functions have been generalized to the space of distributions. For example, the space $L^\infty(\mathbb{R}^n)$ of bounded measurable functions on $\mathbb{R}^n$ has been generalized to the space $\mathcal{D}'_{L^\infty}(\mathbb{R}^n)$ of *bounded distributions* as a subspace of distributions and later the space $\mathcal{D}'_{L^\infty}(\mathbb{R}^n)$ is further generalized to the space $\mathcal{A}'_{L^\infty}(\mathbb{R}^n)$ of *bounded hyperfunctions.* It is very natural to consider the following stability problem for the functional equation in distributions and hyperfunctions $u, v$ with respect to bounded distributions and bounded hyperfunctions

$$u \circ T - u \otimes v + v \otimes u \in \mathcal{D}'_{L^\infty}(\mathbb{R}^{2n}) \text{ [resp. } \mathcal{A}'_{L^\infty}(\mathbb{R}^{2n})], \tag{1.7}$$

where $\mathcal{D}'_{L^\infty}(\mathbb{R}^{2n})$ and $\mathcal{A}'_{L^\infty}(\mathbb{R}^{2n})$ are the spaces of bounded distributions and bounded hyperfunctions, $T : \mathbb{R}^{2n} \to \mathbb{R}^n$ such that $T(x, y) = x - y$ for all $x, y \in \mathbb{R}^n$, and $\circ$, $\otimes$ denote pullback and tensor product of generalized functions respectively. In [10] the distributional version of the stability of (1.2) was proved. In this paper, as a parallel result we prove the stability of (1.7). As in [10] the main tool is *the heat kernel method* initiated by T. Matsuzawa [22] which represents the generalized functions in some class as the initial values of solutions of the heat equation with appropriate growth

conditions [12, 22]. Making use of the heat kernel method we can convert (1.7) to the classical Ulam-Hyers stability problem of the functional inequality; there exist $C > 0$ and $N \geq 0$[ resp. for every $\epsilon > 0$ there exists $C_\epsilon > 0$ ] such that

$$|\tilde{u}(x-y,t+s) - \tilde{u}(x,t)\tilde{v}(y,s) + \tilde{v}(x,t)\tilde{u}(y,s)| \leq C\left(\frac{1}{t}+\frac{1}{s}\right)^N \quad [\text{ resp. } C_\epsilon e^{\epsilon(1/t+1/s)}\,] \tag{1.8}$$

for all $x,\, y \in \mathbb{R}^n,\ t, s > 0$, where $\tilde{u},\, \tilde{v},\, \tilde{w},\, \tilde{k} : \mathbb{R}^n \times (0,\infty) \to \mathbb{C}$ are solutions of the heat equation whose initial values are $u, v, w, k$ respectively. In Section 3, we consider the stability problem (1.8) with a more general setting, which will be used, combined with the heat kernel method [12, 22], to prove the stability problem of (1.7).

## 2. Distributions and hyperfunctions

We first introduce the spaces $\mathcal{S}'$ of Schwartz tempered distributions and $\mathcal{G}'$ of Gelfand hyperfunctions(see [16, 17, 18, 22, 29] for more details of these spaces). We use the notations: $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $|x| = \sqrt{x_1^2 + \cdots x_n^2}$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, for $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0$ is the set of non-negative integers and $\partial_j = \frac{\partial}{\partial x_j}$.

**Definition 2.1.** [29] *We denote by $\mathcal{S}$ or $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of all infinitely differentiable functions $\varphi$ in $\mathbb{R}^n$ such that*

$$\|\varphi\|_{\alpha,\beta} = \sup_x |x^\alpha \partial^\beta \varphi(x)| < \infty \tag{2.1}$$

*for all $\alpha,\ \beta \in \mathbb{N}_0^n$, equipped with the topology defined by the seminorms $\|\cdot\|_{\alpha,\beta}$. The elements of $\mathcal{S}$ are called rapidly decreasing functions and the elements of the dual space $\mathcal{S}'$ are called tempered distributions.*

**Definition 2.2.** [16, 17] *We denote by $\mathcal{G}$ or $\mathcal{G}(\mathbb{R}^n)$ the Gelfand space of all infinitely differentiable functions $\varphi$ in $\mathbb{R}^n$ such that*

$$\|\varphi\|_{h,k} = \sup_{x\in\mathbb{R}^n,\, \alpha,\, \beta\in\mathbb{N}_0^n} \frac{|x^\alpha \partial^\beta \varphi(x)|}{h^{|\alpha|}k^{|\beta|}\alpha!^{1/2}\beta!^{1/2}} < \infty$$

*for some $h,\, k > 0$. We say that $\varphi_j \longrightarrow 0$ as $j \to \infty$ if $||\varphi_j||_{h,k} \longrightarrow 0$ as $j \to \infty$*
*for some $h,\, k$, and denote by $\mathcal{G}'$ the strong dual space of $\mathcal{G}$ and call its elements Gelfand hyperfunctions.*

As a generalization of the space $L^\infty$ of bounded measurable functions, L. Schwartz introduced the space $\mathcal{D}'_{L^\infty}$ of bounded distributions as a subspace of tempered distributions.

**Definition 2.3.** [29] *We denote by $\mathcal{D}_{L^1}(\mathbb{R}^n)$ the space of smooth functions on $\mathbb{R}^n$ such that $\partial^\alpha\varphi \in L^1(\mathbb{R}^n)$ for all $\alpha \in \mathbb{N}_0^n$ equipped with the topology defined by the countable family of seminorms*

$$\|\varphi\|_m = \sum_{|\alpha|\leq m} \|\partial^\alpha \varphi\|_{L^1}, \quad m \in \mathbb{N}_0.$$

*We denote by $\mathcal{D}'_{L^\infty}$ the strong dual space of $\mathcal{D}_{L^1}$ and call its elements bounded distributions.*

As a generalization of bounded distributions, the space $\mathcal{A}'_{L^\infty}$ of bounded hyperfunctions has been introduced as a subspace of $\mathcal{G}'$.

**Definition 2.4.** [12] *We denote by $\mathcal{A}_{L^1}$ the space of smooth functions on $\mathbb{R}^n$ satisfying*

$$\|\varphi\|_h = \sup_\alpha \frac{\|\partial^\alpha \varphi\|_{L^1}}{h^{|\alpha|}\alpha!} < \infty$$

*for some constant $h > 0$. We say that $\varphi_j \to 0$ in $\mathcal{A}_{L^1}$ as $j \to \infty$ if there is a positive constant $h$ such that*

$$\sup_\alpha \frac{\|\partial^\alpha \varphi_j\|_{L^1}}{h^{|\alpha|}\alpha!} \to 0 \quad as \ \ j \to \infty.$$

*We denote by $\mathcal{A}'_{L^\infty}$ the strong dual space of $\mathcal{A}_{L^1}$.*

It is well known that the following topological inclusions hold:

$$\mathcal{G} \hookrightarrow \mathcal{S} \hookrightarrow \mathcal{D}_{L^1}, \qquad \mathcal{D}'_{L^\infty} \hookrightarrow \mathcal{S}' \hookrightarrow \mathcal{G}',$$

$$\mathcal{G} \hookrightarrow \mathcal{A}_{L^1} \hookrightarrow \mathcal{D}_{L^1}, \quad \mathcal{D}'_{L^\infty} \hookrightarrow \mathcal{A}'_{L^\infty} \hookrightarrow \mathcal{G}'.$$

It is known that the space $\mathcal{G}(\mathbb{R}^n)$ consists of all infinitely differentiable functions $\varphi(x)$ on $\mathbb{R}^n$ which can be extended to an entire function on $\mathbb{C}^n$ satisfying

$$|\varphi(x+iy)| \le C\exp(-a|x|^2 + b|y|^2), \quad x, y \in \mathbb{R}^n \tag{2.2}$$

for some $a$, $b$, $C > 0$(see [16]).

**Definition 2.5.** *Let $u_j \in \mathcal{G}'(\mathbb{R}^{n_j})$ for $j = 1, 2$. Then the tensor product $u_1 \otimes u_2$ of $u_1$ and $u_2$, defined by*

$$\langle u_1 \otimes u_2, \varphi(x_1, x_2)\rangle = \langle u_1\,,\, \langle u_2\,,\, \varphi(x_1, x_2)\rangle\,\rangle$$

*for $\varphi(x_1, x_2) \in \mathcal{G}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, belongs to $\mathcal{G}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.*

## 3. Stability of (1.8)

Throughout this paper $\langle G, +\rangle$ is a 2-divisible commutative group, $f, g : G \times (0, \infty) \to \mathbb{C}$ and $N$ denotes a fixed nonnegative real number. We consider the stability problems of each of the following functional inequalities;

$$\begin{gathered} \textit{there exist } C > 0 \textit{ and } d > 0 \textit{ such that} \\ |f(x-y, t+s) - f(x,t)g(y,s) + g(x,t)f(y,s)| \le C\left(\frac{1}{t} + \frac{1}{s}\right)^N + d, \quad \forall\, x, y \in G,\ t, s > 0; \end{gathered} \tag{3.1}$$

$$\begin{gathered} \textit{for every } \epsilon > 0, \textit{ there exists } C_\epsilon > 0 \textit{ which depends on } \epsilon \textit{ such that} \\ |f(x-y, t+s) - f(x,t)g(y,s) + g(x,t)f(y,s)| \le C_\epsilon e^{\epsilon(1/t+1/s)}, \quad \forall\, x, y \in G,\ t, s > 0. \end{gathered} \tag{3.2}$$

From now on, a function $a$ from a semigroup $\langle S, +\rangle$ to the field $\mathbb{C}$ of complex numbers is said to be *an additive function* provided $a(x+y) = a(x) + a(y)$ for all $x, y \in S$ and $m : S \to \mathbb{C}$ is said to be *an exponential function* provided $m(x+y) = m(x)m(y)$ for all $x, y \in S$.

We introduce the following conditions (3.3) and (3.4) on $f : G \times (0,\infty) \to \mathbb{C}$ and $N$;

$$\begin{aligned} &\textit{there exist } C > 0 \textit{ and } d > 0 \textit{ such that} \\ &|f(x,t)| \le Ct^{-N} + d, \quad \forall\, x \in G,\ t > 0; \end{aligned} \tag{3.3}$$

$$\begin{aligned} &\textit{for every } \epsilon > 0,\ \textit{there exists } C_\epsilon > 0 \textit{ which depends on } \epsilon \textit{ such that} \\ &|f(x,t)| \le C_\epsilon e^{\epsilon/t}, \quad \forall\, x \in G,\ t > 0. \end{aligned} \tag{3.4}$$

Using the idea in [20, p. 104] we obtain the following (See [10] for the proofs).

**Lemma 3.1.** *Let $f, g : G \times (0,\infty) \to \mathbb{C}$ satisfy the inequality; for each $y \in G$ and $s > 0$ there exist positive constants $C = C(y,s)$ and $d = d(y,s)$ [resp. for each $y \in G$, $s > 0$ and $\epsilon > 0$ there exists a positive constant $C_\epsilon = C_\epsilon(y,s)$ ] such that*

$$|f(x-y,t+s) - f(x,t)g(y,s)| \le Ct^{-N} + d \ [\textit{resp. } C_\epsilon e^{\epsilon/t}] \tag{3.5}$$

*for all $x \in G, t > 0$. Then either $f$ satisfies* (3.3) [*resp.*(3.4)] *or $g$ is an exponential function.*

**Lemma 3.2.** *Let $m : G \times (0,\infty) \to \mathbb{C}$ be a nonzero exponential function satisfying* (3.3) [*resp.*(3.4)]. *Then $m$ can be written in the form*

$$m(x,t) = m_1(x)m_2(t),$$

*where $m_1 : G \to \mathbb{C}$, $m_2 : (0,\infty) \to \mathbb{C}$ is exponential functions satisfying $|m_1(x)| = 1$ for all $x \in G$.*

**Lemma 3.3.** *Let $m$ be a nonzero exponential function satisfying* (3.3) [*resp.*(3.4)]. *Suppose that $f : G \times (0,\infty) \to \mathbb{C}$ satisfies the inequality; there exist positive constants $C$ and $d$ [resp. for each $\epsilon > 0$, there exists a positive constant $C_\epsilon$ ] such that*

$$|f(x+y,t+s) - f(x,t)m(y,s) - f(y,s)m(x,t)| \le C\left(\frac{1}{t} + \frac{1}{s}\right)^N + d \ [\textit{resp. } C_\epsilon e^{\epsilon/t}] \tag{3.6}$$

*for all $x, y \in G, t, s > 0$. Then we have*

$$f(x,t) = a(x)m_1(x)m_2(t) + 2f\left(0, \frac{t}{2}\right) m_1(x) m_2\left(\frac{t}{2}\right) + R(x,t),$$

*where $a : G \to \mathbb{C}$ is an additive function, $m : (0,\infty) \to \mathbb{C}$ is an exponential function, $\lambda \in \mathbb{C}$ and $R : G \times (0,\infty) \to \mathbb{C}$ satisfies*

$$|R(x,t)| \le Ct^{-N} + d \ [\,resp.(3.4)]$$

*for all $x \in G, t > 0$.*

**Theorem 3.4.** *Suppose that $f, g : G \times (0,\infty) \to \mathbb{C}$ satisfy the inequality* (3.1) [*resp.*(3.2)]. *Then either*

$$f(x-y,t+s) - f(x,t)g(y,s) + g(x,t)f(y,s) = 0 \tag{3.7}$$

*for all $x,\ y \in G,\ t, s > 0$, or else there exist $\lambda_1,\ \lambda_2 \in \mathbb{C}$, not both zero, such that $\lambda_1 f(x,t) - \lambda_2 g(x,t)$ satisfies* (3.1) [*resp.*(3.2)].

*Proof.* It suffices to prove that $f, g$ satisfies (3.7) when $\lambda_1 f(x,t) - \lambda_2 g(x,t)$ satisfies (3.3) [resp.(3.4)] only for $\lambda_1 = \lambda_2 = 0$. Let

$$F(x,y,t,s) = f(x-y,t+s) - f(x,t)g(y,s) + g(x,t)f(y,s). \tag{3.8}$$

Choosing $y_1$ and $s_1$ with $f(y_1,s_1) \neq 0$ we have

$$g(x,t) = k_1 f(x,t) + k_2 f(x-y_1,t+s_1) - k_2 F(x,y_1,t,s_1), \tag{3.9}$$

where $k_1 = \frac{g(y_1,s_1)}{f(y_1,s_1)}$ and $k_2 = \frac{-1}{f(y_1,s_1)}$. From (3.8) and (3.9) we have

$$\begin{aligned}
&f\big((x-y)-z,(t+s)+r\big) \\
=&f(x-y,t+s)g(z,r) - g(x-y,t+s)f(z,r) + F(x-y,z,t+s,r) \\
=&f(x-y,t+s)g(z,r) \\
&-\Big(k_1 f(x-y,t+s) + k_2 f(x-y-y_1,t+s+s_1) - k_2 F(x-y,y_1,t+s,s_1)\Big) f(z,r) \\
&+ F(x-y,z,t+s,r) \\
=&\Big(f(x,t)g(y,s) - g(x,t)f(y,s) + F(x,y,t,s)\Big) g(z,r) \\
&- k_1\Big(f(x,t)g(y,s) - g(x,t)f(y,s) + F(x,y,t,s)\Big) f(z,r) \\
&+ k_2\Big(f(x,t)g(y+y_1,s+s_1) - g(x,t)f(y+y_1,s+s_1) \\
&+ F(x,y+y_1,t,s+s_1) - F(x-y,y_1,t+s,s_1)\Big) f(z,r) \\
&+ F(x-y,z,t+s,r),
\end{aligned} \tag{3.10}$$

and also we have

$$f\big(x-(y+z),t+(s+r)\big) = f(x,t)g(y+z,s+r) - g(x,t)f(y+z,s+r) + F(x,y+z,t,s+r). \tag{3.11}$$

From (3.10) and (3.11) we have

$$\begin{aligned}
&f(x,t)\Big(g(y,s)g(z,r) - k_1 g(y,s)f(z,r) + k_2 g(y+y_1,s+s_1)f(z,r) - g(y+z,s+r)\Big) \\
&+ g(x,t)\Big(-f(y,s)g(z,r) + k_1 f(y,s)f(z,r) - k_2 f(y+y_1,s+s_1)f(z,r) + f(y+z,s+r)\Big) \\
=&F(x,y+z,t,s+r) - F(x-y,z,t+s,r) - F(x,y,t,s)g(z,r) + k_1 F(x,y,t,s)f(z,r) \\
&- k_2\Big(F(x,y+y_1,t,s+s_1) - F(x-y,y_1,t+s,s_1)\Big) f(z,r).
\end{aligned} \tag{3.12}$$

Fixing $y$, $z$, $s$, $r$ in (3.12), using (3.1) and (3.8) we have

$$\begin{aligned}
&\big|F(x,y+z,t,s+r) - F(x-y,z,t+s,r) - F(x,y,t,s)g(z,r) + k_1 F(x,y,t,s)f(z,r) \\
&- k_2\Big(F(x,y+y_1,t,s+s_1) - F(x-y,y_1,t+s,s_1)\Big) f(z,r)\big| \\
&\leq 2C\left(\frac{1}{t}+\frac{1}{r}\right)^N + 2d + C_1\left(\frac{1}{t}+\frac{1}{s}\right)^N + d_1 + C_2\left(\frac{1}{t}+\frac{1}{s_1}\right)^N + d_2 \\
&\leq C' t^{-N} + d',
\end{aligned}$$

where $C' = 2^N(2C + C_1 + C_2)$, $d' = 2^N(2Cr^{-N} + C_1 s^{-N} + C_2 s_1^{-N}) + 2d + d_1 + d_2$.

Similarly, using (3.2) we obtain that for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$\begin{aligned}
&\big|F(x,y+z,t,s+r) - F(x-y,z,t+s,r) - F(x,y,t,s)g(z,r) + k_1F(x,y,t,s)f(z,r)\\
&- k_2\Big(F(x,y+y_1,t,s+s_1) - F(x-y,y_1,t+s,s_1)\Big)f(z,r)\big|\\
&\le 2C_\epsilon e^{\epsilon(1/t+1/r)} + C_1C_\epsilon e^{\epsilon(1/t+1/s)} + C_2C_\epsilon e^{\epsilon(1/t+1/s_1)}\\
&\le C'_\epsilon e^{\epsilon/t},
\end{aligned}$$

where $C'_\epsilon = C_\epsilon(2e^{\epsilon/r} + C_1e^{\epsilon/s} + C_2e^{\epsilon/s_1})$.

Thus, by the assumption that $\lambda_1 f(x,t) - \lambda_2 g(x,t)$ satisfies (3.3) [resp.(3.4)] only for $\lambda_1 = \lambda_2 = 0$ we have

$$\begin{aligned}
&g(y,s)g(z,r) - k_1g(y,s)f(z,r) + k_2g(y+y_1,s+s_1)f(z,r) - g(y+z,s+r)\\
&= f(y,s)g(z,r) - k_1f(y,s)f(z,r) + k_2f(y+y_1,s+s_1)f(z,r) - f(y+z,s+r) = 0.
\end{aligned}$$

Thus, it follows that

$$\begin{aligned}
&F(x,y+z,t,s+r) - F(x-y,z,t+s,r)\\
&=\Big(-k_1F(x,y,t,s) + k_2F(x,y+y_1,t,s+s_1) - k_2F(x-y,y_1,t+s,s_1)\Big)f(z,r)\\
&\quad + F(x,y,t,s)g(z,r).
\end{aligned} \tag{3.13}$$

Now, if we fix $x$, $y$, $t$, $s$, the left hand side of (3.13) satisfies (3.3) [resp. (3.4)] as a function of $z$ and $r$. From the right hand side of (3.13), using the assumption that $\lambda_1 f(x,t) - \lambda_2 g(x,t)$ satisfies (3.3) [resp.(3.4)] only for $\lambda_1 = \lambda_2 = 0$ it follows that $F \equiv 0$. This completes the proof. □

**Theorem 3.5.** *Let $f, g : G \times (0,\infty) \to \mathbb{C}$ satisfy* (3.1) [*resp.* (3.2)]. *Then $(f,g)$ satisfies one of the following* :

(i) *both $f$ and $g$ satisfy* (3.3) [*resp.*(3.4)],

(ii) *$f(x,t) = a(x)m(t) + R(x,t)$, $g(x,t) = \lambda f(x,t) + m(t)$ for all $x \in G, t > 0$, where $a : G \to \mathbb{C}$ is an additive function, $m : (0,\infty) \to \mathbb{C}$ is an exponential function, $\lambda \in \mathbb{C}$ and $R : G \times (0,\infty) \to \mathbb{C}$ satisfies $|R(x,t)| \le Ct^{-2N} + d$ [ resp.*(3.4)] *for all $x \in G, t > 0$ and for some $C, d > 0$,*

(iii) *$f(x-y,t+s) - f(x,t)g(y,s) + g(x,t)f(y,s) = 0$ for all $x, y \in G, t, s > 0$.*

*Proof.* Assume that $(f,g)$ does not satisfy (iii). Then by Lemma 3.4 there exist $\lambda_1$, $\lambda_2 \in \mathbb{C}$, not both zero, such that $\lambda_1 f(x,t) - \lambda_2 g(x,t)$ satisfies (3.3) [resp.(3.4)].

(Case 1) $f(\neq 0)$ satisfies (3.3) [resp.(3.4)].

Assume that $f(\neq 0)$ satisfies (3.3). Choosing $y_0 \in G, s_0 > 0$ such that $f(y_0,s_0) \neq 0$, dividing $|f(y_0,s_0)|$ in both sides of (3.1) and using the triangle inequality we have

$$\begin{aligned}
|g(x,t)| &\le \frac{1}{|f(y_0,s_0)|}\left(|f(x-y_0,t+s_0)| + |f(x,t)g(y_0,s_0)| + C\left(\frac{1}{t}+\frac{1}{s_0}\right)^N + d\right)\\
&\le C_1(t+s_0)^{-N} + d_1 + C_2t^{-N} + d_2 + C_3\left(\frac{1}{t}+\frac{1}{s_0}\right)^N + d_3\\
&\le C't^{-N} + d'
\end{aligned}$$

for all $x \in G, t > 0$ and for some positive constants $C_1, C_2, C_3, d_1, d_2, d_3, C'$ and $d'$. Similarly, if $f$ satisfies (3.4) we can show that for every $\epsilon > 0$ there exists $C'_\epsilon > 0$ such that

$$|g(x,t)| \leq C'_\epsilon e^{\epsilon/t}$$

for all $x \in G, t > 0$. Thus, we obtain the case (ii).

(Case 2) $f$ does not satisfy (3.3) [resp. (3.4)].

Assume that $f$ does not satisfy (3.3). In this case we must have $\lambda_2 \neq 0$ and we can write

$$g(x,t) = -\frac{\lambda_1}{\lambda_2} f(x,t) + B(x,t) := \lambda f(x,t) + B(x,t) \tag{3.14}$$

for all $x \in G, t > 0$, where $R$ satisfies (3.3) [resp. (3.4)]. Putting (3.14) in (3.1) we have

$$|f(x-y,t+s) - f(x,t)B(y,s) + B(x,t)f(y,s)| \leq C\left(\frac{1}{t} + \frac{1}{s}\right)^N + d \tag{3.15}$$

for all $x, y \in G, t, s > 0$. Using the triangle inequality and fixing $y$ and $s$ in (3.15) we have

$$|f(x-y,t+s) - f(x,t)B(y,s)| \leq |B(x,t)f(y,s)| + C\left(\frac{1}{t} + \frac{1}{s}\right)^N + d \leq C' t^{-N} + d'$$

for all $x, y \in G, t, s > 0$ and for some positive constants $C'$ and $d'$. Applying Lemma 3.1 we have

$$B(x,t) = m(x,t) \tag{3.16}$$

for all $x \in G$, $t > 0$, where $m$ is an exponential function on $G \times (0, \infty)$. Now, applying Lemma 3.2 we have

$$R(x,t) = m(x,t) = m_1(x)m_2(t) \tag{3.17}$$

for all $x \in G, t > 0$, where $m_1 : G \to \mathbb{C}$, $m_2 : (0, \infty) \to \mathbb{C}$ are exponential functions. Replacing $(x,t)$ by $(y,s)$ in (3.15) we have

$$|f(-x+y,t+s) - f(y,s)B(x,t) + B(y,s)f(x,t)| \leq C\left(\frac{1}{t} + \frac{1}{s}\right)^N + d \tag{3.18}$$

for all $x, y \in G, t, s > 0$. From (3.15) and (3.18), using the triangle inequality, putting $y = 0$ and replacing $t, s$ by $\frac{t}{2}$ we have

$$|f(x,t) + f(-x,t)| \leq C 2^{2N+1} t^{-N} + 2d \tag{3.19}$$

for all $x \in G, t > 0$. Replacing $x$ by $-x$, $y$ by $-y$ in (3.15), we have

$$|f(-x+y,t+s) - f(-x,t)B(-y,s) + B(-x,t)f(-y,s)| \leq C\left(\frac{1}{t} + \frac{1}{s}\right)^N + d \tag{3.20}$$

for all $x, y \in G, t, s > 0$. From (3.20) and using (3.19) with fixing $y$ and $s$ we have

$$|f(-x+y,t+s) - f(x,t)B(-y,s) + B(-x,t)f(y,s)| \leq C_1 t^{-N} + d_1 \tag{3.21}$$

for all $x \in G, t > 0$. From (3.20) and (3.21) with fixing $y$ and $s$ we have

$$|f(x,t)\left(B(y,s) - B(-y,s)\right) - f(y,s)\left(B(x,t) - B(-x,t)\right)| \leq C_2 t^{-N} + d_2 \tag{3.22}$$

for all $x, y \in G, t, s > 0$. Since $f$ does not satisfy (3.3), it follows from (3.22) that

$$B(y, s) = B(-y, s)$$

for all $y \in G, s > 0$ and hence $m_1(y) = 1$ for all $y \in G$. Thus, we have

$$g(x, t) = \lambda f(x, t) + m_2(t) \tag{3.23}$$

for all $x \in G, t > 0$. From (3.15), (3.17) and (3.19) we have

(3.24)

$$\begin{aligned} |f(x+y, t+s) - f(x, t)m_2(s) - f(y, s)m_2(t)| &\leq |f(y, s) + f(-y, s)||m_2(t)| + C\left(\frac{1}{t} + \frac{1}{s}\right)^N + d \\ &\leq (C2^{2N+1}t^{-N} + 2d)Ct^{-N} + C\left(\frac{1}{t} + \frac{1}{s}\right)^N + d \\ &\leq C'\left(\frac{1}{t} + \frac{1}{s}\right)^{2N} + d' \end{aligned}$$

for all $x, y \in G, t, s > 0$ and for some $C' > 0, d' > 0$.

Similarly, if $f$ satisfies (3.4) we can show that for every $\epsilon > 0$ there exists $C'_\epsilon > 0$ such that

$$|f(x+y, t+s) - f(x, t)m_2(s) - f(y, s)m_2(t)| \leq C'_\epsilon e^{\epsilon/t} \tag{3.25}$$

for all $x \in G, t > 0$. Applying Lemma 3.3 with (3.23) and (3.24) we have

$$f(x, t) = a(x)m_2(t) + 2f\left(0, \frac{t}{2}\right) m_2\left(\frac{t}{2}\right) + R(x, t) \tag{3.26}$$

for all $x \in G, t > 0$, where $a$ is an additive mapping and $R$ satisfies (3.3)[resp. (3.4)]. Replacing $(y, s)$ by $(x, t)$ in (3.1) we see that $f(0, t)$ satisfies (3.3). Thus, $2f\left(0, \frac{t}{2}\right) m_2\left(\frac{t}{2}\right) + R(x, t)$ satisfies (3.3)[resp. (3.4)]. Replacing $2f\left(0, \frac{t}{2}\right) m_2\left(\frac{t}{2}\right) + R(x, t)$ by $R(x, t)$ and $m_2$ by $m$ we get the case (iii). This completes the proof.

□

## 4. Main results

In this section as a main result of the paper we consider the stability of (1.6). The main tools of our proof are based on *structure theorems* for generalized functions and *the heat kernel method* initiated by T. Matsuzawa [22] which represents the generalized functions as initial values of solutions of the heat equation with appropriate growth conditions [8, 9, 11, 12, 22]. For the proof of our theorem we employ the $n$-dimensional heat kernel $E_t(x)$ given by

$$E_t(x) = (4\pi t)^{-n/2} \exp(-|x|^2/4t), \ t > 0.$$

In view of (2.2), we can see that the heat kernel $E_t$ belongs to the Gelfand space $\mathcal{G}(\mathbb{R}^n)$ for each $t > 0$. Thus, for each $u \in \mathcal{G}'(\mathbb{R}^n)$, the convolution $(u * E_t)(x) := \langle u_y, E_t(x - y)\rangle$ is well defined. We call $(u * E_t)(x)$ *the Gauss transform* of $u$. From now on we denote by $\tilde{u}(x, t)$ the Gauss transform of

$u$. It is well known that the Gauss transform $\tilde{u}(x,t)$ is a smooth solution of the heat equation such that $\tilde{u}(x,t) \to u$ in weak star topology as $t \to 0^+$, i.e.,

$$\langle u, \varphi \rangle = \lim_{t \to 0^+} \int \tilde{u}(x,t)\varphi(x)dx$$

for all $\varphi \in \mathcal{G}$.

We first discuss the solutions of the corresponding trigonometric functional equations in the space $\mathcal{G}$ of Gelfand generalized functions.

**Lemma 4.1.** *The solutions $u$, $v \in \mathcal{G}'$ of the equation*

$$u \circ T - u \otimes v + v \otimes u = 0 \tag{4.1}$$

*are either*

$$u = \lambda(e^{c\cdot x} - e^{-c\cdot x}), \quad v = \gamma e^{c\cdot x} + (1-\gamma)e^{-c\cdot x} \tag{4.2}$$

*or else*

$$u = c \cdot x, \quad v = 1 + \lambda c \cdot x. \tag{4.3}$$

*Proof.* As a consequence of the results in [4, 15] the solutions $(u, v)$ of (4.1) are equal to the smooth solutions $(f, g)$ of the equation

$$f(x-y) - f(x)g(y) + f(y)g(x) = 0 \tag{4.4}$$

for all $x, y \in \mathbb{R}^n$. By [2, Theorem 11] all solutions of (4.4) are given by

$$f(x) = \lambda(m(x) - m(-x)), \quad g(x) = \gamma m(x) + (1-\gamma)m(-x) \tag{4.5}$$

or else

$$f(x) = a(x), \quad g(x) = 1 + \lambda a(x), \tag{4.6}$$

where $m$ is an exponential function and $a$ is an additive function. From (4.5) and (4.6) $m$ and $a$ are smooth functions and hence $m(x) = e^{c\cdot x}$ and $a(x) = c \cdot x$ for some $c \in \mathbb{C}^n$. Thus, we get (4.2) and (4.3). This completes the proof. □

The proof of Theorem 2.3 of [11] works even when $p = \infty$, i.e., we obtain the following.

**Lemma 4.2.** [11] *The Gauss transform $\tilde{u}(x,t) := (u * E)(x,t)$ of $u \in \mathcal{D}'_{L^\infty}(\mathbb{R}^n)$ is a smooth solution of the heat equation $(\Delta - \partial/\partial_t)\tilde{u} = 0$ satisfying*:

(i) *There exist constants $C > 0$, $N \geq 0$ such that*

$$|\tilde{u}(x,t)| \leq Ct^{-N} \quad \text{for all } x \in \mathbb{R}^n,\ t > 0. \tag{4.7}$$

(ii) $\tilde{u}(x,t) \to u$ *as* $t \to 0^+$ *in the sense that for every* $\varphi \in \mathcal{D}_{L^1}$,

$$\langle u, \varphi \rangle = \lim_{t \to 0^+} \int \tilde{u}(x,t)\varphi(x)\,dx.$$

*Conversely, every smooth solution $\tilde{u}(x,t)$ of the heat equation satisfying the estimate* (4.7) *can be uniquely expressed as $\tilde{u}(x,t) = (u * E)(x,t)$ for some $u \in \mathcal{D}'_{L^\infty}(\mathbb{R}^n)$.*

The following lemma is a special case of Theorem 3.5 of [12] when $p = \infty$ where the space $\mathcal{A}'_{L^\infty}(\mathbb{R}^n)$ is denoted by $\mathcal{B}_{L^\infty}(\mathbb{R}^n)$.

**Lemma 4.3.** [12] *The Gauss transform $\tilde{u}(x,t) := (u * E)(x,t)$ of $u \in \mathcal{A}'_{L^\infty}(\mathbb{R}^n)$ is a smooth solution of the heat equation $(\Delta - \partial/\partial_t)\tilde{u} = 0$ satisfying:*

(i) *For every $\epsilon > 0$ there exists a constant $C_\epsilon > 0$ such that*

$$|\tilde{u}(x,t)| \le C_\epsilon e^{\epsilon/t} \quad \text{for all } x \in \mathbb{R}^n,\ t > 0. \tag{4.8}$$

(ii) *$\tilde{u}(x,t) \to u$ as $t \to 0^+$ in the sense that for every $\varphi \in \mathcal{A}_{L^1}$,*

$$\langle u, \varphi \rangle = \lim_{t \to 0^+} \int \tilde{u}(x,t)\varphi(x)\, dx.$$

*Conversely, every smooth solution $\tilde{u}(x,t)$ of the heat equation satisfying the estimate* (4.8) *can be uniquely expressed as $\tilde{u}(x,t) = (u * E)(x,t)$ for some $u \in \mathcal{D}'_{L^\infty}(\mathbb{R}^n)$.*

The following structure theorem for bounded distributions is well known.

**Lemma 4.4.** [29] *Every $u \in \mathcal{D}'_{L^\infty}(\mathbb{R}^n)$ can be expressed as*

$$u = \sum_{|\alpha| \le p} \partial^\alpha f_\alpha \tag{4.9}$$

*for some $p \in \mathbb{N}_0$ where $f_\alpha$ are bounded continuous functions on $\mathbb{R}^n$. The equality* (4.9) *implies that*

$$\langle u, \varphi \rangle = \sum_{|\alpha| \le p} (-1)^{|\alpha|} \int f_\alpha(x) \partial^\alpha \varphi(x) dx$$

*for all $\varphi \in \mathcal{D}_{L^1}$.*

As a special case of Theorem 3.4 of [12] when $p = \infty$ where the space $\mathcal{A}'_{L^\infty}(\mathbb{R}^n)$ is denoted by $\mathcal{B}_{L^\infty}(\mathbb{R}^n)$ we obtain the following.

**Lemma 4.5.** [12] *Every $u \in \mathcal{A}'_{L^\infty}(\mathbb{R}^n)$ can be expressed by*

$$u = \left( \sum_{k=0}^{\infty} a_k\, \Delta^k \right) g + h \tag{4.10}$$

*where $\Delta$ denotes the Laplacian, $g, h$ are bounded continuous functions on $\mathbb{R}^n$ and $a_k,\ k = 0, 1, 2, \ldots$ satisfy the following estimates; for every $L > 0$ there exists $C > 0$ such that*

$$|a_k| \le C L^k / k!^2$$

*for all $k = 0, 1, 2, \ldots$.*

The following properties of the heat kernel will be useful, which can be found in [22].

**Proposition 4.6.** [22] *For each $t > 0$, $E_t(\cdot)$ is an entire function and the following estimate holds; there exists $C > 0$ such that*

$$|\partial_x^\alpha E_t(x)| \le C^{|\alpha|} t^{-(n+|\alpha|)/2} \alpha!^{1/2} \exp(-|x|^2/8t). \tag{4.11}$$

*Also for each $t, s > 0$ we have*

$$(E_t * E_s)(x) := \int E_t(x-y) E_s(y) dy = E_{t+s}(x). \tag{4.12}$$

Now, we state and prove the main theorem.

**Theorem 4.7.** *Let $u, v \in \mathcal{G}'(\mathbb{R}^n)$. Then $(u, v)$ satisfies (4.1) if and only if $(u, v)$ satisfies one of the followings*:

(i) $u,\ v \in \mathcal{D}'_{L^\infty}(\mathbb{R}^n)$ *[resp.* $\mathcal{A}'_{L^\infty}(\mathbb{R}^n)$*]*,

(ii) $u = c \cdot x + r,\ v = 1 + \lambda c \cdot x$ *for some* $c \in \mathbb{C}^n,\ \lambda \in \mathbb{C}$ *and* $r \in \mathcal{D}'_{L^\infty}(\mathbb{R}^n)$ *[resp.* $\mathcal{A}'_{L^\infty}(\mathbb{R}^n)$*]*,

(iii) $u = \lambda(e^{c\cdot x} - e^{-c\cdot x}),\quad v = \gamma e^{c\cdot x} + (1-\gamma)e^{-c\cdot x}$ *for some* $c \in \mathbb{C}^n, \lambda, \gamma \in \mathbb{C}$.

*Proof.* We use the same method as in the proof of [10, Theorem 4.6]. Here we give the proof for the reader. Convolving the tensor product $E_t(x)E_s(y)$ of n-dimensional heat kernels in the left hand side of (4.1), in view of the semigroup property $(E_t * E_s)(x) = E_{t+s}(x)$ of the heat kernel we have

$$\begin{aligned}[(u \circ T) * (E_t(\xi)E_s(\eta))](x,\, y) &= \langle u_\xi,\, \int E_t(x-\xi-\eta)E_s(y-\eta)\, d\eta\rangle \\ &= \langle u_\xi,\, (E_t * E_s)(x-y-\xi)\rangle \\ &= \langle u_\xi,\, E_{t+s}(x-y-\xi)\rangle \\ &= \tilde{u}(x-y, t+s).\end{aligned} \tag{4.13}$$

Similarly we have

$$\begin{aligned}[(u \otimes v) * (E_t(\xi)E_s(\eta))](x,\, y) &= \tilde{u}(x,t)\tilde{v}(y,s), \\ [(v \otimes u) * (E_t(\xi)E_s(\eta))](x,\, y) &= \tilde{v}(x,t)\tilde{u}(y,s),\end{aligned} \tag{4.14}$$

where $\tilde{u}(x,t), \tilde{v}(x,t)$ are the Gauss transforms of $u$, $v$, respectively. Let $w := u \circ T - u \otimes v + v \otimes u$. Then $w \in \mathcal{D}'_{L^\infty}(\mathbb{R}^{2n})$ [resp. $\mathcal{A}'_{L^\infty}(\mathbb{R}^{2n})$]. First, we suppose that $w \in \mathcal{D}'_{L^\infty}(\mathbb{R}^{2n})$. Using (4.9) and (4.11) we have

$$\begin{aligned}|[w * (E_t(\xi)E_s(\eta))](x,y)| &\le \sum_{|\alpha|\le p} |[\partial^\alpha f_\alpha * (E_t(\xi)E_s(\eta))](x,y)| \\ &\le \sum_{|\alpha|\le p} |[f_\alpha * \partial^\alpha_{\xi,\eta}(E_t(\xi)E_s(\eta))](x,y)| \\ &\le \sum_{|\alpha|\le p} \|f_\alpha\|_{L^\infty} \|\partial^\alpha_{\xi,\eta}(E_t(\xi)E_s(\eta))\|_{L^1} \\ &\le C_1 \sum_{|\beta|+|\gamma|\le p} \|\partial^\beta_\xi E_t(\xi)\|_{L^1} \|\partial^\gamma_\eta E_s(\eta)\|_{L^1} \\ &\le C_2 \sum_{|\beta|+|\gamma|\le p} t^{-(n+|\beta|)/2} s^{-(n+|\gamma|)/2} \\ &\le C\left(\frac{1}{t} + \frac{1}{s}\right)^N + d,\end{aligned}$$

where $N = n + p/2$ and the constants $C$ and $d$ depend only on $p$. Secondly we suppose that $w \in \mathcal{A}'_{L^\infty}(\mathbb{R}^{2n})$. Then, using (4.11) we have

$$\begin{aligned}
\|\Delta^k(E_t(\xi)E_s(\eta))\|_{L^1} &\le \sum_{|\alpha|=k} \frac{k!}{\alpha!} \|\partial^{2\alpha}(E_t(\xi)E_s(\eta))\|_{L^1} \\
&\le \sum_{|\beta|+|\gamma|=k} \frac{k!}{\beta!\gamma!} \|\partial_\xi^{2\beta} E_t(\xi)\|_{L^1} \|\partial_\eta^{2\gamma} E_s(\eta)\|_{L^1} \\
&\le \sum_{|\beta|+|\gamma|=k} \frac{k!(2\beta)!^{1/2}(2\gamma)!^{1/2} M^{2k}}{\beta!\,\gamma!} t^{-n/2-|\beta|} s^{-n/2-|\gamma|} \\
&\le \sum_{|\beta|+|\gamma|=k} k!(2M)^{2k} t^{-n/2-|\beta|} s^{-n/2-|\gamma|} \\
&\le k!(2\sqrt{n}M)^{2k} \left(1/t + 1/s\right)^{n+k}.
\end{aligned}$$

Now, by the structure (4.10) of bounded hyperfunctions together with the growth condition of $a_k,\ k = 0, 1, 2, \ldots$ we have

$$\begin{aligned}
|[w * (E_t(\xi)E_s(\eta))](x,y)| &\le \sum_{k=0}^{\infty} \|a_k(\Delta^k g) * (E_t(\xi)E_s(\eta))\|_{L^\infty} + \|h * (E_t(\xi)E_s(\eta))\|_{L^\infty} \\
&\le \|g\|_{L^\infty} \sum_{k=0}^{\infty} \|a_k \Delta^k (E_t(\xi)E_s(\eta))\|_{L^1} + \|h\|_{L^\infty} \|E_t(\xi)E_s(\eta)\|_{L^1} \\
&\le C_1 \sum_{k=0}^{\infty} \frac{1}{k!} (4nM^2L)^k \left(1/t + 1/s\right)^{n+k} + \|h\|_{L^\infty} \\
&\le C_2 \sum_{k=0}^{\infty} \frac{1}{k!} \epsilon^k \left(1/t + 1/s\right)^{n+k} + \|h\|_{L^\infty} \\
&\le C_\epsilon\, e^{\epsilon(1/t+1/s)},
\end{aligned}$$

where $L$ is taken so that $4nM^2L < \epsilon$ and the constant $C_\epsilon$ depends only on $w$ and $\epsilon$. Thus, we have the inequality; there exist $C > 0$ and $d > 0$ [resp. for every $\epsilon > 0$ there exists $C_\epsilon > 0$] such that

$$|\tilde{u}(x-y, t+s) - \tilde{u}(x,t)\tilde{v}(y,s) + \tilde{v}(x,t)\tilde{u}(y,s)| \le C\left(\frac{1}{t} + \frac{1}{s}\right)^N + d \ [\text{resp. } C_\epsilon e^{\epsilon(1/t+1/s)}] \tag{4.15}$$

where $\tilde{u}$, $\tilde{v}$ are the Gauss transforms of $u, v$, respectively, given in Lemma 4.2. Replacing $f$ by $\tilde{u}$, $g$ by $\tilde{v}$ in Theorem 3.5 and using the continuity of $\tilde{u}$ and $\tilde{v}$ we obtain one of the followings (I) $\sim$ (III):

(I) both $\tilde{u}$ and $\tilde{v}$ satisfy (3.3) [resp.(3.4)],
(II) $\tilde{u}(x,t) = c \cdot x e^{bt} + R(x,t)$, $\tilde{v}(x,t) = \lambda \tilde{u}(x,t) + e^{bt}$,
where $c \in \mathbb{C}^n$, $b, \lambda \in \mathbb{C}$ and $R : \mathbb{R}^n \times (0,\infty) \to \mathbb{C}$ satisfies

$$|R(x,t)| \le Ct^{-2N} + d \ [resp.(3.4)]$$

for all $x \in \mathbb{R}^n, t > 0$ and for some $C, d > 0$,
(III) $\tilde{u}(x-y, t+s) - \tilde{u}(x,t)\tilde{v}(y,s) + \tilde{v}(x,t)\tilde{u}(y,s) = 0$ for all $x, y \in \mathbb{R}^n, t, s > 0$.

By Lemma 4.2, case (I) implies (i). For the case (II), since $\tilde{u}, \tilde{v}$ are solutions of the heat equation we must have $b = 0$ and so is $R(x,t) = \tilde{u}(x,t) - c \cdot x$. Letting $t \to 0^+$ in (II) we obtain case (ii).

Finally, letting $t \to 0^+$ in (III) we have

$$u \circ T - u \otimes v + v \otimes u = 0. \tag{4.16}$$

The nontrivial solutions of the equation (4.16) are given by (iii) or $u = c \cdot x,\ v = 1 + \lambda c \cdot x$ which is included in the case (ii). This completes the proof. □

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## References

[1] J. Aczél, Lectures on functional equations in several variables, Academic Press, New York-London, 1966

[2] J. Aczél and J. Dhombres, Functional equations in several variables, Cambridge University Press, New York-Sydney, 1989.

[3] T. Aoki, On the stability of the linear transformation in Banach spaces, J. Math. Soc. Japan, 2(1950), 64–66.

[4] J. A. Baker, Distributional methods for functional equations, Aequationes Math. 62(2001), 136–142.

[5] J. A. Baker, On a functional equation of Aczél and Chung, Aequationes Math. 46(1993) 99–111.

[6] J. Chung and J. Chang, On the stability of trigonometric functional equations in distributions and hyperfunctions, Abstract and Applied Analysis Volume 2013, Article ID 275915, 12 pages.

[7] J. Chang and J. Chung, On a generalized Hyers-Ulam stability of trigonometric functional equations, Journal of Applied Mathematics, Volume 2012, Article ID 610714, 14 pages.

[8] J. Chung, S.-Y. Chung and D. Kim, A characterization for Fourier hyperfunctions, Publ. Res. Inst. Math. Sci. 30(1994), 203–208.

[9] J. Chung, S.-Y. Chung and D. Kim, Une caractérisation de l'espace de Schwartz C. R. Acad. Sci. Paris Sér. I Math. 316(1993), 23–25.

[10] J. Chung and D. Kim, Ulam problem for the sine addition formula in hyperfunctions, Publ. RIMS Kyoto Univ. 50 (2014), 227–250.

[11] S.-Y. Chung, An heat equation approach to distributions with $L^p$ growth, Comm. Kor. math. Soc. 9(1994), No. 4, 897–903.

[12] S. Y. Chung, D. Kim and E. G. Lee, Periodic hyperfunctions and Fourier series, Proc. Amer. Math. Soc. 128(2000), 2421–2430.

[13] S. Czerwik, Stability of Functional Equations of Ulam-Hyers-Rassias Type, Hadronic Press, Inc., Palm Harbor, Florida, Florida,2003.

[14] I. Fenyö, On an inequality of P. W. Cholewa, Internat Schriftenreihe Numer. math., Vol 80(1987) 277-280.

[15] G. L. Forti, Hyer-Ulam stability of functional equation in several variables, Aeq. Math. 50(1995), 143–190.

[16] I. M. Gelfand and G.E. Shilov, Generalized functions II, Academic Press, New York, 1968.

[17] I. M. Gelfand and G. E. Shilov, Generalized functions IV, Academic, Press, New York, 1968.

[18] L. Hörmander, The analysis of linear partial differential operators I,Springer-Verlag, Berlin-New York, 1983.

[19] D. H. Hyers, On the stability of the linear functional equations, Proc. Nat. Acad. Sci. USA 27(1941), 222-224.

[20] D. H. Hyers, G. Isac and Th. M. Rassias, Stability of functional equations in several variables, Birkhauser, 1998.

[21] S.M. Jung, Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis. Springer, New York (2011).

[22] T. Matsuzawa, A calculus approach to hyperfunctions II, Trans. Amer. Math. Soc. 313(1989), 619–654.

[23] J. M. Rassias, On Approximation of Approximately Linear Mappings by Linear Mappings, J. Funct. Anal. 46(1982), 126–130.

[24] J. M. Rassias, On Approximation of Approximately Linear Mappings by Linear Mappings, Bull. Sci. Math. 108(1984), 445–446.

[25] J. M. Rassias, Solution of a problem of Ulam, J. Approx. Th. 57(1989), 268–273.

[26] Th. M. Rassias, On the stability of functional equations in Banach spaces, J. Math. Anal. Appl. 251(2000), 264–284.

[27] Th. M. Rassias, On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. 7(1978), 297-300.

[28] K. Ravi and M. Arunkumar, On the Ulam-Gavruta-Rassias stability of the orthogonally Euler-Lagrange type functional equation, Intern. J. Appl. Math. Stat, 7(2007), 143–156.

[29] L. Schwartz, Théorie des distributions, Hermann, Paris, 1966.

[30] L. Székelyhidi, The stability of sine and cosine functional equations, Proc. Amer. Math. Soc. 110(1990), 109–115.

[31] L. Székelyhidi, The stability of d'Alembert type functional equations, Acta Sci. Math. Szeged. 44(1982c), 313–320.

[32] S. M. Ulam, Problems in modern mathematics, Chapter VI, Wiley, New York, 1964.

[33] D.V. Widder, The heat equation, Academic Press, New York, 1975.

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# Effect of cytotoxic T lymphocytes on HIV-1 dynamics

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**Abstract**

The purpose of this paper is to study the effect of the cytotoxic T lymphocytes (CTLs) on an HIV-1 dynamics. The model considers that the virus infects the macrophages in addition to the $CD4^+$ T cells. The role of the CTLs is to kill the infected macrophages and $CD4^+$ T cells. The time delay which accounts the time of infection and the time of producing new active HIV-1 is modeled. The HIV-1 dynamics is modeled as a 6-dimensional nonlinear delay differential equations. The incidence rate of infection and killer rate of infected cells are given by general nonlinear functions. We study the qualitative behavior of the system. The global stability analysis has been established using Lyapunov method and LaSalle invariance principle. We present an example and perform numerical simulations to emphasize our theoretical results.
*Keywords*: Global stability; HIV infection; time delay; Immune response; Direct Lyapunov method.

## 1 Introduction

Recently, the study of Human Immunodeficiency Virus type-1 (HIV-1) and Acquired Immunodeficiency Syndrome (AIDS) has become a topic of interest in the mathematical literature. In the pursuit of understanding the interaction between the HIV-1 and immune system, several mathematical models have been proposed.

The following is the basic model of HIV-1 infection dynamics that has been described and studied in [1]:

$$\begin{aligned}\dot{x} &= \lambda - dx - \beta xv,\\ \dot{y} &= \beta xv - \delta y,\\ \dot{v} &= ky - rv.\end{aligned}$$

Here, the concentrations of uninfected $CD4^+$ T cells, infected $CD4^+$ T cells and virus are represented by $x, y$ and $v$, respectively. The production rate of $CD4^+$ T cells is represented by $\lambda$, while the infection rate, and thus the infected $CD4^+$ T cell production rate, is represented by $\beta xv$, where $\beta$ is the infection rate constant. The uninfected cells and infected cells are die with rate $dx$ and $\delta y$, respectively. $k$ represents the rate constant of virion generation by $CD4^+$ T cells while $r$ represents the rate constant of viral particle emptying from the plasma.

Replication models assume cytotoxic T lymphocyte cells (CTLs) to be the main host defence restricting viral replication in vivo and thus the main determinant of viral load. Nowak and Bangham [2] constructed the first model of HIV taking into account CTLs as:

$$\begin{aligned}\dot{x} &= \lambda - dx - \beta xv,\\ \dot{y} &= \beta xv - \delta y - pyz,\\ \dot{v} &= ky - rv,\\ \dot{z} &= cyz - bz.\end{aligned}$$

where $z$ represent the concentration of CTLs which multiply at a rate $cyz$ when stimulated by infected cells, while $bz$ represents the death rate of this population of CTLs.

Delay differential equations are used to introduce delays into the infection equations and/or equations for virus production to account for the intracellular phase of the viral life cycle. This delay is defined as period between infection of a $CD4^+$ T-cell and the point at which the infected cell begins to produce viral particles (see, e.g. [3], [5], [6], [7], [8], [9], [10]). Complications have been shown to occur ([11], [12], [13]) when time delays are introduced into infection models with immune responses. Such complications include stable periodic solutions and chaos. The use of general kernel function to represent distributed intracellular delays has been motivated by the argument that constant delays may not biologically realistic ([22],[23],[24]). In contrast to Nakata's [17] investigation of the stability of an immunity mediated HIV-1 model with two finite distributed intracellular delays, Wang et al. [16] and Li and Shu [14] examined the stability of an infection model with infinite distributed intracellular delays by constructing Lyapunov functionals. Yuan and Zou [15] proposed and developed an appropriate mathematical model for HIV-1 infection by incorporating distributed delay into the cell infection equation and another virus production equation and nonlinear incidence rate and a nonlinear removal rate for the infected cells. However, the presence of the macrophages has been neglected.

Our aim in this paper is to study the effect of the CTL immune response of the global dynamics of a distributed delayed HIV-1 model which describe the interaction between the virus and two target cells, $CD4^+$ T cells and macrophages. The motivation for considering the two target cell model is the observation that the rate of viral load decline was considerably lower after the rapid first phase of decay during the 1-2 weeks after antiretroviral treatment ([3],[4],[18]). The model is a 6-dimensional nonlinear ODES that takes into account cytotoxic T lymphocyte cells (CTLs) with nonlinear incidence rate and distributed delays using distributed kernels reflecting the variance in time required for viral entry into cells and the variability in time required for intracellular virion reproduction. The positive invariance properties and the boundedness of the solutions for the model are studied. By constructing explicit Lyapunov functionals and using the LaSalle invariance principle, which are extensions and modified forms of the Lyapunov functionals given in [15], we prove that the steady states of the model are globally asymptotically stable (GAS) and the dynamics of the system is fully determined by the basic reproduction number $R_0$.

## 2 Mathematical model

We shall examine a deterministic model of HIV infection, which represents the interaction of HIV with two co-circulation populations of target cells, representing $CD4^+$ T and macrophages cells. The system takes into consideration the distributed invasion and production delays and (i) We assume that the incidence rate is given by a nonlinear form. (ii) The model takes into consideration cytotoxic T lymphocyte cells (CTLs) immune response:

$$\dot{x}_1(t) = \mu_1 - k_1 x_1(t) - \alpha_1 x_1(t) f_1(v(t)), \tag{1}$$

$$\dot{y}_1(t) = \alpha_1 \int_0^\infty e^{-m_1\tau} G_1(\tau) x_1(t-\tau) f_1(v(t-\tau)) d\tau - r y_1(t) - \beta y_1(t) h_1(z(t)), \tag{2}$$

$$\dot{x}_2(t) = \mu_2 - k_2 x_2(t) - \alpha_2 x_2(t) f_2(v(t)), \tag{3}$$

$$\dot{y}_2(t) = \alpha_2 \int_0^\infty e^{-m_2\tau} G_2(\tau) x_2(t-\tau) f_1(v(t-\tau)) d\tau - r y_2(t) - \beta y_2(t) h_2(z(t)), \tag{4}$$

$$\dot{v}(t) = Nr \left( \int_0^\infty e^{-n_1\tau} \Psi_1(\tau) y_1(t-\tau) d\tau + \int_0^\infty e^{-n_2\tau} \Psi_2(\tau) y_2(t-\tau) d\tau \right) - dv(t), \tag{5}$$

$$\dot{z}(t) = \lambda \left( y_1(t) + y_2(t) \right) - qz(t). \tag{6}$$

The state variables describes the plasma concentrations of: $x_1$, $y_1$, represent the uninfected and infected $CD4^+$ T cells; $x_2$, $y_2$, represent the uninfected and infected macrophages. Eq. (1) and (3) describe the populations of target cells, where $\mu_1$ and $\mu_2$ perform the rates of new generations of $CD4^+$ T cell and macrophages from sources within the body, $k_1$, $k_2$ are the death rate constants, and $\alpha_1$, $\alpha_2$ are the infection rate constants. Equation (2) and (4) represent the population dynamics of the infected target cells, where $r$ represent the clearance rate and it killed at rate $\beta y_1(t)h_1(z(t))$ and $\beta y_2(t)h_2(z(t))$, respectively. The CTL cells are produced at a rate $\lambda(y_1+y_2)z$ and are decayed at a rate $qz$ . Assume the kernel functions $G_i$ and $\Psi_i$, $i=1,2$ satisfy $G_i(\tau_i)>0$, $\Psi_i(\tau_i)>0$. Let us denote $a_i=\int\limits_0^\infty e^{-m_i\tau}G_i(\tau)d\tau$, $b_i=\int\limits_0^\infty e_i^{-n_i\tau}\Psi_i(\tau)d\tau$ , $i=1,2$. Thus $0<a_i\leq 1$, $0<b_i\leq 1$

All parameters are assumed to be positive. The function $f_i(v)$ and $h_i(z)$ are continuously differentiable and guarantee this conditions are met:

(C1): $f_i(0)=0$, $f_i'(\xi_i)$ exists and satisfies $f_i'(\xi_i)\geq 0$ and $\left(\frac{f_i(\xi_i)}{\xi_i}\right)'\leqslant 0$ in $(0,\infty)$,

(C2): $h_i(0)=0$, $h_i(\zeta_i)$ is strictly increasing in $(0,\infty)$,

## 2.1 Positively and Boundedness

To prove the positively and the boundedness of the solutions, it is biologically reasonable to consider the following non-negative initial conditions for the system (1-6), define the Banach space of fading memory type

$$C_\alpha=\{\varphi\in C\ ((-\infty,0],R):\varphi(\theta)e^{\alpha\theta}\text{ is uniformly continuous for }\theta\in(-\infty,0]\ and\ \|\varphi\|<\infty\}$$

where $\alpha$ is a positive constant and $\|\varphi\|=\sup_{\theta\leq 0}|\varphi(\theta)|\,e^{\alpha\theta}$. Let $\mathrm{C}_\alpha^+=\{\varphi\in C_\alpha:\varphi(\theta)\geq 0\text{ for }\theta\in(-\infty,0]\}$. The initial conditions for system (1-6) are given as:

$$\begin{aligned}&x_1(\theta)=\varphi_1(\theta),\ y_1(\theta)=\varphi_2(\theta),\ x_2(\theta)=\varphi_3(\theta),\ y_2(\theta)=\varphi_4(\theta),v(\theta)=\varphi_5(\theta),\\&z(\theta)=\varphi_6(\theta)\ \text{ for }\ \theta\in[-\infty,0]\,,\ \varphi_i\in C_\alpha^+,\ i=1,2,...,6.\end{aligned}\tag{7}$$

By the fundamental theory of functional differential equations (see [20] and [21]), model (1-6) with initial conditions (7) has a unique solution and the following lemma establishes the positivity and boundedness of the solutions.

**Lemma 1.** Let $(x_1(t),y_1(t),x_2(t),y_2(t),v(t),z(t))$ be the solution of system (1-6) with the initial conditions (7), then $x_1(t),y_1(t),x_2(t),y_2(t),v(t)$ and $z(t)$ are all positive and bounded for all $t>0$.

**Proof.** First, we will prove that $x_i(t)>0, i=1,2$, for all $t\geq 0$. Assume that $x_i(t)$ loses its nonnegativity on some local existence interval $[0,\upsilon]$ for some constant $\upsilon$ and let $t^*\in[0,\upsilon]$ be such that $x_i(t^*)=0$. From (1) and (3) we have $x_i(t^*)=\mu_i>0$. Hence $x_i(t^*)>0$ for some $t\in(t^*,t^*+\epsilon)$, where $\epsilon>0$ is sufficiently small. This leads to a contradiction and hence $x_i(t)>0$, for all $t\geq 0$. Further by using the variation of parameters method and Eq. (2), (4) and (5) we have

$$\begin{aligned}y_i(t)&=y_i(0)e^{-\int_0^t(r+\beta h(z(s)))ds}\\&\quad+\alpha_i\int_0^t e^{-\int_s^t(r+h(z(\eta)))d\eta}\int_0^\infty e^{-m_i\eta}G_i(\eta)x(s-\eta)f_i(v(s-\eta))d\eta ds;\quad i=1,2.\\v(t)&=v(0)e^{-dt}+Nr\int_0^t e^{-d(t-s)}\int_0^\infty\sum_{i=1}^2 e^{-n_i\eta}\Psi_i(\eta)y_i(s-\eta)d\eta ds,\end{aligned}$$

confirming that $y_i(t)\geq 0$, $i=1,2$, and $v(t)\geq 0$ for all $t\geq 0$. Now from (6) we get

$$z(t)=z(0)e^{-qt}+\lambda\int_0^t e^{-q(t-s)}\sum_{i=1}^2 y_i(s)ds.$$

Then $z(t)\geq 0$, for all $t\geq 0$, and this prove the positively of the solution. Now we shall prove that the solution are bounded, from Eq.(1) and (3), we have $\dot{x}_i(t)\leqslant\mu_i-k_ix_i(t)$, this implies $\limsup_{t\to\infty}x_i(t)\leq\frac{\mu_i}{k_i}, i=1,2$, let

$U_i(t) = \int_0^\infty e^{-m_i\tau} G_i(\tau) x_i(t-\tau)d\tau + y_i(t)$, then

$$\begin{aligned}
\dot{U}_i(t) &= \int_0^\infty e^{-m_i\tau} G_i(\tau)\dot{x}_i(t-\tau)d\tau + \dot{y}_i(t),\\
&= \int_0^\infty e^{-m_i\tau} G_i(\tau)(\mu_i - k_i x_i(t-\tau) - \alpha_i x_i(t-\tau) f_i(v(t-\tau)))d\tau\\
&+ \alpha_i \int_0^\infty e^{-m_i\tau} G_i(\tau) x_i(t-\tau) f_i(v(t-\tau))d\tau - r y_i(t) - \beta y_i(t) h_i(z(t)),\\
&= \mu_i a_i - \alpha_i \int_0^\infty e^{-m_i\tau} G_i(\tau) x_i(t-\tau)d\tau - r y_i(t) - \beta y_i(t) h_i(z(t)),\\
&\leq \mu_i a_i - p_i U_i(t).
\end{aligned}$$

It follows that $\limsup_{t\to\infty} U_i(t) \leq \frac{\mu_i a_i}{p_i}$, where $p_i = \min\{\alpha_1,\ \alpha_2,\ r\}$. Since $x_i(t) > 0$, $y_i(t) \geq 0$ and $z(t) \geq 0$ then lim sup $\sum_{i=1}^2 y_i(t)_{t\to\infty} \leq L_i$, lim sup $y_i(t)_{t\to\infty} \leq L_i$, $i = 1, 2$. On the other hand,

$$\dot{v}(t) \leq Nr(b_1 L_1 + b_2 L_2) - dv(t) \leq NrbL - dv(t).$$

where $b_i = \int_0^\infty e^{-n_i\tau}\Psi_i(\tau)d\tau$. From Eq.(6) we get $\dot{z}(t) \leq \lambda L - qz(t)$. Then lim $\sup_{t\to\infty} v(t) \leq \frac{NrbL}{d}$ and lim $\sup_{t\to\infty} z(t) \leq \frac{\lambda L}{q}$. Therefore, $x_1(t)$, $y_1(t)$, $x_2(t)$, $y_2(t)$, $v(t)$, and $z(t)$ are ultimately bounded and this complete the proof of boundedness of solutions.

## 2.2 Basic reproduction number and steady state

To obtain the equilibrium points of model (1)-(6), we consider the following equations:

$$\begin{aligned}
\mu_1 - k_1 x_1 - \alpha_1 x_1 f_1(v) &= 0,\\
\alpha_1 a_1 x_1 f_1(v) - r y_1 - \beta y_1 h_1(z) &= 0,\\
\mu_2 - k_2 x_2 - \alpha_2 x_2 f_2(v) &= 0,\\
\alpha_2 a_2 x_2 f_2(v) - r y_2 - \beta y_2 h_2(z) &= 0,\\
Nr\,(b_1 y_1 + b_2 y_2) - dv &= 0,\\
\lambda\,(y_1 + y_2) - qz &= 0.
\end{aligned}$$

We find that if $z \neq 0$, there is two steady states $E_0 = (x_1^0, 0, x_2^0, 0, 0, 0)$ where $x_1^0 = \frac{\mu_1}{k_1}$, $x_2^0 = \frac{\mu_2}{k_2}$, and $E^* = (x_1^*, y_1^*, x_2^*, y_2^*, v^*, z^*)$ satisfies the equations:

$$\begin{aligned}
x_i^* &= \frac{\mu_i}{k_i + \alpha_i f_i(v^*)}, \qquad & \textstyle\sum_{i=1}^2 y_i^* &= \frac{d}{Nrb} v^*,\\
z^* &= \frac{\lambda}{q}\textstyle\sum_{i=1}^2 y_i^* = \frac{\lambda d}{Nrbq} v^*, \quad & y_i^* &= \frac{\alpha_i a_i f_i(v^*)}{r + \beta h_i(z^*)} x_i^*.
\end{aligned}$$

The basic reproduction number, $R_0$, for system (1)-(6) is given by:

$$R_0 = \frac{\mu_1 \alpha_1 a_1 N b_1 f_1'(0)}{k_1 d} + \frac{\mu_2 \alpha_2 a_2 N b_2 f_2'(0)}{k_2 d} = R_1 + R_2,$$

where, $R_1$ and $R_2$ are the basic reproduction numbers for $CD4^+$ T cells and macrophages cells, severally. Now, we shall prove that $R_0 > 1$ is a sufficient condition to ensure the existence of an infected steady state $E^* = (x_1^*, y_1^*, x_2^*, y_2^*, v^*, z^*)$. Using the above calculations the existence of an infected equilibrium is equivalent to the existence of a positive root of the equation $L(v) = 0$, where

$$L(v^*) = \alpha_1 a_1 f_1(v^*)\frac{\mu_1}{k_1 + \alpha_1 f_1(v^*)} + \alpha_2 a_2 f_2(v^*)\frac{\mu_2}{k_2 + \alpha_2 f_2(v^*)} - \frac{rd}{Nrb} v^* - \frac{\beta d}{Nrb} v^* h\left(\frac{\lambda d v^*}{Nrbq}\right),$$

and it satisfies $L(0) = 0,\ L(+\infty) = -\infty,$ and

$$\begin{aligned} L'(0) &= \alpha_1 a_1 f_1'(0)\frac{\mu_1}{k_1} + \alpha_2 a_2 f_2'(0)\frac{\mu_2}{k_2} - \frac{rd}{Nrb} \\ &= \frac{d}{Nb}\frac{\alpha_1 a_1 \mu_1 Nb f_1'(0)}{k_1 d} + \frac{\alpha_2 a_2 \mu_2 Nb f_2'(0)}{k_2 d} - 1 = \frac{d}{Nb}\left(R_0 - 1\right) > 0. \end{aligned}$$

It follows from the continuity of the function $L(v)$ in $[0, \infty)$ that $L(v) = 0$ has at least one positive root. Hence, we see that the condition at least has one infected equilibrium $E^*$ when $R_0 > 1$. We can rewrite the model as:

$$\dot{x}_1(t) = \mu_1 - k_1 x_1(t) - \alpha_1 x_1(t) f_1(v(t)), \tag{8}$$

$$\dot{y}_1(t) = \vartheta_1 \int_0^\infty e^{-m_1\zeta} g_1(\zeta) x_1(t-\zeta) f_1(v(t-\zeta)) d\zeta - r y_1(t) - \beta y_1(t) h(z(t)), \tag{9}$$

$$\dot{x}_2(t) = \mu_2 - k_2 x_2(t) - \alpha_2 x_2(t) f_2(v(t)), \tag{10}$$

$$\dot{y}_1(t) = \vartheta_2 \int_0^\infty e^{-m_2\zeta} g_2(\zeta) x_2(t-\zeta) f_1(v(t-\zeta)) d\zeta - r y_2(t) - \beta y_2(t) h(z(t)), \tag{11}$$

$$\dot{v}(t) = \gamma_1 \int_0^\infty e^{-n_1\zeta} \psi_1(\zeta) y_1(t-\zeta) d\zeta + \gamma_2 \int_0^\infty e^{-n_2\zeta} \psi_2(\zeta) y_2(t-\zeta) d\zeta - d v(t), \tag{12}$$

$$\dot{z}(t) = \lambda\left(y_1(t) + y_2(t)\right) - q z(t). \tag{13}$$

For simplify, we taked $h_1 = h_2 = h,\ \vartheta_i = \alpha_i a_i,\ \gamma_i = Nrb_i,\ g_i(\zeta) = \frac{G_i(\zeta)}{a_i},\ \psi_i(\zeta) = \frac{\Psi_i(\zeta)}{b_i}$.

# 3 Global stability

In this section, we going to show that the steady states satisfy the global stability condition:

**Theorem 1.** Let Conditions C1 and C2 hold true and $R_0 \leq 1$, then the infection-free equilibrium $E_0$ is globally asymptotically stable.

**Proof.** Define $H_i(t) = \int_t^\infty g_i(\zeta) d\zeta_i,\ P_i(t) = \int_t^\infty \psi_i(\zeta) d\zeta$, and consider laypunov function $W(t) = \sum_{i=1}^3 W_i(t)$, where,

$$\begin{aligned} W_1(t) &= \sum_{i=1}^2 \frac{1}{2}\left(x_i(t) - \frac{\mu_i}{k_i}\right)^2 + \frac{\alpha_i\mu_i}{\vartheta_i k_i} y_i(t) + \frac{\alpha_i\mu_i r}{\vartheta_1\vartheta_2 k_i} v(t) + \frac{\alpha_i\mu_i\beta}{\vartheta_i k_i \lambda}\int_0^{z(t)} h(\zeta) d\zeta, \\ W_2(t) &= \sum_{i=1}^2 \frac{\alpha_i\mu_i}{k_i} \int_0^\infty H_i(\zeta_i) x_i(t-\zeta_i) f_i(v(t-\zeta_i)) d\zeta_i, \\ W_3(t) &= \sum_{i=1}^2 \frac{\alpha_i\mu_i r}{k_i\vartheta_i} \int_0^\infty P_i(\zeta_i) y_i(t-\zeta_i) d\zeta_i, \end{aligned}$$

It clear that, $W(t) \geq 0$ and $W(t) = 0$ if and only if $x_i(t) = \frac{\mu_i}{k_i}$ and $y_i(t) = v(t) = z(t) = 0$. The derivative $W_i(t)$ of along the solution is:

$$\begin{aligned} \dot{W}_1(t) = \sum_{i=1}^2 \Bigg[ & \left(x_i(t) - \frac{\mu_i}{k_i}\right)\left(\mu_i - k_i x_i(t) - \alpha_i x_i(t) f(v(t))\right) + \frac{\alpha_i\mu_i}{k_i}\int_0^\infty g_i(\zeta_i) x_i(t-\zeta_i) f_i(v(t-\zeta_i)) d\zeta_i \\ & - \frac{\alpha_i\mu_i r}{k_i\vartheta_i} y_i(t) - \frac{\alpha_i\mu_i\beta}{k_i\vartheta_i} y_i(t) h(z(t)) + \frac{\alpha_i\mu_i r}{k_i\vartheta_i}\int_0^\infty \psi_i(\zeta_i) y_i(t-\zeta_i) d\zeta_i - \frac{\alpha_i\mu_i r d}{k_i\vartheta_i\gamma_i} v(t) \\ & + \frac{\alpha_i\mu_i\beta}{k_i\lambda\vartheta_i} h(z(t))\left[\lambda y_i(t) - q z(t)\right] \Bigg], \end{aligned}$$

Note that $H_i(0)=1, H_i(\infty)=0$ and $dH_i(t)=-g_i(t)dt$. Using integration by parts, we calculate the derivative of $W_2$:

$$\begin{aligned}\dot{W}_2(t) &= \sum_{i=1}^{2}\frac{\alpha_i\mu_i}{k_i}\int_0^\infty H_i(\zeta_i)\frac{d\left(x_i(t-\zeta_i)f_i(v(t-\zeta_i))\right)}{dt}d\zeta_i = -\sum_{i=1}^{2}\frac{\alpha_i\mu_i}{k_i}\int_0^\infty H_i(\zeta_i)\frac{d\left(x_i(t-\zeta_i)f_i(v(t-\zeta_i))\right)}{d\zeta_i}d\zeta_i,\\ &= -\sum_{i=1}^{2}\left(\frac{\alpha_i\mu_i}{k_i}H_i(\zeta_i)x_i(t-\zeta_i)f_i(v(t-\zeta_i))\mid_{\zeta=0}^{\infty}+\frac{\alpha_i\mu_i}{k_i}\int_0^\infty x_i(t-\zeta_i)f_i(v(t-\zeta_i))dH_i\left(\zeta_i\right)\right),\\ &= \sum_{i=1}^{2}\left(\frac{\alpha_i\mu_i}{k_i}x_i(t)f_i(v(t))-\frac{\alpha_i\mu_i}{k_i}\int_0^\infty g_i(\zeta_i)x_i(t-\zeta_i)f_i(v(t-\zeta_i))d\zeta_i\right).\end{aligned}$$

Similarly

$$\dot{W}_3(t)=\sum_{i=1}^{2}\left(\frac{\alpha_i\mu_i r}{k_i\vartheta_i}y_i(t)-\frac{\alpha_i\mu_i r}{k_i\vartheta_i}\int_0^\infty\psi_i(\zeta_i)y_i(t-\zeta_i)d\zeta_i\right),$$

Therefore

$$\begin{aligned}\dot{W}(t)=\sum_{i=1}^{2}\Bigg[&-k_i\left(x_i(t)-\frac{\mu_i}{k_i}\right)^2-\alpha_i x_i^2(t)f_i(v(t))+\frac{\alpha_i\mu_i}{k_i}x_i(t)f_i(v(t))\\ &+\frac{\alpha_i\mu_i}{k_i}\int_0^\infty g_i(\zeta_i)x_i(t-\zeta_i)f_i(v(t-\zeta_i))d\zeta_i-\frac{\alpha_i\mu_i r}{k_i\vartheta_i}y_i(t)-\frac{\alpha_i\mu_i\beta}{k_i\eta_i}y_i(t)h(z(t))\\ &+\frac{\alpha_i\mu_i r}{k_i\vartheta_i}\int_0^\infty\psi_i(\zeta_i)y_i(t-\zeta_i)d\zeta_i-\frac{\alpha_i\mu_i rd}{k_i\vartheta_i\gamma_i}v(t)+\frac{\alpha_i\mu_i\beta}{k_i\vartheta_i}y_i(t)h(z(t))\\ &-\frac{\alpha_i\mu_i\beta q}{k_i\lambda\vartheta_i}z(t)h(z(t))+\frac{\alpha_i\mu_i}{k_i}x_i(t)f_i(v(t))-\frac{\alpha_i\mu_i}{k_i}\int_0^\infty g_i(\zeta_i)x_i(t-\zeta_i)f_i(v(t-\zeta_i))d\zeta_i\\ &+\frac{\alpha_i\mu_i r}{k_i\vartheta_i}y_i(t)-\frac{\alpha_i\mu_i r}{k_i\vartheta_i}\int_0^\infty\psi_i(\zeta_i)y_i(t-\zeta_i)d\zeta_i\Bigg].\end{aligned}$$

$$\begin{aligned}\dot{W}(t)=\sum_{i=1}^{2}\Bigg[&-k_i\left(x_i(t)-\frac{\mu_i}{k_i}\right)^2-\alpha_i f_i(v(t))\left(x_i^2(t)-\frac{2\mu_i}{k_i}x_i(t)+\frac{\mu_i^2}{k_i^2}\right)\\ &+\frac{\alpha_i\mu_i^2}{k_i^2}f_i(v(t))-\frac{\alpha_i\mu_i rd}{k_i\eta_i\gamma_i}v(t)-\frac{\alpha_i\mu_i\beta q}{k_i\lambda\eta_i}z(t)h(z(t))\Bigg].\end{aligned}$$

Hence

$$\begin{aligned}\dot{W}(t)=\sum_{i=1}^{2}\Bigg[&-k_i\left(x_i(t)-\frac{\mu_i}{k_i}\right)^2-\alpha_i f_i(v(t))\left(x_i(t)-\frac{\mu_i}{k_i}\right)^2-\frac{\alpha_i\mu_i\beta q}{k_i\lambda\vartheta_i}z(t)h(z(t))\\ &+\frac{\alpha_i\mu_i rd}{k_i\vartheta_i\gamma_i}v(t)\left(\frac{\mu_i\vartheta_i\gamma_i}{k_i rd}\frac{f_i(v(t))}{v(t)}-1\right)\Bigg].\end{aligned}$$

But from Condition (C1), we have $\frac{f_i(v(t))}{v(t)}\leqslant f_i'(0)$. Hence

$$\dot{W}(t)\leq\sum_{i=1}^{2}\left[-k_i\left(x_i(t)-\frac{\mu_i}{k_i}\right)^2-\alpha_i f_i(v(t))\left(x_i(t)-\frac{\mu_i}{k_i}\right)^2+\frac{\alpha_i\mu_i rd}{k_i\vartheta_i\gamma_i}\left(R_0-1\right)v(t)-\frac{\alpha_i\mu_i\beta q}{k_i\lambda\vartheta_i}z(t)h(z(t))\right].$$

If $R_0\leq 1$, then, $\dot{W}_1\leq 0$. To prove the global stability of the infected equilibrium, we need to use this lemma:

**Lemma 2.** If satisfies Condition (C1), then $g\left(F(\sigma)\right) \leq g(\sigma),\ \sigma > 0$ with the equality holding only at $\sigma = 1,$ where $F(\sigma) = \frac{f(v^*\sigma)}{f(v^*)}$, and $g(u) = u - 1 - \ln u$ and with $g : (0,\infty) \to [0,\infty)$ has the global minimum $g(1) = 0$ and positive elsewhere for $\zeta_i \in (0,\infty)$.

**Proof.** Since $F(1) = 1$ and the derivative of $g(\sigma)$ has the same sign as $\sigma - 1$ for $\sigma > 0$, we need only to prove that $\sigma \leq F(\sigma) \leq 1$ for $\sigma \in (0,1)$ and $1 \leq F(\sigma) \leq \sigma$ for $\sigma \in [1,\infty)$. The proof of case $\sigma \in [1,\infty)$ is similar to that case of $\sigma \in (0,1)$, so we will only consider the case when $\sigma \in (0,1)$. Note that $\sigma \leq F\left(\sigma\right) \leq 1$ is equivalent to $\frac{f(v^*)}{v^*} \leq \frac{f(v^*\sigma)}{v^*\sigma} \leq \frac{f(v^*)}{v^*\sigma}$ for $\sigma \in (0,1)$, from Condition (C1) we completed the proof.

**Theorem 2.** Let conditions C1 and C2 hold true and $R_0 > 1$, then the chronic infection equilibrium $E^*$ is globally asymptotically stable for all positive solution.

**Proof.** Define

$$V_1 = \sum_{i=1}^{2} g_i\left(\frac{x_i\left(t\right)}{x_i^*}\right), \qquad V_2 = \sum_{i=1}^{2}\int_0^{\infty} H_i(\zeta_i) g_i\left(\frac{x_i(t-\zeta_i) f_i(v(t-\zeta_i))}{x_i^* f_i(v^*)}\right) d\zeta_i,$$

$$V_3 = \sum_{i=1}^{2} g_i\left(\frac{y_i\left(t\right)}{y_i^*}\right), \qquad V_4 = \sum_{i=1}^{2} g_i\left(\frac{v\left(t\right)}{v^*}\right),$$

$$V_5 = \int_{z^*}^{z(t)} \left[h\left(\zeta_i\right) - h\left(z^*\right)\right] d\zeta_i,, \qquad V_6 = \sum_{i=1}^{2}\int_0^{\infty} \Psi_i\left(\zeta_i\right) g_i\left(\frac{y_i\left(s\right)}{y_i^*}\right) ds.$$

with the infected steady state conditions:

$$\begin{aligned} \mu_i &= k_i x_i^* + \alpha_i x_i^*\ f_i\left(v^*\right), & \gamma_i y_i^* &= d v^*, \\ \vartheta_i x_i^* f_i\left(v^*\right) &= r y_i^* + \beta y_i^* h\left(z^*\right), & \lambda y_i^* &= q z^*. \end{aligned} \tag{14}$$

we will let the function $V\left(t\right)$ and study the derivative of the Lyapunov functional as:

$$\begin{aligned} V\left(t\right) = x_i^* V_{1i}\left(t\right) &+ \alpha_i x_i^*\ f_i\left(v^*\right) V_{2i}\left(t\right) + \frac{\alpha_i y_i^*}{\vartheta_i} V_{3i}\left(t\right) + \frac{\alpha_i r v^*}{\vartheta_i \gamma_i} V_{4i}\left(t\right) + \frac{\alpha_i \beta}{\lambda \vartheta_i} V_{5i}\left(t\right) + \frac{\alpha_i \beta y_i^* h\left(z^*\right)}{d\vartheta_i} V_{4i}\left(t\right) \\ &+ \alpha_i x_i^*\ f_i\left(v^*\right) V_{6i}\left(t\right). \end{aligned}$$

satisfies $V\left(t\right) \geq 0$ with the equality holding if and only $x_i\left(t\right) = x_i^*,\ y_i\left(t\right) = y_i^*,\ v\left(t\right) = v^*,\ z\left(t\right) = z^*$ and $x_i(t-\zeta_i) f_i(v(t-\zeta_i)) = x_i^*\ f_i\left(v^*\right),\ y_i(t-\zeta_i) = y_i^*$. We get

$$\begin{aligned} \dot{V}_{1i}\left(t\right) &= \frac{1}{x_i^*}\left(1 - \frac{x_i^*}{x_i\left(t\right)}\right)\left(k_i x_i^* + \alpha_i x_i^*\ f_i\left(v^*\right) - k_i x_i(t) - \alpha_i x_i(t) f_i(v(t))\right), \\ &= \frac{-k_i\left(x_i\left(t\right) - x_i^*\right)^2}{x_i^* x_i\left(t\right)} + \alpha_i\ f_i\left(v^*\right)\left[1 - \frac{x_i^*}{x_i\left(t\right)} - \frac{x_i(t) f(v(t))}{x_i^*\ f_i\left(v^*\right)} + \frac{f_i(v(t))}{f_i\left(v^*\right)}\right]. \end{aligned} \tag{15}$$

$$\begin{aligned} \dot{V}_{2i}\left(t\right) = \frac{x_i(t) f_i(v(t))}{x_i^* f_i(v^*)} - \ln\left(\frac{x_i(t) f_i(v(t))}{x_i^* f_i(v^*)}\right) &- \int_0^{\infty} g_i\left(\zeta_i\right) \frac{x_i(t-\zeta_i) f_i(v(t-\zeta_i))}{x_i^* f_i(v^*)} d\zeta_i \\ &+ \int_0^{\infty} g_i\left(\zeta_i\right) \ln\left(\frac{x_i(t-\zeta_i) f_i(v(t-\zeta_i))}{x_i^* f_i(v^*)}\right) d\zeta_i, \end{aligned}$$

where

$$H_i(0) = 1, \qquad H_i(\infty) = 0, \qquad dH_i(t) = -g_i\left(t\right) dt.$$

$$\dot{V}_{3i}\left(t\right) = \frac{1}{y_i^*}\left(1 - \frac{y_i^*}{y_i\left(t\right)}\right)\left(\eta_i \int_0^{\infty} g_i(\zeta_i) x_i(t-\zeta_i) f_i(v(t-\zeta_i)) d\zeta_i - r y_i(t) - \beta y_i(t) h(z(t))\right).$$

Using Eq. (14)

$$\dot{V}_{3i}(t) = \frac{\vartheta_i x_i^* f_i(v^*)}{y_i^*}\int_0^\infty g_i(\zeta_i)\frac{x_i(t-\zeta_i)f_i(v(t-\zeta_i))}{x_i^* f_i(v^*)}d\zeta_i - \frac{\vartheta_i x_i^* f_i(v^*)}{y_i^*}\int_0^\infty g_i(\zeta_i)\frac{x_i(t-\zeta_i)y_i^* f_i(v(t-\zeta_i))}{x_i^* y_i(t) f_i(v^*)}d\zeta_i,$$
$$-\frac{\vartheta_i x_i^* f_i(v^*)}{y_i^*}\left[\frac{y_i(t)h(z(t))}{y_i^* h(z^*)} - \frac{h(z(t))}{h(z^*)}\right] + r\left[1 - \frac{y_i(t)}{y_i^*} + \frac{y_i(t)h(z(t))}{y_i^* h(z^*)} - \frac{h(z(t))}{h(z^*)}\right]. \quad (16)$$

$$\dot{V}_{4i}(t) = \frac{1}{y_i^*}\left(1 - \frac{v^*}{v(t)}\right)\left(\gamma_1\int_0^\infty \psi_1(\zeta_1)y_1(t-\zeta_1)d\zeta_1 + \gamma_2\int_0^\infty \psi_2(\zeta_2)y_2(t-\zeta_2)d\zeta_2 - dv(t)\right),$$

Using Eq. (14)

$$\dot{V}_{4i}(t) = d\left[1 - \frac{v(t)}{v^*} + \int_0^\infty \psi_i(\zeta_i)\frac{y_i(t-\zeta_i)}{y_i^*}d\zeta_i - \int_0^\infty \psi_i(\zeta_i)\frac{v^* y_i(t-\zeta_i)}{v(t)y_i^*}d\zeta_i\right],$$
$$\dot{V}_{5i}(t) = [h(z(t)) - h(z^*)][\lambda y_i(t) - qz(t)].$$

Using Eq. (14)

$$\dot{V}_{5i}(t) = -q[h(z(t)) - h(z^*)][z(t) - z^*] + \lambda y_i^* h(z^*)\left[1 - \frac{y_i(t)}{y_i^*} + \frac{y_i(t)h(z(t))}{y_i^* h(z^*)} - \frac{h(z(t))}{h(z^*)}\right].$$

Similar to $\dot{V}_{2i}(t)$ the derivative of $\dot{V}_{6i}(t)$,differentiating gives

$$\dot{V}_{6i}(t) = \frac{y_i(t)}{y_i^*} - \int_0^\infty \psi_i(\zeta_i)\frac{y_i(t-\zeta_i)}{y_i^*}d\zeta_i + \int_0^\infty \psi_i(\zeta_i)\ln\left(\frac{y_i(t-\zeta_i)}{y_i(t)}\right)d\zeta_i,$$

It follows that

$$\dot{V}(t) = \sum_{i=1}^2\left[\frac{-k_i(x_i(t) - x_i^*)^2}{x_i(t)} + \alpha_i\, x_i^* f_i(v^*)\left[1 - \frac{x_i^*}{x_i(t)} + \frac{f_i(v(t))}{f_i(v^*)}\right]\right.$$
$$+\alpha_i\, x_i^* f_i(v^*)\int_0^\infty g_i(\zeta_i)\ln\left(\frac{x_i(t-\zeta_i)f_i(v(t-\zeta_i))}{x_i^* f_i(v^*)}\right)d\zeta_i - \alpha_i\, x_i^* f_i(v^*)\int_0^\infty g_i(\zeta_i)\frac{x_i(t-\zeta_i)y_i^* f_i(v(t-\zeta_i))}{x_i^* y_i(t) f_i(v^*)}d\zeta_i$$
$$+\frac{\alpha_i r y_i^*}{\vartheta_i}\left[1 - \frac{y_i(t)}{y_i^*} + \frac{y_i(t)h(z(t))}{y_i^* h(z^*)} - \frac{h(z(t))}{h(z^*)}\right] - \alpha_i\, x_i^* f_i(v^*)\left[\frac{y_i(t)h(z(t))}{y_i^* h(z^*)} - \frac{h(z(t))}{h(z^*)}\right]$$
$$+\frac{\alpha_i r v^* d}{\vartheta_i \gamma_i}\left[1 - \frac{v(t)}{v^*} + \int_0^\infty \psi_i(\zeta_i)\frac{y_i(t-\zeta_i)}{y_i^*}d\zeta_i - \int_0^\infty \psi_i(\zeta_i)\frac{v^* y_i(t-\zeta_i)}{v(t)y_i^*}d\zeta_i\right]$$
$$+\frac{\alpha_i\beta y_i^* h(z^*)}{\vartheta_i}\left[1 - \frac{v(t)}{v^*} + \int_0^\infty \psi_i(\zeta_i)\frac{y_i(t-\zeta_i)}{y_i^*}d\zeta_i - \int_0^\infty \psi_i(\zeta_i)\frac{v^* y_i(t-\zeta_i)}{v(t)y_i^*}d\zeta_i\right]$$
$$-\frac{\alpha_i\beta q}{\lambda\vartheta_i}[h(z(t)) - h(z^*)][z(t) - z^*] + \frac{\alpha_i\beta y_i^* h(z^*)}{\vartheta_i}\left[1 - \frac{y_i(t)}{y_i^*} + \frac{y_i(t)h(z(t))}{y_i^* h(z^*)} - \frac{h(z(t))}{h(z^*)}\right]$$
$$\left.+\alpha_i\, x_i^* f_i(v^*)\frac{y_i(t)}{y_i^*} - \alpha_i\, x_i^* f_i(v^*)\int_0^\infty \psi_i(\zeta_i)\frac{y_i(t-\zeta_i)}{y_i^*}d\zeta_i + \alpha_i\, x_i^* f_i(v^*)\int_0^\infty \psi_i(\zeta_i)\ln\left(\frac{y_i(t-\zeta_i)}{y_i(t)}\right)d\zeta_i\right],$$

We have $\frac{rdv^*}{\gamma_i} = ry_i^* = \vartheta_i\ x_i^* f_i(v^*) - \beta y_i^* h(z^*)$, then

$$\dot{V}(t) = \sum_{i=1}^{2} \frac{-k_i (x_i(t) - x_i^*)^2}{x_i(t)} + \alpha_i\ x_i^* f_i(v^*) \left[3 - \frac{x_i^*}{x_i(t)} + \frac{f_i(v(t))}{f_i(v^*)}\right] - \alpha_i\ x_i^* f_i(v^*) \frac{v(t)}{v^*}$$
$$- \alpha_i\ x_i^* f_i(v^*) \int_0^\infty g_i(\zeta_i) \frac{x_i(t-\zeta_i) y_i^* f_i(v(t-\zeta_i))}{x_i^* y_i(t) f_i(v^*)} d\zeta_i + \alpha_i\ x_i^* f_i(v^*) \int_0^\infty g_i(\zeta_i) \ln\left(\frac{x_i(t-\zeta_i) f_i(v(t-\zeta_i))}{x_i^* f_i(v^*)}\right) d\zeta_i$$
$$- \alpha_i\ x_i^* f_i(v^*) \left[\frac{y_i(t) h(z(t))}{y_i^* h(z^*)} - \frac{h(z(t))}{h(z^*)}\right] - \alpha_i\ x_i^* f_i(v^*) \int_0^\infty \psi_i(\zeta_i) \frac{y_i(t-\zeta_i)}{y_i^*} d\zeta_i$$
$$- \frac{\alpha_i \beta q}{\lambda \vartheta_i} [h(z(t)) - h(z^*)][z(t) - z^*] + \alpha_i\ x_i^* f_i(v^*) \int_0^\infty \psi_i(\zeta_i) \ln\left(\frac{y_i(t-\zeta_i)}{y_i(t)}\right) d\zeta_i,$$

We can write

$$3 = 2\int_0^\infty g_i(\zeta_i)\, d\zeta_i + \int_0^\infty \psi_i(\zeta_i)\, d\zeta_i.$$

Hence

$$\dot{V}(t) = \sum_{i=1}^{2} \left[\frac{-k_i (x_i(t) - x_i^*)^2}{x_i(t)} - \frac{\alpha_i \beta q}{\lambda \vartheta_i} [h(z(t)) - h(z^*)][z(t) - z^*]\right.$$
$$+\alpha_i\ x_i^* f_i(v^*) \left[\int_0^\infty g_i(\zeta_i) \left(-g\left(\frac{x_i^*}{x_i(t)}\right) - g\left(\frac{x_i(t-\zeta_i) y_i^* f_i(v(t-\zeta_i))}{x_i^* y_i(t) f_i(v^*)}\right) - \ln\frac{x^*}{x(t)}\right.\right.$$
$$\left. - \ln\frac{x(t-\zeta_i) y^* f_i(v(t-\zeta_i))}{x_i^* y_i(t) f_i(v^*)} + \ln\frac{x_i(t-\zeta_i) f(v(t-\zeta_i))}{x_i(t) f_i(v(t))}\right) d\zeta_i + \int_0^\infty \psi_i(\zeta_i) \left(-g\left(\frac{v^* y_i(t-\zeta_i)}{v(t) y_i^*}\right)\right.$$
$$\left.\left. - \ln\frac{v^* y_i(t-\zeta_i)}{v(t) y_i^*} + \ln\frac{y_i(t-\zeta_i)}{y_i(t)}\right) d\zeta_i - \frac{f_i(v(t))}{f_i(v^*)} - \frac{v(t)}{v^*}\right]\Bigg],$$
$$\leq \sum_{i=1}^{2} \left[\alpha_i\ x_i^* f_i(v^*) \left[\frac{f_i(v(t))}{f_i(v^*)} - \ln\left(\frac{f_i(v(t))}{f_i(v^*)}\right) - \frac{v(t)}{v^*} + \ln\left(\frac{v(t)}{v^*}\right)\right]\right],$$
$$= \sum_{i=1}^{2} [\alpha_i\ x_i^* f_i(v^*) (g(F(\sigma)) - g(\sigma))],$$

where $\sigma = \frac{v(t)}{v^*}$ and using Lemma 2, we get $\dot{V}(t) \leq 0$ and $\dot{V}(t) = 0$ if and only if $x_i(t) = x_i^*, z(t) = z^*,\ y_i^* f_i(v(t-\zeta_i)) = y_i(t) f_i(v^*),\ v^* y_i(t-\zeta_i) = v(t) y_i^*$ and $v(t) = v^*$ for $\zeta_i \in [0, \infty)$. Then the solutions converge to $\Gamma$, which is the largest invariant subset of $\{\dot{V}(t) = 0\}$ and by conforming LaSalle's invariance principle, we get that $E^*$ is GAS in $\Gamma$.

# 4 Numerical simulations

In this section, we present an instance to explain the main results given in Theorem 1 and 2 by using the Lyapunov direct method. We have determined a set of conditions which guarantee that the steady states of model (1)-(6) are GAS. Table 1 have the estimate values of model (1)-(6) parameters. The effects of two main factors on the qualitative behavior of the system which include therapy efficacy $\varepsilon$ and time delay $\tau$ will be studied below in details. Using MATLAB we have implemented all computations. This example is obtained from the model (1)-(6) by choosing particular template of the functions $f_i(v(t))$ and $h_i(z(t))$ as follow:

$$f_1(v(t)) = \frac{v}{1+\omega_1 v},\ f_2(v(t)) = \frac{v}{1+\omega_2 v},\ h_1(z(t)) = z(t),\quad h_2(z(t)) = z(t)\ ,$$

where $\omega_1,\ \omega_2 \geq 0$ are constants. Further more, we are going to choose a particular form of the probability distribution functions $G_i(\tau)$ and $\Psi_i(\tau)$ as $G_i(\tau) = \delta(\tau - \tau_i)$, $\Psi_i(\tau) = \delta(\tau - \tau_i)$, $\ i = 1, 2$, where $\delta(.)$ is the

Dirac delta function, $\tau_1$ and $\tau_2$ are constants where $\tau_i \in [0,\infty]$, $i = 1, 2.$ are constants where $\int_0^\infty G_i(\tau)d\tau = \int_0^\infty \Psi_i(\tau)d\tau = 1$. Using Dirac delta function properties we get:

$$g_i = \int_0^\infty e^{-m_i\tau}\delta(\tau-\tau_i)d\tau = e^{-m_i\tau_i}\int_0^\infty \delta(\tau-\tau_i)d\tau = e^{-m_i\tau_i}$$

$$\psi_i = \int_0^\infty e^{-n_i\tau}\delta(\tau-\tau_i)d\tau = e^{-n_i\tau_i}\int_0^\infty \delta(\tau-\tau_i)d\tau = e^{-n_i\tau_i},$$

$$\int_0^\infty \delta(\tau-\tau_i)e^{-m_i\tau}x_i(t-\tau)f_i(v(t-\tau))d\tau = e^{-m_i\tau_i}x_i(t-\tau_i)f_i(t-\tau_i),$$

$$\int_0^\infty \delta(\tau-\tau_i)y_i(t-\tau_i)d\tau = e^{-n_i\tau_i}y_i(t-\tau_i).$$

Referring to the previous relations, we can rewrite model (1)-(6) as follows

$$\dot{x}_1(t) = \mu_1 - k_1x_1(t) - \alpha_1x_1(t)\frac{v(t)}{1+\omega_1v(t)}, \tag{17}$$

$$\dot{y}_1(t) = \alpha_1e^{-m_1\tau_1}x_1(t-\tau_1)\frac{v(t-\tau_1)}{1+\omega_1v(t-\tau_1)} - ry_1(t) - \beta y_1(t)z(t), \tag{18}$$

$$\dot{x}_2(t) = \mu_2 - k_2x_2(t) - \alpha_2x_2(t)\frac{v(t)}{1+\omega_2v(t)} \tag{19}$$

$$\dot{y}_1(t) = \alpha_2e^{-m_2\tau_2}F_2x_2(t-\tau_2)\frac{v(t-\tau_2)}{1+\omega_2v(t-\tau_2)} - ry_2(t) - \beta y_2(t)z(t), \tag{20}$$

$$\dot{v}(t) = Nr\left(e^{-n_1\tau_1}y_1(t-\tau_1) + e^{-n_2\tau_2}y_2(t-\tau_2)\right) - dv(t), \tag{21}$$

$$\dot{z}(t) = \lambda\left(y_1(t)+y_2(t)\right) - qz(t). \tag{22}$$

To study the effect of drug efficacy, we choose $\alpha_1 = (1-\varepsilon)a_0$ and $\alpha_2 = (1-\varepsilon)b_0$. We have chosen the initial conditions:

IC: $\varphi_1(u) = 600,\ \varphi_2(u) = 1,\ \varphi_3(u) = 500,\ \varphi_4(u) = 1,\ \varphi_5(u) = 10$ and $\varphi_6(u) = 40,\ u \in [-\infty, 0]$.

**Table 1:** We define the parameter values of model (17-22) as follow:

| Parameter | Value | Parameter | Value |
|---|---|---|---|
| $\mu_1$ | 10 cells mm$^{-3}$day$^{-1}$ | $\mu_2$ | 6 cells mm$^{-3}$day$^{-1}$ |
| $k_1$ | 0.01 day$^{-1}$ | $k_2$ | 0.01 day$^{-1}$ |
| $a_0$ | 0.004 day$^{-1}$ | $b_0$ | 0.001 day$^{-1}$ |
| $\omega_1$ | 0.05 virus$^{-1}$ mm$^3$ | $\omega_2$ | 0.05 cells$^{-1}$ mm$^3$ |
| $r$ | 0.3 day$^{-1}$ | $m_1$ | 1 day$^{-1}$ |
| $\beta$ | 0.001 | $m_2$ | 1 day$^{-1}$ |
| $N$ | 5 virus cells$^{-1}$ | $n_1$ | 1 day$^{-1}$ |
| $d$ | 3 day$^{-1}$ | $n_2$ | 1 day$^{-1}$ |
| $\lambda$ | 3 day$^{-1}$ | $\tau_1 = \tau_2$ | varied |
| $q$ | 0.1 day$^{-1}$ | $\varepsilon$ | varied |

**Case I**: **Effect of drug efficacy on the dynamical behavior of the system:**

In this case, we fix the delay parameter $\tau_1 = \tau_2 = \tau = 0.5$. Figures 1-6 show the effect of drug efficacy on the stability of the steady states and the evolution of the uninfected and infected for each CD4+ T cells and macrophages, free virus particles and immune response. We observe that, as the drug efficacy is increased from

$\varepsilon = 0$ to $\varepsilon = 0.8$, $E_1$ still exists and is a globally asymptomatically stable. Moreover the concentrations of uninfected CD4$^+$T cells and macrophages are increasing and converging to their normal values $\frac{\mu_1}{k_1} = 1000$ cells mm$^{-3}$, $\frac{\mu_2}{k_2} = 600$ cells mm$^{-3}$, respectively. While the concentrations of CD4$^+$T, macrophages infected cells and free viruses are decaying and tend to zero when $\varepsilon = 0.8$. The concentration of cytotoxic T lymphocytes (CTLs) immune response is increasing for the values of equal to $0, 0.2, 0.5$ and tend to zero when $\varepsilon$ equal to 0.8. It means that the numerical results are consistent with the theoretical results that are given in theorem 1,2. We can see from the simulation results that the treatment with such drug efficacy succeeded to eliminate the HIV virus from the blood.

**Case II**: **Effect of time delay on the dynamical behavior of the system:**

In this case, we confirm the effect of delay parameter in pre-treatment case where $\varepsilon = 0.0$. Figures 7-12 show the effect of time delay on the stability of the steady states and the evolution of the uninfected and infected for each CD4+ T cells and macrophages, free virus particles and immune response. We observe that, as time delay is increased from $\tau = 0.1$ to 0.9, $E_1$ still exists and is a globally asymptomatically stable. Moreover the concentrations of uninfected CD4$^+$T cells and macrophages are increasing for the values of $\tau$ except $\tau = 0.1$. The concentrations of CD4$^+$T, macrophages infected cells and free viruses are decaying with the increasing of time delay values and tend to zero when $\tau = 0.9$. While the concentration of cytotoxic T lymphocytes (CTLs) immune response is increasing for the values of equal to $0.1, 0.3, 0.5$ and it tend to zero when $\tau$ equal to 0.9. It means that the numerical results are consistent with the theoretical results that are given in theorem 1,2. Moreover from a biological point of view, the intracellular delay plays a similar role as an antiviral treatment in eliminating the virus. Where, sufficiently large delay repress viral replication and works on virus clearance. This awaken us to the significance of medications running on the prolong of intracellular delay period.

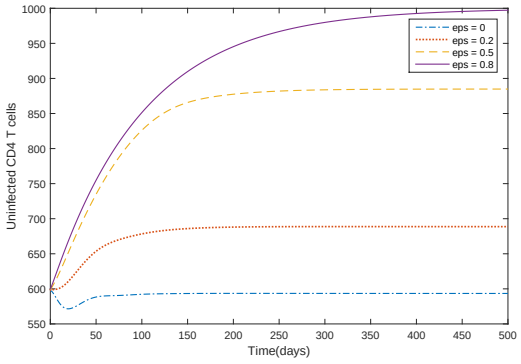



Figure 1: The evolution of uninfected CD4+T cells against time with constant time delay $\tau = 0.5$.

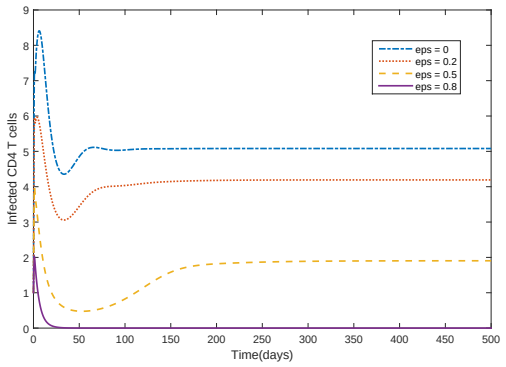



Figure 2: The evolution of infected CD4+T cells against time with constant time delay $\tau = 0.5$.

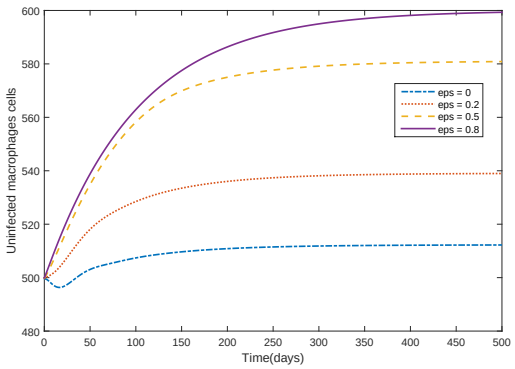



Figure 3: The evolution of uninfected macrophages cells against time with constant time delay $\tau = 0.5$.

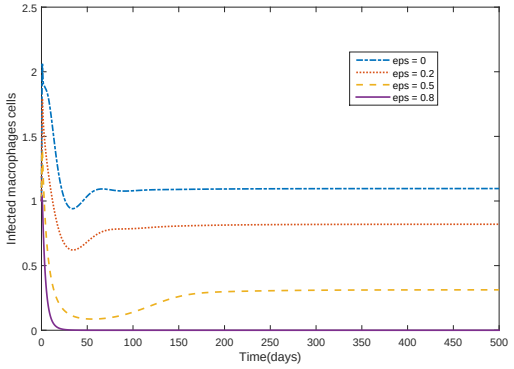



Figure 4: The evolution of infected macrophages cells against time with constant time delay $\tau = 0.5$.

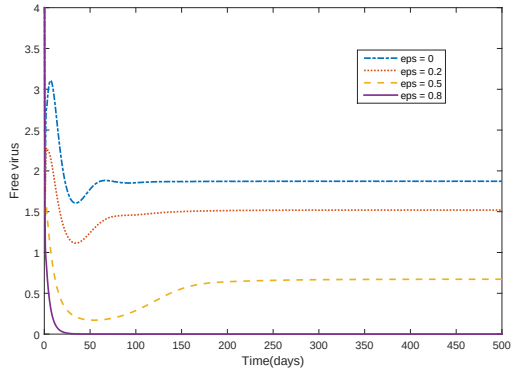



Figure 5: The evolution of free viruses against time with constant time delay $\tau = 0.5$.

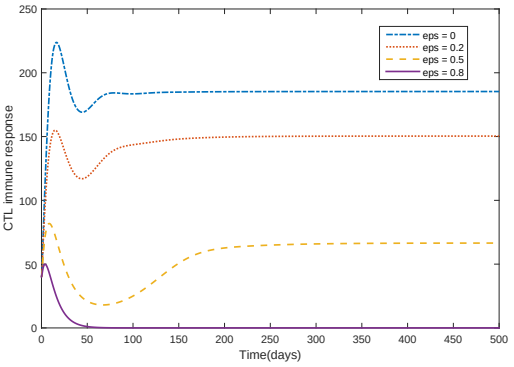



Figure 6: The evolution of immune response against time with constant time delay $\tau = 0.5$.

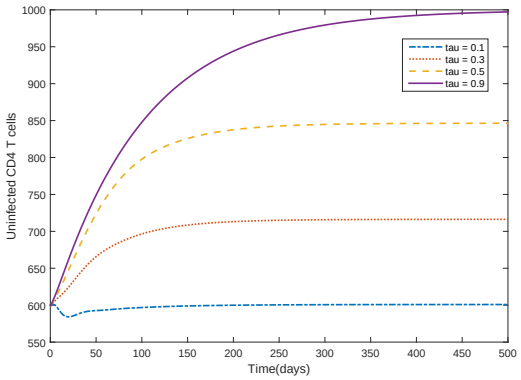



Figure 7: The pre-treatment evolution of uninfected CD4+T cells against time.

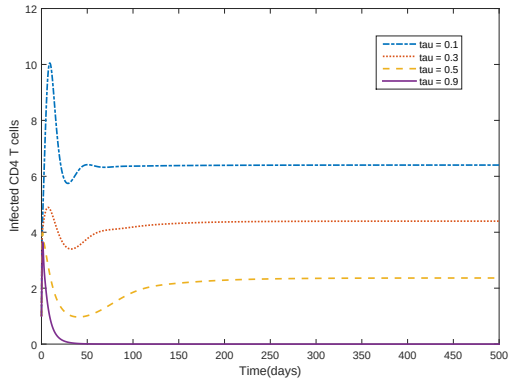



Figure 8: The pre-treatment evolution of infected CD4+T cells against time.

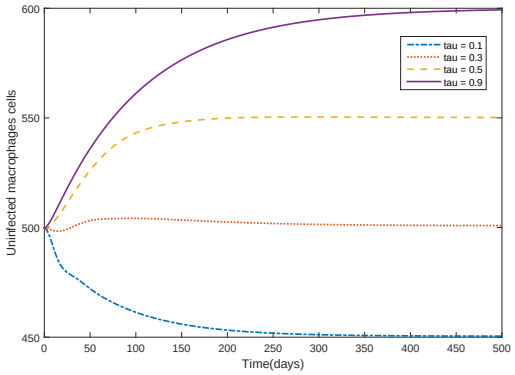



Figure 9: The pre-treatment evolution of uninfected macrophages cells against time.

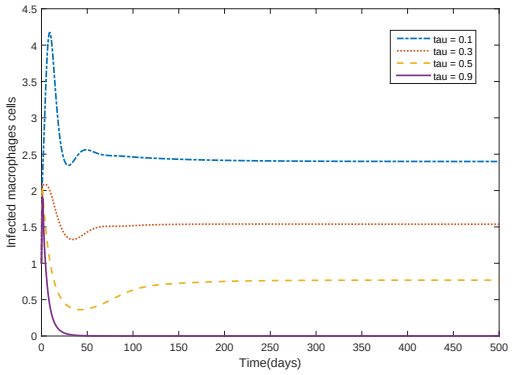



Figure 10: The pre-treatment evolution of infected macrophages cells against time.

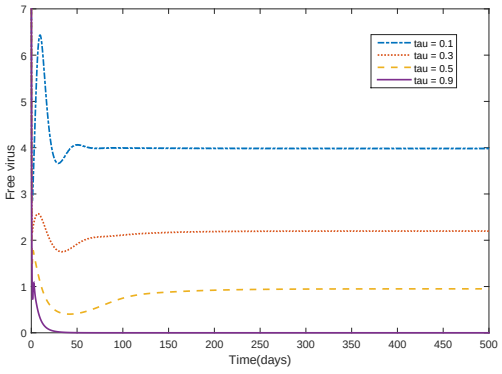



Figure 11: The pre-treatment evolution of free viruses against time.

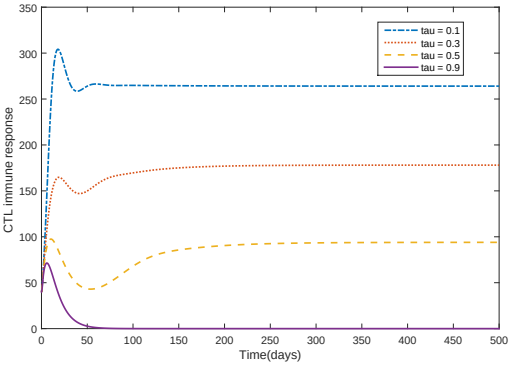



Figure 12: The pre-treatment evolution of immune response against time.

# 5 Conclusion

In this paper, we suggested a distributed delayed human immunodeficiency virus (HIV) models with CTL and two target cells as a system of nonlinear ODES. We demonstrated the positively and boundedness of the solutions and calculate the steady states of the model. Besides we have used suitable Lyapunov functions to set the global asymptotic stability of the steady states. We have derived the basic reproduction number $R_0$ and established that the global dynamics are completely established by the value of the related reproduction number.

# References

[1] M.A. Nowak, R. Anderson, M. Boerlijst, S. Bonhoeffer, R. May and A. McMichael, HIV-1 evolution and disease progression, Science 274 (1996) 1008-1010.

[2] M.A. Nowak and C.R.M. Bangham, Population dynamics of immune responses to persistent viruses, Science 272 (1996) 74-79.

[3] A. M. Elaiw, I. A. Hassanien, and S. A. Azoz, Global stability of HIV infection models with intracellular delays, Journal of the Korean Mathematical Society, 49(4) (2012), 779-794.

[4] A.M. Elaiw and S.A. Azoz, Global properties of a class of HIV infection models with Beddington-DeAngelis functional response, Mathematical Methods in the Applied Sciences, 36 (2013), 383-394.

[5] A. M. Elaiw, N. H. AlShamrani, Mathematical analysis of a cell mediated immunity in a virus dynamics model with nonlinear infection rate and removal, Journal of Computational Analysis and Applications, 21 (3) (2016), 578-586.

[6] A. M. Elaiw, N. H. AlShamrani and M. A. Alghamdi, Global stability analysis of a delayed viral infection model with antibodies and general nonlinear incidence rate, 20 (2) (2016), 277-295.

[7] A.V. Herz, S. Bonhoeffer, R.M. Anderson, R.M. May and M.A. Nowak, Viral dynamics in vivo: Limitations on estimates of intracellular delay and virus decay, Proceedings of the National Academy of sciences USA, 93 (1996), 7247-7251.

[8] J.E. Mittler, B. Markowitz, D.D. Ho, and A.S. Perelson, Improved estimates for HIV-1 clearance rate and intracellular delay,AIDS, 13 (1999), 1415-1417.

[9] P.W. Nelson, J.D. Murray and A.S. Perelson, A model of HIV-1 pathogenesis that includes an intracellular delay, Mathematical Bioscience, 163 (2000), 201-215.

[10] R.V. Culshaw, S. Ruan and G. Webb, A mathematical model of cell-to-cell HIV-1 that includes a time delay, Journal of Mathematical Biology 46 (2003) 425-444.

[11] J. Wang, G. Huang and Y. Takeuchi, Global asymptotic stability for HIV-1 dynamics with two distributed delays, Mathematical Medicine and Biology, IMA.

[12] K. Wang, W. Wang, H. Pang and X. Liu, Complex dynamic behavior in a viral model with delayed immune responses, Physica D 226 (2007) 197-208.

[13] H. Zhu, Y. Luo and M. Chen, Stability and Hopf bifurcation of HIV infection model with CTL-response delay, Comput. Mathematical Applied, 62 (2011), 3091-3102.

[14] M. Y. Li and H. Shu, Global dynamics of a mathematical model for HTLV-1 infection of CD4+ T cells with CTL response, Nonlinear Anal. Real World Applied, 13 (2012), 1080-1092.

[15] Z. Yuan and X. Zou, Global threshold dynamics in an HIV virus model with nonlinear infection rate and distributed invasion and production delays, Math. Biosciences and engineering, 10 (2013), 483-498.

[16] S. Liu and L. Wang, Global stability of an HTLV-1 model with distributed intracellular delays and combination therapy, Math. Biosciences and engineering, 7 (2010), 675-685.

[17] Y. Nakata, Global dynamics of a cell mediated immunity in viral infection models with distributed delays, Math. Biosciences and engineering, 375 (2011), 14-27.

[18] A.S. Perelson, P. Essunger, Y. Cao, M. Vesanen, A. Hurley, K. Saksela, M. Markowitz, D.D. Ho, Decay characteristics of HIV-1- infected compartments during combination therapy, Nature 387 (1997) 188-191.

[19] A.S. Perelson, P.W. Nelson, Mathematical analysis of HIV-1 dynamics in vivo, SIAM Review 41(1) (1999) 3-44.

[20] T.A. Burton, Volterra integral and differential equations,in Mathematics, Science and Engineering, Elsevier, Amsterdam-Boston 202(2) (2005).

[21] J. Hale, S.M. Verduyn Lunel, Introduction to Functional differential equations, Springer-Verlag, (1993).

[22] P. Nelson, and A.S. Perelson, Mathematical analysis of delay differential equation models of HIV-1 infected, Math. Biosciences 179 (2002) 73-94.

[23] H. Zhu, and X. Zou, Impact of delays in cell infection and virus production on HIV-1 dynamics, Math Medicine and Biology, IMA 25 (2008) 99-112.

[24] H. Zhu, and X. Zou, Dynamics of a HIV-1 infection model with cell-mediated immune response and intracellular delay, Discrete Continuous Dynam. Systems-B, 12 (2009) 511-524.

# The pseudo-$T$-direction and pseudo-Nevanlinna direction of $K$-quasi-meromorphic mapping *

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**Abstract**

By applying Ahlfors' theory of covering surfaces, we prove that for quasi-meromorphic mapping $f$ satisfying $\limsup_{r\to\infty}\frac{T(r,f)}{(\log r)^2}=+\infty$, there exists at least one pseudo-$T$-direction of $f$. We also prove that there exists at least one pseudo-Nevanlinna direction of $f$ which is also pseudo-$T$-direction of $f$ under the same condition.
**Key words:** $K$-quasi-meomrophic mapping; pseudo-T-direction; pseudo-Nevanlinna direction
**2000 Mathematics Subject Classification :** 30D 60.

## 1 Introduction, definitions and results

It is very interesting topic on singular directions of meromorphic functions in the fields of complex analysis([3, 6, 8, 13, 10, 14]), such as Julia direction, Borel direction, T-direction, Hayman direction, and so on. In 1997, Sun and Yang [7] extended the value distribution theory of meromorphic functions (see [3, 13] for standard references) to the corresponding theory of quasi-meromorphic mappings [1, 7]. In fact, for value distribution of quasi-meromorphic mappings $f$, the singular direction for $f$ is also one of the main research objects. In [7], Sun and Yang obtained an existence theorem of the Borel direction by using the filling disc theorem of quasi-meromorphic mappings. Later, there were some important results about singular directions for quasi-meromorphic mappings. In 1999, Chen and Sun [1] gave the definition of Nevanlinna directions of quasi-meromorphic mappings on the complex plane and proved that there exists at least one Nevanlinna direction for quasi-meromorphic mappings of infinite order by using type function, and they also obtained that the Nevanlinna direction for quasi-meromorphic mappings of infinite order is also one Borel direction with respect to the type function. In 2004, Liu and Yang [4] studied the relationship between the Julia direction and the Nevanlinna direction of quasi-meromorphic mappings by applying a fundamental inequality of quasi-meromorphic mappings on an angular domain.

For a meromorphic function $f$, Zheng [14] introduced a new singular direction called a $T$-direction conjectured that a transcendental meromorphic function $f$ must have at least one $T$-direction and proved that $\limsup\limits_{r\to\infty}\frac{T(r,f)}{(\log r)^2}=+\infty$. Later, H. Guo, J. H. Zheng and T. W. Ng [2] proved that the conjecture is true by using Ahlfors-Shimizu character $T(r,\Omega)$ of a meromorphic function in an angular domain $\Omega$. Xuan [12] studied the existence of $T$-direction of algebroid function dealing with multiple values. In 2006, Li and Gu [5] proved that there exists at least one Nevanlinna direction for a $K$-quasi-meromorphic mapping $f$ under the condition $\limsup\limits_{r\to\infty}\frac{T(r,f)}{(\log r)^2}=$

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$+\infty$. In this paper we will further investigate some new singular direction of $K$-quasi-meromorphic mapping $f$. Before stating our main results, we will introduce some definitions and notations, which can be found in [7, 11].

**Definition 1.1** *(see [7]). Let $f$ be a complex and continuous functions in a region $D$. If for any rectangle $R = \{x + iy; a < x < b, c < y < d\}$ in $D$, $f(x + iy)$ is an absolutely continuous function of $y$ for almost every $x \in (a, b)$, and $f(x + iy)$ is an absolutely continuous function of $x$ for almost every $y \in (c, d)$, then $f$ is said to be absolutely continuous on lines in the region $D$. We also call that $f$ is $ACL$ in $D$.*

**Definition 1.2** *(see [7, Definition 1.1]). Let $f$ be a homemorphism from $D$ to $D'$. If*

*(i) $f$ is $ACL$ in $D$,*

*(ii) there exists $K \geq 1$ such that $f(z) = u(x, y) + iv(x, y)$ satisfies $|f_z| + |f_{\bar{z}}| \leq K(|f_z| - |f_{\bar{z}}|)$ a. e. in $D$, then $f$ is called an univalent $K$-quasiconformal mapping in $D$. If $D'$ is a region on Riemann sphere $V$, then $f$ is named an univalent $K$-quasi-meromorphic mapping in $D$.*

**Definition 1.3** *(see [7, Definition 1.2]) Let $f$ be a complex and continuous function in the region $D$. For every point $z_0$ in $D$, if there is a neighborhood $U(\subset D)$ and a positive integer $n$ depending on $z_0$, such that*

$$F(z) = \begin{cases} (f(z))^{\frac{1}{n}}, & f(z_0) = \infty, \\ (f(z) - f(z_0))^{\frac{1}{n}} + f(z_0), & f(z_0) \neq \infty. \end{cases}$$

*is an univalent $K$-quasi-meromorphic mapping, then $f$ is named $n$-valent $K$-quasi-meromorphic mapping at point $z_0$. If $f$ is $n$-valent $K$-quasi-meromorphic at every point of $D$, then $f$ is called a $K$-quasi-meromorphic mapping in $D$.*

Let $V$ be the Riemann sphere whose diameter is 1. For any complex number $a$, let $n(r, a)$ be the number of zero points of $f(z) - a$ in disc $|z| < r$, counted according to their multiplicities, $n^{l)}(r, a)$ be the number of zeros of $f(z) - a$ with multiplicity $\leq l$ in disc $|z| < r$, counted according to their multiplicities. Let $F_r$ be the covering surface $f(z) = u(x, y) + iv(x, y)$ on sphere $V$ and $S(r, f)$ be the average covering times of $F_r$ to $V$,

$$S(r, f) = \frac{|F_r|}{|V|} = \frac{1}{\pi} \int_0^r \int_0^{2\pi} \frac{|f_z|^2 - |f_{\bar{z}}|^2}{(1 + |f|^2)^2} r d\varphi dr,$$

where $|F_r|$ and $|V|$ are the areas of $F_r$ and $V$ respectively,

$$T(r, f) = \int_0^r \frac{S(r, f)}{r} dr,$$

$$N(r, a) = \int_0^r \frac{n(t, a) - n(0, a)}{t} dt + n(0, a) \log r,$$

$$N^{l)}(r, a) = \int_0^r \frac{n^{l)}(t, a) - n^{l)}(0, a)}{t} dt + n^{l)}(0, a) \log r.$$

Let $\Omega(\varphi_1, \varphi_2) = \{z \in \mathbb{C} : \varphi_1 < \arg z < \varphi_2\} (0 \leq \varphi_1 < \varphi_2 \leq 2\pi)$, we denote

$$S(r, \varphi_1, \varphi_2; f) = \frac{|F_r|}{|V|} = \frac{1}{\pi} \int_0^r \int_{\varphi_1}^{\varphi_2} \frac{|f_z|^2 - |f_{\bar{z}}|^2}{(1 + |f|^2)^2} r d\varphi dr,$$

$$T(r, \varphi_1, \varphi_2; f) = \int_0^r \frac{S(r, \varphi_1, \varphi_2; f)}{r} dr,$$

when $\varphi_1 = 0, \varphi_2 = 2\pi$, we note $S(r, 0, 2\pi; f) = S(r, f), T(r, 0, 2\pi; f) = T(r, f)$.

For any complex number $a$, let $n(r,\varphi_1,\varphi_2;a)$ be the number of zero points of $f(z)-a$ in sector $\Omega(\varphi_1,\varphi_2)\cap\{z:|z|<r\}$, counted according to their multiplicities, $n^{l)}(r,\varphi_1,\varphi_2;a)$ be the number of zeros of $f(z)-a$ with multiplicity $\leq l$ in sector $\Omega(\varphi_1,\varphi_2)\cap\{z:|z|<r\}$, counted according to their multiplicities. We define

$$N(r,\varphi_1,\varphi_2;a)=\int_0^r\frac{n(t,\varphi_1,\varphi_2;a)-n(0,\varphi_1,\varphi_2;a)}{t}dt+n(0,\varphi_1,\varphi_2;a)\log r,$$

$$N^{l)}(r,\varphi_1,\varphi_2;a)=\int_0^r\frac{n^{l)}(t,\varphi_1,\varphi_2;a)-n^{l)}(0,\varphi_1,\varphi_2;a)}{t}dt+n^{l)}(0,\varphi_1,\varphi_2;a)\log r.$$

Next we give the definitions concerning the Nevanlinna direction of $K$-quasi-meromorphic mappings dealing with multiple values .

**Definition 1.4** *Let $f$ be a $K$-quasi-meromorphic mapping and $l$ be a positive integer. Then we call $\delta^{l)}(a,\varphi_0)$ the deficiency of the value $a$ in the direction $\Delta(\varphi_0)$: $\arg z=\varphi_0, 0\leq\varphi_0<2\pi$. We call $a$ the deficiency value of $f$ in the direction $\Delta(\varphi_0)$ if $\delta^{l)}(a,\varphi_0)>0$, where*

$$\delta^{l)}(a,\varphi_0)=1-\limsup_{\varepsilon\to+0}\limsup_{r\to\infty}\frac{N^{l)}(r,\varphi_0-\varepsilon,\varphi_0+\varepsilon;a)}{T(r,\varphi_0-\varepsilon,\varphi_0+\varepsilon;f)}.$$

**Definition 1.5** *We call $\Delta(\varphi_0):\arg z=\varphi_0$ the pseudo-Nevanlinna direction of $f$ if, for any system $a_j\in\mathbb{C}\cup\{\infty\}(j=1,2,\ldots,q)$ of distinct values and any system $k_j(j=1,2,\ldots,q)$ such that $k_j$ is a positive integer or $+\infty$ such that*

$$\sum_{j=1}^{q}\left(1-\frac{1}{k_j+1}\right)>2, \tag{1}$$

*and*

$$\sum_{j=1}^{q}\frac{k_j}{k_j+1}\delta^{k_j}(a_j,\varphi_0)\leq 2.$$

Similarly, we give the pseudo-T-direction of $K$-quasi-meromorphic mapping as follows.

**Definition 1.6** *Let $f$ be the $K$-quasi-meromorphic mapping. A direction $B:\arg z=\varphi_0(0\leq\varphi_0\leq 2\pi)$ is called a $T$-direction of $f$ if, for any $\varepsilon(0<\varepsilon<\frac{\pi}{2})$, and any system $a_j\in\mathbb{C}\cup\{\infty\}(j=1,2,\ldots,q)$ of distinct values and any system $k_j(j=1,2,\ldots,q)$ such that $k_j$ is a positive integer or $+\infty$ satisfying (2), there exists at least one integer $j(1\leq j\leq q)$ such that*

$$\limsup_{r\to\infty}\frac{N^{k_j)}(r,\varphi_0-\varepsilon,\varphi_0+\varepsilon,a_j)}{T(r,f)}>0.$$

Now, we will give an existence theorem of pseudo-$T$-direction of $K$-quasi-meromorphic mapping $f$ as follows.

**Theorem 1.1** *Let $f$ be the $K$-quasi-meromorphic mapping satisfying*

$$\limsup_{r\to\infty}\frac{T(r,f)}{(\log r)^2}=+\infty, \tag{2}$$

*then there exists at least one pseudo-$T$-direction of $f$.*

We also investigate the problem on the relationship between pseudo-Nevanlinna direction and pseudo-$T$-direction of $f$ under the condition $\limsup\limits_{r\to\infty}\frac{T(r,f)}{(\log r)^2}=+\infty$, and obtain the following result:

**Theorem 1.2** *Let $f$ be the $K$-quasi-meromorphic mapping satisfying (2). Then there exists at least one direction which is both one pseudo-Nevanlinna direction of $f$ and one pseudo-$T$-direction of $f$.*

## 2 Some Lemmas

Let $F$ be a finite covering surface of $F_1$, $F$ is bounded by a finite number of analytic closed Jordan curves, its boundary is denoted by $\partial F$. We call the part of $\partial F$, which lies the interior of $F_1$, the relative boundary of $F$, and denote its length by $L$. Let $D$ be a domain of $F_1$, its boundary consists of finite number of points or analytic closed Jordan curves, and $F(D)$ be the part of $F$, which lies above $D$. We denote the area of $F, F_1, F(D)$ and $D$ by $|F|, |F_1|, |F(D)|$ and $|D|$, respectively. We call

$$S = \frac{|F|}{|F_1|}, \qquad S(D) = \frac{|F(D)|}{|D|}$$

the mean covering numbering of $F$ relative to $F_1, D$, respectively.

**Lemma 2.1** *(see [9, Theorem 3]) Let $F$ be a simply connected finite covering surface on the unit sphere $V$, and let $k_j (j = 1, 2, \ldots, q)$ be $q$ positive integers. Let $D_j (j = 1, 2, \ldots, q)$ be $q(\geq 2)$ disjoint spherical disks with radius $\delta/3 (> 0)$ on $V$ and without a pair of $D_j$ such that their spherical distance is less than $\delta$ and let $n_j^{k_j)}$ be the number of simply connected islands in $F(D_j)$, which consist of not more than $k_j$ sheets, then*

$$\sum_{j=1}^{q} \frac{k_j}{k_j+1} n_j^{k_j)} \geq \left( \sum_{j=1}^{q} \left( 1 - \frac{1}{k_j+1} \right) - 2 \right) S - \frac{C + 9\pi h}{\delta^3} L,$$

*where $L$ is the length of the relative boundary of $F$.*

By applying Lemma 2.1, we can get an important inequality of $K$-quasi-meromorphic mapping in an angular domain as follows.

**Lemma 2.2** *Suppose that $f(z)$ is a $K$-quasi-meromorphic mapping, and let $k_j (j = 1, 2, \ldots, q)$ be $q$ positive integers, and $\{a_j\}$ are $q(\geq 3)$ distinct points on $V$ and without a pair of $\{a_j\}$ such that their spherical distance is less than $\delta + 2\delta/3$, $n_j^{k_j)}$ be the number of zeros of $f(z) - a_j$, which are consisted of not more than $k_j$ multiplicities, then*

$$\sum_{j=1}^{q} \frac{k_j}{k_j+1} n_j^{k_j)} \geq \left( \sum_{j=1}^{q} \left( 1 - \frac{1}{k_j+1} \right) - 2 \right) S - \frac{C + 9\pi h}{\delta^3} L.$$

**Lemma 2.3** *(see [5, Lemma 2.2]). Let $f(z)$ be a $K$-quasi-meromorphic mapping on the angular domain $\Omega(\varphi_0 - \delta, \varphi_0 + \delta)$, $a_1, \ldots, a_q (q \geq 3)$ are distinct points on the unit sphere $V$ and the spherical distance of any two points is no smaller than $\gamma \in (0, \frac{1}{2})$. Let $F_0 = V \setminus \{a_1, a_2, \ldots, a_q\}$, $D = \Omega(r, \varphi_0 - \varphi, \varphi_0 + \varphi) \cap \{z : |z| > 1\} \setminus \{f^{-1}(a_1), f^{-1}(a_2), \ldots, f^{-1}(a_1)\}$ and $D_r = D \cap \{z : |z| < r\} (r > 1)$, $F_r = f(D_r) \subset V$, then for any positive number $\varphi$ satisfying $0 < \varphi < \delta$, we have*

$$\begin{aligned} L(\partial f(D_r)) \leq & \sqrt{2K}\pi \left[ \frac{d(S(r, \varphi_0 - \varphi, \varphi_0 + \varphi; f) - S(1, \varphi_0 - \varphi, \varphi_0 + \varphi; f))}{d\varphi} \right]^{\frac{1}{2}} (\log r)^{\frac{1}{2}} \\ & + \sqrt{2K\delta r}\mu^{\frac{1}{2}}(r, \varphi_0 - \delta, \varphi_0 + \delta) + \sqrt{2K\delta}\mu^{\frac{1}{2}}(1, \varphi_0 - \delta, \varphi_0 + \delta). \end{aligned} \tag{3}$$

*where $F_r$ is the covering surface of $F_0$ and $L(\partial f(D_r))$ is the length of the relative boundary of $F_r$ relative to $F_0$, and*

$$\mu(r, \varphi_0 - \delta, \varphi_0 + \delta) = \int_{\varphi_0 - \delta}^{\varphi_0 + \delta} \frac{|f_z|^2 - |f_{\bar{z}}|^2}{(1 + |f(re^{i\varphi}|^2)^2} r d\varphi.$$

**Lemma 2.4** *Let $f(z)$ be a $K$-quasi-meromorphic mapping on the angular domain $\Omega(\varphi_0-\delta,\varphi_0+\delta)$, and $k_j(j=1,2,\ldots,q)$ $q$ positive integers. If $a_1,\ldots,a_q(q\geq 3)$ are distinct points on the unit sphere $V$ and the spherical distance of any two points is no small than $\gamma\in(0,\frac{1}{2})$. Then*

$$\begin{aligned}
&\left(\sum_{j=1}^{q}\left(1-\frac{1}{k_j+1}\right)-2\right)S(r,\varphi_0-\varphi,\varphi_0+\varphi;f)\\
\leq&\sum_{j=1}^{q}\frac{k_j}{k_j+1}n^{k_j)}(r,\varphi_0-\delta,\varphi_0+\delta;a_j)+\frac{2C^2\gamma^{-6}\pi^2K}{\left(\sum_{j=1}^{q}\left(1-\frac{1}{k_j+1}\right)-2\right)(\delta-\varphi)}\log r\\
&+\left(\sum_{j=1}^{q}\left(1-\frac{1}{k_j+1}\right)-2\right)S(1,\varphi_0-\varphi,\varphi_0+\varphi;f)+2C\gamma^{-3}\delta^{\frac{1}{2}}K^{\frac{1}{2}}r^{\frac{1}{2}}\mu^{\frac{1}{2}}(r,\varphi_0-\delta,\varphi_0+\delta)\\
&+2C\gamma^{-3}\delta^{\frac{1}{2}}K^{\frac{1}{2}}\mu^{\frac{1}{2}}(1,\varphi_0-\delta,\varphi_0+\delta) \qquad (4)
\end{aligned}$$

*and*

$$\begin{aligned}
&\left(\sum_{j=1}^{q}\left(1-\frac{1}{k_j+1}\right)-2\right)T(r,\varphi_0-\varphi,\varphi_0+\varphi;f)\\
\leq&\sum_{j=1}^{q}\frac{k_j}{k_j+1}N^{k_j)}(r,\varphi_0-\delta,\varphi_0+\delta;a_j)+\frac{2C^2\gamma^{-6}\pi^2K}{\left(\sum_{j=1}^{q}\left(1-\frac{1}{k_j+1}\right)-2\right)(\delta-\varphi)}(\log r)^2\\
&+\left(\sum_{j=1}^{q}\left(1-\frac{1}{k_j+1}\right)-2\right)T(1,\varphi_0-\varphi,\varphi_0+\varphi;f)\\
&+\left(\sum_{j=1}^{q}\left(1-\frac{1}{k_j+1}\right)-2\right)S(1,\varphi_0-\varphi,\varphi_0+\varphi;f)\log r\\
&+2C\gamma^{-3}\delta^{\frac{1}{2}}K^{\frac{1}{2}}\mu^{\frac{1}{2}}(1,\varphi_0-\delta,\varphi_0+\delta)\log r+\lambda(r,\varphi_0-\delta,\varphi_0+\delta) \qquad (5)
\end{aligned}$$

*for any $\varphi,0<\varphi<\delta$, where $C$ is a constant depending only on $\{a_1,a_2,\ldots,a_q\}$. $\lambda(r,\varphi_0-\delta,\varphi_0+\delta)=2C\gamma^{-3}\delta^{\frac{1}{2}}K^{\frac{1}{2}}\int_1^r(\frac{\mu(r,\varphi_0-\delta,\varphi_0+\delta)}{r})^{\frac{1}{2}}dr$, $(\mu(r,\varphi_0-\delta,\varphi_0+\delta)=\int_{\varphi_0-\delta}^{\varphi_0+\delta}\frac{|f_z|^2-|f_{\bar{z}}|^2}{(1+|f(re^{i\varphi}|^2)^2}rd\varphi)$*

$$\lambda(r,\varphi_0-\delta,\varphi_0+\delta)\leq 2C\gamma^{-3}\delta^{\frac{1}{2}}\pi^{\frac{1}{2}}K^{\frac{1}{2}}(T(r,\varphi_0-\delta,\varphi_0+\delta;f))^{\frac{1}{2}}\log T(r,\varphi_0-\delta,\varphi_0+\delta;f) \qquad (6)$$

*outside a set $E_\delta$ of $r$ at most, where $E_\delta$ consists of a series of intervals and satisfies $\int_{E_\delta}(r\log r)^{-1}dr<+\infty$.*

*Proof:* Under the condition of Lemma 2.3 and Lemma 2.2, we have

$$S(D_r)=S(r,\varphi_0-\varphi,\varphi_0+\varphi;f)-S(1,\varphi_0-\varphi,\varphi_0+\varphi;f). \qquad (7)$$

Using Lemma 2.1, we easily obtain

$$\begin{aligned}
&\left(\sum_{j=1}^{q}\left(1-\frac{1}{k_j+1}\right)-2\right)[S(r,\varphi_0-\varphi,\varphi_0+\varphi;f)-S(1,\varphi_0-\varphi,\varphi_0+\varphi;f)]\\
\leq&\sum_{j=1}^{q}\frac{k_j}{k_j+1}n^{k_j)}(r,\varphi_0-\delta,\varphi_0+\delta;a_j)+C\gamma^{-3}L(\partial(D_r)). \qquad (8)
\end{aligned}$$

where $C$ is a constant depending only on $\{a_1, a_2, \ldots, a_q\}$.

Taking (3) into (8), we have

$$\begin{aligned}
&\left(\sum_{j=1}^{q}\left(1-\frac{1}{k_j+1}\right)-2\right)[S(r,\varphi_0-\varphi,\varphi_0+\varphi;f)-S(1,\varphi_0-\varphi,\varphi_0+\varphi;f)]\\
&-\sum_{j=1}^{q}\frac{k_j}{k_j+1}n^{k_j)}(r,\varphi_0-\delta,\varphi_0+\delta;a_j)-C\gamma^{-3}\sqrt{2K\delta r}\mu^{\frac{1}{2}}(r,\varphi_0-\delta,\varphi_0+\delta)\\
&-C\gamma^{-3}\sqrt{2K\delta}\mu^{\frac{1}{2}}(1,\varphi_0-\delta,\varphi_0+\delta)\\
\leq& C\gamma^{-3}\sqrt{2K}\pi\left[\frac{d(S(r,\varphi_0-\varphi,\varphi_0+\varphi;f)-S(1,\varphi_0-\varphi,\varphi_0+\varphi;f))}{d\varphi}\right]^{\frac{1}{2}}(\log r)^{\frac{1}{2}}. \qquad (9)
\end{aligned}$$

We denote

$$\begin{aligned}
A(r,\varphi)=&\left(\sum_{j=1}^{q}\left(1-\frac{1}{k_j+1}\right)-2\right)[S(r,\varphi_0-\varphi,\varphi_0+\varphi;f)-S(1,\varphi_0-\varphi,\varphi_0+\varphi;f)]\\
&-\sum_{j=1}^{q}\frac{k_j}{k_j+1}n^{k_j)}(r,\varphi_0-\delta,\varphi_0+\delta;a_j)-C\gamma^{-3}\sqrt{2K\delta r}\mu^{\frac{1}{2}}(r,\varphi_0-\delta,\varphi_0+\delta)\\
&-C\gamma^{-3}\sqrt{2K\delta}\mu^{\frac{1}{2}}(1,\varphi_0-\delta,\varphi_0+\delta). \qquad (10)
\end{aligned}$$

By (9) and (10), we have

$$A(r,\varphi)\leq C\gamma^{-3}\sqrt{2K}\pi\left[\frac{d(S(r,\varphi_0-\varphi,\varphi_0+\varphi;f)-S(1,\varphi_0-\varphi,\varphi_0+\varphi;f))}{d\varphi}\right]^{\frac{1}{2}}(\log r)^{\frac{1}{2}}. \qquad (11)$$

And from (10), it follows that $A(r,\varphi)$ is an increasing function of $\varphi$. Thus, there exists $\delta_0>0$, such that $A(r,\varphi)\leq 0$ for $0<\varphi\leq\delta_0$ and $A(r,\varphi)>0$ for $\varphi>\delta_0$.

Now, two following cases will be considered:

**Case 1.** For $\varphi>\delta_0$, by (11) we have

$$[A(r,\varphi)]^2\leq 2C^2\gamma^{-6}K\pi^2\frac{d(S(r,\varphi_0-\varphi,\varphi_0+\varphi;f)-S(1,\varphi_0-\varphi,\varphi_0+\varphi;f))}{d\varphi}\log r. \qquad (12)$$

By (10) we have

$$\frac{dA(r,\varphi)}{d\varphi}=\left(\sum_{j=1}^{q}\left(1-\frac{1}{k_j+1}\right)-2\right)\frac{d(S(r,\varphi_0-\varphi,\varphi_0+\varphi;f)-S(1,\varphi_0-\varphi,\varphi_0+\varphi;f))}{d\varphi}. \qquad (13)$$

From (12) and (13) we have

$$[A(r,\varphi)]^2\leq\frac{2C^2\gamma^{-6}K\pi^2\log r}{\sum_{j=1}^{q}\left(1-\frac{1}{k_j+1}\right)-2}\cdot\frac{dA(r,\varphi)}{d\varphi},$$

*i.e.*,

$$d\varphi\leq\frac{2C^2\gamma^{-6}K\pi^2\log r}{\sum_{j=1}^{q}\left(1-\frac{1}{k_j+1}\right)-2}\cdot\frac{dA(r,\varphi)}{[A(r,\varphi)]^2}.$$

For the above inequality, by integrating its two sides, we have

$$\delta-\varphi=\int_{\varphi}^{\delta}d\varphi\leq\frac{2C^2\gamma^{-6}K\pi^2\log r}{\sum_{j=1}^{q}\left(1-\frac{1}{k_j+1}\right)-2}\int_{\varphi}^{\delta}\frac{dA(r,\varphi)}{[A(r,\varphi)]^2}\leq\frac{2C^2\gamma^{-6}K\pi^2\log r}{\sum_{j=1}^{q}\left(1-\frac{1}{k_j+1}\right)-2}\cdot\frac{1}{A(r,\varphi)}.$$

Thus

$$A(r,\varphi) \le \frac{2C^2\gamma^{-6}K\pi^2\log r}{\left(\sum_{j=1}^{q}\left(1-\frac{1}{k_j+1}\right)-2\right)(\delta-\varphi)}. \tag{14}$$

**Case 2.** Because $A(r,\varphi) \le 0$ when $0 < \varphi \le \delta_0$, the above inequality also holds.

From Case 1 and Case 2, we can easily get

$$A(r,\varphi) \le \frac{2C^2\gamma^{-6}K\pi^2\log r}{\left(\sum_{j=1}^{q}\left(1-\frac{1}{k_j+1}\right)-2\right)(\delta-\varphi)},$$

for any $\varphi, 0 < \varphi < \delta$. Thus, from (10) we can get (4) easily.

Then, by dividing $r$ and integrating from 1 to $r$ on each sides of (4) we get

$$\begin{aligned}
&\left(\sum_{j=1}^{q}\left(1-\frac{1}{k_j+1}\right)-2\right)T(r,\varphi_0-\varphi,\varphi_0+\varphi;f)\\
\le&\sum_{j=1}^{q}\frac{k_j}{k_j+1}N^{k_j)}(r,\varphi_0-\delta,\varphi_0+\delta;a_j)+\frac{2C^2\gamma^{-6}\pi^2K}{\left(\sum_{j=1}^{q}\left(1-\frac{1}{k_j+1}\right)-2\right)(\delta-\varphi)}(\log r)^2\\
&+\left(\sum_{j=1}^{q}\left(1-\frac{1}{k_j+1}\right)-2\right)T(1,\varphi_0-\varphi,\varphi_0+\varphi;f)\\
&+\left(\sum_{j=1}^{q}\left(1-\frac{1}{k_j+1}\right)-2\right)S(1,\varphi_0-\varphi,\varphi_0+\varphi;f)\log r\\
&+2C\gamma^{-3}\delta^{\frac{1}{2}}K^{\frac{1}{2}}\mu^{\frac{1}{2}}(1,\varphi_0-\delta,\varphi_0+\delta)\log r+2C\gamma^{-3}\delta^{\frac{1}{2}}K^{\frac{1}{2}}\int_1^r\left[\frac{\mu(r,\varphi_0-\delta,\varphi_0+\delta)}{r}\right]^{\frac{1}{2}}dr.
\end{aligned}$$

From the definitions of $S(r,\varphi_1,\varphi_2;f)$, $\mu(r,\varphi_0-\delta,\varphi_0+\delta)$ and $\lambda(r,\varphi_0-\delta,\varphi_0+\delta)$, and Schwarz's inequality we get

$$\begin{aligned}
(\lambda(r,\varphi_0-\delta,\varphi_0+\delta))^2 &= 4C^2\gamma^{-6}\delta K\left[\int_1^r\left(\frac{\mu(r,\varphi_0-\delta,\varphi_0+\delta)}{r}\right)^{\frac{1}{2}}dr\right]^2\\
&\le 4C^2\gamma^{-6}\delta K\int_1^r\mu(r,\varphi_0-\delta,\varphi_0+\delta)dr\int_1^r r^{-1}dr\\
&\le 4C^2\gamma^{-6}\pi\delta K\log r\int_1^r dS(r,\varphi_0-\delta,\varphi_0+\delta;f)\\
&\le 4C^2\gamma^{-6}\pi\delta KS(r,\varphi_0-\delta,\varphi_0+\delta;f)\log r\\
&= 4C^2\gamma^{-6}\pi\delta K\frac{dT(r,\varphi_0-\delta,\varphi_0+\delta;f)}{dr}r\log r.
\end{aligned}\tag{15}$$

Choosing $r_0, r_0 > 0$ such that $T(r_0,\varphi_0-\delta,\varphi_0+\delta;f) > 1$, and setting $E_\delta = \{r_0 < r < \infty : (\lambda(r,\varphi_0-\delta,\varphi_0+\delta))^2 > 4C^2\gamma^{-6}\pi\delta KT(r,\varphi_0-\delta,\varphi_0+\delta;f)(\log T(r,\varphi_0-\delta,\varphi_0+\delta;f))^2\}$, thus we have

$$\begin{aligned}
\int_{E_\delta}\frac{dr}{r\log r} &\le \int_{E_\delta}\frac{dT(r,\varphi_0-\delta,\varphi_0+\delta;f)}{T(r,\varphi_0-\delta,\varphi_0+\delta;f)[\log T(r,\varphi_0-\delta,\varphi_0+\delta;f)]^2}\\
&\le [\log T(r_0,\varphi_0-\delta,\varphi_0+\delta;f)]^{-1} < +\infty.
\end{aligned}\tag{16}$$

Then for $r > r_0$ and $r \notin E_\delta$, we have (5).

Thus, the proof of Lemma 2.4 is completed. □

**Lemma 2.5** *(see [11, Lemma 2.4] or [12]). Let $F(r)$ be a positive nondecreasing function defined for $1 < r < +\infty$ and satisfy*

$$\limsup_{r\to\infty} \frac{F(r)}{(\log r)^2} = +\infty. \tag{17}$$

*Then, for any subset $E \subset (1, +\infty)$ satisfying $\int_E \frac{dr}{r\log r} < \frac{1}{\wp}(\wp \geq 2)$,*

$$\limsup_{r\to\infty, r\in(1,+\infty)\setminus E} \frac{F(r)}{(\log r)^2} = +\infty.$$

**Lemma 2.6** *Let $f(z)$ be the $K$-quasi-meromorphic mapping and $m(m > 1)$ be a positive integer. Put $\varphi_0 = 0, \varphi_1 = \frac{2\pi}{m}, \ldots, \varphi_{m-1} = (m-1)\frac{2\pi}{m}$. Let*

$$\Delta(\varphi_i) = \left\{ z \mid |\arg z - \varphi_i| < \frac{2\pi}{m} \right\} \qquad (0 \leq i \leq m-1).$$

*Then among these $m$ angular domains $\{\Delta(\varphi_i)\}$, there exists at least an angular domain $\Delta(\varphi_i)$ such that for any system $a_j(j = 1, 2, \ldots, q)$ of distinct values and any system $k_j(j = 1, 2, \ldots, q$ such that $k_j$ is a positive integer or $+\infty$ and that*

$$\sum_{j=1}^{q} \left(1 - \frac{1}{k_j + 1}\right) > 2,$$

*there exists at least one integer $j(1 \leq j \leq q)$ such that*

$$\limsup_{r\to\infty} \frac{N^{k_j)}(r, \Delta(\varphi_i), a)}{T(r, f)} > 0.$$

*Proof:* Suppose that the conclusion is false. Then for any $\Delta(\varphi_i)(i = 0, 1, \ldots, m-1)$, there is a system $a_j^i(j = 1, 2, \ldots, q)$ of distinct values and a system $k_j^i(j = 1, 2, \ldots, q)$ such that $k_j^i$ is a positive integer or $+\infty$ and that

$$\sum_{j=1}^{q} (1 - \frac{1}{k_j^i + 1}) > 2,$$

for any $j(1 \leq j \leq q)$, we have

$$\limsup_{r\to\infty} \frac{N^{k_j^i)}(r, \Delta(\varphi_i), a_j^i)}{T(r, f)} = 0. \tag{18}$$

Let $\beta$ be any positive integer. Put $\varphi_{i,k} = \frac{2\pi}{m}i + \frac{2k\pi}{\beta m}, 0 \leq i \leq m-1, 0 \leq k \leq \beta - 1$. For any given number $r > 1$, writing

$$\Delta_{i,k}(r) = \{z || z| < r, \varphi_{i,k} < \varphi_{i,k+1}\}.$$

Then

$$\{|z| < r\} = \sum_{k=0}^{\beta-1} \sum_{i=0}^{m-1} \Delta_{i,k}(r).$$

Put

$$\overline{\Delta}_i = \left\{ z | \frac{\varphi_{i,0} + \varphi_{i,1}}{2} \leq argz \leq \frac{\varphi_{i+1,0} + \varphi_{i+1,1}}{2} \right\},$$

$$\Delta_i^0 = \{z | \varphi_{i,0} < argz < \varphi_{i+1,1}\}, \quad 0 \leq i \leq m-1,$$

$$\sum_{j=1}^{q} \left(1 - \frac{1}{k_j + 1}\right) = \min_{1\leq i\leq m} \left\{ \sum_{j=1}^{q} \left(1 - \frac{1}{k_j^i + 1}\right) \right\} > 2.$$

From Lemma 2.4 we have

$$\left(\sum_{j=1}^{q}\left(1-\frac{1}{k_j+1}\right)-2\right)S(r,\overline{\Delta}_i,f)\leq\sum_{j=1}^{q}\frac{k_j^i}{k_j^i+1}n^{k_j^i)}(r,\Delta_i^0,a_j^i)+O(\log r)+h_i r^{\frac{1}{2}}\mu^{\frac{1}{2}}(r,\varphi_{i,0},\varphi_{i+1,1}).$$

Add from $i=0$ to $m-1$ and divide both sides of this inequality by $r$ and integrate both sides from 1 to $r$, and since $T(r,f)=\sum_{i=0}^{m-1}T(r,\overline{\Delta}_i,f)$, then the following inequality can be obtained

$$\begin{aligned}&\left(\sum_{j=1}^{q}\left(1-\frac{1}{k_j+1}\right)-2\right)T(r,f)\\&\leq\sum_{i=0}^{m-1}\sum_{j=1}^{q}\frac{k_j^i}{k_j^i+1}N^{k_j^i)}(r,\Delta_i^0,a_j^i)+O((\log r)^2)+\sum_{i=0}^{m-1}\lambda(r,\varphi_{i,0},\varphi_{i+1,1}),\end{aligned}\tag{19}$$

where

$$\lambda(r,\varphi_{i,0},\varphi_{i+1,1})\leq h_i\left[\frac{2\pi}{m}(1+\frac{1}{\beta})\right]^{\frac{1}{2}}(T(r,\varphi_{i,0},\varphi_{i+1,1};f))^{\frac{1}{2}}\log T(r,\varphi_{i,0},\varphi_{i+1,1};f)$$

at most outside a set $E_i$ of $r$, where $E_i$ satisfies $\int_{E_i}(r\log r)^{-1}dr<+\infty(i=0,1,\ldots,m-1)$.

For any $i\in\{0,1,\ldots,m-1\}$ and $\wp\geq 2$, there exists $r_i>0$ such that $T(r_i,\varphi_{i,0},\varphi_{i+1,1};f)>e^{\wp m}$ for $r>r_i$. Then it follows from (16) that

$$\int_{E_i}\frac{1}{r\log r}dr\leq\frac{1}{\log T(r,\varphi_{i,0},\varphi_{i+1,1};f)}<\frac{1}{\wp m}<\frac{1}{\wp}.$$

Put $E=\cup_{i=0}^{m-1}E_i$, then

$$\int_E\frac{1}{r\log r}dr\leq\sum_{i=0}^{m-1}\int_{E_i}\frac{1}{r\log r}dr\leq m\max_{0\leq i\leq m-1}\int_{E_i}\frac{1}{r\log r}dr<m\cdot\frac{1}{\wp m}<\frac{1}{\wp}.$$

By applying Lemma 2.5 to this set $E$ and $T(r,f)$, we obtain that

$$\limsup_{r\to\infty,r\in(1,\infty)\backslash E}\frac{T(r,f)}{(\log r)^2}=+\infty.$$

There exists $\{r_n\}\in(r,+\infty)\backslash E$,

$$\lim_{n\to\infty}\frac{T(r_n,f)}{(\log r_n)^2}=+\infty.$$

For this sequence $\{r_n\}$, by (19) we have

$$\begin{aligned}&\left(\sum_{j=1}^{q}\left(1-\frac{1}{k_j+1}\right)-2\right)T(r_n,f)\\&\leq\sum_{i=0}^{m-1}\sum_{j=1}^{q}\frac{k_j^i}{k_j^i+1}N^{k_j^i)}(r_n,\Delta_i^0,a_j^i)+O((\log r_n)^2)+\sum_{i=0}^{m-1}\lambda(r_n,\varphi_{i,0},\varphi_{i+1,1}).\end{aligned}$$

From (18), by dividing both sides of the above inequality by $T(r_n,f)$ and letting $n\to\infty$, we obtain $\sum_{j=1}^{q}\left(1-\frac{1}{k_j+1}\right)-2\leq 0$ that is, $\sum_{j=1}^{q}\left(1-\frac{1}{k_j+1}\right)\leq 2$, a contradict.

Thus, this completes the proof of Lemma 2.6. □

# 3 The Proof of Theorem 1.1

*Proof:* By Lemma 2.6, we can choose subsequence of $\{\theta_m\}$, assume that $\theta_m \to \theta_0$ when $m \to \infty$. Then $B : \arg z = \theta_0$ is a pseudo-T-direction of $f$.

In fact, for any $\varepsilon(0 < \varepsilon < \frac{\pi}{2})$, when $m$ is sufficiently large, we have $\Delta(\theta_m) \subset \Omega(\theta_0, \varepsilon)$. By Lemma 2.6, we have

$$\limsup_{r\to\infty} \frac{N^{k_j)}(r, \theta_0, \varepsilon, a_j)}{T(r, f)} \geq \limsup_{r\to\infty} \frac{N^{k_j)}(r, \Delta(\theta_m), a_j)}{T(r, f)} > 0.$$

Thus, we complete the proof of Theorem 1.1. □

# 4 The Proof of Theorem 1.2

*Proof:* Suppose $\delta \in (0, 2\pi)$, we can choose $r_0 > 0$ such that $T(r_0, \varphi_0 - \delta, \varphi + \delta; f) > e^{\wp}$. Then it follows from (16) that

$$\int_{E_\delta} \frac{1}{r \log r} dr \leq \frac{1}{\log T(r_0, \varphi_0 - \delta, \varphi + \delta; f)} < \frac{1}{\wp}.$$

By applying Lemma 2.4 for the set $E_\delta$ and $T(r, f)$, it follows that

$$\limsup_{r\to\infty, r\in(1,\infty)\backslash E_\delta} \frac{T(r, f)}{(\log r)^2} = +\infty.$$

So, there exists a sequence $\{r_n\} \in (r, +\infty)\backslash E_\delta$,

$$\lim_{n\to\infty} \frac{T(r_n, f)}{(\log r_n)^2} = +\infty. \tag{20}$$

By applying the finite covering theorem at $[0, 2\pi]$, there exists some $\varphi_0$ such that $\varphi_0 \in [0, 2\pi]$ and

$$\limsup_{n\to\infty} \frac{T(r_n, \varphi_0 - \varphi, \varphi_0 + \varphi; f)}{T(r_n, f)} > 0 \tag{21}$$

for an arbitrary $\varphi, 0 < \varphi < \varphi_0$. Thus, we will prove that the direction $\Delta(\varphi_0) : \arg z = \varphi_0$ is one pseudo-Nevanlinna direction of $f(z)$ which is also the pseudo-$T$-direction of $f(z)$.

**Step one.** We firstly prove that the direction $\Delta(\varphi_0) : \arg z = \varphi_0$ is one pseudo-Nevanlinna direction of $f(z)$.

Otherwise, for an arbitrary positive number $\varepsilon_0 > 0$, there exists a system $a_j \in \mathbb{C} \cup \{\infty\}(j = 1, 2, \ldots, q)$ of distinct values and a system $k_j(j = 1, 2, \ldots, q)$ such that $k_j$ is a positive integer or $+\infty$ and that

$$\sum_{j=1}^{q} \left(1 - \frac{1}{k_j + 1}\right) > 2, \tag{22}$$

the following inequality holds

$$\sum_{j=1}^{q} \frac{k_j}{k_j + 1} \delta^{k_j}(a_j, \varphi_0) > 2 + \varepsilon_0.$$

From the definition of $\delta^{k_j}(a_j, \varphi_0)$, we get

$$\limsup_{\varphi\to+0} \limsup_{r\to+\infty} \sum_{j=1}^{q} \frac{k_j}{k_j + 1} \frac{N^{k_j)}(r, \varphi_0 - \varphi, \varphi_0 + \varphi; a_j)}{T(r, \varphi_0 - \varphi, \varphi_0 + \varphi; f)} < \sum_{j=1}^{q} (1 - \frac{1}{k_j + 1}) - 2 - \varepsilon_0.$$

Thus, there exists some $\varphi'(>0)$, and for any $\varphi, 0<\varphi<\varphi'$, we have

$$\limsup_{r\to+\infty}\sum_{j=1}^{q}\frac{k_j}{k_j+1}\frac{N^{k_j)}(r,\varphi_0-\varphi,\varphi_0+\varphi;a_j)}{T(r,\varphi_0-\varphi,\varphi_0+\varphi;f)}<\sum_{j=1}^{q}(1-\frac{1}{k_j+1})-2-\varepsilon_0. \tag{23}$$

Then for any $\varphi, 0<\varphi<\varphi'$, set

$$T(\varphi)=\limsup_{n\to+\infty}\frac{T(r_n,\varphi_0-\varphi,\varphi_0+\varphi;f)}{T(r_n,f)}. \tag{24}$$

Obviously, $T(\varphi)$ is an increasing function in interval $[0,\varphi']$. From (21) we have $T(\varphi)>0$. So, $0<T(\varphi)\leq 1$. Since the increasing of $T(\varphi)$ in interval $[0,\varphi']$ and the continuous theorem for monotonous functions, we can see that all discontinuous points of $T(\varphi)$ constitute a countable set at most. Then, by Lemma 2.4, we can get

$$\begin{aligned}&\left(\sum_{j=1}^{q}(1-\frac{1}{k_j+1})-2\right)T(r_n,\varphi_0-\varphi,\varphi_0+\varphi;f)\\ \leq&\sum_{j=1}^{q}\frac{k_j}{k_j+1}N^{k_j)}(r_n,\varphi_0-\delta,\varphi_0+\delta;a_j)+O(\log r_n)^2\\ &+O((T(r_n,\varphi_0-\delta,\varphi_0+\delta;f))^{\frac{1}{2}}\log T(r_n,\varphi_0-\delta,\varphi_0+\delta;f))\end{aligned} \tag{25}$$

for $0<\varphi<\delta<\varphi'$ and $r_n\notin E_\delta$.

Thus, it follows from (23)-(25) that

$$\left(\sum_{j=1}^{q}(1-\frac{1}{k_j+1})-2\right)T(\varphi)<\left(\sum_{j=1}^{q}(1-\frac{1}{k_j+1})-2-\varepsilon_0\right)T(\delta). \tag{26}$$

Then, we get from (26)

$$T(\varphi)\to T(\delta),\qquad \varphi\to\delta. \tag{27}$$

By combining (26) with (27), we can obtain $T(\delta)=0$, which is a contradiction to $T(\delta)>0$. Then $\Delta(\varphi_0):argz=\varphi_0$ is the pseudo-Nevanlinna direction of $f(z)$.

**Step two.** We will prove that $\Delta(\varphi_0):argz=\varphi_0$ is the pseudo-$T$-direction of $f(z)$.

Otherwise, there exists $\varepsilon_0>0$ and there is a system $a_j(j=1,2,\ldots,q)$ of distinct values and a system $k_j(j=1,2,\ldots,q)$ such that $k_j$ is a positive integer or $+\infty$ and that

$$\sum_{j=1}^{q}(1-\frac{1}{k_j+1})>2,$$

for any $j(1\leq j\leq q)$, we have

$$\limsup_{r\to\infty}\frac{N^{k_j)}(r,\varphi_0-\varepsilon_0,\varphi_0+\varepsilon_0,a_j)}{T(r,f)}=0.$$

Then there exists a sequence $\{r_n\}$ such that

$$\lim_{r\to\infty}\frac{N^{k_j)}(r_n,\varphi_0-\varepsilon_0,\varphi_0+\varepsilon_0,a_j)}{T(r_n,f)}=0. \tag{28}$$

For $\varphi\in(0,\varepsilon_0)$, similar to (24), we define $T(\varphi)$, then $0<T(\varphi)\leq 1$. By Lemma 2.4, for the above sequence $\{r_n\}\subset(1,+\infty)\backslash E_\delta$ and $0<\varphi<\varphi'<\delta$, we have

$$\left(\sum_{j=1}^{q}(1-\frac{1}{k_j^i+1})-2\right)T(r_n,\varphi_0-\varphi,\varphi_0+\varphi;f)\leq\sum_{j=1}^{q}\frac{k_j}{k_j+1}N^{k_j)}(r_n,\varphi_0-\varepsilon_0,\varphi_0+\varepsilon_0;a_i^j)+O((\log r_n)^2)$$

$$+O((T(r_n,\varphi_0-\varepsilon_0,\varphi_0+\varepsilon_0;f))^{\frac{1}{2}}\log T(r_n,\varphi_0-\varepsilon_0,\varphi_0+\varepsilon_0;f)). \tag{29}$$

By (21), (28), (29) and $\sum_{j=1}^{q}(1-\frac{1}{k_j+1})>2$, we can obtain $T(\varphi)\leq 0$ which is a contradiction with $T(\varphi)>0$. Therefore $\Delta(\varphi_0):argz=\varphi_0$ is the pseudo-$T$-direction of $f(z)$.

Thus, this completes the proof of Theorem 1.2.

□

# References


[1] T. W. Chen, D. C. Sun, *Singular directions of quasi-meromorphic mappings*, Acta Math. Sci. Ser. A Chin. Ed. 19(4) (1999): 472-478.

[2] H. Guo, J. H. Zheng, T. W. Ng, *On a new singular direction of meromorphic functions*, Bull. Austral. Math. Soc. 69 (2004): 277-287.

[3] W. Hayman, Meromorphic Functions, Clarendon, Oxford,1964.

[4] M. S. Liu, Y. Yang, *The Nevanlinna direction and Julia direction of quasi-meromorphic mappings*, Acta Math. Sci. Ser. A Chin. Ed. 24(5) (2004): 578-582.

[5] C. H. Li, Y. X. Gu, *A fundamental inequality for $K$-quasi-meromorphic mappings in an angular domain and its application*, Acta Math. Sinica Ser. A 49(6) (2006): 1279-1287.

[6] W. C. Lu, On the $\lambda^*$-Logarithmic Type of Analytic Functions epresented by Laplace-Stieltjes Transformation, J. Jiangxi Norm. Univ., Nat. Sci. 40 (6) (2016), 591-594.

[7] D. C. Sun, L. Yang, *Value distribution of $K$-quasi-meromorphic mappings*, Sci. China Ser. A 27(2) (1997): 132-139.

[8] J. Wang, K. Xia, F. Long, The Poles of Meromorphic Solutions of Fermat Type Differential-Difference Equations, J. Jiangxi Norm. Univ., Nat. Sci. 40 (5) (2016), 497-499.

[9] Z. J. Wu, Y. Q. Chen, Z. X. Xuan, An inequality of meromorphic functions and its application, The Scientific World Journal 2014(2014) Art. 242851, 9 pages.

[10] Z. J. Wu, B. Wang, The Characteristic Function of E-Valued Meromorphic Functions, J. Jiangxi Norm. Univ., Nat. Sci. 40 (5) (2016), 500-504.

[11] H. Y. Xu, T. S. Zhan, On the existence of $T$-direction and Nevanlinna direction of $K$-quasi-meromorphic mapping dealing with multiple values, Bull. Malays. Math. Sci. Soc. 33 (2) (2010): 281-294.

[12] Z. X. Xuan, *On the existence of $T$-direction of algebroid functions: A problem of J.H. Zheng*, J. Math. Anal. Appl. 341 (1) (2008): 540-547.

[13] L. Yang, Value distribution theory, Springer-Verlag, Belin,1993.

[14] J. H. Zheng, *On transcendental meromorphic functions with radially distributed values*, Sci. China 47 (2004): 401-416.

# Some inequalities on small functions and derivatives of meromorphic functions on annuli *

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**Abstract**

In this paper, we firstly establish the second main theorem about meromorphic functions on annuli concerning small functions. Then, by using this theorem, we deal with the uniqueness of meromophic functions sharing some small functions on annuli and obtain the results of meromporphic functions sharing five small functions on annuli, which is an answer to the question of Cao and Yi. In addition, we investigate the properties of meromorphic functions on annuli, and obtain a form of Yang's inequality on annuli by reducing the coeffcients of Hayman's inequality. Moreover, we also study Hayman's inequality in different forms, and obtain accurate estimates of sums of deficiencies.

**Key words:** Small function, Nevanlinna theory, the annulus.

**Mathematical Subject Classification (2010):** 30D30, 30D35.

## 1 Introduction

Firstly, we always assumed that the reader is familiar with the notations of the Nevanlinna theory such as $T(r,f), m(r,f), N(r,f)$ and so on (see [6, 22, 23]).

In 1920s, R. Nevanlinna (see [17]) first established the famous Nevanlinna characteristic of meromorphic functions. It is well known that the Nevanlinna characteristic is powerful, and Nevanlinna theory of value distribution play an important role in the research of complex analysis, which has been used to deal with various complex problems, such as: complex differential equation, complex difference equation, uniqueness of meromorphic functions, complex dynamic systems, etc. Among many basic theorems in Nevanlinna theory, the second main theorem is very important to study the value distribution, uniqueness, singular direction, which is listed as follows.

**Theorem 1.1** *(see [6, 23]). Let $f(z)$ be a non-identically-constant meromorphic function, let $a_1,\ldots,a_q$ be distinct complex numbers, one of which can be equal to $\infty$. Then*

$$\sum_{j=1}^{q} m(r,\frac{1}{f-a_j}) < 2T(r,f) - N_1(r,f) + S(r,f),$$

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$$(q-2)T(r,f) < \sum_{j=1}^{q} \overline{N}(r, \frac{1}{f-a_j}) + S(r,f),$$

*where*

$$N_1(r,f) = N(r, \frac{1}{f'}) + 2N(r,f) - N(r,f'),$$

*and* $S(r,f) = O(\log r)$ *as* $r \to \infty$, *if* $f$ *is of finite order,* $S(r,f) = O(\log(rT(r,f))$ *as* $r \to \infty$, *excluding, possibly, some set of segments in* $[0,\infty)$ *with finite total length, if* $f$ *is of infinite order.*

As a corollary we get the following result about deficiencies

$$\sum_{a \in \overline{\mathbb{C}}} \delta(a,f) \leq 2,$$

where

$$\delta(a,f) = \liminf_{r\to\infty} \frac{m(r, \frac{1}{f-a})}{T(r,f)} = 1 - \limsup_{r\to\infty} \frac{N(r, \frac{1}{f-a})}{T(r,f)}.$$

Nevanlinna asked whether Theorem 1.1 is still true when we replace constants $a_i$ to arbitrary collection of distinct small functions $a_i(z)$ with respect to $f$. This question is very interesting and attracted many investigations (for references, see [5, 18]). In 2004, Yamanoi [21] extended the second main theorem to the case of small functions and obtained the following result

**Theorem 1.2** *(see [21]). Let* $f(z)$ *be a non-constant meromorphic function and* $a_1(z)$, $a_2(z), \ldots,$ $a_q(z)$ *be* $q$ *distinct small functions of* $f(z)$. *Then, for all* $\varepsilon > 0$

$$(q-2-\varepsilon)T(r,f) < \sum_{j=1}^{q} \overline{N}(r, \frac{1}{f-a_j}),$$

*as* $r \to \infty, r \notin E$, *where* $E$ *is a subset of* $[0,\infty)$ *such that* $E$ *is of finite linear measure, and the defect relation:*

$$\sum \delta(a(z), f) \leq 2.$$

After theirs theorems, there are vast references on the value distribution of meromorphic functions in the whole complex plane (see [6, 13, 14, 19, 23]). Moreover, it is an interesting topic how to extend some important results of Nevanlinna value distribution in the complex plane to some subset of the whole complex plane, such as, the unit disc, the angular domain, the annuli. In 2003, J. H. Zheng firstly took into account the value distribution of meromorphic functions in an angular domain and extended the second main theorem in the complex plane to an angular domain (see [24, 25]).

**Theorem 1.3** *(see [26, pp.59 and pp.85]). Let* $f$ *be a nonconstant meromorphic functions in an angular domain* $\Omega(\alpha,\beta) = \{z : \alpha < \arg z < \beta\}(0 < \beta - \alpha < 2\pi)$ *let* $a_1, \ldots, a_q$ *be distinct complex numbers, one of which can be equal to* $\infty$. *Then*

$$(q-2)\mathfrak{T}_{\alpha,\beta}(r,f) < \sum_{j=1}^{q} \overline{\mathfrak{N}}_{\alpha,\beta}(r, \frac{1}{f-a_j}) + Q_{\alpha,\beta}(r,f),$$

*where* $Q_{\alpha,\beta}(r,f) = O(\log r + \log^+ \mathfrak{T}_{\alpha,\beta}(r,f))$ *as* $r \to \infty$, *possibly except some set of* $r$ *with finite linear measure.*

**Theorem 1.4** *(see [26, Theorem 2.3.2]). Let* $f$ *be a nonconstant meromorphic functions in an angular domain* $\Omega(\alpha,\beta) = \{z : \alpha < \arg z < \beta\}(0 < \beta - \alpha < 2\pi)$ *let* $a_1, \ldots, a_q (q \geq 3)$ *be distinct small functions with respect to* $f(z)$. *Then, for any positive number* $\varepsilon$, *we have*

$$(q-2-\varepsilon)\mathfrak{T}_{\alpha,\beta}(r,f) < \sum_{j=1}^{q} \mathfrak{N}_{\alpha,\beta}(r, \frac{1}{f-a_j}) + Q_{\alpha,\beta}(r,f).$$

In recent, there have some results on the Nevannlina Theory of meromorphic functions on annulus (see [8, 9, 11, 15, 16, 20]). In 2005, Khrystiyanyn and Kondratyuk [8, 9] proposed the Nevanlinna theory for meromorphic functions on annuli (see also [10]). Later, the other forms of the second fundamental theorem on annuli were given by Cao, Yi and Xu [3].

**Theorem 1.5** *(see [3, Theorem 2.3]). (The second fundamental theorem) Let $f$ be a nonconstant meromorphic function on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \leq +\infty$. Let $a_1$, $a_2$, ..., $a_q$ be $q$ distinct complex numbers in the extended complex plane $\overline{\mathbb{C}}$. Let $k_1$, $k_2$, ..., $k_q$ be $q$ positive integers, and let $\lambda \geq 0$. Then*

$$(q-2)T_0(r,f) < \sum_{j=1}^{q} N_0(r, \frac{1}{f-a_j}) - N_0^{(1)}(r,f) + S_1(r,f),$$

$$(q-2)T_0(r,f) < \sum_{j=1}^{q} \overline{N}_0(r, \frac{1}{f-a_j}) + S_1(r,f),$$

*where*

$$N_0^{(1)}(r,f) = N_0(r, \frac{1}{f'}) + 2N_0(r,f) - N_0(r,f'),$$

*and $S_1(r,f)$ is stated as in Lemma 2.1.*

*The basic notions of the Nevanlinna theory on annuli will be showed in the next section.* Lund and Ye [15] in 2009 studied functions meromorphic on the annuli with the form $\{z : R_1 < |z| < R_2\}$, where $R_1 \geq 0$ and $R_2 \leq \infty$. In 2009 and 2011, Cao [2, 3] investigated the uniqueness of meromorphic functions on annuli sharing some values and some sets, and obtained an analog of Nevanlinna's famous five-value theorem. From Theorems 1.1-1.4, we can pose the following question

**Question 1.1** *Whether Theorem 1.5 is still true when we replace constants $a_i$ to arbitrary collection of distinct small functions $a_i(z)$ with respect to $f$?*

In [6], W. K. Hayman obtained the following well-known theorem by investigating the characteristic functions of meromorphic function and its derivative in the complex plane.

**Theorem 1.6** *(see [6, Hayman inequality]). Let $f$ be a transcendental meromorphic function on complex plane, then for any positive integer $k$, we have*

$$T(r,f) < \left(2+\frac{1}{k}\right) N\left(r, \frac{1}{f}\right) + \left(2+\frac{2}{k}\right) N\left(r, \frac{1}{f^{(k)}-1}\right) + S(r,f).$$

W. K. Hayman [6] gave a question: whether the coefficients of two counting functions $N(r, \frac{1}{f})$ and $N(r, \frac{1}{f^{(k)}-1})$ are best in Theorem 1.1? Hayman's question attracted many investigations (for references, see [23, 27, 4]). In [23], Yang Lo studied the above question and established the well-known Yang Lo's inequality, in which the coefficients of the counting functions is more precise than the ones of Hayman inequality.

**Theorem 1.7** *(see [23]). Let $f$ be a transcendental meromorphic function on the complex plane, then for any $\varepsilon > 0$ and positive integer $k$, we have*

$$\begin{aligned} T(r,f) \quad < \quad & \left(1+\frac{1}{k}\right) N(r, \frac{1}{f}) + \left(1+\frac{1}{k}\right) N(r, \frac{1}{f^{(k)}-1}) - N\left(r, \frac{1}{f^{(k+1)}}\right) \\ & + \varepsilon T(r,f) + S(r,f). \end{aligned}$$

## 2 Conclusions

The main purpose of this paper is to extend Theorem 1.5 to the case of small functions and obtained the following result.

**Theorem 2.1** *Let $f$ be a nonconstant meromorphic function on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \leq +\infty$. let $a_1, \ldots, a_q (q \geq 3)$ be distinct small functions with respect to $f(z)$. Then, for any positive number $\varepsilon$, we have*

$$(q - 2 - \varepsilon)T_0(r, f) < \sum_{j=1}^{q} N_0(r, \frac{1}{f - a_j}) + S_1(r, f).$$

From Theorem 2.1, we can get the following result immediately.

**Theorem 2.2** *Let $f_1$ and $f_2$ be two transcendental or admissible meromorphic functions on the annulus $\mathbb{A} = \{z : 0 < |z| < \infty\}$. Let $a_j(z)$ $(j = 1, 2, 3, 4, 5)$ be five distinct small functions with respect to $f_1$ and $f_2$. If $f_1, f_2$ share $a_j(z)$ $CM$ for $j = 1, 2, 3, 4, 5$, then $f_1(z) \equiv f_2(z)$.*

The other purpose of this paper is to study the Hayman inequality of meromorphic function on annuli. We obtain:

**Theorem 2.3** *Let $f$ be a transcendental or admissible meromorphic function on the annulus $\mathbb{A} = \{z : 0 < |z| < \infty\}$, then for any $\varepsilon > 0$ and positive integer $k$, we have*

$$\begin{aligned} T_0(r, f) \quad < \quad & \left(1 + \frac{1}{k}\right) N_0\left(r, \frac{1}{f}\right) + \left(1 + \frac{1}{k}\right) N_0\left(r, \frac{1}{f^{(k)} - 1}\right) \\ & - N_0\left(r, \frac{1}{f^{(k+1)}}\right) + \varepsilon T_0(r, f) + S_1(r, f). \end{aligned}$$

*Furthermore, when $a, b$ are two finite complex number, $a \neq b$ and $b \neq 0$. Then we have*

$$\delta_0(a, f) + \delta_0^k(b, f^{(k)}) \leq \frac{k + 2}{k + 1}.$$

**Remark 2.1** *For $a \in \widehat{\mathbb{C}}$, we define*

$$\delta_0(a, f) = 1 - \limsup_{r \to \infty} \frac{N_0\left(r, \frac{1}{f - a}\right)}{T_0(r, f)}, \quad \delta_0^k(a, f^{(k)}) = 1 - \limsup_{r \to \infty} \frac{N_0(r, \frac{1}{f^{(k)} - a})}{T_0(r, f)}.$$

**Definition 2.1** *Let $f(z)$ be a non-constant meromorphic function on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \leq +\infty$. The function $f$ is called a transcendental or admissible meromorphic function on the annulus $\mathbb{A}$ provided that*

$$\limsup_{r \to \infty} \frac{T_0(r, f)}{\log r} = \infty, \quad 1 < r < R_0 = +\infty$$

*or*

$$\limsup_{r \to R_0} \frac{T_0(r, f)}{-\log(R_0 - r)} = \infty, \quad 1 < r < R_0 < +\infty,$$

*respectively.*

Moreover, we also investigated other kind of precise inequalities, and obtained accurate estimation of the sum of deficiencies as follows.

**Theorem 2.4** *Let $f$ be a transcendental or admissible meromorphic function on the annulus $\mathbb{A} = \{z : 0 < |z| < \infty\}$, then for any finite complex numbers $a, b(a \neq b), \varepsilon > 0$ and positive integer $k$, we have*

$$\begin{aligned} T_0(r, f^{(k)}) \quad < \quad & \left(1 + \frac{1}{2k}\right) N_0\left(r, \frac{1}{f^{(k)} - a}\right) + \left(1 + \frac{1}{2k}\right) N_0\left(r, \frac{1}{f^{(k)} - b}\right) \\ & - N_0\left(r, \frac{1}{f^{(k)}}\right) + \varepsilon T_0(r, f) + S_1(r, f^{(k)}), \end{aligned}$$

*Furthermore, we have*

$$\delta_0(a, f^{(k)}) + \delta_0(b, f^{(k)}) \leq 1 + \frac{1}{2k+1}.$$

# 3 Preliminaries and some lemmas

Now, we will introduce the basic notations and conclusion about meromorphic functions on annuli (see [8, 9, 10]). From the Doubly Connected Mapping Theorem [1], we can get that each doubly connected domain is conformally equivalent to the annulus $\{z : r < |z| < R\}$, $0 \leq r < R \leq +\infty$. For two cases: $r = 0, R = +\infty$ simultaneously and $0 < r < R < +\infty$, the latter case the homothety $z \mapsto \frac{z}{\sqrt{rR}}$ reduces the given domain to the annulus $\{z : \frac{1}{R_0} < |z| < R_0\}$, where $R_0 = \sqrt{\frac{R}{r}}$. Thus, every annulus is invariant with respect to the inversion $z \mapsto \frac{1}{z}$ in two cases.

Let $f$ be a meromorphic function on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < r < R_0 \leq +\infty$, the Nevanlinna characteristic of $f$ on the annulus $\mathbb{A}$ is defined by

$$T_0(r, f) = m_0(r, f) + N_0(r, f).$$

Some basic conclusions and properties of $T_0(r, f), N_0(r, f), m_0(r, f)$ had been introduced in (see [3, 8, 9, 10]).

In 2005, the lemma on the logarithmic derivative on the the annulus $\mathbb{A}$ was obtained by Khrystiyanyn and Kondratyuk [9].

**Lemma 3.1** *(see [9, Lemma on the logarithmic derivative]). Let $f$ be a nonconstant meromorphic function on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $R_0 \leq +\infty$, and let $\lambda > 0$. Then*

$$m_0\left(r, \frac{f'}{f}\right) = S_1(r, f),$$

*where (i) in the case $R_0 = +\infty$,*

$$S_1(r, *) = O(\log(rT_0(r, *)))$$

*for $r \in (1, +\infty)$ except for the set $\triangle_r$ such that $\int_{\triangle_r} r^{\lambda-1} dr < +\infty$;*
*(ii) if $R_0 < +\infty$, then*

$$S_1(r, *) = O(\log(\frac{T_0(r, *)}{R_0 - r}))$$

*for $r \in (1, R_0)$ except for the set $\triangle_r^{'}$ such that $\int_{\triangle_r'} \frac{dr}{(R_0 - r)^{\lambda-1}} < +\infty$.*

**Remark 3.1** *If $f$ is a transcendental or admissible meromorphic function on the annulus $\mathbb{A}$, then $S_1(r, f) = o(T_0(r, f))$ holds for all $1 < r < R_0$ except for the set $\triangle_r$ or the set $\triangle_r^{'}$ mentioned in Theorem 3.1, respectively.*

By using the same argument as in (Valiron-Mohon'ko) ([12]), we can get the following lemma.

**Lemma 3.2** *Let $f$ be a nonconstant meromorphic function on the annulus $\mathbb{A}$. Then for all irreducible rational functions in $f$,*

$$R(z,f(z))=\frac{\sum_{i=0}^{m}a_i(z)f(z)^i}{\sum_{j=0}^{n}b_j(z)f(z)^j},$$

*where meromorphic coefficients $a_i(z),b_j(z)$ are small functions with respect of $f$, then the characteristic function of $R(z,f(z))$ satisfies that*

$$T_0(r,R(z,f(z)))=dT_0(r,f)+S_1(r,f),$$

*where $d=\max\{m,n\}$ .*

In order to prove the following lemma, we firstly give the definition of Wronskian determinant of $a_1(z),\dots,a_p(z)$

$$W(a_1(z),\dots,a_p(z))=\begin{vmatrix} a_1(z) & a_2(z) & \cdots & a_p(z)\\ a_1'(z) & a_2'(z) & \cdots & a_p'(z)\\ \vdots & \vdots & & \vdots\\ a_1^{(p-1)}(z) & a_2^{(p-1)}(z) & \cdots & a_p^{(p-1)}(z)\end{vmatrix},$$

where $a_1(z),\dots,a_p(z)$ are meromorphic in annuli.

**Lemma 3.3** *Let $f$ be a transcendental or admissible meromorphic function on the annulus $\mathbb{A}=\{z:0<|z|<\infty\}$ and $\varepsilon>0$. Then for $p=1,2,\dots.$, we have*

$$(p-1)\overline{N}_0(r,f)\le(1+\varepsilon)N_0\left(r,\frac{1}{f^{(p)}}\right)+(1+\varepsilon)N_0^1(r,f)+S_1(r,f), \tag{1}$$

*where $N_0^1(r,f)=N_0(r,f)-\overline{N}_0(r,f)$.*

*Proof:* For any given $\varepsilon>0$ and positive integer $n$, we choose a positive integer $n(>\frac{p}{\varepsilon})$, and consider for all $z\in\mathbb{A}$. Let $W(z)=W(1,z,z^2,\cdots,z^{p+n-1},\ f,zf,\cdots,z^nf)$ as the Wronskian determinant of $1,z,z^2,\cdots,z^{p+n-1},f,zf,\cdots,\ z^nf$. Since $f$ is a transcendental meromorphic function, we can suppose that $W(z)\neq0$. It is easy to see that $W(z)$ is a homogeneous differential polynomial of degree $p+1$ in $f$ with polynomial coefficients of $z$ and without $f^{(j)}(z)(j<p)$ in each term of $W(z)$.

Let $B(z)=W(z)\cdot(f^{(p)}(z))^{-n-1}$, from Lemma 3.1, it follows

$$m_0(r,B)=S_1(r,f).$$

From the first fundamental theorem for meromorphic function on annuli, we have

$$\begin{aligned}N_0(r,\frac{1}{B})&\le T_0(r,B)+O(1)=N_0(r,B)+m_0(r,B)+O(1)\\&\le N_0(r,B)+S_1(r,f).\end{aligned} \tag{2}$$

Next, we will estimate the number of zeros and poles of $B$ on $\mathbb{A}$. From the definition of $W(z)$, we have

$$W(z)=f^{p+2n+1}W(f^{-1},zf^{-1},\cdots,z^{p+n-1}f^{-1},1,z,\cdots,z^n).$$

If $z_0$ is a pole of $f$ of order $t$, then

$$W(z)=O((z-z_0)^{-t(p+2n+1)}),\qquad z\to z_0.$$

Hence

$$\begin{aligned}B(z)&=O\left((z-z_0)^{(n+1)(p+t)-t(p+2n+1)}\right)\\&=O\left((z-z_0)^{n(p-1)-(p+n)(t-1)}\right),\end{aligned} \tag{3}$$

as $z \to z_0$.

Let $\overline{N}_t^0(r), \overline{N}_t^\infty(r)$ and $\overline{N}_t^*(r)$ be the counting functions for those poles of $f$ of order $t$ on $\mathbb{A}$, where $B(z)$ has a zero, pole or finite nonzero value, respectively, each pole being counted only once. From (2) and (3), we get

$$\begin{aligned}\sum_{t=1}^{\infty}(n(p-1)-(p+n)(t-1))\overline{N}_p^0(r) &\le N_0(r,\frac{1}{B}) \le N_0(r,B)+S_1(r,f)\\ &\le \sum_{t=1}^{\infty}((p+n)(t-1)-n(p-1))\overline{N}_t^\infty(r)\\ &\quad +(n+1)N_0\left(r,\frac{1}{f^{(p)}}\right)+S_1(r,f). \end{aligned} \tag{4}$$

If a pole of $f$ contributes to $\overline{N}_t^*(r)$, then from (3) it follows $n(p-1)-(p+n)(t-1)\le 0$ and

$$n(p-1)\overline{N}_t^*(r) \le (p+n)(t-1)\overline{N}_t^*(r).$$

Summing for $t=1,2,\cdots$ in above and substituting to (4), we obtain

$$n(p-1)\sum_{t=1}^{\infty}\overline{N}_t(r) \le (p+n)\sum_{t=1}^{\infty}(t-1)\overline{N}_t(r)+(n+1)N_0\left(r,\frac{1}{f^{(p)}}\right)+S_1(r,f), \tag{5}$$

where $\overline{N}_t(r)=\overline{N}_t^0(r)+\overline{N}_t^\infty(r)+\overline{N}_t^*(r)$.

Noting

$$\begin{aligned}\sum_{t=1}^{\infty}(t-1)\overline{N}_t(r) &= \sum_{t=1}^{\infty}\left[t\overline{N}_t(r)-\overline{N}_t(r)\right]\\ &= \sum_{t=1}^{\infty}\left[N_t(r)-\overline{N}_t(r)\right]=N_0(r,f)-\overline{N}_0(r,f),\end{aligned}$$

since $n>\frac{p}{\varepsilon}$ and (5), we have proved Lemma 3.3. □

By using the same argument as in the proof of Lemma 3.3, we can get the following lemma.

**Lemma 3.4** *Let $f(z)$ and $a_j(z)(j=1,2,\ldots,p;p\ge 3)$ be stated as in Theorem 2.1. Set $W(f)=W(a_1(z),a_2(z),\ldots,a_p(z),f(z))$. If $a_j(z)(j=1,2,\ldots,p;p\ge 3)$ are linearly independent, then for $\varepsilon>0$, we have*

$$p\overline{N}_0(r,f)\le N_0\left(r,\frac{1}{W(f)}\right)+(1+\varepsilon)N_0(r,f)+S_1(r,f).$$

**Lemma 3.5** *Let $f$ be a transcendental or admissible meromorphic function on the annulus $\mathbb{A}=\{z:0<|z|<\infty\}$. Then for any $\varepsilon>0$ and positive integer $k$, we have*

$$\overline{N}_0(r,f)<\frac{1}{k}N_0\left(r,\frac{1}{f^{(k)}}\right)+\frac{1}{k}N_0(r,f)+\varepsilon T_0(r,f)+S_1(r,f). \tag{6}$$

*Proof:* Replacing $\varepsilon$ with $\frac{\varepsilon}{3}$ in Lemma 3.3, it follows

$$\begin{aligned}\overline{N}_0(r,f)<&\frac{1}{k}N_0\left(r,\frac{1}{f^{(k)}}\right)+\frac{1}{k}N_0(r,f)+\frac{\varepsilon}{3k}N_0\left(r,\frac{1}{f^{(k)}}\right)\\ &+\frac{\varepsilon}{3k}N_0(r,f)+S_1(r,f).\end{aligned} \tag{7}$$

Since

$$\begin{aligned} N_0\left(r,\frac{1}{f^{(k)}}\right) &\leq T_0(r,f^{(k)})+O(1)\\ &\leq m_0\left(r,\frac{f^{(k)}}{f}\right)+m_0(r,f)+N_0(r,f^{(k)})+O(1)\\ &\leq m_0(r,f)+N_0(r,f)+k\overline{N}_0(r,f)+S_1(r,f)\\ &\leq (k+1)T_0(r,f)+S_1(r,f), \end{aligned}$$

from (7), we have

$$\begin{aligned} \frac{\varepsilon}{3k}N_0\big(r,\frac{1}{f^{(k)}}\big)+\frac{\varepsilon}{3k}N_0(r,f) &\leq \frac{k+2}{3k}\varepsilon T_0(r,f)+S_1(r,f) \\ &\leq \varepsilon T_0(r,f)+S_1(r,f). \end{aligned} \tag{8}$$

From (7) and (8), we get (6) easily. □

**Lemma 3.6** *(see [7, Theorem 2]). Let $f$ be a transcendental or admissible meromorphic function on the annulus $\mathbb{A}=\{z:0<|z|<\infty\}$, and $k$ be a positive integers. Then*

$$T_0(r,f)<\overline{N}_0(r,f)+N_0(r,\frac{1}{f})+N_0(r,\frac{1}{f^{(k)}-1})-N_0(r,\frac{1}{f^{(k+1)}})+S_1(r,f).$$

**Lemma 3.7** *Let $f$ be a transcendental or admissible meromorphic function on the annulus $\mathbb{A}=\{z:0<|z|<\infty\}$. Then for any $\varepsilon>0$ and positive integer $k$, we have*

$$\overline{N}_0(r,f)<\frac{1}{k}N_0\left(r,\frac{1}{f}\right)+\frac{1}{k}N_0\left(r,\frac{1}{f^{(k)}-1}\right)+\varepsilon T_0(r,f)+S_1(r,f). \tag{9}$$

*Proof:* From Lemma 3.5 we have

$$N_0\left(r,\frac{1}{f^{(k+1)}}\right)>(k+1)\overline{N}_0(r,f)-N_0(r,f)-\frac{k+1}{2}\varepsilon T_0(r,f)-S_1(r,f).$$

Substituting the above inequality back into Lemma 3.6, we obtain

$$\begin{aligned} k\overline{N}_0(r,f) &< N_0\left(r,\frac{1}{f}\right)+N_0\left(r,\frac{1}{f^{(k)}-1}\right)+(N_0(r,f)-T_0(r,f))\\ &\quad+\frac{k+1}{2}\varepsilon T_0(r,f)+S_1(r,f). \end{aligned}$$

Therefore

$$\begin{aligned} \overline{N}_0(r,f) &< \frac{1}{k}N_0\left(r,\frac{1}{f}\right)+\frac{1}{k}N_0\left(r,\frac{1}{f^{(k)}-1}\right)+\frac{k+1}{2k}\varepsilon T_1(r,f)+S_1(r,f)\\ &< \frac{1}{k}N_0\left(r,\frac{1}{f}\right)+\frac{1}{k}N_0\left(r,\frac{1}{f^{(k)}-1}\right)+\varepsilon' T_0(r,f)+S_1(r,f). \end{aligned}$$

Thus, this completes the proof of Lemma 3.7. □

From Theorem 1.5, we get the following conclusion easily.

**Lemma 3.8** *Let $f$ be a transcendental or admissible meromorphic function on the annulus $\mathbb{A}=\{z:0<|z|<\infty\}$. Then for any finite complex number $a,b(a\neq b)$, we have*

$$T_0(r,f)\leq N_0(r,f)+N_0\left(r,\frac{1}{f-a}\right)+N_0\left(r,\frac{1}{f-b}\right)-N_0^0(r)+S_1(r,f),$$

*where $N_0^0(r)=2N_0(r,f)-N_0(r,f')+N_0\left(r,\frac{1}{f'}\right)$.*

**Lemma 3.9** *Let $f$ be a transcendental or admissible meromorphic functions on the annulus $\mathbb{A} = \{z : 0 < |z| < \infty\}$. Then for any finite complex numbers $a, b(a \neq b)$, positive number $\varepsilon > 0$ and positive integer $k$, we have*

$$\overline{N}_0(r,f) < \frac{1}{2k}N_0\left(r,\frac{1}{f^{(k)}-a}\right) + \frac{1}{2k}N_0\left(r,\frac{1}{f^{(k)}-b}\right) + \varepsilon T_0(r,f) + S_1(r,f).$$

*Proof:* By using Lemma 3.8 for $f^{(k)}$ and three distinct complex numbers $a, b, \infty$, we have

$$\begin{aligned} T_0\left(r,f^{(k)}\right) \leq & N_0\left(r,f^{(k)}\right) + N_0\left(r,\frac{1}{f^{(k)}-a}\right) + N_0\left(r,\frac{1}{f^{(k)}-b}\right) \\ & - N_0^0(r) + S_1\left(r,f^{(k)}\right), \end{aligned}$$

where $N_0^0(r) = 2N_0\left(r,f^{(k)}\right) - N_0\left(r,f^{(k+1)}\right) + N_0\left(r,\frac{1}{f^{(k+1)}}\right)$.

Thus, we get

$$\begin{aligned} T_0\left(r,f^{(k)}\right) \leq & \overline{N}_0(r,f) + N_0\left(r,\frac{1}{f^{(k)}-a}\right) + N_0\left(r,\frac{1}{f^{(k)}-b}\right) \\ & - N_0\left(r,\frac{1}{f^{(k+1)}}\right) + S_1\left(r,f^{(k)}\right). \end{aligned} \tag{10}$$

Since $T_0(r,f^{(k)}) = m_0(r,f^{(k)}) + N_0(r,f) + k\overline{N}_0(r,f)$, then by applying Lemma 3.7 for $f^{(k+1)}$, it follows

$$N_0\left(r,\frac{1}{f^{(k+1)}}\right) > (k+1)\overline{N}_0(r,f) - N_0(r,f) - (k+1)\varepsilon T_0(r,f) - (k+1)S_1(r,f).$$

Substituting the two above inequalities back into (10) , we get

$$\begin{aligned} \overline{N}_0(r,f) &< \frac{1}{2k}N_0\left(r,\frac{1}{f^{(k)}-a}\right) + \frac{1}{2k}N_0\left(r,\frac{1}{f^{(k)}-b}\right) + \frac{k+1}{2k}\varepsilon T_0(r,f) \\ &\quad + \frac{k+2}{2k}S_1(r,f^{(k)}) \\ &< \frac{1}{2k}N_0\left(r,\frac{1}{f^{(k)}-a}\right) + \frac{1}{2k}N_0\left(r,\frac{1}{f^{(k)}-b}\right) + \varepsilon' T_0(r,f) \\ &\quad + S_1(r,f^{(k)}). \end{aligned}$$

From the definition of $S_1(r,*)$ and $T_0(r,f) \leq T_0(r,f^{(k)}) \leq (k+1)T_0(r,f) + S_1(r,f)$, where $S_1(r,f)$ is as stated in Lemma 3.1, we can get the conclusion of Lemma 3.9.

Thus, we can complete the proof of Lemma 3.9. □

# 4 Proofs of Theorem 2.1 and Theorem 2.2

## 4.1 The proof of Theorem 2.1

Without any loss of generalities, suppose that $\{a_1(z), a_2(z), \ldots, a_p(z)\}$ is a maximum linearly independent subset of $a_j(z)(j = 1, 2, \ldots, q)$, then $p \leq q$ and each $a_j(z)(j = 1, 2, \ldots, q)$ can be linearly expressed in terms of $a_j(z)(j = 1, 2, \ldots, p)$. Set $W(f) = W(a_1, a_2, \ldots, a_p(z), f)$, then

$$W(f) = b_p f^{(p)} + b_{p-1}f^{(p-1)} + \cdots + b_1 f' + b_0 f, \tag{11}$$

where $b_j(j = 1, 2, \ldots, p)$ are small functions with respect to $f$. It follows from (11) that

$$N_0(r,W(f)) = p\overline{N}_0(r,f) + N_0(r,f) + S_1(r,f) \tag{12}$$

and

$$m_0(r,W(f)) \le m_0(r,f) + m_0(r,\frac{W(f)}{f}) = m_0(r,f) + S_1(r,f). \tag{13}$$

Thus from (12), (13) it follows that

$$T_0(r,W(f)) \le p\overline{N}_0(r,f) + T_0(r,f) + S_1(r,f). \tag{14}$$

From the definition of $W(f)$, we have $W(f-a_j) = W(f)$ for $j = 1,2,\ldots,q$. Thus, it follows by Lemma 3.1 that

$$m_0(r,\frac{W(f)}{f-a_j}) = m_0(r,\frac{W(f-a_j)}{f-a_j}) = S_1(r,f). \tag{15}$$

Set

$$F(z) = \sum_{j=1}^{q} \frac{1}{f(z)-a_j(z)}.$$

Then it follows from (14),(15) and by Lemma 3.4 that

$$\begin{aligned} m_0(r,F) &\le m_0(r,\frac{1}{W(f)}) + m_0(r,FW(f)) \\ &\le T_0(r,W(f)) - N_0(r,\frac{1}{W(f)}) + S_1(r,f) \\ &\le p\overline{N}_0(r,f) + T_0(r,f) - N_0(r,\frac{1}{W(f)}) + S_1(r,f) \\ &\le T_0(r,f) + (1+\varepsilon)N_0(r,f) + S_1(r,f). \end{aligned}$$

Then it follows by Lemma 3.2 that

$$\begin{aligned} qT_0(r,f) &= T_0(r,F) + S_1(r,f) \\ &\le \sum_{j=1}^{q} N_0(r,\frac{1}{f-a_j}) + T_0(r,f) + (1+\varepsilon)N_0(r,f) + S_1(r,f) \\ &\le \sum_{j=1}^{q} N_0(r,\frac{1}{f-a_j}) + (2+\varepsilon)T_0(r,f) + S_1(r,f). \end{aligned}$$

Hence, this completes the proof of Theorem 2.1.

## 4.2 The proof of Theorem 2.2

Suppose $f(z) \not\equiv g(z)$. By applying Theorem 2.1, since $f$ and $g$ share $a_1,\ldots,a_5$ $CM$, we have

$$\begin{aligned} (3-\varepsilon)T_0(r,f) &\le \sum_{j=1}^{5} N_0(r,\frac{1}{f-a_j}) + S_1(r,f) \le N_0(r,\frac{1}{f-g}) + S_1(r,f) \\ &\le T_0(r,f) + T_0(r,g) + S_1(r,f), \end{aligned}$$

that is,

$$(2-\varepsilon)T_0(r,f) \le T_0(r,g) + S_1(r,f). \tag{16}$$

Similarly, we have

$$(2-\varepsilon)T_0(r,g) \le T_0(r,f) + S_1(r,g). \tag{17}$$

Thus for any small number $\varepsilon(>0)$, it follows from (16) and (17) that

$$(1-\varepsilon)\left[T_0(r,f) + T_0(r,g)\right] \le S_1(r,f) + S_1(r,g),$$

which is a contradiction with the assumption that $f,g$ are transcendental or admission.

Therefore, we have $f \equiv g$.

# 5 Proofs of Theorem 2.3 and Theorem 3.4

## 5.1 The proof of Theorem 2.3

From Lemma 3.4 and Lemma 3.7, we get

$$\begin{aligned} T_0(r,f) &< \left(1+\frac{1}{k}\right)N_0\left(r,\frac{1}{f}\right)+\left(1+\frac{1}{k}\right)N_0\left(r,\frac{1}{f^{(k)}-1}\right) \\ &\quad -N_0\left(r,\frac{1}{f^{(k+1)}}\right)+\varepsilon T_0(r,f)+S_1(r,f). \end{aligned}$$

Now, we will prove the inequality of the sum of deficiencies as follows. First, by using the above inequality for the function $\frac{f-a}{b}$, then it follows

$$\begin{aligned} T_0(r,f) &< \left(1+\frac{1}{k}\right)N_0\left(r,\frac{1}{f-a}\right)+\left(1+\frac{1}{k}\right)N_0\left(r,\frac{1}{f^{(k)}-b}\right)-N_0\left(r,\frac{1}{f^{(k+1)}}\right) \\ &\quad +\varepsilon T_0(r,f)+S_1(r,f) \\ &< \left(1+\frac{1}{k}\right)N_0\left(r,\frac{1}{f-a}\right)+\left(1+\frac{1}{k}\right)N_0\left(r,\frac{1}{f^{(k)}-b}\right)+\varepsilon T_0(r,f)+S_1(r,f). \end{aligned}$$

Dividing the both sides of the above inequality by $T_0(r,f)$, we have

$$\left(1+\frac{1}{k}\right)\left(1-\frac{N_0\left(r,\frac{1}{f-a}\right)}{T_0(r,f)}+1-\frac{N_0\left(r,\frac{1}{f^{(k)}-b}\right)}{T_0(r,f)}\right)<1+\frac{2}{k}+\varepsilon+\frac{S_1(r,f)}{T_0(r,f)}. \tag{18}$$

From the definitions of $\delta_0(a,f),\delta_0^k(a,f^{(k)})$, then it follows from (18) that

$$\begin{aligned} &\left(1+\frac{1}{k}\right)\left(\delta_0(a,f)+\delta_0^k(b,f^{(k)})\right) \\ &\leq \left(1+\frac{1}{k}\right)\liminf_{r\to\infty}\left(1-\frac{N_0\left(r,\frac{1}{f-a}\right)}{T_0(r,f)}+1-\frac{N_0\left(r,\frac{1}{f^{(k)}-b}\right)}{T_0(r,f)}\right) \\ &\leq \limsup_{r\to\infty}\left(1+\frac{2}{k}+\varepsilon\right)+\liminf_{r\to\infty}\frac{S_1(r,f)}{T_0(r,f)}. \end{aligned}$$

Since $f$ is a transcendental or admission on annuli, we have

$$\lim_{r\to\infty}\frac{S_1(r,f)}{T_0(r,f)}=0. \tag{19}$$

Hence,

$$\delta_0(a,f)+\delta_0^k(b,f^{(k)})\leq\frac{k+2}{k+1}.$$

Thus, this completes the proof of Theorem 2.3.

## 5.2 The proof of Theorem 2.4

By Lemma 3.9, it follows from (10) that

$$\begin{aligned} T_0\left(r,f^{(k)}\right) &\leq \left(1+\frac{1}{2k}\right)N_0\left(r,\frac{1}{f^{(k)}-a}\right)+\left(1+\frac{1}{2k}\right)N_0\left(r,\frac{1}{f^{(k)}-b}\right) \\ &\quad -N_0\left(r,\frac{1}{f^{(k+1)}}\right)+\varepsilon T_0(r,f)+S_1\left(r,f^{(k)}\right). \end{aligned}$$

The above inequality implies

$$\left(1+\frac{1}{2k}\right)\left(2-\frac{N_0\left(r,\frac{1}{f^{(k)}-a}\right)}{T_0(r,f^{(k)})}-\frac{N_0\left(r,\frac{1}{f^{(k)}-b}\right)}{T_0(r,f^{(k)})}\right)<1+\frac{1}{k}+\varepsilon+\frac{S_1(r,f^{(k)})}{T_0(r,f^{(k)})}. \tag{20}$$

Thus, it follows from (20) and the definition of $\delta_{\alpha,\beta}(a,f)$ that

$$\begin{aligned}
&\left(1+\frac{1}{2k}\right)\left(\delta_0(a,f^{(k)})+\delta_0(b,f^{(k)})\right)\\
\leq\ &\left(1+\frac{1}{2k}\right)\liminf_{r\to\infty}\left(1-\frac{N_0\left(r,\frac{1}{f^{(k)}-a}\right)}{T_0(r,f^{(k)})}+1-\frac{N_0\left(r,\frac{1}{f^{(k)}-b}\right)}{T_0(r,f^{(k)})}\right)\\
\leq\ &\limsup_{r\to\infty}\left(1+\frac{1}{k}+\varepsilon\right)+\liminf_{r\to\infty}\frac{S_1(r,f^{(k)})}{T_0(r,f^{(k)})}.
\end{aligned}$$

Since $f$ is a transcendental or admission on annuli, we have the following equalities easily

$$\liminf_{r\to\infty}\frac{S_1(r,f^{(k)})}{T_0(r,f^{(k)})}=0. \tag{21}$$

Since $\varepsilon$ is arbitrary, it follows from (21) that

$$\delta_0(a,f^{(k)})+\delta_0(b,f^{(k)})\leq 1+\frac{1}{2k+1}.$$

Therefore, we complete the proof of Theorem 2.4.

# References


[1] S. Axler, *Harmomic functions from a complex analysis viewpoit*, Amer. Math. Monthly **93** (1986), 246-258.

[2] T. B. Cao and H. X. Yi, *Uniqueness theorems of meromorphic functions sharing sets IM on annuli*, Acta Mathematica Sinica (Chinese Series), **54** (4) (2011), 623-632. (in Chinese).

[3] T. B. Cao, H. X. Yi and H. Y. Xu, *On the multiple values and uniqueness of meromorphic functions on annuli*, Comput. Math. Appl. **58** (2009), 1457-1465.

[4] H. H. Chen, Singular directions related to the Hayman inequality, Progress in Math., 16(1) (1987), 73-79, (in Chinese).

[5] C.T. Chuang, Une généralisation d'une inégalité de Nevanlinna, Sci. Sinica **13** (1964), 887-895.

[6] W. Hayman, *Meromorphic functions*, Oxford: Clarendon Press, 1964.

[7] Y. Hu, H. Y. Xu, The Milloux's inequality of meromorphic functions on annulis, J. Jiangxi Norm. Univ., Nat. Sci. **33** (3) (2009), 293-296.

[8] A. Ya. Khrystiyanyn and A. A. Kondratyuk, *On the Nevanlinna theory for meromorphic functions on annuli. I*, Mat. Stud. **23**(2005), No.1, 19-30.

[9] A. Ya. Khrystiyanyn and A. A. Kondratyuk, *On the Nevanlinna theory for meromorphic functions on annuli. II*, Mat. Stud. **24** (2005), No.2, 57-68.

[10] A. A. Kondratyuk and I. Laine, *Meromorphic functions in multiply connected domains*, Laine, Ilpo (ed.), fourier series methods in complex analysis. Proceedings of the workshop, Mekrijärvi, Finland, July 24-29, 2005. Joensuu: University of Joensuu, Department of Mathematics (ISBN 952-458-888-9/pbk). Report series. Department of mathematics, University of Joensuu 10, 9-111 (2006).

[11] R. Korhonen, *Nevanlinna theory in an annulus, value distribution theory and related topics*, Adv. Complex Anal. Appl. **3** (2004), 167-179.

[12] I. Laine, *Nevanlinna Theory and Complex Differential Equations*, Walter de Gruyter, Berlin, 1993.

[13] X. M. Li, H. X. Li, X. Zhang, Normal Families of Meromorphic Functions Concerning Composite Meromorphic Functions and Fixed Points, J. Jiangxi Norm. Univ. Nat. Sci. 40(6) (2016), 578-594.

[14] L. W. Liao, The new developments in the research of nonlinear complex differential equations. J. Jiangxi Norm. Univ. Nat. Sci. 39(4) (2015), 331-339.

[15] M. Lund and Z. Ye, *Logarithmic derivatives in annuli*, J. Math. Anal. Appl. **356** (2009), 441-452

[16] M. Lund and Z. Ye, *Nevanlinna theory of meromorphic functions on annuli*, Sci. China. Math. **53** (2010), 547-554.

[17] R. Nevanlinna, *Eindentig keitssätze in der theorie der meromorphen funktionen*, Acta. Math. **48** (1926), 367-391.

[18] N. Steinmetz, Eine Varallgemeinerung des zweiten Nevanlinnaschen Hauptsatzes, J. Rein Angew. Math. **368** (1986), 131-141.

[19] Z. J. Wu and B. Wang, The Characteristic Function of E-Valued Meromorphic Functions, J. Jiangxi Norm. Univ., Nat. Sci. 40 (5) (2016), 500-504.

[20] H. Y. Xu and Z. X. Xuan, The uniqueness of analytic functions on annuli sharing some values, Abstract and Applied Analysis **2012** (2012), Art. 896596, 1-13.

[21] K. Yamanoi, The second main theorem for small functions and related problems, Acta Math. **192** (2004), 225-294.

[22] H. X. Yi and C. C. Yang, *Uniqueness theory of meromorphic functions*, Science Press 1995/Kluwer 2003.

[23] L. Yang, *Value distribution theory*, Berlin: Springer-Verlag/ Beijing: Science Press, 1993.

[24] J. H. Zheng, *On uniqueness of meromorphic functions with shared values in some angular domains*, Canad J. Math. **47** (2004), 152-160.

[25] J. H. Zheng, *On uniqueness of meromorphic functions with shared values in one angular domains*, Complex Var. Elliptic Equ. **48** (2003), 777-785.

[26] J. H. Zheng, *Value Distribution of Meromorphic Functions*, Tsinghua University Press, Beijing, Springer, Heidelberg, 2010.

[27] J. H. Zheng, N. Wu, Hayman T directions of meromorphic functions, Taiwanese J Math. 14 (6) (2010), 2219-2228.

# WEIGHTED COMPOSITION OPERATORS BETWEEN WEIGHTED HILBERTIAN BERGMAN SPACES IN THE UNIT POLYDISK

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ABSTRACT. In this paper, we prove that the topology spaces of nonzero weighted composition operators acting on some Hilbert spaces of holomorphic functions in the unit polydisk are path connected, which generalized Hosokawa, Izuchi and Ohno's results in single complex variables' case [9].
KEYWORDS: Weighted Hilbertian Bergman spaces, weighted composition operator, polydisk, norm topology, Hilbert-Schmidt topology.

## 1. INTRODUCTION

Let $H(\mathbb{D}^N)$ be the space of analytic functions on the open unit polydisk

$$\mathbb{D}^N := \{z = (z_1, \ldots, z_N) \in \mathbb{C}^N : |z_i| < 1, i = 1, 2, \ldots, N\}$$

and $H^\infty$ the space of bounded analytic functions on $\mathbb{D}^N$ with the supremum norm $\|\cdot\|_\infty$. When $N = 1$, the unit polydisk reduces to the unit open disc $\mathbb{D}$ in the complex plane $\mathbb{C}$. Let $\mathcal{S}(\mathbb{D}^N)$ be the set of analytic self-maps of $\mathbb{D}^N$. Every $\varphi = (\varphi_1, \cdots, \varphi_N) \in \mathcal{S}(\mathbb{D}^N)$ induces the composition operator $C_\varphi$ defined by $C_\varphi f = f \circ \varphi$ for $f \in H\left(\mathbb{D}^N\right)$. If $u \in H(\mathbb{D}^N)$, the multiplication, $M_u : H(\mathbb{D}^N) \to H(\mathbb{D}^N)$, is defined by

$$M_u(f)(z) = u(z) \cdot f(z)$$

for any $f \in H(\mathbb{D}^N)$ and $z \in \mathbb{D}^N$. If $u \in H(\mathbb{D}^N)$ and $\varphi \in \mathcal{S}(\mathbb{D}^N)$, we call the operator $M_u C_\varphi$ to be the weighted composition operator.

Much effort has been expended on characterizing those analytic maps which induce bounded or compact composition operators between those classic spaces of analytic functions. Readers interested in this topic can refer to the books [16] by Shapiro, [7] by Cowen and MacCluer, and [23, 24] by Zhu.

An active topic is the topological structure of the space of composition operators acting on function spaces. If $X$ is a Banach space of analytic functions,

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we employ the symbol $\mathcal{C}(X)$ to represent the space of composition operators on $X$ equipped with the operator norm topology. In 1981, Berkson [3] firstly studied the topological structure of $\mathcal{C}(H^2(\mathbb{D}))$. Central problem focuses on both the structure of $\mathcal{C}(H^2(\mathbb{D}))$ and the compact differences of its members.

In 1989, MacCluer [12] showed that, on the weighted Bergman space $A^2_s(\mathbb{D})$ for $s \geq -1$, all the compact composition operators can be connected by paths, and she gave necessary conditions for two composition operators to have compact difference. At about the same time, Shapiro and Sundberg [17] gave further results on compact difference and isolation and they believe that the compact composition operators should form a connected component of the set $\mathcal{C}(H^2(\mathbb{D}))$.

In 2008, Gallardo-Gutiërrez and co-workers [8] showed that there exists noncompact composition operators in the component generated by all compact composition operators. In 2005, Moorhouse [13] characterized compact difference for composition operators acting on $A^2_\lambda(\mathbb{D})$, $\lambda > -1$, and gave a partial answer to the component structure of $\mathcal{C}(A^2_\lambda(\mathbb{D}))$. Later, Kriete and Moorhouse [11] extended that results to linear combinations. In 2012, Saukko [14, 15] obtained a complete characterization of bounded and compact differences between standard weighted Bergman spaces. Recently, Choe, Koo and Park [5, 6] extend Moorhouses characterization to the Bergman spaces in unit polydisk and unit ball. In 2015, Hosokawa, Izuchi and Ohno [9] investigate the topology space of weighted composition operators acting between some Hilbert spaces on $\mathbb{D}$ in general, and they also consider the Hilbert-Schmidt norm topology. Readers interested in those related topic can refer recent papers [19, 20, 21, 22] and the references therein.

Generally speaking, theory of composition operators on the spaces of holomorphic functions in the unit polydisk are far from complete. To completely characterize the boundedness and compactness of composition operators on Hardy spaces and weighted Bergman spaces is still open. In [2], Bayart showed that the study of boundedness of composition operators on the polydisk is a difficult problem, and many obstacles are caused by differences between the torus of $\mathbb{D}^N$ (distinguishing boundary of $\mathbb{D}^N$) and the whole boundary. Stessin and Zhu [18] characterized the boundedness of composition operators between different weighted Bergman spaces in the polydisk. Inspired by [2, 9, 18], we continue to investigate the topology spaces of weighted composition operators between different weighted Bergman spaces in the unit polydisk. On those spaces, we will also consider the Hilbert Schmidt topology spaces of weighed composition operators.

## 2. Preliminaries

Let $\mathrm{d}A(z) = \mathrm{d}x\mathrm{d}y/\pi$ denote the normalized area measure of $\mathbb{D}$. For $s > -1$ the weighted Hilbertian Bergman space $A^2_s(\mathbb{D}^N)$ consists of all functions

$f \in H(\mathbb{D}^N)$ such that

$$\|f\|_s^2 = \int_{\mathbb{D}^N} |f(z)|^2 \mathrm{d}v_s(z) < \infty,$$

where $\mathrm{d}v_s(z) = \mathrm{d}A_s(z_1)\cdots\mathrm{d}A_s(z_N)$ and $\mathrm{d}A_s(z_j) = (s+1)(1-|z_j|^2)^s \mathrm{d}A(z_j)$.

The inner product of $A_s^2(\mathbb{D}^N)$ is given by

$$\langle f, g\rangle_s = \int_{\mathbb{D}^N} f(z)\overline{g(z)}\mathrm{d}v_s(z),$$

where $f, g \in A_s^2$. And the reproducing kernel of $A_s^2(\mathbb{D}^N)$ is given by

$$k_{A_s^2,w}(z) = \prod_{j=1}^N \frac{1}{(1-\bar{w_j}z_j)^{2+s}}.$$

When $s = 0$, $A_0^2(\mathbb{D}^N)$ is the classical Bergman space. It is well known that

$$\|z^\alpha\|_s^2 = \int_{\mathbb{D}^N} |z^\alpha|^2 \mathrm{d}v_s(z) = \prod_{j=1}^N \int_{\mathbb{D}} |z_j^{\alpha_j}|^2 \mathrm{d}A_s(z_j) = \Gamma^N(s+1)\prod_{j=1}^N \frac{\Gamma(\alpha_j+1)}{\Gamma(\alpha_j+s+2)}$$

where $\alpha = (\alpha_1, \ldots, \alpha_N)$ and $z^\alpha = z_1^{\alpha_1}\cdots z_N^{\alpha_N}$.

We believe the following pointwise esitmate is well known, and we list it with a simple proof for the completeness.

**Lemma 2.1.** *Let $p > 0$ and $s > -1$. If $f \in A_s^p(\mathbb{D}^N)$, then*

$$|f(w)| \le \frac{\|f\|_s}{\prod_{i=1}^N (1-|w_i|^2)^{\frac{2+\alpha}{p}}}$$

*for each $w = (w_1, \ldots, w_N) \in \mathbb{D}^N$.*

*Proof.* Fixing $w_2, \ldots, w_N$, function $f(\zeta_1, w_2, \ldots, w_N)$ is analytic with respect to $\zeta_1 \in \mathbb{D}$. It is well known that

$$\begin{aligned} |f(w_1, w_2, \ldots, w_N)|^p &\le \frac{\|f(\zeta_1, w_2, \ldots, w_N)\|_s^p}{(1-|w_1|^2)^{2+s}} \\ &= \frac{\int_{\mathbb{D}} |f(\zeta_1, w_2, \ldots, w_N)|^p \mathrm{d}A_s(\zeta_1)}{(1-|w_1|^2)^{2+s}}. \end{aligned}$$

We can also estimate $|f(\zeta_1, w_2, \ldots, w_N)|^p$ by the similar inequality with respect to $w_2$, and then up to $w_N$, hence we have

$$\begin{aligned} & |f(w_1, w_2, \ldots, w_N)|^p \\ \leq & \frac{\int_{\mathbb{D}} |f(\zeta_1, w_2, \ldots, w_N)|^p \mathrm{d}A_s(\zeta_1)}{(1-|w_1|^2)^{2+s}} \\ \leq & \frac{\int_{\mathbb{D}} \int_{\mathbb{D}} |f(\zeta_1, \zeta_2, \ldots, w_N)|^p \mathrm{d}A_s(\zeta_2) \mathrm{d}A_s(\zeta_1)}{(1-|w_1|^2)^{2+s}(1-|w_2|^2)^{2+s}} \\ \cdots & \\ \leq & \frac{\int_{\mathbb{D}} \cdots \int_{\mathbb{D}} |f(\zeta_1, \zeta_2, \ldots, \zeta_N)|^p \mathrm{d}A_s(\zeta_N) \ldots \mathrm{d}A_s(\zeta_2) \mathrm{d}A_s(\zeta_1)}{(1-|w_1|^2)^{2+s} \ldots (1-|w_N|^2)^{2+s}} \end{aligned}$$

That is

$$|f(w)| \leq \frac{\|f\|_s}{\prod_{i=1}^{N} (1-|w_i|^2)^{\frac{2+\alpha}{p}}}.$$

□

It is an obvious consequence that the point evaluation $\epsilon_w : f \mapsto f(w) = f(w_1, \ldots, w_N)$ is a bounded linear functional on $A_s^p(\mathbb{D}^N)$, and

$$\max_{i=1,\ldots,N} \sup_{|w_i| \leq r} \|\epsilon_w\|_{A_s^p} < \infty$$

for every $0 < r < 1$.

If $-1 < s' < s$, we can immediately have that $A_{s'}^2 \subset A_s^2$ by a direct computation $\|f\|_s \leq C\|f\|_{s'}$ where the constant $C$ depends only on $s$ and $s'$. Let $\mathcal{C}_w(A_{s'}^2, A_s^2)$ be the space of nonzero bounded weighted composition operators from a weighted Hilbertian Bergman space $A_{s'}^2$ to another $A_s^2$ with the operator norm topology, that is,

$$\mathcal{C}_w(A_{s'}^2, A_s^2) = \{M_u C_\varphi : M_u C_\varphi : A_{s'}^2 \to A_s^2 \text{ is bounded, } u \neq 0\}.$$

And $\mathcal{C}_w(A_s^2) = \mathcal{C}_w(A_s^2, A_s^2)$. For a bounded linear operator $T : X' \to X$, we write $\|T\|_{X',X}$ its operator norm. If $-1 < s' < s$, for $M_u C_\varphi \in \mathcal{C}_w(A_s^2)$, we have

$$\|M_u C_\varphi f\|_{A_s^2} \leq \|M_u C_\varphi\|_{A_s^2} \|f\|_s \leq C \cdot \|M_u C_\varphi\|_{A_s^2} \|f\|_{s'}$$

for every $f \in A_{s'}^2$. Hence $M_u C_\varphi : A_{s'}^2 \to A_s^2$ is bounded and

$$\|M_u C_\varphi\|_{A_{s'}^2, A_s^2} \leq \|M_u C_\varphi\|_{A_s^2} \quad \text{for every } M_u C_\varphi \in \mathcal{C}_w(A_s^2). \tag{2.1}$$

Restricting $M_u C_\varphi \in \mathcal{C}_w(A_s^2)$ on $A_{s'}^2$, we may consider that $M_u C_\varphi$ is also a bounded linear operator from $A_{s'}^2$ to $A_s^2$. We note that

$$\mathcal{C}_w(A_{s'}^2, A_s^2) = \mathcal{C}_w(A_s^2)$$

as sets, and if $M_u C_\varphi \in \mathcal{C}_w(A_{s'}^2, A_s^2)$, then $u \in A_s^2$.

## 3. Operator norm topology spaces

Now we discuss the operator norm topology spaces $C_w(A^2_{s'}, A^2_s)$.

**Lemma 3.1.** *If $\varphi = (\varphi_1, \dots, \varphi_N) \in \mathcal{S}(\mathbb{D}^N)$ and $\|\varphi\|_\infty := \max\limits_{i=1,\dots,N} \|\varphi_i\|_\infty < 1$, then $C_\varphi f \in H^\infty$ for every $f \in A^2_s$ and*

$$\|C_\varphi f\|_\infty \le \|f\|_s \max_{i=1,\dots,N} \sup_{|w_i| \le \|\varphi\|_\infty} \|\epsilon_w\|_{A^2_s}.$$

*Proof.* For $f \in \mathcal{H}$ and $z \in \mathbb{D}^N$, we have

$$|(C_\varphi f)(z)| = |f(\varphi(z))| \le \|f\|_s \|\epsilon_{\varphi(z)}\|_{A^2_s} \le \|f\|_s \max_{i=1,\dots,N} \sup_{|w_i| \le \|\varphi\|_\infty} \|\epsilon_w\|_{A^2_s},$$

so we get the assertion. □

The main result is following.

**Theorem 3.2.** *If $-1 < s' < s$, then the space $\mathcal{C}_w(A^2_{s'}, A^2_s)$ is path connected.*

*Proof.* Let $M_u C_\varphi \in \mathcal{C}_w(A^2_{s'}, A^2_s)$. Since $\mathcal{C}_w(A^2_{s'}, A^2_s) = \mathcal{C}_w(A^2_s)$ as sets, we have $u \in A^2_s$ and $\|M_u C_\varphi\|_{A^2_s} < \infty$. Let $0 \le r < 1$. For $f \in A^2_s$, by Lemma 3.1 we have $f \circ r\varphi = f(r\varphi_1, \dots, r\varphi_N) \in H^\infty$ and by Lemma 2.1,

$$\begin{aligned} \|M_u C_{r\varphi} f\|_s &= \|u(f \circ r\varphi)\|_s \le \|f \circ r\varphi\|_\infty \|u\|_s \\ &\le \|u\|_s \|f\|_s \max_{i=1,\dots,N} \sup_{|w_i| \le r} \|\epsilon_w\|_{A^2_s}. \end{aligned}$$

Hence $M_u C_{r\varphi} \in \mathcal{C}_w(A^2_s)$, so $M_u C_{r\varphi} \in \mathcal{C}_w(A^2_{s'}, A^2_s)$.

Fixing $0 \le t_0 \le 1$, we apply the similar method in [9] to show that $\|M_u C_{t_0\varphi} - M_u C_{t\varphi}\|_{A^2_{s'}, A^2_s} \to 0$ as $t \to t_0$. Let $g(z) = \sum_\alpha c_\alpha z^\alpha \in A^2_{s'}$. For each $0 \le t \le 1$, let

$$g_t(z) = \sum_\alpha c_\alpha (t_0^{|\alpha|} - t^{|\alpha|}) z^\alpha.$$

Since $A^2_{s'} \subset A^2_s$, we have $g \in A^2_s$. Note that

$$\begin{aligned} \|g_t\|_s^2 &= \langle g_t, g_t \rangle_s = \int_{\mathbb{D}^N} \sum_\alpha c_\alpha (t_0^{|\alpha|} - t^{|\alpha|}) z^\alpha \overline{\sum_\alpha c_\alpha (t_0^{|\alpha|} - t^{|\alpha|}) z^\alpha} \mathrm{d}v_s(z) \\ &= \sum_\alpha (t_0^{|\alpha|} - t^{|\alpha|})^2 \int_{\mathbb{D}^N} |c_\alpha|^2 |z|^\alpha \mathrm{d}v_s(z) \\ &\le 4 \sum_\alpha \int_{\mathbb{D}^N} |c_\alpha|^2 |z|^\alpha \mathrm{d}v_s(z) = 4\|g\|_s^2, \end{aligned}$$

we have $g_t \in A_s^2$. Hence

$$\begin{aligned}
& \|(M_u C_{t_0\varphi} - M_u C_{t\varphi})g\|_s^2 \\
= & \left\| u \sum_\alpha c_\alpha (t_0^{|\alpha|} - t^{|\alpha|})\varphi^\alpha \right\|_s^2 \\
= & \|M_u C_\varphi g_t\|_s^2 \le \|M_u C_\varphi\|_{A_s^2}^2 \|g_t\|_s^2 \\
= & \|M_u C_\varphi\|_{A_s^2}^2 \sum_\alpha |c_\alpha|^2 |t_0^{|\alpha|} - t^{|\alpha|}|^2 \|z^\alpha\|_s^2 \\
\le & \|M_u C_\varphi\|_{A_s^2}^2 \sup_{|\alpha|>0} \left( |t_0^{|\alpha|} - t^{|\alpha|}|^2 \frac{\|z^\alpha\|_s^2}{\|z^\alpha\|_{s'}^2} \right) \sum_\alpha |c_\alpha|^2 \|z^\alpha\|_{s'}^2 \\
\le & \|M_u C_\varphi\|_{A_s^2}^2 \sup_{|\alpha|>0} \left( |t_0^{|\alpha|} - t^{|\alpha|}| \frac{\|z^\alpha\|_s}{\|z^\alpha\|_{s'}} \right)^2 \|g\|_{s'}^2.
\end{aligned}$$

Then

$$\|M_u C_{t_0\varphi} - M_u C_{t\varphi}\|_{A_{s'}^2, A_s^2} \le \|M_u C_\varphi\|_{A_s^2} \sup_{|\alpha|>0} \left( |t_0^{|\alpha|} - t^{|\alpha|}| \frac{\|z^\alpha\|_s}{\|z^\alpha\|_{s'}} \right).$$

For any positive integer $n$, we have

$$\sup_{|\alpha|>0} \left( \left| t_0^{|\alpha|} - t^{|\alpha|} \right| \frac{\|z^\alpha\|_s}{\|z^\alpha\|_{s'}} \right) \le \sum_{|\alpha|<n^2} \left( \left| t_0^{|\alpha|} - t^{|\alpha|} \right| \frac{\|z^\alpha\|_s}{\|z^\alpha\|_{s'}} \right) + 2 \sup_{|\alpha|\ge n^2} \frac{\|z^\alpha\|_s}{\|z^\alpha\|_{s'}}.$$

Hence

$$\limsup_{t\to t_0} \|M_u C_{t_0\varphi} - M_u C_{t\varphi}\|_{A_{s'}^2, A_s^2} \le 2\|M_u C_\varphi\|_{A_s^2} \sup_{|\alpha|\ge n^2} \frac{\|z^\alpha\|_s}{\|z^\alpha\|_{s'}}. \tag{3.1}$$

According to the Stirling's approximation, we have

$$\begin{aligned}
\sup_{|\alpha|\ge n^2} \frac{\|z^\alpha\|_s}{\|z^\alpha\|_{s'}} &= \sup_{|\alpha|\ge n^2} \frac{\Gamma^N(s+1) \prod_{j=1}^N \frac{\Gamma(\alpha_j+1)}{\Gamma(\alpha_j+s+2)}}{\Gamma^N(s'+1) \prod_{j=1}^N \frac{\Gamma(\alpha_j+1)}{\Gamma(\alpha_j+s'+2)}} \\
\le & \; C \cdot \sup_{|\alpha|\ge n^2} \prod_{j=1}^N \left( \frac{\alpha_j + s' + 2}{\alpha_j + s + 2} \right)^{\alpha_j+s'+2} \cdot \left( \frac{1}{\alpha_j + s + 2} \right)^{s-s'}
\end{aligned}$$

Since $|\alpha| \ge n$, there is at least one $\alpha_j \ge n$. Considering that

$$\left( \frac{\alpha_j + s' + 2}{\alpha_j + s + 2} \right)^{\alpha_j+s'+2}$$

is bounded for $j = 1, \ldots, N$, we get $M_u C_{t\varphi} \to M_u C_{t_0\varphi}$ as $t \to t_0$ in $\mathcal{C}_w(A_{s'}^2, A_s^2)$ by letting $n \to \infty$ in (3.1).

□

## 4. Hilbert-Schmidt norm topology spaces

We will consider the topology spaces of Hilbert-Schmidt weighted composition operators. If $X$ is a separable Hilbert spaces with orthonormal bases $\{e_m\}$ and $X'$ is another Hilbert space, recall that a linear operator $T : X \to X'$ is said to be Hilbert-Schmidt if

$$\|T\|^2_{X,X',HS} := \sum_{m=0}^{\infty} \|Te_m\|^2_{X'} \leq \infty$$

Since $\{z^\alpha/\|z^\alpha\|_s\}$ is an orthonomal bases of $A^2_s$, $M_uC_\varphi \in \mathcal{C}_w(A^2_s)$ is Hilbert-Schmidt if and only if

$$\|M_uC_\varphi\|^2_{A^2_s,HS} := \sum_{|\alpha|\geq 0} \frac{\|M_uC_\varphi(z^\alpha)\|^2_s}{\|z^\alpha\|^2_s} = \sum_{|\alpha|\geq 0} \frac{\|u\varphi_1^{\alpha_1}\cdots\varphi_N^{\alpha_N}\|^2_s}{\|z_1^{\alpha_1}\cdots z_N^{\alpha_N}\|^2_s} < \infty.$$

We denote by $\mathcal{C}_{w,HS}(A^2_s)$ the space of Hilbert-Schmidt operators $M_uC_\varphi$ in $\mathcal{C}_w(A^2_s)$ with the Hilbert-Schmidt norm topology.

We have that $M_uC_\varphi \in \mathcal{C}_w(A^2_{s'}, A^2_s)$ is Hilbert-Schmidt if and only if

$$\|M_uC_\varphi\|^2_{A^2_{s'},A^2_s,HS} := \sum_{\alpha} \frac{\|u\varphi^\alpha\|^2_s}{\|z^\alpha\|^2_{s'}} < \infty.$$

We denote by $\mathcal{C}_{w,HS}(A^2_{s'}, A^2_s)$ the space of $M_uC_\varphi \in \mathcal{C}_w(A^2_{s'}, A^2_s)$ which are Hilbert-Schmidt operators. We consider the Hilbert-Schmidt norm topology on $\mathcal{C}_{w,HS}(A^2_{s'}, A^2_s)$. The topology on $\mathcal{C}_{w,HS}(A^2_{s'}, A^2_s)$ is stronger than the operator norm one. So a path connected set in $\mathcal{C}_{w,HS}(A^2_{s'}, A^2_s)$ is so in $\mathcal{C}_w(A^2_{s'}, A^2_s)$. Since

$$\|M_uC_\varphi\|^2_{A^2_{s'},A^2_s,HS} \leq C\sum_{\alpha} \frac{\|u\varphi^\alpha\|^2_s}{\|z^\alpha\|^2_s} = C\|M_uC_\varphi\|^2_{A^2_s,HS},$$

we have $\mathcal{C}_{w,HS}(\mathcal{H}) \subset \mathcal{C}_{w,HS}(A^2_{s'}, A^2_s)$. Next lemma will be employed in the main result of this section.

**Lemma 4.1.** *Let $s', s > -1$. For each $0 \leq r \leq 1$, if the operator $M_uC_\varphi : A^2_{s'} \to A^2_s$ is a Hilbert-Schmidt operator for some nonzero $u \in A^2_s$ and $\varphi \in \mathcal{S}(\mathbb{D}^N)$, then $M_uC_{r\varphi}$ is also a Hilbert-Schmidt operator from $A^2_{s'}$ to $A^2_s$.*

*Proof.* The Lemma follows immediately from the computations that

$$\|M_uC_{r\varphi}\|^2_{A^2_{s'},A^2_s,HS} = \sum_{\alpha\geq 0} \frac{\|M_uC_{r\varphi}(z^\alpha)\|^2_s}{\|z^\alpha\|^2_{s'}}$$

$$= \sum_{\alpha\geq 0} r^{2|\alpha|} \frac{\|u\varphi^\alpha\|^2_s}{\|z^\alpha\|^2_{s'}} \leq \|M_uC_\varphi\|^2_{A^2_{s'},A^2_s,HS} < \infty.$$

□

**Theorem 4.2.** *If $-1 < s' < s$, then the topology space $\mathcal{C}_{w,HS}(A^2_{s'}, A^2_s)$ is path connected.*

*Proof.* If $M_uC_\varphi \in \mathcal{C}_{w,HS}(A^2_{s'}, A^2_s)$, recall that $\{z^\alpha/\|z^\alpha\| : \alpha\}$ is an orhonomal basis, and we have

$$\|M_uC_\varphi\|^2_{A^2_{s'},A^2_s,HS} = \sum_{|\alpha|\geq 0} \frac{\|u\varphi^\alpha\|^2_s}{\|z^\alpha\|^2_{s'}} < \infty. \tag{4.1}$$

Besides, $M_uC_{t\varphi} \in \mathcal{C}_{w,HS}(A^2_{s'}, A^2_s)$ for every $0 \leq t \leq 1$ by Lemma 4.1.

If we fix $0 \leq t_0 \leq 1$. we can also prove $\|M_uC_{t_0\varphi} - M_uC_{t\varphi}\|_{A^2_{s'},A^2_s,HS} \to 0$ as $t \to t_0$ as following statements. For any positive integer $N$, we have

$$\begin{aligned}\|M_uC_{t_0\varphi} - M_uC_{t\varphi}\|^2_{A^2_{s'},A^2_s,HS} &= \sum_{|\alpha|\geq 0} \frac{\|u(t_0^{|\alpha|} - t^{|\alpha|})\varphi^\alpha\|^2_s}{\|z^\alpha\|^2_{s'}} \\ \leq \sum_{|\alpha|\leq N} \left|t_0^{|\alpha|} - t^{|\alpha|}\right|^2 \frac{\|u\varphi^\alpha\|^2_s}{\|z^\alpha\|^2_{s'}} &+ \sum_{|\alpha|>N} \frac{\|u\varphi^\alpha\|^2_s}{\|z^\alpha\|^2_{s'}}.\end{aligned}$$

Take $\varepsilon > 0$ arbitrary. Then by (4.1), we may take $N$ large enough so that

$$\sum_{|\alpha|>N} \frac{\|u\varphi^\alpha\|^2_s}{\|z^\alpha\|^2_{s'}} < \varepsilon.$$

Hence

$$\|M_uC_{t_0\varphi} - M_uC_{t\varphi}\|^2_{A^2_{s'},A^2_s,HS} < \varepsilon + \sum_{|\alpha|\leq N} \left|t_0^{|\alpha|} - t^{|\alpha|}\right|^2 \frac{\|u\varphi^\alpha\|^2_s}{\|z^\alpha\|^2_{s'}}$$

By letting $t \to t_0$, we have

$$\limsup_{t\to t_0} \|M_uC_{t_0\varphi} - M_uC_{t\varphi}\|^2_{A^2_{s'},A^2_s,HS} < \varepsilon.$$

Let $\varepsilon \to 0$ then, we have the topology space $\mathcal{C}_{w,HS}(A^2_{s'}, A^2_s)$ is path connected. □

## 5. Final remarks

**Lemma 5.1.** *Let $s > -1$. If $\|\varphi\|_\infty < 1$ and $u \in A^2_s$, then $M_uC_\varphi \in \mathcal{C}_w(A^2_s)$ and is compact.*

*Proof.* By the first paragraph of the proof in Theorem 3.2, we have $M_uC_\varphi \in \mathcal{C}_w(A^2_s)$.

To show that $M_uC_\varphi$ is compact, let $\{f_n\}$ be a sequence in $A^2_s$ such that there is a positive constant $K$ satisfying $\|f_n\|_s < K$ for every $n$. By the pointwise estimate (Lemma 2.1), we may assume that $f_n$ converges to some $f \in H(\mathbb{D}^N)$ uniformly on compact subsets of $\mathbb{D}^N$. By the assumption, $f_n \circ \varphi \to f \circ \varphi$ in $H^\infty$. Hence both $u(f_n \circ \varphi)$ and $u(f \circ \varphi)$ are in $A^2_s$, and

$$\|M_uC_\varphi f_n - u(f \circ \varphi)\|_s \leq \|u\|_s \|f_n \circ \varphi - f \circ \varphi\|_\infty \to 0, \quad n \to \infty.$$

Thus $M_uC_\varphi \in \mathcal{C}_w(A^2_s)$ is compact. □

**Proposition 5.2.** *If $-1 < s' < s$, then $\mathcal{C}_w(A^2_{s'}, A^2_s)$ consists of all compact weighted composition operators.*

*Proof.* For $0 < r < 1$, by Lemma 5.1 $M_u C_{r\varphi} \in \mathcal{C}_w(A^2_s)$ is compact. Hence $M_u C_{r\varphi} : A^2_s \to A^2_s$, which can be regarded as the composition of $\mathrm{id} : A^2_{s'} \to A^2_s$ and $M_u C_{r\varphi} : A^2_s \to A^2_s$, is compact. Since the algebra of compact operators is closed in norm topology, we get $M_u C_\varphi$ is compact since it can be approximated by compact operators $M_u C_{r\varphi}$ by Theorem 3.2. □

We note that on many spaces, the compact (weighted) composition operators form a path connected subset in the topology space of bounded (weighted) composition operators. The compactness has played an important role in the proof of the main result.

## References


[1] R. F. Allen, K. C. Heller and M. A. Pons, Compact differences of composition operators on weighted Dirichlet spaces, Cent. Eur. J. Math., 12(7), 1040-1051(2014).

[2] F. Bayart, Composition oeprators on the polydisk induced by affine maps, J. Funct. Anal., 260, 1969-2003(2011).

[3] E. Berkson, Composition opertors isolated in the uniform operator topology, Proc. Amer. Math. Soc., 81, 230-232(1981).

[4] B. Berndtsson, Interpolating sequences for $H^\infty$ in the ball, Indag. Math. (Proc.), 88(1), 1-10(1985).

[5] B. R. Choe, H. Koo and I. Park, Compact differences of composition operators over polydisk. Integr. Equ. Oper. Theory, 73, 57-91(2012).

[6] B. R. Choe, H. Koo and I. Park, Compact differences of composition operators on the Bergman spaces over the Ball, Potential Anal., 40, 81-102(2014).

[7] C. Cowen and B.D. MacCluer, Composition Operators on Spaces of Analytic Functions, CRC Press, Boca Raton, 1995.

[8] E. A. Gallardo-Gutiérrez, M. J. González, P. J. Nieminen and E. Saksman, On the connected component of compact composition operators on the Hardy space, Advances in Mathematics, 219, 986-1001(2008).

[9] T. Hosokawa, K. J. Izuchi, and S. Ohno, Weighted composition operators between Hilbert spaces of analytic functions in the operator norm and Hilbert-Schmidt norm topologies, J. Math. Anal. Appl., 421, 1546-1558(2015).

[10] K. Heller, B. MacCluer, R. Weir, Compact differences of composition operators in several variables. Integr. Equ. Oper. Theory, 69, 247-268(2011).

[11] T. Kriete and J. Moorhouse, Linear relations in the Calkin algebra for composition operators, Trans. Amer. Math. Soc., 359, 2915-2944(2007).

[12] B. D. MacCluer, Components in the space of composition operators, Integral Equations Operator Theory, 12, 725-738(1989).

[13] J. Moorhouse, Compact differences of composition operators, Journal of Functional Analysis, 219, 70-92(2005).

[14] E. Saukko, Difference of composition operators between standard weighted Bergman spaces, J. Math. Anal. Appl., 381, 789-798(2011).

[15] E. Saukko, An application of atomic decomposition in Bergman spaces to the study of differences of composition operators, 262, 3872-3890(2012).

[16] J. H. Shapiro, Composition Operators and Classical Function Theory, Springer Verlag, New York, 1993.

[17] J. H. Shapiro and C. Sundberg, Isolation amongst the composition operators, Pacific J. Math., 145, 117-152(1990).
[18] M. Stessin and K. Zhu, Composition operators induced by symbols defined on a polydisk, J. Math. Anal. Appl., 319, 815-829(2006).
[19] C. Tong, and Z. Zhou, The compactness of the sum of weighted composition operators on the ball algebra, J. Ineq. Appl., 2011(45), 1-10(2011).
[20] C. Z. Tong and Z. H. Zhou, Intertwining relations for Volterra operators on the Bergman space, Illinois J. Math. 57(1), 195-211(2013).
[21] C. Z. Tong and Z. H. Zhou, Compact intertwining relations for composition operators between the weighted Bergman spaces and the weighted Bloch spaces, J. Korean Math. Soc., 51(1), 125-135(2014).
[22] C. Tong, C. Yuan and Z. Zhou, Topological structures of derivative weighted composition operators on the Bergman space, J. Function Spaces, 2015(8), 1-8(2015).
[23] K. Zhu, Operator Theory in Function Spaces, Marcel Dekker. Inc, New York, 1990.
[24] K. Zhu, Spaces of Holomorphic Functions in the Unit Ball. Grad. Texts in Math, Springer, 2005.

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# On the $L_\infty$ convergence of a nonlinear difference scheme for Schrödinger equations

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**Abstract**

In this article, a nonlinear difference scheme for Schrödinger equations is studied. The existence of the difference solution is proved by Brouwer fixed point theorem. With the aid of the fact that the difference solution satisfies two conservation laws, the difference solution is proved to be bounded in the $L_\infty$ norm. Then, the difference solution is shown to be unique and second order convergent in the $L_\infty$ norm. Finally, a convergent iterative algorithm is presented.

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**Keywords:** Schrödinger equations, Nonlinear difference scheme, Solvability, Convergence

## 1 Introduction

The Schrödinger equation is one of the most important equations in quantum mechanics. This model equation also arises in many other branches of science and technology, e.g. optics, seismology and plasma physics. Recently, a growing interest is on the numerical solution to the nonlinear Schrödinger equations. Many authors investigated the finite difference methods for solving this kind of equations, including the conservation, solvability, stability, convergence and the symplectic geometry (see [1] − [8]).

Consider nonlinear Schrödinger equations

$$i\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial x^2}+q|u|^2u=0,\quad 0<x<1, 0<t\le T, \tag{1.1}$$

$$u(x,0)=\varphi(x),\qquad 0\le x\le 1, \tag{1.2}$$

$$u(0,t)=0,\ u(1,t)=0,\qquad 0<t\le T, \tag{1.3}$$

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where $q$ is a real constant, $\varphi(x)$ is a known function and $\varphi(0)=\varphi(1)=0, u(x,t)$ is an unknown function. Take two positive integers $m$ and $n$. Denote $h=1/m, \tau=T/n$, so we have

$$x_j=jh,\ 0\le j\le m,\ t_k=k\tau,\ 0\le k\le n.$$

Denote $\Omega_{h\tau}=\{(x_j,t_k)\mid 0\le j\le m, 0\le k\le n\}$. Suppose $u=\{u_j^k\mid 0\le j\le m, 0\le k\le n\}$ be a discrete grid function on $\Omega_{h\tau}$. Introduce the following notations:

$$u_j^{k+\frac{1}{2}}=\frac{1}{2}\left(u_j^{k+1}+u_j^k\right),\ \delta_t u_j^{k+\frac{1}{2}}=\frac{1}{\tau}\left(u_j^{k+1}-u_j^k\right),\ D_t u_j^k=\frac{1}{2\tau}\left(u_j^{k+1}-u_j^{k-1}\right),$$

$$\delta_x u_{j+\frac{1}{2}}^k=\frac{1}{h}\left(u_{j+1}^k-u_j^k\right),\ \delta_x^2 u_j^k=\frac{1}{h}\left(\delta_x u_{j+\frac{1}{2}}^k-\delta_x u_{j-\frac{1}{2}}^k\right).$$

The author of [9] developed the following nonlinear difference scheme for (1.1)-(1.3)

$$\begin{aligned} i\delta_t u_j^{k+\frac{1}{2}}+\delta_x^2 u_j^{k+\frac{1}{2}}+\tfrac{q}{2}\left(\left|u_j^k\right|^2+\left|u_j^{k+1}\right|^2\right)u_j^{k+\frac{1}{2}}=0,\\ 1\le j\le m-1, 0<k\le n-1, \end{aligned} \tag{1.4}$$

$$u_j^0=\varphi(x_j),\quad 1\le j\le m-1, \tag{1.5}$$

$$u_0^k=0, u_m^k=0,\qquad 0<k\le n. \tag{1.6}$$

The contents in [9] pointed out that the difference scheme preserves the densities and the energy of the solution, and the author also proved that the difference scheme is uniquely solvable and convergent with the convergence order of $(\tau^2+h^2)$ in $L_2$ norm under some constraints on the stepsizes. On this basis, we proof further that this difference scheme is convergent with the convergence order of $(\tau^2+h^2)$ in $L_\infty$ norm.

In this paper, we will analyze the difference scheme (1.4)-(1.6). The remainder of the paper is arranged as follows. In Section 2, the existence of the difference solution is shown by the Brouwer fixed point theorem. Then with the aid of the conversations of the difference solution, the boundedness and uniqueness of difference solution are proved. In Section 3, the convergence of the difference scheme is discussed. The difference scheme is proved to be convergent with the convergence order of $\mathbf{O}(\tau^2+h^2)$ in $L_\infty$ norm. In Section 4, an iterative algorithm for the difference scheme with the proof of the convergence is given. A short conclusion section ends the paper.

## 2 The existence of the difference solution

In this section, we will prove that the finite difference scheme (1.4)-(1.6) exists a solution.

Let $H = \{v \mid v = (v_0, v_1, \ldots, v_m), v_j \in C, 0 \le j \le m, v_0 = v_m = 0\}$ be the space of complex grid functions on $\Omega_h$. Given any complex grid functions $v, w \in H$, denote the inner product

$$(v, w) = h \sum_{j=1}^{m-1} v_j \bar{w}_j.$$

The discrete $L_p$ norm $\|\cdot\|_p$, maximum-norm $\|\cdot\|_\infty$ and $H_0^1$ norm $|\cdot|_1$ are defined respectively as follows

$$\|v\|_p = \sqrt[p]{h \sum_{j=1}^{m-1} |v_j|^p}, p \ge 1, \quad \|v\|_\infty = \max_{0 \le i \le m} |v_i|,$$

$$|v|_1 = \sqrt{h \sum_{j=0}^{m-1} \left| \frac{v_{j+1} - v_j}{h} \right|^2}.$$

For abbreviation, we write $\|\cdot\|_2$ as $\|\cdot\|$.

In order to illustrate the existence of the difference solution, we need the following lemma.

**Lemma 2.1.** *(Brouwer Fixed Point Theorem [10]) Let $(H, (\cdot, \cdot))$ be a finite dimensional inner product space, $\|\cdot\|$ the associated norm, and $\Pi : H \to H$ be continuous. Assume moreover that*

$$\exists \alpha > 0, \ \forall z \in H, \ \|z\| = \alpha, \ Re(\Pi(z), z) \ge 0.$$

*Then, there exists an element $z^* \in H$ such that $\Pi(z^*) = 0$ and $\|z^*\| \le \alpha$.*

**Theorem 2.2.** *The solution of difference scheme (1.4) − (1.6) exists.*

*Proof.* Suppose $\left\{u_j^k \mid 0 \le j \le m\right\}$ be the numerical solution. Using the notation introduced before, we rewrite the difference scheme (1.4)-(1.6) in the following form

$$i\tfrac{2}{\tau}\left(u_j^{k+\frac{1}{2}} - u_j^k\right) + \delta_x^2 u_j^{k+\frac{1}{2}} + \tfrac{q}{2}\left(\left|u_j^k\right|^2 + \left|2u_j^{k+\frac{1}{2}} - u_j^k\right|^2\right) u_j^{k+\frac{1}{2}} = 0, \quad 1 \le j \le m-1, \tag{2.1}$$

$$u_0^{k+\frac{1}{2}} = 0, \ u_m^{k+\frac{1}{2}} = 0. \tag{2.2}$$

Let

$$v_j = u_j^{k+\frac{1}{2}}, \ 0 \le j \le m,$$

then (2.1)-(2.2) can be written as

$$i\frac{2}{\tau}\left(v_j - u_j^k\right) + \delta_x^2 v_j + \frac{q}{2}\left(|u_j^k|^2 + |2v_j - u_j^k|^2\right) v_j = 0, \ 1 \le j \le m-1, \tag{2.3}$$

$$v_0 = 0,\ v_m = 0. \tag{2.4}$$

Define the mapping $\Pi : H \to H$,

$$(\Pi(v))_j = \begin{cases} 0, & j = 0, m, \\ v_j - u_j^k - i\frac{2}{\tau}\left[\delta_x^2 v_j + \frac{q}{2}(|u_j^k|^2 + |2v_j - u_j^k|^2)v_j\right], & 1 \le j \le m-1. \end{cases}$$

Computing the inner product of $\Pi(v)$ and $v$, we obtain

$$\begin{aligned} (\Pi(v), v) &= h\sum_{j=1}^{m-1}\left[v_j - u_j^k - i\tfrac{\tau}{2}\delta_x^2 v_j - i\tfrac{\tau q}{4}(|u_j^k|^2 + |2v_j - u_j^k|^2)v_j\right]\bar{v}_j \\ &= \|v\|^2 - (u^k, v) - i\tfrac{\tau}{2}h\sum_{j=1}^{m-1}\left(\delta_x^2 v_j\right)\bar{v}_j - i\tfrac{\tau q}{4}\left(|u_j^k|^2 + |2v_j - u_j^k|^2\right)|v_j|^2 \\ &= \|v\|^2 - (u^k, v) - i\tfrac{\tau}{2}h\sum_{j=1}^{m}\left|\delta_x v_{j-\frac{1}{2}}\right|^2 - i\tfrac{\tau q}{4}\left(|u_j^k|^2 + |2v_j - u_j^k|^2\right)|v_j|^2. \end{aligned}$$

So taking the real part of the inner product

$$\begin{aligned} Re(\Pi(v), v) &= \|v\|^2 - Re(u^k, v) \\ &= \|v\|^2 - Re\left(h\sum_{j=1}^{m-1} u_j^k \bar{v}_j\right) \\ &\ge \|v\|^2 - Re\left[\tfrac{h}{2}\left(\sum_{j=1}^{m-1} u_j^k\right)^2 + \left(\sum_{j=1}^{m-1}\bar{v}_j\right)^2\right] \\ &= \|v\|^2 - \tfrac{1}{2}\left(\|u^k\|^2 + \|v\|^2\right) \\ &= \tfrac{1}{2}\left(\|v\|^2 - \|u^k\|^2\right) \end{aligned}$$

When $\|v\| = \|u^k\|$, $Re(\Pi(v), v) \ge 0$. Using Lemma 2.1, we have $\forall v \in H, \|v\| = \|u^{k+\frac{1}{2}}\| = \|u^k\| > 0, Re(\Pi(v), v) \ge 0$. Then there exists an element $v^* \in H$ such that $\Pi(v^*) = 0$ and $\|v^*\| \le \|u^k\|$. Hence, it is easily seen that the solution $\{v_j \mid 0 \le j \le m\}$ satisfies the difference scheme (2.3)-(2.4). The proof is complete. □

# 3 The uniqueness of the difference solution

**Theorem 3.1.** *([9])The solution of difference scheme* $(1.4) - (1.6)$ *is conservative. In more precisely, let* $\left\{u_j^k \mid 0 \le j \le m, 0 \le k \le n\right\}$ *be the solution of* $(1.4) - (1.6)$*, we have*

$$\|u^k\|^2 = \|u^0\|^2,\ E^k = E^0,\ 1 \le k \le n,$$

*where*

$$E^k = h\sum_{j=0}^{m-1}\left|\delta_x u_{j+\frac{1}{2}}^k\right|^2 - \frac{q}{2}h\sum_{j=1}^{m-1}\left|u_j^k\right|^4.$$

**Theorem 3.2.** *([9])The difference solution of* $(1.4)-(1.6)$ *is bounded in the discrete* $L_\infty$ *norm. In more precisely, let* $\left\{u_j^k \mid 0\le j\le m, 0\le k\le n\right\}$ *be the solution of* $(1.4)-(1.6)$*, we have*

$$\|u^k\|_\infty^2 \le \frac{1}{2}\left(\frac{1}{8}|q|^2\|u^0\|^6 + |u^0|_1^2 - \frac{q}{2}\|u^0\|_4^4\right),\ 1\le k\le n,$$

Using Theorems 3.1 and 3.2, we can obtain

**Theorem 3.3.** *The difference solution of* $(1.4)-(1.6)$ *is unique.*

*Proof.* To proof this theorem, we can prove the solution of difference scheme (2.3)-(2.4) is unique.

Let $\{v_j \mid 0\le j\le m\}$ and $\{w_j \mid 0\le j\le m\}$ be the solutions of (2.3)-(2.4). Then we have

$$i\frac{2}{\tau}(w_j-u_j^k)+\delta_x^2 w_j+\frac{q}{2}\left(|u_j^k|^2+|2w_j-u_j^k|^2\right)w_j=0, 1\le j\le m-1, \quad (3.1)$$

$$w_0=0,\ w_m=0. \quad (3.2)$$

Denote

$$\theta_j = v_j - w_j, 0\le j\le m.$$

Subtracting (3.1)-(3.2) from (2.3)-(2.4) respectively, we obtain the following equations

$$\begin{aligned} &i\tfrac{2}{\tau}\theta_j+\delta_x^2\theta_j+\tfrac{q}{2}|u_j^k|^2\theta_j+\tfrac{q}{2}\left(|2v_j-u_j^k|^2v_j-|2w_j-u_j^k|^2w_j\right)=0,\\ &\qquad\qquad 1\le j\le m-1, \end{aligned} \quad (3.3)$$

$$\theta_0=0,\ \theta_m=0. \quad (3.4)$$

Multiplying (3.3) by $-ih\bar{\theta}_j$, summing up for $j$ from 1 to $m-1$, we can obtain

$$\begin{aligned} &\tfrac{2}{\tau}h\sum_{j=1}^{m-1}|\theta_j|^2 - ih\sum_{j=1}^{m-1}\left(\delta_x^2\theta_j\right)\bar{\theta}_j - i\tfrac{q}{2}h\sum_{j=1}^{m-1}|u_j^k|^2|\theta_j|^2\\ &-i\tfrac{q}{2}h\sum_{j=1}^{m-1}\left(|2v_j-u_j^k|^2v_j-|2w_j-u_j^k|^2w_j\right)\bar{\theta}_j=0. \end{aligned} \quad (3.5)$$

Adding the term $-|2v_j-u_j^k|^2w_j$ to the part of the forth term in (3.5). Meanwhile, noticing the equality $|a|^2-|b|^2=(a-b)\bar{a}+b\overline{(a-b)}$ where both $a$ and $b$ are complex functions, we have

$$\begin{aligned} & |2v_j-u_j^k|^2v_j-|2w_j-u_j^k|^2w_j\\ = & |2v_j-u_j^k|^2(v_j-w_j)+(|2v_j-u_j^k|^2-|2w_j-u_j^k|^2)w_j\\ = & |2v_j-u_j^k|^2\theta_j+\left[2(v_j-w_j)\overline{2v_j-u_j^k}+2\overline{(v_j-w_j)}(2w_j-u_j^k)\right]w_j\\ = & |2v_j-u_j^k|^2\theta_j+2\left[\theta_j\overline{2v_j-u_j^k}+\bar{\theta}_j(2w_j-u_j^k)\right]w_j. \end{aligned}$$

Thus, taking the real part of (3.5) with the rewritten forth term, we obtain

$$\frac{2}{\tau}\|\theta\|^2 - Im\left\{\frac{q}{2}h\sum_{j=1}^{m-1}2\left[|\theta_j|^2\overline{(2v_j-u_j^k)}+(2w_j-u_j^k)w_j\right]\right\}=0.$$

According to Theorem 3.2 which illustrates that the solution $v, w$ are boundedness, we easily get that $|v|, |w| \le \|u^k\|_\infty$. So using Theorem 3.2 and Cauchy-Schwarz inequality, we have

$$\begin{aligned}\tfrac{2}{\tau}\|\theta\|^2 &\le |q|h\sum_{j=1}^{m-1}\left(|2v_j-u_j^k|+|2w_j-u_j^k|\right)|\theta_j|^2|w_j| \\ &\le |q|h\sum_{j=1}^{m-1}\left(2|v_j|+|u_j^k|+2|w_j|+|u_j^k|\right)\|\theta\|^2|w_j| \\ &\le 6|q|\cdot\|u^k\|_\infty^2\|\theta\|^2\end{aligned}$$

Denote the right term of the inequality in Theorem 3.2 be a constant $c_1$, we have

$$\frac{2}{\tau}\|\theta\|^2 \le 6c_1^2|q|\cdot\|\theta\|^2.$$

When $\tau < \frac{1}{3c_1^2|q|}$, we get $\|\theta\|^2 = 0$. Hence, $v_j = w_j, 0 \le j \le m$. The proof is complete. □

# 4 The convergence of the finite difference scheme

Suppose that the continuous problem (1.1)-(1.3) has a smooth solution $u$, and $U_j^k = \{u(x_j, t_k) \mid 0 \le j \le m, 0 \le k \le n\}$ is the solution $u$ under the mapping $\Omega_{h\tau}$. In this section, we will illustrate that the solution $u_j^k$ of the difference scheme (1.4)-(1.6) is convergent to the solution $U_j^k$ with the convergence order of $\mathbf{O}(\tau^2+h^2)$ in the $L_\infty$ norm.

Denote

$$c_0 = \max_{0\le t\le T}\|u(\cdot,t)\|_\infty, \tag{4.1}$$

$$e_j^k = U_j^k - u_j^k,\ 0 \le j \le m,\ 0 \le k \le n.$$

**Lemma 4.1.** *(Gronwall Inequality [9]) Assume $\{G^n \mid n \ge 0\}$ is a nonnegative sequence, and satisfies*

$$G^{n+1} \le (1+c\tau)G^n + \tau g,\ n = 0, 1, 2, \ldots,$$

*where $c$ and $g$ are nonnegative constants. Then $G$ satisfies*

$$G^n \le e^{cn\tau}\left(G^0 + \frac{g}{c}\right),\ n = 0, 1, 2, \ldots.$$

**Lemma 4.2.** *([11]) For any complex functions $U, V, u, v$, one has*

$$|\,|U|^2V - |u|^2v| \le (max\,\{|U|, |V|, |u|, |v|\})^2\cdot(2|U-u| + |V-v|).$$

**Lemma 4.3.** *([9]) Denote*

$$V_h = \{v \mid v = \{v_i \mid 0 \le i \le m\} \text{ is the grid function on } \Omega_h\},$$
$$\dot{V}_h = \{v \mid v = \{v_i \mid 0 \le i \le m\} \in V_h, v_0 = v_m = 0\}.$$

*(1) Suppose $v \in \dot{V}_h$, so there is*

$$\|v\|_\infty \le \frac{1}{2}|v|_1.$$

*(2) Suppose $v \in \dot{V}_h$. For any $\varepsilon > 0$, there is*

$$\|v\|_\infty^2 \le \varepsilon |v|_1^2 + \frac{1}{4\varepsilon}\|v\|^2.$$

**Theorem 4.4.** *Suppose that the difference scheme $(1.4)-(1.6)$ has the solution $u_j^k$ and the equations $(1.1)-(1.3)$ has the solution $U_j^k$. When $\tau$ is small enough, there exists a constant $C$ independent of $h, \tau$ such that*

$$\|e^k\|_\infty \le C(\tau^2 + h^2),\ 0 \le k \le n. \tag{4.2}$$

*Proof.* Subtracting(1.4)-(1.6) from (1.1)-(1.3) respectively, we obtain the error equations

$$i\delta_t e_i^{k+\frac{1}{2}} + \delta_x^2 e_i^{k+\frac{1}{2}} + \frac{q}{2}\left[(|U_j^k|^2 + |U_j^{k+1}|^2)U_j^{k+\frac{1}{2}} - (|u_j^k|^2 + |u_j^{k+1}|^2)u_j^{k+\frac{1}{2}}\right] = R_j^k,\ 1 \le j \le m-1,\ 0 \le k \le n-1, \tag{4.3}$$

$$e_j^0 = 0, \quad 1 \le j \le m-1, \tag{4.4}$$

$$e_0^k = 0, e_m^k = 0, \qquad 0 \le k \le n. \tag{4.5}$$

In using the Taylor expansion with Lagrange remainder, we can get

$$R_j^k = \frac{\tau^2}{24}\frac{\partial^3 u}{\partial t^3}(x_j, \eta_{jk}) + \frac{h^2}{24}\left[\frac{\partial^4 u}{\partial x^4}(\xi_{jk}, t_k) + \frac{\partial^4 u}{\partial x^4}(\xi'_{j,k+1}, t_{k+1})\right] + \frac{\tau^2}{8}\frac{\partial^4 u}{\partial x^2 \partial t^2}(x_j, \zeta_{jk}),$$

where

$$\eta_{jk}, \zeta_{jk} \in (t_k, t_{k+1}),\ \xi_{jk}, \xi'_{j,k+1} \in (x_{j-1}, x_{j+1}).$$

Therefore there exists a constant $c_2$ such that

$$|R_j^k| \le c_2(\tau^2 + h^2),\ 1 \le j \le m-1,\ 0 < k \le n-1. \tag{4.6}$$

Let $G_j^{k+\frac{1}{2}} = (|U_j^k|^2 + |U_j^{k+1}|^2)U_j^{k+\frac{1}{2}} - (|u_j^k|^2 + |u_j^{k+1}|^2)u_j^{k+\frac{1}{2}}$, and add the term $-(|U_j^k|^2 + |U_j^{k+1}|^2)u_j^{k+\frac{1}{2}}$ to the $G_j^{k+\frac{1}{2}}$, we obtain

$$\begin{aligned} G_j^{k+\frac{1}{2}} &= (|U_j^k|^2 + |U_j^{k+1}|^2)e_j^{k+\frac{1}{2}} + \left(|U_j^k|^2 + |U_j^{k+1}|^2 - |u_j^k|^2 - |u_j^{k+1}|^2\right)u_j^{k+\frac{1}{2}} \\ &= (|U_j^k|^2 + |U_j^{k+1}|^2)e_j^{k+\frac{1}{2}} + [(U_j^k - u_j^k)\bar{U}_j^k + u_j^k\overline{(U_j^k - u_j^k)} \\ &\quad + (U_j^{k+1} - u_j^{k+1})\bar{U}_j^{k+1} + u_j^{k+1}\overline{(U_j^{k+1} - u_j^{k+1})}]u_j^{k+\frac{1}{2}} \\ &= (|U_j^k|^2 + |U_j^{k+1}|^2)e_j^{k+\frac{1}{2}} + (e_j^k\bar{U}_j^k + u_j^k\bar{e}_j^k + e_j^{k+1}\bar{U}_j^{k+1} + u_j^{k+1}\bar{e}_j^{k+1})u_j^{k+\frac{1}{2}}. \end{aligned}$$

Noticing the initial-boundary value conditions (1.2)-(1.3) and (1.5)-(1.6), we have

$$G_0^{k+\frac{1}{2}} = \left(|U_0^k|^2 + |U_0^{k+1}|^2\right) U_0^{k+\frac{1}{2}} - \left(|u_0^k|^2 + |u_0^{k+1}|^2\right) u_0^{k+\frac{1}{2}} = 0,$$
$$G_m^{k+\frac{1}{2}} = \left(|U_m^k|^2 + |U_m^{k+1}|^2\right) U_m^{k+\frac{1}{2}} - \left(|u_m^k|^2 + |u_m^{k+1}|^2\right) u_m^{k+\frac{1}{2}} = 0.$$

According to Lemma 4.2, we get

$$|G_j^{k+\frac{1}{2}}| \le \left(\max\left\{|U_j^k|, |U_j^{k+\frac{1}{2}}|, |u_j^k|, |u_j^{k+\frac{1}{2}}|\right\}\right)^2 \cdot \left(2|e_j^k| + |e_j^{k+\frac{1}{2}}|\right),$$

or we can say there exists a positive constant $c_3$ such that

$$\|G^{k+\frac{1}{2}}\|^2 \le c_3\left(\|e^k\|^2 + \|e^{k+1}\|^2\right),\ 0 < k \le n-1, \tag{4.7}$$
$$|G^{k+\frac{1}{2}}|_1^2 \le c_3\left(\|e^k\|^2 + \|e^{k+1}\|^2 + |e^k|_1^2 + |e^{k+1}|_1^2\right),\ 0 < k \le n-1. \tag{4.8}$$

Multiplying the (4.3) by $h\bar{e}_j^{k+\frac{1}{2}}$, summing $j$ from 1 to $m-1$, we have

$$ih\sum_{j=1}^{m-1}(\delta_t e_j^{k+\frac{1}{2}})\bar{e}_j^{k+\frac{1}{2}} + h\sum_{j=1}^{m-1}(\delta_x^2 e_j^{k+\frac{1}{2}})\bar{e}_j^{k+\frac{1}{2}} + \tfrac{q}{2}h\sum_{j=1}^{m-1} G_j^{k+\frac{1}{2}}\bar{e}_j^{k+\frac{1}{2}} = h\sum_{j=1}^{m-1} R_j^k\bar{e}_j^{k+\frac{1}{2}},$$
$$i\tfrac{1}{2\tau}(\|e^{k+1}\|^2 - \|e^k\|^2) - h\sum_{j=1}^{m}|\delta_x e_{j-\frac{1}{2}}^{k+\frac{1}{2}}|^2 + \tfrac{q}{2}h\sum_{j=1}^{m-1}\left(|U_j^k|^2 + |U_j^{k+1}|^2\right)|e_j^{k+\frac{1}{2}}|^2$$
$$+\tfrac{q}{2}h\sum_{j=1}^{m-1}\left(e_j^k\bar{U}_j^k + u_j^k\bar{e}_j^k + e_j^{k+1}\bar{U}_j^{k+1} + u_j^{k+1}\bar{e}_j^{k+1}\right)u_j^{k+\frac{1}{2}}\bar{e}_j^{k+\frac{1}{2}} = h\sum_{j=1}^{m-1} R_j^k\bar{e}_j^{k+\frac{1}{2}}.$$

Taking the imaginary part and then using (4.1), (4.6) and Theorem 3.2, we can get

$$\begin{aligned}
\tfrac{1}{2\tau}\left(\|e^{k+1}\|^2 - \|e^k\|^2\right) &\le \tfrac{|q|}{2}h\sum_{j=1}^{m-1}|\left(e_j^k\bar{U}_j^k + u_j^k\bar{e}_j^k + e_j^{k+1}\bar{U}_j^{k+1} + u_j^{k+1}\bar{e}_j^{k+1}\right)\\
&\quad\cdot u_j^{k+\frac{1}{2}}\bar{e}_j^{k+\frac{1}{2}}| + h\sum_{j=1}^{m-1}|R_j^k\bar{e}_j^{k+\frac{1}{2}}|\\
&\le \tfrac{|q|}{2}h\sum_{j=1}^{m-1}\left(|e_j^k|c_0 + c_1|e_j^k| + |e_j^{k+1}|c_0 + c_1|e_j^{k+1}|\right)\\
&\quad\cdot c_1|e_j^{k+\frac{1}{2}}| + h\sum_{j=1}^{m-1}|R_j^k||\bar{e}_j^{k+\frac{1}{2}}|\\
&\le \tfrac{|q|}{2}h(c_0+c_1)c_1\sum_{j=1}^{m-1}(|e_j^k| + |e_j^{k+1}|)\cdot\tfrac{1}{2}(|e_j^k| + |e_j^{k+1}|)\\
&\quad+\tfrac{1}{2}c_2^2(\tau^2+h^2)^2 + \tfrac{h}{2}\sum_{j=1}^{m-1}\left[\tfrac{1}{2}(|e_j^k|^2 + |e_j^{k+1}|^2)\right]\\
&\le \left[\tfrac{|q|}{2}(c_0+c_1)c_1 + \tfrac{1}{4}\right](\|e^k\|^2 + \|e^{k+1}\|^2)\\
&\quad+\tfrac{1}{2}c_2^2(\tau^2+h^2)^2,\ 0 \le k \le n-1.
\end{aligned}$$

Thus,

$$\left(1 - \tau\left(|q|(c_0+c_1)c_1 + \tfrac{1}{2}\right)\right)\|e^{k+1}\|^2$$
$$\le \left(1 + \tau\left(|q|(c_0+c_1)c_1 + \tfrac{|q|}{2}\right)\right)\|e^k\|^2 + \tau c_2^2(\tau^2+h^2)^2,\ 0 \le k \le n-1.$$

Let $\beta = \tau\left(|q|(c_0+c_1)c_1+\frac{1}{2}\right)$. When $\beta \le \frac{1}{3}$, we have

$$\|e^{k+1}\|^2 \le (1+3\beta)\|e^k\|^2 + \frac{1}{1-\beta}\tau c_2^2(\tau^2+h^2)^2,$$

or we can say

$$\|e^{k+1}\|^2 \le \left[1+3\tau\left(|q|(c_0+c_1)c_1+\frac{1}{2}\right)\right]\|e^k\|^2 + \frac{3}{2}\tau c_2^2(\tau^2+h^2)^2,\ 0\le k\le n-1.$$

According to Gronwall Inequality in Lemma 4.1, we obtain

$$\|e^k\|^2 \le \exp\left[3k\tau\left(|q|(c_0+c_1)c_1+\frac{1}{2}\right)\right]\cdot\left[\|e^0\|^2 + \frac{c_2^2(\tau^2+h^2)^2}{2|q|(c_0+c_1)c_1+1}\right], \quad 1\le k\le n.$$

By the initial-boundary value conditions, we could easily know $\|e^0\| = 0$, so

$$\begin{aligned} \|e^k\| &\le \exp\left[\tfrac{3}{2}T(|q|(c_0+c_1)c_1+\tfrac{1}{2})\right]\frac{c_2}{\sqrt{2|q|(c_0+c_1)c_1+1}}(\tau^2+h^2) \\ &= c_4(\tau^2+h^2),\ 0\le k\le n. \end{aligned} \tag{4.9}$$

Multiplying the (4.3) by $-h\delta_t\bar{e}_j^{k+\frac{1}{2}}$, summing $j$ from 1 to $m-1$ and taking the real part, we have

$$\begin{aligned} & -Re\left\{h\sum_{j=1}^{m-1}\left(\delta_x^2 e_j^{k+\frac{1}{2}}\right)\left(\delta_t\bar{e}_j^{k+\frac{1}{2}}\right)\right\} \\ = & \ \frac{q}{2}Re\left\{h\sum_{j=1}^{m-1}G_j^{k+\frac{1}{2}}\left(\delta_t\bar{e}_j^{k+\frac{1}{2}}\right)\right\} - Re\left\{h\sum_{j=1}^{m-1}R_j^k\left(\delta_t\overline{e_j}^{k+\frac{1}{2}}\right)\right\}. \end{aligned} \tag{4.10}$$

Now, we estimate each term of (4.10).

Firstly, simplifying the left of (4.10), we obtain

$$\begin{aligned} & -Re\left\{h\sum_{j=1}^{m-1}\left(\delta_x^2 e_j^{k+\frac{1}{2}}\right)\left(\delta_t\bar{e}_j^{k+\frac{1}{2}}\right)\right\} \\ = & \ h\sum_{j=1}^{m}\left(\delta_x e_{j-\frac{1}{2}}^{k+\frac{1}{2}}\right)\cdot\delta_x\left(\delta_t\bar{e}_{j-\frac{1}{2}}^{k+\frac{1}{2}}\right) \\ = & \ h\sum_{j=1}^{m}\left(\delta_x e_{j-\frac{1}{2}}^{k+\frac{1}{2}}\right)\cdot\delta_t\left(\delta_x\bar{e}_{j-\frac{1}{2}}^{k+\frac{1}{2}}\right) \\ = & \ h\sum_{j=1}^{m}\frac{1}{2}\left(\delta_x e_{j-\frac{1}{2}}^{k+1}+\delta_x e_{j-\frac{1}{2}}^{k}\right)\cdot\frac{1}{\tau}\left(\delta_x\bar{e}_{j-\frac{1}{2}}^{k+1}-\delta_x\bar{e}_{j-\frac{1}{2}}^{k}\right) \\ = & \ \frac{1}{2\tau}h\sum_{j=1}^{m}\left(|\delta_x e_{j-\frac{1}{2}}^{k+1}|^2-|\delta_x e_{j-\frac{1}{2}}^{k}|^2\right), \end{aligned}$$

that is

$$-Re\left\{h\sum_{j=1}^{m-1}\left(\delta_x^2 e_j^{k+\frac{1}{2}}\right)\left(\delta_t\bar{e}_j^{k+\frac{1}{2}}\right)\right\} = \frac{1}{2\tau}\left(|e^{k+1}|_1^2-|e^k|_1^2\right). \tag{4.11}$$

Let the right term of (4.10) be $A_1, A_2$ separately. By the error equation (4.3), we have

$$\delta_t \bar{e}_j^{k+\frac{1}{2}} = -i\delta_x^2 \bar{e}_j^{k+\frac{1}{2}} - \frac{q}{2} i \bar{G}_j^{k+\frac{1}{2}} + i\bar{R}_j^k. \tag{4.12}$$

Substituting (4.12) into $A_1$, we obtain

$$\begin{aligned}
& Re\left\{ h\sum_{j=1}^{m-1} G_j^{k+\frac{1}{2}} \left(\delta_t \bar{e}_j^{k+\frac{1}{2}}\right)\right\} \\
= \; & Re\left\{ h\sum_{j=1}^{m-1} G_j^{k+\frac{1}{2}} \left(-i\delta_x^2 \bar{e}_j^{k+\frac{1}{2}} - \tfrac{q}{2} i \bar{G}_j^{k+\frac{1}{2}} + i\bar{R}_j^k\right)\right\} \\
= \; & Re\left\{ -ih\sum_{j=1}^{m-1} \delta_x^2 \bar{e}_j^{k+\frac{1}{2}} G_j^{k+\frac{1}{2}} - i\tfrac{q}{2}\|G^{k+\frac{1}{2}}\|^2 + ih\sum_{j=1}^{m-1} \bar{R}_j^k G_j^{k+\frac{1}{2}}\right\} \\
= \; & Im\left\{ -h\sum_{j=1}^{m-1} \delta_x^2 \bar{e}_j^{k+\frac{1}{2}} G_j^{k+\frac{1}{2}} + h\sum_{j=1}^{m-1} \bar{R}_j^k G_j^{k+\frac{1}{2}}\right\} \\
= \; & B_1 + B_2,
\end{aligned}$$

where

$$B_1 \le |h\sum_{j=1}^{m-1} \delta_x^2 \bar{e}_j^{k+\frac{1}{2}} G_j^{k+\frac{1}{2}}| \le |-h\sum_{j=1}^{m} \delta_x \bar{e}_{j-\frac{1}{2}}^{k+\frac{1}{2}} \delta_x G_{j-\frac{1}{2}}^{k+\frac{1}{2}}| \le \frac{1}{2}\left(|e^{k+\frac{1}{2}}|_1^2 + |G^{k+\frac{1}{2}}|_1^2\right),$$

$$B_2 \le |h\sum_{j=1}^{m-1} \bar{R}_j^k G_j^{k+\frac{1}{2}}| \le \frac{h}{2}\left(\sum_{j=1}^{m-1} |\bar{R}_j^k|^2 + \sum_{j=1}^{m-1} |G_j^{k+\frac{1}{2}}|^2\right) \le \frac{1}{2}\left(\|\bar{R}^k\|^2 + \|G^{k+\frac{1}{2}}\|^2\right).$$

Then according to (4.6)-(4.8), we can estimate the first right term $A_1$ as follow

$$\begin{aligned}
A_1 \; \le \; & \tfrac{q}{4}\left(|e^{k+\frac{1}{2}}|_1^2 + |G^{k+\frac{1}{2}}|_1^2 + \|\bar{R}^k\|^2 + \|G^{k+\frac{1}{2}}\|^2\right) \\
\le \; & \tfrac{q}{4}|e^{k+\frac{1}{2}}|_1^2 + \tfrac{q}{4}c_3\left(\|e^k\|^2 + \|e^{k+1}\|^2 + |e^k|_1^2 + |e^{k+1}|_1^2\right) \\
& + \tfrac{q}{4}c_2^2(\tau^2+h^2)^2 + \tfrac{q}{4}c_3\left(\|e^k\|^2 + \|e^{k+1}\|^2\right) \\
= \; & \tfrac{q}{4}\left[|e^{k+\frac{1}{2}}|_1^2 + c_2^2(\tau^2+h^2)^2\right] + \tfrac{q}{2}c_3\left(\|e^k\|^2 + \|e^{k+1}\|^2\right) \\
& + \tfrac{q}{4}c_3\left(|e^k|_1^2 + |e^{k+1}|_1^2\right) \\
\le \; & \tfrac{q}{4}\left[|e^{k+\frac{1}{2}}|_1^2 + c_2^2(\tau^2+h^2)^2\right] + qc_3c_4^2(\tau^2+h^2)^2 + \tfrac{q}{4}c_3\left(|e^k|_1^2 + |e^{k+1}|_1^2\right) \\
= \; & \tfrac{q}{4}c_3\left(|e^k|_1^2 + |e^{k+1}|_1^2\right) + \tfrac{q}{4}|e^{k+\frac{1}{2}}|_1^2 + \left(qc_3c_4^2 + \tfrac{q}{4}c_2^2\right)(\tau^2+h^2)^2.
\end{aligned}$$

According to (4.6), we can also estimate the second right term $A_2$ as follow

$$A_2 \le \frac{1}{2}\left(\|R^k\|^2 + |e^{k+\frac{1}{2}}|_1^2\right) \le \frac{1}{2}\left(c_1^2(\tau^2+h^2)^2 + |e^{k+\frac{1}{2}}|_1^2\right).$$

Now, substituting the three estimations just represented before into (4.10),

we obtain

$$\begin{aligned}
& \tfrac{1}{2\tau}\left(|e^{k+1}|_1^2-|e^k|_1^2\right)\\
\leq\ & \tfrac{q}{4}c_3\left(|e^k|_1^2+|e^{k+1}|_1^2\right)+\tfrac{q}{4}|e^{k+\frac{1}{2}}|_1^2+\left(qc_3c_4^2+\tfrac{q}{4}c_2^2\right)(\tau^2+h^2)^2\\
& +\tfrac{1}{2}c_2^2(\tau^2+h^2)^2+\tfrac{1}{2}|e^{k+\frac{1}{2}}|_1^2\\
\leq\ & \tfrac{q}{4}c_3\left(|e^k|_1^2+|e^{k+1}|_1^2\right)+\tfrac{q+2}{4}|e^{k+\frac{1}{2}}|_1^2+\left(qc_3c_4^2+\tfrac{q+2}{4}c_2^2\right)(\tau^2+h^2)^2\\
\leq\ & \tfrac{q}{4}c_3\left(|e^k|_1^2+|e^{k+1}|_1^2\right)+\tfrac{q+2}{4}\tfrac{1}{2}(|e^k|_1^2+|e^{k+1}|_1^2)+\left(qc_3c_4^2+\tfrac{q+2}{4}c_2^2\right)(\tau^2+h^2)^2\\
=\ & \tfrac{2qc_3+q+2}{8}\left(|e^k|_1^2+|e^{k+1}|_1^2\right)+\left(qc_3c_4^2+\tfrac{q+2}{4}c_2^2\right)(\tau^2+h^2)^2,
\end{aligned}$$

that is

$$\begin{aligned}
\left(1-\tau\tfrac{2qc_3+q+2}{4}\right)|e^{k+1}|_1^2 \leq\ & \left(1+\tau\tfrac{2qc_3+q+2}{4}\right)|e^k|_1^2\\
& +\left(2qc_3c_4^2+\tfrac{q+2}{2}c_2^2\right)(\tau^2+h^2)^2,\ 0\leq k\leq n-1.
\end{aligned}$$

Let $\beta=\tau\frac{2qc_3+q+2}{4}$. When $\beta\leq\frac{1}{3}$, we have

$$|e^{k+1}|_1^2\leq\left(1+3\tau\frac{2qc_3+q+2}{4}\right)|e^k|_1^2+\frac{3}{2}\tau\left(2qc_3c_4^2+\frac{q+2}{2}c_2^2\right)(\tau^2+h^2)^2,\quad 0\leq k\leq n-1.$$

Denote

$$c_5=\frac{2qc_3+q+2}{4},\ c_6=2qc_3c_4^2+\frac{q+2}{2}c_2^2,$$

then we rewrite the inequality as follow

$$|e^{k+1}|_1^2\leq(1+3\tau c_5)|e^k|_1^2+\frac{3}{2}\tau c_6(\tau^2+h^2)^2,\ 0\leq k\leq n-1.$$

Using Gronwall inequality, we get

$$|e^k|_1^2\leq\exp(3k\tau c_5)\cdot\left(|e^0|_1^2+\frac{c_6(\tau^2+h^2)^2}{2c_5}\right),\ 1\leq k\leq n.$$

By the initial-boundary value conditions, we also know $|e^0|_1^2=0$, so

$$\begin{aligned}
|e^k|_1^2 &\leq \exp(3k\tau c_5)\cdot\tfrac{c_6(\tau^2+h^2)^2}{2c_5}\\
&\leq \exp(3c_5T)\tfrac{c_6}{2c_5}(\tau^2+h^2)^2,\ 0\leq k\leq n.
\end{aligned}\tag{4.13}$$

According to (1) in Lemma 4.3, we have

$$\begin{aligned}
\|e^k\|_\infty^2 &\leq \tfrac{1}{4}|e^k|_1^2\\
&\leq \tfrac{c_6}{8c_5}\exp(3c_5T)(\tau^2+h^2)^2,\ 0\leq k\leq n.
\end{aligned}$$

Denote

$$C=\sqrt{\frac{c_6}{8c_5}\exp(3c_5T)}.$$

Therefore, when $\tau$ is small enough, there exists a constant $C$ independent of $h,\tau$ such that

$$\|e^k\|_\infty\leq C(\tau^2+h^2),0\leq k\leq n.\tag{4.14}$$

This completes the proof.

□

# 5 Iterative algorithm

There are some discrete methods about the nonlinear Schrödinger equations [12]-[13]. In this section, we use an iterative method [9] to compute the solution of the difference scheme (2.3)-(2.4).

Define the following iterative method

$$i\frac{2}{\tau}\left(v_j^{(l)}-u_j^k\right)+\delta_x^2 v_j^{(l)}+\frac{q}{2}\left(|u_j^k|^2+|2v_j^{(l-1)}-u_j^k|^2\right)v_j^{(l)}=0, \quad 1\le j\le m-1, 0\le k\le n-1, \tag{5.1}$$

$$v_0^{(l)}=0,\ v_m(l)=0, \tag{5.2}$$

where $v_j^{(0)}=u_j^k, 0\le j\le m,\ l=1,2,\ldots.$

Multiplying the (5.1) by $h\bar{v}_j^{(l)}$, summing $j$ from 1 to $m-1$ and taking the imaginary part, we have

$$\frac{2}{\tau}\|v^{(l)}\|^2-Reh\sum_{j=1}^{m-1}u_j^k\bar{v}_j^{(l)}=0,$$

that is

$$\begin{aligned}\tfrac{2}{\tau}\|v^{(l)}\|^2 &= Reh\sum_{j=1}^{m-1}u_j^k\bar{v}_j^{(l)}\\ &\le h\left(\sum_{j=1}^{m-1}(u_j^k)^2\cdot\sum_{j=1}^{m-1}(\bar{v}_j^{(l)})^2\right)^{\frac{1}{2}}\\ &= \left(h\sum_{j=1}^{m-1}(u_j^k)^2\right)^{\frac{1}{2}}\cdot\left(h\sum_{j=1}^{m-1}(\bar{v}_j^{(l)})^2\right)^{\frac{1}{2}}\\ &= \|u^k\|\cdot\|v^{(l)}\|.\end{aligned}$$

Thus,

$$\|v^{(l)}\|\le\|u^k\|,\ l=1,2,\ldots. \tag{5.3}$$

Denote

$$\varepsilon_j^{(l)}=v_j-v_j^{(l)},\ 0\le j\le m.$$

**Theorem 5.1.** *Suppose that the solution is $\{u_j^k\mid 0\le j\le m\}$, $\tau$ is sufficiently small enough, then the iterative method (5.1) − (5.2) is convergent.*

*Proof.* Subtracting (5.1)-(5.2) from (2.3)-(2.4), we obtain

$$i\frac{2}{\tau}\varepsilon_j^{(l)}+\delta_x^2\varepsilon_j^{(l)}+\frac{q}{2}\left[(|u_j^k|^2+|2v_j-u_j^k|^2)v_j-(|u_j^k|^2+|2v_j^{(l-1)}-u_j^k|^2)v_j^{(l)}\right]=0,\ 1\le j\le m-1, \tag{5.4}$$

$$\varepsilon_0^{(l)}=\varepsilon_m^{(l)}=0. \tag{5.5}$$

Multiplying the (5.4) by $h\bar{\varepsilon_j}^{(l)}$, summing $j$ from 1 to $m-1$, we have

$$\begin{aligned} & i\tfrac{2}{\tau}h\sum_{j=1}^{m-1}|\varepsilon_j^{(l)}|^2+h\sum_{j=1}^{m-1}(\delta_x^2\varepsilon_j^{(l)})\bar{\varepsilon_j}^{(l)}+\tfrac{q}{2}h\sum_{j=1}^{m-1}|u_j^k|^2|\varepsilon_j^{(l)}|^2 \\ & +\tfrac{q}{2}h\sum_{j=1}^{m-1}\left[|2v_j-u_j^k|^2v_j-|2v_j^{(l-1)}-u_j^k|^2v_j^{(l)}\right]\bar{\varepsilon_j}^{(l)}=0. \end{aligned} \tag{5.6}$$

Noticing the term in brackets, we add $|2v_j^{(l-1)}-u_j^k|^2v_j$ to this term as follow

$$\begin{aligned} & |2v_j-u_j^k|^2v_j-|2v_j^{(l-1)}-u_j^k|^2v_j^{(l)} \\ = \;& |2v_j^{(l-1)}-u_j^k|^2\left(v_j-v_j^{(l)}\right)+\left(|2v_j-u_j^k|^2-|2v_j^{(l-1)}-u_j^k|^2\right)v_j \\ = \;& |2v_j^{(l-1)}-u_j^k|^2\varepsilon_j^{(l)}+\left(2\varepsilon_j^{(l-1)}\overline{(2v_j-u_j^k)}+(2v_j^{(l-1)}-u_j^k)2\bar{\varepsilon_j}^{(l-1)}\right)v_j. \end{aligned}$$

Then, substituting this rewritten term into (5.6) and taking the imaginary part, we have

$$\begin{aligned} & \tfrac{2}{\tau}\|\varepsilon^{(l)}\|^2 \\ \le \;& \tfrac{|q|}{2}h\sum_{j=1}^{m-1}\left|\left(2\varepsilon_j^{(l-1)}\overline{(2v_j-u_j^k)}+(2v_j^{(l-1)}-u_j^k)2\bar{\varepsilon_j}^{(l-1)}\right)v_j\bar{\varepsilon_j}^{(l)}\right| \\ \le \;& |q|\cdot\|\varepsilon^{(l-1)}\|_\infty\cdot\|\varepsilon^{(l)}\|_\infty\cdot h\sum_{j=1}^{m-1}\left(|2v_j-u_j^k|+|2v_j^{(l-1)}-u_j^k|\right)|v_j| \\ \le \;& |q|\cdot\|\varepsilon^{(l-1)}\|_\infty\cdot\|\varepsilon^{(l)}\|_\infty\cdot\left(\|2v\|+\|u^k\|+\|2v^{(l-1)}\|+\|u^k\|\right)\|v\| \\ \le \;& |q|\cdot\|\varepsilon^{(l-1)}\|_\infty\cdot\|\varepsilon^{(l)}\|_\infty\cdot 6\|v\|^2. \end{aligned}$$

According to (5.3) and Theorem 3.1, we obtain

$$\begin{aligned} \|\varepsilon^{(l)}\|^2 &\le 3\tau|q|\cdot\|\varepsilon^{(l-1)}\|_\infty\cdot\|\varepsilon^{(l)}\|_\infty\cdot\|u^k\|^2 \\ &= 3\tau|q|\cdot\|\varepsilon^{(l-1)}\|_\infty\cdot\|\varepsilon^{(l)}\|_\infty\cdot\|u^0\|^2. \end{aligned} \tag{5.7}$$

Similarly, taking the real part of (5.6), we have

$$\begin{aligned} & |\varepsilon^{(l)}|_1^2 \\ \le \;& \tfrac{|q|}{2}\|u^k\|^2\cdot\|\varepsilon^{(l)}\|_\infty^2+\tfrac{|q|}{2}\|\varepsilon^{(l)}\|_\infty^2\cdot h\sum_{j=1}^{m-1}\left[4(v_j^{(l-1)})^2+(u_j^k)^2-4v_j^{(l-1)}u_j^k\right] \\ & +|q|\cdot\|\varepsilon^{(l-1)}\|_\infty\cdot\|\varepsilon^{(l)}\|_\infty\cdot 6\|v\|^2 \\ \le \;& \tfrac{|q|}{2}\|u^k\|^2\|\varepsilon^{(l)}\|_\infty^2+\tfrac{|q|}{2}\|\varepsilon^{(l)}\|_\infty^2\cdot\left(4\|u^k\|^2+\|u^k\|^2+4\|u^k\|^2\right) \\ & +6|q|\cdot\|u_k\|^2\|\varepsilon^{(l-1)}\|_\infty\|\varepsilon^{(l)}\|_\infty \\ \le \;& 5|q|\cdot\|u^k\|^2\|\varepsilon^{(l)}\|_\infty^2+6|q|\cdot\|u^k\|^2\|\varepsilon^{(l-1)}\|_\infty\|\varepsilon^{(l)}\|_\infty \\ \le \;& 5|q|\cdot\|u^0\|^2\|\varepsilon^{(l)}\|_\infty^2+6|q|\cdot\|u^0\|^2\|\varepsilon^{(l-1)}\|_\infty\|\varepsilon^{(l)}\|_\infty. \end{aligned}$$

According to (2) in Lemma 4.3, for any $\alpha>0$, there is

$$\begin{aligned} \|\varepsilon^{(l)}\|_\infty^2 &\le \alpha|\varepsilon^{(l)}|_1^2+\tfrac{1}{4\alpha}\|\varepsilon^{(l)}\|^2 \\ &\le \alpha\left(5|q|\cdot\|u^0\|^2\|\varepsilon^{(l)}\|_\infty^2+6|q|\cdot\|u^0\|^2\|\varepsilon^{(l-1)}\|_\infty\|\varepsilon^{(l)}\|_\infty\right) \\ &\quad +\tfrac{1}{4\alpha}\cdot 3\tau|q|\cdot\|u_0\|^2\|\varepsilon^{(l-1)}\|_\infty\|\varepsilon^{(l)}\|_\infty. \end{aligned}$$

That is

$$\begin{aligned}\|\varepsilon^{(l)}\|_\infty \quad \leq \quad & \alpha\left(5|q|\cdot\|u^0\|^2\|\varepsilon^{(l)}\|_\infty + 6|q|\cdot\|u^0\|^2\|\varepsilon^{(l-1)}\|_\infty\right)\\ & +\tfrac{1}{4\alpha}\cdot 3\tau|q|\cdot\|u_0\|^2\|\varepsilon^{(l-1)}\|_\infty.\end{aligned}$$

Taking $\alpha = 1/(12|q|\cdot\|u_0\|^2)$, we get

$$\|\varepsilon^{(l)}\|_\infty \leq \frac{5}{12}\|\varepsilon^{(l)}\|_\infty + \frac{1}{2}\|\varepsilon^{(l-1)}\|_\infty + 9\tau q^2\|u_0\|^4\|\varepsilon^{(l-1)}\|_\infty,$$

that is

$$\|\varepsilon^{(l)}\|_\infty \leq \frac{12}{7}\left(\frac{1}{2} + 9\tau q^2\|u_0\|^4\right)\|\varepsilon^{(l-1)}\|_\infty.$$

When $\tau q^2\|u_0\|^4 \leq \frac{1}{144}$, we have

$$\|\varepsilon^{(l)}\|_\infty \leq \frac{12}{7}\left(\frac{1}{2} + \frac{9}{144}\right)\|\varepsilon^{(l-1)}\|_\infty = \frac{27}{28}\|\varepsilon^{(l-1)}\|_\infty.$$

This completes the proof. □

# 6 Conclusion

In this paper, we consider a nonlinear finite difference scheme for the Schrödinger equations. We prove that the difference scheme has a unique and bounded solution and the finite difference solution is convergent with the convergence order of $\mathbf{O}(\tau^2 + h^2)$ in $L_\infty$ norm. Finally we give a convergent iterative method to compute the solution of the difference scheme.

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# References

[1] T. Kato, On nonlinear Schrödinger equations, Ann. Inst. H. Poincaté Phys. Théor. 46, 113-129(1987).

[2] J. Zhang, On the finite-time behaviour for nonlinear Schrödinger equations, Commun. Math. Phys. 162, 249-260(1994).

[3] Q. Chang, E. Jia and W. Sun, Difference schemes for solving the generalized nonlinear Schrödinger equation, J. Comput. Phys. 148, 397-415(1999).

[4] A. Kurtinaitis and F. Ivanauska, Finite difference solution methods for a system of the nonlinear Schrödinger equations, Nonlinear Anal. Mode. Control. 9, 247-258(2004).

[5] W. Bao and Y. Cai, Uniform error estimates of finite difference methods for the nonlinear Schrödinger equation with wave operator, SIAM J. Numer. Anal. 50, 492-521(2012).

[6] L. Ignat and E. Zuazua, Convergence rates for dispersive approximation schemes to nonlinear Schrödinger equations, J. Math. Pures Appl. 98, 479-517(2012).

[7] Z. Sun, X. Wu, J. Zhang and D. Wang, A linearized difference scheme for semilinear parabolic equations with nonlinear absorbing boundary conditions, Appl. Math. Comput. 218, 5187-5201(2012).

[8] N. Saito and T. Sasaki, Finite difference approximation for nonlinear Schrödinger equations with application to blow-up computation, Japan J. Indust. Appl. Math. 33, 427-470(2016).

[9] Z. Sun, Numerical Methods of the Partial Differential Equations, second ed., Science Press, Beijing, 2005.

[10] G. Akrivis, Finite difference discretization of the cubic Schrödinger equation, IMA J. Numer. Anal. 13, 115-124(1993).

[11] Z. Sun and D. Zhao, On the $L_\infty$ convergence of a difference scheme for coupled nonlinear Schrödinger equations, Comput. Math. Appl. 59, 3286-3300(2010).

[12] A. Mai and Z. Zhou, Ground state solutions for the periodic discrete nonlinear Schrödinger equations with superlinear nonlinearities, Abstr. Appl. Anal. 2013, 1-11(2013).

[13] G. Sun and A. Mai, Ground state solutions for discrete nonlinear Schrödinger equations with potentials, J. Comput. Anal. Appl. 19, 39-51(2015).

# SINGLE POINT V.S. TOTAL BLOW-UP FOR A REACTION DIFFUSION EQUATION WITH NONLOCAL SOURCE

DENGMING LIU

ABSTRACT. In this paper, we consider the following initial-boundary value problem of a semilinear parabolic equation with local and nonlocal sources

$$u_t = \Delta u + u^p + \int_B u^q(x,t)\,dx, \quad (x,t) \in B \times (0,T),$$

where $p,\ q > 0,\ B = \{x \in \mathbb{R}^N : |x| < R\}$. We completely classify blow-up solutions into total blow-up case and single point blow-up case according to the different values of the nonlinear parameters, and give the blow-up rates of solutions near the blow-up time.

## 1. INTRODUCTION

In this paper, we deal with the property of the blow-up solution of the following reaction-diffusion equation with local and nonlocal sources

$$\begin{cases} u_t = \Delta u + u^p + \int_B u^q(x,t)\,dx, & (x,t) \in B \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial B \times (0,T), \\ u(x,0) = u_0(x), & x \in \overline{B}, \end{cases} \tag{1.1}$$

where $p,\ q > 0,\ B = \{x \in \mathbb{R}^N : |x| < R\}$. Throughout this paper, we assume that the initial data $u_0 \in C^2(B) \cap C(\overline{B})$, $u_0(x) = u(r) \geq 0$ with $r = |x|$, and $u_0'(r) < 0$ for $r \in (0, R]$. Moreover, we assume that there exists a positive constant $\delta$ such that $\Delta u_0 + u_0^p + \int_B u_0^q dx \geq \delta$. When $\min\{p,q\} \geq 1$, we can easily show the local existence and uniqueness of classical solution of problem (1.1). If $\min\{p,q\} < 1$, the existence of maximal solution can be proved. Moreover, if $\max\{p,q\} > 1$, we can prove that the solution of (1.1) blows up in finite time for large initial data. In this paper, we consider the blow-up set of problem (1.1) and denote the blow-up time by $T$. We now begin with the definition of the blow-up point for a blow-up solution.

**Definition 1.1.** *A point $x \in \overline{B}$ is called a blow-up point if there exists a sequence $(x_n, t_n)$ such that $x_n \to x$, $t_n \nearrow T$ and $u(x_n, t_n) \to \infty$ as $n \to \infty$.*

The set of all blow-up points is called the blow-up set. For simplification, we denote the blow-up set by $S$. When $S = \overline{B}$, we call this phenomenon "total blow-up" and when the blow-up set include only one point, we call this "single point blow-up".

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In 1984, Weissler(see [1]) considered the property of the blow-up solution for the following one-dimensional initial-boundary value problem

$$\begin{cases} u_t = \Delta u + u^p, & (x,t) \in (-R,R) \times (0,T), \\ u(x,t) = 0, & (x,t) \in \{-R,R\} \times (0,T), \\ u(x,0) = u_0(x), & x \in [-R,R], \end{cases} \tag{1.2}$$

where $p > 1$, and obtained the single point blow-up phenomenon under some suitable conditions. In [2], Friedman and McLeod generalized Weissler's results to $N$-dimensional case, and showed that the blow-up point is only the origin, namely, $S = \{0\}$.

Chadam et al. in [3] studied the following problem with localized reaction term

$$\begin{cases} u_t = \Delta u + u^q(x^*,t), & (x,t) \in B \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial B \times (0,T), \\ u(x,0) = u_0(x), & x \in \overline{B}, \end{cases} \tag{1.3}$$

and proved that total blow-up occurs whenever a solution blows up, that is, $S = \overline{B}$. Souplet [4,5] extended the results in [3] to the case for the moving source $x^*(t)$ and obtained the precise blow-up profiles of the total blow-up solution.

Recently, Okada and Fukuda in [7] dealt with the single point and total blow-up for the following problem

$$\begin{cases} u_t = \Delta u + u^p + u^q(x^*,t), & (x,t) \in B \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial B \times (0,T), \\ u(x,0) = u_0(x), & x \in \overline{B}. \end{cases} \tag{1.4}$$

They showed that $p = q + 1$ is a cut off between the single point blow-up and the total blow-up for $x^* = 0$, and $p = q$ is the critical exponent of the single point blow-up and the total blow-up for $x^* \neq 0$.

Motivated by above works, we investigate problem (1.1). Similar to [7], the main purpose of this article is to evaluate the effect of the competition between $u^p$ and $\int_B u^q dx$ on the single blow-up and total blow-up. Motivated by the idea of Souplet in [6], through modifying the construction of auxiliary functions used in [7], we completely classify blow-up solutions into total blow-up case and single point blow-up case according to the different values of $p$ and $q$, and give the blow-up rates of solutions near the blow-up time.

In order to state our results, we first let $\varphi$ be a solution of

$$\begin{cases} \varphi_t = \Delta \varphi, & (x,t) \in B \times (0,T), \\ \varphi(x,t) = 0, & (x,t) \in \partial B \times (0,T), \\ \varphi(x,0) = \varphi_0(x) \geq 0, & x \in \overline{B}, \end{cases} \tag{1.5}$$

where $\varphi_0 \in C^2(B) \cap C(\overline{B})$, $\varphi_0(x) = \varphi(r)$ with $r = |x|$, and $\varphi_0'(r) < 0$ for $r \in (0,R]$. The main results of this article are stated as follows.

**Theorem 1.2.** *Suppose $q > p$ and $q > 1$, and let $u$ be a solution of (1.1) with $u_0 = \lambda\varphi_0\,(\lambda > 0)$, then there exists a positive constant $\lambda_0\,(\varphi_0)$ such that if $\lambda > \lambda_0\,(\varphi_0)$, then $u$ blows up on the whole domain, that is, $S = \overline{B}$; Moreover, the following estimate*

$$C_1\,(T-t)^{-\frac{1}{q-1}} \le u\,(x,t) \le C_2\,(T-t)^{-\frac{1}{q-1}}, \quad t \to T \tag{1.6}$$

*holds for any compact subset of $B$, here $C_1$, $C_2$ are positive constants.*

**Theorem 1.3.** *Suppose $p \ge q$ and $p > 1$, then all blow-up solutions of problem (1.1) blow up only at the origin, namely, $S = \{0\}$; Moreover, there exist positive constants $C_3$ and $C_4$ such that*

$$C_3\,(T-t)^{-\frac{1}{p-1}} \le u\,(0,t) \le C_4\,(T-t)^{-\frac{1}{p-1}}, \quad t \to T. \tag{1.7}$$

**Remark 1.4.** *From Theorems 1.1 and 1.2, we know that $p = q$ is the critical exponent for single point blow-up and total blow-up.*

This paper is organized as follows. In the next section, we will give some lemmas. In section 3, we concern with the single point blow-up and the total blow-up, and give the proofs of Theorems1.1 and 1.2, respectively.

## 2. Preliminary

In this section, we will state two important lemmas, which will be used in the sequel.

**Lemma 2.1.** *Suppose $q > 1$ and $q > p$, let $u\,(x,t)$ be a solution of (1.1) with $u_0\,(x) = \lambda\varphi_0\,(x)$, then there exists a positive constant $\lambda_0\,(\varphi_0)$ such that*

$$u\,(x,t) \ge \frac{\varphi\,(x,t)}{2\varphi_0\,(0)} u\,(0,t) \equiv \psi\,(x,t)\,u\,(0,t), \quad (x,t) \in B \times [0,T), \tag{2.1}$$

*holds if $\lambda > \lambda_0\,(\varphi_0)$.*

*Proof.* Using maximum principle (see [4]), we have

$$0 \le \varphi\,(x,t) \le \varphi_0\,(x) \le \varphi_0\,(0).$$

Because of $q > p$ and $u_0\,(0) = \lambda\varphi_0\,(0)$, we can choose $\lambda$ large enough such that

$$2\psi\,(x,t)\,u_0^{p-q}\,(0) \le \{\lambda\varphi_0\,(0)\}^{p-q} \le \int_B \psi^q\,(x,t)\,dx. \tag{2.2}$$

Now, letting

$$U = u\,(x,t) - \psi\,(x,t)\,u\,(0,t),$$

after a series of simple computation, we have

$$\begin{aligned} U_t - \Delta U &= u^p + \int_B u^q dx - \psi\,(x,t)\left(\Delta u\,(0,t) + u^p\,(0,t) + \int_B u^q dx\right) \\ &\ge \frac{1}{2}\int_B u^q dx - \psi\,(x,t)\,u^p\,(0,t). \end{aligned} \tag{2.3}$$

On the other hand, $\Delta u_0 + u_0^p + \int_B u_0^q dx \ge \delta$ means $u_t \ge 0$, thus, for any $t \in [0,T)$, we see

$$u\,(0,t) \ge u_0\,(0). \tag{2.4}$$

Combining (2.2), (2.3) with (2.4), we know

$$\begin{aligned} U_t - \Delta U &\geq \frac{1}{2}\left(\int_B u^q dx - 2\psi(x,t)\, u_0^{p-q}(0)\, u^q(0,t)\right) \\ &\geq \frac{1}{2}\left(\int_B u^q dx - \int_B \psi^q u^q(0,t)\, dx\right) \\ &= \frac{q}{2}\int_B U\Phi dx, \end{aligned} \tag{2.5}$$

where

$$\Phi = \int_0^1 \left[\theta u + (1-\theta)\,\psi u(0,t)\right]^{q-1} d\theta.$$

In addition, for any $(x,t) \in \partial B \times (0,T)$, we have

$$U(x,t) = 0, \tag{2.6}$$

and

$$U(x,0) = u_0(x) - \psi(x,0)\, u_0(0) = \lambda\varphi_0(x) - \frac{\varphi_0(x)}{2\varphi_0(0)}\lambda\varphi_0(0) = \frac{\lambda\varphi_0(x)}{2} \geq 0. \tag{2.7}$$

From (2.5), (2.6), (2.7) and maximum principle, it follows that

$$U(x,t) \geq 0, \quad (x,t) \in B \times [0,T),$$

which leads to (2.1). The proof of Lemma 2.1 is complete. □

**Lemma 2.2.** *Suppose $p > 1$ and $q \geq 0$, let $u(x,t)$ be a solution of (1.1), then there exists a positive constant $\eta$ such that*

$$u_t \geq \eta\varphi(x,t)\left(u^p + \int_B u^q dx\right), \quad (x,t) \in B \times [0,T). \tag{2.8}$$

*Proof.* Putting

$$J(x,t) = u_t(x,t) - \eta\varphi(x,t)\left(u^p + \int_B u^q dx\right),$$

where $\eta$ will be chosen later. Computing directly, we obtain

$$\begin{aligned} J_t - \Delta J - pu^{p-1}J \geq\ & \eta p\varphi u^{p-1}\left(u^p + \int_B u^q dx - u_t + \Delta u\right) \\ & - \eta(\varphi_t - \Delta\varphi)\left(u^p + \int_B u^q dx\right) + q(1-\eta\varphi)\int_B u^{q-1}u_t dx \\ & + 2\eta p u^{p-1}\nabla u\cdot\nabla\varphi + \eta p(p-1)\varphi u^{p-2}|\nabla u|^2 \\ \geq\ & 2\eta p u^{p-1}\nabla u\cdot\nabla\varphi + q(1-\eta\varphi)\int_B u^{q-1}u_t dx. \end{aligned} \tag{2.9}$$

Since $u$ and $\varphi$ are radially symmetric and monotone decreasing with respect to $r$, we have

$$\nabla u\cdot\nabla\varphi = u_r\left(\frac{x_1}{r}, \frac{x_2}{r}, \cdots, \frac{x_n}{r}\right)\cdot\varphi_r\left(\frac{x_1}{r}, \frac{x_2}{r}, \cdots, \frac{x_n}{r}\right) = u_r\varphi_r \geq 0.$$

On the other hand, by maximum principle, the assumption $\Delta u_0 + u_0^p + \int_B u_0^q dx \geq \delta$ implies that $u_t \geq 0$. Choosing $\eta$ small enough such that

$$1 - \eta\varphi(x,t) \geq 0, \tag{2.10}$$

we then have

$$J_t - \Delta J - pu^{p-1}J \geq 0. \tag{2.11}$$

Moreover, we can verify that

$$\begin{aligned} J(x,0) &= u_t(x,0) - \eta\varphi_0(x)\left(u_0^p(x) + \int_B u_0^q(x)\,dx\right) \\ &= \Delta u_0(x) + u_0^p(x) + \int_B u_0^q(x)\,dx - \eta\varphi_0(x)\left(u_0^p(x) + \int_B u_0^q(x)\,dx\right) \\ &\geq \mu - \eta\varphi_0(0)\left(u_0^p(0) + \int_B u_0^q(0)\,dx\right) \\ &\geq 0, \end{aligned} \tag{2.12}$$

holds for sufficiently small $\eta$. In addition, for any $(x,t) \in \partial B \times (0,T)$, we have

$$J(x,t) = 0. \tag{2.13}$$

From (2.11), (2.12), (2.13) and maximum principle, it follows that

$$J(x,t) \geq 0, \quad (x,t)\, B \times [0,T),$$

which yields (2.8). The proof of Lemma 2.2 is complete. □

## 3. Proof of Theorems 1.1 and 1.2

In this section, we will discuss the single point and total blow-up phenomena according to the different values of $p$ and $q$. Firstly, we give the proof of Theorem 1.1.

*Proof of Theorem 1.1.* Since

$$\lim_{t\to T} u(0,t) = +\infty,$$

we can easily show the total blow-up result under the condition $q > p$ from Lemma 2.1.

Moreover, noticing the fact that

$$\max_{x\in\overline{B}} u(x,t) = u(0,t),$$

we have

$$\Delta u(0,t) \leq 0.$$

On the other hand, thanks to $q > p$ and $q > 1$, there exists $t_1 \in (0,T)$ such that

$$\begin{aligned} u_t(0,t) &= \Delta u(0,t) + u^p(0,t) + \int_B u^q(0,t)\,dx \\ &\leq u^p(0,t) + \int_B u^q(0,t)\,dx \\ &\leq (|B|+1)\,u^q(0,t), \quad t \in (t_1,T). \end{aligned} \tag{3.1}$$

Combining (3.1) with Lemma 2.1, we obtain

$$C_1(T-t)^{-\frac{1}{q-1}} \leq Cu(0,t) \leq u(x,t), \quad (x,t) \in K \times (t_1,T), \tag{3.2}$$

where $K$ is any compact subset of $B$.

Furthermore, it follows that, from (2.1) and (2.8),

$$u_t(0,t) \geq \eta\varphi(0,t)\int_B u^q dx \geq \eta|B|\varphi(0,t)\psi^q(0,t)u^q(0,t). \tag{3.3}$$

Integrating (3.3) from 0 to $t$, we conclude

$$u(x,t) \leq u(0,t) \leq C_2(T-t)^{-\frac{1}{q-1}}, \quad (x,t) \in B\times(0,T). \tag{3.4}$$

Combining (3.2) with (3.3), we get (1.6). The proof of Theorem 1.1 is complete. □

Now, by the method of contradiction, we give the proof of Theorem 1.2.

*Proof of Theorem 1.2.* Using mean value theorem, and noticing the fact that $u(x,t)$ is radially symmetric and monotone decreasing with respect to $r$, we know that there exists a unique point $x^*$ such that

$$\int_B u^q dx = |B|u^q(x^*,t), \quad x^* \neq 0 \text{ and } x^* \notin \partial B.$$

We suppose that, on the contrary, there is a blow-up point $x_0 \neq 0$ ($|x_0| \leq |x^*| = r_0$). Since $u(x,t)$ is radially symmetric and monotone decreasing on $r$, then for any $r_1 \in [0, r_0]$, we see

$$\lim_{t\to T} u(r_1,t) = \infty.$$

Letting $0 < \mu_1 < \mu < \mu_2 < r_0$, and

$$S_0(\mu,\gamma) = \left\{x = (x_1, x_2, \cdots, x_N) \in \mathbb{R}^N : \mu < x_1 < \mu+\gamma, 0 < x_j < \gamma\,(j = 2, \cdots N)\right\},$$

here $\gamma$ is a sufficiently small constant such that

$$\overline{S_0(\mu,\gamma)} \in B(\mu_2)\setminus\overline{B}(\mu_1).$$

Defining an auxiliary function as the form

$$F(x,t) = u_{x_1}(x,t) + \epsilon b(x)u^m(x,t), \quad (x,t) \in S_0\times[0,T), \tag{3.5}$$

where $\epsilon$, $m$ will be determined later, and

$$b(x) = \sin\frac{\pi(x_1-\mu)}{\gamma}\prod_{j=2}^{N}\sin\frac{\pi x_j}{\gamma}.$$

Calculating directly, we obtain

$$\begin{aligned}
&F_t - \Delta F - \left(pu^{p-1} - \frac{2\epsilon m\nabla b\cdot\nabla u}{u_{x_1}}u^{m-1}\right)F \\
&= \epsilon(m-p)bu^{p+m-1} + \epsilon mbu^{m-1}\int_B u^q dx + \frac{\epsilon\pi^2 N}{\gamma^2}bu^m \\
&= \epsilon(m-p)bu^{p+m-1} + \epsilon mb|B|u^{m-1}u^q(x^*,t) + \frac{\epsilon\pi^2 N}{\gamma^2}bu^m \\
&\quad - \epsilon m(m-1)bu^{m-2}|\nabla u|^2 + \frac{2m\epsilon^2\nabla b\cdot\nabla u}{u_{x_1}}bu^{2m-1} \\
&\leq \epsilon(m-p)bu^{p+m-1} + \epsilon mb|B|u^{m+q-1} + \frac{\epsilon\pi^2 N}{\gamma^2}bu^m + \frac{2m\epsilon^2\nabla b\cdot\nabla u}{u_{x_1}}bu^{2m-1}.
\end{aligned}$$

On the other hand, it is easy to verify that

$$0 < \frac{2m\epsilon\nabla b\cdot\nabla u}{u_{x_1}} < \frac{2m\pi\epsilon N\mu_2}{\gamma\mu_1}.$$

For the case $p = q$, we can choose $m < \frac{p}{1+|B|}$ and $\tau_1$ large enough such that

$$\begin{aligned} & F_t - \Delta F - \left( pu^{p-1} - \frac{2\epsilon m \nabla b \cdot \nabla u}{u_{x_1}} u^{m-1} \right) F \\ & \leq -\epsilon b u^{m-1} \left\{ \left[ p - m\left(1 + |B|\right) \right] u^p - \frac{\pi^2 N}{\gamma^2} u - \frac{2m\pi\epsilon N \mu_2}{\gamma \mu_1} u^m \right\} \\ & \leq 0, \end{aligned} \tag{3.6}$$

holds for every $(x,t) \in S_0 \times [\tau_1, T)$.

For the case $p > q$, we can take $m < p$ and $\tau_2$ large enough such that, for any $(x,t) \in S_0 \times [\tau_2, T)$, the following inequality holds

$$\begin{aligned} & F_t - \Delta F - \left( pu^{p-1} - \frac{2\epsilon m \nabla b \cdot \nabla u}{u_{x_1}} u^{m-1} \right) F \\ & \leq -\epsilon b u^{m-1} \left\{ \left( p - m \right) u^p - m\left|B\right| u^q - \frac{\pi^2 N}{\gamma^2} u - \frac{2m\pi\epsilon N \mu_2}{\gamma \mu_1} u^m \right\} \\ & \leq 0. \end{aligned} \tag{3.7}$$

Next, putting $\tau = \max\left\{\tau_1, \tau_2\right\}$, and taking $\epsilon$ small enough, such that

$$\begin{aligned} F\left(x,\tau\right) &= u_{x_1}\left(x,\tau\right) + \epsilon b\left(x\right) u^m\left(x,\tau\right) \\ &\leq \max_{x \in S_0} u_{x_1}\left(x,\tau\right) + \epsilon \max_{x \in S_0} u^m\left(x,\tau\right) < 0. \end{aligned} \tag{3.8}$$

In addition, we can easily check that,

$$F\left(x,t\right) = u_{x_1}\left(x,t\right) < 0, \quad (x,t) \in \partial S_0 \times [\tau, T). \tag{3.9}$$

Combining (3.6), (3.7) and (3.8) with (3.9), we conclude immediately that

$$F\left(x,t\right) \leq 0, \quad \text{for any } (x,t) \in S_0 \times [\tau, T),$$

which implies

$$-u^{-m} u_{x_1} \geq \epsilon b\left(x\right). \tag{3.10}$$

Fixing

$$a' = \left(a_2, \cdots a_N\right) \in \mathbb{R}^{N-1},$$

and denoting

$$a_1 = \left(\mu + \gamma, a_2, \cdots a_N\right).$$

Integrating (3.10) by $x_1$ from $\mu$ to $\mu + \gamma$, we have

$$0 < \int_{\mu}^{\mu+\gamma} \epsilon b\left(x\right) dx_1 = \frac{2\epsilon\gamma}{\pi} \prod_{j=2}^{N} \sin\frac{\pi x_j}{\gamma} \leq \frac{1}{\left(m-1\right) u^{m-1}\left(a_1,t\right)}. \tag{3.11}$$

Since

$$\lim_{t \to T} u\left(a_1,t\right) = +\infty,$$

from (3.11), we have a contradiction. Hence, $u\left(x,t\right)$ blows up only at the origin.

In light of

$$\max_{x \in \overline{B}} u\left(x,t\right) = u\left(0,t\right),$$

then similar to the process of the derivation of (3.1) and (3.2), we find that

$$C_3\left(T-t\right)^{-\frac{1}{p-1}} \leq u\left(0,t\right), \quad t \to T. \tag{3.12}$$

Now, using Lemma 2.2, we get

$$u_t(0,t) \geq \eta\varphi(0,t)\, u^p(0,t). \tag{3.13}$$

From (3.13), it follows immediately that

$$u(0,t) \leq C_4 (T-t)^{-\frac{1}{p-1}}, \quad t \to T. \tag{3.14}$$

Combining (3.12) with (3.14), we arrive at (1.7). The proof of Theorem 1.2 is complete. □

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## References

[1] F. B. Weissler, Single point blow-up for a semilinear initial value problem, J. Diff. Equns., (55)1984, 204-224.
[2] A. Friedman and B. McLeod, Blow-up of positive solutions of semilinear heat equations, Indiana Univ. Math. J., (34)1985, 425-447.
[3] J. M. Chadam, A. Pierce and H. M. Yin, The blowup property of solutions to some diffusion equations with localized nonlinear reactions, J. Math. Anal. Appl., (169)1992, 313-328.
[4] P. Souplet, Blow-up in nonlocal reaction diffusion equations, SIAM J. Math. Anal., (29)1998, 1301-1334.
[5] P. Souplet, Uniform blow-up profiles and boundary behavior for diffusion equations with nonlocal nonlinear source, J. Diff. Equns., (153)1999, 374-406.
[6] P. Souplet, Single-point blow-up for a semilinear parabolic system, J. Eur. Math. Soc., 11(2009), 169-188.
[7] A. Okada and I. Fukuda, Total versus single point blow-up of solutions of a semilinear parabolic equation with localized reaction, J. Math. Anal. Appl., (281)2003, 485-500.

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# COMMON FIXED POINT RESULTS FOR WEAKLY COMPATIBLE MAPPINGS USING $C$-CLASS FUNCTIONS

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ABSTRACT. In this paper, using the concept of $C$-class fuction, we prove the existence of common fixed point for generalized Zamfirescu-type mappings and generalized weakly Zamfirescu-type mappings. Our results generalize so many results in the literature.

## 1. INTRODUCTION

In 1922, Banach proved the existence of fixed point on complete metric space $(X, d)$. A mapping $f$ has been considered to be a contraction and a self-mapping.

**Definition 1.1.** Let $(X, d)$ be a metric space. A mapping $f : X \to X$ is said to be a contraction mapping if there exists $k \in [0, 1)$ such that

$$d(f(x), f(y)) \leq kd(x, y).$$

Later many authours have proved fixed point existence on several type of generalized contractions. Kannan type and Chatterjea type mappings were significant type of mappings since they provided existence of fixed point for non-continuous mappings in literature (see [4, 6]).

In 1972, Zamfirescu [7] generalized functions of Banach, Kannan and Chatterjea by introducing a new kind of mapping and proved the existence of fixed points for mappings.

**Definition 1.2.** Let $(X, d)$ be a metric space. A mapping $f : D \to X$ is said to be a Zamfirescu mapping if for all $x, y \in X$ it satisfies the condition

$$d(f(x), f(y)) \leq kM_f(x, y)$$

for some $k \in [0, 1)$, where

$$M_f(x, y) := \max\Big\{d(x, y), \frac{1}{2}\big[d(x, f(x)) + d(y, f(y))\big], \frac{1}{2}\big[d(x, f(y)) + d(y, f(x))\big]\Big\}.$$

Apart all these generalizations, Dugundji and Granas [5] in 1978 generalized the contraction mapping as follows.

**Definition 1.3.** [5] Let $(X, d)$ be a metric space and $D \subset X$. A mapping $f : D \to X$ is said to be a weakly contraction mapping if there exists $\alpha : D \times D \to [0, 1]$ such that $\Theta(a, b) := sup\big\{\alpha(x, y) : a \leq d(x, y) \leq b\big\} < 1$ for every $0 < a \leq b$ and for all $x, y \in D$,

$$d(f(x), f(y)) \leq \alpha(x, y)d(x, y).$$

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In 2014, the concept of $C$-class functions was introduced by Ansari [2]. By using this concept, we can generalize many fixed point theorems in the literature.

**Definition 1.4.** [2] A function $F : [0,\infty)^2 \to \mathbb{R}$ is called a *$C$-class* function (also denoted as $\mathcal{C}$ ) if it is continuous and satisfies the following:
(1) $F(s,t) \leq s$;
(2) $F(s,t) = s$ implies that either $s = 0$ or $t = 0$
for all $s, t \in [0,\infty)$.

**Example 1.5.** [2] The following functions $f : [0,\infty)^2 \to \mathbb{R}$ are elements of $\mathcal{C}$, for all $s, t \in [0,\infty)$:
(1) $f(s,t) = s - t$, $f(s,t) = s \Rightarrow t = 0$;
(2) $f(s,t) = ms$, $0<m<1$, $f(s,t) = s \Rightarrow s = 0$;
(3) $f(s,t) = \frac{s}{(1+t)^r}$; $r \in (0,\infty)$, $f(s,t) = s \Rightarrow s = 0$ or $t = 0$;
(4) $f(s,t) = \log(t + a^s)/(1 + t)$, $a > 1$, $f(s,t) = s \Rightarrow s = 0$ or $t = 0$;
(5) $f(s,t) = \ln(1 + a^s)/2$, $a > e$, $f(s,1) = s \Rightarrow s = 0$.

**Definition 1.6.** Let $\Psi$ denote all the functions $\psi : [0,\infty) \to [0,\infty)$ which satisfy
(i) $\psi(t) = 0$ if and only if $t = 0$,
(ii) $\psi$ is continuous,
(iii) $\psi(s) \leq s$, $\forall s > 0$.

**Definition 1.7.** Let $(X,d)$ be a metric space. Then $f, g : X \to X$ are said to be weakly compatible if $fg(x) = gf(x)$ for $x \in X$ whenever $f(x) = g(x)$.

**Lemma 1.8.** ( [3]) *Suppose that $(X,d)$ is a metric space. Let $\{x_n\}$ be a sequence in $X$ such that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence, then there exist an $\varepsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k) > n(k) > k$ such that $d(x_{m(k)}, x_{n(k)}) \geq \varepsilon$, $d(x_{m(k)-1}, x_{n(k)}) < \varepsilon$ and*
*(i)* $\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)+1}) = \varepsilon$;
*(ii)* $\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \varepsilon$;
*(iii)* $\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}) = \varepsilon$.

In this paper, we prove the existence of common fixed point for two weakly compatible mappings on a complete metric space.

## 2. Main results

**Definition 2.1.** Let $(X,d)$ be a metric space. Consider two self-mappings $f$ and $g$ on $X$ and let $\alpha : X \times X \to [0,1]$ be a function. Then $g$ is an $f$-weakly generalized Zamfirescu type mapping if, for all $F \in \mathcal{C}$, $\psi \in \Psi$ and for all $x, y \in X$,

$$\begin{aligned} d(g(x),g(y)) \leq & F\Big(\alpha\big(f(x),f(y)\big) max\Big\{\big(d(f(x),f(y)), \frac{1}{2}\big[d(f(x),g(x)) + d(f(y),g(y))\big], \\ & \qquad \frac{1}{2}\big[d(f(x),g(y)) + d(f(y),g(x))\big]\Big\}, \\ & \psi\Big(\alpha\big(f(x),f(y)\big) max\Big\{\big(d(f(x),f(y)), \frac{1}{2}\big[d(f(x),g(x)) + d(f(y),g(y))\big], \\ & \qquad \frac{1}{2}\big[d(f(x),g(y)) + d(f(y),g(x))\big]\Big\}\Big)\Big). \end{aligned}$$

**Theorem 2.2.** *Let $(X,d)$ be a complete metric space and $f,g:X\to X$ mappings such that $g$ is an $f$-weakly generalized Zamfirescu type mapping. Then $f$ and $g$ have a unique common fixed point on $X$ if the following conditions are satisfied:*

(1) $g(X)\subset f(X)$;
(2) $f(X)$ *is complete;*
(3) $f,g$ *are weakly compatible.*

*Proof.* Choose $x_0\in Y$ arbitrarily and $x_n\in X$ such that $f(x_n)=g(x_{n-1})$. Then

$$
\begin{aligned}
&d(f(x_n),f(x_{n+1}))=d(g(x_{n-1}),g(x_n)) \qquad (2.1)\\
&\le F\Big(\alpha\big(f(x_{n-1}),f(x_n)\big)max\Big\{d(f(x_{n-1}),f(x_n)),\frac{1}{2}\big[d(f(x_{n-1}),g(x_{n-1}))+d(f(x_n),g(x_n))\big],\\
&\qquad\qquad \frac{1}{2}\big[d(f(x_n),g(x_{n-1}))+d(f(x_{n-1}),g(x_n))\big]\Big\},\\
&\quad \psi\Big(\alpha\big(f(x_{n-1}),f(x_n)\big)max\Big\{d(f(x_{n-1}),f(x_n)),\frac{1}{2}\big[d(f(x_{n-1}),g(x_{n-1}))+d(f(x_n),g(x_n))\big],\\
&\qquad\qquad \frac{1}{2}\big[d(f(x_n),g(x_{n-1}))+d(f(x_{n-1}),g(x_n))\big]\Big\}\Big)\Big)\\
&\le \alpha\big(f(x_{n-1}),f(x_n)\big)max\Big\{d(f(x_{n-1}),f(x_n)),\frac{1}{2}\big[d(f(x_{n-1}),g(x_{n-1}))+d(f(x_n),g(x_n))\big],\\
&\qquad\qquad \frac{1}{2}\big[d(f(x_n),g(x_{n-1}))+d(f(x_{n-1}),g(x_n))\big]\Big\}\\
&\le \alpha\big(f(x_{n-1}),f(x_n)\big)max\Big\{d(f(x_{n-1}),f(x_n)),\frac{1}{2}\big[d(f(x_{n-1}),f(x_n))+d(f(x_n),f(x_{n+1}))\big],\\
&\qquad\qquad \frac{1}{2}\big[d(f(x_n),f(x_n))+d(f(x_{n-1}),f(x_{n+1}))\big]\Big\}\\
&\le \alpha\big(f(x_{n-1}),f(x_n)\big)max\Big\{d(f(x_{n-1}),f(x_n)),\frac{1}{2}\big[d(f(x_{n-1}),f(x_n))+d(f(x_n),f(x_{n+1}))\big],\\
&\qquad\qquad \frac{1}{2}\big[d(f(x_{n-1}),f(x_n))+d(f(x_n),f(x_{n+1}))\big]\Big\}\\
&\le \alpha\big(f(x_{n-1}),f(x_n)\big)max\Big\{d(f(x_{n-1}),f(x_n)),\frac{1}{2}\big[d(f(x_{n-1}),f(x_n))+d(f(x_n),f(x_{n+1}))\big]\Big\}.
\end{aligned}
$$

**Claim:** $d(f(x_n),f(x_{n+1}))\le\alpha(f(x_{n-1}),f(x_n))d(f(x_{n-1}),f(x_n))$.
Suppose that

$$
\begin{aligned}
&d(f(x_n),f(x_{n+1}))\le\tfrac{\alpha(f(x_{n-1}),f(x_n))}{2}[d(f(x_{n-1}),f(x_n))+d(f(x_n),f(x_{n+1}))]\\
&\qquad\Longrightarrow d(f(x_n),f(x_{n+1}))\le\frac{\alpha(f(x_{n-1}),f(x_n))}{2-\alpha(f(x_{n-1}),f(x_n))}d(f(x_{n-1}),f(x_n))\\
&\qquad\qquad\le\alpha(f(x_{n-1}),f(x_n))d(f(x_{n-1}),f(x_n)).
\end{aligned}
$$

Then $\{d(f(x_n),f(x_{n+1}))\}$ is positive, decreasing and converges to some $d\in[0,\infty)$.

Now, letting $n\to\infty$ in (2.1), we get

$$
\begin{aligned}
d\le&\lim_{n\to\infty}\alpha\big(f(x_{n-1}),f(x_n)\big)max\Big\{d(f(x_{n-1}),f(x_n)),\frac{1}{2}\big[d(f(x_{n-1}),g(x_{n-1}))+d(f(x_n),g(x_n))\big],\\
&\qquad\qquad\frac{1}{2}\big[d(f(x_n),g(x_{n-1}))+d(f(x_{n-1}),g(x_n))\big]\Big\}\\
\le&d,
\end{aligned}
$$

which implies that

$$d = \lim_{n\to\infty} \alpha\big(f(x_{n-1}), f(x_n)\big) max\Big\{ d(f(x_{n-1}), f(x_n)), \tag{2.2}$$
$$\frac{1}{2}\big[d(f(x_{n-1}), g(x_{n-1})) + d(f(x_n), g(x_n))\big], \frac{1}{2}\big[d(f(x_n), g(x_{n-1})) + d(f(x_{n-1}), g(x_n))\big]\Big\}.$$

Again, letting $n \to \infty$ in (2.1) and using (2.2), we get

$$d \le F(d, \psi(d)) \le d.$$

Thus $F(d, \psi(d)) = d$, which implies that $d = 0$.

Next, we prove that $\{f(x_n)\}$ is Cauchy. Suppose not. Then by Lemma 1.8, there exist sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $m_k > n_k \ge k$ such that $d(f(x_{m_{k-1}}), f(x_{n_k}))$ and $d(f(x_{m_k}), f(x_{n_k}))$ converge to some $\delta > 0$. So

$$\begin{aligned}
&d(f(x_{m_k}), f(x_{n_k})) = d(g(x_{m_k-1}), g(x_{n_{k-1}})) \qquad (2.3)\\
&\le F\Big(\alpha\big(f(x_{m_k-1}), f(x_{n_k-1})\big) max\Big\{ d(f(x_{m_k-1}), f(x_{n_k-1})),\\
&\qquad \frac{1}{2}\big[d(f(x_{m_k-1}), g(x_{m_k-1})) + d(f(x_{n_k-1}), g(x_{n_k-1}))\big],\\
&\qquad \frac{1}{2}\big[d(f(x_{m_k-1}), g(x_{n_k-1})) + d(f(x_{n_k-1}), g(x_{m_k-1}))\big]\Big\},\\
&\quad \psi\Big(\alpha\big(f(x_{m_k-1}), f(x_{n_k-1})\big) max\Big\{ d(f(x_{m_k-1}), f(x_{n_k-1})),\\
&\qquad \frac{1}{2}\big[d(f(x_{m_k-1}), g(x_{m_k-1})) + d(f(x_{n_k-1}), g(x_{n_k-1}))\big],\\
&\qquad \frac{1}{2}\big[d(f(x_{m_k-1}), g(x_{n_k-1})) + d(f(x_{n_k-1}), g(x_{m_k-1}))\big]\Big\}\Big)\Big)\\
&\le \alpha\big(f(x_{m_k-1}), f(x_{n_k-1})\big) max\Big\{ d(f(x_{m_k-1}), f(x_{n_k-1})),\\
&\qquad \frac{1}{2}\big[d(f(x_{m_k-1}), g(x_{m_k-1})) + d(f(x_{n_k-1}), g(x_{n_k-1}))\big],\\
&\qquad \frac{1}{2}\big[d(f(x_{m_k-1}), g(x_{n_k-1})) + d(f(x_{n_k-1}), g(x_{m_k-1}))\big]\Big\}\\
&\le max\Big\{ d(f(x_{m_k-1}), f(x_{n_k-1})), \frac{1}{2}\big[d(f(x_{m_k-1}), g(x_{m_k-1})) + d(f(x_{n_k-1}), g(x_{n_k-1}))\big],\\
&\qquad \frac{1}{2}\big[d(f(x_{m_k-1}), g(x_{n_k-1})) + d(f(x_{n_k-1}), g(x_{m_k-1}))\big]\Big\}\\
&\le max\Big\{ \Big[d(f(x_{m_k-1}), f(x_{n_k})) + d(f(x_{n_k}), f(x_{n_{k-1}}))\Big], d(f(x_{n_{k-1}}), f(x_{n_k})),\\
&\qquad \frac{1}{2}\big[d(f(x_{m_k-1}), f(x_{n_k})) + d(f(x_{n_{k-1}}), f(x_{n_k})) + d(f(x_{n_k}), f(x_{m_k}))\big]\Big\}.
\end{aligned}$$

Letting $n \to \infty$ in (2.3), we get

$$\begin{aligned}
\delta = \lim_{k\to\infty} \alpha\big(f(x_{m_k-1}), f(x_{n_k-1})\big) max\Big\{ & d(f(x_{m_k-1}), f(x_{n_k})),\\
& \frac{1}{2}\big[d(f(x_{m_k-1}), g(x_{m_k-1})) + d(f(x_{n_k}), g(x_{n_k}))\big], \qquad (2.4)\\
& \frac{1}{2}\big[d(f(x_{m_k-1}), g(x_{n_k})) + d(f(x_{n_k}), g(x_{m_k-1}))\big]\Big\}.
\end{aligned}$$

Again, letting $n \to \infty$ in (2.3) and using (2.4),

$$\delta \leq F(\delta, \psi(\delta)) \leq \delta.$$

Thus $F(\delta, \psi(\delta)) = \delta$, which implies that $\delta = 0$, which is a contradiction.

Therefore, $\{f(x_m)\}$ is Cauchy and converges to $x = f(u)$ for some $x \in X$.

Next, we prove that $d(f(u), g(u)) = 0$.

$$\begin{aligned}
d(g(u), x) &= \lim_{n\to\infty} d(g(u), g(x_n)) \\
&\leq \lim_{n\to\infty} F\Big(\alpha(f(u), f(x_n))max\Big\{d(f(u), f(x_n)), \frac{1}{2}\big[d(f(u), g(u)) + d(f(x_n), g(x_n))\big], \\
&\qquad \frac{1}{2}\big[d(f(x_n), g(u)) + d(f(u), g(x_n))\big]\Big\}, \\
&\qquad \psi\Big(\alpha(f(u), f(x_n))max\Big\{d(f(u), f(x_n)), \frac{1}{2}\big[d(f(u), g(u)) + d(f(x_n), g(x_n))\big], \\
&\qquad \frac{1}{2}\big[d(f(x_n), g(u)) + d(f(u), g(x_n))\big]\Big\}\Big)\Big) \\
&\leq \lim_{n\to\infty} \alpha(f(u), f(x_n))max\Big\{d(f(u), f(x_n)), \frac{1}{2}\big[d(f(u), g(u)) + d(f(x_n), g(x_n))\big], \\
&\qquad \frac{1}{2}\big[d(f(x_n), g(u)) + d(f(u), g(x_n))\big]\Big\} \\
&= \lim_{n\to\infty} \alpha(f(u), f(x_n))max\Big\{d(f(u), f(x_n)), \frac{1}{2}\big[d(f(u), g(u)) + d(f(x_n), f(x_{n+1}))\big], \\
&\qquad \frac{1}{2}\big[d(f(x_n), g(u)) + d(f(u), f(x_{n+1}))\big]\Big\} \\
&\leq \frac{1}{2}d(f(u), g(u)) \leq \frac{1}{2}d(x, g(u)).
\end{aligned}$$

So $x = g(u) = f(u)$ on $X$. Therefore, by the weak compatiblity of $f$ and $g$, we have $f(x) = fg(u) = gf(u) = g(x)$.

**Claim:** $x$ is a common fixed point of $f$ and $g$.

$$\begin{aligned}
d(x, g(x)) &= d(g(u), g(x)) \\
&\leq F\Big(\alpha(f(u), f(x))max\Big\{d(f(u), f(x)), \frac{1}{2}\big[d(f(u), g(u)) + d(f(x), g(x)), \\
&\qquad \frac{1}{2}\big[d(f(u), g(x)) + d(f(x), g(u))\big]\Big\}, \\
&\qquad \psi\Big(\alpha(f(u), f(x))max\Big\{d(f(u), f(x)), \frac{1}{2}\big[d(f(u), g(u)) + d(f(x), g(x)), \\
&\qquad \frac{1}{2}\big[d(f(u), g(x)) + d(f(x), g(u))\big]\Big\}\Big)\Big) \\
&\leq \alpha(f(u), f(x))max\Big\{d(f(u), f(x)), \frac{1}{2}\big[d(f(u), g(u)) + d(f(x), g(x)), \\
&\qquad \frac{1}{2}\big[d(f(u), g(x)) + d(f(x), g(u))\big]\Big\} \\
&\leq d(x, g(x)).
\end{aligned}$$

Thus $F(d(x, g(x)), \psi(d(x, g(x)))) = d(x, g(x))$, which implies $d(x, g(x)) = 0$. So $x$ is a common fixed point of $f$ and $g$ on $X$.

**Uniqueness of common fixed point:**

Suppose that $x$ and $x'$ are common fixed points of $f$ and $g$. Then

$$\begin{aligned}
d(x,x') &= (f(x),f(x')) = d(g(x),g(x')) \\
&\le F\Big(\alpha(f(x),f(x'))max\Big\{d(f(x),f(x')),\frac{1}{2}\big[d(f(x),g(x))+d(f(x'),g(x'))\big], \\
&\qquad\qquad \frac{1}{2}\big[d(f(x),g(x'))+d(f(x'),g(x))\big]\Big\}, \\
&\quad \psi\Big(\alpha(f(x),f(x'))max\Big\{d(f(x),f(x')),\frac{1}{2}\big[d(f(x),g(x))+d(f(x'),g(x'))\big], \\
&\qquad\qquad \frac{1}{2}\big[d(f(x),g(x'))+d(f(x'),g(x))\big]\Big\}\Big)\Big) \\
&\le \alpha(f(x),f(x'))max\Big\{d(f(x),f(x')),\frac{1}{2}\big[d(f(x),g(x))+d(f(x'),g(x'))\big], \\
&\qquad\qquad \frac{1}{2}\big[d(f(x),g(x'))+d(f(x'),g(x))\big]\Big\} \\
&\le \alpha(f(x),f(x'))\max\Big\{d(x,x'),\frac{1}{2}\big[d(x,x')+d(x',x)\big]\Big\} \\
&\le d(x,x').
\end{aligned}$$

Therefore, $F(d(x,x'),\psi(d(x,x')) = d(x,x')$, which implies $d(x,x') = 0$. So $x$ is the unique common fixed point of $f$ and $g$ on $X$. □

**Definition 2.3.** Let $(X,d)$ be a metric space. Consider two self-mappings $f$ and $g$ on $X$ and let $\alpha : X \times X \to [0,1]$ be a function. Then $g$ is said to satisfy condition $(A)$ on $f$ if, for all $F \in \mathcal{C}$, $\psi \in \Psi$, $k \in [0,1)$ and for all $x,y \in X$,

$$\begin{aligned}
d(g(x),g(y)) \le k\ \alpha\big(f(x),f(y)\big)max\Big\{&(d(f(x),f(y)),\frac{1}{2}\big[d(f(x),g(x))+d(f(y),g(y))\big], \\
&\frac{1}{2}\big[d(f(x),g(y))+d(f(y),g(x))\big]\Big\}.
\end{aligned}$$

**Corollary 2.4.** *Let $(X,d)$ be a complete metric space and $f,g : X \to X$ mappings such that $g$ satisfies the condition $(A)$ on $f$. Then $f$ and $g$ have a unique common fixed point on $X$ if the following conditions are satisfied:*

(1) *$g(X) \subset f(X)$;*
(2) *$f(X)$ is complete;*
(3) *$f,g$ are weakly compatible.*

*Proof.* Choose $x_0 \in Y$ arbitrarily. Let $x_n \in X$ be the element such that $f(x_n) = g(x_{n-1})$ and define a function $F_1 : [0,\infty)^2 \to \mathbb{R}$ as $F_1(s,t) = ks$ for all $k \in [0,1)$ which is a *C-class* function. Since $f$ and $g$ satisfy the condition $(A)$,

$$\begin{aligned}
d(g(x),g(y)) \le\ k\ \alpha\big(f(x),f(y)\big)max\Big\{&(d(f(x),f(y)),\frac{1}{2}\big[d(f(x),g(x))+d(f(y),g(y))\big], \\
&\frac{1}{2}\big[d(f(x),g(y))+d(f(y),g(x))\big]\Big\}
\end{aligned}$$

$$= F_1\Big(\alpha\big(f(x),f(y)\big)max\Big\{(d(f(x),f(y)),\frac{1}{2}\big[d(f(x),g(x))+d(f(y),g(y))\big],$$
$$\frac{1}{2}\big[d(f(x),g(y))+d(f(y),g(x))\big]\Big\}$$
$$\psi\Big(\alpha\big(f(x),f(y)\big)max\Big\{(d(f(x),f(y)),\frac{1}{2}\big[d(f(x),g(x))+d(f(y),g(y))\big],$$
$$\frac{1}{2}\big[d(f(x),g(y))+d(f(y),g(x))\big]\Big\}\Big)\Big).$$

Hence by Theorem 2.2, $f$ and $g$ have a unique common fixed point in $X$. □

**Definition 2.5.** Let $(X,d)$ be a metric space. Let $f$ and $g$ be two self-mappings on $X$. Then $g$ is said to satisfy condition $(B)$ on $f$ if, for all $F\in\mathcal{C}$, $\psi\in\Psi$ and for all $x,y\in X$,

$$d(g(x),g(y))$$
$$\leq F\Big(max\Big\{(d(f(x),f(y)),\frac{1}{2}\big[d(f(x),g(x))+d(f(y),g(y))\big],\frac{1}{2}\big[d(f(x),g(y))+d(f(y),g(x))\big]\Big\},$$
$$\psi\Big(max\Big\{(d(f(x),f(y)),\frac{1}{2}\big[d(f(x),g(x))+d(f(y),g(y))\big],\frac{1}{2}\big[d(f(x),g(y))+d(f(y),g(x))\big]\Big\}\Big)\Big).$$

**Corollary 2.6.** *Let $(X,d)$ be a complete metric space and $f,g:X\to X$ mappings such that $g$ satisfies the condition $(B)$ on $f$. Then $f$ and $g$ have a unique common fixed point on $X$ if the following conditions are satisfied:*

(1) $g(X)\subset f(X)$;
(2) $f(X)$ *is complete;*
(3) $f,g$ *are weakly compatible.*

*Proof.* By Theorem 2.2, if $\alpha(x,y)=1$ for all $x,y\in X$, then the mappings $f$ and $g$ have a unique common fixed point on $X$ . □

**Definition 2.7.** Let $(X,d)$ be a metric space and $f,g$ be two self-mappings on $X$. Then $g$ is said to satisfy condition $(C)$ on $f$ if, for all $x,y\in X$ and $a,b,c\in[0,1)$,

$$d(g(x),g(y))\leq\max\Big\{a(d(f(x),f(y)),\frac{b}{2}\big[d(f(x),g(x))+d(f(y),g(y))\big],$$
$$\frac{c}{2}\big[d(f(x),g(y))+d(f(y),g(x))\big]\Big\}.$$

*Remark* 2.8. If we choose $f=I^X$ ($I^X$ is the identity mapping in the condition $(C)$, then we obtain the definition of Zamfierscu mapping [7].

**Corollary 2.9.** *Let $(X,d)$ be a complete metric space and $f,g:X\to X$ mappings such that $g$ satisfies the condition $(C)$ on $f$. Then $f$ and $g$ have a unique common fixed point on $X$ if the following conditions are satisfied:*

(1) $g(X)\subset f(X)$;
(2) $f(X)$ *is complete;*
(3) $f,g$ *are weakly compatible.*

*Proof.* Choose $x_0 \in Y$ arbitrarily. Let $x_n \in X$ be elements such that $f(x_n) = g(x_{n-1})$ and define a function $F_1 : [0,\infty)^2 \to \mathbb{R}$ as $F_1(s,t) = ks$ for all $k \in [0,1)$ which is a *C-class* function. Since $f$ and $g$ satisfy the condition $(C)$,

$$\begin{aligned} d(g(x),g(y)) &\le \max\Big\{a(d(f(x),f(y)), \frac{b}{2}d(f(x),g(x)) + d(f(y),g(y))\big], \\ &\qquad \frac{c}{2}\big[d(f(x),g(y)) + d(f(y),g(x))\big]\Big\} \\ &\le k\ \max\Big\{(d(f(x),f(y)), \frac{1}{2}\big[d(f(x),g(x)) + d(f(y),g(y))\big], \\ &\qquad \frac{1}{2}\big[d(f(x),g(y)) + d(f(y),g(x))\big]\Big\} \\ &= F_1\Big(max\Big\{(d(f(x),f(y)), \frac{1}{2}\big[d(f(x),g(x)) + d(f(y),g(y))\big], \\ &\qquad \frac{1}{2}\big[d(f(x),g(y)) + d(f(y),g(x))\big]\Big\}, \\ &\quad \psi\Big(max\Big\{(d(f(x),f(y)), \frac{1}{2}\big[d(f(x),g(x)) + d(f(y),g(y))\big], \\ &\qquad \frac{1}{2}\big[d(f(x),g(y)) + d(f(y),g(x))\big]\Big\}\Big)\Big). \end{aligned}$$

Hence by Corollary 2.6, $f$ and $g$ have a unique common fixed point in $X$. □

**Definition 2.10.** Let $(X,d)$ be a metric space and $f$ and $g$ two self-mappings on $X$. Then $g$ is said to satisfy condition $(D)$ on $f$ if, for all $x, y \in X$,

$$\begin{aligned} d(g(x),g(y)) &\le \max\Big\{(d(f(x),f(y)), \frac{1}{2}\big[d(f(x),g(x)) + d(f(y),g(y))\big], \\ &\qquad \frac{1}{2}\big[d(f(x),g(y)) + d(f(y),g(x))\big]\Big\} \\ &- \Psi\Big(\max\Big\{(d(f(x),f(y)), \frac{1}{2}\big[d(f(x),g(x)) + d(f(y),g(y))\big], \\ &\qquad \frac{1}{2}\big[d(f(x),g(y)) + d(f(y),g(x))\big]\Big\}\Big) \end{aligned}$$

**Corollary 2.11.** *Let $(X,d)$ be a complete metric space and $f, g : X \to X$ mappings such that $g$ satisfies the condition $(D)$ on $f$. Then $f$ and $g$ have a unique common fixed point on $X$ if the following conditions are satisfied:*

(1) $g(X) \subset f(X)$;
(2) $f(X)$ *is complete;*
(3) $f, g$ *are weakly compatible.*

*Proof.* Choose $x_0 \in Y$ arbitrarily. Let $x_n \in X$ be elements such that $f(x_n) = g(x_{n-1})$ and define a function $F_2 : [0,\infty)^2 \to \mathbb{R}$ as $F_2(s,t) = s - t$ which is a *C-class* function. Since $f$ and $g$ satisfies the condition $(D)$,

$$\begin{aligned} d(g(x),g(y)) &\le \max\Big\{(d(f(x),f(y)), \frac{1}{2}\big[d(f(x),g(x)) + d(f(y),g(y))\big], \\ &\qquad \frac{1}{2}\big[d(f(x),g(y)) + d(f(y),g(x))\big]\Big\} \end{aligned}$$

$$
\begin{aligned}
&- \Psi\Big( \max\Big\{(d(f(x),f(y)), \frac{1}{2}\big[d(f(x),g(x))+d(f(y),g(y))\big], \\
&\qquad\qquad \frac{1}{2}\big[d(f(x),g(y))+d(f(y),g(x))\big]\Big\}\Big) \\
&= F_2\Big( \max\Big\{(d(f(x),f(y)), \frac{1}{2}\big[d(f(x),g(x))+d(f(y),g(y))\big], \\
&\qquad\qquad \frac{1}{2}\big[d(f(x),g(y))+d(f(y),g(x))\big]\Big\}, \\
&\quad \psi\Big( \max\Big\{(d(f(x),f(y)), \frac{1}{2}\big[d(f(x),g(x))+d(f(y),g(y))\big], \\
&\qquad\qquad \frac{1}{2}\big[d(f(x),g(y))+d(f(y),g(x))\big]\Big\}\Big)\Big).
\end{aligned}
$$

Hence by Corollary 2.6, $f$ and $g$ have a unique common fixed point in $X$. □

**Definition 2.12.** [1] Let $X$ be a normed linear space. Then a set $Y \in X$ is called $q$-starshaped with $q \in Y$ if the segment $[q,x] = \{(1-k)q + kx : 0 \le k \le 1\}$ joining $q$ to $x$ is contained in $Y$ for all $x \in Y$.

**Definition 2.13.** Let $(X,d)$ be a metric space, $f, T$ two self-mappings on $X$ and let $\alpha : X \times X \to [0,1]$ be a function. Then $T$ is said to be a $f$-weakly generalized almost Zamfirescu mapping if, for all $x, y \in X$ and $a, b, c \in (0,1)$,

$$
\begin{aligned}
&\|T(x) - T(y)\| \\
&\le F\Big(\alpha\big(f(x)-f(y)\big) \max\Big\{a\|f(x)-f(y)\|, \frac{b}{2}\big[dist(f(x),[q,T(x)]) + dist(f(y),[q,T(y)])\big], \\
&\qquad\qquad \frac{c}{2}\big[dist(f(x),[q,T(y)]) + dist(f(y),[q,T(x)])\big]\Big\}, \\
&\quad \Psi\Big(\alpha\big(f(x)-f(y)\big) \max\Big\{a\|f(x)-f(y)\|, \frac{b}{2}\big[dist(f(x),[q,T(x)]) + dist(f(y),[q,T(y)])\big], \\
&\qquad\qquad \frac{c}{2}\big[dist(f(x),[q,T(y)]) + dist(f(y),[q,T(x)])\big]\Big\}\Big)\Big).
\end{aligned}
$$

**Theorem 2.14.** *Let $f$ and $T$ be self-mappings on a nonempty $q$-starshaped subset $Y$ of a Banach space $X$, where $T$ is a $f$-weakly generalized almost Zamfirescu mapping and satisfies the following conditions:*

1. *$f$ is linear and $q = f(q)$;*
2. *$T(X) \subset f(X)$;*
3. *$f(X)$ is complete;*
4. *$f, T$ are weakly compatible.*

*Define a mapping $T_n$ on $Y$ by*

$$T_n(x) = (1-\beta_n)q + \beta_n T(x),$$

*where $\{\beta_n\}$ is a sequence of numbers in $(0,1)$. Then for each $n$, $T_n$ and $f$ have exactly one common fixed point $x_n$ in $Y$ such that $f(x_n) = x_n = (1-\beta_n)q + \beta_n T(x_n)$. Also $T$ and $f$ have a common fixed point $x \in Y$. Moreover, if $\{x_n\}$ is Cauchy and $\lim_{n\to\infty} \beta_n = 1$, then $x_n \to x$.*

*Proof.* By definition,

$$
\begin{aligned}
&\|T(x) - T(y)\| \\
&\le F\Big(\alpha\big(f(x)-f(y)\big) \max\Big\{a\|f(x)-f(y)\|, \frac{b}{2}\big[dist(f(x),[q,T(x)]) + dist(f(y),[q,T(y)])\big],
\end{aligned}
$$

$$\frac{c}{2}\big[dist(f(x),[q,T(y)])+dist(f(y),[q,T(x)])\big]\Big\},$$

$$\Psi\Big(\alpha\big(f(x)-f(y)\big)\max\Big\{a\|f(x)-f(y)\|,\frac{b}{2}\big[dist(f(x),[q,T(x)])+dist(f(y),[q,T(y)])\big],$$

$$\frac{c}{2}\big[dist(f(x),[q,T(y)])+dist(f(y),[q,T(x)])\big]\Big\}\Big)\Big)$$

$$\leq\alpha\big(f(x)-f(y)\big)\max\Big\{a\|f(x)-f(y)\|,\frac{b}{2}\big[dist(f(x),[q,T(x)])+dist(f(y),[q,T(y)])\big],$$

$$\frac{c}{2}\big[dist(f(x),[q,T(y)])+dist(f(y),[q,T(x)])\big]\Big\},$$

$$\leq\max\Big\{a\|f(x)-f(y)\|,\frac{b}{2}\big[dist(f(x),[q,T(x)])+dist(f(y),[q,T(y)])\big],$$

$$\frac{c}{2}\big[dist(f(x),[q,T(y)])+dist(f(y),[q,T(x)])\big]\Big\},$$

$$\leq\max\Big\{a\|f(x)-f(y)\|,\frac{b}{2}\big[\|f(x)-T(x)\|+\|f(y)-T(y)\|\big],$$

$$\frac{c}{2}\big[\|f(x)-T(y)\|+\|f(y)-T(x)\|\big]\Big\}.$$

Therefore, by Corollary 2.9, $T$ and $f$ have a common fixed point $x\in Y$.

By definition,

$$\|T_n(x)-T_n(y)\|=\beta_n\|T(x)-T(y)\|$$

$$\leq\beta_n F\Big(\alpha\big(f(x)-f(y)\big)\max\Big\{a\|f(x)-f(y)\|,\frac{b}{2}\big[dist(f(x),[q,T(x)])+dist(f(y),[q,T(y)])\big],$$

$$\frac{c}{2}\big[dist(f(x),[q,T(y)])+dist(f(y),[q,T(x)])\big]\Big\},$$

$$\Psi\Big(\alpha\big(f(x)-f(y)\big)\max\Big\{a\|f(x)-f(y)\|,\frac{b}{2}\big[dist(f(x),[q,T(x)])+dist(f(y),[q,T(y)])\big],$$

$$\frac{c}{2}\big[dist(f(x),[q,T(y)])+dist(f(y),[q,T(x)])\big]\Big\}\Big)\Big)$$

$$\leq\beta_n\alpha\big(f(x)-f(y)\big)\max\Big\{a\|f(x)-f(y)\|,\frac{b}{2}\big[dist(f(x),[q,T(x)])+dist(f(y),[q,T(y)])\big],$$

$$\frac{c}{2}\big[dist(f(x),[q,T(y)])+dist(f(y),[q,T(x)])\big]\Big\},$$

$$\leq\beta_n\alpha\big(f(x)-f(y)\big)\max\Big\{\|f(x)-f(y)\|,\frac{1}{2}\big[\|f(x)-T(x)\|+\|f(y)-T(y)\|\big],$$

$$\frac{1}{2}\big[\|f(x)-T(y)\|+\|f(y)-T(x)\|\big]\Big\}.$$

Therefore, by Corollary 2.4, $T_n$ and $f$ have a common fixed point $x\in Y$.

By the assumption that $\{x_n\}$ is Cauchy, let us consider $\{x_n\}\to y$. If $\lim_{n\to\infty}\beta_n=1$, then

$$\begin{aligned}\|x_n-x\|&=\|T_n(x_n)-T(x)\|\\&=\|(1-\beta_n)p+\beta_nT(x_n)-T(x)\|\\&\leq\|(1-\beta_n)p\|+\beta_n\|T(x_n)-T(x)\|\end{aligned}$$

$$\begin{aligned}
&\leq \|(1-\beta_n)p\| + \beta_n F\Big(\alpha\big(f(x)-f(y)\big)\max\Big\{a\|f(x_n)-f(x)\|,\\
&\qquad\qquad \frac{b}{2}\big[dist(f(x_n),[q,T(x_n)])+dist(f(x),[q,T(x)])\big],\\
&\qquad\qquad \frac{c}{2}\big[dist(f(x_n),[q,T(x)])+dist(f(x),[q,T(x_n)])\big]\Big\},\\
&\qquad \Psi\Big(\alpha\big(f(x)-f(y)\big)\max\Big\{a\|f(x_n)-f(x)\|,\\
&\qquad\qquad \frac{b}{2}\big[dist(f(x_n),[q,T(x_n)])+dist(f(x),[q,T(x)])\big],\\
&\qquad\qquad \frac{c}{2}\big[dist(f(x_n),[q,T(x)])+dist(f(x),[q,T(x_n)])\big]\Big\}\Big)\Big)\\
&\leq \|(1-\beta_n)p\| + \beta_n \max\Big\{a\|f(x_n)-f(x)\|,\\
&\qquad\qquad \frac{b}{2}\big[\|f(x_n)-T(x_n)\|+\|f(x)-T(x)\|\big],\\
&\qquad\qquad \frac{c}{2}\big[\|f(x_n)-T(x)\|+\|f(x)-T(x_n)\|\big]\Big\}\\
&\leq \|(1-\beta_n)p\| + \beta_n \max\Big\{a\|x_n-x\|, \frac{b}{2}\Big(\frac{1-\beta_n}{\beta_n}\Big)\|x_n-p\|,\\
&\qquad\qquad \frac{c}{2}\big[\|x_n-x\|+\|x-\frac{1}{\beta_n}x_n+\Big(\frac{1-\beta_n}{\beta_n}\Big)p\|\big]\Big\}.
\end{aligned}$$

Letting $n\to\infty$, we obtain

$$\|y-x\| \leq \max\Big\{a\|y-x\|, 0, \frac{c}{2}\big[\|y-x\|+\|x-y\|\big]\Big\} \leq k\|y-x\|,$$

where $k=\max\{a,c\}$. Therefore, $x=y$ and so $x_n\to x$. □

## References


[1] M. Abbas, D. Ilic, Common fixed points of generalised almost nonexpasive mappings, *Filomat* **24:3** (2010), 11-18.

[2] A.H. Ansari, Note on $\varphi$-$\psi$-contractive type mappings and related fixed point, *The 2nd Regional Conference on Mathematics and Applications. Payame Noor University.* 2014, pp. 377–380.

[3] G.V.R. Babu, P.D. Sailaja, A fixed point theorem of generalized weakly contractive maps in orbitally complete metric spaces, *Thai J. Math.* **9** (2011), 1–10.

[4] S.K. Chatterjea, Fixed-point theorems, C. R. Acad. Bulgare Sci. **25** (1972), 727–730.

[5] J. Dugundji. A. Granas, Weakly contractive maps and elementary domain invariance theorem, *Bull. Soc. Math. Crèce* **19** (1978), 141–151.

[6] R. Kannan, Some results on fixed points, *Bull. Calcutta Math. Soc.* **60** (1968), 71–76.

[7] T. Zamfirescu, Fixed point theorems in metric spaces, *Arch. Math.* **23** (1972), 292–298.



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# COUPLED FIXED POINT THEOREM IN PARTIALLY ORDERED MODULAR METRIC SPACES AND ITS AN APPLICATION

ALİ MUTLU, KÜBRA ÖZKAN, AND UTKU GÜRDAL

Abstract. In this article, we extend certain coupled fixed theorem which was introduced for mappings having the mixed monotone property in various metric spaces to partially ordered modular metric spaces. In addition to this, we express some results about this theorem. Finally, we show using our main theorem that there exists a unique solution for a given nonlinear integral equation.

## 1. Introduction

The first time in literature, Guo and Lakshmikantham [12] introduced the concept of coupled fixed point in 1987. After that, Bhaskar and Lakshmikantham [5] introduced the concept of the mixed monotone property and expressed certain coupled fixed point theorems which are considered as the most interesting fixed point theorems for mappings having this property in ordered metric spaces. They showed the existence of a unique solution for a periodic boundary value problem. Since the coupled fixed point theorems in the study of nonlinear integral equations and differential equations are important tools, many researcher have studied them in various partially ordered metric spaces, e.g. [3, 4, 6, 13, 16, 18, 20, 21, 22, 23, 24].

Lately, a lot of significant results related to fixed point theorems have been extended to modular metric spaces which was introduced by Chistyakov via F-modular [7] in 2008 and developed the theory of this spaces in 2010 [8]. And then, many authors made various studies on these structures, e.g. [1, 2, 9, 10, 11, 14, 15, 17].

In this article, we extend certain coupled fixed theorem which was introduced for mappings having the mixed monotone property in various metric spaces to partially ordered modular metric spaces. In addition to this, we investigate some results about this theorem. Finally, we show using our main theorem that there exists a unique solution for a given nonlinear integral equation.

## 2. Modular Metric Spaces

Here, we express a series of definitions of some fundamental notions related to modular metric spaces.

**Definition 2.1.** [19] Let $X$ be a vector space on $\mathbb{R}$ and $\rho : X \to [0, \infty]$ be a function. If $\rho$ satisfies the following conditions, we call that $\rho$ is a modular on $X$:

(1) $\rho(0) = 0$;
(2) If $a \in X$ and $\rho(\gamma a) = 0$ for all numbers $\gamma > 0$, then $a = 0$;

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(3) $\rho(-a) = \rho(a)$, for all $a \in X$;
(4) $\rho(\gamma a + \theta b) \leq \rho(a) + \rho(b)$ for all $\gamma, \theta \geq 0$ with $\gamma + \theta = 1$ and $a, b \in X$.

Let $X \neq \emptyset$ and $\omega : (0, \infty) \times X \times X \to [0, \infty]$ be a function where $\lambda \in (0, \infty)$. Throughout this article, the value $\omega(\lambda, a, b)$ is denoted by $\omega_\lambda(a, b)$ for all $a, b \in X$ and $\lambda > 0$.

**Definition 2.2.** [8] Let $X \neq \emptyset$ and $\omega : (0, \infty) \times X \times X \to [0, \infty]$ be a function. If $\omega$ satisfies the following conditions for all $a, b, c \in X$, we call that $\omega$ is a metric modular on $X$:
**(m1)** $\omega_\lambda(a, b) = 0$ for all $\lambda > 0 \Leftrightarrow a = b$;
**(m2)** $\omega_\lambda(a, b) = \omega_\lambda(b, a)$ for all $\lambda > 0$;
**(m3)** $\omega_{\lambda+\mu}(a, b) \leq \omega_\lambda(a, c) + \omega_\mu(c, b)$ for all $\lambda, \mu > 0$.

From [8, 9], we know that as fix $a_0 \in X$, the two sets

$$X_\omega = X_\omega(a_0) = \{a \in X : \omega_\lambda(a, a_0) \to 0 \text{ as } \lambda \to \infty\}$$

and

$$X_\omega^* = X_\omega^*(a_0) = \{a \in X : \exists \lambda = \lambda(a) > 0 \text{ such that } \omega_\lambda(a, a_0) < \infty\}$$

are said to be modular spaces.

From [8, 9], the modular space $X_\omega$ can be equipped by a metric

$$d_\omega(a, b) = \inf\{\lambda > 0 : \omega_\lambda(a, b) \leq \lambda\}$$

which is generated by $\omega$ for any $a, b \in X_\omega$ where $\omega$ is a modular on $X$.

**Definition 2.3.** [15] Let $X_\omega$ be a modular metric space, $\{a_n\}_{n\in\mathbb{N}}$ be a sequence in $X_\omega$ and $C \subseteq X_\omega$. Then,

(1) $\{a_n\}_{n\in\mathbb{N}}$ is called a modular convergent sequence such that $a_n \to a,\ a \in X_\omega$ if
$$\omega_\lambda(a_n, a) \to 0 \text{ as } n \to \infty$$
for all $\lambda > 0$.
(2) $\{a_n\}_{n\in\mathbb{N}}$ is called a modular Cauchy sequence if and only if for all $\epsilon > 0$ there exists $n(\epsilon) \in \mathbb{N}$ such that $\omega_\lambda(a_n, a_m) < \epsilon$ for each $m, n \geq n(\epsilon)$ and $\lambda > 0$.
(3) $C$ is called complete modular if every modular Cauchy sequence $\{a_n\}$ in $C$ is a modular convergent in $C$.

## 3. Main Results

Let $\leq$ be a ordered relation and $X_\omega$ be a modular metric space. Throughout this article, $(X_\omega, \leq)$ denotes partially ordered modular metric space.

**Definition 3.1.** Let $(X_\omega, \leq)$ be a partially ordered modular metric space. The mapping $F : X_\omega \times X_\omega \to X_\omega$ has the mixed monotone property if $F$ holds the following conditions for any $a, b \in X_\omega$

$$a_1 \leq a_2 \Rightarrow F(a_1, b) \leq F(a_2, b),\ a_1, a_2 \in X_\omega$$

and

$$b_1 \geq b_2 \Rightarrow F(a, b_1) \leq F(a, b_2),\ b_1, b_2 \in X_\omega.$$

These imply that $F$ is monotone non-decreasing in $a$ and monotone non-increasing in $b$.

**Definition 3.2.** Let $X_\omega$ be a modular metric space, $(a,b) \in X_\omega \times X_\omega$ and $F : X_\omega \times X_\omega \to X_\omega$ be a mapping. $(a,b)$ is called a coupled fixed point of the mapping if

$$F(a,b) = a \text{ and } F(b,a) = b.$$

**Theorem 3.3.** *Let $(X_\omega, \leq)$ be a partially ordered complete modular metric space, the mapping $F : X_\omega \times X_\omega \to X_\omega$ has the mixed monotone property in $X_\omega$ and $k$, $l$ be non-negative constants such that $k + l < 1$. Suppose that we have the following condition for all $a, b, p, q \in X_\omega$ and $\lambda > 0$*

$$\omega_\lambda(F(a,b), F(p,q)) \leq k\omega_\lambda(a,p) + l\omega_\lambda(b,q) \tag{3.1}$$

*where $a \geq p$, $q \geq b$.*

*If there exist $a_0, b_0 \in X_\omega$ with $a_0 \leq F(a_0,b_0)$ and $b_0 \geq F(b_0,a_0)$, then $F$ has a unique coupled fixed point.*

*Proof.* Let $a_0, b_0 \in X_\omega$ with $a_0 \leq F(a_0,b_0)$ and $b_0 \geq F(b_0,a_0)$. We take $a_1, b_1 \in X_\omega$ with $a_1 = F(a_0,b_0)$ and $b_1 = F(b_0,a_0)$. Again we take $a_2, b_2 \in X_\omega$ with $a_2 = F(a_1,b_1)$ and $b_2 = F(b_1,a_1)$. Repeating this way, we obtain sequences $\{a_n\}$ and $\{b_n\}$ in $X_\omega$ with

$$a_{n+1} = F(a_n,b_n) \text{ and } b_{n+1} = F(b_n,a_n)$$

for all $n \in \mathbb{N}^+$. In view of mixed monotone property of $F$, we get

$$a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq a_{n+1} \leq \cdots$$

and

$$b_0 \geq b_1 \geq b_2 \geq \cdots \geq b_n \geq b_{n+1} \geq \cdots .$$

Then, by (3.1), we get

$$\begin{aligned} \omega_\lambda(a_n, a_{n+1}) &= \omega_\lambda(F(a_{n-1},b_{n-1}), F(a_n,b_n)), \\ &\leq k\omega_\lambda(a_{n-1},a_n) + l\omega_\lambda(b_{n-1},b_n) \end{aligned} \tag{3.2}$$

for all $n \in \mathbb{N}^+$, $\lambda > 0$ and $k + l < 1$. Similarly,

$$\begin{aligned} \omega_\lambda(b_n, b_{n+1}) &= \omega_\lambda(F(b_{n-1},a_{n-1}), F(b_n,a_n)), \\ &\leq k\omega_\lambda(b_{n-1},b_n) + l\omega_\lambda(a_{n-1},a_n) \end{aligned} \tag{3.3}$$

for all $n \in \mathbb{N}^+$, $\lambda > 0$ and $k + l < 1$. Therefore, letting

$$e_n = \omega_\lambda(a_n, a_{n+1}) + \omega_\lambda(b_n, b_{n+1})$$

for all $n \in \mathbb{N}^+$, $\lambda > 0$. Using equations (3.2) and (3.3), we get

$$\begin{aligned} e_n &= \omega_\lambda(a_n, a_{n+1}) + \omega_\lambda(b_n, b_{n+1}) \\ &\leq k\omega_\lambda(a_{n-1},a_n) + l\omega_\lambda(b_{n-1},b_n) + k\omega_\lambda(b_{n-1},b_n) + l\omega_\lambda(a_{n-1},a_n) \\ &= (k+l)(\omega_\lambda(a_{n-1},a_n) + \omega_\lambda(b_{n-1},b_n)) \\ &= (k+l)e_{n-1}. \end{aligned}$$

Then, we obtain that

$$0 \leq e_n \leq (k+l)e_{n-1} \leq (k+l)^2 e_{n-2} \leq \cdots \leq (k+l)^n e_0. \tag{3.4}$$

If $e_0 = 0$, then $e_0 = \omega_\lambda(a_0,a_1) + \omega_\lambda(b_0,b_1) = 0$. Therefore, we get $\omega_\lambda(a_0,a_1) = 0$ and $\omega_\lambda(b_0,b_1) = 0$. So, from condition (m2) of modular metric spaces, we get

$$a_0 \leq a_1 = F(a_0,b_0) \quad \text{and} \quad b_0 \geq b_1 = F(b_0,a_0).$$

This implies that $(a_0,b_0)$ is a coupled fixed point of $F$.

Now, let $e_0 > 0$. Preserving the generality, suppose that for $m, n \in \mathbb{N}$ and $n < m$, there exist $n_{\frac{\lambda}{m-n}} \in \mathbb{N}$ satisfying

$$\omega_{\frac{\lambda}{m-n}}(a_n, a_{n+1}) + \omega_{\frac{\lambda}{m-n}}(b_n, b_{n+1}) = e_n$$

for all $\frac{\lambda}{m-n} > 0$ and $n \geq n_{\frac{\lambda}{m-n}}$. We get

(3.5)
$$\begin{aligned} \omega_\lambda(a_n, a_m) &\leq \omega_{\frac{\lambda}{m-n}}(a_n, a_{n+1}) + \omega_{\frac{\lambda}{m-n}}(a_{n+1}, a_{n+2}) + \cdots + \omega_{\frac{\lambda}{m-n}}(a_{m-1}, a_m), \\ \omega_\lambda(b_n, b_m) &\leq \omega_{\frac{\lambda}{m-n}}(b_n, b_{n+1}) + \omega_{\frac{\lambda}{m-n}}(b_{n+1}, b_{n+2}) + \cdots + \omega_{\frac{\lambda}{m-n}}(b_{m-1}, b_m) \end{aligned}$$

for each $n < m$. Thus, from (3.4) and (3.5), we get

$$\begin{aligned} \omega_\lambda(a_n, a_m) + \omega_\lambda(b_n, b_m) &\leq (\omega_{\frac{\lambda}{m-n}}(a_n, a_{n+1}) + \omega_{\frac{\lambda}{m-n}}(b_n, b_{n+1})) + \cdots \\ &\quad + (\omega_{\frac{\lambda}{m-n}}(a_{m-1}, a_m) + \omega_{\frac{\lambda}{m-n}}(b_{m-1}, b_m)), \\ &= e_n + e_{n+1} + \cdots + e_{m-1}, \\ &\leq (k+l)^n e_0 + (k+l)^{n+1} e_0 + \cdots + (k+l)^{m-1} e_0, \\ &= ((k+l)^n + (k+l)^{n+1} + \cdots + (k+l)^{m-1}) e_0, \\ &\leq \frac{(k+l)^n}{1-(k+l)} e_0 \end{aligned} \tag{3.6}$$

for $n < m$ and $\lambda > 0$. Let $k + l = \delta$. Since there exists $n_0$ such that $\frac{\delta^{n_0}}{1-\delta} e_0 < \epsilon$, from (3.6), we have for each $n, m \geq n_0$ that

$$\omega_\lambda(a_n, a_m) + \omega_\lambda(b_n, b_m) < \epsilon$$

for an arbitrary $\epsilon > 0$. Then, $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences in $X_\omega$. Using completeness of $X_\omega$, we can talk about existence of $a, b \in X_\omega$ with

$$\lim_{n\to\infty} a_n = a \text{ and } \lim_{n\to\infty} b_n = b.$$

There exists $n_0 \in \mathbb{N}$ with $\omega_{\frac{\lambda}{2}}(a_n, a) < \frac{\epsilon}{2}$ and $\omega_{\frac{\lambda}{2}}(b_n, b) < \frac{\epsilon}{2}$ for all $n \geq n_0$, $\lambda > 0$ and every $\epsilon > 0$. So, from (3.1), we get

$$\begin{aligned} \omega_\lambda(F(a,b), a) &\leq \omega_{\frac{\lambda}{2}}(F(a,b), a_{n+1}) + \omega_{\frac{\lambda}{2}}(a_{n+1}, a) \\ &= \omega_{\frac{\lambda}{2}}(F(a,b), F(a_n, b_n)) + \omega_{\frac{\lambda}{2}}(a_{n+1}, a) \\ &\leq k\omega_{\frac{\lambda}{2}}(a_n, a) + l\omega_{\frac{\lambda}{2}}(b_n, b) + \omega_{\frac{\lambda}{2}}(a_{n+1}, a) \\ &< k\frac{\epsilon}{2} + l\frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= (k+l)\frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon \end{aligned}$$

for all $\lambda > 0$ and each $n \in \mathbb{N}$. Then, $\omega_\lambda(F(a,b), a) = 0$. So, $F(a,b) = a$. In similar way, we get $F(b,a) = b$. These imply that $(a,b)$ is a coupled fixed point of $F$. We assume that $F$ has an another coupled fixed point $(a^*, b^*)$. Then, for $\lambda > 0$ we get

$$\omega_\lambda(a^*, a) = \omega_\lambda(F(a^*, b^*), F(a,b)) \leq k\omega_\lambda(a^*, a) + l\omega_\lambda(b^*, b)$$

and

$$\omega_\lambda(b^*, b) = \omega_\lambda(F(b^*, a^*), F(b,a)) \leq k\omega_\lambda(b^*, b) + l\omega_\lambda(a^*, a).$$

Therefore, we get

$$\omega_\lambda(a^*, a) + \omega_\lambda(b^*, b) \leq (k+l)(\omega_\lambda(a^*, a) + \omega_\lambda(b^*, b)). \tag{3.7}$$

Since $k+l<1$, from (3.7) we get $\omega_\lambda(a^*,a)+\omega_\lambda(b^*,b)=0$ for all $\lambda>0$. Hence, we obtain that

$$\omega_\lambda(a^*,a)=0 \Leftrightarrow a^*=a \text{ and } \omega_\lambda(b^*,b)=0 \Leftrightarrow b^*=b.$$

Therefore, $(a,b)$ is a unique coupled fixed point of $F$. □

If we take equal the constants $k$, $l$ in Theorem (3.3), the following corollary is obtained.

**Corollary 3.4.** Let $(X_\omega,\leq)$ be a partially ordered complete modular metric space, the mapping $F: X_\omega \times X_\omega \to X_\omega$ has the mixed monotone property in $X_\omega$ and $k\in[0,1)$. Suppose that we have the following condition for all $a,b,p,q\in X_\omega$ and $\lambda>0$

$$\omega_\lambda(F(a,b),F(p,q)) \leq \frac{k}{2}(\omega_\lambda(a,p)+\omega_\lambda(b,q)) \tag{3.8}$$

where $a\geq p$, $q\geq b$.

If there exist $a_0,b_0\in X_\omega$ with

$$a_0\leq F(a_0,b_0) \text{ and } b_0\geq F(b_0,a_0),$$

then $F$ has a unique coupled fixed point.

**Theorem 3.5.** *Let $(X_\omega,\leq)$ be a partially ordered complete modular metric space. Suppose that $X_\omega$ satisfies the following conditions*

***(i)*** *if a non-decreasing sequence $a_n\to a$, then $a_n\leq a$ for all $n$,*

***(ii)*** *if a non-increasing sequence $b_n\to b$, then $b_n\geq b$ for all $n$.*

*Let the mapping $F: X_\omega\times X_\omega\to X_\omega$ has the mixed monotone property in $X_\omega$ and $k$, $l$ be non-negative constants such that $k+l<1$. Suppose that we have the following condition for all $a,b,p,q\in X_\omega$ and $\lambda>0$*

$$\omega_\lambda(F(a,b),(p,q)) \leq k\omega_\lambda(a,p)+l\omega_\lambda(b,q)$$

*where $a\geq p$, $q\geq b$.*

*If there exist $a_0,b_0\in X_\omega$ with*

$$a_0\leq F(a_0,b_0) \text{ and } b_0\geq F(b_0,a_0),$$

*then $F$ has a unique coupled fixed point.*

*Proof.* This proof can be made in analogy to the proof of Theorem (3.3). Here, it will be enough only to show that $F(a,b)=a$ and $F(b,a)=b$ for proof. Let $\epsilon>0$. Since $F(a_{n-1},b_{n-1})=a_n\to a$ and $F(b_{n-1},a_{n-1})=b_n\to b$, there exists $n_0\in\mathbb{N}$ with

$$\begin{array}{rcl}\omega_\lambda(a_n,a) & = & \omega_\lambda(F(a_{n-1},b_{n-1}),a)<\frac{\epsilon}{3}\\ \omega_\lambda(b_n,b) & = & \omega_\lambda(F(b_{n-1},a_{n-1}),b)<\frac{\epsilon}{3}\end{array} \tag{3.9}$$

for all $n\geq n_0$. Letting $n\geq n_0$ and using the equations (3.1) and (3.9), we get

$$\begin{array}{rcl}\omega_\lambda(F(a,b),a) & \leq & \omega_{\frac{\lambda}{2}}(F(a,b),a_{n+1})+\omega_{\frac{\lambda}{2}}(a_{n+1},a)\\ & = & \omega_{\frac{\lambda}{2}}(F(a,b),F(a_n,b_n))+\omega_{\frac{\lambda}{2}}(a_{n+1},a)\\ & \leq & k\omega_{\frac{\lambda}{2}}(a_n,a)+l\omega_{\frac{\lambda}{2}}(b_n,b)+\omega_{\frac{\lambda}{2}}(a_{n+1},a)\\ & < & k\frac{\epsilon}{3}+l\frac{\epsilon}{3}+\frac{\epsilon}{3}\\ & = & (k+l)\frac{\epsilon}{3}+\frac{\epsilon}{3}<\epsilon.\end{array}$$

Then, we get $F(a,b) = a$. In a similar way, we obtain that $\omega_\lambda(F(a,b),a) < \epsilon$ implies $F(b,a) = b$. On the other hand, uniqueness of the coupled fixed point of $F$ can be shown in a similar way with the proof of Theorem (3.3). □

**Corollary 3.6.** Let $(X_\omega, \leq)$ be a partially ordered complete modular metric space. Suppose that $X_\omega$ satisfies the following conditions;

**(i)** if a non-decreasing sequence $a_n \to a$, then $a_n \leq a$ for all $n$,

**(ii)** if a non-increasing sequence $b_n \to b$, then $b_n \geq b$ for all $n$.

Let the mapping $F : X_\omega \times X_\omega \to X_\omega$ has the mixed monotone property in $X_\omega$ and $k \in [0,1)$. Suppose that we have the following condition for all $a,b,p,q \in X_\omega$ and $\lambda > 0$

$$\omega_\lambda(F(a,b),(p,q)) \leq \frac{k}{2}(\omega_\lambda(a,p) + \omega_\lambda(b,q))$$

where $a \geq p$, $q \geq b$. If there exist $a_0, b_0 \in X_\omega$ with

$$a_0 \leq F(a_0,b_0) \text{ and } b_0 \geq F(b_0,a_0),$$

then $F$ has a unique coupled fixed point.

**Example 3.7.** Let $X_\omega = \mathbb{R}$. If we take the usual partial order $\leq$ in $\mathbb{R}$, then $(\mathbb{R}, \leq)$ is a partially ordered set. We define a mapping $\omega : (0,\infty) \times \mathbb{R} \times \mathbb{R} \to [0,\infty)$ by $\omega_\lambda(a,b) = \frac{|a-b|}{\lambda}$ for all $a,b \in \mathbb{R}$ and $\lambda > 0$. It can be said that $X_\omega$ is a complete modular metric space. We take a mapping $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $F(a,b) = \frac{a+b}{6}$. We easily see that $F$ has the mixed monotone property. Then, we have

$$\begin{aligned}
\omega_\lambda(F(a,b),F(p,q)) &= \omega_\lambda(\frac{a+b}{6},\frac{p+q}{6}) \\
&= \frac{|\frac{a+b}{6} - \frac{p+q}{6})|}{\lambda} \\
&= \frac{1}{6}\frac{|a-p+b-q|}{\lambda} \\
&\leq \frac{1}{6}(\frac{|a-p|}{\lambda} + \frac{|b-q|}{\lambda}) \\
&= \frac{1}{6}(\omega_\lambda(a,p) + \omega_\lambda(b,q))
\end{aligned}$$

for any $a,b,p,q \in X_\omega$ So, the equation (3.8) is satisfied for $k = \frac{1}{3}$. Therefore, from Corollary (3.4), $F$ has a unique coupled fixed point. Also, there are $a_0 = 0 \leq F(0,0) = F(a_0,b_0)$ and $b_0 = 0 \geq F(0,0) = F(b_0,a_0)$. It is obvious that $(0,0)$ is the coupled fixed point of $F$.

On the other hand, if $F : X_\omega \times X_\omega \to X_\omega$ is defined by $F(a,b) = \frac{a+b}{2}$, then $F$ satisfies the condition (3.8) for $k = 1$. Then, we get

$$\begin{aligned}
\omega_\lambda(F(a,b),F(p,q)) &= \omega_\lambda(\frac{a+b}{2},\frac{p+q}{2}) \\
&= \frac{|\frac{a+b}{2} - \frac{p+q}{2})|}{\lambda} \\
&= \frac{1}{2}\frac{|a-p+b-q|}{\lambda} \\
&\leq \frac{1}{2}(\frac{|a-p|}{\lambda} + \frac{|b-q|}{\lambda}) \\
&= \frac{1}{2}(\omega_\lambda(a,p) + \omega_\lambda(b,q)).
\end{aligned}$$

Therefore, coupled fixed points of $F$ are both $(0,0)$ and $(1,1)$. Namely, $F$ has not a unique the coupled fixed point. Then, the conditions $k<1$ in Corollary (3.4) and $k+l<1$ in Theorem (3.3) are the most appropriate conditions for satisfying the uniqueness of coupled fixed point.

## 4. Application

Here, we show that there exists a unique solution of a nonlinear integral equation using the Theorem (3.3).

We consider the following nonlinear integral equations:

$$\begin{aligned} a(s) &= \int_0^S f(s,a(t),b(t))dt,\ s\in[0,S]=I \\ b(s) &= \int_0^S f(s,b(t),a(t))dt,\ s\in[0,S]=I \end{aligned} \tag{4.1}$$

where $S\in\mathbb{R}^+$ $(S>0)$ and $f: I\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$.

Let $X_\omega=C(I,\mathbb{R})$ and $X_\omega$ be a partially ordered set. We define the order relation as follows:

$$a\leq b\Leftrightarrow a(s)\leq b(s)$$

for $a,b\in C(I,\mathbb{R})$ and all $s\in I$. We can easily see that $X_\omega$ is a complete modular metric space such that

$$\omega_\lambda(a,b)=\sup_{s\in I}\frac{|a(s)-b(s)|}{\lambda},$$

for all $a,b\in X$ and $\lambda>0$.

**Assumption 4.1.** There exist two non-negative constants $k$ and $l$ with $k+l<1$ such that

$$0\leq f(s,a,b)-f(s,p,q)\leq\frac{1}{S}\left(\frac{k(a-p)+l(q-b)}{\lambda}\right) \tag{4.2}$$

for all $s\in I$, $a,b,p,q\in X_\omega$ and $\lambda>0$ where $a\geq p$, $q\geq b$.

**Definition 4.2.** $(\alpha,\beta)\in C(I,\mathbb{R})\times C(I,\mathbb{R})$ is called a coupled lower and upper solution of the integral equations (4.1) if $\alpha(s)\leq\beta(s)$ and

$$\alpha(s)\leq\int_0^S f(s,\alpha(t),\beta(t))dt$$

$$\beta(s)\leq\int_0^S f(s,\beta(t),\alpha(t))dt$$

for all $s\in I$.

**Theorem 4.3.** *We suppose that the Assumption* (4.1) *is satisfied. The integral equations* (4.1) *have a unique solution in* $C(I,\mathbb{R})$ *if there exists a coupled lower and upper solution for equations* (4.1).

*Proof.* Let $X_\omega=C(I,\mathbb{R})$. $X_\omega$ is a partially ordered set if we define the order relation such that for $a,b\in C(I,\mathbb{R})$ and all $s\in I$

$$a\leq b\Leftrightarrow a(s)\leq b(s).$$

It is obvious that $X_\omega$ is a complete modular metric space such that

$$\omega_\lambda(a,b) = \sup_{s\in I} \frac{|a(s)-b(s)|}{\lambda} \tag{4.3}$$

for $a, b \in C(I,\mathbb{R})$ and all $\lambda > 0$. Also, we define a partial order on $X_\omega \times X_\omega = C(I,\mathbb{R}) \times C(I,\mathbb{R})$ such that

$$(a,b) \leq (p,q) \Rightarrow a(s) \leq p(s) \text{ and } q(s) \leq b(s)$$

for $(a,b),(p,q) \in X_\omega \times X_\omega$ and all $s \in I$.

Now, we define $F : X_\omega \times X_\omega \to X_\omega$ with

$$F(a,b)(s) = \int_0^S f(s,a(t),b(t))dt \tag{4.4}$$

for $a, b \in C(I,\mathbb{R})$ and $s \in I$. We need to show that $F$ has the mixed monotone property. If $a_1 \leq a_2$, that is, $a_1(s) \leq a_2(s)$ for all $s \in I$, by Assumption (4.1) we get

$$\begin{aligned} F(a_1,b)(s) - F(a_2,b)(s) &= \int_0^S f(s,a_1(t),b(t))dt - \int_0^S f(s,a_2(t),b(t))dt \\ &= \int_0^S (f(s,a_1(t),b(t)) - f(s,a_2(t),b(t)))dt \\ &\leq 0. \end{aligned}$$

Then, $F(a_1,b)(s) \leq F(a_2,b)(s)$ for all $s \in I$. That is, $F(a_1,b) \leq F(a_2,b)$.

Similarly, if $b_1 \geq b_2$, that is, $b_1(s) \geq b_2(s)$ for all $s \in I$, by Assumption (4.1), we get

$$\begin{aligned} F(a,b_1)(s) - F(a,b_2)(s) &= \int_0^S f(s,a(t),b_1(t))dt - \int_0^S f(s,a(t),b_2(t))dt \\ &= \int_0^S (f(s,a(t),b_1(t)) - f(s,a(t),b_2(t)))dt \\ &\leq 0. \end{aligned}$$

Then, $F(a,b_1)(s) \leq F(a,b_2)(s)$ for all $s \in I$. That is, $F(a,b_1) \leq F(a,b_2)$. Therefore, $F(a,b)$ is monotone nondecreasing in $a$ and monotone nonincreasing in $b$.

Now, we show that $F$ has a coupled fixed point. Let $a \geq p$ and $q \geq b$. Then, $a(s) \geq p(s)$ and $q(s) \geq b(s)$ for all $s \in I$. From equation (4.2), (4.3) and (4.4), we

obtain that

$$\begin{aligned}
\frac{|F(a,b)(s)-F(p,q)(s)|}{\lambda} &= \frac{|\int_0^S f(s,a(t),b(t))dt-\int_0^S f(s,p(t),q(t))dt|}{\lambda}\\
&= \frac{|\int_0^S (f(s,a(t),b(t))-f(s,p(t),q(t)))dt|}{\lambda}\\
&\leq \frac{\int_0^S |f(s,a(t),b(t))-f(s,p(t),q(t))|dt}{\lambda}\\
&\leq \frac{1}{S}\int_0^S (\frac{|k(a(t)-p(t))+l(q(t)-b(t))|}{\lambda})dt\\
&\leq \frac{1}{S}\int_0^S (k\frac{|a(t)-p(t)|}{\lambda}+l\frac{|b(t)-q(t)|}{\lambda})dt\\
&\leq \frac{1}{S}\int_0^S (k\sup_{z\in I}\frac{|a(z)-p(z)|}{\lambda}+l\sup_{z\in I}\frac{|b(z)-q(z)|}{\lambda})dt\\
&= \frac{1}{S}\cdot S\cdot (k\sup_{z\in I}\frac{|a(z)-p(z)|}{\lambda}+l\sup_{z\in I}\frac{|b(z)-q(z)|}{\lambda})\\
&= k\sup_{z\in I}\frac{|a(z)-p(z)|}{\lambda}+l\sup_{z\in I}\frac{|b(z)-q(z)|}{\lambda}
\end{aligned}$$

for all $\lambda>0$ which implies that

$$\frac{|F(a,b)(s)-F(p,q)(s)|}{\lambda}\leq k\sup_{z\in I}\frac{|a(z)-p(z)|}{\lambda}+l\sup_{z\in I}\frac{|b(z)-q(z)|}{\lambda}.$$

Therefore, we obtain that

$$\omega_\lambda(F(a,b),F(p,q))\leq k\omega_\lambda(a,p)+l\omega_\lambda(b,q)$$

for $a,b,p,q\in X_\omega$ and all $\lambda>0$. On the other hand, let $(\alpha,\beta)$ be a coupled lower and upper solution of the integral equations (4.1), then we get

$$\alpha(s)\leq F(\alpha,\beta)(s) \text{ and } \beta(s)\geq F(\beta,\alpha)(s)$$

for all $s\in I$ where $\alpha,\beta\in I$. Then,

$$\alpha\leq F(\alpha,\beta) \text{ and } \beta\geq F(\beta,\alpha).$$

We obtain that Theorem (3.3) is satisfied. Therefore, from Theorem (3.3), we get a unique solution $(a,b)\in X_\omega\times X_\omega$ of integral equations (4.1). □

## References


1. A.A.N. Abdou and M.A. Khamsi, *Fixed point results of pointwise contractions in modular metric spaces*, Fixed Point Theory Appl. **2013** (2013), Article ID 163.
2. A.A.N. Abdou and M.A. Khamsi, *Fixed points of multivalued contraction mappings in modular metric spaces*, Fixed Point Theory Appl. **2014** (2014), Article ID 249 .
3. I. Altun and G. Durmaz, *Some fixed point theorems on ordered cone metric spaces*, Rend. Circ. Mat. Palermo **58** (2009), 319-325.
4. I. Altun and H. Simsek, *Some fixed point theorems on ordered metric spaces and application*, Fixed Point Theory Appl. **2010** (2010), 17 pages, Article ID 621492.
5. T.G. Bhaskar and V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal. **65** (2006), 1379-1393.
6. L.j. Círíc and V. Lakshmikantham, *Coupled random fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Stoch. Anal. Appl. 27 (2009), 1246-1259.

7. V.V. Chistyakov, *Modular metric spaces generated by F-modulars*, Folia Math. **14** (2008), 3-25.
8. V.V. Chistyakov, *Modular metric spaces I basic concepts*, Nonlinear Anal. **72** (2010), 1-14.
9. V.V. Chistyakov, *Fixed points of modular contractive maps*, Doklady Mathematics **86** (2012), no.1, 515-518.
10. Y.J. Cho, R. Saadati and G. Sadeghi, *Quasi-contractive mappings in modular metric spaces*, J. Appl. Math. **2012** (2012), 5 pages Article ID 907951.
11. M.E. Ege and C. Alaca, *Some properties of modular S-metric spaces and its fixed point results*, Journal of Computational Analysis and Applications. **20** (2016), no.1, 24–33.
12. D. Guo and V. Lakshmikantham, *Coupled fixed points of nonlinear operators with applications*, Nonlinear Anal. **11** (1987), 623-632.
13. E. Karapinar, *Coupled fixed point theorems for nonlinear contractions in cone metric spaces*, Comput. Math. Appl. **59** (2010), 3656-3668.
14. E. Kilinc and C. Alaca, *A Fixed point theorem in modular metric spaces*, Adv. Fixed Point Theory **4** (2014), 199–206.
15. C. Mongkolkeha, W. Sintunavarat and P. Kumam, *Fixed point theorems for contraction mappings in modular metric spaces*, Fixed Point Theory Appl. **2011** (2011), 2011:93.
16. V. Lakshmikantham and L.j. Círíc, *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Anal. **70** (2009), 4341-4349.
17. P. Kumam, *Fixed point theorems for nonexpansive mapping in modular spaces*, Arch Math. (Brno) **40** (2004), 345–353.
18. N.V. Luong and N.X. Thuan, *Coupled fixed points in partially ordered metric spaces and application*, Nonlinear Anal. **74** (2011), 983992.
19. J. Musielak and W. Orlicz, *On modular spaces*, Studia Math. **18** (1959), 49–65.
20. A. Mutlu, N. Yolcu, B. Mutlu and N. Bildik, *On Common Coupled Fixed Point Theorems for Comparable Mappings in Ordered Partially Metric Spaces*, Abstract and Applied Analysis **2014** (2014), 6 pages Article ID 486384.
21. A. Mutlu, N. Yolcu and B. Mutlu, *Coupled Fixed Point Theorem for Mixed Monotone Mappings on Partially Ordered Dislocated Quasi Metric Spaces*, Global Journal of Mathematics **1** (2015), no.1, 12-17.
22. A. Mutlu, N. Yolcu and B. Mutlu, *Fixed Point Theorems in Partially Ordered Rectangular Metric Spaces*, British Journal of Mathematics & Computer Science **15** (2016), no.2, 1-9.
23. F. Sabetghadam, H.P. Masiha and A.H. Sanatpour, *Some coupled fixed point theorems in cone metric spaces*, Fixed Point Theory Appl. (2009) 8 pages Article ID 125426 doi:10.1155/2009/125426.
24. B. Samet, *Coupled fixed point theorems for a generalized MeirKeeler contraction in partially ordered metric spaces*, Nonlinear Anal. **72** (2010), 4508-4517.



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# Connected $m - K_n-$residual graph and its application in cryptology

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**Abstract**

The minimum order and extremal graph of a connected $m-K_n-$residual graph were first proposed by Erdös, Harary and Klawe, in addition to two important conjectures related to the graph. They addressed said conjectures solely from the perspective of $K_n-$residual graphs, and did not study the $m - K_n-$residual graphs $m > 1$. In this study, we found that with $n > 2m + 2$, the minimum order of connected $m - K_n-$residual graph was $(m + n)(m + 1)$ and the unique extremal graph was $K_{m+n} \times K_{m+1}$. This conclusion agreed with the conjectures proposed by Erdös, Harary and Klawe. Moreover, with $n = 2m + 2$, we identified at least two non isomorphic $m - K_n-$residual graphs; this did not correspond with the conjecture. The $m - K_n-$residual graph was determined only by $m, n$ and the maximum connected branch $r$. The relationship among $m, n$ and $r$ was similar to that among the parameters of Hill cryptosystem implementation steps. According to these observations and principle knowledge regarding Hill cryptosystem implementation, the novel binary cryptosystem related $m-K_n-$residual graphs was established. We also built a Hill password encryption algorithm that ensures the binary cryptosystem is effective. The complexity of the minimum order and extremal graphs of connected $m-K_n-$residual graphs make the ciphertext, plaintext, and relationship between the keys highly complex and give the binary cryptosystem favorable performance.

*Keywords:* Residual graph; Minimum order; Extremal graph; Isomorphic.

## 1 Introduction

The residual graph, which represents an important branch of graph theory, was first proposed by Erdös, Harary and Klawe[1], and has since been widely applied throughout information science, networking, computer science, and other fields[2-7]. By definition, a residual graph is built by removing points in the closed neighborhood $N(u)$ that are isomorphic with the original graph; each removed point the closed neighborhood $N(u)$ in the graph has the same nature as its counterpart the original graph. For example, $K_3$ is a highly stable triangle that retains its original shape but with even higher stability after the adjacent edges and vertices of $m - K_3-$residual graph are removed for $m$ times. In cryptology, an important component of information security[8-9], known information is denoted by graphs and kept in cipher form. Corresponding residual graphs are constructed according to their relevant definitions. Many residual graphs can be constructed, but it is difficult to select only one as the ciphertext: Only a sufficiently complex(i.e.,unique)representative residual graph can ensure information security. The complete graph $K_n$ is often utilized in the computer networking and computer aided design fields[10-12], making it a useful research object in regards to the minimum order and extremal graph of connected $m - K_n-$residual graph.

Erdös, Harary and Klawe originally defined this concept[1] and they concluded that as $n > 1, n \neq 2$, the minimum order of connected $m - Kn-$residual graph is $2(n + 1)$; as $n \neq 2, 3, 4$, $K_{n+1} \times K_2$ is the unique connected $K_n-$residual graph with the minimum order. They also proposed the following two conjectures for connected $m - K_n-$residual graphs.

**Conjecture 1:** *If $n \neq 2$, then every connected $m - K_n-$residual graph has at least min$\{2n(m + 1), (n + m)(m + 1)\}$ vertices.*

**Conjecture 2:** *If $n$ is large, there is a unique smallest connected $m - K_n-$residual graph.*

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It is difficult to study the minimum order and residua graph of connected $m-K_n-$residual graph. As shown in Fig.1, for example, we successfully constructed a residual graph with order of 6 from $m=2$ and $n=1$, but it was very difficult to prove that the minimum order was 6 or even that the graph as-drawn was the unique extremal graph.

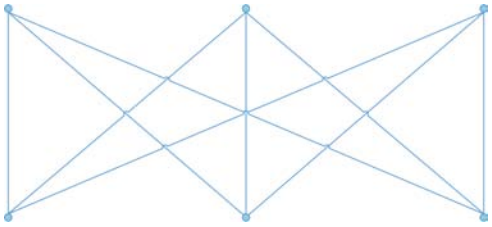


Fig.**1** $2-K_1-$residual graph.

There has been notable progress towards resolving problems related to the minimum order and extremal graph of connected $m-K_n-$residual graphs. Author [13], for example, investigated complete residual graphs of odd order to find that there is no $K_n-$residual graph of odd order for any odd n. Author [14] studied the $K_n-$residual graph to determine several other important properties and ultimately obtain the minimum order and extremal graph with $n=2,3,4$ . Author [15] explored the $2-K_n-$residual graph to find that when $n>5, n\neq 6$ , the results are in accordance with the two conjectures listed above. Other researchers [16] found that when $n\geq 9$ , the $3-K_n-$residual graph is also in accordance with the conjectures. Author [17] examined the nature of a connected residual graph when $n=3,4$. In this study, we explored a connected $m-K_n-$residual graph to find that when $n>2m+2$, the minimum order was $(m+n)(m+1)$ and the unique extremal graph was $K_{m+n}\times K_{m+1}$. This also agreed well with the conjectures described above. In addition, when $n=2m+2$, there were two isomorphic connected $m-K_n-$residual graphs as least−this does not align with the conjectures.

Recent years have seen rapid advancements in modern cryptography [18-23]. Information security has become especially important with the advent of the internet. Protecting information via secure, efficient cryptosystems is a popular research subject, to this effect. Hill cryptosystems [23-24] are widely used in encryption and digital signature applications, but do contain loopholes. Researchers are still looking for a more effective cryptosystem. At present, the relationship between graph theory and cryptography is understood primarily as a relationship between the block cipher and DNA algorithm components of graph theory [25-27]. In other words, graph theory is an important theoretical basis for cryptography.

Again, there are some loopholes in the security of the traditional Hill cryptosystem [28-30]. Despite notable achievements in improving cryptosystem security, there is much room for further improvement. In this paper, plaintext is denoted by $m, n$, and the maximum connected branch $r$. The parameters of the Hill cryptosystem and minimum order and extremal graph of the connected residual graph were carefully assessed, and a novel Hill password encryption algorithm was established. The binary cryptosystem was found to be secure due to the complexity of the minimum order and extremal graphs of connected $m-K_n-$residual graphs, which makes the ciphertext, plaintext, and relationship between the keys highly complex. The security of this binary cryptosystem was also found to be adjustable.

The remainder of this paper is organized as follows. Section 2 provides some background information on the residual graph concept, and Section 3 introduces the minimum order and extremal graph of the connected $m-K_n-$residual graph. In Section 4, the binary cipher is proposed according to several parameters of the connected $m-K_n-$residual graph and the security of the resulting system is discussed. Section 5 concludes the paper.

## 2 Preliminaries

The following concepts and results for solving the $m-K_n-$residual graphs.

**Definition 2.1** *$V(G)$ is the number of vertices in a graph $G$, $N(u)$ is the closed neighborhood of vertex $u\in V(G)$, and $N^*(u)$ are the original neighborhood and closed neighborhood of $u$ in $G$.*

**Definition 2.2** *For $u\in V(G)$, define $G_u=G-N[u]$. For convenience, we use notation $\langle S\rangle$ to mean $G[S]$(the subgraph induced by $S$ in $G$).*

**Definition 2.3** *Let $F \subset G$, then the degree of $F$ in $G$ is the cardinality of its boundary $d_G(F) = \sum d_G(x) - d_F(x), x \in F$.*

**Definition 2.4** *A graph $G$ is said to be a $F-$residual graph if for every vertex $v$ in $G$, the graph obtained from $G$ by removing the closed neighborhood of $v$ is isomorphic to $F$. We inductively define a multiply $-F-$residual graph by saying that $G$ is an $m-F-$residual graph if the removal of the closed neighborhood of any vertex of $G$ results in a $(m-1)-F-$residual graph, where of course a $1-F-$residual graph is simply said to be a $F-$residual graph.*

**Definition 2.5** *Let $X, Y \subset V(G)$, $X$ is said to be adjacent to $Y$, and viceversa, if there exist $x \in X$ and $y \in Y$, then $xy \in E(G)$. If $xy \in E(G)$ for all $x \in X$ and $y \in Y$, then $X$ is said to be complete adjacent to $Y$, and viceversa, for example, $X$ and $Y$ are said to be nonadjacent if there are no edges between them.*

**Definition 2.6** *Let $G_1$ and $G_2$ are two disjoint graphs the join, $G = G_1 + G_2$, of $G_1$ and $G_2$ is defined as follows $V(G) = V(G_1) \cup V(G_2)$ two vertex $u$ and $v$ are adjacent to each other, if and only if, $u \in V(G_1)$ and $v \in V(G_2)$, or $uv \in E(G_1)$ or $uv \in E(G_2)$.*

The known supporting results are summarized in the following Lemma.

**Lemma 2.1** *[1] If $G$ is a connected $F$-residual graph, then for any vertex $u$ in $G$, the degree $d(u) = \nu(G) - \nu(F) - 1$.*

**Lemma 2.2** *[17] If $G = G_1 + G_2$, then $G$ is $m - F-$residual graph, if and only if, both $G_1$ and $G_2$ are $m - F-$residual graphs.*

**Lemma 2.3** *[15] If $G$ is a $2 - K_n-$residual graph, when $\nu(G) = 3n + t$, $1 \leq 2n$, then it hasn't $u \in G$, which makes $d(u) = n + t - 1$, when $n \geq 5, n \neq 6$, then $\nu(G) = 3n + 6$, $G \cong K_{n+2} \times K_3$.*

**Lemma 2.4** *[16] If $G$ is a $3 - K_n-$residual graph, when $n \geq 11$, then $\nu(G) = 4n + 12$, $G \cong K_{n+3} \times K_4$.*

**Lemma 2.5** *[17] Assume $G$ is an $mK_n-$residual graph, $G \neq (m+1)K_n$, then $\nu(G) \geq 2(m+1)n$, and $K_{m+1,m+1}[K_n]$ is the unique extremal graphs.*

# 3 On connected $m - K_n-$residual graphs

In order to obtain the minimum order and extremal graph of the connected $m - K_n-$residual graph, we need the following Lemma.

**Lemma 3.1** *Assume $G$ is an $m - K_n-$residual graph, $m \geq 2$, $u \in G$, and $F \subset G$, $F$ is the maximum connected subgraph, and $F$ is a $r - K_n-$residual graph, $(0 \leq r \leq m-2)$, if $\nu(G) < \nu(F) + 2(m-r)n$, then there certainly a $v \in G_u$, which makes $G_v = G - N[v]$ connected.*

**Proof.** In the following, by reductio absurdum, we suppose there don't exist vertexes $v \in G_u$, which make $G_v = G - N[v]$ connected, then we need to prove $\nu(G) \geq \nu(F_1) + 2(m-r)n$. Set $H = \{F | F \subset G_v, v \in G_u, u \in F\}$. Let $F_1 \in H$, $\nu(F_1) = max\{\nu(F) | F \in H\}$, and $G_v = F_1 \cup G_1$, then $F_1$ is a $r - K_n-$residual graph. When $r = 0$, $F_1 \cong K_n$, then $G_1$ is an $(m-r-2) - K_n-$residual graph.

For every $w \in (G - N(F_1)) \subset G_u$, let $G_w = F_2 \cup G_2$, where $F_1 \subset G_w$, hence $F_1 \subset F_2$, because $F_1$ is the biggest connected components, then $F_2 = F_1$, $G_w = F_1 \cup G_2$, $N(F) \cap G_2 = \emptyset$, $G_2 \subset [G - N(F_1)]$, $G_2$ is an $(m-r-2) - K_n-$residual graph too, $G - N(F_1)$ is a $(m-r-1) - K_n-$residual graph, and $\nu(G - N(F_1)) \geq (m-r)n$.

Let $N(F_1) - F_1 = X$, then $G - N(F_1)$ is complete adjacent to $X$, in the following, we have discussed $F_1$ according to three conditions.

**Case 1.** $F_1$ is complete adjacent to $X$, then

$$\begin{aligned} G &= \langle N(F_1) \cup (G - N(F_1)) \rangle = \langle X \cup F_1 \cup (G - N(F_1)) \rangle \\ &= \langle X \rangle + \langle F_1 \cup (G - N(F_1)) \rangle = \langle X \rangle + \langle G - X \rangle, \end{aligned}$$

because $|X| \geq (m+1)n$, then

$$\nu(G) \geq (m+1)n + \nu(F_1) + (m-r)n \geq \nu(F_1) + 2(m-r)n.$$

**Case 2.** $F_1$ is a $(r+1)$ independent, let $\{u_0, u_1, \cdots, u_r\} \subset F_1$, let $X - N(u_0, u_1, \cdots, u_r) = X_1$, then $X_1 \neq \emptyset$, hence we have

$$G^{'} = G - N(u_0, u_1, \cdots, u_r) = \langle X \rangle + \langle G - N(F_1) \rangle,$$

by Lemma 2.2, then $G^{'}$ is a $(m-r-1) - K_n-$residual graph, and $|X_1| \geq (m-r)n$, according to the proving in Case 1, we know that $\nu(G) \geq \nu(F_1) + 2(m-r)n$.

**Case 3.** There is a $l$ independent set and $l < r+1$. Set $u_1, u_2, \cdots, u_l\} \subset F_1$, let $X - N(u_1, u_2, \cdots, u_l) = X_2$, then $X_2 \neq \emptyset$. Let

$$G^{''} = G - N(u_1, u_2, \cdots, u_l) = \langle X_1 \rangle + \langle G^{''} \cup (G - N(F_1)) \rangle,$$

because$|X| \geq |X_2| \geq (m-l+1)n > (m-r)n$, according to the proving in Case 1, we know that $\nu(G) \geq \nu(F_1) + 2(m-r)n$.

From the proves above, we know that if $\nu(G) < \nu(F) + 2(m-r)n$, then there certainly exist a vertices $v \in G_u$, which makes $G_v = G - N[v]$ connected.

**Theorem 3.1** *Assume $G$ is an $m-K_n-$residual graph, $m \geq 2$, if $\delta(G) = n$, then $\nu(G) \geq (m+3)n+m-1$, and the Fig.3 is the only extremal graph.*

**Proof.** We take induction to m, let $G_1$ be a $2 - K_n-$residual graph, if exist $w \in G_{1u}$, which makes $G_w = G_1 - N(w)$ connected, then $d_{G_1}(w) = n + t - 2$, $d(w) = n + t - 1$, $G_w = H_1 \cup H_2 \cong 2K_n$ and $u \in H_1 \cong K_n, i = 1, 2$. By Lemma 3.1,

$$\nu(G_1) \geq \delta(G) + \nu(G_w) + 1 \geq n + 4n + 1 = 5n + 1,$$

$G_w$ is a $K_n-$residual graph, when $\nu(G_1) = 5n+1$, we can specify the Fig.1 is the minimum $2-K_n-$residual graph.

Suppose that Theorem is true for $m \geq 2$, and assume $G$ is an $(m+1) - K_n-$residual graph, and $\delta(G) = n$, let $u \in G$, $d(u) = n$, and assume $\nu(G) \leq (m+4)n + m$. In the following, we prove that it must exist a vertex $v \in G_u$ with the case of the above assumptions, which makes $G_v$ connected. By contrary, by Lemma 3.1, we have $\nu(G) \geq \nu(F) + 2(m-r+1)n$, and $F$ is a connected $r - K_n-$residual graph, if $F \cong K_n$. We have

$$\begin{aligned} \nu(G) &\geq n + 2(m+1)n = (m+4)n + (m-1)n \\ &= (m+4)n + m + (m-1)n - 1) - 1 > (m+4)n + m, \end{aligned}$$

this is a contradiction. Since $u \in F$, $d(w) = n$, we have $r \neq 1$. If $r \geq 2$, by induction hypothesis of $F$ is a connected $r - K_n-$residual graph with $\delta(F) = n$, hence $\nu(F) \geq (r+3)n + r - 1$ and

$$\begin{aligned} \nu(G) &\geq (r+3)n + r - 1 + 2(m+1-r)n \\ &= (m+4)n + (m+1-r)n + r - 1 \\ &> (m+4)n + m, \end{aligned}$$

which contradicts that $\nu(G) \leq (m+4)n+m$. So we have there must be a vertex $v \in G_u$, which makes $G_v$ connected. Let $u \in G_v$, then $n \leq \delta(G_v) \leq d_{G_v}(u) \leq d(u) = n$, so we have $\delta(G_v) = n$, and by induction hypothesis, we have $\nu(G_u) \leq (m+3)n + m - 1$, and by Lemma 2.1,

$$\begin{aligned} \nu(G) &= \nu(G_u) + d(v) + 1 \\ &\geq (m+3)n + m - 1 + n + 1 \\ &= (m+4)n + m. \end{aligned}$$

So we have $\nu(G) = (m+4)n + m$, and $\nu(G) < (m+4)n + m$ is not true, then $\nu(G) \geq (m+4)n + m$. When $\nu(G) = (m+4)n+m$, we can construct $(m-1)-K_n-$residual graph and $m-K_n-$residual graph, the Fig.2 is the minimum $(m-1)-K_n-$residual graph, and the Fig.3 is the minimum $m-K_n-$residual graph.

In the following, we have proved the Fig.3 is the only extremal graph. Let $v \in G_u$, $G_v = X_1 \cup X_2 \cup Y_1 \cup Y_2 \cup V_j$, $j = 3, 4, \cdots, m+1$, $X_i \cong Y_i, i = 1, 2$, $V_j \cong K_{n+1}$. Let $V_j = \{x_j, y_j\} \cup C_j$, $j = 3, 4, \cdots, m+1$, and let $X = X_1 \cup X_2 \cup \{x_3, x_4, \cdots, x_{m+1}\}$, $Y = Y_1 \cup Y_2 \cup \{y_3, y_4, \cdots, y_{m+1}\}$, then $X$ is complete adjacent to $Y$. Suppose $u \in C_3$, $G_u$ is the Fig.2, we have $G = \langle N(v) \cup G_v \rangle$, $G_u = \langle (G_v - N(u)) \cup N(v) \rangle = \langle ((G_u) - V_3) \cup V_{m+2} \rangle$, where $V_{m+2} = N(u) = \{x_{m+2}, y_{m+2}\} \subset C_{m+2}$, $x_{m+2}$ is complete adjacent to $Y - \{y_3\}$, $y_{m+2}$ is complete adjacent to $X - \{x_3\}$, hence we have let $X^* = X \cup \{x_{m+2}\}$, $Y^* = Y \cup \{y_{m+2}\}$, then $X^*$ is complete adjacent to $Y^*$, so that the Fig.3 of $G$ is the only extremal graph. □

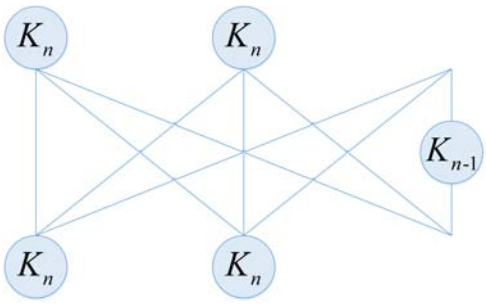



Fig.**1** $2-K_n-$residual graph with order $5n+1$

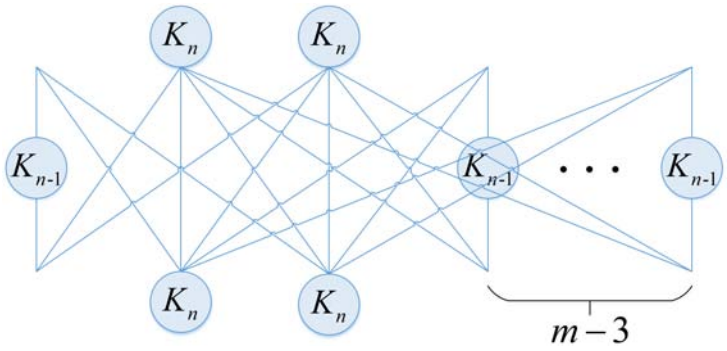



Fig.**2** $(m-1)-K_n$-residual graph with order $(m+2)n+m-2$

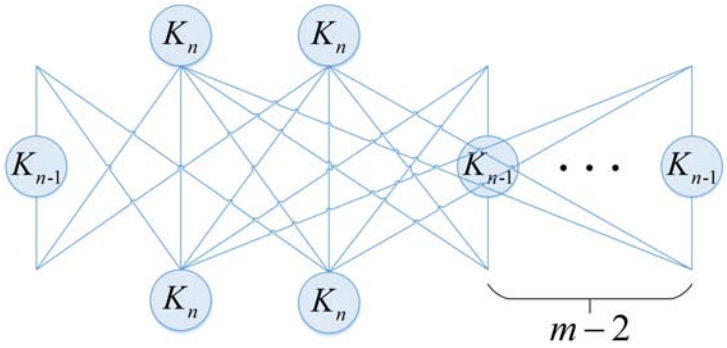



Fig.**3** $m-K_n$-residual graph with order $(m+3)n+m-1$

On the basis of the conclusion from literature [15] and the definition of $m-K_n-$residual graph, we have

**Theorem 3.2** *Let $G$ be a connected $m-K_n-$residual graph, $\nu(G)=(m+1)n+t$, $1\leq t\leq 2n$, $m>2$, $n>4$, then $G$ has no vertex of degree $n+t-1$.*

**Proof.** We take induction to $m$, when $m=2$, by Lemma 2.3, $G$ has no vertex of degree $n+t-1$. When $m=3$, $G$ is a connected $3-K_n-$residual graph $\nu(G)=4n+t, 1\leq t\leq 2n$. Supposing there exist such vertices like $u\in G$, which make $d(u)=n+t-1$ and $G_u=H_1\cup H_2\cup H_3=3K_n$, because $\nu(G)=4n+t, 1\leq t\leq 2n$, then $\nu(G)\leq 6n\leq 6n+2$. By Lemma 3.1, there exists at least $v\in G_u$, and we might as well suppose $v\in H_3$, which makes $G_v$ connected, then $G_v-N(u)=H_1\cup H_2\cong 2K_n$, and by Lemma 2.1, $\nu(G_v)=\nu(G)-d(v)-1$, let $d(v)=n-r$, $\nu(G_v)=4n+t-n-r-1=3n+t_1$, let $t_1=n-r-1$, $\nu(G_v)=3n+t_1, 1\leq t_1\leq 2n$, because $G_v$ is $2-K_n-$residual graph, it is contradictory to the conclusion of literature [15] that there are no vertexes makes $d(u)=n+t-1$.

We suppose let $G$ be a connected $m-K_n-$residual graph, and there doesn't exist such vertices like $u\in G$, which make $d(u)=n+t-1$. For $G$ be a connected $(m+1)-K_n-$residual graph, $\nu(G)=(m+2)n+t$, $1\leq t\leq 2n$, $m>2$, $n>4$. Supposing there exists $u\in G$, which makes $d(u)=n+t-1$, then $G_u=H_1\cup H_2\cup\cdots\cup H_{m+1}=(m+1)K_n$, because $\nu(G)=(m+2)n+t, 1\leq t\leq 2n$, then

$$\begin{aligned}\nu(G)&\leq (m+2)n+2n=mn+4n\\&<mn+4n+m=(m+4)n+m,\end{aligned}$$

by Lemma 2.3 and Lemma 3.1, there exists $v\in G_u$, let $v\in H_{m+1}$, which makes $G_v$ is connected then

$$G_v - N(u) = H_1 \cup H_2 \cup \cdots \cup H_m \cong mK_n,$$

and by Lemma2.1, $\nu(G_v) = \nu(G) - d(v) - 1$, let $d(v) = n - r$, $\nu(G_v) = (m+2)n + t - n - r - 1 = (m+1)n + t_1$, $t_1 = n - r - 1$ , it is contradictory to the induction and assumption that $G_v$ is an $m - K_n-$residual graph, then there don't exist vertices, which make $d(u) = n + t - 1$, $u \in G$.□

**Theorem 3.3** *There are at least two non isomorphic minimum $m-K_{n_1}-$residual graph, One is $K_{m+n_1} \times K_{m+1}$, and the other is $G_m[K_{\frac{n_1}{2}}]$, where $G_m \cong m - K_2-$residual graph, $n_1 = 2m + 2$, and this example doesn't meet the conclusion Erdös and Harary and Klawe made in [1].*

**Proof.** Let $G \cong K_{m+n_1} \times K_{m+1}$, $\nu(G) = (m + n_1)(m + 1)$, and $G_m \cong m - K_2-$residual graph, by literature [1], $\nu(G_m) \geq 3m + 2$, $\nu(G_m[K_{\frac{n_1}{2}}]) = \frac{n_1}{2}(3m + 2)$, and $n_1 = 2m + 2$, so when $n = n_1$, it doesn't meet the conclusion Erdös and Harary and Klawe made that only one $K_{m+n_1} \times K_{m+1}$ is an $m - K_{n_1}-$residual graph. □

**Theorem 3.4** *Let $G$ be a connected $m - K_n-$residual graph, if $n > 2m + 2$, $m \geq 3$, then $\nu(G) \geq (m+n)(m+1)$, and when $\nu(G) = (m+n)(m+1)$, $G \cong K_{m+n} \times K_{m+1}$ is a connected $m - K_n-$residual graph of minimum order, it is only such graph, so we show that the conjectures are true, when $n > 2m+2$.*

**Proof.** At first, we prove $\nu(G) \geq (m + n)(m + 1)$ and construct $m - K_n-$residual graph, and show the conjecture [1] is true.

Let $G_1$ be $2 - K_n-$residual graph, when $n > 6$, by Lemma 2.3, $G_1 \cong K_{n+2} \times K_3$, and $G_1$ is the Fig.4, all points in the same square are mutually adjacent, and two vertices, which are joined by a line, are adjacent. Let $u \in G$, we have $G_u$ is $(m - 1) - K_n-$residual graph, suppose $\nu(G_u) = (m + n - 1)m$, and $G_u \cong K_{m+n-1} \times K_m$. Let

$$G_u = \langle \overline{H_1} \cup \overline{H_2} \cdots \cup \overline{H_m} \rangle = \{x_r^j |_{r=1,2,\cdots,m}^{j=1,2,\cdots,n+m-1}\}, \tag{3.1}$$

where $\overline{H_r} = \langle x_r^1, x_r^2, \cdots, x_r^{n+m-1} \rangle$, and if $i = j$, then $x_l^i$ is adjacent to $x_k^j$, if $i \neq j$, $x_l^i$ is nonadjacent to $x_m^j$, $l \neq k, k, l = 1, 2, \cdots, m$, adjacent to the Fig.5, all points in the same square are mutually adjacent, and two vertices, which are joined by a line, are adjacent. According to the induction and assumption that $\nu(G_u) \geq (m + n - 1)m$. When $G_u$ is disconnected, by Lemma 2.5, let $G_u = (m - 1)nK_n$, and $\nu(G_u) \geq 2(m - 1)n$. Because $n \geq 2m + 2, m \geq 3$, then $\nu(G_u) \geq 2(m - 1)n > (m + n - 1)m$, by definition of residual graph, if and only if, when $G_u$ is connected, $\nu(G)$ is minimum. According to the Fig.5, we can construct $m - K_n-$residual graph in the Fig.6, when $\nu(G) = (m + n)(m + 1)$. The Fig.6, all points in the same square are mutually adjacent, and two vertices, which are joined by a line, are adjacent. By $G_u$ is connected and Lemma 2.1, $\nu(G) = \nu(G_u) + d(v) + 1$, according to the induction and assumption that $\nu(G_u) \geq (m + n - 1)m$, and by the Fig.6, $d(u) \geq n + 2m - 1$, and by Lemma 2.1, hence

$$\begin{aligned} \nu(G) &= \nu(G_u) + d(v) + 1 \\ &\geq (m + n - 1)m + n + 2m - 1 + 1 \\ &= (m + n)(m + 1). \end{aligned}$$

In the following, exist a vertex $u \in G$, which makes $G_u$ connected. Because $\nu(G) = (m + n)(m + 1)$, by Lemma 3.1, when $F$ is $r - K_n-$residual graph, $0 \leq r \leq (m - 2)$, $\nu(G) = \nu(F) + 2(m - r)n$. In the following, we prove $\nu(G) = \nu(F) + 2(m - r)n \geq (m + n)(m + 1)$, when $F \cong K_n$, and $n > 2m + 2$, we have $\nu(G) \geq n + 2mn > (m + n)(m + 1)$, if $r \geq 1$ by induction hypothesis $F$ is a connected $r - K_n-$residual graph. So $\nu(F) \geq (r + n)(r + 1)$, and

$$\nu(G) = \nu(F) + 2(m - r)n = (r + n)(r + 1) + 2(m - r)n,$$

because of $(r + n)(r + 1) + 2(m - r)n - (m + n)(m + 1) > (m - r)^2 + m - r > 0$, then

$$\begin{aligned} \nu(G) &= \nu(F) + 2(m - r)n \\ &= (r + n)(r + 1) + 2(m - r)n \\ &> (m + n)(m + 1). \end{aligned}$$

So it must exist a vertex $u \in G$, which makes $G_u$ connected.

From the proves above, we know $\nu(G) \geq (m + n)(m + 1)$, when $n > 2m + 2$, because $2n(m + 1) > (m + n)(m + 1)$, so we show the conjecture [1] is true.

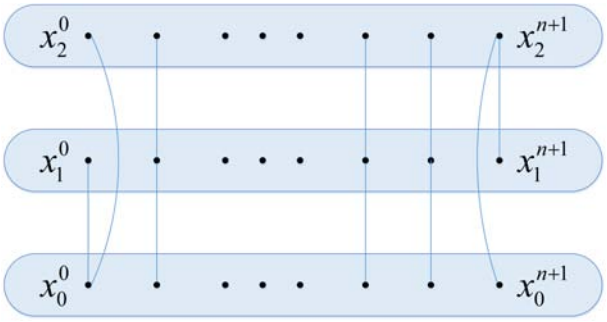



Fig.**4** $2-K_n-$residual graph with minimum order $3n+6$

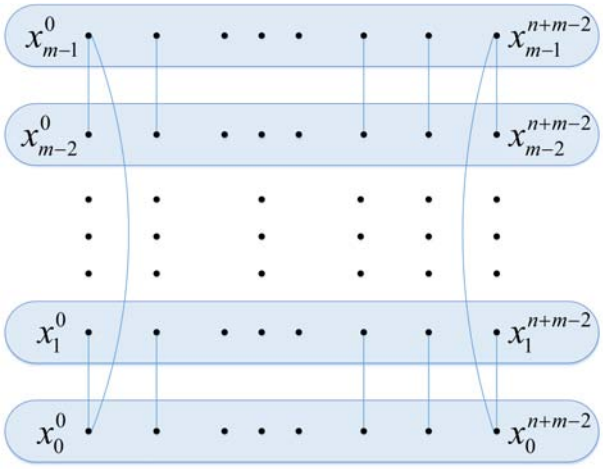



Fig.**5** $(m-1)-K_n-$residual graph with minimum order $(m+n-1)m$

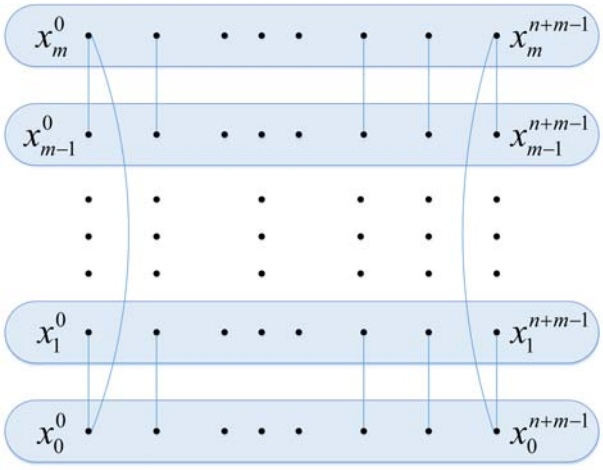



Fig.**6** $m-K_n-$residual graph with minimum order $(m+n)(m+1)$

In the following, we prove uniqueness, we have a discussion according to five conditions, so we show that the conjecture [2] is true.

**Fact 1.** Let

$$F = H_1 \cup H_2 \cup \cdots \cup H_m \cong K_{m+n-1} \times K_m,$$

where $H_i \cong K_{n+m-1}, i = 1, 2, \cdots, m$, then $H_i$ and $H_j$ have bijection

$$\theta : V(H_i) \to V(H_j),\ i, j = 1, 2, \cdots, m,$$

and $u_i \in H_i$ is adjacent to $\theta(u_i) \in H_j$, where $i \neq j$, $i, j = 1, 2, \cdots, m$. If $H \subset F$, and $H \subset K_s$, $3 \leq s \leq n+m-1$, then $H \subset H_w$, $1 \leq w \leq n$.

**Fact 2.** Let $u \in G$, $\nu(G_u) = (m+n-1)(m-1)$, hence $G_u \cong K_{m+n-1} \times K_m$. Adjacent to (3.1), let

$G_2 = G - N[x_2^{n+m-1}] = \langle H_0^* \cup H_1^* \cup H_3^* \cup \cdots \cup H_m^* \rangle \cong K_{m+n-1} \times K_m$,
we have

$$K_{m+n-2} \cong \overline{H_1} - x_1^{n+m-1} = \langle x_1^1, x_1^2, \cdots, x_1^{n+m-2} \rangle \subset G_2$$

by Fact 1, without loss of generality, we may assume that

$$\begin{gathered}\langle x_1^1, x_1^1, \cdots, x_1^{n+m-2} \rangle \subset H_1^* = \langle x_0^0, x_1^1, \cdots, x_1^{n+m-2} \rangle \\ \langle x_2^1, x_2^2, \cdots, x_2^{n+m-2} \rangle \subset H_2^* = \langle x_2^0, x_2^1, \cdots, x_2^{n+m-2} \rangle \\ \cdots\cdots \\ \langle x_m^1, x_m^2, \cdots, x_m^{n+m-2} \rangle \subset H_m^* = \langle x_m^0, x_m^1, \cdots, x_m^{n+m-2} \rangle \end{gathered} \tag{3.2}$$

If $x_0^j$ is adjacent to $x_1^j$, where $j = 0, 1, \cdots, n+m-1$. Obvious $x_0^0 = u$, then $H_0^* = \langle x_0^0, x_0^1, \cdots, x_0^{n+m-2} \rangle$.

We now prove $x_1^0$ is adjacent to $x_1^{n+m-1}$. Suppose the contrary, let $G_3 = G - N[x_1^{n+m-1}]$, $x_1^0 \in G_3$, by (3.1), (3.2), we have $x_1^0$ is adjacent to $\{x_1^1, x_1^2, \cdots, x_1^{n+m-1}\} \subset N[x_1^{n+m-1}]$. Thus

$$\begin{aligned} d(x_1^0) &\geq d_{G_3}(x_1^0) + n + m - 2 \\ &= n + m - 1 + n + m - 2 \\ &> n + m - 1 - m = n + 2m - 1. \end{aligned}$$

So $x_1^0$ is adjacent to $x_1^{n+m-1}$, $x_1^0$ is adjacent to $\overline{H_1}$. Set

$$H_1 = \langle x_1^0, x_1^1, \cdots, x_1^{n+m-1} \rangle \cong K_{m+n}.$$

Similarly, $x_w^0$ is adjacent to $\overline{H_w}$, set

$$H_w = \langle x_w^0, x_w^1, \cdots, x_w^{n+m-1} \rangle \cong K_{m+n}, w = 1, 2, \cdots, m.$$

Similarly, in

$$G - N^*[x_2^{n+m-1}] = \langle H_0^* \cup H_1^* \cup H_3^* \cup \cdots \cup H_m^* \rangle,$$

we have $x_0^{n+m-1} \in N^*[x_2^{n+m-1}] = \langle H_0^* \cup H_1^* \cup H_3^* \cup \cdots \cup H_m^* \rangle$ complete adjacent to $\overline{H_0^*}$. Obvious,

$$x_0^{n+m-1} \in (H_2 \cup x_1^{n+m-1} \cup x_3^{n+m-1} \cup \cdots \cup x_m^{n+m-1}) \subset N^*[x_2^{n+m-1}],$$

hence $H_0 = \langle x_0^0, x_0^1, \cdots, x_0^{n+m-1} \rangle \cong K_{m+n}$.

**Fact 3.** Any vertex in $H_r$ is adjacent to single vertex in $H_s, r \neq s$. Suppose the contrary, let $x_0^j \in H_0$ be nonadjacent to $H_m$, then

$$G^* = G - N[x_0^j] \cong K_{n+m-1} \times K_m,$$

but $H_m \cong K_{m+n}$, $H_m^* \subset G^*$, contrary to $G^* \cong K_{n+m-1} \times K_m$, hence $x_0^j$ is adjacent to $H_m$. If $H_m$ has two vertices adjacent to $x_0^j$, by

$$\begin{aligned} d_{H_0}(x_0^j) &= n + m + m - 1 = n + 2m - 1, \\ d(x_0^j) &= n + m + m + 1 = n + 2m + 1, \end{aligned}$$

we have this $x_0^j$ is nonadjacent to one of $H_w$, which is a contradiction. Similarly, $x_{i-1}^j$ is adjacent to $H_i, i = 1, 2, \cdots, m$, and is adjacent to only one vertex in $H_i$.

**Fact 4.** By Fact 3, we have $x_1^0$ adjacent to $H_2$. If $x_1^0$ is adjacent to $x_2^j, j \neq 0$, by $x_2^j, j \neq 0$ is adjacent to $x_1^j$, thus $H_1$ has two vertices adjacent to $x_2^j, j \neq 0$, contrary to Fact 3, so $x_1^0$ adjacent to $x_2^0$. Similar, so $x_i^0$ adjacent to $x_j^0$, and $x_i^{n+m-1}$ adjacent to $x_0^{n+m-1}, i = 1, 2, \cdots, m$.

**Fact 5.** Since $x_0^j$ is adjacent to $x_m^j$ for $j = 0, n+m-1$, let $x_0^j$ be nonadjacent to $x_m^j$ for $j \neq 0, n+m-1$, by Fact 3, we have $x_0^j$ is adjacent to $x_m^i, i \neq j$. Since $x_l^j$ is adjacent to $x_k^j, l \neq k$, $l, k = 1, 2, \cdots, m$, set

$$G - N[x_0^j] = \langle (H_1 - x_1^j) \cup (H_2 - x_2^j) \cup \cdots \cup (H_m - x_m^j) \rangle \cong K_{n+m-1} \times K_m \tag{3.3}$$

By Fact 4, we have $x_1^t$ adjacent to $x_m^t$, $t \neq i, j$, by (3.3), we have $x_1^i$ adjacent to $x_m^j$, contrary (3.1), hence $x_0^j$ is adjacent to $x_m^j$. Similarly, $x_0^j$ is adjacent to $x_m^j$.

Hence

$$G = \langle X \rangle = \langle x_r^j |_{r=0,1,2,\cdots,m}^{j=0,1,2,\cdots,n+m-1} \rangle,$$

where $x_r^i$ is adjacent to $x_s^j$, if and only if $r = s, i \neq j$, or $i = j, r \neq s$, thus $G \cong K_{m+n} \times K_{m+1}$.□

# 4 Binary cryptosystem of connected $m - K_n-$residual graphs

The implementation principle and steps of the traditional Hill cryptosystem were followed to establish the novel, binary cryptosystem proposed here. The Hill cryptosystem is a symmetrical cipher that is effectively resistant to frequency analysis that was first proposed by Prof. Lester S. Hill in 1929. It is implemented in the following steps.

(1) The digitization of plaintext $M = [m_1, m_2, \cdots, m_k]$ (41 characters, including 26 letters, the figures from 0 to 9 and punctuation corresponded to the figures from 0 to 40, respectively), with $t$ components taken as a row vector (if the amount could not reach $t$ in the last row, the space is required for supplementation.) that constitute a $nt$ matrix, written as $M$ .

(2) Matrix $A$ of $t \cdot t$ is constructed in $Z_{41}$ where $gcd(det(A), 41) = 1$ is required as the encryption key.

(3) Encryption operation $C = E(M) \equiv MA(mod41)$ is carried out to obtain the ciphertext.

(4) Decryption operation $M = D(C) \equiv CA^{-1}(mod41)$ is carried out to recover the plaintext.

It is difficult for the Hill cryptosystem to withstand a plaintext attack. After attackers intercept the ciphertext $C$, they are able to guess certain words used in the plaintext to attempt to ascertain the key $K$, then can calculate $MK$ to determine whether ciphertext $C$ can be generated. A large amount of information is stored in a computer system in a form of figure and transmitted through a public signal channel. Unfortunately, these computer systems and signal channels are very susceptible to attack in an open environment. The complexities of the minimum order and extremal graph of the connected $m-K_n-$residual graph can be used to denote the plaintext with different $K_n$, thus forming the proposed binary cryptosystem of the connected $m - K_n-$residual graph. The corresponding encryption algorithm is as follows.

(1) Each character in plaintext $M$ is translated into the corresponding figure $n_i$; 41 characters, (26 letters, figures from 0 to 9, and punctuation corresponded to the figures from 0 to 40, respectively, where $n_i \in Z_{41}$ .)

(2)For encryption operation, each figure $n_i$ corresponds to a $K_{n_i}$. The sequence $[m_1, m_2, \cdots, m_k]$ represents the multiples, where $m_k \in Z$. Finally, the largest connected branch $r$ can be identified and assigned to an extremal graph, i.e., the ciphertext$C$.

(3) For decryption operation, each ciphertext $C$ equals an extremal graph and the complete graph $K_{n_j}$ is determined according to the multiple sequences $[m_1, m_2, \cdots, m_k]$ and the largest connected branch $r$. The corresponding $n_j$ can also be identified, $n_j \in Z_{41}$(i.e., $j(j = 1, 2, \cdots, 41)$); the resulting graph is the plaintext $M$.

These implementation steps are depicted in Fig. 8.

The binary cryptosystem of the connected $m - K_n-$residual graph makes full use of all available complexity in constructing the minimum order and extremal graph of the connected $m - K_n-$residual graph, which ensures optimal binary cryptosystem security. Even if attackers intercept the ciphertext $C$ and knew that the minimum order and extremal graph of the connected $m - K_n-$residual graph have been used for encryption, they are unable to solve the decryption operation and thus cannot obtain the plaintext.

# 5 Conclusions

(1) The minimum order and extremal graphs of connected $m - K_n-$residual graphs form an open question put forward by Erdös, Harary, and Klawe which was addressed in this study.

(2) It is difficult to determine the minimum order and extremal graphs of connected $m-K_n-$residual graphs.

(3) We confirmed that the $m$ and $n$ values identified are in accordance with the conjectures of Erdös, Harary, and Klawe; namely, that the minimum order and extremal graphs of connected $m-K_n-$residual graphs do exist. This had important practical significance for establishing the proposed binary cryptosystem.

(4) A new binary password system comprised of an image-encryption-based password system and encryption algorithm was proposed here. This system represents an enhanced relationship between graph theory and cryptography.

(5) According to the complexity of the minimum order and extremal graph of the connected $m - K_n-$residual graph, the ciphertext, plaintext, and relationship between the keys is highly complex and the binary cryptosystem performs well. The security of the binary cryptosystem can be effectively adjusted according to these factors.

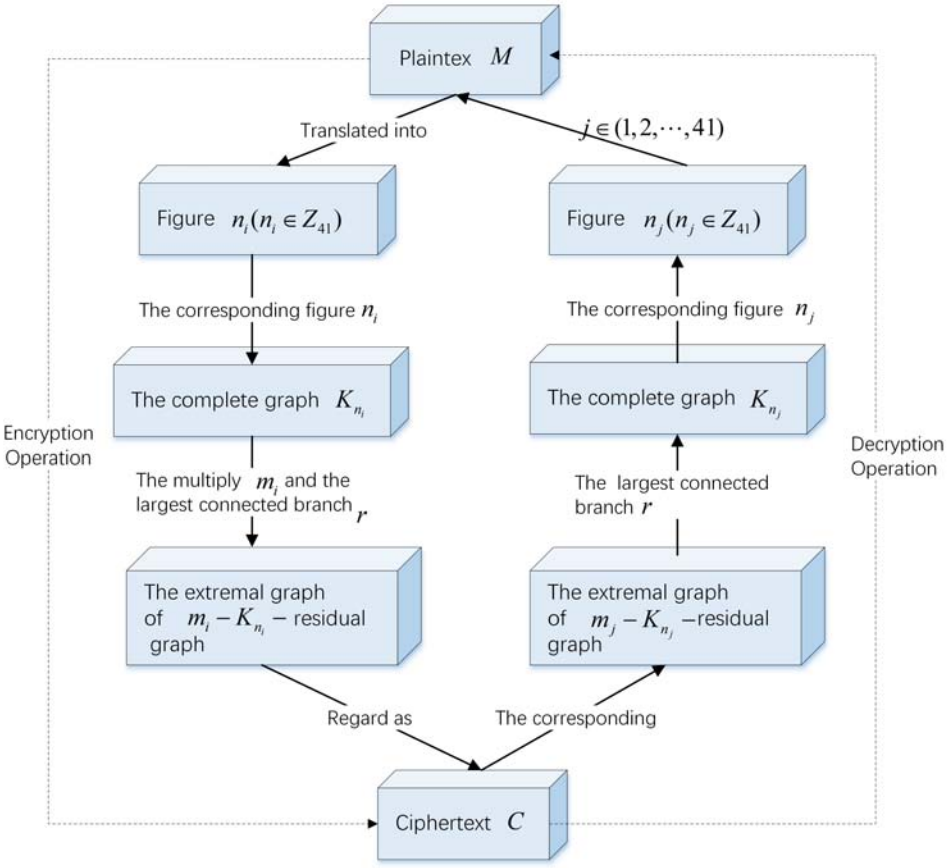



Fig.**8** Encryption and decryption operation

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# References

[1] P. Erdös, F. Harary and M. Klawe, Residually-Complete Graphs, Annals of Discrete Mathematics 6 (1980) 117-123.

[2] A.Matthias, S. Friedl, Residual properties of graph manifold groups, Topology and its Applications 158(10) (2011) 1179-1191.

[3] M.S. Lin, C.C. Ting, A polynomial-time algorithm for computing K-terminal residual reliability of d-trapezoid graphs, Information Processing Letters 115(2)(2015)371-376.

[4] A.A.Chernyak, Residual reliability of P-threshold graphs, Discrete Applied Mathematics 135(1) (2004) 83–95.

[5] B. Trotta, Residual Properties of Simple Graphs, Bulletin of the Australian Mathematical Society 82(3) (2008) 488-504.

[6] M.S.Lin, C.C.Ting, A polynomial-time algorithm for computing K-terminal residual reliability of d-trapezoid graphs, Information processing letters 115 (2015) 371-376.

[7] P. Luksic, T. Pisanski, Distance-residual graphs, Mathematics 9(3) (2006) 104-111.

[8] K. Patra, B. Nemade, D. P. Mishra and P. P. Satapathy, Cued-Click Point Graphical Password Using Circular Tolerance to Increase Password Space and Persuasive Features, Procedia Computer Science 79 (2016) 490-498.

[9] T.Y. Chang, C.J. Tsai and J.H. Lin, A graphical-based password keystroke dynamic authentication system for touch screen handheld mobile devices, Journal of Systems and Software 85(5) (2012) 1157-1165.

[10] F. Septyanto, K. A. Sugeng, On the Rainbow and Strong Rainbow Connection Numbers of the m-Splitting of the Complete Graph $K_n$, Procedia Computer Science 74(2015)155-161.

[11] R. El-Shanawany, H. O.F. Gronau, M. Grttmller, Orthogonal double covers of $K_n$,n by small graphs, Discrete Applied Mathematics 138(1-2) (2004) 47-63.

[12] A. Aguado, Structures, relative stabilities, and electronic properties of potassium clusters $K_n$ $(13 \leq n \leq 80)$ 1021(1) (2013) 135-143.

[13] S. Yang, H. Duan, Residually Complete Graph with odd order, Acta Mathematics Applicatae Sinica 34(5) 2011 778-785.

[14] H. Duan, Y. Li, On connected $K_n-$ Residual Graphs, Operations Research Transactions 20(2) (2016) 38-48.

[15] J. Liao, S. Yang, Y. deng, On connected $2-K_n-$residual graphs, Mediterranean Journal of Mathematics 6 (2012) 12–27.

[16] H. Duan, B. Zeng, Z. Dou, On connected three multiply $K_n-$ Residual Graphs, Operations Research Transactions 18(2) (2014) 59-68.

[17] J. Liao, S. Yang, Two improvement on the Erdös, Harary and Klawe conjecture, Mediterranean Journal of Mathematics 3 (2014) 1-16.

[18] A. Thomas, E. M. Manuel, Embedment of Montgomery Algorithm on Elliptic Curve Cryptography over RSA Public Key Cryptography, Procedia Technology 24 (2016) 911-917.

[19] E. Bellini, N. Murru, An efficient and secure RSA-like cryptosystem exploiting Rdei rational functions over conics, Finite Fields and Their Applications 39 (2016) 179-194.

[20] R. Rizk, Y. Alkady, Two-phase hybrid cryptography algorithm for wireless sensor networks, Journal of Electrical Systems and Information Technology 2(3) (2015) 296-313.

[21] P. Singh, V. Neema, S. Daulatabad and A. P. Shah, Subthreshold Circuit Designing and Implementation of Finite Field Multiplier for Cryptography Application, Procedia Computer Science 79 (2016) 597-602.

[22] S. Chandra, B. Mandal, Sk. S. Alam and S. Bhattacharyya, Content Based Double Encryption Algorithm Using Symmetric Key Cryptography, Procedia Computer Science 57 (2015) 1228-1234.

[23] K. Adinarayana Reddy, B. Vishnuvardhan, Madhuviswanatham, A.V.N. Krishna, A Modified Hill Cipher Based on Circulant Matrices,Procedia Technology 4 (2012) 114-118.

[24] B. Acharya, M. Dhar Sharma, S. Tiwari, V. K. Minz, Privacy protection of biometric traits using modified hill cipher with involutory key and robust cryptosystem, Procedia Computer Science 2 (2010) 242-247.

[25] Bakhtiari, M. Aizaini, Serious Security Weakness in RSA Cryptosystem, International Journal of Computer Science Issues 9(1) (2012) 175-178.

[26] Bassam J. Mohd, Thaier Hayajneh, Athanasios V. Vasilakos, A survey on lightweight block ciphers for low-resource devices: Comparative study and open issues, Journal of Network and Computer Applications 58 (2015) 73-93.

[27] X.J. Tong, Z. Wang, Y. Liu, M. Zhang, L.J. Xu, A novel compound chaotic block cipher for wireless sensor networks,Communications in Nonlinear Science and Numerical Simulation. 22 (2015) 120-133.

[28] X. Xu, L. He, A. Shimada, R. Taniguchi, H.M. Lu, Learning unified binary codes for cross-modal retrieval via latent semantic hashing, Neurocomputing 213(12) (2016) 191-203.

[29] Y.C. Wan, M.W. Wang, Z.W. Ye, X.D. Lai, A feature selection method based on modified binary coded ant colony optimization algorithm, Applied Soft Computing 49 (2016) 248-258.

[30] V. Thayananthan, A. Albeshri, Big Data Security Issues Based on Quantum Cryptography and Privacy with Authentication for Mobile Data Center, Procedia Computer Science 50 (2015) 149-156.

# A comparative analysis of the Harry Dym model with and without singular kernel.

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**Abstract**
The question that raised after the recent introduction of the derivative with no singular kernel was: **Does this non-singularity have a comprehensive impact on real life phenomena like wave motion or other related motions?** We comprehensively analyze the Harry Dym model generalized with two types of derivatives, namely a derivative with singular kernel, the Caputo derivative and the other one without singular kernel called the Caputo-Fabrizio derivative. Using Picard $L$-stability combined with the fixed-point theorem, the well-posedness of both models are proved together with their existence and uniqueness results existence. Techniques to approximate numerical solutions are provided for each of the two models with graphical representations performed and compared for several values given to the derivative order $\alpha$. Similar behaviors are noticed for soliton waves related to close values of $\alpha$ and are compared to the soliton wave of the standard first order ($\alpha = 1$) Harry Dym model.

**Keywords**: Harry Dym equation; existence, uniqueness, derivative with and without singular kernel, approximated solutions

**AMS Mathematics Subject Classification:** 35F10, 26A33, 35D05.

## 1. Introduction

It is well known that one of the most popularly used derivative with fractional order is the one introduced in 1967 by Michele Caputo and called the Caputo derivative [1] given by

$$ {}^{c}D_{t}^{\alpha} r(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{d}{d\tau} r(x,\tau)\, d\tau, \tag{1.1} $$

$0 < \alpha \leq 1$, with its associated anti-derivative well known to be defined by

$$ I_0^{-\alpha} r(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{r(x,\tau)}{(t-\tau)^{1-\alpha}} d\tau. \tag{1.2} $$

However in April 2015, Caputo and Fabrizio [5] observed that the Caputo fractional derivative is unable to properly described some features related to some behavior happening in the fields of classical thermal media, classical viscoelastic materials or electromagnetic. Hence they proposed a new definition, the fractional derivative with no singular kernel in order to address the noticed unsolved

issues. The new version called the Caputo-Fabrizio derivative differentiate from the Caputo derivative by having no singular kernel in the integral part. It reads as

$$ {}^{cf}D_t^{\alpha} r(x,t) = \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)} \int_0^t \frac{d}{d\tau} r(x,\tau) \exp\left(-\frac{\alpha(t-\tau)}{1-\alpha}\right) d\tau. \tag{1.3} $$

$0 < \alpha \leq 1$, where $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$. Its associated fractional integral (anti-derivative) is given by [8]:

$$ {}^{cf}I_t^{\alpha} r(x,t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} r(x,t) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t r(x,\tau)\, d\tau, \tag{1.4} $$

$\alpha \in [0,1]$ $t \geq 0$. This expression shows the integral as a sort of average between the function $r$ and its integral. Since then, many authors have improved the concepts. Starting with Losada and Nieto [8] who developed the associated fractional integral (1.3) and Doungmo Goufo [11] or Doungmo Goufo and Atangana [12] who proposed the related Riemann-Liouville version.

Nevertheless, the literature is full of several other definitions of fractional derivative. General formulation is done in [9] and many useful properties are intensively analyzed in [1–3, 10], especially in the analysis of the spread of diseases [14–17]. The main goal of this work is to apply both Caputo and Caputo-Fabrizio derivatives to the same Harry Dym equation and see their impact on the output behavior of the solutions. In other words, does the non-singularity have a significant influence on a real life process like wave motion or other motion? Recall that unlike Caputo derivative, there is no singularity at $t = \tau$ for Caputo-Fabrizio derivative and the following equality for Caputo-Fabrizio fractional derivative is satisfied:

$$ \lim_{\alpha \to 1} {}^{cf}D_t^{\alpha} r(x,t) = \frac{\partial}{\partial t} r(x,t) \tag{1.5} $$

and

$$ \lim_{\alpha \to 0} {}^{cf}D_t^{\alpha} r(x,t) = r(x,t) - r(x,0). \tag{1.6} $$

This paper investigates the non-linear third-order Harry Dym differential equation within the scope of the two types of derivatives mentioned above. Note that the traditional Harry Dym model can be solved using the Lax operator [18–20] and is associated with the Sturm-Liouville operator.

## 2. Solvability with Caputo fractional derivative

Let $\Omega = (a,b)$, $\mathbb{R} \ni T > 0\, \mathbb{R} \ni b > a \in \mathbb{R}$ and $r \in C^0[\Omega \times [0,T]]$. Let $\alpha \in [0;1], \beta \in (0,+\infty)$ then, the non-linear Dym equation expressed with the Caputo time fractional derivative is investigated in this section. Existence and uniqueness of the exact solution are shown for the model under investigation that reads as

$$ {}^{C}D_t^{\alpha} r(x,t) = r^3 r_{xxx}(x,t), \tag{2.1} $$

subject to the initial condition

$$ r(x,0) = g(x) \tag{2.2} $$

with ${}^{C}D_t^{\alpha} r(x,t)$ the Caputo derivative as defined in (1.1) and $g:\ \Omega \longmapsto \mathbb{R}_+$. We start by transforming (2.1) into an integral form by applying the anti-derivative integral (1.2) on both sides to get

$$r(x,t)-r(x,0)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-y)^{\alpha-1}(r^3 r_{xxx}(x,t))dy \tag{2.3}$$

Let us proceed by simplicity and consider the operator with three variables

$$\Xi(x,t,r)=r^3 r_{xxx}(x,t). \tag{2.4}$$

The next goal is to show that operator $\Xi$ with respect to variable $r$ verifies the Lipschitz condition. For that,

$$\|\Xi(x,t,r)-\Xi(x,t,v)\|=\left\|r^3 r_{xxx}(x,t)-v^3 v_{xxx}(x,t)\right\|_{C^0[\Omega\times[0,T]]} \tag{2.5}$$

Assuming that $r$ and $v$ are bounded functions, there is a positive constants $k_1>0$ and $k_2>0$ such that

$$\|r\|^3_{C^0[\Omega\times[0,T]]}\leq k_1 \qquad \text{and} \qquad \|v\|^3_{C^0[\Omega\times[0,T]]}\leq k_2. \tag{2.6}$$

Furthermore, using the properties of the norm and the Lipschitz condition for the first order derivative function $\partial_x$ there is a positive constant $\vartheta$ such that (2.5) becomes

$$\|\Xi(x,t,r)-\Xi(x,t,v)\|_{C^0[\Omega\times[0,T]]}\leq k_1 k_2 \vartheta^3 \|r-v\|_{C^0[\Omega\times[0,T]]}. \tag{2.7}$$

Putting

$$K=k_1 k_2 \vartheta^3,$$

we finally get

$$\|\Xi(x,t,r)-\Xi(x,t,v)\|_{C^0[\Omega\times[0,T]]}\leq K\|r-v\|_{C^0[\Omega\times[0,T]]},$$

which therefore proves the desired Lipschitz condition. This enables us to evaluate the following norm

$$\|r(x,t)\|_{C^0[\Omega\times[0,T]]},\ t\in[0,T].$$

Assuming the existence and the boundedness of the initial condition $g$, there is a positive constant $C$ such that $\|g(x)\|_{C^0[\Omega\times[0,T]]}\leq C$ for any $x\in\Omega$. Whence,

$$\begin{array}{l}\|r(x,t)\|_{C^0[\Omega\times[0,T]]}\leq \|r(x,0)\|_{C^0[\Omega\times[0,T]]}+\frac{1}{\Gamma(\alpha)}\int_0^t (t-y)^{\alpha-1}\Xi(x,y,r(x,y))dy \\ \leq \|g(x)\|_{C^0[\Omega\times[0,T]]}+\frac{K}{\Gamma(\alpha)}\int_0^t (t-y)^{\alpha-1}dy \\ \leq C+\frac{KT^\alpha}{\alpha\Gamma(\alpha)}\end{array}, \tag{2.8}$$

which yields the following propositions:

**Proposition 2.1.** *Assuming that g given in* (2.2) *is bounded, let* $0<\alpha<1$ *and* $\Xi[x,t,r(x,t)]:[\Omega\times[0,T]]\times B\longrightarrow A$ *(with* $A\supset B$*) be a continuous function with respect to t for any fixed* $x\in\Omega$, $r\in\overline{B}$. *If* $r(x,t)\in C^0[\Omega\times[0,T]]$, *then the function* $r(x,t)$ *verifies the model* (2.1)-(2.2) *if and only if* $r(x,t)$ *verifies the corresponding Volterra integral equation* (2.3).

**Proof.** To prove the necessity condition we assume that $r(x,t) \in C^0[\Omega \times [0,T]]$ satisfies the equations (2.1)-(2.2). Because $\Xi(x,y,r(x,y) \in C[\Omega \times [0,T] \times B]$ for any $r \in \bar{B}$ then (2.1) means there exists the Caputo fractional derivative of $r$ in $C[\Omega \times [0,T]]$. However

$$ {}_0^C D_t^\alpha r = \frac{\partial}{\partial t}\left(I_0^{-\alpha}\right)[r(x,t) - r(x,0)]. \tag{2.9}$$

Exploiting

$$I_0^{-\alpha}[r(x,t) - r(x,0)] \in C^0[\Omega \times [0,T]]$$

and applying the results in [13] for $\gamma = 0$ to

$$V(x,t) = r(x,t) - r(x,o),$$

yields

$$\begin{aligned} I_0^\alpha \, {}_0^C D_t^\alpha r(x,t) &= I_0^\alpha \, {}_0^C D_0^\alpha [r(x,t) - r(x,0)] \\ &= r(x,t) - r(x,0) - \sum_{j=1}^{1} \frac{r_{1-\alpha}^{i-j}(x,0)_{\alpha-j}}{\Gamma(\alpha - j + 1)} \end{aligned} \tag{2.10}$$

with $r_{1-\alpha}(x,t) = I_0^{1-\alpha}[r(x,t) - r(x,0)]$. Using the integration by parts in (2.10) and differentiating the resulting expression give

$$r_{1-\alpha}^{(1-j)}(x,t) = \frac{\partial}{\partial t}\left(I_0^{2-\alpha}[\partial_t r(x,t) - r(x,0)]\right) \tag{2.11}$$

Changing of variable $t = \beta + \rho(y - \beta)$ leads to

$$r_{1-\alpha}^{(1-j)}(x,t) = \frac{(y-\beta)^{1-\alpha}}{\Gamma(1-\alpha)} \int_0^1 (1-\rho)^{-\alpha}\left(r^{1-j}[\beta + \rho(y-\beta)]\right). \tag{2.12}$$

Recalling $0 < \alpha < 1$ and $r^{1-j}(x,t) \in C[\Omega \times [0,T]]$, equation (2.12) and (2.10) take the form

$$I_0^\alpha \, {}_0^C D_t^\alpha r(x,t) = r(x,t) - r(x,0) \tag{2.13}$$

Since $I_0^\alpha \Xi(x,\tau,r(x,\tau)) \in C^0[\Omega \times [0,T]]$ and using the Lipschitz condition of $\Xi$, we obtain

$$\|I_0^\alpha \Xi(x,\tau,r(x,\tau)\|_{C^0[\Omega \times [0,T]]} \leq \frac{KT^\alpha}{\alpha\Gamma(\alpha)}.$$

Nonetheless, applying $I_0^\alpha$ on both sides of (2.1) and making use of the initial condition we prove the necessity condition by recovering the Volterra version (2.3).

Conversely for the sufficient condition, assume $r(x,t) \in C[\Omega \times [0,T]]$ verifies the Volterra version (2.3) of equations (2.1)-(2.2). It suffices to show that $r(x,t)$ satisfies the initial condition (2.2).

The differentiation of the two sides of the Volterra version yields

$$\partial_t r(x,t) = \frac{1}{\Gamma(\alpha-1)} \int_0^t \Xi(x,\tau,r(x,\tau))(t-\tau)^{\alpha} d\tau.$$

Changing again the variable $t = \beta + \rho(y-\beta)$ in the Volterra expression for $k=1$, leads to

$$r^k(x,t) = \frac{(y-\beta)}{\Gamma(\alpha-k)} \int_0^t \frac{\Xi(x,\beta+\rho(y-\beta),r(x,\beta+\rho(y-\beta)))}{(1-\beta)^{1-\alpha+k}} dy$$

Passing to the limit as $y \longrightarrow \beta^+$ and making use of the continuity of $K$ show that the sought initial condition is verified, and the sufficient condition is proved. □

**Proposition 2.2.** *Considering* $0 < \alpha < 1$ *and the Lipschitz condition for* $\Xi$ *then, there is a unique solution for equation* (2.3) *in the space*

$$C^{0,\alpha}[0,T] \times \Omega$$

**Proof.** From the above analysis in Theorem 2.1, it is sufficient to show the existence of the unique solution $r(x,t) \in C^0[\Omega \times [0,T]]$ of the Volterra equation (2.3). Indeed, the model (2.3) holds in any interval $[0,\tau] \subseteq [0,T]$. Hence, we select the adequate $t_1 \in [0,T]$ so that

$$\left| \frac{Kt_1^{\alpha}}{\alpha\Gamma(\alpha)} \right| < 1$$

and then, prove the desired existence result of a unique $r(x,t) \in C^0[\Omega \times [0,t_1]]$. We can exploit the technique of successive approximation and set

$$\begin{aligned} r_0(x,t) &= r(x,0) \\ r_n(x,t) &= r_{n-1}(x,0) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\Xi(x,\tau,r_{n-1}(x,\tau))}{(t-\tau)^{1-\alpha}} d\tau, \quad n \in \mathbb{N}. \end{aligned} \tag{2.14}$$

Obviously, $r(x,0) \in C^0[0,T]$ and the differentiation of (2.14) with respect to $t$ yields

$$\partial_t r_n(x,t) = \frac{1}{\Gamma(\alpha-1)} \int_0^t \frac{\Xi(x,\tau,r_{n-1}(x,\tau))}{(t-\tau)^{-\alpha}} d\tau.$$

Using the differentiability of $r(x,0)$ with respect to $t$ leads to $r_n(x,t) \in C^0[0,T]$.
The quantity $\|r_n(x,t) - r_{n-1}(x,t)\|_{C^0[\Omega\times[0,t_1]]}$ can be evaluated for $n \in \mathbb{N}$. Whence,

$$\|r_n(x,t) - r_{n-1}(x,t)\|_{C^0[\Omega\times[0,t_1]]} \leq \frac{Kt_1^{\alpha}}{\alpha\Gamma(\alpha)}.$$

Furthermore,

$$\begin{aligned}\|r_2(x,t)-r_1(x,t)\|_{C^0[\Omega\times[0,t_1]]} &\leq \|I_0^\alpha \Xi(x,\tau,r_1(x,\tau))-\Xi(x,\tau,r_0(x,\tau))\|_{C^0[\Omega\times[0,t_1]]}\\ &\leq \frac{t_1^\alpha}{\alpha\Gamma(\alpha)}\|r_1(x,t)-r_0(x,t)\|_{C^0[\Omega\times[0,t_1]]}\\ &\leq \frac{Kt_1^\alpha}{\alpha\Gamma(\alpha)}\cdot\frac{t_1^\alpha}{\alpha\Gamma(\alpha)}\end{aligned}$$

The above iteration is repeated $n$-times and yields

$$\|r_2(x,t)-r_1(x,t)\|_{C^0[\Omega\times[0,t_1]]} \leq \left(\frac{Kt_1^\alpha}{\alpha\Gamma(\alpha)}\right)^{n-1}\cdot\frac{t_1^\alpha}{\alpha\Gamma(\alpha)}$$

Hence, the sequence $\{r_n(x,t)\}_{n\in\mathbb{N}}$ has $r(x,t)\in C^0[\Omega\times[0,t_1]]$ as its limit function. Moreover, the assumption

$$\left|\frac{t_1^\alpha}{\alpha\Gamma(\alpha)}\right|<1$$

gives $\lim_{n\longrightarrow\infty}\|r_n(x,t)-r_{n-1}(x,t)\|_{C^0[\Omega\times[0,t_1]]}=0$. However by considering $t_1=T$, we can estimate

$$\Xi(x,t,r_n(x,\tau))-\Xi(x,t,r(x,t)).$$

Taking into account the Lipschitz condition of $\Xi$ leads to

$$\left\|\frac{1}{\Gamma(\alpha)}\int_0^t\frac{\Xi(x,\tau,r_n(x,\tau))}{(t-\tau)^{1-\alpha}}d\tau-\frac{1}{\Gamma(\alpha)}\int_0^t\frac{\Xi(x,\tau,r(x,\tau))}{(t-\tau)^{1-\alpha}}d\tau\right\|_{C^0[\Omega\times[0,T]]}$$

$$\leq\left(\frac{Lt_1^\alpha}{\alpha\Gamma(\alpha)}\right)\|r_n(x,t)-r(x,t)\|_{C^0[\Omega\times[0,t_1]]}$$

and

$$\lim_{n\longrightarrow\infty}\left\|\frac{1}{\Gamma(\alpha)}\int_0^t\frac{\Xi(x,\tau,r_n(x,\tau))}{(t-\tau)^{1-\alpha}}d\tau-\frac{1}{\Gamma(\alpha)}\int_0^t\frac{\Xi(x,\tau,r(x,\tau))}{(t-\tau)^{1-\alpha}}d\tau\right\|_{C^0[\Omega\times[0,T]]}=0,$$

which prove that $r(x,t)$ satisfies (2.3) in the space $C^0[\Omega\times[0,T]]$.

**Uniqueness result**

Considering now that there are two separate solutions $r_1(x,t)$ and $r_2(x,t)$ verifying (2.3) on $[0,t_1]$ then,

$$\|r_1(x,t)-r_2(x,t)\|_{C^0[\Omega\times[0,t_1]]}\leq\frac{Kt_1^\alpha}{\alpha\Gamma(\alpha)}\|r_1(x,t)-r_2(x,t)\|_{C^0[\Omega\times[0,t_1]]}.$$

This gives $1\leq\frac{Kt_1^\alpha}{\alpha\Gamma(\alpha)}$ which is a contradiction. The solution is unique in $C^0[\Omega\times[0,t_1]]$
We consider now the closed interval $[t_1,t_2]$ with $t_2=t_1+h_1$ where $h_1>0$ and $t_2<T$. For $t\in[t_1,t_2]$,

we have

$$r(x,t)=\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}\frac{\Xi(x,\tau,r_n(x,\tau))}{(t-\tau)^{1-\alpha}}d\tau+r(x,0)+\frac{1}{\Gamma(\alpha)}\int_{0}^{t_1}\frac{\Xi(x,\tau,r_n(x,\tau))}{(t-\tau)^{1-\alpha}}d\tau.$$

Taking into account the uniqueness result on $[0,t_1]$, then

$$r(x,t)=r_1^*(x,t)+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}\frac{\Xi(x,\tau,r_n(x,\tau))}{(t-\tau)^{1-\alpha}}d\tau$$

where $r_1^*(x,t)=r(x,0)+\frac{1}{\Gamma(\alpha)}\int_{0}^{t_1}\frac{\Xi(x,\tau,r_n(x,\tau))}{(t-\tau)^{1-\alpha}}d\tau$ is a given function.

we repeat the same analysis presented above and there is an unique solution $r(x,t)\in C^0\left[\Omega\times[t_1,t_2]\right]$ for (2.3). Taking another interval $[t_2,t_3]$ so that $t_3=t_2+h_2$ with $h_2>0$ and $t_3<T$, the same analysis is performed to finally obtain the existence of an unique solution $r(x,t)\in C^0\left[\Omega\times[0,T]\right]$ of equation (2.3) and therefore, proves the existence of an unique solution of equation (2.1) in the space $C^{0,\alpha}\left[\Omega\times[0,T]\right]$. □

**Corollary 2.3.** *Assume $\Xi$ satisfies Theorem 2.1. If the inequality*

$$\left|\frac{KT^{\alpha}}{\alpha\Gamma(\alpha)}\right|\leq 1$$

*holds, then the sequence $r_n(x,t),\ (n\in\mathbb{N})$ tends to the exact solution $r(x,t)$. Moreover, we have for any $n\in\mathbb{N}$,*

$$\|r(x,t)-r_n(x,t)\|_{C^0[\Omega\times[0,t_1]]}\leq\frac{T^{\alpha}}{\alpha\Gamma(\alpha)}\frac{K^n}{1-\frac{T^{\alpha}}{\alpha\Gamma(\alpha)}}$$

**Proof.** The proof is done by the mathematical induction on $n$.

For

$$\|r_1(x,t)-r_0(x,t)\|_{C^0[\Omega\times[0,t_1]]}\leq\frac{KT^{\alpha}}{\alpha\Gamma(\alpha)}$$

$$\|r_2(x,t)-r_1(x,t)\|_{C^0[\Omega\times[0,t_1]]}\leq K\left(\frac{T}{\alpha\Gamma(\alpha)}\right)^2$$

Thus by induction,

$$\|r_n(x,t)-r_{n-1}(x,t)\|_{C^0[\Omega\times[0,t_1]]}\leq\frac{T^{\alpha}}{\alpha\Gamma(\alpha)}\left(\frac{KT^{\alpha}}{\alpha\Gamma(\alpha)}\right)^{n-1}$$

but,

$$\begin{aligned}
&\|r(x,t)-r_n(x,t)\|_{C^0[\Omega\times[0,t_1]]} = \lim_{j\longrightarrow\infty}\|r_{n+j}(x,t)-r_n(x,t)\|_{C^0[\Omega\times[0,t_1]]}\\
&= \|r_{n+1}(x,t)-r_n(x,t)\|_{C^0[\Omega\times[0,t_1]]} + \|r_{n+2}(x,t)-r_{n+1}(x,t)\|_{C^0[\Omega\times[0,t_1]]} + \cdots\\
&\leq K^n\left(\frac{T^\alpha}{\alpha\Gamma(\alpha)}\right)^{n+1} + K^{n+1}\left(\frac{T^\alpha}{\alpha\Gamma(\alpha)}\right)^{n+2} + \cdots\\
&= K^n\left(\frac{T^\alpha}{\alpha\Gamma(\alpha)}\right)^{n+1}\left[\sum_{k=0}^{\infty}\left(\frac{KT^\alpha}{\alpha\Gamma(\alpha)}\right)^k\right]
\end{aligned} \tag{2.15}$$

and the corollary is proved. □

This implies the following existence and uniqueness results for our model (2.1)-(2.2):

**Corollary 2.4.** *The function $r(x,t)$ is the strong solution of the sequence $r_n(x,t)$ given by* (2.14).

**Proof.** This result is an immediate consequence of Propositions 2.2 and 2.1 □

## 3. Analyzis with Caputo-Fabrizio fractional derivative

### 3.1. Introduction and formulation

**Definition 3.1 (Piccard's $L$-stability).**
Consider the Banach space $(B,\|\|\|)$, the self-map $L$ of $B$ and the recursive technique $\sigma_{n+1}=g(L,\sigma_n)$. Let us assume that $B(L)$, containing all the fixed points of $L$ has at least one element and $\sigma_n$ converges to an element $b$ of $B(L)$. Let $\{u_n\}\subseteq B$ and set $d_n=\|u_{n+1}-g(L,u_n)\|$. Therefore $\lim_{n\longrightarrow\infty} d_n=0$ leads to $\lim_{n\longrightarrow\infty} u_n=b$. In this case, we say that the reccurence formula $\sigma_{n+1}=g(L,\sigma_n)$ is $L$-stable.

**Remark 3.1.** Assuming that $\{u_n\}$ has a upper boundary, $\sigma_{n+1}=L\sigma_n$ is called Piccard's iteration if the conditions of Definition 3.1 are satisfied and therefore, will be $L$-stable.

**Lemma 3.1.** *Let $(B,\|\|\|)$ be a Banach space and L a self-map of B verifying*

$$\|Lx-Ly\|\leq C\|x-Lx\|+C\|x-y\|$$

*for all $x,y\in B$, where $0\leq C$, $0\leq\alpha\leq 1$. If L admits a fixed point $f$ then, L is Picard L-stable.*

**Proof.** [4, Theorem 3.1] □

**Proposition 3.2.** *The self-map L expressed as*

$$\begin{aligned}
L(r_n(x,t)) &= r_{n+1}(x,t)\\
&= r_n(x,t) + {}^{cf}I_t^\alpha[r^3 r_{xxx}(x,t))]
\end{aligned}$$

*is L-stable in $L^2(a,b)$*

**Proof.** The first step is to prove that $L$ has a fixed-point. For that set $i, j \in \mathbb{N}$ then,

$$\|Lr_i(x,t) - Lr_j(x,t)\| = \|r_{i+1}(x,t) - r_{j+1}(x,t)\| = \left\|r_i(x,t) + {}^{cf}I_t^{\alpha}\left[r_i^3 \partial_{x^3}^3 r_i\right] - r_j(x,t) - {}^{cf}I_t^{\alpha}\left[r_j^3 \partial_{x^3}^3 r_j\right]\right\|$$

with

$${}^{cf}I_t^{\alpha} r(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} r(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t r(\tau)\, d\tau,$$

the anti-derivative associated to the Caputo-Fabrizio derivative as given in (1.4). Then, making use of the boundedness the function $u$, the Lipschitz condition for the first order operator $\partial_x$ with the same constants $k_1,\ k_2,\ \vartheta$ as in the previous section, we have

$$\|Lr_i(x,t) - Lr_j(x,t)\| \leq \|r_i(x,t) - r_j(x,t)\| + \|{}^{cf}I_t^{\alpha}[r_i^3 \partial_{x^3}^3 r_i - r_j^3 \partial_{x^3}^3 r_j]\|$$

$$\leq \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} k_1 k_2 \vartheta^3 \|r_i - r_j\| + \frac{2\alpha}{(2-\alpha)M(\alpha)} k_1 k_2 \vartheta^3 \|r_i - r_j\|.$$

Thus,

$$\|Lr_i(x,t) - Lr_j(x,t)\| \leq \mathscr{K} \|r_i(x,t) - r_j(x,t)\|$$

With

$$\mathscr{K} = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} k_1 k_2 \vartheta^3 \|r_i - r_j\| + \frac{2\alpha}{(2-\alpha)M(\alpha)} k_1 k_2 \vartheta^3.$$

Consequently $L$ is Lipschitz continuous with respect to $r$ and this means the non-linear operator $L$ has a fixed point. To conclude the proof, it is necessary to remark that taking $C = 0$ and $C = P$, the conditions of Lemma 3.1 are verified for $L$ and then, $L$ is Picard $L$-stable. □

## 4. Numerical Solvability

This section deals with some numerical schemes associated with both models. So a technique to determine the solution for each model using integral iterative methods is presented. We The model with Caputo derivative is iteratively solved making use of Laplace transform while the model with the Caputo-Fabrizio derivative exploit the Sumudu transform. Similar results are obtained as shown below.

### 4.1. Numerical Approximations with Caputo derivative

Applying the Laplace transform $\mathscr{L}$ on both sides of the model (2.1)-(2.2) iteratively yields

$$pr(x,p) - g(x) = \mathscr{L}\left[r^3 r_{xxx}(x,t)\right](p),$$

equivalently

$$r(x,p)$$

$$= \frac{g(x)}{p} + \frac{1}{p}\mathscr{L}\left[r^3 r_{xxx}(x,t)\right](p).$$

Taking the inverse Laplace transform $\mathscr{L}^{-1}$ yields

$$r(x,t)$$

$$= g(x) + \mathscr{L}^{-1}\left(\frac{1}{p}\mathscr{L}\left[r^3 r_{xxx}(x,t)\right]\right)(t).$$

Now, we can introduce the following iterative formula

$$r_0(x,t) = g(x)$$
$$r_{n+1}(x,t) = r_n(x,t) + \mathscr{L}^{-1}\left(\frac{1}{p}\mathscr{L}\left[r^3 r_{xxx}(x,t)\right]\right)(t).$$

the above formula leads to a numerical approximation with Caputo derivative and the approximate solution reads as

$$r(x,t) = \lim_{n\longrightarrow\infty} r_n(x,t)$$

### 4.2. Numerical Approximations Caputo-Fabrizio derivative

Here we first recall the following important relation

$$\mathscr{S}\left({}^{cf}_{0}D^{\alpha}_{t} f(t)\right) = M(\alpha)\frac{pF(p) - f(o)}{1-\alpha+\alpha p}$$

where $F(p) = \mathscr{S}(f(t))$ is the Sumudu transform of $f(t)$. Applying the Sumudu transform $\mathscr{S}$ on both sides of equation (2.1) yields

$$\frac{pr(x,p) - g(x)}{1-\alpha+\alpha p} = \mathscr{S}\left[r^3 r_{xxx}(x,t)\right](p),$$

equivalently

$$r(x,p)$$

$$= \frac{g(x)}{p} + (1-\alpha+\alpha p)\mathscr{S}\left[r^3 r_{xxx}(x,t)\right](p).$$

The inverse Sumudu transform $\mathscr{S}^{-1}$ yields

$$r(x,t)$$

$$= g(x) + \mathscr{S}^{-1}\left((1-\alpha+\alpha p)\mathscr{S}\left[r^3 r_{xxx}(x,t)\right]\right)(t).$$

repeating as above, the following iterative formula introduced

$$r_0(x,t) = g(x)$$
$$r_{n+1}(x,t) = r_n(x,t) + \mathscr{S}^{-1}\left((1-\alpha+\alpha p)\mathscr{S}\left[r^3 r_{xxx}(x,t)\right]\right)(t).$$

which leads to a numerical approximation with Caputo-Fabrizio derivative and the approximate solution reads as

$$r(x,t) = \lim_{n\longrightarrow\infty} r_n(x,t).$$

Those recurrence schemes are used and numerical representations of both models can be depicted in Fig.1 to Fig.5 for different values of the order $\alpha$. The graphics in Fig.1, Fig.2, Fig.3, Fig.4 and Fig.5, performed respectively for $\alpha = 0.2,\ \alpha = 0.3,\ \alpha = 0.4,\ \alpha = 0.8$ and $\alpha = 1.0$ using the Caputo derivative and compared to one using the Caputo-Fabrizio derivative show the standard well-known wave solution of the Harry Dym equation. It is clear that the figures show similar behavior for solutions of both models.

## 5. Concluding remarks

We have proved the existence and uniqueness of the solution to for the nonlinear Harry Dym equation modelled with both the classical Caputo derivative and the newly introcuced derivative of fractional order with no singular kernel. It is the first time that the same model of Harry Dym is analyzed using both derivatives in the same work. This proves that there is a possible way to extent the nonlinear Harry Dym model to the scope of fractional calculus.Two numerical methods suitable to approximate the solutions of model with both derivatives have been presented with numerical simulations performed for $\alpha = 0.2,\ \alpha = 0.3,\ \alpha = 0.4,\ \alpha = 0.8$ and $\alpha = 1.0$. Each figure exhibits solution with similar behavior for the involved wave associated to the Dym equation. This paper innovates by pointing out another concrete application of the Caputo-Fabrizio derivative, new in the literature and till under investigation. More complex investigation will certainly follow.

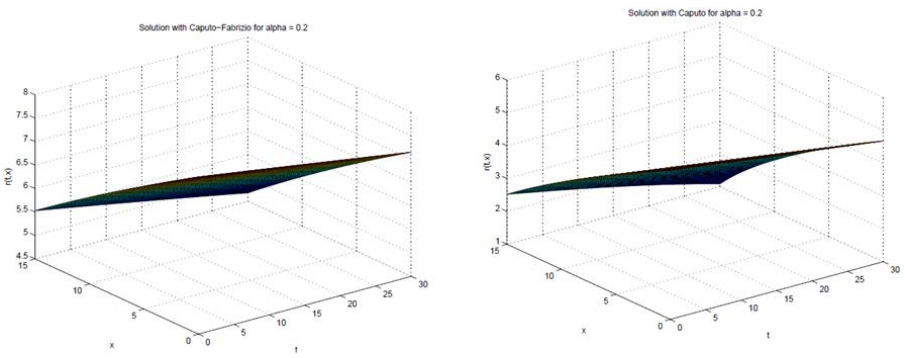



Fig. 1: Representation of the solution $r(x,t)$ when $\alpha = 0.2$ with both derivatives.

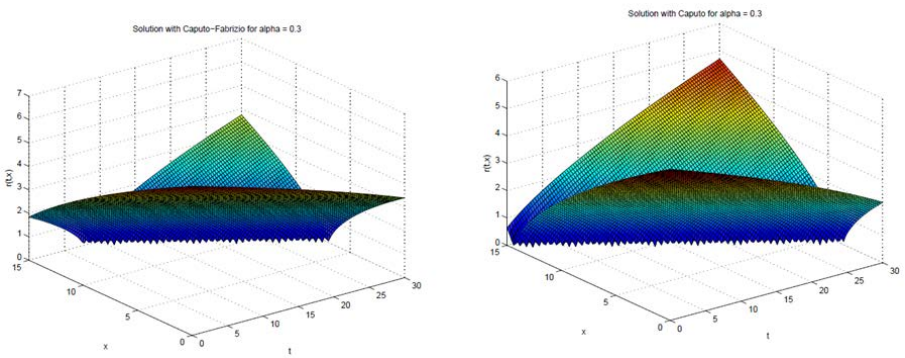



Fig. 2: Representation of the solution $r(x,t)$ when $\alpha = 0.3$ with both derivatives.

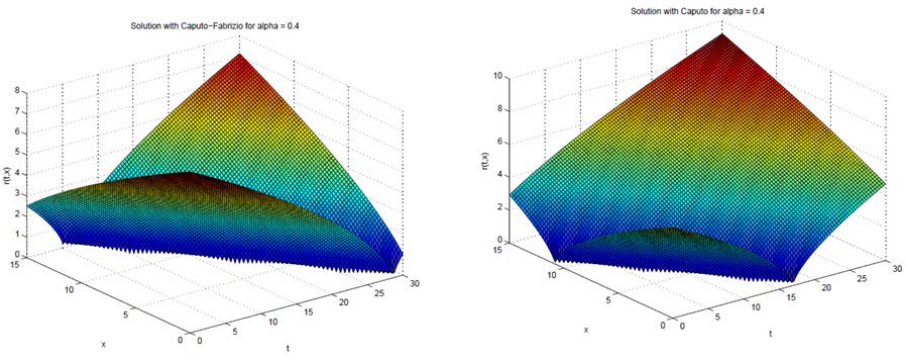



Fig. 3: Representation of the solution $r(x,t)$ when $\alpha = 0.4$ with both derivatives.

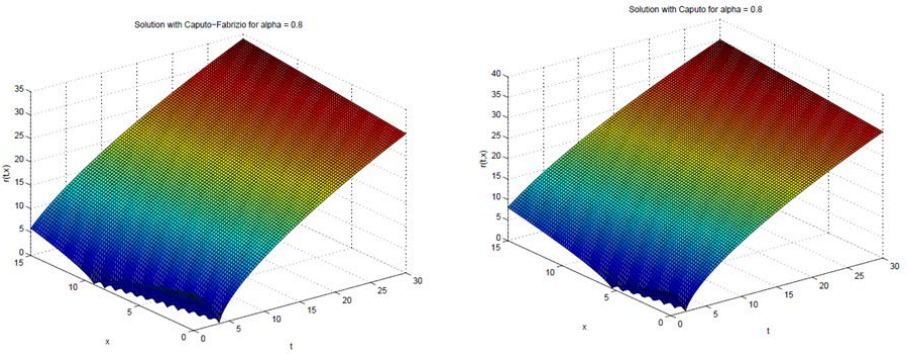



Fig. 4: Representation of the solution $r(x,t)$ when $\alpha = 0.8$ with both derivatives.

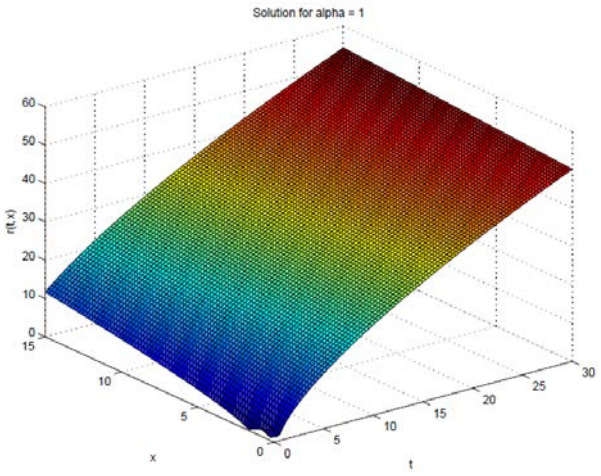



Fig. 5: Representation of the solution $r(x,t)$ when $\alpha = 1.0$ with both derivatives.

## References

[1] Caputo M., Linear models of dissipation whose Q is almost frequency independent II, Geophys. J. R. Ast. Soc. 13 (5), 529-539, 1967; Reprinted in: Fract. Calc. Appl. Anal. 11 (1), 3-14, (2008)

[2] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, vol. 204, Elsevier Science B.V., Amsterdam, The Netherlands, 2006.
[3] Atangana A. and Doungmo Goufo E.F., "Extension of Matched Asymptotic Method to Fractional Boundary Layers Problems," Mathematical Problems in Engineering, vol. 2014, Article ID 107535, 7 pages, 2014. doi:10.1155/2014/107535
[4] D. Eberly, Stability Analysis for Systems of Differential Equations, Geometric Tools, LLC, (2008). http://www.geometrictools.com.
[5] Caputo M, Fabrizio M, A new definition of fractional derivative without singular kernel, Progr. Fract. Differ. Appl. 1 (2), (2015) 1–13
[6] Caputo Michele , Carcione José M., Botelho Marco A. B., Modeling Extreme-Event Precursors with the Fractional Difusion Equation, Fractional Calculus and Applied Analysis, 18 (1) 208–222, (2015).
[7] Doungmo Goufo EF, A biomathematical view on the fractional dynamics of cellulose degradation, Fractional Calculus and Applied Analysis, Vol. 18, No 3, 554–564, (2015). DOI: 10.1515/fca-2015-0034.
[8] Losada J, Nieto JJ, Properties of the new fractional derivative without singular Kernel. Progr. Fract. Differ. Appl, 1(2), (2015) 87–92
[9] Tenreiro Machado J., Mainardi F., and Kiryakova V. Fractional calculus: quo vadimus? (Where are we going?). Fract. Calc. Appl. Anal., 18(2):495–526, 2015.
[10] Doungmo Goufo EF, A biomathematical view on the fractional dynamics of cellulose degradation, Fractional Calculus and Applied Analysis, Vol. 18, No 3, 554–564, (2015). DOI: 10.1515/fca-2015-0034
[11] Doungmo Goufo EF, Application of the Caputo-Fabrizio fractional derivative without singular kernel to Korteweg–de Vries–Bergers equation, Mathematical Modelling and Analysis, Vol. 21, Issue 2, 188-198, 2016, http://dx.doi.org/10.3846/13926292.2016.1145607.
[12] Doungmo Goufo EF, Atangana A, Analytical and numerical schemes for a derivative with filtering property and no singular kernel with applications to diffusion, The European Physical Journal Plus, Vol. 131, Issue 8, 2016. DOI: 10.1140/epjp/i2016-16269-1
[13] Luchko Y. and Gorenflo R., The initial value problem for some fractional differential equations with the Caputo derivative. Preprint Series A0898, Freic Universitat Berlin, 1998, Fachbreich Mathematik and Informatik, 1998.
[14] Area I, Batarfi H, Losada J, Nieto JJ, Shammakh W, Torres A. On a Fractional Order Ebola Epidemic Model. Adv Difference Equ. 2015; Art. ID 278, 12 pp.
[15] Rachah A, Torres DFM. Mathematical modelling, simulation, and optimal control of the 2014 Ebola outbreak in West Africa. Discrete Dyn Nat Soc. 2015; Art. ID 842792, 9 pp.
[16] Area I, Losada J, Ndaïrou F, Nieto JJ, Tcheutia DD. Mathematical modeling of 2014 Ebola outbreak. Math Method Appl Sci.; in press.
[17] Du, M, Wang, Z, Hu, H: Measuringmemorywith the order of fractional derivative. Sci. Rep. 3, 3431 (2013)
[18] Beals R. and Coifman R., Scattering and Inverse Scattering for First Order Systems, Commun. Pure and Appl. Math. 37 (1984) 39-90
[19] Gerdjikov V. and Yanovski A., Completeness of the Eigenfunctions for the Caudrey-Beals-Coifman System, J. Math. Phys. 35 (1994) 3687-3725
[20] Z. Yan, The Riccati equation with variable coefficients expansion algorithm to find more exact solutions of nonlinear differential equation, MMRC, AMSS, Academis Sinica, Beijing 22(2003), 275-284.
[21] Khan Y., Sayevand K., Fardi M., Ghasemi M., A novel computing multi-parametric homotopy approach for system of linear and nonlinear Fredholm integral equations, Applied Mathematics and Computation 2014, 249, 229–236.

# Product-type Operators from Weighted Bergman Spaces to Bloch-Orlicz Spaces

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**Abstract:** Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the class of all analytic functions on $\mathbb{D}$. Let $\varphi$ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. By constructing some suitable test functions in weighted Bergman space, in this paper the boundedness and compactness of the product-type operators $D^n M_u C_\varphi$, $D^n C_\varphi M_u$, $C_\varphi D^n M_u$, $M_u D^n C_\varphi$, $M_u C_\varphi D^n$ and $C_\varphi M_u D^n$ from weighted Bergman space to Bloch-Orlicz space are characterized in terms of the symbol functions $u$ and $\varphi$.

**Keywords:** Weighted Bergman-type space; Bloch-Orlicz space; product-type operator; boundedness; compactness

**MR (2010) Subject Classification:** 47B38; 47B33, 47B37

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## 1 Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the class of all analytic functions on $\mathbb{D}$. Let $\varphi$ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. The weighted composition operator $W_{\varphi,u}$ on $H(\mathbb{D})$ is defined by

$$W_{\varphi,u} f(z) = u(z) f(\varphi(z)), \ z \in \mathbb{D}.$$

If $u \equiv 1$, it becomes the composition operator, usually denoted by $C_\varphi$. If $\varphi(z) = z$, it becomes the multiplication operator, usually denoted by $M_u$. Since $W_{\varphi,u} = M_u C_\varphi$, it is a product-type operator. A standard problem is to provide function theoretic characterizations when $\varphi$ and $u$ induce a bounded or compact weighted composition operator (see, for example, [2, 4, 8, 13, 15, 23, 26, 27] and the references therein).

Let $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The $n$th differentiation operator $D^n$ on $H(\mathbb{D})$ is defined by

$$D^n f(z) = f^{(n)}(z), \ \ z \in \mathbb{D},$$

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where $f^{(0)} = f$. If $n = 1$, it is the differentiation operator $D$. A systematic study of some other product-type operators started by Stević et al. since the publication of papers [11] and [12]. Before that there were a few papers in the topic, e.g., [5]. The publication of paper [11] first attracted some attention in product-type operators $DC_\varphi$ and $C_\varphi D$ (see, e.g., [14, 18, 20] and the references therein). The publication of paper [12] attracted some attention in product-type operators involving integral-type ones (see, e.g., [9, 19, 21] and the references therein). Now there is a great interest in various product-type operators (see, e.g., [6, 7, 10, 16, 30, 31] and the references therein).

By using multiplication, composition and the $n$th differentiation operators, we define the product-type operators in the following six ways

$$D^n M_u C_\varphi,\ D^n C_\varphi M_u,\ C_\varphi D^n M_u,\ M_u D^n C_\varphi,\ M_u C_\varphi D^n,\ C_\varphi M_u D^n. \tag{1.1}$$

When $n = 1$, they were studied by Sharma in [17]. They were also studied on the weighted Bergman space in a unified manner by Stević et al. in [24] and [25].

By constructing some test functions in weighted Bergman space, here we characterize the boundedness and compactness of the product-type operators in (1.1) from weighted Bergman space to Bloch-Orlicz space. Because some more suitable test functions were not found in weighted Bergman space, before this work we didn't find any result on these operators from weighted Bergman space to Bloch-Orlicz space.

Let $dA(z) = \frac{1}{\pi}dxdy$ be the normalized Lebesgue measure on $\mathbb{D}$. For $\alpha > -1$, let $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z)$ be the weighted Lebesgue measure on $\mathbb{D}$. For $p \geq 1$, the famous weighted Bergman space $A^p_\alpha$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|^p_{A^p_\alpha} = \int_{\mathbb{D}} |f(z)|^p dA_\alpha(z) < \infty.$$

It is well known that the weighted Bergman space $A^p_\alpha$ with the norm $\|\cdot\|_{A^p_\alpha}$ is a Banach space. For some results of the weighted Bergman space, see, for example, [28, 29].

Let $\Psi$ be a strictly increasing convex function on $[0, +\infty)$ such that $\Psi(0) = 0$. The Bloch-Orlicz space $\mathcal{B}^\Psi$ was introduced in [15] by Ramos Fernández, is the class of all $f \in H(\mathbb{D})$ such that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)\Psi(\lambda|f'(z)|) < \infty$$

for some $\lambda > 0$ depending on $f$. Ramos Fernández in [15] proved that $\mathcal{B}^\Psi$ is isometrically equal to $\mu_\Psi$-Bloch space, where

$$\mu_\Psi(z) = \frac{1}{\Psi^{-1}(\frac{1}{1-|z|^2})},\ \ z \in \mathbb{D}.$$

Hence, $\mathcal{B}^\Psi$ is a Banach space with the norm given by

$$\|f\|_{\mathcal{B}^\Psi} = |f(0)| + \sup_{z\in\mathbb{D}} \mu_\Psi(z)|f'(z)|.$$

This space generalizes some other spaces. For example, if $\Psi(t) = t^p$ with $p > 0$, then the space $\mathcal{B}^\Psi$ coincides with the weighted Bloch space $\mathcal{B}^\alpha$, where $\alpha = 1/p$. Also, if $\Psi(t) = t\log(1+t)$, then $\mathcal{B}^\Psi$ coincides with the Log-Bloch space (see [1]).

Let $X$ and $Y$ be Banach spaces. A linear operator $L : X \to Y$ is bounded if there exists a positive constant $K$ such that

$$\|Lf\|_Y \leq K\|f\|_X$$

for all $f \in X$. The operator $L : X \to Y$ is compact if it maps bounded sets into relatively compact sets.

In this paper, the letter $C$ denotes a positive constant which may differ from one occurrence to the other. The notation $a \lesssim b$ means that there exists a positive constant $C$ such that $a \leq Cb$. When $a \lesssim b$ and $b \lesssim a$, we write $a \asymp b$.

## 2 Prerequisites

The first result is a alternative to Proposition 3.11 in [3], which characterizes the compactness in terms of sequential convergence. So the proof is omitted.

**Lemma 2.1.** *Let $T$ be one of the operators in (1.1). Then the bounded operator $T : A^p_\alpha \to \mathcal{B}^\Psi$ is compact if and only if for every bounded sequence $\{f_j\}_{j\in\mathbb{N}}$ in $A^p_\alpha$ such that $f_j \to 0$ uniformly on every compact subset of $\mathbb{D}$ as $j \to \infty$, it follows that*

$$\lim_{j\to\infty} \|Tf_j\|_{\mathcal{B}^\Psi} = 0.$$

For $k = 0$, the following lemma was proved in [29], while for $k \geq 1$ it essentially follows from the Jensen's inequality (see [6]).

**Lemma 2.2.** *Let $\alpha > -1$ and $p \geq 1$. Then for each $k \in \mathbb{N}_0$, there exists a positive constant $C_k = C(\alpha, p, k)$ independent of $f \in A^p_\alpha$ and $z \in \mathbb{D}$ such that*

$$|f^{(k)}(z)| \leq \frac{C_k\|f\|_{A^p_\alpha}}{(1-|z|^2)^{k+\frac{\alpha+2}{p}}}.$$

In order to construct some test functions in weighted Bergman space, for a fixed $w \in \mathbb{D}$ and $i \in \mathbb{N}_0$ we define the following function

$$k_{w,i}(z) = \frac{(1-|w|^2)^{i+\frac{\alpha+2}{p}}}{(1-\overline{w}z)^{i+\frac{2\alpha+4}{p}}},\ z \in \mathbb{D}.$$

Then from [6], we know that $f_{w,i} \in A^p_\alpha$ and

$$\sup_{w\in\mathbb{D}} \|k_{w,i}\|_{A^p_\alpha} \lesssim 1. \tag{2.1}$$

By using some suitable linear combinations of the functions $k_{w,i}$, we obtain the test function in $A^p_\alpha$ in the following result, which will be used in the proofs of our main results.

**Lemma 2.3.** *Let $w \in \mathbb{D}$ and $n \in \mathbb{N}$. Then for each fixed $k \in \{0, 1, \ldots, n+1\}$, there exist constants $a_{0,k}$, $a_{1,k}$, $\ldots$, $a_{n+1,k}$ such that the function*

$$f_{w,k}(z) = \sum_{i=0}^{n+1} a_{i,k} k_{w,i}(z)$$

*satisfies*

$$f^{(k)}_{w,k}(w) = \frac{\overline{w}^k}{(1-|w|^2)^{k+\frac{\alpha+2}{p}}} \quad \text{and} \quad f^{(j)}_{w,k}(w) = 0 \tag{2.2}$$

*for each $j \in \{0, 1, \ldots, n+1\} \setminus \{k\}$. Moreover,*

$$\sup_{w\in\mathbb{D}} \|f_{w,k}\|_{A^p_\alpha} \lesssim 1. \tag{2.3}$$

*Proof.* We write $a = (2\alpha+4)/p$. From a direct calculation, it follows that the system (2.2) is equivalent to the following system

$$\begin{cases} \sum\limits_{i=0}^{n+1} (a+i) a_{i,k} = 0 \\ \sum\limits_{i=0}^{n+1} (a+i)(\alpha+i+1) a_{i,k} = 0 \\ \qquad \cdots\cdots \\ \sum\limits_{i=0}^{n+1} \prod\limits_{j=0}^{k-1} (a+i+j) a_{i,k} = 1 \\ \qquad \cdots\cdots \\ \sum\limits_{i=0}^{n+1} \prod\limits_{j=0}^{n} (a+i+j) a_{i,k} = 0. \end{cases} \tag{2.4}$$

Hence we only need to prove that there exist constants $a_{0,k}$, $a_{1,k}$, $\ldots$, $a_{n+1,k}$ such that the system (2.4) holds. By Lemma 3 in [22], the determinant of the system (2.4) equals $\prod_{j=1}^{n+1} j!$, which is different from zero. So there exist constants $a_{0,k}$, $a_{1,k}$, $\ldots$, $a_{n+1,k}$ such that the system (2.4) holds. From (2.1) the asymptotic expression of $\sup_{w\in\mathbb{D}} \|f_{w,k}\|_{A^p_\alpha} \lesssim 1$ is obvious. □

**Remark 2.1.** *It is not hard to see that $f_{w,k} \to 0$ uniformly on every compact subset of $\mathbb{D}$ as $|w| \to 1^-$.*

Stević in [22] used the Faà di Bruno's formula of the following version

$$(f\circ\varphi)^{(n)}(z) = \sum_{k=0}^{n} f^{(k)}(\varphi(z)) B_{n,k}(\varphi'(z), \ldots, \varphi^{(n-k+1)}(z)), \tag{2.5}$$

where $B_{n,k}(x_1, ..., x_{n-k+1})$ is the Bell polynomial. For $n \in \mathbb{N}$ the sum can go from $k = 1$ since $B_{n,0}(\varphi'(z), ..., \varphi^{(n-k+1)}(z)) = 0$, however we will keep the summation since for $n = 0$ the only existing term $B_{0,0}$ is equal to 1. From (2.5) and the Leibnitz formula the next Lemma 2.4 follows.

**Lemma 2.4.** *Let $f$, $u \in H(\mathbb{D})$ and $\varphi$ be an analytic self-map of $\mathbb{D}$. Then*

$$\big(u(z)f(\varphi(z))\big)^{(n+1)} = \sum_{k=0}^{n+1} f^{(k)}(\varphi(z)) \sum_{j=k}^{n+1} C_{n+1}^{j} u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z), \ldots, \varphi^{(j-k+1)}(z)\big).$$

## 3 Boundedness of the product-type operators

First we characterize the boundedness of the operator $D^n M_u C_\varphi : A^p_\alpha \to \mathcal{B}^\Psi$.

**Theorem 3.1.** *Let $\alpha > -1$, $p \geq 1$, $\varphi$ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. Then the following statements hold.*

*(i) The operator $D^n M_u C_\varphi : A^p_\alpha \to \mathcal{B}^\Psi$ is bounded.*

*(ii) The functions $u$ and $\varphi$ satisfy the following conditions:*

$$I_k := \sup_{z\in\mathbb{D}} \frac{\mu_\Psi(z)\Big|\sum\limits_{j=k}^{n+1} C_{n+1}^{j} u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z), \ldots, \varphi^{(j-k+1)}(z)\big)\Big|}{(1-|\varphi(z)|^2)^{k+\frac{\alpha+2}{p}}} < \infty$$

*for each $k \in \{0, 1, \ldots, n+1\}$.*

*Proof.* $(i) \Rightarrow (ii)$. Let $h_0(z) \equiv 1 \in A^p_\alpha$. Then we get

$$L_0 = \sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big|\sum_{j=0}^{n+1} C_{n+1}^{j} u^{(n+1-j)}(z) B_{j,0}(\varphi'(z), \ldots, \varphi^{(j+1)}(z))\Big| \leq C\|D^n M_u C_\varphi\|. \tag{3.1}$$

Let $h_k(z) = z^k \in A^p_\alpha$, $k = 1, 2, \ldots, n+1$. Assume now that we have proved the following inequalities

$$\sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big|\sum_{j=l}^{n+1} C_{n+1}^{j} u^{(n+1-j)}(z) B_{j,l}(\varphi'(z), \ldots, \varphi^{(j-l+1)}(z))\Big| \leq C\|D^n M_u C_\varphi\| \tag{3.2}$$

for each $l \in \{0, 1, ..., k-1\}$ and a $k \leq n+1$. Applying Lemma 2.4 to the function $h_k$, and noticing that $h_k^{(s)}(z) \equiv 0$ for $s > k$, we get

$$\begin{aligned}\big(D^n M_u C_\varphi h_k\big)'(z) &= \sum_{j=0}^{k} h_k^{(j)}(\varphi(z)) \sum_{i=j}^{n+1} C_{n+1}^{i} u^{(n+1-i)}(z) B_{i,j}(\varphi'(z), \ldots, \varphi^{(i-j+1)}(z)) \\ &= \sum_{j=0}^{k} k \cdots (k-j+1)(\varphi(z))^{k-j} \sum_{i=j}^{n+1} C_{n+1}^{i} u^{(n+1-i)}(z) B_{i,j}(\varphi'(z), \ldots, \varphi^{(i-j+1)}(z)). \end{aligned} \tag{3.3}$$

From (3.3), the boundedness of function $\varphi$ and the triangle inequality, by noticing that the coefficient at

$$\sum_{j=k}^{n+1} C_{n+1}^{j} u^{(n+1-j)}(z) B_{j,k}(\varphi'(z), \ldots, \varphi^{(j-k+1)}(z))$$

is independent of $z$ and finally using hypothesis (3.2) we easily obtain

$$L_k := \sup_{z\in\mathbb{D}} \mu_\Psi(z) \Big| \sum_{j=k}^{n+1} C_{n+1}^{j} u^{(n+1-j)}(z) B_{j,k}(\varphi'(z), \ldots, \varphi^{(j-k+1)}(z)) \Big| \leq C \|D^n M_u C_\varphi\|. \quad (3.4)$$

By induction we see that (3.4) holds for each $k \in \{0, 1, \ldots, n+1\}$.

For a fixed $w \in \mathbb{D}$ and $k \in \{0, 1, \ldots, n+1\}$, by Lemma 2.3 there exist constants $a_{0,k}$, $a_{1,k}$, $\ldots$, $a_{n+1,k}$ such that the function

$$f_{\varphi(w),k}(z) = \sum_{i=0}^{n+1} a_{i,k} k_{\varphi(w),i}(z),$$

satisfies

$$f^{(k)}_{\varphi(w),k}(\varphi(w)) = \frac{\overline{\varphi(w)}^k}{(1-|\varphi(w)|^2)^{k+\frac{\alpha+2}{p}}} \quad \text{and} \quad f^{(j)}_{\varphi(w),k}(\varphi(w)) = 0 \quad (3.5)$$

for each $j \in \{0, 1, \ldots, n+1\} \setminus \{k\}$. Moreover,

$$\sup_{w\in\mathbb{D}} \|f_{\varphi(w),k}\|_{A^p_\alpha} \leq C. \quad (3.6)$$

Then from (3.5), (3.6) and the boundedness of $D^n M_u C_\varphi : A^p_\alpha \to \mathcal{B}^\Psi$, we have

$$\begin{aligned} I_k(w) &:= \frac{\mu_\Psi(w)|\varphi(w)|^k \big| \sum_{j=k}^{n+1} C_{n+1}^{j} u^{(n+1-j)}(w) B_{j,k}\big(\varphi'(w), \ldots, \varphi^{(j-k+1)}(w)\big)\big|}{(1-|\varphi(w)|^2)^{k+\frac{\alpha+2}{p}}} \\ &\leq \|D^n M_u C_\varphi f_{\varphi(w),k}\|_{\mathcal{B}^\Psi} \leq C\|D^n M_u C_\varphi\|. \end{aligned} \quad (3.7)$$

From (3.7) we see that

$$\sup_{z\in\mathbb{D}} I_k(z) \leq C\big\|D^n M_u C_\varphi\big\|,$$

from which we obtain

$$\sup_{|\varphi(z)|>1/2} \frac{\mu_\Psi(z) \big| \sum_{j=k}^{n+1} C_{n+1}^{j} u^{(n+1-j)}(z) B_{j,k}(\varphi'(z), \ldots, \varphi^{(j-k+1)}(z))\big|}{(1-|\varphi(z)|^2)^{k+\frac{\alpha+2}{p}}} \leq C\|D^n M_u C_\varphi\|. \quad (3.8)$$

On the other hand, from (3.4) we get

$$\sup_{|\varphi(z)|\le 1/2} \frac{\mu_\Psi(z)\Big|\sum\limits_{j=k}^{n+1} C_{n+1}^{j} u^{(n+1-j)}(z) B_{j,k}(\varphi'(z),\dots,\varphi^{(j-k+1)}(z))\Big|}{(1-|\varphi(z)|^2)^{k+\frac{\alpha+2}{p}}} \le CL_k \le C\|D^n M_u C_\varphi\|. \tag{3.9}$$

Hence from (3.8) and (3.9) we see that $I_k < \infty$ for each $k \in \{0, 1, ..., n+1\}$.

$(ii) \Rightarrow (i)$. From Lemma 2.2 and Lemma 2.4, for all $f \in A_\alpha^p$ we have

$$\begin{aligned}
&\sup_{z\in\mathbb{D}} \mu_\Psi(z)\big|\big(D^n M_u C_\varphi f\big)'(z)\big| \\
&= \sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big|\sum_{k=0}^{n+1} f^{(k)}(\varphi(z)) \sum_{j=k}^{n+1} C_{n+1}^{j} u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z),\dots,\varphi^{(j-k+1)}(z)\big)\Big| \\
&\le \sup_{z\in\mathbb{D}} \mu_\Psi(z) \sum_{k=0}^{n+1} \big|f^{(k)}(\varphi(z))\big| \Big|\sum_{j=k}^{n+1} C_{n+1}^{j} u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z),\dots,\varphi^{(j-k+1)}(z)\big)\Big| \\
&\le \sum_{k=0}^{n+1} C_k I_k \|f\|_{A_\alpha^p}.
\end{aligned} \tag{3.10}$$

It is clear that

$$|(D^n M_u C_\varphi f)(0)| \le C\|f\|_{A_\alpha^p}. \tag{3.11}$$

Hence from (3.10) and (3.11) it follows that $D^n M_u C_\varphi : A_\alpha^p \to \mathcal{B}^\Psi$ is bounded. □

**Remark 3.1.** *If $D^n C_\varphi M_u : A_\alpha^p \to \mathcal{B}^\Psi$ is a zero operator, then it is obvious that $\|D^n C_\varphi M_u\| = 0$. Hence, the case is usually excluded from such considerations.*

**Remark 3.2.** *Since $D^n C_\varphi M_u = D^n M_{u\circ\varphi} C_\varphi$, the characterization of the boundedness of $D^n C_\varphi M_u : A_\alpha^p \to \mathcal{B}^\Psi$ can be directly obtained from Theorem 3.1. So we omit here.*

Noticing that

$$(C_\varphi D^n M_u f)'(z) = \sum_{k=0}^{n+1} C_{n+1}^{k} u^{(n+1-k)}(\varphi(z))\varphi'(z) f^{(k)}(\varphi(z)),$$

we can obtain the following result whose proof is similar to that of Theorem 3.1. So we also omit.

**Theorem 3.2.** *Let $\alpha > -1$, $p \ge 1$, $\varphi$ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. Then the following statements hold.*

*(i) The operator $C_\varphi D^n M_u : A_\alpha^p \to \mathcal{B}^\Psi$ is bounded.*

*(ii) The functions $u$ and $\varphi$ satisfy the following conditions:*

$$J_k := \sup_{z\in\mathbb{D}} \frac{\mu_\Psi(z)|u^{(n+1-k)}(\varphi(z))||\varphi'(z)|}{(1-|\varphi(z)|^2)^{k+\frac{\alpha+2}{p}}} < \infty$$

*for each $k \in \{0, 1, \ldots, n+1\}$.*

From a calculation, we have

$$(M_u D^n C_\varphi f)'(z) = \sum_{k=0}^{n} f^{(k)}(\varphi(z))\big[u'(z)B_{n,k}(\varphi'(z),\ldots,\varphi^{(n-k+1)}(z)) + u(z)B_{n+1,k}(\varphi'(z),\ldots,\varphi^{(n-k+2)}(z))\big] + u(z)(\varphi'(z))^{n+1}f^{(n+1)}(\varphi(z)),$$

and then we have the next result.

**Theorem 3.3.** *Let $\alpha > -1$, $p \geq 1$, $\varphi$ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. Then the following statements hold.*

*(i) The operator $M_u D^n C_\varphi : A^p_\alpha \to \mathcal{B}^\Psi$ is bounded.*

*(ii) The functions $u$ and $\varphi$ satisfy the following conditions:*

$$M_k := \sup_{z\in\mathbb{D}} \frac{\mu_\Psi(z)\big|u'(z)B_{n,k}(\varphi'(z),\ldots,\varphi^{(n-k+1)}(z)) + u(z)B_{n+1,k}(\varphi'(z),\ldots,\varphi^{(n-k+2)}(z))\big|}{(1-|\varphi(z)|^2)^{k+\frac{\alpha+2}{p}}} < \infty$$

*for each $k \in \{0, 1, \ldots, n\}$, and*

$$M_{n+1} := \sup_{z\in\mathbb{D}} \frac{\mu_\Psi(z)|u(z)||\varphi'(z)|^{n+1}}{(1-|\varphi(z)|^2)^{n+1+\frac{\alpha+2}{p}}} < \infty.$$

Since $(M_u C_\varphi D^n f)'(z) = u'(z)f^{(n)}(\varphi(z)) + u(z)\varphi'(z)f^{(n+1)}(\varphi(z))$, we have the following result.

**Theorem 3.4.** *Let $\alpha > -1$, $p \geq 1$, $\varphi$ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. Then the following statements hold.*

*(i) The operator $M_u C_\varphi D^n : A^p_\alpha \to \mathcal{B}^\Psi$ is bounded.*

*(ii) The functions $u$ and $\varphi$ satisfy the following conditions:*

$$R_{\cdot} = \sup_{z\in\mathbb{D}} \frac{\mu_\Psi(z)|u'(z)|}{(1-|\varphi(z)|^2)^{n+\frac{\alpha+2}{p}}} < \infty,$$

*and*

$$S := \sup_{z\in\mathbb{D}} \frac{\mu_\Psi(z)|u(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{n+1+\frac{\alpha+2}{p}}} < \infty.$$

**Remark 3.3.** *Noticing that $C_\varphi M_u D^n = M_{u\circ\varphi} C_\varphi D^n$, we can obtain the characterization of the boundedness of $C_\varphi M_u D^n : A^p_\alpha \to \mathcal{B}^\Psi$ from Theorem 3.4. Here we omit.*

## 4 Compactness of the product-type operators

We first characterize the compactness of the operator $D^n M_u C_\varphi : A^p_\alpha \to \mathcal{B}^\Psi$.

**Theorem 4.1.** *Let $\alpha > -1$, $p \geq 1$, $\varphi$ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. Then the following statements hold.*

*(i) The operator $D^n M_u C_\varphi : A^p_\alpha \to \mathcal{B}^\Psi$ is compact.*

*(ii) The functions $u$ and $\varphi$ satisfy $L_k < \infty$ and*

$$\lim_{|\varphi(z)|\to 1^-} \frac{\mu_\Psi(z)\Big|\sum_{j=k}^{n+1} C^j_{n+1} u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z),\dots,\varphi^{(j-k+1)}(z)\big)\Big|}{(1-|\varphi(z)|^2)^{k+\frac{\alpha+2}{p}}} = 0$$

*for each $k \in \{0,1,\dots,n+1\}$.*

*Proof.* $(i) \Rightarrow (ii)$. Suppose that the operator $D^n M_u C_\varphi : A^p_\alpha \to \mathcal{B}^\Psi$ is compact. Then it is clear that the operator $D^n M_u C_\varphi : A^p_\alpha \to \mathcal{B}^\Psi$ is bounded. Hence from the proof of Theorem 3.1 it follows that $L_k < \infty$ for each $k \in \{0,1,\dots,n+1\}$. Consider a sequence $\{\varphi(z_i)\}_{i\in\mathbb{N}}$ in $\mathbb{D}$ such that $|\varphi(z_i)| \to 1^-$ as $i \to \infty$. If such a sequence does not exist, then the last condition in $(ii)$ obviously holds. Without loss of generality, we may suppose that $|\varphi(z_i)| > 1/2$ for all $i \in \mathbb{N}$. For each fixed $k \in \{0,1,\dots,n+1\}$, by using this sequence we define the function sequence $f_{i,k}(z) = f_{\varphi(z_i),k}(z)$, $i \in \mathbb{N}$. Then from Lemma 2.3 and Remark 2.1, we see that $\sup_{i\in\mathbb{N}} \|f_{i,k}\|_{A^p_\alpha} \leq C$ and $f_{i,k} \to 0$ uniformly on every compact subset of $\mathbb{D}$ as $i \to \infty$, moreover

$$f^{(k)}_{i,k}(\varphi(z_i)) = \frac{\overline{\varphi(z_i)}^k}{(1-|\varphi(z_i)|^2)^{k+\frac{\alpha+2}{p}}} \quad \text{and} \quad f^{(j)}_{i,k}(\varphi(z_i)) = 0 \tag{4.1}$$

for each $j \in \{0,1,\dots,n+1\} \setminus \{k\}$. Then from Lemma 2.1 we have

$$\lim_{i\to\infty} \|D^n M_u C_\varphi f_{i,k}\|_{\mathcal{B}^\Psi} = 0. \tag{4.2}$$

Combing (4.1) and (4.2), for each fixed $k \in \{0,1,\dots,n+1\}$ we get

$$\lim_{i\to\infty} \frac{\mu_\Psi(z_i)\Big|\sum_{j=k}^{n+1} C^j_{n+1} u^{(n+1-j)}(z_i) B_{j,k}(\varphi'(z_i),\dots,\varphi^{(j-k+1)}(z_i))\Big|}{(1-|\varphi(z_i)|^2)^{k+\frac{\alpha+2}{p}}} = 0. \tag{4.3}$$

$(ii) \Rightarrow (i)$. We first prove that $D^n M_u C_\varphi : A^p_\alpha \to \mathcal{B}^\Psi$ is bounded. We observe that the last condition in $(ii)$ implies that for every $\varepsilon > 0$, there is an $\eta \in (0,1)$, such that for any $z \in K = \{z \in \mathbb{D} : |\varphi(z)| > \eta\}$

$$I_k(z) = \frac{\mu_\Psi(z)\Big|\sum\limits_{j=k}^{n+1} C^j_{n+1} u^{(n+1-j)}(z) B_{j,k}(\varphi'(z), \ldots, \varphi^{(j-k+1)}(z))\Big|}{(1-|\varphi(z)|^2)^{k+\frac{\alpha+2}{p}}} < \varepsilon \tag{4.4}$$

for each $k \in \{0, 1, \ldots, n+1\}$. From the fact $L_k < \infty$ for each $k \in \{0, 1, \ldots, n+1\}$ and (4.4), we have

$$I_k \le \varepsilon + \frac{L_k}{(1-\eta^2)^{k+\frac{\alpha+2}{p}}}. \tag{4.5}$$

From (4.5) and Theorem 3.1, we see that the operator $D^n M_u C_\varphi : A^p_\alpha \to \mathcal{B}^\Psi$ is bounded.

In order to prove that $D^n M_u C_\varphi : A^p_\alpha \to \mathcal{B}^\Psi$ is compact, by Lemma 2.1 we only need to prove that, if $\{f_i\}_{i\in\mathbb{N}}$ is a sequence in $A^p_\alpha$ such that $\sup_{i\in\mathbb{N}} \|f_i\|_{A^p_\alpha} \le M$ and $f_i \to 0$ uniformly on any compact subset of $\mathbb{D}$ as $i \to \infty$, then

$$\lim_{i\to\infty} \|D^n M_u C_\varphi f_i\|_{\mathcal{B}^\Psi} = 0.$$

For such chosen $\varepsilon$ and $\eta$, by using (4.4), Lemma 2.2 and Lemma 2.4, we have

$$\begin{aligned}
&\sup_{z\in\mathbb{D}} \mu_\Psi(z)\big|(D^n M_u C_\varphi f_i)'(z)\big| \\
&= \sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big|\sum_{k=0}^{n+1} f_i^{(k)}(\varphi(z)) \sum_{j=k}^{n+1} C^j_{n+1} u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z), \ldots, \varphi^{(j-k+1)}(z)\big)\Big| \\
&\le \sup_{z\in\mathbb{D}} \mu_\Psi(z) \sum_{k=0}^{n+1} \big|f_i^{(k)}(\varphi(z))\big| \Big|\sum_{j=k}^{n+1} C^j_{n+1} u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z), \ldots, \varphi^{(j-k+1)}(z)\big)\Big| \\
&\le \big(\sup_{z\in K} + \sup_{z\in\mathbb{D}\setminus K}\big)\mu_\Psi(z) \sum_{k=0}^{n+1} \big|f_i^{(k)}(\varphi(z))\big| \Big|\sum_{j=k}^{n+1} C^j_{n+1} u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z), \ldots, \varphi^{(j-k+1)}(z)\big)\Big| \\
&\le \sum_{k=0}^{n+1} L_k \sup_{|z|\le\eta} |f_i^{(k)}(z)| + M \sum_{k=0}^{n+1} C_k \varepsilon.
\end{aligned} \tag{4.6}$$

From (4.6), Lemma 2.1 and the fact $f_i \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $i \to \infty$ implies that for each $k \in \mathbb{N}$, $f_i^{(k)} \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $i \to \infty$, we finally get

$$\lim_{i\to\infty} \sup_{z\in\mathbb{D}} \mu_\Psi(z)\big|(D^n M_u C_\varphi f_i)'(z)\big| = 0. \tag{4.7}$$

It is clear that

$$\lim_{i\to\infty} \big|(D^n M_u C_\varphi f_i)(0)\big| = 0. \tag{4.8}$$

From (4.7) and (4.8) we obtain

$$\lim_{i\to\infty} \|D^n M_u C_\varphi f_i\|_{\mathcal{B}^\Psi} = 0.$$

This shows that the operator $D^n M_u C_\varphi : A^p_\alpha \to \mathcal{B}^\Psi$ is compact. □

**Remark 4.1.** *Since $D^n C_\varphi M_u = D^n M_{u\circ\varphi} C_\varphi$, the characterization of the compactness of $D^n C_\varphi M_u : A^p_\alpha \to \mathcal{B}^\Psi$ can be directly obtained from Theorem 4.1. So we omit here.*

Similar to Theorems 3.2, 3.3 and 3.4, we have the following results.

**Theorem 4.2.** *Let $\alpha > -1$, $p \geq 1$, $\varphi$ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. Then the following statements hold.*

*(i) The operator $C_\varphi D^n M_u : A^p_\alpha \to \mathcal{B}^\Psi$ is compact.*

*(ii) The functions $u$ and $\varphi$ satisfy the following conditions:*

$$\sup_{z\in\mathbb{D}} \mu_\Psi(z)|u^{(n+1-k)}(\varphi(z))||\varphi'(z)| < \infty$$

*and*

$$\lim_{|\varphi(z)|\to 1^-} \frac{\mu_\Psi(z)|u^{(n+1-k)}(\varphi(z))||\varphi'(z)|}{(1-|\varphi(z)|^2)^{k+\frac{\alpha+2}{p}}} = 0$$

*for each $k \in \{0, 1, \ldots, n+1\}$.*

**Theorem 4.3.** *Let $\alpha > -1$, $p \geq 1$, $\varphi$ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. Then the following statements hold.*

*(i) The operator $M_u D^n C_\varphi : A^p_\alpha \to \mathcal{B}^\Psi$ is compact.*

*(ii) The functions $u$ and $\varphi$ satisfy the following conditions:*

$$\sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big|u'(z)B_{n,k}(\varphi'(z),\ldots,\varphi^{(n-k+1)}(z)) + u(z)B_{n+1,k}(\varphi'(z),\ldots,\varphi^{(n-k+2)}(z))\Big| < \infty,$$

$$\lim_{|\varphi(z)|\to 1^-} \frac{\mu_\Psi(z)\Big|u'(z)B_{n,k}(\varphi'(z),\ldots,\varphi^{(n-k+1)}(z)) + u(z)B_{n+1,k}(\varphi'(z),\ldots,\varphi^{(n-k+2)}(z))\Big|}{(1-|\varphi(z)|^2)^{k+\frac{\alpha+2}{p}}} = 0$$

*for each $k \in \{0, 1, \ldots, n\}$,*

$$\sup_{z\in\mathbb{D}} \mu_\Psi(z)|u(z)||\varphi'(z)|^{n+1} < \infty,$$

*and*

$$\lim_{|\varphi(z)|\to 1^-} \frac{\mu_\Psi(z)|u(z)||\varphi'(z)|^{n+1}}{(1-|\varphi(z)|^2)^{n+1+\frac{\alpha+2}{p}}} = 0.$$

**Theorem 4.4.** *Let $\alpha > -1$, $p \geq 1$, $\varphi$ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. Then the following statements hold.*

*(i) The operator $M_u C_\varphi D^n : A^p_\alpha \to \mathcal{B}^\Psi$ is compact.*

*(ii) The functions $u$ and $\varphi$ are such that $u \in \mathcal{B}^\Psi$,*

$$\sup_{z\in\mathbb{D}} \mu_\Psi(z)|u(z)||\varphi'(z)| < \infty,$$

$$\lim_{|\varphi(z)|\to 1^-} \frac{\mu_\Psi(z)|u'(z)|}{(1-|\varphi(z)|^2)^{n+\frac{\alpha+2}{p}}} = 0,$$

*and*

$$\lim_{|\varphi(z)|\to 1^-} \frac{\mu_\Psi(z)|u(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{n+1+\frac{\alpha+2}{p}}} = 0.$$

**Remark 4.2.** *Noticing that $C_\varphi M_u D^n = M_{u\circ\varphi} C_\varphi D^n$, we can obtain the characterization of the compactness of $C_\varphi M_u D^n : A^p_\alpha \to \mathcal{B}^\Psi$ from Theorem 4.4. Here we omit.*

## References


[1] Attele, K., Toeplitz and Hankel operators on Bergman spaces, *Hokkaido Math. J.,* 1992, 21:279-293.

[2] Colonna, F. and Li, S., Weighted composition operators from the minimal Möbius invariant space into the Bloch space, *Mediter. J. Math.,* 2013, 10(1):395-409.

[3] Cowen, C. C. and MacCluer, B. D., Composition operators on spaces of analytic functions, Boca Roton: CRC Press, 1995.

[4] Esmaeili, K. and Lindström, M., Weighted composition operators between Zygmund type spaces and their essential norms, *Integral Equ. Oper. Theory.,* 2013, 75:473-490.

[5] Hibschweiler, R. A. and Portnoy, N., Composition followed by differentiation between Bergman and Hardy spaces, *Rocky Mountain J. Math.,* 2005, 35(3):843-855.

[6] Jiang, Z. J., On a class of operators from weighted Bergman spaces to some spaces of analytic functions, *Taiwan. J. Math.,* 2011, 15(5):2095-2121.

[7] Jiang, Z. J., On a product-type operator from weighted Bergman-Orlicz space to some weighted type spaces, *Appl. Math. Comput.,* 2015, 256:37-51.

[8] Jiang, Z. J., Completely continuous composition operators on Orlicz spaces, *Adv. Math. (China),* 2015, 44(1):111-116 (in Chinese).

[9] Krantz, S. and Stević, S., On the iterated logarithmic Bloch space on the unit ball, *Nonlinear Anal.,* 2009, 71:1772-1795.

[10] Liu, Y. and Yu, Y., Products of composition, multiplication and radial derivative operators from logarithmic Bloch spaces to weighted-type spaces on the unit ball, *J. Math. Anal. Appl.,* 2015, 423(1):76-93.

[11] Li, S. and Stević, S., Composition followed by differentiation between Bloch type spaces, *J. Comput. Anal. Appl.,* 2007, 9(2):195-205.

[12] Li, S. and Stević, S., Products of composition and integral type operators from $H^\infty$ to the Bloch space, *Complex Var. Elliptic Equ.,* 2008, 53(5):463-474.

[13] Madigan, K. and Matheson, A., Compact composition operators on the Bloch space, *Trans. Amer. Math. Soc.,* 1995, 347:2679-2687.

[14] Ohno, S., Products of composition and differentiation on Bloch spaces, *Bull. Korean Math. Soc.,* 2009, 46(6):1135-1140.

[15] Ramos Fernández, J. C., Composition operators on Bloch-Orlicz type spaces, *Appl. Math. Comput.,* 2010, 217:3392-3402.

[16] Sehba, B. and Stević, S., On some product-type operators from Hardy-Orlicz and Bergman-Orlicz spaces to weighted-type spaces, *Appl. Math. Comput.,* 2014, 233:565-581.

[17] Sharma, A. K., Products of composition multiplication and differentiation between Bergman and Bloch type spaces, *Turkish. J. Math.,* 2011, 35:275-291.

[18] Stević, S., Norm and essential norm of composition followed by differentiation from $\alpha$-Bloch spaces to $H^\infty_\mu$, *Appl. Math. Comput.,* 2009, 207:225-229.

[19] Stević, S., On an integral-type operator from logarithmic Bloch-type and mixed-norm spaces to Bloch-type spaces, *Nonlinear Anal.,* 2009, 71:6323-6342.

[20] Stević, S., Products of composition and differentiation operators on the weighted Bergman space, *Bull. Belg. Math. Soc.,* 2009, 16:623-635.

[21] Stević, S., Products of integral-type operators and composition operators from the mixed norm space to Bloch-type spaces, *Siberian Math. J.,* 2009, 50(4):726-736.

[22] Stević, S., Weighted differentiation composition operators from $H^\infty$ and Bloch spaces to $n$th weighted-type spaces on the unit disk, *Appl. Math. Comput.,* 2010, 216:3634-3641.

[23] Stević, S., Chen, R. and Zhou, Z., Weighted composition operators between Bloch type spaces in the polydisc, *Mat. Sb.,* 2010, 201(2):289-319.

[24] Stević, S., Sharma, A. K. and Bhat, A., Products of multiplication composition and differentiation operators on weighted Bergman spaces, *Appl. Math. Comput.,* 2011, 217:8115-8125.

[25] Stević, S., Sharma, A. K. and Bhat, A., Essential norm of multiplication composition and differentiation operators on weighted Bergman spaces, *Appl. Math. Comput.,* 2011, 218:2386-2397.

[26] Zhang, X.J., Guan, Y. and Li, M., Composition operators between logarithmic weighted Bloch type spaces on the unit ball, *Adv. Math. (China),* 2015, 44(3):553-561 (in Chinese).

[27] Zhao, L. K., Fredholm weighted composition operators on the weighted Dirichlet space, *Adv. Math. (China),* 2014, 43(3):419-424 (in Chinese).

[28] Zhu, K., Operator theory in function space, New York: Marcel Dekker, 1990.

[29] Zhu, K., Spaces of holomorphic functions in the unit ball, New York: Springer-Verlag, 2005.

[30] Zhu, X., Products of differentiation, composition and multiplication operator from Bergman type spaces to Bers spaces, *Integral Transforms Spec. Funct.,* 2007, 18:223-231.

[31] Zhu, X., Generalized weighted composition operators from Bloch spaces into Bers-type spaces, *Filomat.,* 2012, 26:1163-1169.

# On some recent fixed point results for $\alpha$-admissible mappings in $b$-metric spaces

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**Abstract**: The purpose of this paper is to present some fixed point theorems for weak $\alpha$-admissible mappings type in the setting of $b$-metric spaces. The results greatly optimize and improve some fixed point results in the existing literature. Moreover, we highlight our assertions by utilizing an example. In addition, we use our results to obtain the existence of solution for a class of nonlinear integral equations.

**Keywords**: $\alpha$-admissible mapping, $\alpha$-contraction mapping, rational $\alpha$-Geraghty contraction of type, fixed point, integral equation

## 1 Introduction

Since Polish mathematician Banach proved the well-known Banach contraction mapping principle in metric spaces in 1922 (see [1]), fixed point theory occupies a prominent place in strong research activity. Due to its applications in finding the existence of solutions for the nonlinear Volterra integral equations, nonlinear integro-differential equations and existence of equilibria in game theory as well, it has become the most celebrated tool in nonlinear analysis. During the past decades, scholars extend this principle towards different spaces, such as $G$-metric spaces, 2-metric spaces, fuzzy metric spaces, probabilistic metric spaces, cone metric spaces, partial metric spaces, modular metric spaces, $b$-metric spaces, *etc* (see [2-10]). Whereas, the most influential spaces among them, *i.e.*, $b$-metric spaces, or metric type spaces called by some authors, introduced by Bakhtin [9] or Czerwik [10], have a rapid development. Compared with other spaces, people are willing to deal with fixed point problems or the variational principle for single-valued or multi-valued operators in $b$-metric spaces, based on the fact that $b$-metrics have no continuity in general.

On the other hand, people fascinate fixed point results by substituting the Banach contractive mapping, such as Kannan contraction mapping, Chatterjea contraction mapping, $\alpha$-$\psi$-contractive type mapping, cyclic contractive mapping, multivalued contraction

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mapping, and so on (see [6-14]). Recently, Samet *et al.* [12] introduced the notion of $\alpha$-admissible mapping in the framework of metric spaces, and very recently, Sintunavarat [15] introduced the concepts of $\alpha$-admissible mapping type $S$, weak $\alpha$-admissible mapping, weak $\alpha$-admissible mapping type $S$, as some generalizations of $\alpha$-admissible mapping. Moreover, [15] proved fixed point theorems based on his new types of $\alpha$-admissibility in the setup of $b$-metric spaces. In this paper, inspired by [15], we introduce the notion of $\alpha$-admissibility mapping, and obtain some fixed point theorems, as compared to the main results of [15], with much simpler conditions and more straightforward proofs. Furthermore, we cope with some fixed point results for the mappings on rational $\alpha$-Geraghty contraction of type in terms of $\alpha$-admissibility in $b$-metric spaces. In addition, we give an application in the existence of a solution for a class of nonlinear integral equations. Our conditions are weak and applicable compared to the applications from [15].

For the sake of reader, the following definitions and results will be needed in the sequel.

**Definition 1.1**([16]). A mapping $\varphi : [0,\infty) \to [0,\infty)$ is said to be an altering distance function if it holds:

(1) $\varphi$ is nondecreasing and continuous;

(2) $\varphi(t) = 0$ if and only if $t = 0$.

**Definition 1.2**([15]). Let $X$ be a nonempty set and $s \geq 1$ a given real number. Let $\alpha : X \times X \to [0,\infty)$ and $f : X \to X$ be mappings. We say $f$ is an $\alpha$-admissible mapping type $S$ if for all $x, y \in X$, $\alpha(x,y) \geq s$ leads to $\alpha(fx, fy) \geq s$. In particular, $f$ is called $\alpha$-admissible mapping if $s = 1$.

**Remark 1.3** Usually, use $\mathcal{A}(X,\alpha)$ and $\mathcal{A}_s(X,\alpha)$ to denote the collection of all $\alpha$-admissible mappings on $X$ and the collection of all $\alpha$-admissible mappings type $S$ on $X$. It is worth reminding that the class of $\alpha$-admissible mappings and the class of $\alpha$-admissible mappings type $S$ are independent, in other words, $\mathcal{A}(X,\alpha) \neq \mathcal{A}_s(X,\alpha)$ in general case.

**Definition 1.4**([15]). Let $X$ be a nonempty set and $s \geq 1$ a given real number. Let $\alpha : X \times X \to [0,\infty)$ and $f : X \to X$ be mappings. We say $f$ is a weak $\alpha$-admissible mapping type $S$ if for all $x \in X$, $\alpha(x, fx) \geq s$ leads to $\alpha(fx, ffx) \geq s$. In particular, $f$ is called weak $\alpha$-admissible mapping if $s = 1$.

**Remark 1.5**. Customarily, utilize $\mathcal{WA}(X,\alpha)$ and $\mathcal{WA}_s(X,\alpha)$ to denote the collection of all weak $\alpha$-admissible mappings on $X$ and the collection of all weak $\alpha$-admissible mappings type $S$ on $X$. Clearly, $\mathcal{A}(X,\alpha) \subseteq \mathcal{WA}(X,\alpha)$ and $\mathcal{A}_s(X,\alpha) \subseteq \mathcal{WA}_s(X,\alpha)$.

**Definition 1.6**([10]). Let $X$ be a nonempty set and $s \geq 1$ a real number. A mapping $d : X \times X \to [0,\infty)$ is called a $b$-metric if for all $x, y, z \in X$, the following conditions are satisfied:

(b1) $d(x,y) = 0$ if and only if $x = y$;

(d2) $d(x,y) = d(y,x)$;

(d3) $d(x,z) \leq s[d(x,y) + d(y,z)]$.

In this case, $(X,d)$ is called a $b$-metric space.

**Definition 1.7**([17]). Let $(X,d)$ be a $b$-metric space, $x \in X$ and $\{x_n\}$ a sequence in $X$.

Then we say

(i) $\{x_n\}$ $b$-converges to $x$ if $d(x_n, x) \to 0$ as $n \to \infty$. In this case, we write $\lim_{n\to\infty} x_n = x$.

(ii) $\{x_n\}$ is a $b$-Cauchy sequence if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

(iii) $(X, d)$ is $b$-complete if every $b$-Cauchy sequence is $b$-convergent in $X$.

(iv) a function $f : X \to Y$ is $b$-continuous at a point $x \in X$ if $\{x_n\} \subset X$ $b$-converges to $x$, then $\{fx_n\}$ $b$-converges to $fx$, where $(Y, \rho)$ is a $b$-metric space.

Throughout this paper, unless otherwise specified, $X$ is a nonempty set, $f : X \to X$ is a mapping, $\mathrm{Fix}(f)$ denotes the set of all fixed points of $f$ on $X$, that is,

$$\mathrm{Fix}(f) := \{x \in X | fx = x\}.$$

Also, for each elements $x$ and $y$ in a $b$-metric space $(X, d)$ with coefficient $s \geq 1$, let

$$M_s(x, y) := \max\left\{d(x, y), d(x, fx), d(y, fy), \frac{d(x, fy) + d(y, fx)}{2s}\right\}.$$

**Lemma 1.8**([18]). Let $(X, d)$ be a $b$-metric space with coefficient $s \geq 1$ and let $\{x_n\}$ and $\{y_n\}$ be $b$-convergent to points $x, y \in X$, respectively. Then we have

$$\frac{1}{s^2} d(x, y) \leq \liminf_{n\to\infty} d(x_n, y_n) \leq \limsup_{n\to\infty} d(x_n, y_n) \leq s^2 d(x, y).$$

In particular, if $x = y$, then we have $\lim_{n\to\infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$, we have

$$\frac{1}{s} d(x, z) \leq \liminf_{n\to\infty} d(x_n, z) \leq \limsup_{n\to\infty} d(x_n, z) \leq s d(x, z).$$

**Definition 1.9**([15]). Let $(X, d)$ be a $b$-metric space with coefficient $s \geq 1$, let $\alpha : X \times X \to [0, \infty)$ be a mapping and let $\psi, \varphi : [0, \infty) \to [0, \infty)$ be two altering distance functions. A mapping $f : X \to X$ is said to be an $(\alpha, \psi, \varphi)_s$-contraction mapping if

$$x, y \in X \text{ with } \alpha(x, y) \geq s \Longrightarrow \psi(s^3 d(fx, fy)) \leq \psi(M_s(x, y)) - \varphi(M_s(x, y)). \tag{1.1}$$

In this case, write $\Omega_s(X, \alpha, \psi, \varphi)$ as the collection of all $(\alpha, \psi, \varphi)_s$-contraction mappings.

**Theorem 1.10**([15]). Let $(X, d)$ be a $b$-complete $b$-metric space with coefficient $s \geq 1$, let $\psi, \varphi : [0, \infty) \to [0, \infty)$ be two altering distance functions and let $\alpha : X \times X \to [0, \infty)$ and $f : X \to X$ be given mappings. Suppose that the following conditions hold:

$(S_1)$ $f \in \Omega_s(X, \alpha, \psi, \varphi) \cap \mathcal{WA}_s(X, \alpha)$;

$(S_2)$ there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq s$;

$(S_3)$ $\alpha$ has a transitive property type $S$, that is, for $x, y, z \in X$,

$$\alpha(x, y) \geq s \text{ and } \alpha(y, z) \geq s \Rightarrow \alpha(x, z) \geq s;$$

$(S_4)$ $f$ is $b$-continuous.

Then $\mathrm{Fix}(f) \neq \emptyset$.

**Theorem 1.11**([15]). Let $(X, d)$ be a $b$-complete $b$-metric space with coefficient $s \geq 1$, let $\psi, \varphi : [0, \infty) \to [0, \infty)$ be two altering distance functions and let $\alpha : X \times X \to [0, \infty)$ and $f : X \to X$ be given mappings. Suppose that the following conditions hold:

$(S_1)$ $f \in \Omega_s(X, \alpha, \psi, \varphi) \cap \mathcal{WA}_s(X, \alpha)$;
$(S_2)$ there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq s$;
$(S_3)$ $\alpha$ has a transitive property type $S$;
$(\widetilde{S}_4)$ $X$ is $\alpha_s$-regular, that is, if $\{x_n\}$ is a sequence in $X$ such that

$$\alpha(x_n, x_{n+1}) \geq s$$

for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq s$ for all $n \in \mathbb{N}$.
Then $\text{Fix}(f) \neq \emptyset$.

**Corollary 1.12**([15]). Let $(X, d)$ be a $b$-complete $b$-metric space with coefficient $s \geq 1$, let $\psi, \varphi : [0, \infty) \to [0, \infty)$ be two altering distance functions and let $\alpha : X \times X \to [0, \infty)$ and $f : X \to X$ be given mappings. Suppose that the following conditions hold:
$(\widetilde{S}_1)$ $f \in \Omega_s(X, \alpha, \psi, \varphi) \cap \mathcal{A}_s(X, \alpha)$;
$(S_2)$ there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq s$;
$(S_3)$ $\alpha$ has a transitive property type $S$;
$(S_4)$ $f$ is $b$-continuous.
Then $\text{Fix}(f) \neq \emptyset$.

**Corollary 1.13**([15]). Let $(X, d)$ be a $b$-complete $b$-metric space with coefficient $s \geq 1$, let $\psi, \varphi : [0, \infty) \to [0, \infty)$ be two altering distance functions and let $\alpha : X \times X \to [0, \infty)$ and $f : X \to X$ be given mappings. Suppose that the following conditions hold:
$(\widetilde{S}_1)$ $f \in \Omega_s(X, \alpha, \psi, \varphi) \cap \mathcal{A}_s(X, \alpha)$;
$(S_2)$ there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq s$;
$(S_3)$ $\alpha$ has a transitive property type $S$;
$(\widetilde{S}_4)$ $X$ is $\alpha_s$-regular.
Then $\text{Fix}(f) \neq \emptyset$.

# 2 Main results

**Definition 2.1**. Let $(X, d)$ be a $b$-metric space with coefficient $s \geq 1$, let $\alpha : X \times X \to [0, \infty)$ be a mapping and $\varepsilon > 1$ be a constant. A mapping $f : X \to X$ is said to be an $\alpha$-contraction mapping if

$$x, y \in X \text{ with } \alpha(x, y) \geq s \Longrightarrow s^{\varepsilon} d(fx, fy) \leq M_s(x, y). \tag{2.1}$$

In this case, write $\Omega_s(X, \alpha)$ as the collection of all $\alpha$-contraction mappings.

**Theorem 2.2**. Let $(X, d)$ be a $b$-complete $b$-metric space with coefficient $s > 1$. Let $\alpha : X \times X \to [0, \infty)$ and $f : X \to X$ be given mappings. Suppose that the following conditions hold:
$(S_1)$ $f \in \Omega_s(X, \alpha) \cap \mathcal{WA}_s(X, \alpha)$;
$(S_2)$ there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq s$;
$(S_3)$ $f$ is $b$-continuous.
Then $\text{Fix}(f) \neq \emptyset$.

**Proof**. By $(S_2)$, for $x_0 \in X$, construct a Picard iteration sequence $\{x_n\}$ satisfying $x_{n+1} = fx_n$, $n \in \mathbb{N}$. Assume that $x_{n_0} = x_{n_0+1}$ for some $n_0$, then $\mathrm{Fix}(f) = \{x_{n_0}\} \neq \emptyset$, in this case, the conclusion is satisfied. So set $x_n \neq x_{n+1}$ for all $n$, that is, $d(x_n, x_{n+1}) > 0$ for all $n$. Let us prove the following inequality:

$$d(x_{n+1}, x_{n+2}) \leq \lambda d(x_n, x_{n+1}), \tag{2.2}$$

where $\lambda \in [0, \frac{1}{s})$ is a constant.

Indeed, in view of $f \in \mathcal{WA}_s(X, \alpha)$ and $\alpha(x_0, fx_0) \geq s$, it implies that

$$\alpha(x_1, x_2) = \alpha(fx_0, ffx_0) \geq s.$$

Repeating this process, we make a conclusion that

$$\alpha(x_n, x_{n+1}) \geq s$$

for all $n$. Making the most of (2.1), we have

$$\begin{aligned}
s^{\varepsilon} d(x_{n+1}, x_{n+2}) &= s^{\varepsilon} d(fx_n, fx_{n+1}) \\
&\leq M_s(x_n, x_{n+1}) \\
&= \max\Bigg\{ d(x_n, x_{n+1}), d(x_n, fx_n), d(x_{n+1}, fx_{n+1}), \\
&\qquad \frac{d(x_n, fx_{n+1}) + d(x_{n+1}, fx_n)}{2s} \Bigg\} \\
&= \max\Bigg\{ d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \\
&\qquad \frac{d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1})}{2s} \Bigg\} \\
&\leq \max\left\{ d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \frac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2} \right\} \\
&= \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}.
\end{aligned} \tag{2.3}$$

If $d(x_n, x_{n+1}) \leq d(x_{n+1}, x_{n+2})$, then by (2.3), it follows that

$$s^{\varepsilon} d(x_{n+1}, x_{n+2}) \leq d(x_{n+1}, x_{n+2}).$$

Hence, $d(x_{n+1}, x_{n+2}) = 0$, it is a contradiction. If $d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1})$, then by (2.3), it establishes that

$$s^{\varepsilon} d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1}).$$

As a result, (2.2) holds, where $\lambda = \frac{1}{s^{\varepsilon}} \in [0, \frac{1}{s})$.

Now by [11, Lemma 3.1], taking advantage of (2.2), we claim that $\{x_n\}$ is a $b$-Cauchy sequence. Since $(X, d)$ is $b$-complete, we know that $\{x_n\}$ $b$-converges to some point $x \in X$. Finally, we show $x \in \mathrm{Fix}(f)$. Actually, by using $(S_3)$, it is not hard to verify that

$$d(fx, x) \leq s[d(fx, fx_n) + d(fx_n, x)] = s[d(fx, fx_n) + d(x_{n+1}, x)] \to 0 \text{ as } n \to \infty.$$

Therefore, $d(fx, x) = 0$, that is to say, $x \in \mathrm{Fix}(f)$. □

**Theorem 2.3**. Let $(X, d)$ be a $b$-complete $b$-metric space with coefficient $s > 1$. Let $\alpha : X \times X \to [0, \infty)$ and $f : X \to X$ be given mappings. Suppose that the following conditions hold:

$(S_1)$ $f \in \Omega_s(X, \alpha) \cap \mathcal{WA}_s(X, \alpha)$;

$(S_2)$ there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq s$;

$(\widetilde{S}_3)$ $X$ is $\alpha_s$-regular.

Then $\mathrm{Fix}(f) \neq \emptyset$.

**Proof**. Making full use of the proof of Theorem 2.2, we obtain a sequence $\{x_n\}$ satisfying $x_{n+1} = fx_n \to x \in X$ as $n \to \infty$. Then by $(\widetilde{S}_3)$, we get $\alpha(x_n, x) \geq s$ for all $n \in \mathbb{N}$. By virtue of $(S_1)$, we have

$$\begin{aligned} s^{\varepsilon} d(fx_n, fx) &\leq M_s(x_n, x) \\ &= \max\Bigg\{ d(x_n, x), d(x_n, fx_n), d(x, fx), \\ &\qquad \frac{d(x_n, fx) + d(x, fx_n)}{2s} \Bigg\} \\ &\leq \max\Bigg\{ d(x_n, x), s[d(x_n, x) + d(x_{n+1}, x)], d(x, fx), \\ &\qquad \frac{d(x_n, x) + d(x, fx)}{2} + \frac{d(x, x_{n+1})}{2s} \Bigg\} \\ &\to \max\left\{ 0, 0, d(x, fx), \frac{d(x, fx)}{2} \right\} = d(x, fx) \ (n \to \infty), \end{aligned}$$

which implies that

$$\lim_{n \to \infty} d(fx_n, fx) \leq \frac{1}{s^{\varepsilon}} d(x, fx). \tag{2.4}$$

Note that

$$\frac{1}{s} d(x, fx) \leq d(x, fx_n) + d(fx_n, fx) = d(x, x_{n+1}) + d(fx_n, fx). \tag{2.5}$$

Taking the limit as the above inequality (2.5) and utilizing (2.4), we speculate that

$$\frac{1}{s} d(x, fx) \leq \frac{1}{s^{\varepsilon}} d(x, fx),$$

which follows that $d(x, fx) = 0$, that is, $x \in \mathrm{Fix}(f)$. □

**Corollary 2.4**. Let $(X, d)$ be a $b$-complete $b$-metric space with coefficient $s > 1$. Let $\alpha : X \times X \to [0, \infty)$ and $f : X \to X$ be given mappings. Suppose that the following conditions hold:

$(\widetilde{S}_1)$ $f \in \Omega_s(X, \alpha) \cap \mathcal{A}_s(X, \alpha)$;

$(S_2)$ there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq s$;

$(S_3)$ $f$ is $b$-continuous.

Then $\mathrm{Fix}(f) \neq \emptyset$.

**Corollary 2.5**. Let $(X,d)$ be a $b$-complete $b$-metric space with coefficient $s>1$. Let $\alpha: X\times X\to[0,\infty)$ and $f: X\to X$ be given mappings. Suppose that the following conditions hold:

$(\widetilde{S}_1)$ $f\in\Omega_s(X,\alpha)\cap\mathcal{A}_s(X,\alpha)$;

$(S_2)$ there exists $x_0\in X$ such that $\alpha(x_0,fx_0)\geq s$;

$(\widetilde{S}_3)$ $X$ is $\alpha_s$-regular.

Then $\mathrm{Fix}(f)\neq\emptyset$.

**Remark 2.6**. Theorem 2.2, Theorem 2.3, Corollary 2.4 and Corollary 2.5 greatly optimize and improve Sintunavarat's theorems, *i.e.*, Theorem 1.10, Theorem 1.11, Corollary 1.12 and Corollary 1.13, respectively. Actually, on the one hand, compared with (1.1), (2.1) not only deletes the limitation of the altering distance functions $\psi$ and $\varphi$, but also it dispenses with an item $\varphi(M_s(x,y))$ which makes the condition become much wider. These are some great improvements. Moreover, our index $\varepsilon>1$ is arbitrary, and it clearly contains $\varepsilon=3$. Hence, our range $\varepsilon>1$ is much larger and more applicable. On the other hand, our theorems dismiss the condition of transitive property type $S$ for the mapping $\alpha$. That is to say, the conditions of our theorems are weaker than Sintunavarat's theorems. Therefore, our conclusions may be more convenient than Sintunavarat's in applications.

**Remark 2.7**. From the proofs of our theorems, it is easy to see that we do not use Lemma 1.8. Our proofs are much simpler since we do not refer to $b$-discontinuity of $b$-metric. Whereas, in order to overcome the difficulty of the $b$-discontinuity of $b$-metric, the proofs of Sintunavarat's theorems are very comprehensive based on the fact of depending on Lemma 1.8 strongly.

**Example 2.8**. Let $X=\mathbb{R}$ and define

$$d(x,y)=|x-y|^2$$

for all $x,y\in X$. Then $(X,d)$ is a $b$-complete $b$-metric space with coefficient $s=2$. Define mappings $f: X\to X$ and $\alpha: X\times X\to[0,\infty)$ by

$$fx=\begin{cases}\frac{x}{4}, & x\in[0,\frac{16}{3}],\\ \frac{1}{8}x+\frac{2}{3}, & x\in(\frac{16}{3},\infty),\end{cases}$$

and

$$\alpha(x,y)=\begin{cases}\frac{5}{4}+\frac{1}{|x-y|}, & x,y\in[0,\frac{16}{3}],\\ 0, & \text{otherwise.}\end{cases}$$

Let us prove $f\in\Omega_s(X,\alpha)$. Actually, assume that $x,y\in X$ with $\alpha(x,y)\geq s=2$ and hence $x,y\in[0,\frac{16}{3}]$ with $|x-y|\leq\frac{4}{3}$. Let $1<\varepsilon\leq 4$ be a constant. Then

$$\begin{aligned}2^{\varepsilon}d(fx,fy)&=2^{\varepsilon}|fx-fy|^2\\&=2^{\varepsilon}\left|\frac{x}{4}-\frac{y}{4}\right|^2\\&=2^{\varepsilon-4}|x-y|^2\\&\leq M_s(x,y).\end{aligned}$$

As a consequence, $f \in \Omega_s(X, \alpha)$.

We also can verify $f \in \mathcal{WA}_s(X, \alpha)$. Indeed, if $x \in X$ such that

$$\alpha(x, fx) \geq s = 2,$$

then $x, fx \in [0, \frac{16}{3}]$ and $|x - fx| \leq \frac{4}{3}$. Thus $x \in [0, \frac{16}{9}]$. This indicates that $ffx \in [0, \frac{1}{9}]$ and hence

$$\alpha(fx, ffx) = \frac{5}{4} + \frac{16}{3|x|} \geq \frac{17}{4} > s = 2.$$

Otherwise, it is obvious that $f$ is $b$-continuous and there exists $x_0 = 1$ such that

$$\alpha(x_0, fx_0) = \alpha(1, f1) = \frac{5}{4} + \frac{1}{|1 - f1|} = \frac{31}{12} \geq 2 = s.$$

Consequently, all the conditions of Theorem 2.2 hold. Thus $\mathrm{Fix}(f) = \{0\} \neq \emptyset$.

However, we cannot use Theorem 1.10 to get $\mathrm{Fix}(f) \neq \emptyset$, since $\alpha$ is unsuitable for the condition $(S_3)$ of this theorem. Indeed, put $x = 4$, $y = 3$, $z = 2$. Though $\alpha(x, y) = \frac{9}{4} > 2$ and $\alpha(y, z) = \frac{9}{4} > 2$, whereas, $\alpha(x, z) = \frac{7}{4} < 2$. So $(S_3)$ does not hold in this example. In other words, Theorem 2.2 is more superior than Theorem 1.10. □

In the sequel, let $s \geq 1$ be a constant and let $\mathcal{F}_s$ denote the class of all functions $\beta : [0, \infty) \to [0, \frac{1}{s})$ satisfying the following condition:

$$\limsup_{n \to \infty} \beta(t_n) = \frac{1}{s} \text{ implies that } t_n \to 0 \text{ as } n \to \infty.$$

**Definition 2.9**. Let $(X, d)$ be a $b$-metric space with coefficient $s \geq 1$, and let $\alpha : X \times X \to [0, \infty)$ be a function. A mapping $f : X \to X$ is called a rational $\alpha$-Geraghty contraction of type $\mathbf{I}_{\varepsilon,\beta}$ if there exist $\varepsilon > 0$ and $\beta \in \mathcal{F}_s$ such that

$$x, y \in X \text{ with } \alpha(x, y) \geq s \Longrightarrow \alpha(x, y) s^{\varepsilon} d(fx, fy) \leq \beta(M_I(x, y)) M_I(x, y), \tag{2.6}$$

where

$$M_I(x, y) = \max \left\{ d(x, y), \frac{d(x, fx)\, d(y, fy)}{1 + d(x, y)}, \frac{d(x, fx)\, d(y, fy)}{1 + d(fx, fy)} \right\}.$$

**Definition 2.10**. Let $(X, d)$ be a $b$-metric space with coefficient $s \geq 1$, and let $\alpha : X \times X \to [0, \infty)$ be a function. A mapping $f : X \to X$ is called a rational $\alpha$-Geraghty contraction of type $\mathbf{II}_{\varepsilon,\beta}$ if there exist $\varepsilon > 0$ and $\beta \in \mathcal{F}_s$ such that

$$x, y \in X \text{ with } \alpha(x, y) \geq s \Longrightarrow \alpha(x, y) s^{\varepsilon} d(fx, fy) \leq \beta(M_{II}(x, y)) M_{II}(x, y), \tag{2.7}$$

where

$$M_{II}(x, y) = \max \left\{ d(x, y), \frac{d(x, fx)\, d(x, fy) + d(y, fy)\, d(y, fx)}{1 + s\left[d(x, fx) + d(y, fy)\right]}, \right.$$
$$\left. \frac{d(x, fx)\, d(x, fy) + d(y, fy)\, d(y, fx)}{1 + s\left[d(x, fy) + d(y, fx)\right]} \right\}.$$

**Definition 2.11**. Let $(X,d)$ be a $b$-metric space with coefficient $s \geq 1$, and let $\alpha : X \times X \to [0,\infty)$ be a function. A mapping $f : X \to X$ is called a rational $\alpha$-Geraghty contraction of type $\mathbf{III}_{\varepsilon,\beta}$ if there exist $\varepsilon > 0$ and $\beta \in \mathcal{F}_s$ such that

$$x,y \in X \text{ with } \alpha(x,y) \geq s \Longrightarrow \alpha(x,y)s^{\varepsilon}d(fx,fy) \leq \beta(M_{III}(x,y))M_{III}(x,y), \tag{2.8}$$

where

$$M_{III}(x,y) = \max\left\{ d(x,y), \frac{d(x,fx)\,d(y,fy)}{1+s\left[d(x,y)+d(x,fy)+d(y,fx)\right]}, \right.$$
$$\left. \frac{d(x,fy)\,d(x,y)}{1+sd(x,fx)+s^3\left[d(y,fx)+d(y,fy)\right]} \right\}.$$

**Theorem 2.12**. Let $(X,d)$ be a $b$-complete $b$-metric space with coefficient $s > 1$, and let $\alpha : X \times X \to [0,\infty)$ be a function and $f : X \to X$ be a mapping. Suppose that the following conditions hold:

(i) $f$ is a rational $\alpha$-Geraghty contraction of type $\mathbf{I}_{\varepsilon,\beta}$ (resp. type $\mathbf{II}_{\varepsilon,\beta}$ or type $\mathbf{III}_{\varepsilon,\beta}$);

(ii) $f \in \mathcal{A}_s(X,\alpha)$ and there exists $x_0 \in X$ such that $\alpha(x_0,fx_0) \geq s$;

(iii) $f$ is $b$-continuous or $X$ is $\alpha_s$-regular.

Then $\mathrm{Fix}(f) \neq \emptyset$.

**Proof**. By (ii) and the proof of Theorem 2.2, we can construct a Picard iteration sequence $\{x_n\}$ satisfying $x_{n+1} = fx_n$ and

$$\alpha(x_n,x_{n+1}) \geq s$$

for all $n \in \mathbb{N}$. Let us prove that

$$d(x_{n+1},x_{n+2}) \leq \lambda d(x_n,x_{n+1}) \tag{2.9}$$

for all $n \in \mathbb{N}$, where $\lambda \in [0,\frac{1}{s})$.

First of all, let $f$ be a rational $\alpha$-Geraghty contraction of type $\mathbf{I}_{\varepsilon,\beta}$. Then by (2.6), we have

$$\begin{aligned} s^{\varepsilon}d(x_{n+1},x_{n+2}) &= s^{\varepsilon}d(fx_n,fx_{n+1}) \\ &\leq \alpha(x_n,x_{n+1})s^{\varepsilon}d(fx_n,fx_{n+1}) \\ &\leq \beta(M_I(x_n,x_{n+1}))M_I(x_n,x_{n+1}) \\ &\leq \frac{1}{s}M_I(x_n,x_{n+1}), \end{aligned} \tag{2.10}$$

which follows that

$$\begin{aligned} & s^{\varepsilon+1}d(x_{n+1},x_{n+2}) \leq M_I(x_n,x_{n+1}) \\ &= \max\left\{ d(x_n,x_{n+1}), \frac{d(x_n,x_{n+1})\,d(x_{n+1},x_{n+2})}{1+d(x_n,x_{n+1})}, \frac{d(x_n,x_{n+1})\,d(x_{n+1},x_{n+2})}{1+d(x_{n+1},x_{n+2})} \right\} \\ &\leq \max\left\{ d(x_n,x_{n+1}), \frac{d(x_n,x_{n+1})\,d(x_{n+1},x_{n+2})}{d(x_n,x_{n+1})}, \frac{d(x_n,x_{n+1})\,d(x_{n+1},x_{n+2})}{d(x_{n+1},x_{n+2})} \right\} \\ &= \max\left\{ d(x_n,x_{n+1}), d(x_{n+1},x_{n+2}) \right\}. \end{aligned} \tag{2.11}$$

If $d(x_n, x_{n+1}) \leq d(x_{n+1}, x_{n+2})$, then from (2.11), it leads to

$$d(x_{n+1}, x_{n+2}) \leq \frac{1}{s^{\varepsilon+1}} d(x_{n+1}, x_{n+2}) < d(x_{n+1}, x_{n+2}).$$

This is a contradiction. So

$$d(x_{n+1}, x_{n+2}) \leq \frac{1}{s^{\varepsilon+1}} d(x_n, x_{n+1}).$$

In this case, (2.9) is satisfied, where $\lambda = \frac{1}{s^{\varepsilon+1}} \in [0, \frac{1}{s})$.

Secondly, let $f$ be a rational $\alpha$-Geraghty contraction of type $\mathbf{II}_{\varepsilon,\beta}$. Then similarly by (2.10), we have

$$s^{\varepsilon} d(x_{n+1}, x_{n+2}) \leq \frac{1}{s} M_{II}(x_n, x_{n+1}),$$

which establishes that

$$\begin{aligned}
&s^{\varepsilon+1} d(x_{n+1}, x_{n+2}) \leq M_{II}(x_n, x_{n+1}) \\
&= \max \left\{ d(x_n, x_{n+1}), \frac{d(x_n, x_{n+1}) d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+2}) d(x_{n+1}, x_{n+1})}{1 + s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})]}, \right. \\
&\qquad \left. \frac{d(x_n, x_{n+1}) d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+2}) d(x_{n+1}, x_{n+1})}{1 + s[d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1})]} \right\} \\
&= \max \left\{ d(x_n, x_{n+1}), \frac{d(x_n, x_{n+1}) d(x_n, x_{n+2})}{1 + s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})]}, \frac{d(x_n, x_{n+1}) d(x_n, x_{n+2})}{1 + sd(x_n, x_{n+2})} \right\} \\
&\leq \max \left\{ d(x_n, x_{n+1}), \frac{sd(x_n, x_{n+1}) [d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})]}{1 + s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})]}, \frac{d(x_n, x_{n+1}) d(x_n, x_{n+2})}{sd(x_n, x_{n+2})} \right\} \\
&\leq \max \left\{ d(x_n, x_{n+1}), \frac{sd(x_n, x_{n+1}) [d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})]}{s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})]}, \frac{d(x_n, x_{n+1})}{s} \right\} \\
&\leq d(x_n, x_{n+1}).
\end{aligned}$$

Accordingly, (2.9) is also satisfied, where $\lambda = \frac{1}{s^{\varepsilon+1}} \in [0, \frac{1}{s})$.

Thirdly, let $f$ be a rational $\alpha$-Geraghty contraction of type $\mathbf{III}_{\varepsilon,\beta}$. Then similarly by (2.13), we have

$$s^{\varepsilon} d(x_{n+1}, x_{n+2}) \leq \frac{1}{s} M_{III}(x_n, x_{n+1}),$$

which implies that

$$\begin{aligned}
&s^{\varepsilon+1} d(x_{n+1}, x_{n+2}) \leq M_{III}(x_n, x_{n+1}) \\
&= \max \left\{ d(x_n, x_{n+1}), \frac{d(x_n, x_{n+1}) d(x_{n+1}, x_{n+2})}{1 + s[d(x_n, x_{n+1}) + d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1})]}, \right. \\
&\qquad \left. \frac{d(x_n, x_{n+2}) d(x_n, x_{n+1})}{1 + sd(x_n, x_{n+1}) + s^3 [d(x_{n+1}, x_{n+1}) + d(x_{n+1}, x_{n+2})]} \right\} \\
&\leq \max \left\{ d(x_n, x_{n+1}), \frac{d(x_n, x_{n+1}) d(x_{n+1}, x_{n+2})}{1 + s[d(x_n, x_{n+1}) + d(x_n, x_{n+2})]}, \right. \\
&\qquad \left. \frac{s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] d(x_n, x_{n+1})}{1 + sd(x_n, x_{n+1}) + s^3 d(x_{n+1}, x_{n+2})} \right\} \\
&\leq \max \left\{ d(x_n, x_{n+1}), \frac{d(x_n, x_{n+1}) d(x_{n+1}, x_{n+2})}{d(x_n, x_{n+1})}, \frac{s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] d(x_n, x_{n+1})}{s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})]} \right\} \\
&= \max \{ d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}) \}.
\end{aligned}$$

A similar discussion as to (2.11), we can get (2.9), too.

In a word, under the conditions of (i) and (ii), we always acquire (2.9). As a consequence, by using [11, Lemma 3.1] and the $b$-completeness of $(X,d)$, there exists a point $x\in X$ such that $x_n\to x$ as $n\to\infty$. Now by (iii), if $f$ is $b$-continuous, then

$$x=\lim_{n\to\infty}x_{n+1}=\lim_{n\to\infty}f\left(x_n\right)=f\left(\lim_{n\to\infty}x_n\right)=fx,$$

that is, $x\in \mathrm{Fix}(f)$. If $X$ is $\alpha_s$-regular, then $\alpha(x_n,x)\geq s$. Put

$$M(x_n,x)\in\{M_I(x_n,x),M_{II}(x_n,x),M_{III}(x_n,x)\},$$

it follows immediately from (2.6), (2.7) and (2.8) that

$$\alpha(x_n,x)s^{\varepsilon}d(fx_n,fx)\leq\beta(M(x_n,x))M(x_n,x). \tag{2.12}$$

We show that

$$\lim_{n\to\infty}M(x_n,x)=0. \tag{2.13}$$

Indeed, for one thing,

$$\begin{aligned}M_I\left(x_n,x\right)&=\max\left\{d\left(x_n,x\right),\frac{d\left(x_n,x_{n+1}\right)d\left(x,fx\right)}{1+d\left(x_n,x\right)},\frac{d\left(x_n,x_{n+1}\right)d\left(x,fx\right)}{1+d\left(x_{n+1},fx\right)}\right\}\\&\to\max\left\{0,0,0\right\}=0,\quad\text{as } n\to\infty.\end{aligned}$$

For another thing,

$$\begin{aligned}M_{II}\left(x_n,x\right)&=\max\left\{d\left(x_n,x\right),\frac{d\left(x_n,x_{n+1}\right)d\left(x_n,fx\right)+d\left(x,fx\right)d\left(x,x_{n+1}\right)}{1+s\left[d\left(x_n,x_{n+1}\right)+d\left(x,fx\right)\right]},\right.\\&\qquad\left.\frac{d\left(x_n,x_{n+1}\right)d\left(x_n,fx\right)+d\left(x,fx\right)d\left(x,x_{n+1}\right)}{1+s\left[d\left(x_n,fx\right)+d\left(x,x_{n+1}\right)\right]}\right\}\\&\leq\max\left\{d\left(x_n,x\right),d\left(x_n,x_{n+1}\right)s\left[d\left(x_n,x\right)+d\left(x,fx\right)\right]+d\left(x,fx\right)d\left(x,x_{n+1}\right),\right.\\&\qquad\left.d\left(x_n,x_{n+1}\right)s\left[d\left(x_n,x\right)+d\left(x,fx\right)\right]+d\left(x,fx\right)d\left(x,x_{n+1}\right)\right\}\\&\to\max\left\{0,0,0\right\}=0,\quad\text{as } n\to\infty.\end{aligned}$$

For the third thing,

$$\begin{aligned}M_{III}(x_n,x)&=\max\left\{d(x_n,x),\frac{d(x_n,x_{n+1})d(x,fx)}{1+s[d(x_n,x)+d(x_n,fx)+d(x,x_{n+1})]},\right.\\&\qquad\left.\frac{d(x_n,fx)d(x_n,x)}{1+sd(x_n,x_{n+1})+s^3[d(x,x_{n+1})+d(x,fx)]}\right\}\\&\to\max\left\{0,0,0\right\}=0,\quad\text{as } n\to\infty.\end{aligned}$$

Thus (2.13) holds.

Using (2.12) and (2.13), we speculate that

$$\begin{aligned}s^{\varepsilon}d(x,fx)&\leq s^{\varepsilon+1}[d(x,fx_n)+d(fx_n,fx)]\\&\leq s^{\varepsilon+1}d(x,x_{n+1})+s^{\varepsilon+1}\alpha(x_n,x)d(fx_n,fx)\\&\leq s^{\varepsilon+1}d(x,x_{n+1})+s\beta(M(x_n,x))M(x_n,x)\\&\leq s^{\varepsilon+1}d(x,x_{n+1})+M(x_n,x)\to 0,\quad\text{as } n\to\infty,\end{aligned}$$

which establishes that $d(x,fx)=0$, that is to say, $x\in\mathrm{Fix}(f)$. □

# 3 Application

In this section, we prove an existence theorem for a solution of the following nonlinear integral equation by using our results in the previous section:

$$x(c) = \phi(c) + \int_a^b K(c, r, x(r))\mathrm{d}r, \tag{3.1}$$

where $a, b \in \mathbb{R}$, $x \in C[a, b]$ (the set of all continuous functions from $[a, b]$ into $\mathbb{R}$, $\phi : [a, b] \to \mathbb{R}$ and $K : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ are given mappings.

The following theorem greatly improves Theorem 3.1 of [15] with simpler conditions, which illustrates the superiority of our results.

**Theorem 3.1**. Consider the nonlinear integral equation (3.1). Suppose that the following conditions hold:

(i) $K : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing in the third order;

(ii) there exists $p > 1$ satisfying the following condition: for each $r, c \in [a, b]$ and $x, y \in C[a, b]$ with $x(w) \leq y(w)$ for all $w \in [a, b]$, we have

$$|K(c, r, x(r)) - K(c, r, y(r))| \leq \zeta(c, r)|x(r) - y(r)|, \tag{3.2}$$

where $\zeta : [a, b] \times [a, b] \to [0, \infty)$ is a continuous function satisfying

$$\sup_{c\in[a,b]} \left( \int_a^b \zeta(c, r)^p \mathrm{d}r \right) \leq \frac{1}{2^{\varepsilon p-\varepsilon}(b - a)^{p-1}}$$

and $\varepsilon > 1$ is a constant.

(iii) there exists $x_0 \in C[a, b]$ such that $x_0(c) \leq \phi(c) + \int_a^b K(c, r, x_0(r))\mathrm{d}r$ for all $c \in [a, b]$.

Then the nonlinear integral equation (3.1) has a solution.

**Proof**. Put $X = C[a, b]$ and define a mapping $f : X \to X$ by

$$(fx)(c) = \phi(c) + \int_a^b K(c, r, x(r))\mathrm{d}r$$

for all $x \in X$ and $c \in [a, b]$. Define a mapping $d : X \times X \to [0, \infty)$ by

$$d(x, y) = \sup_{c\in[a,b]} |x(c) - y(c)|^p \quad (p > 1)$$

for all $x, y \in X$. Then $(X, d)$ is a $b$-complete $b$-metric space with coefficient $s = 2^{p-1}$. Define a mapping $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 2^{p-1}, & x(c) \leq y(c) \text{ for all } c \in [a, b], \\ \tau, & \text{otherwise}, \end{cases}$$

where $0 < \tau < 2^{p-1}$. Since $K$ is nondecreasing in the third order, we get $f \in \mathcal{A}_s(X, \alpha) \subset \mathcal{WA}_s(X, \alpha)$. By (iii), it infers $(S_2)$ in Theorem 2.3 is satisfied. Also, we get that condition $(\widetilde{S}_3)$ in Theorem 2.3 also holds (see [21]).

Finally, we show $f \in \Omega_s(X, \alpha)$. To prove this fact, we first choose $q \in \mathbb{R}$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Assume that $x, y \in X$ such that $\alpha(x, y) \geq s = 2^{p-1}$, that is, $x(c) \leq y(c)$ for all $c \in [a, b]$. From (ii) and the Hölder inequality, for each $c \in [a, b]$ we get

$$\begin{aligned}
&2^{\varepsilon p-\varepsilon}|(fx)(c) - (fy)(c)|^p \\
&\leq 2^{\varepsilon p-\varepsilon}\left(\int_a^b |K(c, r, x(r)) - K(c, r, y(r))|\mathrm{d}r\right)^p \\
&\leq 2^{\varepsilon p-\varepsilon}\left[\left(\int_a^b 1^q \mathrm{d}r\right)^{\frac{1}{q}}\left(\int_a^b |K(c, r, x(r)) - K(c, r, y(r))|^p \mathrm{d}r\right)^{\frac{1}{p}}\right]^p \\
&\leq 2^{\varepsilon p-\varepsilon}(b-a)^{\frac{p}{q}}\left(\int_a^b \zeta(c, r)^p |x(r) - y(r)|^p \mathrm{d}r\right) \\
&\leq 2^{\varepsilon p-\varepsilon}(b-a)^{\frac{p}{q}}\left(\int_a^b \zeta(c, r)^p d(x, y)\mathrm{d}r\right) \\
&\leq 2^{\varepsilon p-\varepsilon}(b-a)^{p-1} M_s(x, y)\left(\int_a^b \zeta(c, r)^p \mathrm{d}r\right) \\
&\leq M_s(x, y).
\end{aligned}$$

This implies that $s^{\varepsilon} d(fx, fy) \leq M_s(x, y)$. Hence $f \in \Omega_s(X, \alpha)$. Thus all the conditions of Theorem 2.3 are satisfied and hence $f$ has a fixed point in $X$. It follows that the nonlinear integral equation (3.1) has a solution. □

**Remark 3.2**. Compared with [15, Theorem 3.1], our Theorem 3.1 has many superiorities. First, our condition (ii) is much simpler than (ii) from [15, Theorem 3.1]. Indeed, our condition (3.2) is weaker than the corresponding condition of [15, Theorem 3.1]. Moreover, we delete the function $\Upsilon(t)$. Whereas, $\Upsilon(t)$ is a complex function with very strong conditions. Otherwise, our function $\zeta(c, r)$ satisfies the wider condition since $\varepsilon$ is arbitrary. Further, even if $\varepsilon = 3$, our condition for $\zeta(c, r)$ is also much weaker.

**Remark 3.3**. In [15, Theorem 3.1], there exist some mistakes. For instance, the incorrect equality from the proof of [15, Theorem 3.1] appears as follows:

$$\left(s^3 d(fx, fy)\right)^p = \left(2^{3p-3} \sup_{t\in[a,b]} |(fx)(t) - (fy)(t)|\right)^p.$$

In fact, it should be the following:

$$\left(s^3 d(fx, fy)\right)^p = \left(2^{3p-3} \sup_{t\in[a,b]} |(fx)(t) - (fy)(t)|^p\right)^p.$$

Due to such mistake, the conditions from [15, Theorem 3.1] need some revisions. Similar revisions should be done in Corollary 3.2 and Corollary 3.3 from [15, Theorem 3.1].

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# References

[1] S. Banach, Sur les opérations dans les ensembles abstraits et leurs applications aux équations intégrales., Fund. Math., **3** (1922), 133-181.

[2] Z. Mustafa, F. Awawdeh, W. Shatanawi, Fixed point theorem for expansive mappings in $G$-metric Spaces, Int. J. Contemp. Math. Sciences, **5**(50) (2010), 2463-2472.

[3] Mona. S. Bakry, H. M. Abu-Donia, Fixed-point theorems for a probabilistic 2-metric spaces, J. King Saud University (Science), **22** (2010), 217-221.

[4] C. Ionescu, S. Rezapour, M.-E. Samei, Fixed points of some new contractions on intuitionistic fuzzy metric spaces, Fixed Point Theory Appl., **2013**, 2013: 168.

[5] G. Cain, R. Kasriel, Fixed and periodic points of local contraction mappings on probabilistic metric spaces, Theory Comput. Systems, **9**(3) (1975), 289-297.

[6] L. -G. Huang, X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl., **332**(2) (2007), 1468-1476.

[7] T. Abdeljawad, J. Alzabut, A. Mukheimer, Y Zaidan, Banach contraction principle for cyclical mappings on partial metric spaces, Fixed Point Theory Appl., **2012**, 2012: 154.

[8] A.-A. Abdou, M.-A Khamsi, Fixed points of multivalued contraction mappings in modular metric spaces, Fixed Point Theory Appl., **2014**, 2014: 249.

[9] I. A. Bakhtin, The contraction principle in quasi-metric spaces, Funct. Anal., **30** (1989), 26-37.

[10] S. Czerwik, Contraction mappings in $b$-metric spaces, Acta Math. Inform., Univ. Ostrav., **1** (1993), 5-11.

[11] M. Jovanović, Z. Kadelburg, S. Radenović, Common fixed point results in metric type spaces, Fixed Point Theory Appl., **2010**, Article ID 978121, 15 pages.

[12] B. Samet, C. Vetro, P. Vetro, Fixed point theorems for $\alpha$-$\psi$-contractive type mappings., Nonlinear Anal., **75** (2012), 2154-2165.

[13] R. Kannan, Some results on fixed points. II. Amer. Math. Monthly, **76** (1969), 405-408.

[14] M. Imdad, A. Erduran, Suzuki-type generalization of Chatterjea contraction mappings on complete partial metric spaces, J. Oper., **2013** (2013), Article ID 923843, 5 pages.

[15] W. Sintunavarat, Nonlinear integral equations with new admissibility types in $b$-metric spaces, J. Fixed Point Theory Appl., DOI 10.1007/s11784-015-0276-6, 20 pages.

[16] H. Huang, S. Radenović, J. Vujaković, On some recent coincidence and immediate consequences in partially ordered $b$-metric spaces, Fixed Point Theory and Applications, **2015**, 2015: 63.

[17] M. Boriceanu, M. Bota and A. Petrusel, Mutivalued fractals in $b$-metric spaces, Cent. Eur. J. Math., **8** (2010), 367-377.

[18] A. Aghajani, M. Abbas, J. R. Roshan, Common fixed point of generalized weak contractive mappings in partially ordered $b$-metric spaces, Math. Slovaca, **64**(4) (2014), 941-960.

[19] H. Huang, L. Paunović, S. Radenović, On some new fixed point results for rational Geraghty contractive mappings in ordered $b$-metric spaces, J. Nonlinear Science Appl., **8**(5) (2015), 800-807.

[20] N. Hussain, E. Karapınar, P. Salimi, F. Akbar, $\alpha$-admissible mappings and related fixed point theorems, Fixed Point Theory Appl., **2013**, 2013: 114.

[21] J. J. Nieto, R. Rodíguez-López, Existence and uniqueness of fixed points in partially ordered sets and applications to ordinary differential equations. Acta Math. Sin. (Engl. Ser.) **23** (2007), 2205-2212.

# Anti Implicative IF-Ideals in BCK/BCI-algebras

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(Dedicated to the 79th birthday of Ivo G. Rosenberg.)

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**Abstract:**Using triangular norms, we present a new classification of fuzzy subalgebras, ideals and implicative ideals in BCK/BCI-algebras.

**Keywords:** t-norm; anti if-ideal; anti implicative if-ideal; BCK/BCI-algebra.

## 1 Introduction

BCK/BCI-algebras are an important class of logical algebras introduced by Imai and Iseki [7], and was extensively investigated by several researches. BCK/BCI-algebras generalize, on the one hand, the notion of the algebra of sets with the set subtraction as the only fundemental non-nullary operation and, on the other hand, the notion of the implication algebra (see [7]). In 1965, Zadeh [16] introduced the notion of fuzzy sets and in 1991, Xi [15] applied this notion to BCK/BCI-algebras. In 1990, Biswas [4] introduced the notion of anti fuzzy subgroups of groups and in 2008, modifying Biswas' idea, Kutukcu and Sharma [10] introduced the notion of anti fuzzy ideals in BCC-algebras.

In the present paper, we introduce the notions of anti if-subalgebras, anti if-ideals and anti implicative if-ideals of BCK/BCI-algebras with respect to arbitrary t-conorms and t-norms. Illustrating with examples, we prove that our definitions are more general than the classical ones. We also prove that an if-subset of a BCK/BCI-algebra is an anti if-ideal if and only if the complement of this if-subset is an anti if-ideal. We also discuss some relationships between such notions.

Let us recall [7,8] that a BCI-algebra is an algebra $(X, *, 0)$ of type $(2, 0)$ which satisfies the following conditions, for all $x, y, z \in X$: (i) $((x * y) * (x * z)) * (z * y) = 0$; (ii) $(x * (x * y)) * y = 0$; (iii) $x * x = 0$; (iv) $x * y = 0$ and $y * x = 0$ imply $x = y$. A BCI-algebra $X$ satisfying the additional condition (v) for all $x \in X$, $0 * x = 0$ is called a BCK-algebra.

We can define a partial ordering $\leq$ on $X$ by $x \leq y$ if and only if $x * y = 0$. Furthermore, in any BCK/BCI-algebra $X$, the following properties hold, for all $x, y, z \in X$: (i) $(x * y) * z = (x * z) * y$; (ii)

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$x*(x*(x*y))=x*y$; (iii) $x*y\leq x$; (iv) $x*0=x$; (v) $(x*z)*(y*z)\leq x*y$; (vi) $x\leq y$ implies $x*z\leq y*z$ and $z*y\leq z*x$.

A non-empty subset $A$ of a BCK/BCI-algebra $X$ is called an ideal of $X$ if $0\in A$, and $x*y\in A$ and $y\in A$ imply $x\in A$ for all $x,y\in X$.

A non-empty subset $A$ of a BCK/BCI-algebra $X$ is called an implicative ideal of $X$ if $0\in A$, and $(x*(y*x))*z\in A$ and $z\in A$ imply $x\in A$ for all $x,y,z\in X$. Any implicative ideal is an ideal, but not conversely.

A mapping $f$ of a BCK/BCI-algebra $X$ into a BCK/BCI-algebra $Y$ is called a homomorphism if $f(x*y)=f(x)*f(y)$ for all $x,y\in X$.

By a triangular conorm (shortly t-conorm) $S$ [14], we mean a binary operation on the unit interval $[0,1]$ which satisfies the following conditions, for all $x,y,z\in[0,1]$: (i) $S(x,0)=x$; (ii) $S(x,y)\leq S(x,z)$ if $y\leq z$; (iii) $S(x,y)=S(y,x)$; (iv) $S(x,S(y,z))=S(S(x,y),z)$. Some important examples of t-conorms are $S_L(x,y)=\min\{x+y,1\}$, $S_P(x,y)=x+y-xy$ and $S_M(x,y)=\max\{x,y\}$.

By a triangular norm (shortly t-norm) $T$ [14], we mean a binary operation on the unit interval $[0,1]$ which satisfies the following conditions, for all $x,y,z\in[0,1]$: (i) $T(x,1)=x$; (ii) $T(x,y)\leq T(x,z)$ if $y\leq z$; (iii) $T(x,y)=T(y,x)$; (iv) $T(x,T(y,z))=T(T(x,y),z)$. Some important examples of t-norms are $T_L(x,y)=\max\{x+y-1,0\}$, $T_P(x,y)=xy$ and $T_M(x,y)=\min\{x,y\}$.

A t-conorm $S$ and a t-norm $T$ are called associated [11], i.e. $S(x,y)=1-T(1-x,1-y)$ for all $x,y\in[0,1]$. For example, t-conorm $S_M$ and t-norm $T_M$ are associated [6,9-11]. Also, it is well known [6,9] that if $S$ is a t-conorm and $T$ is a t-norm, then $\max\{x,y\}\leq S(x,y)$ and $\min\{x,y\}\geq T(x,y)$ for all $x,y\in[0,1]$, respectively.

Note that, the concepts of t-conorms and t-norms are known as the axiomatic skeletons that we use for characterizing fuzzy unions and intersections, respectively. These concepts were originally introduced by Menger [13] and several properties and examples for these concepts were proposed by many authors (see [6,9-11,13,14]).

A fuzzy subset $A$ in an arbitrary non-empty set $X$ is a function $\mu_A:X\to[0,1]$. The complement of $\mu_A$, denoted by $\mu_A^c$, is the fuzzy subset in $X$ given by $\mu_A^c(x)=1-\mu_A(x)$ for all $x\in X$.

**Definition 1.1 ([15])** *A fuzzy subset $A$ in a BCK/BCI-algebra $X$ is called a fuzzy BCK/BCI-subalgebra of $X$ if*

$$\mu_A(x*y)\geq\min\{\mu_A(x),\mu_A(y)\}$$

*for all $x,y\in X$.*

**Definition 1.2 ([15])** *A fuzzy subset $A$ in a BCK/BCI-algebra $X$ is called a fuzzy ideal of $X$ if*

$$\mu_A(0)\geq\mu_A(x)\geq\min\{\mu_A(x*y),\mu_A(y)\}$$

*for all $x,y\in X$.*

**Definition 1.3 ([10,12])** *A fuzzy subset $A$ in a BCK/BCI-algebra $X$ is called a implicative fuzzy ideal of $X$ if*

$$\mu_A(0)\geq\mu_A(x)\geq\min\{\mu_A((x*(y*x))*z),\mu_A(z)\}$$

*for all $x,y,z\in X$.*

**Definition 1.4 ([12])** *A fuzzy subset $A$ in a BCK/BCI-algebra $X$ is called an anti fuzzy BCK/BCI-subalgebra of $X$ if*

$$\mu_A(x*y)\leq\max\{\mu_A(x),\mu_A(y)\}$$

*for all $x,y\in X$.*

**Definition 1.5 ([12])** *A fuzzy subset $A$ in a BCK/BCI-algebra $X$ is called an anti fuzzy ideal of $X$ if*

$$\mu_A(0) \leq \mu_A(x) \leq \max\{\mu_A(x*y), \mu_A(y)\}$$

*for all $x, y \in X$.*

As a generalization of the notion of fuzzy subsets in $X$, Atanassov [2] introduced the concept of intuitionistic fuzzy subsets (or simply if-sets) defined on $X$ as objects having the form

$$A = \{(x, \mu_A(x), \lambda_A(x)) : x \in X\}$$

where the functions $\mu_A : X \to [0,1]$ and $\lambda_A : X \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\lambda_A(x)$) of each element $x$ in $X$ to the set $A$, respectively, and $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$ for all $x$ in $X$.

In [3], for every two if-subsets $A$ and $B$ in $X$, we have

(i) $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x)$ and $\lambda_A(x) \geq \lambda_B(x)$ for all $x \in X$,

(ii) $\Box A = \{(x, \mu_A(x), \mu_A^c(x)) : x \in X\}$,

(iii) $\Diamond A = \{(x, \lambda_A^c(x), \lambda_A(x)) : x \in X\}$.

For the sake of simplicity, we shall use the symbol $A = (\mu_A, \lambda_A)$ for the if-subset $A = \{(x, \mu_A(x), \lambda_A(x)) : x \in X\}$.

# 2 Anti IF-Ideals

**Definition 2.1** *An if-subset $A = (\mu_A, \lambda_A)$ in a BCK/BCI-algebra $X$ is said to be an anti intuitionistic fuzzy BCK/BCI-subalgebra of $X$ (or simply, an anti if-BCK/BCI-subalgebra of $X$) if*

(i) $\mu_A(x*y) \leq \max\{\mu_A(x), \mu_A(y)\}$,

(ii) $\lambda_A(x*y) \geq \min\{\lambda_A(x), \lambda_A(y)\}$

*for all $x, y \in X$.*

**Definition 2.2** *An if-subset $A = (\mu_A, \lambda_A)$ in a BCK/BCI-algebra $X$ is said to be an anti intuitionistic fuzzy BCK/BCI-subalgebra of $X$ with respect to a t-conorm $S$ and a t-norm $T$ (or simply, an $(S,T)$-anti if-BCK/BCI-subalgebra of $X$) if*

(i) $\mu_A(x*y) \leq S(\mu_A(x), \mu_A(y))$

(ii) $\lambda_A(x*y) \geq T(\lambda_A(x), \lambda_A(y))$

*for all $x, y \in X$.*

**Remark 2.3** *Every anti if-BCK/BCI-subalgebra of a BCK/BCI-algebra is an $(S,T)$-anti if-BCK/BCI-subalgebra of $X$ such that $S = S_M$ and $T = T_M$, but it is clear that the converse is not true. If $\lambda_A(x) = 1 - \mu_A(x)$ for all $x \in X$, then every anti if-BCK/BCI-subalgebra of a BCK/BCI-algebra $X$ is an anti fuzzy BCK/BCI-subalgebra of $X$. Also, if $\lambda_A(x) = 1 - \mu_A(x)$ for all $x \in X$, $S = S_M$ and $T = T_M$, then every $(S,T)$-anti if-BCK/BCI-subalgebra of a BCK/BCI-algebra $X$ is an anti fuzzy BCK/BCI-subalgebra of $X$.*

**Definition 2.4** *An if-subset $A = (\mu_A, \lambda_A)$ in a BCK/BCI-algebra $X$ is said to be an anti if-ideal of $X$ if*

(i) $\mu_A(0) \leq \mu_A(x)$ and $\lambda_A(0) \geq \lambda_A(x)$,

(ii) $\mu_A(x) \leq \max\{\mu_A(x * y), \mu_A(y)\}$,

(iii) $\lambda_A(x) \geq \min\{\lambda_A(x * y), \lambda_A(y)\}$

*for all $x, y \in X$.*

**Definition 2.5** *An if-subset $A = (\mu_A, \lambda_A)$ in a BCK/BCI-algebra $X$ is said to be an anti if-ideal of $X$ with respect to a t-conorm $S$ and a t-norm $T$ (or simply, an $(S, T)$-anti if-ideal of $X$) if*

(i) $\mu_A(0) \leq \mu_A(x)$ and $\lambda_A(0) \geq \lambda_A(x)$,

(ii) $\mu_A(x) \leq S(\mu_A(x * y), \mu_A(y))$,

(iii) $\lambda_A(x) \geq T(\lambda_A(x * y), \lambda_A(y))$

*for all $x, y \in X$.*

**Remark 2.6** *Every anti if-ideal of a BCK/BCI-algebra is an $(S, T)$-anti if-ideal of $X$ such that $S = S_M$ and $T = T_M$, but it is clear that the converse is not true. If $\lambda_A(x) = 1 - \mu_A(x)$ for all $x \in X$, then every anti if- ideal of a BCK/BCI-algebra $X$ is an anti fuzzy ideal of $X$. Also, if $\lambda_A(x) = 1 - \mu_A(x)$ for all $x \in X$, $S = S_M$ and $T = T_M$, then every $(S, T)$-anti if-ideal of a BCK/BCI-algebra $X$ is an anti fuzzy ideal of $X$.*

**Example 2.7** *Let $X = \{0, 1, 2, 3\}$ be a BCK-algebra with the Cayley table as follows*

| $*$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 1 |
| 2 | 2 | 1 | 0 | 2 |
| 3 | 3 | 3 | 3 | 0 |

*Define an if-set $A = (\mu_A, \lambda_A)$ in $X$ by*

$$\mu_A(x) = \begin{cases} 0, & x = 0 \\ 1/2, & x = 1 \text{ or } 2 \\ 1, & x = 3 \end{cases} \quad \text{and} \quad \lambda_A(x) = \begin{cases} 1, & x = 0 \\ 1/3, & x = 1 \text{ or } 2 \\ 0, & x = 3 \end{cases}$$

*It is easy to check that $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$, $\mu_A(0) \leq \mu_A(x)$ and $\lambda_A(0) \geq \lambda_A(x)$. Also, $\mu_A(x) \leq S_M(\mu_A(x*y), \mu_A(y))$ and $\lambda_A(x) \geq T_L(\lambda_A(x*y), \lambda_A(y))$ for all $x, y \in X$. Hence $A = (\mu_A, \lambda_A)$ is an $(S_M, T_L)$-anti if-ideal of $X$. Also note that t-conorm $S_M$ and t-norm $T_L$ are not associated.*

**Remark 2.8** *Note that, the above example holds even with the t-conorm $S_M$ and t-norm $T_M$, and hence $A = (\mu_A, \lambda_A)$ is also an $(S_M, T_M)$-anti if-ideal of $X$. Therefore, every anti if-ideal of $X$ is an $(S, T)$-anti if-ideal but the converse is not true.*

**Example 2.9** *Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the Cayley table as follows*

| $*$ | 0 | a | b | c | d |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| a | a | 0 | a | 0 | 0 |
| b | b | b | 0 | 0 | 0 |
| c | c | b | a | 0 | 0 |
| d | d | d | d | d | 0 |

*Define an if-set* $A = (\mu_A, \lambda_A)$ *in* $X$ *by*

$$\mu_A(x) = \begin{cases} 1/2, & x \in \{0, a, b\} \\ 3/4, & \text{otherwise} \end{cases} \quad \text{and} \quad \lambda_A(x) = \begin{cases} 1/3, & x \in \{0, a, b\} \\ 1/4, & \text{otherwise.} \end{cases}$$

*It is easy to check that* $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$, $\mu_A(0) \leq \mu_A(x)$ and $\lambda_A(0) \geq \lambda_A(x)$. Also, $\mu_A(x) \leq S_L(\mu_A(x*y), \mu_A(y))$ *and* $\lambda_A(x) \geq T_P(\lambda_A(x*y), \lambda_A(y))$ *for all* $x, y \in X$. *Hence* $A = (\mu_A, \lambda_A)$ *is an* $(S_L, T_P)$*-anti if-ideal of* $X$. *But* $A = (\mu_A, \lambda_A)$ *is not an anti if-ideal of* $X$.

**Lemma 2.10** *If* $A = (\mu_A, \lambda_A)$ *is an* $(S, T)$*-anti if-ideal of a BCK/BCI-algebra* $X$*, then so is* $\Box A = (\mu_A, \mu_A^c)$ *such that t-conorm* $S$ *and t-norm* $T$ *are associated.*

**Proof.** Since $A = (\mu_A, \lambda_A)$ is an $(S, T)$-anti if-ideal of $X$, then $\mu_A(0) \leq \mu_A(x)$ for all $x \in X$ and so $1 - \mu_A^c(0) \leq 1 - \mu_A^c(x)$, hence $\mu_A^c(0) \geq \mu_A^c(x)$. Also, for all $x, y \in X$, we have

$$\mu_A(x) \leq S(\mu_A(x * y), \mu_A(y))$$

and so

$$1 - \mu_A^c(x) \leq S(1 - \mu_A^c(x * y), 1 - \mu_A^c(y))$$

which implies

$$\mu_A^c(x) \geq 1 - S(1 - \mu_A^c(x * y), 1 - \mu_A^c(y)).$$

Since $S$ and $T$ are associated, we have

$$\mu_A^c(x) \geq T(\mu_A^c(x * y), \mu_A^c(y)).$$

Thus, $\Box A = (\mu_A, \mu_A^c)$ is an $(S, T)$-anti if-ideal of $X$. ■

**Lemma 2.11** *If* $A = (\mu_A, \lambda_A)$ *is an* $(S, T)$*-anti if-ideal of a BCK/BCI-algebra* $X$*, then so is* $\Diamond A = (\lambda_A^c, \lambda_A)$ *such that t-conorm* $S$ *and t-norm* $T$ *are associated.*

**Proof.** The proof is similar to the proof of Lemma 2.10. ■

Combining the above two lemmas, it is easy to see that the following theorem is valid.

**Theorem 2.12** $A = (\mu_A, \lambda_A)$ *is an* $(S, T)$*-anti if-ideal of a BCK/BCI-algebra* $X$ *if and only if* $\Box A$ *and* $\Diamond A$ *are* $(S, T)$*-anti if-ideals of* $X$ *such that t-conorm* $S$ *and t-norm* $T$ *are associated.*

**Corollary 2.13** $A = (\mu_A, \lambda_A)$ *is an* $(S, T)$*-anti if-ideal of a BCK/BCI-algebra* $X$ *if and only if* $\mu_A$ *and* $\lambda_A^c$ *are anti fuzzy ideals of* $X$ *such that t-conorm* $S$ *and t-norm* $T$ *are associated.*

**Lemma 2.14** *Let* $A = (\mu_A, \lambda_A)$ *be an* $(S, T)$*-anti if-ideal of a BCK/BCI-algebra* $X$. *If* $\leq$ *is a partial ordering on* $X$ *then* $\mu_A(x) \leq \mu_A(y)$ *and* $\lambda_A(y) \leq \lambda_A(x)$ *for all* $x, y \in X$ *such that* $x \leq y$.

**Proof.** Let $X$ be a BCK/BCI-algebra. It is known [8] that $\leq$ is a partial ordering on $X$ defined by $x \leq y$ if and only if $x * y = 0$ for all $x, y \in X$. Let $A = (\mu_A, \lambda_A)$ be an $(S,T)$-anti if-ideal of $X$. Then

$$\mu_A(x) \leq S(\mu_A(x * y), \mu_A(y)) = S(\mu_A(0), \mu_A(y)) \leq \mu_A(y)$$

and

$$\lambda_A(x) \geq T(\lambda_A(x * y), \lambda_A(y)) = T(\lambda_A(0), \lambda_A(y)) \geq \lambda_A(y).$$

These complete the proof. ■

**Theorem 2.15** *Let $A = (\mu_A, \lambda_A)$ be an $(S,T)$-anti if-ideal of a BCK/BCI-algebra $X$. $A$ is an $(S,T)$-anti if-BCK/BCI-subalgebra of $X$.*

**Proof.** Let $A = (\mu_A, \lambda_A)$ be an $(S,T)$-anti if-ideal *of* $X$. Since $x * y \leq x$ for all $x, y \in X$, it follows from Lemma 2.14 that $\mu_A(x * y) \leq \mu_A(x)$ and $\lambda_A(x) \leq \lambda_A(x * y)$. Then

$$\mu_A(x * y) \leq \mu_A(x) \leq S(\mu_A(x * y), \mu_A(y)) \leq S(\mu_A(x), \mu_A(y))$$

and

$$\lambda_A(x * y) \geq \lambda_A(x) \geq T(\lambda_A(x * y), \lambda_A(y)) \geq T(\lambda_A(x), \lambda_A(y))$$

and so $A$ is an $(S,T)$-anti if-BCK/BCI-subalgebra of $X$. ■

**Remark 2.16** *The converse of the above theorem does not hold in general. In fact, suppose that $X$ be the BCK-algebra in Example 2.7. Define an if-set $A = (\mu_A, \lambda_A)$ in $X$ by*

$$\mu_A(x) = \begin{cases} 0, & x = 0 \\ 1/2, & x = 1 \\ 1, & x = 2 \text{ or } 3 \end{cases} \quad \text{and} \quad \lambda_A(x) = \begin{cases} 1, & x = 0 \\ 1/3, & x = 1 \\ 0, & x = 2 \text{ or } 3 \end{cases}$$

*By routine calculations, we know that $A = (\mu_A, \lambda_A)$ is an $(S_M, T_M)$-anti if-BCK-subalgebra of $X$ but not an $(S_M, T_M)$-anti if-ideal of $X$ because $\mu_A(2) = 1 > \max\{\mu_A(2 * 1), \mu_A(1)\}$ and $\lambda_A(2) = 0 < \min\{\lambda_A(2 * 1), \lambda_A(1)\}$.*

If $A = (\mu_A, \lambda_A)$ is an if-subset in a BCK/BCI-algebra $X$ and $f$ is a self mapping of $X$, we define mappings $\mu_A[f] : X \to [0,1]$ by $\mu_A[f](x) = \mu_A(f(x))$ and $\lambda_A[f] : X \to [0,1]$ by $\lambda_A[f](x) = \lambda_A(f(x))$ for all $x \in X$, respectively.

**Proposition 2.17** *If $A = (\mu_A, \lambda_A)$ is an $(S,T)$-anti if-ideal of a BCK/BCI-algebra $X$ and $f$ is an increasing endomorphism of $X$, then $(\mu_A[f], \lambda_A[f])$ is an $(S,T)$-anti if-ideal of $X$.*

**Proof.** For any given $x, y \in X$, we have

$$\begin{aligned} \mu_A[f](x) &= \mu_A(f(x)) \leq S(\mu_A(f(x) * f(y)), \mu_A(f(y))) \\ &= S(\mu_A(f(x * y)), \mu_A(f(y))) \\ &= S(\mu_A[f](x * y), \mu_A[f](y)), \end{aligned}$$

$$\begin{aligned} \lambda_A[f](x) &= \lambda_A(f(x)) \geq T(\lambda_A(f(x) * f(y)), \lambda_A(f(y))) \\ &= T(\lambda_A(f(x * y)), \lambda_A(f(y))) \\ &= T(\lambda_A[f](x * y), \lambda_A[f](y)). \end{aligned}$$

Also, since $X$ is a BCK/BCI-algebra, we have $0 * x = 0$ and so $0 \leq x$ for all $x \in X$. Since $f$ is increasing, we have $f(0) \leq f(x)$ for all $x \in X$ and, from Lemma 2.14, $\mu_A(f(0)) \leq \mu_A(f(x))$ and $\lambda_A(f(x)) \leq \lambda_A(f(0))$ i.e., $\mu_A[f](0) \leq \mu_A[f](x)$ and $\lambda_A[f](x) \leq \lambda_A[f](0)$ for all $x \in X$. This completes the proof. ■

If $f$ is a self mapping of a BCK/BCI-algebra $X$ and $B = (\mu_B, \lambda_B)$ is an if-subset in $f(X)$, then the if-subset $A = (\mu_A, \lambda_A)$ in $X$ defined by $\mu_A = \mu_B \circ f$ and $\lambda_A = \lambda_B \circ f$ (i.e., $\mu_A(x) = \mu_B(f(x))$ and $\lambda_A(x) = \lambda_B(f(x))$ for all $x \in X$) is called the *preimage* of $B$ under $f$.

**Theorem 2.18** *An onto increasing homomorphic preimage of an $(S,T)$-anti if-ideal is an $(S,T)$-anti if-ideal.*

**Proof.** Let $f : X \to Y$ be an onto homomorphism of BCK/BCI-algebras, $B = (\mu_B, \lambda_B)$ be an $(S,T)$-anti if-ideal of $Y$, and $A = (\mu_A, \lambda_A)$ be preimage of $B$ under $f$. Then, we have

$$\begin{aligned} \mu_A(x) &= \mu_B(f(x)) \leq S(\mu_B(f(x) * f(y)), \mu_B(f(y))) \\ &= S(\mu_B(f(x * y)), \mu_B(f(y))) \\ &= S(\mu_A(x * y), \mu_A(y)), \end{aligned}$$

$$\begin{aligned} \lambda_A(x) &= \lambda_B(f(x)) \geq T(\lambda_B(f(x) * f(y)), \lambda_B(f(y))) \\ &= T(\lambda_B(f(x * y)), \lambda_B(f(y))) \\ &= T(\lambda_A(x * y), \lambda_A(y)) \end{aligned}$$

for all $x, y \in X$. Also, $\mu_A(0) = \mu_B(f(0)) \leq \mu_B(f(x)) = \mu_A(x)$ and $\lambda_A(0) = \lambda_B(f(0)) \geq \lambda_B(f(x)) = \lambda_A(x)$ for all $x \in X$. Hence, $A = (\mu_A, \lambda_A)$ is an $(S,T)$-anti if-ideal of $X$. ■

**Lemma 2.19 ([9])** *Let $S$ and $T$ be a t-conorm and a t-norm, respectively. Then*

$$S(S(x,y), S(z,t)) = S(S(x,z), S(y,t)),$$

$$T(T(x,y), T(z,t)) = T(T(x,z), T(y,t))$$

*for all $x, y, z, t \in [0,1]$.*

**Theorem 2.20** *Let $S$ be a t-conorm, $T$ be a t-norm and $X = X_1 \times X_2$ be the direct product BCK/BCI-algebra of BCK/BCI-algebras $X_1$ and $X_2$. If $A_1 = (\mu_{A_1}, \lambda_{A_1})$ (resp. $A_2 = (\mu_{A_2}, \lambda_{A_2})$) is an $(S,T)$-anti if-ideal of $X_1$ (resp. $X_2$), then $A = (\mu_A, \lambda_A)$ is an $(S,T)$-anti if-ideal of $X$ defined by $\mu_A = \mu_{A_1} \times \mu_{A_2}$ and $\lambda_A = \lambda_{A_1} \times \lambda_{A_2}$ such that*

$$\mu_A(x_1, x_2) = (\mu_{A_1} \times \mu_{A_2})(x_1, x_2) = S(\mu_{A_1}(x_1), \mu_{A_2}(x_2)),$$

$$\lambda_A(x_1, x_2) = (\lambda_{A_1} \times \lambda_{A_2})(x_1, x_2) = T(\lambda_{A_1}(x_1), \lambda_{A_2}(x_2))$$

*for all $(x_1, x_2) \in X$.*

**Proof.** Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be any elements of $X$. Since $X$ is a BCK/BCI-algebra, we have

$$\begin{aligned}
\mu_A(x) &= (\mu_{A_1} \times \mu_{A_2})(x_1, x_2) \\
&= S(\mu_{A_1}(x_1), \mu_{A_2}(x_2)) \\
&\leq S(S(\mu_{A_1}(x_1 * y_1), \mu_{A_1}(y_1)), S(\mu_{A_2}(x_2 * y_2), \mu_{A_2}(y_2))) \\
&= S(S(\mu_{A_1}(x_1 * y_1), \mu_{A_2}(x_2 * y_2)), S(\mu_{A_1}(y_1), \mu_{A_2}(y_2))) \\
&= S((\mu_{A_1} \times \mu_{A_2})(x_1 * y_1, x_2 * y_2), (\mu_{A_1} \times \mu_{A_2})(y_1, y_2)) \\
&= S((\mu_{A_1} \times \mu_{A_2})((x_1, x_2) * (y_1, y_2)), (\mu_{A_1} \times \mu_{A_2})(y_1, y_2)) \\
&= S(\mu_A(x * y), \mu_A(y)),
\end{aligned}$$

$$\begin{aligned}
\lambda_A(x) &= (\lambda_{A_1} \times \lambda_{A_2})(x_1, x_2) \\
&= T(\lambda_{A_1}(x_1), \lambda_{A_2}(x_2)) \\
&\geq T(T(\lambda_{A_1}(x_1 * y_1), \lambda_{A_1}(y_1)), T(\lambda_{A_2}(x_2 * y_2), \lambda_{A_2}(y_2))) \\
&= T(T(\lambda_{A_1}(x_1 * y_1), \lambda_{A_2}(x_2 * y_2)), T(\lambda_{A_1}(y_1), \lambda_{A_2}(y_2))) \\
&= T((\lambda_{A_1} \times \lambda_{A_2})(x_1 * y_1, x_2 * y_2), (\lambda_{A_1} \times \lambda_{A_2})(y_1, y_2)) \\
&= T((\lambda_{A_1} \times \lambda_{A_2})((x_1, x_2) * (y_1, y_2)), (\lambda_{A_1} \times \lambda_{A_2})(y_1, y_2)) \\
&= T(\lambda_A(x * y), \lambda_A(y))
\end{aligned}$$

Also,

$$\begin{aligned}
\mu_A(0) &= (\mu_{A_1} \times \mu_{A_2})(0, 0) = S(\mu_{A_1}(0), \mu_{A_2}(0)) \\
&\leq S(\mu_{A_1}(x_1), \mu_{A_2}(x_2)) = (\mu_{A_1} \times \mu_{A_2})(x_1, x_2) \\
&= \mu_A(x),
\end{aligned}$$

$$\begin{aligned}
\lambda_A(0) &= (\lambda_{A_1} \times \lambda_{A_2})(0, 0) = T(\lambda_{A_1}(0), \lambda_{A_2}(0)) \\
&\geq T(\lambda_{A_1}(x_1), \lambda_{A_2}(x_2)) = (\lambda_{A_1} \times \lambda_{A_2})(x_1, x_2) \\
&= \lambda_A(x).
\end{aligned}$$

This completes the proof. ■

**Definition 2.21** *Let $S$ be a t-conorm and $T$ be a t-norm, and let $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ be if-sets in a BCK/BCI-algebra $X$. Then $S$-product of $\mu_A$ and $\mu_B$, and $T$-product of $\lambda_A$ and $\lambda_B$, written $[\mu_A.\mu_B]_S$ and $[\lambda_A.\lambda_B]_T$, are defined by*

$$[\mu_A.\mu_B]_S(x) = S(\mu_A(x), \mu_B(x)),$$

$$[\lambda_A.\lambda_B]_T(x) = T(\lambda_A(x), \lambda_B(x))$$

*for all $x \in X$, respectively.*

**Theorem 2.22** *Let $S$ be a t-conorm and $T$ be a t-norm, and let $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ be $(S, T)$-anti if-ideals of a BCK/BCI-algebra $X$. If $S_1$ is a t-conorm which dominates $S$, that is,*

$$S_1(S(x, y), S(z, t)) \leq S(S_1(x, z), S_1(y, t))$$

*and $T_1$ is a t-norm which dominates $T$, that is,*

$$T_1(T(x,y),T(z,t)) \geq T(T_1(x,z),T_1(y,t))$$

*for all $x,y,z,t \in [0,1]$, then $([\mu_A.\mu_B]_{S_1},[\lambda_A.\lambda_B]_{T_1})$ is an $(S,T)$-anti if-ideal of $X$.*

**Proof.** For any $x,y \in X$, we have

$$\begin{aligned}
[\mu_A.\mu_B]_{S_1}(x) &= S_1(\mu_A(x),\mu_B(x)) \\
&\leq S_1(S(\mu_A(x*y),\mu_A(y)),S(\mu_B(x*y),\mu_B(y))) \\
&\leq S(S_1(\mu_A(x*y),\mu_B(x*y)),S_1(\mu_A(y),\mu_B(y))) \\
&= S([\mu_A.\mu_B]_{S_1}(x*y),[\mu_A.\mu_B]_{S_1}(y)),
\end{aligned}$$

$$\begin{aligned}
[\lambda_A.\lambda_B]_{T_1}(x) &= T_1(\lambda_A(x),\lambda_B(x)) \\
&\geq T_1(T(\lambda_A(x*y),\lambda_A(y)),T(\lambda_B(x*y),\lambda_B(y))) \\
&\geq T(T_1(\lambda_A(x*y),\lambda_B(x*y)),T_1(\lambda_A(y),\lambda_B(y))) \\
&= T([\lambda_A.\lambda_B]_{T_1}(x*y),[\lambda_A.\lambda_B]_{T_1}(y)).
\end{aligned}$$

Also,

$$[\mu_A.\mu_B]_{S_1}(0) = S_1(\mu_A(0),\mu_B(0)) \leq S_1(\mu_A(x),\mu_B(x)) = [\mu_A.\mu_B]_{S_1}(x),$$
$$[\lambda_A.\lambda_B]_{T_1}(0) = T_1(\lambda_A(0),\lambda_B(0)) \geq T_1(\lambda_A(x),\lambda_B(x)) = [\lambda_A.\lambda_B]_{T_1}(x)$$

This completes the proof. ■

# 3 Anti Implicative IF-Ideals

**Definition 3.1** *A fuzzy subset $A$ in a BCK/BCI-algebra $X$ is said to be an anti implicative fuzzy ideal of $X$ if*

(i) $\mu_A(0) \leq \mu_A(x)$,

(ii) $\mu_A(x) \leq \max\{\mu_A((x*(y*x))*z),\mu_A(z)\}$

*for all $x,y,z \in X$.*

**Definition 3.2** *An if-subset $A=(\mu_A,\lambda_A)$ in a BCK/BCI-algebra $X$ is said to be an anti implicative if-ideal of $X$ if*

(i) $\mu_A(0) \leq \mu_A(x)$ and $\lambda_A(0) \geq \lambda_A(x)$,

(ii) $\mu_A(x) \leq \max\{\mu_A((x*(y*x))*z),\mu_A(z)\}$,

(iii) $\lambda_A(x) \geq \min\{\lambda_A(x*(y*x))*z),\lambda_A(z)\}$

*for all $x,y,z \in X$.*

**Definition 3.3** *An if-subset $A = (\mu_A, \lambda_A)$ in a BCK/BCI-algebra $X$ is said to be an anti implicative if-ideal of $X$ with respect to a t-conorm $S$ and a t-norm $T$ (or simply, an $(S,T)$-anti implicative if-ideal of $X$) if*

(i) $\mu_A(0) \leq \mu_A(x)$ and $\lambda_A(0) \geq \lambda_A(x)$,

(ii) $\mu_A(x) \leq S(\mu_A((x*(y*x))*z), \mu_A(z))$,

(iii) $\lambda_A(x) \geq T(\lambda_A(x*(y*x))*z), \lambda_A(z))$

*for all $x, y, z \in X$.*

**Remark 3.4** *Every anti implicative if-ideal of a BCK/BCI-algebra is an $(S,T)$-anti implicative if-ideal of $X$ such that $S = S_M$ and $T = T_M$, but it is clear that the converse is not true. If $\lambda_A(x) = 1 - \mu_A(x)$ for all $x \in X$, then every anti implicative if-ideal of a BCK/BCI-algebra $X$ is an anti implicative fuzzy ideal of $X$. Also, if $\lambda_A(x) = 1 - \mu_A(x)$ for all $x \in X$, $S = S_M$ and $T = T_M$, then every $(S,T)$-anti implicative if-ideal of a BCK/BCI-algebra $X$ is an anti implicative fuzzy ideal of $X$.*

**Example 3.5** *In Example 2.9, it is easy to see that $A = (\mu_A, \lambda_A)$ is also an $(S_L, T_P)$-anti implicative if-ideal of $X$.*

**Remark 3.6** *An $(S,T)$-anti if-ideal of a BCK/BCI-algebra $X$ need not to be $(S,T)$-anti implicative if-ideal. For instance, in Example 2.7, we know that $A = (\mu_A, \lambda_A)$ is an $(S_M, T_L)$-anti if-ideal of $X$ but it is not an $(S_M, T_L)$-anti implicative if-ideal of $X$, because $\mu_A(1) > S_M(\mu_A((1*(2*1))*0), \mu_A(0))$ and $\lambda_A(1) < T_L(\lambda_A((1*(2*1))*0), \lambda_A(0))$.*

**Theorem 3.7** *Any $(S,T)$-anti implicative if-ideal of a BCK/BCI-algebra $X$ is an $(S,T)$-anti if-ideal of $X$.*

**Proof.** In Definition 3.3, let $z = y$ and $y = x$. Hence $\mu_A(x) \leq S(\mu_A((x*(x*x))*y), \mu_A(y))$ and $\lambda_A(x) \geq T(\lambda_A((x*(x*x))*y), \lambda_A(y))$. Since $x*x = 0$ and $x*0 = x$, we obtain (ii) and (iii) in Definition 2.5. This completes the proof. ■

**Theorem 3.8** *Let $A = (\mu_A, \lambda_A)$ be an $(S,T)$-anti if-ideal of a BCK/BCI-algebra $X$. Then $A = (\mu_A, \lambda_A)$ is an $(S,T)$-anti implicative if-ideal of $X$ iff $\mu_A(x) \leq \mu_A(x*(y*x))$ and $\lambda_A(x) \geq \lambda_A(x*(y*x))$ for all $x, y \in X$.*

**Proof.** Assume that $A = (\mu_A, \lambda_A)$ is an $(S,T)$-anti implicative if-ideal. Taking $z = 0$ in (ii) and (iii), and using (i) in Definition 3.3, we get the inequalities. Conversely, since $A = (\mu_A, \lambda_A)$ is an $(S,T)$-anti if-ideal, hence

$$\mu_A(x) \leq \mu_A(x*(y*x)) \leq S(\mu_A((x*(y*x))*z), \mu_A(z)),$$
$$\lambda_A(x) \geq \lambda_A(x*(y*x)) \geq T(\lambda_A((x*(y*x))*z), \lambda_A(z)).$$

This completes the proof. ■

**Lemma 3.9** *Let $A = (\mu_A, \lambda_A)$ be an $(S,T)$-anti if-ideal of a BCK/BCI-algebra $X$. If $x*y \leq z$ holds in $X$, then $\mu_A(x) \leq S(\mu_A(y), \mu_A(z))$ and $\lambda_A(x) \geq T(\lambda_A(y), \lambda_A(z))$ for all $x, y \in X$.*

**Proof.** Since $x*y \leq z$ holds *for all* $x, y \in X$, we have

$$\begin{aligned} \mu_A(x*y) &\leq S(\mu_A((x*y)*z), \mu_A(z)) \\ &\leq S(\mu_A(z*z), \mu_A(z)) \\ &= S(\mu_A(0), \mu_A(z)) \\ &\leq \mu_A(z) \end{aligned}$$

it follows that

$$\mu_A(x) \leq S(\mu_A(x*y), \mu_A(y)) \leq S(\mu_A(y), \mu_A(z)),$$

$$\begin{aligned} \lambda_A(x*y) &\geq T(\lambda_A((x*y)*z), \lambda_A(z)) \\ &\geq T(\lambda_A(z*z), \lambda_A(z)) \\ &= T(\lambda_A(0), \lambda_A(z)) \\ &\geq \lambda_A(z) \end{aligned}$$

hence

$$\lambda_A(x) \geq T(\lambda_A(x*y), \lambda_A(y)) \geq T(\lambda_A(y), \lambda_A(z)).$$

This completes the proof. ■

**Theorem 3.10 ([12])** *A BCK-algebra is implicative iff it is both commutative and positive implicative.*

**Theorem 3.11 ([12])** *If $X$ is an implicative BCK-algebra, then $x*((x*(y*x))*z) \leq z$ for all $x, y, z \in X$.*

**Theorem 3.12** *In an implicative BCK/BCI-algebra, every $(S,T)$-anti if-ideal is an $(S,T)$-anti implicative if-ideal.*

**Proof.** The proof is easily follows from Lemma 3.9 and Theorem 3.11. ■

**Theorem 3.13** *The intersection of any set of $(S,T)$-anti implicative if-ideals of a BCK/BCI-algebra $X$ is also an $(S,T)$-anti implicative if-ideal whenever $S$ and $T$ are continuous norms.*

**Proof.** Let $\left\{A_i = (\mu_{A_i}, \lambda_{A_i})\right\}_{i \in I}$ be a family of $(S,T)$-anti implicative if-ideals of $X$. Then, for any $x, y, z \in X$,

$$\begin{aligned} \left(\cap\, \mu_{A_i}\right)(0) &= \inf\left\{\mu_{A_i}(0)\right\} \leq \inf\left\{\mu_{A_i}(x)\right\} = \left(\cap\, \mu_{A_i}\right)(x), \\ \left(\cap\, \lambda_{A_i}\right)(0) &= \sup\left\{\lambda_{A_i}(0)\right\} \geq \sup\left\{\lambda_{A_i}(x)\right\} = \left(\cap\, \lambda_{A_i}\right)(x). \end{aligned}$$

Also,

$$\begin{aligned} \left(\cap\, \mu_{A_i}\right)(x) &= \inf\left\{\mu_{A_i}(x)\right\} \leq \inf\left\{S(\mu_{A_i}((x*y)*z), \mu_{A_i}(z))\right\} \\ &= S(\inf\left\{\mu_{A_i}((x*y)*z)\right\}, \inf\left\{\mu_{A_i}(z)\right\}) \\ &= S(\left(\cap\, \mu_{A_i}\right)((x*y)*z), \left(\cap\, \mu_{A_i}\right)(z)), \end{aligned}$$

$$\begin{aligned} \left(\cap\, \lambda_{A_i}\right)(x) &= \sup\left\{\mu_{A_i}(x)\right\} \geq \sup\left\{T(\lambda_{A_i}((x*y)*z), \lambda_{A_i}(z))\right\} \\ &= T(\sup\left\{\lambda_{A_i}((x*y)*z)\right\}, \sup\left\{\lambda_{A_i}(z)\right\}) \\ &= T((\cap\, \lambda_{A_i})((x*y)*z), (\cap\, \lambda_{A_i})(z)). \end{aligned}$$

This completes the proof. ■

# 4 Conclusions

In this work, we introduce the notions of anti intuitionistic fuzzy BCK/BCI- subalgebras, anti intuitionistic fuzzy ideals and anti implicative intuitionistic fuzzy ideals with the help of arbitrary t-conorms and t-norms, and discuss some properties such as product, direct product and relations between them. But there are still some open problems. How can we define the notions of anti intuitionistic fuzzy filters and anti intuitionistic fuzzy congruences with recpect to arbitrary t-conorms and t-norms on a BCK/BCI-algebra? What are the relations between such notions, between the cosets of an anti intuitionistic fuzzy filter and anti intuitionistic fuzzy congruences? These could be a topic of further research. Furthermore, using that generalizations, one could define the notion anti intuitionistic fuzzy subgroups in BCK/BCI-algebras with respect to arbitrary t-conorms and t-norms in the sense of [1] and [7]. Using the idea of Dudek et al. [5], one could also generalize the notion of fuzzy topological anti BCK/BCI-algebras to intuitionistic fuzzy structures.

The notions given in this paper can be fundamental to other sciences. For instance, in the last decade, most of researchers are focused on Content Based Image Retrieval, shortly CBIR, and managing uncertainty becomes a fundamental topic in image database. Intuitionistic fuzzy set theory can be ideally suited to deal with this kind of uncertainty. This fuzziness is mainly due to similarity of media features, imperfection in the feature extraction algorithms, esc. Using the concept of this paper, one could develop an anti intuitionistic fuzzy model for image data and provide an anti intuitionistic fuzzy subalgebra for dealing with such data. Moreover, new anti intuitionistic fuzzy algebraic operators could be defined in order to capture the fuzziness related to the semantic descriptors of an image, and built thematic categorizations of multimedia documents using ontological information and anti intuitionistic fuzzy subalgebra in triangular norm systems.

**Problem.** Can we replace in the statement of Theorem 3.13, the condition "$S$ and $T$ are continuous" with "$\inf_{a>0} S(a,a) = 0$ and $\sup_{b<1} T(b,b) = 1$"?

# References

[1] M.T. Abu Osman, On some product of fuzzy subgroups, Fuzzy Sets and Systems 24(1987), 79-86.

[2] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20(1986), 87-96.

[3] K. Atanassov, New operations defined over the intuitionistic fuzzy sets, Fuzzy Sets and Systems 61(1994), 137-142.

[4] R. Biswas, Fuzzy subgroups and anti fuzzy subgroups, Fuzzy Sets and Systems 35(1990), 121-124.

[5] W.A. Dudek, J.B. Jun, S.M. Hong, On fuzzy topological BCC-algebras, Discuss Math. Gen. Algebra Appl. 20(1)(2000), 77-86.

[6] O. Hadzic, E. Pap, Fixed point theory in probabilistic metric spaces, Dordrecht, Kluwer Acad. Publishers, 2001.

[7] Y. Imai, K. Iseki, On axiom systems of propositional calculi, Proc. Japan Academy 42(1966), 19-22.

[8] K. Iseki, S. Tanaka, An introduction to the theory of BCK-algebras, Math. Japon. 22(1978), 1-26.

[9] E.P. Klement, R. Mesiar, E. Pap, Triangular norms, Dordrecht, Kluwer Acad. Publishers, 2000.

[10] S. Kutukcu, S. Sharma, On Anti Fuzzy Structures in BCC-Algebras, J. Concr. Appl. Math. 6(1)(2008), 9-19.

[11] R. Lowen, Fuzzy set theory, Dordrecht, Kluwer Acad. Publishers, 1996.

[12] M.X. Luo, Fuzzy implicative ideals, J. Shaanxi Normal Univ. nat. Sci. Ed. 29(2001), 24-28.

[13] K. Menger, Statistical metrics, Proc. Nat. Acad. Sci. 28(1942), 535-537.

[14] B. Schweizer, A. Sklar, Statistical metric spaces, Pacific J. Math. 10(1960), 314-334.

[15] O.G. Xi, Fuzzy BCK-algebras, Math. Japon. 36(5)(1991), 935-942.

[16] L.A. Zadeh, Fuzzy sets, Inform. Control 8(1965), 338-353.r

# On (IT)-commutativity condition in fixed point consideration of set-valued and single-valued mappings

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**Abstract:**The concept of (IT)-commutativity and property (E.A) between set-valued mappings and single-valued mappings are used to prove some common fixed point theorems on metric spaces without taking condition of continuity.

**Keywords:** (IT)-commutativity, (E.A) property, fixed point.

## 1 Introduction

Banach contraction principle or Banach's fixed point theorem provides a technique for solving a variety of applied problems in mathematical science and engineering. Many authors have extended, generalized and improved Banach's fixed point theorem in different ways. In [9] Jungck introduced more generalized commuting mappings, called compatible mappings, which are more general than commuting and weakly commuting mappings. This concept has been useful for obtaining more comprehensive fixed point theorems (Jeong and Rhoades [8], Jungck [9,10], Jungck and Rhoades [11], Kang and Rhoades [13]).

Recently, Jungck and Rhoades [11,12] defined the concepts of $\delta-$compatible mappings and weakly compatible mappings, respectively, which extend the concept of compatible mappings in single-valued settings to set-valued mappings. They showed that compatible mappings and $\delta-$compatible mappings are weakly compatible but the converse does not need to be true. Several authors used these concepts to prove some common fixed point theorems (Ahmed [2], Chang[3], Rashwan and Ahmed [20,21], Rhoades [22]). Pant [16-19] initiated the study of noncompatible maps and introduced pointwise R-weak commutativity of mappings. He also showed that pointwise R- weak commutativity is equivalent to weak compatibility at coincidence points for single-valued mappings. Following Itoh and Takahshi [7], Singh and Mishra [23] introduced the notion of (IT)-commutativity for single-valued and multi-valued mappings. They showed that (IT)-commutativity of hybrid pair is more general than their

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weak compatibility at the same point. More recently, Aamri and Moutawakil [1] defined a property (E.A) for self single valued maps and obtained some fixed point theorems for such mappings under strict contractive conditions. The class of (E.A) maps contains the class of noncompatible maps.

In this paper, the concept of (IT)-commutativity and property (E.A) between single-valued mappings and set-valued mappings are used to prove some common fixed point theorems on metric spaces without taking any mapping continuous. We improve and generalize the results of Ahmed [2], Fisher [4] and Khan et al. [15].

## 2 Preliminaries

In the sequel, $(X, d)$ denotes a metric space and $B(X)$ is the set of all non empty bounded subsets of $X$. As in [3] and [6], we define

$$\begin{aligned} \delta(A,B) &= \sup\{d(a,b) : a \in A, b \in B\}, \\ D(A,B) &= \inf\{d(a,b) : a \in A, b \in B\}, \\ H(A,B) &= \inf\{r > 0 : A_r \supset B, B_r \supset A\} \end{aligned}$$

for all $A, B$ in $B(X)$ where $A_r = \{x \in X : d(x,a) < r \text{ for some } a \in A\}$ and $B_r = \{y \in X : d(y,b) < r \text{ for some } b \in B\}$.

If $A = \{a\}$ for some $a \in A$, we denote $\delta(a,B)$, $D(a,B)$ and $H(a,B)$ for $\delta(A,B)$, $D(A,B)$ and $H(A,B)$ respectively. Also, if $B = \{b\}$ and $A = \{a\}$, one can deduce that $\delta(A,B) = D(A,B) = H(A,B) = d(a,b)$.

It follows immediately from the definition of $\delta(A,B)$ that $\delta(A,B) = \delta(B,A) \geq 0$, $\delta(A,B) \leq \delta(A,C) + \delta(C,B)$, $\delta(A,B) = 0$ iff $A = B = \{a\}$, $\delta(A,A) = diamA$ for all $A, B, C \in B(X)$.

**Definition 2.1 ([6])** *A sequence $\{A_n\}$ of nonempty subsets of $X$ is said to be convergent to a subset $A$ of $X$ if*

(i) Each point $a$ in $A$ is the limit of a convergent sequence $\{a_n\}$, where $a_n$ is in $A_n$ for all $n \in N$.

(ii) For arbitrary $\epsilon > 0$, there exists an integer $m$ such that $A_n \subseteq A_\epsilon$ for $n > m$, where $A_\epsilon$ denotes the set of all points $x$ in $X$ for which there exists a point $a$ in $A$, depending on $x$, such that $d(x,a) < \epsilon$.

$A$ is said to be the limit of the sequence $\{A_n\}$.

**Lemma 2.2 ([6])** *If $\{A_n\}$ and $\{B_n\}$ are sequences in $B(X)$ converging to $A$ and $B$ in $B(X)$, respectively, then the sequence $\{\delta(A_n, B_n)\}$ converges to $\delta(A,B)$.*

**Lemma 2.3 ([6])** *If $\{A_n\}$ is a sequence in $B(X)$ and $y$ is a point in $X$ such that $\delta(A_n, y) \to 0$, then the sequence $\{A_n\}$ converges to the set $\{y\}$ in $B(X)$.*

**Lemma 2.4 ([3])** *For $A, B, C, D \in B(X)$, we have*

$$\delta(A,B) \leq H(A,C) + \delta(C,D) + H(D,B).$$

**Definition 2.5 ([6])** *The mappings $I : X \to X$ and $F : X \to B(X)$ are said to be weakly commuting if $IFx \in B(X)$ and $\delta(FIx, IFy) \leq \max\{\delta(Ix, Fx), diamIFx\}$ for all $x$ in $X$.*

**Definition 2.6 ([11])** *The mappings $I : X \to X$ and $F : X \to B(X)$ are $\delta$-compatible if the limit $lim_{n\to\infty}\delta(FIx_n, IFx_n) = 0$ whenever $\{x_n\}$ is a sequence in $X$ such that $IFx_n \in B(X)$, $Fx_n \to \{t\}$ and $Ix_n \to t$ for some $t$ in $X$.*

**Definition 2.7 ([12])** *The mappings $I : X \to X$ and $F : X \to B(X)$ are weakly compatible if they commute at coincidence points, i.e. for each point $u \in X$ such that $Fu = \{Iu\}$, we have $FIu = IFu$ (Note that the equation $Fu = \{Iu\}$ implies that $Fu$ is singleton).*

Itoh and Takahsi [7], and Singh and Mishra [23] defined the (IT)-commutativity for single-valued and set-valued mappings is as follows:

**Definition 2.8** *The mappings $I : X \to X$ and $F : X \to B(X)$ are said to be (IT)-commuting (Itoh-Takahasi commutativity is simply called (IT)-commuting) at $x \in X$ if $IFx \subset FIx$. $I$ and $F$ are (IT)-commuting on $X$ if they are (IT)-commuting at each point of $X$.*

In [14], the property (E.A) for single-valued and set-valued mappings is defined as follows:

**Definition 2.9** *The mappings $I : X \to X$ and $F : X \to B(X)$ are said to satisfy the property (E.A) if there exists a sequence $\{x_n\}$ in $X$ such that $lim_{n\to\infty}Ix_n = t$ and $lim_{n\to\infty}Fx_n = \{t\}$ for some $t \in X$.*

**Remark 2.10** *Let $X$ is a metric space, $I : X \to X$ and $F : X \to B(X)$. Then it is clear from Jungck and Rhoade's [11] definition that $I$ and $F$ will not be $\delta$-compatible if there exists at least one sequence $\{x_n\}$ in $X$ such that $lim_{n\to\infty}Ix_n = t$ and $lim_{n\to\infty}Fx_n = \{t\}$ for some $t \in X$, but $\delta(FIx_n, IFx_n)$ is either non- zero or non existent. Thus two non $\delta$-compatible maps satisfy the property (E.A).*

**Example 2.11** *Let $X = [1, \infty]$ with the usual metric. Define $I : X \to X$, $F : X \to B(X)$ by $Ix = x + 1$ and $Fx = [1, x + 1]$ for all $x \in X$. Then $IF_1 \subset FI_1$. Therefore, $I$ and $F$ are IT-commuting at $x = 1$. But $I$ and $F$ are not weakly compatible since $IF_1 \neq FI_1$. Consider the sequence $\{x_n\} = \{1/n\}$. Clearly $lim_{n\to\infty}Ix_n = 1$ and $lim_{n\to\infty}Fx_n = \{1\}$. Thus $I$ and $F$ satisfy the property (E.A). But $I$ and $F$ are not $\delta$-compatible.*

**Remark 2.12** *It is clear from Remark 1 and Example 1 that if $I : X \to X$ and $F : X \to B(X)$ are (IT)-commuting maps then $I$ and $F$ satisfy the property (E.A).*

Let $I : X \to X$ and $F : X \to B(X)$. In all that follows, $C(F, I)$ stands for the set of coincidence points of the maps $F$ and $I$, that is $C(F, I) = \{z : \{Iz\} = Fz\}$.

# 3 Main Results

**Theorem 3.1** *Let $(X, d)$ be a complete metric space. Let $I, J$ be mappings of $X$ into itself and $F, G$ of $X$ into $B(X)$ such that*

(1.1) $\delta(Fx, Gy) \leq \max\{cd(Ix, Jy), c\delta(Ix, Fx), c\delta(Jy, Gy), aD(Ix, Gy)+bD(Jy, Fx)\}$ for all $x, y \in X$, where $0 \leq c < 1$, $a, b \geq 0$, $a + b < 1$, $c\max\{\frac{a}{1-a}, \frac{b}{1-b}\} < 1$,

(1.2) $\cup F(X) \subset J(X)$ and $\cup G(X) \subset I(X)$, then

(i) $F$ and $I$ have a coincidence point.
(ii) $G$ and $J$ have a coincidence point.
Further if

(1.3) $F$ and $I$ are (IT)- commuting at $p \in C(F,I)$, $G$ and $J$ are (IT)- commuting at $q \in C(G,J)$, then

(iii) $I, J, F$ and $G$ have a unique common fixed point $u \in X$.

**Proof.** Let $x_0$ be an arbitrary point in $X$. By (1.2), we choose a point $x_1$ in $X$ such that $Jx_1 \in Fx_0 = Z_0$ and for this point $x_1$, there exists a point $x_2$ in $X$ such taht $Ix_2 \in Gx_1 = Z_1$ and so on. Continuing in this manner, we can define a sequence $\{x_n\}$ as follows:

(1.4) $Jx_{2n+1} \in Fx_{2n} = Z_{2n}$ and $Ix_{2n+2} \in Gx_{2n+1} = Z_{2n+1}$, $n \in N \cup \{0\}$.

For simplicity, we put $V_n = \delta(Z_n, Z_{n+1})$ for $n \in N \cup \{0\}$. By (1.1) and (1.4), we have

$$\begin{aligned} V_{2n} &= \delta(Z_{2n}, Z_{2n+1}) = \delta(Fx_{2n}, Gx_{2n+1}) \\ &\leq \max\{cd(Ix_{2n}, Jx_{2n+1}), c\delta(Ix_{2n}, Fx_{2n}), c\delta(Jx_{2n+1}, Gx_{2n+1}), \\ &\quad aD(Ix_{2n}, Gx_{2n+1}) + bD(Jx_{2n+1}, Fx_{2n})\} \\ &\leq \max\{cV_{2n-1}, cV_{2n}, a(V_{2n-1} + V_{2n})\} \\ &\leq \max\{c, \frac{a}{1-a}\}V_{2n-1} \end{aligned}$$

for $n \in N$. Similarly, one can show that

$$V_{2n+1} = \delta(Z_{2n+1}, Z_{2n+2}) = \delta(Gx_{2n+1}, Fx_{2n+2}) \leq \max\{c, \frac{b}{1-b}\}V_{2n}$$

for $n \in N$. If we put $\beta = \max\{c, \frac{a}{1-a}\}.\max\{c, \frac{b}{1-b}\}$, then by hypothesis it can be easily seen that $0 \leq \beta < 1$. So we deduce that

(1.5) $V_{2n} \leq \beta V_{2n-2} \leq .... \leq \beta^n V_0$ and $V_{2n+1} \leq \beta V_{2n-1} \leq .... \leq \beta^n V_1$

for $n \in N$. Put $M = \max\{V_0, V_1\}$. It follows from the inequality (1.5) that if $z_n$ is an arbitrary point in the set $Z_n$ for $n \in N$, then we obtain that $d(z_{2n}, z_{2n+1}) \leq \delta(Z_{2n}, Z_{2n+1}) \leq \beta^n M$ and $d(z_{2n+1}, z_{2n+2}) \leq \delta(Z_{2n+1}, Z_{2n+2}) \leq \beta^n M$. This implies that $\{z_n\}$ is a Cauchy sequence in the complete metric space $X$. Hence it converges to a point $u \in X$, which does not depend upon the particular choice of each $z_n$. In particular, the sequences $\{Ix_{2n}\}$ and $\{Jx_{2n+1}\}$ converge to $u$ and the sequences of sets $\{Fx_{2n}\}$ and $\{Gx_{2n+1}\}$ converge to the set $\{u\}$.

Since $\cup G(X) \subset I(X)$, there exists a point $p \in X$ such that $\{u\} = \{Ip\}$. Then using (1.1), we have

$$\begin{aligned} \delta(Fp, Gx_{2n+1}) &\leq \max\{cd(Ip, Jx_{2n+1}), c\delta(Ip, Fp), c\delta(Jx_{2n+1}, Gx_{2n+1}), \\ &\quad aD(Ip, Gx_{2n+1}) + bD(Jx_{2n+1}, Fp)\} \\ &\leq \max\{cd(u, Jx_{2n+1}), c\delta(u, Fp), c\delta(Jx_{2n+1}, Gx_{2n+1}), \\ &\quad a\delta(u, Gx_{2n+1}) + b\delta(Jx_{2n+1}, Fp)\}. \end{aligned}$$

Taking the limit as $n \to \infty$, we have

$$\delta(Fp, u) \leq \max\{c\delta(u, Fp), b\delta(u, Fp)\} = \max\{c, b\}\delta(Fp, u)$$

and hence $Fp = \{u\}$, since $\max\{c, b\} < 1$. Therefore $\{Ip\} = \{u\} = Fp$. Thus $Fp$ is singleton and $p \in C(F, I)$.

Since $\cup F(X) \subset J(X)$, there exists a point $q \in X$ such that $\{u\} = \{Jq\}$. Then using (1.1), we have

$$\begin{aligned}\delta(u, Gq) &= \delta(Fp, Gq)\\ &\leq \max\{cd(u,u), c\delta(u,u), c\delta(u,Gq), a\delta(u,Gq) + b\delta(u,u)\}\\ &= \max\{c\delta(u,Gq), a\delta(u,Gq)\}\\ &= \max\{c,a\}\delta(u,Gq).\end{aligned}$$

Since $\max\{c,a\} < 1$, then $\{u\} = Gq = \{Jq\}$. Thus $Gq$ is singleton and $q \in C(G,J)$.

Since $F$ and $I$ are (IT)-commuting at $p \in C(F,I)$, so $IFp \subset FIp$, that is, $\{Iu\} \subset Fu$. Using (1.1), we have

$$\begin{aligned}\delta(Fu, Gx_{2n+1}) &\leq \max\{cd(Iu, Jx_{2n+1}), c\delta(Iu,Fu), c\delta(Jx_{2n+1}, Gx_{2n+1}),\\ &\quad aD(Iu, Gx_{2n+1}) + bD(Jx_{2n+1}, Fu)\}\\ &\leq \max\{c\delta(Fu, Jx_{2n+1}), c\delta(Jx_{2n+1}, Gx_{2n+1}),\\ &\quad a\delta(Fu, Gx_{2n+1}) + b\delta(Jx_{2n+1}, Fu)\}.\end{aligned}$$

Taking the limit as $n \to \infty$, we have

$$\begin{aligned}\delta(Fu, u) &\leq \max\{c\delta(Fu,u), a\delta(Fu,u) + b\delta(u,Fu)\}\\ &= \max\{c, a+b\}\delta(Fu,u).\end{aligned}$$

Since $\max\{c, a+b\} < 1$, then $\{u\} = Fu$. Since $\{u\} = \{Iu\}$, $\{Iu\} = Fu = \{u\}$. Similarly we can prove that $Gu = \{Ju\} = \{u\}$. Thus $u$ is a common fixed point of $I, J, F$ and $G$.

Now suppose there exists $w \in X, u \neq w$ such that $Fw = \{w\} = \{Iw\}$. Using (1.1), we obtain that

$$\begin{aligned}d(w,u) &\leq \delta(Fw, Gu)\\ &\leq \max\{cd(w,u), ad(w,u) + bd(u,w)\}\\ &= \max\{c, a+b\}d(w,u).\end{aligned}$$

Since $\max\{c, a+b\} < 1$, it follows that $u = w$. So $u$ is unique common fixed point of $F$ and $I$ such that $Fu = \{u\} = \{Iu\}$. Similarly, it can be shown that $u$ is the unique common fixed point of $G$ and $J$ such that $Gu = \{u\} = \{Ju\}$. This completes the proof. ■

By using the property (E.A) and by removing the completeness of the space with a set of alternative conditions in Theorem 3.1, we prove the following:

**Theorem 3.2** *Let $(X,d)$ be a metric space. Let $I, J$ be mappings of $X$ into itself and $F, G$ of $X$ into $B(X)$ satisfying condition (1.1), (1.2) and*

(2.1) $\{F, I\}$ or $\{G, J\}$ satisfy the property (E.A),

(2.2) if the range of one of $I(X)$, $J(X)$, $F(X)$ or $G(X)$ is a complete subspace of $X$, then

(i) F and I have a coincidence point
(ii) G and J have a coincidence point.
Further if (1.3) is satisfied then

(iii) $I, J, F$ and $G$ have a unique common fixed point.

**Proof.** Suppose that the pair $\{F, I\}$ satisfies the property (E.A). Then there exists a sequence $\{x_n\}$ in $X$ such that $lim_{n\to\infty} Ix_n = u$ and $lim_{n\to\infty} Fx_n = \{u\}$ for some $u \in X$. Since $\cup F(X) \subset J(X)$, there exists a sequence $\{y_n\}$ in $X$ such that $lim_{n\to\infty} Fx_n = lim_{n\to\infty} Jy_n$. Hence $lim_{n\to\infty} Jy_n = u$. Let us now show that $lim_{n\to\infty} Gy_n = \{u\}$. Indeed in view of (1.1), we have

$$\begin{aligned}\delta(Fx_n, Gy_n) &\leq \max\{cd(Ix_n, Jy_n), c\delta(Ix_n, Fx_n), c\delta(Jy_n, Gy_n),\\ &\quad aD(Ix_n, Gy_n) + bD(Jy_n, Fx_n)\}.\end{aligned}$$

Taking the limit as $n \to \infty$ and since $\max\{a, c\} < 1$, we have $lim_{n\to\infty} Gy_n = \{u\}$. Suppose that $J(X)$ is complete subspace of $X$. Then $u = Jq$ for some $q \in X$. Subsequently, we have $lim_{n\to\infty} Gy_n = \{Jq\} = lim_{n\to\infty} Fx_n$ and $lim_{n\to\infty} Ix_n = lim_{n\to\infty} Jy_n = Jq$. Using (1.1), we have

$$\begin{aligned}\delta(Fx_n, Gq) &\leq \max\{cd(Ix_n, Jq), c\delta(Ix_n, Fx_n), c\delta(Jq, Gq),\\ &\quad aD(Ix_n, Gq) + bD(Jq, Fx_n)\}.\end{aligned}$$

Taking the limit as $n \to \infty$ and since $\max\{a, c\} < 1$, we have $Gq = \{Jq\} = \{u\}$. Thus $Gq$ is singleton and $q \in C(G, q)$. On the other hand $\cup G(X) \subset I(X)$, there exists $p \in X$ such that $\{Ip\} = Gq$. Using (1.1), we have

$$\begin{aligned}\delta(Fp, Gq) &\leq \max\{cd(Ip, Jq), c\delta(Ip, Fp), c\delta(Jq, Gq),\\ &\quad aD(Ip, Gq) + bD(Jq, Fp)\}\\ &\leq \max\{c, b\}\delta(Ip, Fp).\end{aligned}$$

Since $\max\{c, b\} < 1$, so $\{Ip\} = Fp$ that is $p \in C(F, I)$ and $\{Ip\} = Fp = \{u\} = \{Jq\} = Gq$. Since $F$ and $I$ are (IT)-commuting at $p \in X$. Therefore $IFp \subset FIp$ which implies $\{Iu\} \subset Fu$. Similarly $\{Ju\} \subset Gu$. By using (1.1), we have

$$\begin{aligned}\delta(Fu, Gy_n) &\leq \max\{cd(Iu, Jy_n), c\delta(Iu, Fu), c\delta(Jy_n, Gy_n),\\ &\quad aD(Iu, Gy_n) + bD(Jy_n, Fu)\}\\ &\leq \max\{c\delta(Fu, Jy_n), c\delta(Fu, Fu), c\delta(Jy_n, Gy_n),\\ &\quad a\delta(Fu, Gy_n) + b\delta(Jy_n, Fu)\}.\end{aligned}$$

Taking the limit as $n \to \infty$, we have $\delta(Fu, u) \leq \max\{c, a + b\}\delta(Fu, u)$, then $Fu = \{u\}$. Since $\{Iu\} \subset Fu$ and $Fu$ is singleton, $Iu = u$. Therefore $u$ is common fixed point of $F$ and $I$. Similarly we can prove that $u$ is common fixed point of $G$ and $J$.

The proof is similar when $I(X)$ is assumed to be complete subspace of $X$. The cases in which $F(X)$ or $G(X)$ is a complete subspace of $X$ are similar to the cases in which $J(X)$ or $I(X)$, respectively, is complete since $\cup F(X) \subset J(X)$ and $\cup G(X) \subset I(X)$. Hence $u$ is common fixed point of $I, J, F$ and $G$. The uniqueness of common fixed point $u \in X$ follows from (1.1). This completes the proof. ■

With the help of Remark 2.12 and Theorem 3.2, we get the following:

**Corollary 3.3** *Let $(X, d)$ be a metric space. Let $I, J$ be mappings of $X$ into itself and $F, G$ of $X$ into $B(X)$ satisfying condition (1.1), (1.2), (2.2) and the pairs $\{F, I\}$, $\{G, J\}$ are (IT)-commuting. Then $I, J, F$ and $G$ have a unique common fixed point.*

The conditions (1.2) and (2.2) can be removed by taking $I(X)$ and $J(X)$ as closed subspaces of $X$. We prove the following:

**Theorem 3.4** *Let $(X,d)$ be a metric space. Let $I,J$ be mappings of $X$ into itself and $F,G$ of $X$ into $B(X)$ satisfying condition (1.1) and*

(3.1) the pairs $\{F,I\}$ and $\{G,J\}$ satisfy the property (E.A),

(3.2) $I(X)$ and $J(X)$ are closed subspaces of $X$, then

(i) F and I have a coincidence point,
(ii) G and J have a coincidence point.
Further if (1.3) is satisfied then
(iii) $I,J,F$ and $G$ have a unique common fixed point.

**Proof.** Since the pair $\{F,I\}$ satisfies the property (E.A), then there exists a sequence $\{x_n\}$ in $X$ such that $lim_{n\to\infty}Ix_n = u$ and $lim_{n\to\infty}Fx_n = \{u\}$ for some $u \in X$. Again since the pair $\{G,J\}$ satisfies the property (E.A), then there exists a sequence $\{y_n\}$ in $X$ such that $lim_{n\to\infty}Jy_n = w$ and $lim_{n\to\infty}Gy_n = \{w\}$ for some $w \in X$. If $w \neq u$, then by using (1.1), we have

$$\begin{aligned}\delta(Fx_n,Gy_n) &\leq \max\{cd(Ix_n,Jy_n), c\delta(Ix_n,Fx_n), c\delta(Jy_n,Gy_n),\\ &\quad aD(Ix_n,Gy_n)+bD(Jy_n,Fx_n)\}.\end{aligned}$$

Taking the limit as $n\to\infty$, we have

$$\begin{aligned}\delta(u,w) &\leq \max\{cd(u,\ w), ad(u,w)+bd(u,w)\}\\ &= \max\{c,a+b\}d(u,w).\end{aligned}$$

Since $\max\{c,a+b\}<1$, it follows that $u=w$. Since $I(X)$ is closed, we have $lim_{n\to\infty}Ix_n = Ip$ for some $p\in X$. Thus $Ip=u$. Subsequently, we have $lim_{n\to\infty}Fx_n = \{Ip\}$. From (1.1), we have

$$\begin{aligned}\delta(Fp,Gy_n) &\leq \max\{cd(Ip,Jy_n), c\delta(Ip,Fp), c\delta(Jy_n,Gy_n),\\ &\quad aD(Ip,Gy_n)+bD(Jy_n,Fp)\}\end{aligned}$$

Taking the limit as $n\to\infty$, we have $\delta(Fp,u)\leq\max\{c,b\}\delta(u,Fp)$ and hence $Fp=\{u\}=\{Ip\}$, since $\max\{c,b\}<1$. Since $J(X)$ is closed, we have $lim_{n\to\infty}Jy_n = Jq$ for some $q\in X$. Thus $Jq=u$. Subsequently, we have $lim_{n\to\infty}Gy_n=\{Jq\}=\{u\}$. Using (1.1), we have

$$\begin{aligned}\delta(u,Gq) &= \delta(Fp,Gq)\leq\max\{cd(Ip,Jq), c\delta(Ip,Fp), c\delta(Jq,Gq),\\ &\quad aD(Ip,Gq)+bD(Jq,Fp)\}\\ &= \max\{c\delta(u,Gq), a\delta(u,Gq)\}\\ &= \max\{c,a\}\delta(u,Gq).\end{aligned}$$

Since $\max\{c,a\}<1$ then $\{u\}=Gq$. Thus $Gq$ is singleton and $q\in C(G,J)$. The remaining part of the proof is same as that of Theorem 3.1. This completes the proof. ■

With the help of Remark 2.12 and Theorem 3.4, we get the following:

**Corollary 3.5** *Let $(X,d)$ be a metric space. Let $I,J$ be mappings of $X$ into itself and $F,G$ of $X$ into $B(X)$ satisfying condition (1.1), (3.2) and the pairs $\{F,I\}$ and $\{G,J\}$ are (IT)-commuting then $I,J,F$ and $G$ have a fixed point.*

# References

[1] M. Aamri, D. El. Moutawakil, Some new common fixed point theorems under strict contractive conditions. J. Math. Anal. Appl. 270(2002), 181-188.

[2] M. A. Ahmed, Common fixed point theorems for set- valued and single-valued. Demonstratio Math. Vol. 36(2)(2003), 471-481.

[3] T. H. Chang, Fixed point theorems for contractive type set- valued mappings, Math. Japonica 38(4)(1993), 675-690.

[4] B. Fisher, Common fixed points of mappings and set-valued mappings on a metric spaces.Kyungpook Math. J. 25(1985), 35-42.

[5] B. Fisher and S. Sessa, Common fixed points of two pairs of weakly commutings, Univ. U. Novom Sadu. Zb. Rad. Period Mat. Fak. Ser. Mat 18(1986), 45-59.

[6] B. Fisher and S. Sessa, Two common fixed point theorems for weakly commuting mappings, Periodica Math. Hungraica, 20(3)(1989), 207-218.

[7] S. Itoh, W. Takahashi, Single-valued mappings, Multivalued mappings and fixed point theorems, J. Math. Anal. Appl. 59(1977), 514-521.

[8] G. S. Jeong and B. E. Rhoades, Some remarks for involving fixed point theorems for more than two maps, Ind. J. Pure Appl. Math. 28(9)(1997), 1171-1196.

[9] G. Jungck, Compatible mappings and common fixed points, Internat J. Math. Sci., 9(1986), 771-779.

[10] G. Jungck, Common fixed points of commuting and compatible maps on compacta, Proc. Amar. Math. Soc. 103(1988), 977-983.

[11] G. Jungck and B. E. Rhoades, Some fixed point theorems for compatible maps, Internat J. Math. and Math. Sci. 16(1993), 417-428.

[12] G. Jungck and B. E. Rhoades, Fixed points for set-valued functions without continuity, Ind. J. Pure and Appl. Math. 16(3)(1998), 227-238.

[13] S. M. Kang and B. E. Rhoades, Fixed points for four mappings, Math. Japonica 37(1992), 1053-1059.

[14] T. Kamran, Coincidence and fixed points for hybrid strict contraction, J. Math. Anal. Appl. 299(2004), 235-241.

[15] M. S. Khan, I. Kubiaczyk and S. Sessa, On set-valued and single-valued mappings with common fixed points, Bull. Malaysian Math. Soc. 2(10)(1987), 71-81.

[16] R. P. Pant, Common fixed points of non-commuting mappings, J. Math. Anal. Appl.. 188(1994), 436-440.

[17] R. P. Pant, Common fixed point theorems for contractive maps, J. Math. Anal. Appl. 226-(1998), 251-258.

[18] R. P. Pant, Common fixed points of Lipschitz type mapping pairs, J. Math. Anal. Appl. 240(1999), 280-283.

[19] R. P. Pant, Discontinuity and fixed points, J. Math. Anal. Appl. 240(1999), 284-289.

[20] R. A. Rashwan and M. A. Ahmed, Common fixed points for $\delta$-compatible mappings, Southwest J. Pure Appl. Math. 1(1996), 51-61.

[21] R. A. Rashwan and M. A. Ahmed, Fixed points of single and set-valued mappings, Kyungpook Math. J. 38(1998), 29-37.

[22] B. E. Rhoades, Common fixed points of compatible set-valued mappings, Publ. Math. Debrecen 48, No. 3-4(1996), 237-240.

[23] S. L. Singh and S. N. Mishra, Coincidences and fixed points of nonself hybrid contraction, J. Math. Anal. Appl. 256(2001), 486-497.

# Some Generalizations in Intuitionistic Fuzzy Metric Spaces

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**Abstract:**The aim of this paper is to give some new fixed point theorems for contractive type mappings in intuitionistic fuzzy metric spaces. The results presented improve and generalize some well known results in literature.

**Keywords:** Intuitionistic fuzzy metric space; fuzzy metric space; complete; compact; fixed point.

## 1 Introduction

In [6], the well-known fixed point theorems of Banach [1] and Edelstein [3] were extended to fuzzy metric spaces obtaining the following two theorems,

**Theorem 1.1** *Let $(X, M, *)$ be a complete fuzzy metric space. Let $T : X \to X$ be a mapping satisfying*

$$M(Tx, Ty, kt) \geq M(x, y, t)$$

*for all $x, y \in X$ where $0 < k < 1$. Then $T$ has a unique fixed point.*

**Theorem 1.2** *Let $(X, M, *)$ be a compact fuzzy metric space. Let $T : X \to X$ be a mapping satisfying*

$$M(Tx, Ty, .) > M(x, y, .)$$

*for all $x \neq y$ (i.e. $M(Tx, Ty, .) \geq M(x, y, .)$ and $M(Tx, Ty, .) \neq M(x, y, .)$ for all $x \neq y$). Then $T$ has a unique fixed point.*

In this paper, we give some new fixed point theorems in intuitionistic fuzzy metric spaces. The results not only improve and generalize Theorems 1.1 and 1.2, but also unify and extend some main results of [5-8]. Let us call some basic definitions,

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**Definition 1.3 ([2])** *The 5-tuble $(X, M, N, *, \lozenge)$ is an intuitionistic fuzzy metric space if $X$ is an arbitrary set, $*$ is a continuous t-norm, $\lozenge$ is a continuous t-conorm and $M, N$ are fuzzy sets on $X^2 \times [0, \infty)$ satisfying the following conditions: for all $x, y, z \in X$,*

(i) $M(x, y, t) + N(x, y, t) \leq 1$;

(ii) $M(x, y, 0) = 0$;

(iii) $M(x, y, t) = 1$ for all $t > 0$ iff $x = y$;

(iv) $M(x, y, t) = M(y, x, t)$;

(v) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$ for all $t, s > 0$;

(vi) $M(x, y, .) : [0, \infty) \to [0, 1]$ is left continuous;

(vii) $\lim_{t\to\infty} M(x, y, t) = 1$;

(viii) $N(x, y, 0) = 1$;

(ix) $N(x, y, t) = 0$ for all $t > 0$ iff $x = y$;

(x) $N(x, y, t) = N(y, x, t)$;

(xi) $N(x, z, t + s) \leq N(x, y, t) \lozenge N(y, z, s)$ for all $t, s > 0$;

(xii) $N(x, y, .) : [0, \infty) \to [0, 1]$ is right continuous;

(xiii) $\lim_{t\to\infty} N(x, y, t) = 0$.

**Remark 1.4** *By (iii) and (v), it is easy to show that $M(x, y, .)$ is non-decreasing and by (ix) and (xi), it is easy to show that $N(x, y, .)$ is non-increasing for all $x, y \in X$.*

**Remark 1.5** *Every fuzzy metric space $(X, M, *)$ is an intuitionistic fuzzy metric space of the form $(X, M, 1-M, *, \lozenge)$ such that t-norm $*$ and t-conorm $\lozenge$ are assosiated, i.e. $x \lozenge y = 1-((1-x)*(1-y))$ for any $x, y \in [0, 1]$. Moreover, Theorem 1.1 is a fuzzy generalization of Theorem 2 in [4].*

**Definition 1.6 ([2])** *A sequence $\{x_n\}$ in an intuitionistic fuzzy metric space $(X, M, N, *, \lozenge)$ is a Cauchy sequence iff $\lim_n M(x_{n+p}, x_n, t) = 1$ and $\lim_n N(\ x_{n+p}, x_n, t) = 0$ for each $t > 0$ and $p \in \mathbb{N}$. A sequence $\{x_n\}$ in $X$ is convergent to $x \in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1$ and $\lim_{n\to\infty} N(x_n, x, t) = 0$ for each $t > 0$. An intuitionistic fuzzy metric space $(X, M, N, *, \lozenge)$ is called complete, if every Cauchy sequence in $X$ is convergent. It is called compact, if every sequence in $X$ contains a convergent subsequence.*

**Lemma 1.7 ([2])** *Let $\lim_n x_n = x$ and $\lim_n y_n = y$. Then, for all $t > 0$,*

$$\lim_{n\to\infty} \inf M(x_n, y_n, t) \geq M(x, y, t) \text{ and } \lim_{n\to\infty} \sup N(x_n, y_n, t) \leq N(x, y, t),$$

$$\lim_{n\to\infty} \sup M(x_n, y_n, t) \leq M(x, y, t) \text{ and } \lim_{n\to\infty} \inf N(x_n, y_n, t) \geq N(x, y, t).$$

*Particularly, if $M(x, y, .)$ and $N(x, y, .)$ are continuous at point $t$, then*

$$\lim_{n\to\infty} M(x_n, y_n, t) = M(x, y, t) \text{ and } \lim_{n\to\infty} N(x_n, y_n, t) = N(x, y, t).$$

## 2 Main Results

**Definition 2.1** *Let the function $\phi : [0,\infty) \to [0,\infty)$ satisfies the following conditions:*

($\phi_1$) *$\phi(t)$ is strictly increasing, $\phi(0) = 0$ and $\lim_n \phi^n(t) = \infty$ for all $t > 0$, where $\phi^n(t)$ denotes the $n$-th iterative function of $\phi(t)$. Then $\phi(t) > t$* and *$\phi^n(t) > \phi^{n-1}(t)$ for $t > 0$* and *$n = 1, 2, ...$*

($\phi_2$) $\lim_{t\to\infty}[\phi(t) - t] = \infty$.

**Lemma 2.2** *Let $*$ and $\Diamond$ be continuous t-norm and t-conorm, respectively. Then for each $\lambda \in (0,1)$, there is a sequence $\{\lambda_n\}$ in $(0,1)$ such that*

$$(1-\lambda_n) * (1-\lambda_n) > 1-\lambda_{n-1} \text{ and } \lambda_n \Diamond \lambda_n < \lambda_{n-1},\ n = 1, 2, ...,$$

*where $\lambda_0 = \lambda$ (obviously, the sequence $\{\lambda_n\}$ satisfying the condition is decreasing).*

**Proof.** Since $*$ is continuous at point $(1,1)$ and $a * b \le 1 * 1 = 1$, and since $\Diamond$ is continuous at point $(0,0)$ and $a \Diamond b \ge 0 \Diamond 0 = 0$ for all $a, b \in [0,1]$, we have

$$\sup_{0<\mu<1}[(1-\mu) * (1-\mu)] = 1 \text{ and } \inf_{0<\mu<1} \mu \Diamond \mu = 0,$$

respectively. Thus, for each $\lambda \in (0,1)$, there exists $\lambda_1 \in (0,1)$ such that

$$(1-\lambda_1) * (1-\lambda_1) > 1-\lambda \text{ and } \lambda_1 \Diamond \lambda_1 < \lambda.$$

Similarly, there exists $\lambda_2 \in (0,1)$ such that

$$(1-\lambda_2) * (1-\lambda_2) > 1-\lambda_1 \text{ and } \lambda_2 \Diamond \lambda_2 < \lambda_1.$$

Continuing this procedure, we can obtain a sequence $\{\lambda_n\} \subset (0,1)$ satisfying the condition. ■

**Lemma 2.3** *Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space. Let $T : X \to X$ be a mapping satisfying*

$$M(Tx, Ty, t_1) > M(x, y, t_1) \text{ and } N(Tx, Ty, t_1) < N(x, y, t_1),$$

*where $t_1$ is a fixed positive number. Then there exist a continuity point $t_0$ of $M(x, y, .)$ and $N(x, y, .)$ such that*

$$M(Tx, Ty, t_0) > M(x, y, t_0) \text{ and } N(Tx, Ty, t_0) < N(x, y, t_0).$$

**Proof.** Since $M(Tx, Ty, .) - M(x, y, .)$ is left continuous and $N(Tx, Ty, .) - N(x, y, .)$ is right continuous at point $t_1$, there exists $t_2 > 0$ such that

$$M(Tx, Ty, t) > M(x, y, t) \text{ and } N(Tx, Ty, t) < N(x, y, t)$$

for all $t \in [t_2, t_1]$. Note that the set of discontinuous points of $M(x, y, .)$ and $N(x, y, .)$ is countable at most. Hence, there exists $t_0 \in [t_2, t_1]$ such that $M(x, y, .)$ and $N(x, y, .)$ are continuous at $t_0$. ■

**Theorem 2.4** *Let $(X, M, N, *, \Diamond)$ be a complete intuitionistic fuzzy metric space. Let $T : X \to X$ be a mapping satisfying the following conditions:*

(i) there exists $x_0 \in X$ such that

$$\lim_{n\to\infty} M(x_0, T^i x_0, t) = 1, \ \lim_{n\to\infty} N(x_0, T^i x_0, t) = 0, i = 1, 2, ...; \tag{2.1}$$

(ii) there exists a mapping $m : X \to \mathbb{N}$ such that for any $x, y \in X$

$$\begin{aligned} M(T^{m(x)}x, T^{m(x)}y, t) &\geq M(x, y, \phi(t)) \text{ and} \\ N(T^{m(x)}x, T^{m(x)}y, t) &\leq N(x, y, \phi(t)) \end{aligned} \tag{2.2}$$

where the function $\phi(t)$ satisfies conditions $(\phi_1)$ and $(\phi_2)$.

Then $T$ has a unique fixed point $x_*$ and the quasi-iterative sequence $\{x_n = T^{m(x_{n-1})}x_{n-1}\}$ converges to $x_*$.

**Proof.** First, we prove that

$$\sup_{s>0} \inf_{x\in O_T(x_0)} M(x_0, x, s) = 1 \text{ and } \inf_{s>0} \sup_{x\in O_T(x_0)} N(x_0, x, s) = 0 \tag{2.3}$$

where $O_T(x_0) = \left\{x_0, Tx_0, T^2x_0, ...\right\}$ is called the orbit of $x_0$ for $T$. For any $n \in \mathbb{N}$ with $n > m(x_0)$, we can denote

$$n = km(x_0) + s \text{ where } 0 \leq s < m(x_0).$$

Note that $\phi(t) > t$ for all $t > 0$ and $\lim_{t\to\infty}[\phi(t) - t] = \infty$. By (2.1), we have

$$\lim_{t\to\infty} M(x_0, T^i x_0, \phi(t)) = 1 \text{ and } \lim_{t\to\infty} N(x_0, T^i x_0, \phi(t)) = 0 \tag{2.4}$$

for $i = 1, 2, ..., m(x_0)$ and

$$\lim_{t\to\infty} M(x_0, T^i x_0, \phi(t) - t) = 1 \text{ and } \lim_{t\to\infty} N(x_0, T^i x_0, \phi(t) - t) = 0. \tag{2.5}$$

Moreover, by Lemma 2.2, for any $\lambda \in (0, 1)$, there is a sequence $\{\lambda_n\}$ in $(0, 1)$ such that

$$(1 - \lambda_n) * (1 - \lambda_n) > 1 - \lambda_{n-1} \text{ and } \lambda_n \Diamond \lambda_n < \lambda_{n-1}, \ n = 1, 2, ....$$

Thus, it follows from (2.4) and (2.5) that for given $\lambda_k$, there exists $t_0 > 0$ such that

$$\min_{1\leq i\leq m(x_0)} M(x_0, T^i x_0, \phi(t)) > 1 - \lambda_k \text{ and } \max_{1\leq i\leq m(x_0)} N(x_0, T^i x_0, \phi(t)) < \lambda_k,$$

$$M(x_0, T^{m(x_0)}x_0, \phi(t) - t) > 1 - \lambda_k \text{ and } N(x_0, T^{m(x_0)}x_0, \phi(t) - t) < \lambda_k, \forall t > t_0.$$

So, by (2.2), for all $t > t_0$, we have

$$\begin{aligned} M(x_0, T^n x_0, \phi(t)) &= M(x_0, T^{km(x_0)+s}x_0, \phi(t)) \\ &\geq M(x_0, T^{m(x_0)}x_0, \phi(t) - t) \\ &\quad * M(T^{m(x_0)}x_0, T^{km(x_0)+s}x_0, t) \\ &\geq M(x_0, T^{m(x_0)}x_0, \phi(t) - t) \\ &\quad * M(x_0, T^{(k-1)m(x_0)+s}x_0, \phi(t)) \\ &\geq ... \geq M(x_0, T^{m(x_0)}x_0, \phi(t) - t) \\ &\quad *^{(k)}_{...} * M(x_0, T^{m(x_0)}x_0, \phi(t) - t) * M(x_0, T^s x_0, \phi(t)) \\ &> (1 - \lambda_k) *^{(k+1)}_{...} * (1 - \lambda_k) \\ &> (1 - \lambda_{k-1}) *^{(k)}_{...} * (1 - \lambda_{k-1}) \\ &> ... > (1 - \lambda_1) * (1 - \lambda_1) > 1 - \lambda, \end{aligned}$$

$$\begin{aligned}
N(x_0,T^n x_0,\phi(t)) &= N(x_0,T^{km(x_0)+s}x_0,\phi(t))\\
&\le N(x_0,T^{m(x_0)}x_0,\phi(t)-t)\\
&\quad \lozenge N(T^{m(x_0)}x_0,T^{km(x_0)+s}x_0,t)\\
&\le N(x_0,T^{m(x_0)}x_0,\phi(t)-t)\\
&\quad \lozenge N(x_0,T^{(k-1)m(x_0)+s}x_0,\phi(t))\\
&\le \ldots \le N(x_0,T^{m(x_0)}x_0,\phi(t)-t)\\
&\quad \lozenge^{(k)}_{\ldots}\lozenge N(x_0,T^{m(x_0)}x_0,\phi(t)-t)\lozenge N(x_0,T^{s}x_0,\phi(t))\\
&< \lambda_k \lozenge^{(k+1)}_{\ldots}\lozenge \lambda_k\\
&< \lambda_{k-1}\lozenge^{(k+1)}_{\ldots}\lozenge\lambda_{k-1}\\
&< \ldots < \lambda_1\lozenge\lambda_1<\lambda.
\end{aligned}$$

Therefore, for all $t>t_0$,

$$\inf_{x\in O_T(x_0)} M(x_0,x,\phi(t))\ge 1-\lambda \text{ and } \sup_{x\in O_T(x_0)} N(x_0,x,\phi(t))\le\lambda$$

and hence

$$\sup_{s>0}\inf_{x\in O_T(x_0)} M(x_0,x,s)\ge 1-\lambda \text{ and } \inf_{s>0}\sup_{x\in O_T(x_0)} N(x_0,x,s)\le\lambda.$$

By the arbitrariness of $\lambda$, we have

$$\sup_{s>0}\inf_{x\in O_T(x_0)} M(x_0,x,s)= 1 \text{ and } \inf_{s>0}\sup_{x\in O_T(x_0)} N(x_0,x,s)= 0.$$

Next, we prove that the quasi-iterative sequence $\left\{x_n=T^{m(x_{n-1})}x_{n-1}\right\}_{n=1}^{\infty}$ is a Cauchy sequence. For convenience, put $m_i=m(x_i), i=0,1,2,\ldots.$ Then, by (2.1), for all $t>0$, we have

$$\begin{aligned}
M(x_n,x_{n+p},t) &= M(T^{m_{n-1}}x_{n-1},T^{m_{n+p-1}+m_{n+p-2}+\ldots+m_{n-1}}x_{n-1},t)\\
&\ge M(x_{n-1},T^{m_{n+p-1}+\ldots+m_n}x_{n-1},\phi(t))\\
&\ge M(x_{n-2},T^{m_{n+p-1}+\ldots+m_n}x_{n-2},\phi^2(t))\\
&\ge \ldots \ge M(x_0,T^{m_{n+p-1}+\ldots+m_n}x_0,\phi^n(t))\\
&\ge \inf_{x\in O_T(x_0)} M(x_0,x,\phi^n(t))\\
&\ge \sup_{0<s<\phi^n(t)}\inf_{x\in O_T(x_0)} M(x_0,x,s),
\end{aligned}$$

$$\begin{aligned}
N(x_n,x_{n+p},t) &= N(T^{m_{n-1}}x_{n-1},T^{m_{n+p-1}+m_{n+p-2}+\ldots+m_{n-1}}x_{n-1},t)\\
&\le N(x_{n-1},T^{m_{n+p-1}+\ldots+m_n}x_{n-1},\phi(t))\\
&\le N(x_{n-2},T^{m_{n+p-1}+\ldots+m_n}x_{n-2},\phi^2(t))\\
&\le \ldots \le N(x_0,T^{m_{n+p-1}+\ldots+m_n}x_0,\phi^n(t))\\
&\le \sup_{x\in O_T(x_0)} N(x_0,x,\phi^n(t))\\
&\le \inf_{0<s<\phi^n(t)}\sup_{x\in O_T(x_0)} N(x_0,x,s).
\end{aligned}$$

Then, by condition $(\phi_1)$ and (2.3)

$$\lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1 \text{ and } \lim_{n\to\infty} N(x_n, x_{n+p}, t) = 0$$

for all $t > 0$. This means that $\{x_n\}$ is a Cauchy sequence in $X$. By the completeness of $X$, there exists $\lim_n x_n = x_* \in X$. Now, we prove that $x_*$ is the unique fixed point of $T^{m_*}$, where $m_* = m(x_*)$. By (v) and (xi) in Definition 1.3, and (2.2), we have

$$\begin{aligned} M(x_*, T^{m_*}x_*, t) &\geq M(x_*, T^{m_*}x_n, \frac{t}{2}) * M(T^{m_*}x_n, T^{m_*}x_*, \frac{t}{2}) \\ &\geq M(x_*, T^{m_*}x_n, \frac{t}{2}) * M(x_n, T^{m_*}x_*, \phi(\frac{t}{2})), \end{aligned} \tag{2.6}$$

$$\begin{aligned} N(x_*, T^{m_*}x_*, t) &\leq N(x_*, T^{m_*}x_n, \frac{t}{2}) \Diamond N(T^{m_*}x_n, T^{m_*}x_*, \frac{t}{2}) \\ &\leq N(x_*, T^{m_*}x_n, \frac{t}{2}) \Diamond N(x_n, T^{m_*}x_*, \phi(\frac{t}{2})). \end{aligned} \tag{2.7}$$

It is easy to prove that

$$\lim_{n\to\infty} M(x_*, T^{m_*}x_n, u) = 1 \text{ and } \lim_{n\to\infty} N(x_*, T^{m_*}x_n, u) = 0$$

for all $u > 0$. In fact,

$$\begin{aligned} M(x_*, T^{m_*}x_n, u) &\geq M(x_*, x_n, \frac{u}{2}) * M(x_n, T^{m_*}x_n, \frac{u}{2}) \\ &= M(x_*, x_n, \frac{u}{2}) * M(T^{m_{n-1}}x_{n-1}, T^{m_{n-1}+m_*}x_{n-1}, \frac{u}{2}) \\ &\geq M(x_*, x_n, \frac{u}{2}) * M(x_{n-1}, T^{m_*}x_{n-1}, \phi(\frac{u}{2})) \\ &\geq ... \geq M(x_*, x_n, \frac{u}{2}) * M(x_0, T^{m_*}x_0, \phi^n(\frac{u}{2})) \to 1, \end{aligned}$$

$$\begin{aligned} N(x_*, T^{m_*}x_n, u) &\leq N(x_*, x_n, \frac{u}{2}) \Diamond N(x_n, T^{m_*}x_n, \frac{u}{2}) \\ &= N(x_*, x_n, \frac{u}{2}) \Diamond N(T^{m_{n-1}}x_{n-1}, T^{m_{n-1}+m_*}x_{n-1}, \frac{u}{2}) \\ &\leq N(x_*, x_n, \frac{u}{2}) \Diamond N(x_{n-1}, T^{m_*}x_{n-1}, \phi(\frac{u}{2})) \\ &\leq ... \leq N(x_*, x_n, \frac{u}{2}) \Diamond N(x_0, T^{m_*}x_0, \phi^n(\frac{u}{2})) \to 0. \end{aligned}$$

Thus, letting $n \to \infty$ on the right sides of (2.6) and (2.7), and noting the continuity of $*$ and $\Diamond$, we have

$$M(x_*, T^{m_*}x_*, t) = 1 \text{ and } N(x_*, T^{m_*}x_*, t) = 0$$

for all $t > 0$. This implies that $T^{m_*}x_* = x_*$, i.e. $x_*$ is a fixed point of $T^{m(x_*)}$. To show uniqueness, assume that $T^{m(x_*)}y = y$ for some $y \in X$. Then

$$\begin{aligned} M(x_*, y, t) &= M(T^{m(x_*)}x_*, T^{m(x_*)}y, t) \geq M(x_*, y, \phi(t)), \\ N(x_*, y, t) &= N(T^{m(x_*)}x_*, T^{m(x_*)}y, t) \leq N(x_*, y, \phi(t)). \end{aligned}$$

On the other hand, as $M(x_*, y, t)$ is non-decreasing and $N(x_*, y, t)$ is non-increasing, we have $M(x_*, y, t) \leq M(x_*, y, \phi(t))$ and $N(x_*, y, t) \geq N(x_*, y, \phi(t))$, respectively. Thus

$$\begin{aligned} M(x_*, y, t) &= M(x_*, y, \phi(t)) = M(x_*, y, \phi^n(t)), \\ N(x_*, y, t) &= N(x_*, y, \phi(t)) = N(x_*, y, \phi^n(t)) \end{aligned}$$

for all $t > 0$. Hence, by condition $(\phi_1)$, and (vii) and (xiii) in Definition 1.3, we have

$$M(x_*, y, t) = 1 \text{ and } N(x_*, y, t) = 0,$$

i.e. $x_* = y$. Finally, we prove $x_*$ is the unique fixed point of $T$, too. In fact, since $T^{m(x_*)}x_* = x_*$, it follows that $Tx_* = T(T^{m_*}x_*) = T^{m_*}(Tx_*)$. Hence, $Tx_* = x_*$. Uniqueness is obvious. This completes the proof. ■

From Theorem 2.4, we can obtain the following consequence immediately.

**Corollary 2.5** *Let $(X, M, N, *, \Diamond)$ be a complete intuitionistic fuzzy metric space. Let $T : X \to X$ be a mapping satisfying the following conditions:*

(i) there exists $x_0 \in X$ such that

$$\lim_{n\to\infty} M(x_0, T^i x_0, t) = 1, \lim_{n\to\infty} N(x_0, T^i x_0, t) = 0, i = 1, 2, ...;$$

(ii) there exists a mapping $m : X \to \mathbb{N}$ such that for any $x, y \in X$

$$M(T^{m(x)}x, T^{m(x)}y, t) \geq M(x, y, t/k), N(T^{m(x)}x, T^{m(x)}y, t) \leq N(x, y, t/k)$$

where $0 < k < 1$.

Then the conclusion of Theorem 2.4 remains true.

**Proof.** Taking $\phi(t) = t/k$. Obviously, $\phi(t)$ satisfies the conditions $(\phi_1)$ and $(\phi_2)$. Theorefore the conclusion follows from Theorem 2.4 directly. ■

**Corollary 2.6** *Let $(X, M, N, *, \Diamond)$ be a complete intuitionistic fuzzy metric space. Let $T : X \to X$ be a mapping. If there exists a mapping $m : X \to \mathbb{N}$ such that for any $x, y \in X$,*

$$M(T^{m(x)}x, T^{m(x)}y, t) \geq M(x, y, \phi(t)), N(T^{m(x)}x, T^{m(x)}y, t) \leq N(x, y, \phi(t)),$$

*where the function $\phi(t)$ satisfies the conditions $(\phi_1)$ and $(\phi_2)$. Then $T$ has a unique fixed point $x_*$ and the iterative sequence $\{T^n x\}$ converges to $x_*$ for every $x \in X$.*

**Proof.** By Theorem 2.4, we need only to show that the iterative sequence $\{T^n x\}$ converges to $x_*$. For $n \in \mathbb{N}$ with $n > m(x_*)$,

$$n = km(x_*) + s, 0 \leq s < m(x_*).$$

Since

$$\begin{aligned} M(x_*, T^n x, t) &= M(T^{m(x_*)}x_*, T^{km(x_*)+s}x, t) \\ &\geq M(x_*, T^{(k-1)m(x_*)+s}x, \phi(t)) \\ &\geq ... \geq M(x_*, T^s x, \phi^k(t)) \to 1, \end{aligned}$$

$$\begin{aligned} N(x_*, T^n x, t) &= N(T^{m(x_*)} x_*, T^{km(x_*)+s} x, t) \\ &\leq N(x_*, T^{(k-1)m(x_*)+s} x, \phi(t)) \\ &\leq \ldots \leq N(x_*, T^s x, \phi^k(t)) \to 0 \end{aligned}$$

so $\lim_n M(x_*, T^n x, t) = 1$ and $\lim_n N(x_*, T^n x, t) = 0$ for all $t > 0$, i.e. $T^n x \to x_*$. ■

**Remark 2.7** *Taking $\phi(t) = t/k$ $(0 < k < 1)$ and $m(x) = 1$ in Corollary 2.6, we at once obtain Theorem 1.1. Hence Theorem 1.1 is a special case of Corollary 2.6. In the view of Remark 1.5, taking $\phi(t) = t/k$, $N = 1 - M$, and $* = \min$ and $\Diamond = \max$ (i.e. $a * b = \min(a, b)$ and $a \Diamond b = \max(a, b)$) in Corollary 2.6, we obtain that the main results of [5-8] are special cases of Corollary 2.6.*

**Theorem 2.8** *Let $(X, M, N, *, \Diamond)$ be a compact intuitionistic fuzzy metric space. Let $T : X \to X$ be a continuous mapping satisfying*

$$M(Tx, Ty, .) > \min\{M(x, Tx, .), M(y, Ty, .), M(x, y, .)\}, \quad (2.8)$$

$$N(Tx, Ty, .) < \max\{N(x, Tx, .), N(y, Ty, .), N(x, y, .)\} \quad (2.9)$$

*for all $x \neq y$. If there exists $x_0 \in X$ such that $\{T^n x_0\}_{n=0}^{\infty}$ has an accumulation point $x_* \in X$ and for all $t > 0, n = 1, 2, ...$*

$$\begin{aligned} M(T^{n-1}x_0, T^n x_0, t) &\leq M(T^n x_0, T^{n+1} x_0, t), \\ N(T^{n-1}x_0, T^n x_0, t) &\geq N(T^n x_0, T^{n+1} x_0, t) \end{aligned}$$

*then $x_*$ is the unique fixed point of $T$ and $\lim_n T^n x_0 = x_*$.*

**Proof.** Assume $T^n x_0 \neq T^{n+1} x_0$ for each $n \in \mathbb{N}$. (If not, there is $n_0 \in \mathbb{N}$ such that $T^{n_0} x_0 = T^{n_0+1} x_0$. This means that $x_* = T^{n_0} x_0$ is a fixed point of $T$ and $\lim_n T^n x_0 = x_*$). Since $\{T^n x_0\}_{n=0}^{\infty}$ has an accumulation point $x_* \in X$, there exists a subsequence $\{T^{n_i} x_0\}$, $\lim_i T^{n_i} x_0 = x_*$. $\{M(T^n x_0, T^{n+1} x_0, t)\}$ and $\{N(T^n x_0, T^{n+1} x_0, t)\}$ are non-decreasing and non-increasing, respectively, and also bounded, thus $\{M(T^{n_i} x_0, T^{n_i+1} x_0, t)\}$, $\{M(T^{n_i+1} x_0, T^{n_i+2} x_0, t)\}$ and $\{N(T^{n_i} x_0, T^{n_i+1} x_0, t)\}$, $\{N(T^{n_i+1} x_0, T^{n_i+2} x_0, t)\}$ are convergent to a common limit, i.e.

$$\begin{aligned} \lim_{i\to\infty} M(T^{n_i} x_0, T^{n_i+1} x_0, t) &= \lim_{i\to\infty} M(T^{n_i+1} x_0, T^{n_i+2} x_0, t), \\ \lim_{i\to\infty} N(T^{n_i} x_0, T^{n_i+1} x_0, t) &= \lim_{i\to\infty} N(T^{n_i+1} x_0, T^{n_i+2} x_0, t) \end{aligned}$$

for all $t > 0$. By the continuity of $T$, we have

$$\lim_{i\to\infty} T^{n_i+1} x_0 = \lim_{i\to\infty} T(T^{n_i} x_0) = Tx_*.$$

Suppose $x_* \neq Tx_*$. Putting $y = Tx$ in (2.8) and (2.9), we have

$$M(x, Tx, .) < M(Tx, T^2 x, .) \text{ and } N(x, Tx, .) > N(Tx, T^2 x, .)$$

for every $x \neq Tx$. So, by Lemma 2.3, there exists a continuous point $t_0$ of $M(x_*, Tx_*, .)$ and $N(x_*, Tx_*, .)$ such that $M(Tx_*, T^2 x_*, t_0) > M(x_*, Tx_*, t_0)$ and $N(Tx_*, T^2 x_*, t_0) < N(x_*, Tx_*, t_0)$. On the other hand, by Lemma 1.7

$$\begin{aligned} M(x_*, Tx_*, t_0) &= \lim_{i\to\infty} M(T^{n_i} x_0, T(T^{n_i} x_0), t_0) \\ &= \lim_{i\to\infty} M(T^{n_i+1} x_0, T^{n_i+2} x_0, t_0) \\ &\geq M(Tx_*, T^2 x_*, t_0), \end{aligned}$$

$$\begin{aligned} N(x_*, Tx_*, t_0) &= \lim_{i\to\infty} N(T^{n_i}x_0, T(T^{n_i}x_0), t_0) \\ &= \lim_{i\to\infty} N(T^{n_i+1}x_0, T^{n_i+2}x_0, t_0) \\ &\leq N(Tx_*, T^2x_*, t_0) \end{aligned}$$

are contradictions. Therefore $x_* = Tx_*$, i.e. $x_*$ is a fixed point of $T$. Uniqueness follows from (2.8) and (2.9). Finally, we prove that $\lim_n T^n x_0 = x_*$. Since $\lim_i T^{n_i}x_0 = x_*$ and $\lim_i T^{n_i+1}x_0 = Tx_* = x_*$, by Lemma 1.7,

$$\begin{aligned} \lim_{i\to\infty} \inf M(T^{n_i}x_0, T^{n_i+1}x_0, t) &\geq M(x_*, x_*, t) = 1, \\ \lim_{i\to\infty} \sup N(T^{n_i}x_0, T^{n_i+1}x_0, t) &\leq N(x_*, x_*, t) = 0 \end{aligned}$$

for all $t > 0$. So, $\lim_i M(T^{n_i}x_0, T^{n_i+1}x_0, t) = 1$ and $\lim_i N(T^{n_i}x_0, T^{n_i+1}x_0, t) = 0$ for all $t > 0$. For any $n \in \mathbb{N}$ with $n > n_1$, there exists $n_i$ with $n_{i+1} \geq n > n_i$. By (2.8) and (2.9), we have

$$\begin{aligned} M(T^n x_0, x_*, t) &\geq \min\left\{M(T^{n-1}x_0, T^n x_0, t), 1, M(T^{n-1}x_0, x_*, t)\right\} \\ &\geq \min\left\{\begin{array}{c} M(T^{n-1}x_0, T^n x_0, t), M(T^{n-2}x_0, T^{n-1}x_0, t) \\ , M(T^{n-2}x_0, x_*, t) \end{array}\right\} \\ &= \min\left\{M(T^{n-2}x_0, T^{n-1}x_0, t), M(T^{n-2}x_0, x_*, t)\right\} \\ &\geq \ldots \geq \min\left\{M(T^{n_i}x_0, T^{n_i+1}x_0, t), M(T^{n_i}x_0, x_*, t)\right\}, \end{aligned}$$

$$\begin{aligned} N(T^n x_0, x_*, t) &\leq \max\left\{N(T^{n-1}x_0, T^n x_0, t), 0, N(T^{n-1}x_0, x_*, t)\right\} \\ &\leq \max\left\{\begin{array}{c} N(T^{n-1}x_0, T^n x_0, t), N(T^{n-2}x_0, T^{n-1}x_0, t) \\ , N(T^{n-2}x_0, x_*, t) \end{array}\right\} \\ &= \max\left\{N(T^{n-2}x_0, T^{n-1}x_0, t), N(T^{n-2}x_0, x_*, t)\right\} \\ &\leq \ldots \leq \max\left\{N(T^{n_i}x_0, T^{n_i+1}x_0, t), N(T^{n_i}x_0, x_*, t)\right\}. \end{aligned}$$

Letting $n \to \infty$ ($n_i \to \infty$), we have

$$\lim_{n\to\infty} M(T^n x_0, x_*, t) \geq 1 \text{ and } \lim_{n\to\infty} N(T^n x_0, x_*, t) \leq 0$$

for all $t > 0$. Hence, $\lim_n T^n x_0 = x_*$. ■

**Remark 2.9** *Theorem 1.2 is the immediate consequence of Theorem 2.8. In fact, by Theorem 1.2, it is easy to see that $T$ is continuous and (2.8), (2.9) hold for any $x_0 \in X$. In addition, by the compactness of $X$, $\{T^n x_0\}$ has an accumulation point. Hence Theorem 1.2 follows immediately from Theorem 2.8.*

# References

[1] Banach S. Theorie des operations lineaires. Warsaw: Monografje Mathematyczne; 1932.

[2] Chauhan S, Kutukcu S, Dhiman N, Kumar S. Variants of R-weakly commuting mappings and common fixed point theorems in intuitionistic fuzzy metric spaces. Kuwait J Sci 2014;41:49-64.

[3] Edelstein M. On fixed and periodic points under contractive mapping. J London Math Soc 1962;37:74-9.

[4] Fang J. A fixed point theorem of $\phi$-contraction mapping on probabilistic metric spaces. J Xinjiang Univ 1988;5(3):21-8.

[5] Fang J. On fixed point theorems in fuzzy metric spaces. Fuzzy Sets Syst 1992;46:107-13.

[6] Grabiec M. Fixed point in fuzzy metric spaces. Fuzzy Sets Syst 1988;27:385-9.

[7] Istratescu I. A fixed point theorem for mappings with a probabilistic contractive iterate. Rev Roumaine Math Pures Appl 1981;26:431-5.

[8] Sehgal VM, Bharucha-Reid AT. Fixed points of contraction mappings on probabilistic metric spaces. Math Systems Theory 1972;6:97-102.

# Existence and uniqueness of positive solutions for singular higher order fractional differential equations with infinite-point boundary value conditions*

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**Abstract**

In this paper, we consider the singular problem for higher order fractional differential equations with infinite-point boundary value conditions. First, we get some properties of the Green function, then by a monotone iterative technique, we establish the existence and uniqueness of the positive solutions for singular higher order fractional differential equations with infinite-point boundary value conditions. Moreover the iterative sequences of positive solution and error estimation are also given. In the end, an example is given to illustrate one of the main results.

**Keywords:** fractional differential equation, positive solution, monotone iterative technique, Green function, existence and uniqueness,completely continuous operator

## 1 Introduction

In this paper, we consider the following singular problem for a higher order fractional differential equation with infinite-point boundary value conditions.

$$\begin{cases} -\mathcal{D}^{\alpha}x(t) = q(t)f(x(t),\mathcal{D}^{\mu_1}x(t),\mathcal{D}^{\mu_2}x(t),\cdots,\mathcal{D}^{\mu_{n-1}}x(t)), \quad t\in(0,1) \\ x(0) = \mathcal{D}^{\mu_i}x(0) = 0, \mathcal{D}^{\mu}x(1) = \sum\limits_{j=1}^{\infty}\alpha_j\mathcal{D}^{\mu}x(\xi_j), \quad 1\le i\le n-1, \end{cases} \tag{1.1}$$

where $n\ge 3$, $n-1<\alpha<n$, $n-k-1<\alpha-\mu_k<n-k$, for $k=1,2,\cdots,n-2$, $\mu-\mu_{n-1}>0$, $\alpha-\mu_{n-1}\le 2$, $\alpha-\mu>1$, $\alpha_j\ge 0$, $0<\xi_1<\xi_2<\cdots<\xi_{j-1}<\xi_j<\cdots<1(j=1,2,\ldots)$, $0<\sum\limits_{j=1}^{\infty}\alpha_j\xi_j^{\alpha-\mu-1}<1$, $\mathcal{D}^{\alpha}$ is the standard Riemann-Liouville derivative. Recently, fractional differential equations have been applied in variety of different areas such as chemical physics, engineering, electrical networks, mechanics, see [1, 3–10] and references cited therein for details. Zhang et.al [11], by establishing eigenvalue interval for the existence of positive solutions from Schauder's fixed point theorem and the upper and lower solutions, obtained a multiple positive solution of the following fractional differential equation.

$$\begin{cases} -\mathcal{D}^{\alpha}x(t) = \lambda f(x(t),\mathcal{D}^{\mu_1}x(t),\mathcal{D}^{\mu_2}x(t),\cdots,\mathcal{D}^{\mu_{n-1}}x(t)), t\in(0,1) \\ x(0) = \mathcal{D}^{\mu_i}x(0) = 0, \mathcal{D}^{\mu}x(1) = \sum\limits_{j=1}^{p-2}\alpha_j\mathcal{D}^{\mu}x(\xi_j), 1\le i\le n-1, \end{cases} \tag{1.2}$$

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where $n \geq 3$, $n-1 < \alpha < n$, $n-k-1 < \alpha - \mu_k < n-k$, for $k = 1, 2, \cdots, n-2$, $\mu - \mu_{n-1} > 0$, $\alpha - \mu_{n-1} \leq 2$, $\alpha - \mu > 1$, $\alpha_j \geq 0$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{p-3} < \xi_{p-2} < 1 (j = 1, 2, \ldots, p-2)$, $0 < \sum_{j=1}^{p-2} \alpha_j \xi_j^{\alpha-\mu-1} < 1$, $\mathcal{D}^\alpha$ is the standard Riemann-Liouville derivative. In [12], Zhang and Han concerned the following singular nonlinear $(n-1, 1)$ conjugate-type fractional differential equation with one nonlocal term

$$\begin{cases} -\mathcal{D}^\alpha x(t) = f(t, x(t)), t \in (0,1), \alpha \in (n-1, n], \\ x^k(0) = 0, 0 \leq k \leq n-2, x(1) = \int_0^1 x(s) dA(s), \end{cases} \tag{1.3}$$

where $\alpha \geq 2$, $\mathcal{D}^\alpha$ is the standard Riemann-Liouville derivative,$A$ is a function of bounded variation and $\int_0^1 x(s)dA(s)$ denotes the Riemann-Stieltjes integral of $x$ with respect to $A$, $dA$ is a signed measure. By using a monotone iterative technique, Zhang and Han [12] established the existence and uniqueness of this positive solution.

Motivated by the above articles, we study the existence and uniqueness of positive solution of fractional differential equation (1.1). By means of a monotone iterative technique, we obtain the existence and uniqueness of the positive solutions of (1.1).

In the rest of this article, we suppose that following assumptions hold:

(L1) $q(t) : (0,1) \to [0, +\infty)$ is continuous and does not vanish identically of $(0,1)$ and

$$0 < \int_0^1 q(s) ds < +\infty;$$

(L2) $f : (0, +\infty)^n \to [0, +\infty)$ is continuous and is nondecreasing at $x_i > 0$ for $i = 1, 2, \cdots, n$;

(L3) for all $r \in (0,1)$, there exists a constant $0 < \lambda < 1$ such that for each $(x_1, x_2, \cdots, x_n) \in (0, +\infty)^n$,

$$f(rx_1, rx_2, \cdots, rx_n) \geq r^\lambda f(x_1, x_2, \cdots, x_n).$$

**Remark 1.1.** *If (L3) holds, then for $r \geq 1$, there exists a constant $0 < \lambda < 1$ such that for each $(x_1, x_2, \cdots, x_n) \in (0, +\infty)^n$,*

$$f(rx_1, rx_2, \cdots, rx_n) \leq r^\lambda f(x_1, x_2, \cdots, x_n).$$

## 2 Preliminaries

For the convenience of the reader, we present the necessary definitions and lemmas from the fractional calculus theory. These definitions and lemmas can be found in monograph [3, 5, 6, 8, 10, 11].

**Definition 2.1.** *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, +\infty) \to \mathbb{R}$ is given by*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{(\alpha-1)} f(s) ds,$$

*provided that the right-hand side is point wise defined on $(0, +\infty)$.*

**Definition 2.2.** *The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, +\infty) \to \mathbb{R}$ is given by*

$$\mathcal{D}^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} (\frac{d}{dt})^n \int_0^t (t-s)^{(n-\alpha-1)} f(s) ds,$$

*where $n-1 \leq \alpha < n$, provided that the right-hand side is point wise defined on $(0, +\infty)$.*

**Lemma 2.1.** *(1) If $x \in C(0,1)\bigcap L(0,1)$, $\nu > \sigma > 0$, then*

$$I^{\nu}I^{\sigma}x(t) = I^{\nu+\sigma}x(t), \mathcal{D}^{\sigma}I^{\nu}x(t) = I^{\nu-\sigma}x(t), \mathcal{D}^{\sigma}I^{\sigma}x(t) = x(t).$$

*(2)If $\nu > 0$,$\sigma > 0$, then*

$$\mathcal{D}^{\nu}t^{\sigma-1} = \frac{\Gamma(\sigma)}{\Gamma(\sigma-\nu)}t^{\sigma-\nu-1}.$$

**Lemma 2.2.** *Assume that $x \in C(0,1)\bigcap L(0,1)$, with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1)\bigcap L^1(0,1)$. Then*

$$I^{\alpha}\mathcal{D}^{\alpha}x(t) = x(t) + c_1t^{\alpha-1} + c_1t^{\alpha-2} + \cdots + c_nt^{\alpha-n},$$

*where $c_i \in \mathbb{R}(i = 1, 2, \cdots, n)$, $n$ is the smallest integer greater than or equal to $\alpha$.*

**Remark 2.1.** *There are many kinds of functions satisfying conditions $(L2)$ and $(L3)$. In fact, let*
*$K = \{f(x_1, x_2, \cdots, x_n) : f(x_1, x_2, \cdots, x_n)$ satisfies conditions (L2) and (L3)$\}$*
*and $a_i(t)$ be nonnegative continuous on $(0, +\infty)$, $(i = 1, 2, \cdots, n)$.*
*Then it is easily verified directly that the following facts hold:*
*(1) $\sum_{1}^{n} a_i(t)x_i^{b_i} \in K$, where $b_i \in (0,1)$ are constants, $i = 1, 2, \cdots, n$.*
*(2)$\sum_{1}^{n}[a_i(t)x_i^{\mu_i}]^{\frac{1}{\mu}} \in K$ $(i = 1, 2, \cdots, n)$ and $\mu \geq \max_{1\leq i\leq n}\{\mu_i\}$,*
*(3)If $f(x_1, x_2, \cdots, x_n) \in K$ then $a_i(t)f(x_1, x_2, \cdots, x_n) \in K(i = 1, 2, \cdots, n)$.*
*(4)If $f_i(x_1, x_2, \cdots, x_n) \in K$ $(i = 1, 2, \cdots, n)$, then*

$$\max_{1\leq i\leq n}\{f_i(x_1, x_2, \cdots, x_n)\} \in K, \min_{1\leq i\leq n}\{f_i(x_1, x_2, \cdots, x_n)\} \in K$$

*and*

$$\max_{1\leq i\leq n}\{f_i(x_1, x_2, \cdots, x_n)\} + \min_{1\leq i\leq n}\{f_i(x_1, x_2, \cdots, x_n)\} \in K.$$

Let

$$G_1(t,s) = \begin{cases} \dfrac{t^{\alpha-\mu_{n-1}-1}(1-s)^{\alpha-\mu-1} - (t-s)^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})}, 0 \leq s \leq t \leq 1 \\ \dfrac{t^{\alpha-\mu_{n-1}-1}(1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu_{n-1})}, 0 \leq t \leq s \leq 1 \end{cases} \tag{2.1}$$

and

$$G_2(t,s) = \begin{cases} \dfrac{t^{\alpha-\mu-1}(1-s)^{\alpha-\mu-1} - (t-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu_{n-1})}, 0 \leq s \leq t \leq 1 \\ \dfrac{t^{\alpha-\mu-1}(1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu_{n-1})}, 0 \leq t \leq s \leq 1. \end{cases} \tag{2.2}$$

Obviously for $t, s \in [0,1]$, we have

$$0 \leq G_1(t,s) \leq \frac{t^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})}, \tag{2.3}$$

$$0 \leq \frac{t^{\alpha-\mu-1}(1-t)s(1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu_{n-1})} \leq G_2(t,s) \leq \frac{t^{\alpha-\mu-1}}{\Gamma(\alpha-\mu_{n-1})}. \tag{2.4}$$

**Lemma 2.3.** *If $h(t) \in L^1(0,1)$ then the boundary value problem*

$$\begin{cases} -\mathcal{D}^{\alpha-\mu_{n-1}} w(t) = h(t), t \in (0,1) \\ w(0) = 0, \mathcal{D}^{\alpha-\mu_{n-1}} w(1) = \sum_{j=1}^{\infty} \alpha_j \mathcal{D}^{\alpha-\mu_{n-1}} w(\xi_j) \end{cases} \tag{2.5}$$

*has the unique solution*

$$w(t) = \int_0^1 G(t,s)h(s)ds,$$

*where*

$$G(t,s) = G_1(t,s) + \frac{t^{\alpha-\mu_{n-1}-1}}{1 - \sum_{j=1}^{\infty} \alpha_j \xi_j^{\alpha-\mu-1}} \sum_{j=1}^{\infty} \alpha_j G_2(\xi_j, s) \tag{2.6}$$

*is the Green function of the boundary value problem (2.5)*

*Proof.* The proof is similar to Lemma 2.3 of [11] if replacing $\sum_{j=1}^{p-2} \alpha_j \mathcal{D}^{\alpha-\mu_{n-1}} w(\xi_j)$ with $\sum_{j=1}^{\infty} \alpha_j \mathcal{D}^{\alpha-\mu_{n-1}} w(\xi_j)$, so we omit the details. □

**Lemma 2.4.** *The function $G(t,s)$ has the following properties;*
*(1) $G(t,s) > 0$, for $t, s \in (0,1)$;*
*(2) $ct^{\alpha-\mu_{n-1}-1}(1-s)^{\alpha-\mu-1}s \le G(t,s) \le dt^{\alpha-\mu_{n-1}-1}$, where*

$$c = \frac{\sum_{j=1}^{\infty} \alpha_j \xi_j^{\alpha-\mu-1}(1-\xi_j)}{(1 - \sum_{j=1}^{\infty} \alpha_j \xi_j^{\alpha-\mu-1})\Gamma(\alpha-\mu_{n-1})}, \quad d = \frac{1}{(1 - \sum_{j=1}^{\infty} \alpha_j \xi_j^{\alpha-\mu-1})\Gamma(\alpha-\mu_{n-1})}.$$

*Proof.* The first result is obvious so we only prove the second one.

From (2.3), (2.4), (2.6) we have

$$\begin{aligned} G(t,s) &\ge \frac{t^{\alpha-\mu_{n-1}-1}}{1 - \sum_{j=1}^{\infty} \alpha_j \xi_j^{\alpha-\mu-1}} \sum_{j=1}^{\infty} \alpha_j G_2(\xi_j, s) \\ &\ge t^{\alpha-\mu_{n-1}-1} \frac{\sum_{j=1}^{\infty} \alpha_j \xi_j^{\alpha-\mu-1}(1-\xi_j)(1-s)^{\alpha-\mu-1}s}{(1 - \sum_{j=1}^{\infty} \alpha_j \xi_j^{\alpha-\mu-1})\Gamma(\alpha-\mu_{n-1})} \\ &= ct^{\alpha-\mu_{n-1}-1}(1-s)^{\alpha-\mu-1}s \end{aligned}$$

and

$$\begin{aligned} G(t,s) &= G_1(t,s) + \frac{t^{\alpha-\mu_{n-1}-1}}{1-\sum_{j=1}^{\infty}\alpha_j\xi_j^{\alpha-\mu-1}}\sum_{j=1}^{\infty}\alpha_j G_2(\xi_j,s) \\ &\leq \frac{t^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})} + \frac{t^{\alpha-\mu_{n-1}-1}}{1-\sum_{j=1}^{\infty}\alpha_j\xi_j^{\alpha-\mu-1}}\sum_{j=1}^{\infty}\alpha_j G_2(\xi_j,s) \\ &\leq \frac{t^{\alpha-\mu_{n-1}-1}}{(1-\sum_{j=1}^{\infty}\alpha_j\xi_j^{\alpha-\mu-1})\Gamma(\alpha-\mu_{n-1})} = dt^{\alpha-\mu_{n-1}-1}. \end{aligned}$$

□

For the convenience of expression in the rest of the paper, we let $\mu_0 = 0$, $E = C[0,1]$. Now let us consider the following modified problem of (1.1):

$$\begin{cases} -\mathcal{D}^{\alpha-\mu_{n-1}}v(t) = q(t)f(I^{\mu_{n-1}-\mu_0}v(t), I^{\mu_{n-1}-\mu_1}v(t), \cdots, I^{\mu_{n-1}-\mu_{n-2}}v(t), v(t)), t \in (0,1) \\ v(0) = 0, \mathcal{D}^{\mu-\mu_{n-1}}v(1) = \sum_{j=1}^{\infty}\alpha_j\mathcal{D}^{\mu-\mu_{n-1}}v(\xi_j), \end{cases} \tag{2.7}$$

**Lemma 2.5.** *Let $x(t) = I^{\mu_{n-1}-\mu_0}v(t)$, $v(t) \in E$. Then we can transform (1.1) into (2.7). Moreover, if $v \in C[(0,1),(0,+\infty)]$ is a solution of (2.7), then the function $x(t) = I^{\mu_{n-1}-\mu_0}v(t)$ is a positive solution of (1.1).*

*Proof.* The proof is similar to Lemma 2.5 of [11], so we omit the details. □

Let $e_{n-1}(t) = t^{\alpha-\mu_{n-1}-1}$, and for $i = 0, 1, 2, \cdots, n-2$, define

$$e_i(t) = I^{\mu_{n-1}-\mu_i}e_{n-1}(t) = \frac{\Gamma(\alpha-\mu_{n-1})}{\Gamma(\alpha-\mu_i)}t^{\alpha-\mu_i-1}.$$

Let $P = \{v \in E : v(t) \geq 0, t \in [0,1]\}$. Clearly $P$ is a normal cone in $E$. Now let us define a sub-cone of $P$ as follows:

$D = \{v(t) \in P$: there exist two positive numbers $L_v \geq l_v$ such that $l_v e_{n-1}(t) \leq v(t) \leq L_v e_{n-1}(t)\}$. Obviously, $D$ is a nonempty set since $e_{n-1}(t) \in P$.

**Lemma 2.6.** *Let the operator $T : D \to E$ be defined by*

$$(Tv)(t) = \int_0^1 G(t,s)q(s)f(I^{\mu_{n-1}-\mu_0}v(s), I^{\mu_{n-1}-\mu_1}v(s), \cdots, I^{\mu_{n-1}-\mu_{n-2}}v(s), v(s))ds,$$

*and $0 < \int_0^1 q(s)f(e_0(s), e_1(s), \cdots, e_{n-2}(s), e_{n-1}(s))ds < +\infty$, then any fixed point of $T$ is a solution of (2.7), and $T$ is a well defined completely continuous operator and $T : D \to D$.*

*Proof.* For each $v \in D$, there exist two positive numbers $L_v > 1 > l_v$ such that

$$l_v e_{n-1}(t) \leq v(t) \leq L_v e_{n-1}(t), t \in [0,1]. \tag{2.8}$$

According to (2.8), (L2), (L3), Lemma 2.1(2) and Remark 1.1, we have

$$\begin{aligned}(Tv)(t) &\leq \int_0^1 G(t,s)q(s)f(I^{\mu_{n-1}-\mu_0}L_v e_{n-1}(s), I^{\mu_{n-1}-\mu_1}L_v e_{n-1}(s), \cdots, I^{\mu_{n-1}-\mu_{n-2}}L_v e_{n-1}(s), L_v e_{n-1}(s))ds \\ &\leq L_v^\lambda \int_0^1 G(t,s)q(s)f(I^{\mu_{n-1}-\mu_0}e_{n-1}(s), I^{\mu_{n-1}-\mu_1}e_{n-1}(s), \cdots, I^{\mu_{n-1}-\mu_{n-2}}e_{n-1}(s), e_{n-1}(s))ds \\ &\leq dL_v^\lambda \int_0^1 q(s)f(e_0(s), e_1(s), \cdots, e_{n-2}(s), e_{n-1}(s))ds e_{n-1}(t) < +\infty,\end{aligned}$$

and

$$\begin{aligned}(Tv)(t) &\geq \int_0^1 G(t,s)q(s)f(I^{\mu_{n-1}-\mu_0}l_v e_{n-1}(s), I^{\mu_{n-1}-\mu_1}l_v e_{n-1}(s), \cdots, I^{\mu_{n-1}-\mu_{n-2}}l_v e_{n-1}(s), l_v e_{n-1}(s))ds \\ &\geq l_v^\lambda \int_0^1 G(t,s)q(s)f(I^{\mu_{n-1}-\mu_0}e_{n-1}(s), I^{\mu_{n-1}-\mu_1}e_{n-1}(s), \cdots, I^{\mu_{n-1}-\mu_{n-2}}e_{n-1}(s), e_{n-1}(s))ds \\ &\geq cl_v^\lambda \int_0^1 s(1-s)^{\alpha-\mu-1}q(s)f(e_0(s), e_1(s), \cdots, e_{n-2}(s), e_{n-1}(s))ds e_{n-1}(t).\end{aligned}$$

From above it is not difficulty to prove that $T : D \to D$ is a completely continuous operator. □

# 3 Main results

**Theorem 3.1.** *Suppose that (L1)-(L3) hold and*

$$0 < \int_0^1 q(s)f(e_0(s), e_1(s), \cdots, e_{n-2}(s), e_{n-1}(s))ds < +\infty.$$

*Then the equation (2.7) has a unique positive solution $\nu^*$ in $D$, and for each initial value $\nu_0 \in D$, the iterative sequence $\nu_n = T\nu_{n-1}(n = 1, 2, \cdots)$ converges to $\nu^*$ as $n \to +\infty$. Meanwhile $I^{\mu_{n-1}-\mu_0}v^*$ is the unique solution of (1.1). Furthermore we have the following error estimation*

$$\|I^{\mu_{n-1}-\mu_0}\nu_n - \nu^*\| \leq 2(1-(t_0^2)^{\lambda^n})\|I^{\mu_{n-1}-\mu_0}\omega_0\|,$$

*and the rate of convergence is $o(1-(t_0^2)^{\lambda^n})$, where $t_0$ is determined by $\nu_0$, and $t_0 \in (0,1)$.*

*Proof.* From Lemma 2.6 for any $\nu_0 \in D$, then there exist four positive constants $l_{\nu_0}$, $L_{\nu_0}$, $l_{\nu_0}^-$, $L_{\nu_0}^-$ such that

$$l_{\nu_0}e_{n-1}(t) \leq \nu_0 \leq L_{\nu_0}e_{n-1}(t), l_{\nu_0}^- e_{n-1}(t) \leq T\nu_0 \leq L_{\nu_0}^- e_{n-1}(t).$$

So

$$\frac{l_{\nu_0}^-}{L_{\nu_0}}\nu_0 \leq T\nu_0 \leq \frac{L_{\nu_0}^-}{l_{\nu_0}}\nu_0.$$

Let

$$t_0 = \min\left\{\left(\frac{l_{\nu_0}^-}{L_{\nu_0}}\right)^{\frac{1}{1-\lambda}}, \left(\frac{l_{\nu_0}}{L_{\nu_0}^-}\right)^{\frac{1}{1-\lambda}}\right\},$$

then $t_0 \in (0,1)$ and

$$t_0^{1-\lambda}\nu_0 \leq T\nu_0 \leq \left(\frac{1}{t_0}\right)^{1-\lambda}\nu_0. \tag{3.1}$$

Let $\mu_0 = t_0\nu_0$, $\omega_0 = \frac{1}{t_0}\nu_0$, then $\mu_0 \leq \nu_0 \leq \omega_0$. Now we define

$$\mu_n = T\mu_{n-1}, \omega_n = T\omega_{n-1}(n = 1, 2, \cdots). \tag{3.2}$$

According to (L3)and Remark 1.1 we have

$$\begin{aligned}(Trv)(t) &= \int_0^1 G(t,s)q(s)f(I^{\mu_{n-1}-\mu_0}rv(s), I^{\mu_{n-1}-\mu_1}rv(s), \cdots, I^{\mu_{n-1}-\mu_{n-2}}rv(s), rv(s))ds \\ &\geq r^{\lambda}T(v), \text{if } \ 0<\lambda<1 \\ (Trv)(t) &= \int_0^1 G(t,s)q(s)f(I^{\mu_{n-1}-\mu_0}rv(s), I^{\mu_{n-1}-\mu_1}rv(s), \cdots, I^{\mu_{n-1}-\mu_{n-2}}rv(s), rv(s))ds \\ &\leq r^{\lambda}T(v), \text{if } \ \lambda>1.\end{aligned}$$

By a careful calculation and (L2) we can obtain

$$T\mu_0 \geq t_0^{\lambda}T\nu_0 \geq t_0\nu_0 = \mu_0, T\omega_0 \leq \left(\frac{1}{t_0}\right)^{\lambda} T\nu_0 \leq \frac{1}{t_0}\nu_0 = \omega_0. \tag{3.3}$$

By induction we can get

$$\mu_0 \leq \mu_1 \leq \cdots \leq \mu_n \leq \cdots \leq \omega_n \leq \cdots \leq \omega_1 \leq \omega_0. \tag{3.4}$$

Notice that $\mu_0 = t_0^2\omega_0$, by induction it is easy to get $\mu_n \geq (t_0^2)^{\lambda^n}\omega_n, (n=1,2,\cdots)$. Since $P$ is a normal cone with the normality constant 1, and $\mu_{n+m}-\mu_n \leq \omega_n - \mu_n$ for each $m \in \mathbb{N}$, we obtain

$$\|\mu_{n+m}-\mu_n\| \leq \|\omega_n - \mu_n\| \leq (1-(t_0^2)^{\lambda^n})\|\omega_0\| \to 0, (n \to +\infty). \tag{3.5}$$

It means that $\{\mu_n\}$ is a Cauchy-sequence, so $\mu_n$ converges to some $\nu^* \in D$, and $T\nu^* \geq \nu^*$ in the other hand we have

$$\|\omega_n - \nu^*\| \leq \|\omega_n - \mu_n\| + \|\mu_n - \nu^*\| \to 0, (n \to +\infty).$$

This implies that $\omega_n$ converges to $\nu^*$, and and $T\nu^* \leq \nu^*$. So $\nu^*$ is a fixed point of $T$, and $\nu^* \in [\mu_0, \omega_0]$. For each initial value $\nu_0 \in D$, we have $\mu_0 \leq \nu_0 \leq \omega_0$ and $\mu_n \leq \nu_n \leq \omega_n$, $(n=1,2,\cdots)$. So

$$\|\nu_n - \nu^*\| \leq \|\nu_n - \mu_n\| + \|\mu_n - \nu^*\| \leq 2\|\omega_n - \mu_n\| \leq 2(1-(t_0^2)^{\lambda^n})\|\omega_0\|,$$

which implies that $\nu_n$ converges to $\nu^*$. From Lemma 2.5 we get $I^{\mu_{n-1}-\mu_0}\nu^*$ is a positive solution of equation (1.1) satisfying

$$I^{\mu_{n-1}-\mu_0}\nu_n - I^{\mu_{n-1}-\mu_0}\nu^* = I^{\mu_{n-1}-\mu_0}(\nu_n - \nu^*) \leq 2(1-(t_0^2)^{\lambda^n})I^{\mu_{n-1}-\mu_0}\omega_0.$$

Furthermore we have the error estimation

$$\|I^{\mu_{n-1}-\mu_0}\nu_n - I^{\mu_{n-1}-\mu_0}\nu^*\| \leq 2(1-(t_0^2)^{\lambda^n})\|I^{\mu_{n-1}-\mu_0}\omega_0\|.$$

So the rate of convergence is $o(1-(t_0^2)^{\lambda^n})$, where $t_0$ is determined by $\nu_0$.

Now we shall prove the uniqueness of positive solution of(2.7).

For any fixed point $\nu^- \in D$ of $T$, from the definition of $D$, let

$$t_1 = inf\{t>0 : \nu^- \leq t\nu^*\}.$$

Then $0 < t_1 < +\infty$. Now we prove $t_1 \leq 1$. Otherwise, then

$$\nu^- = T\nu^- \leq T(t_1\nu^*) \leq t_1^{\lambda}T(\nu^*) = t_1^{\lambda}\nu^*.$$

It implies $t_1^{\lambda} < t_1$, which contradicts the definition of $t_1$. So $t_1 \leq 1$ and $\nu^- \leq \nu^*$. In the other hand, let $t_2 = inf\{t>0 : \nu^* \leq t\nu^-\}$. In the same way we obtain $\nu^* \leq \nu^-$, thus $\nu^- = \nu^*$, that is $\nu^*$ is the unique fixed point of $T \in D$. Of course, it is also the unique positive solution of (2.7).

Thus according to Lemma 2.5, $I^{\mu_{n-1}-\mu_0}v^*$ is the unique solution of (1.1). □

**Example 3.1** Consider the following equation

$$\begin{cases} \mathcal{D}^{\frac{7}{2}}x(t)+t^2(x^{\frac{4}{9}}(t)+(\mathcal{D}^{\frac{3}{4}}x(t))^{\frac{1}{3}}+(\mathcal{D}^{\frac{7}{4}}x(t))^{\frac{3}{7}}+(\mathcal{D}^{\frac{7}{3}}x(t))^{\frac{3}{8}})=0, t\in[0,1] \\ x(0)=\mathcal{D}^{\frac{3}{4}}x(0)=\mathcal{D}^{\frac{7}{4}}x(0)=\mathcal{D}^{\frac{7}{3}}x(0)=0 \\ \mathcal{D}^{\frac{12}{5}}x(1)=\sum_{j=1}^{\infty}\frac{3}{4}(\frac{1}{3})^j\mathcal{D}^{\frac{12}{5}}x([1-(\frac{1}{3})^j]^{10}), \end{cases} \tag{3.6}$$

where $n=4$, $\mu_1=\frac{3}{4}$, $\mu_2=\frac{7}{4}$, $\mu_3=\frac{7}{3}$, $q(t)=t^2$, $\alpha=\frac{7}{2}$,

$$f(x_1,x_2,x_3,x_4)=x_1^{\frac{4}{9}}(t)+x_2^{\frac{1}{3}}(t)+x_3^{\frac{3}{7}}(t)+x_4^{\frac{3}{8}}(t).$$

By a careful calculation we get $\alpha-\mu-1=\frac{1}{10}$, $\alpha-\mu_{n-1}-1=\frac{1}{6}$,

$$f(x_1,x_2,x_3,x_4)=x_1^{\frac{4}{9}}(t)+x_2^{\frac{1}{3}}(t)+x_3^{\frac{3}{7}}(t)+x_4^{\frac{3}{8}}(t),$$

is increasing on $[0,1]$.

And $f(rx_1,rx_2,rx_3,rx_4)=r^{\frac{4}{9}}f(x_1,x_2,x_3,x_4)$, $0<\frac{4}{9}<1$, $1>\frac{3}{4}=\sum_{j=1}^{\infty}\alpha_j\xi_j^{\alpha-\mu-1}>0$.

$$e_0=I^{\frac{7}{3}}t^{\frac{1}{6}}=\frac{\Gamma(\frac{7}{6})}{\Gamma(\frac{7}{2})}t^{\frac{5}{2}}, e_1=I^{\frac{19}{12}}t^{\frac{1}{6}}=\frac{\Gamma(\frac{7}{6})}{\Gamma(\frac{11}{4})}t^{\frac{7}{4}}, e_2=I^{\frac{7}{12}}t^{\frac{1}{6}}=\frac{\Gamma(\frac{7}{6})}{\Gamma(\frac{7}{4})}t^{\frac{3}{4}}, e_3=t^{\frac{1}{6}}.$$

$$\begin{aligned} 0 &< \int_0^1 q(s)f(e_0(s),e_1(s),\cdots,e_{n-2}(s),e_{n-1}(s))ds \\ &= \int_0^1 s^2 f(\frac{\Gamma(\frac{7}{6})}{\Gamma(\frac{7}{2})}s^{\frac{5}{2}},\frac{\Gamma(\frac{7}{6})}{\Gamma(\frac{11}{4})}s^{\frac{7}{4}},\frac{\Gamma(\frac{7}{6})}{\Gamma(\frac{7}{4})}t^{\frac{3}{4}},s^{\frac{1}{6}}) \\ &\leq \int_0^1 f(\frac{\Gamma(\frac{7}{6})}{\Gamma(\frac{7}{2})},\frac{\Gamma(\frac{7}{6})}{\Gamma(\frac{11}{4})},\frac{\Gamma(\frac{7}{6})}{\Gamma(\frac{7}{4})},1)<+\infty. \end{aligned}$$

From Theorem 3.1, the equation (3.6) has a unique positive solution.

# References


[1] B. Ahmad, A. Alsaedi. Existence and uniqueness of solutions for coupled systems of higher-order nonlinear fractional differential equations. Fixed Point Theory Appl.2010(2010)(Article ID364560,17pages).

[2] Y. Cui. Uniqueness of solution for boundary value problems for fractional differential equations. Appl. Math. Lett. 51(2016)48-54.

[3] C. Goodrich. Existence of a positive solution to systems of differential equations of fractional order. Comput. Math. Appl. 62(2011)1251-128.

[4] J.Li,B.Liu,L.Liu.Solutions for a boundary value problem at resonance on $[0,+\infty)$. Math. Comput. Model. 58(2013) 1769-1776.

[5] R. MillerK. An introduction to the fractional calculus and fractional differential equations. NewYork: Wiley; 1993.

[6] I. Podlubny. Fractional Differential Equations. Mathematics in Science and Engineering, Academic Press, NewYork: London, Toronto. 1999.

[7] H. Qu, X. Liu. Existence of nonnegative solutions for a fractional m-point boundary value problem at resonance. Bound.ValueProbl. 127(2013)415-425.

[8] Y. Wang, L. Liu, Y. Wu. Positive solutions for a nonlocal fractional differential equation. Nonlinear. Anal. 74(2011)3599-3605.

[9] L. Yang, W. Zhang. Positive solution for second order multi-point boundary value problem at resonance.Fixed Point Theory.2(2011)155-166

[10] S. Zhang. Positive solutions to singular boundary value problem for nonlinear fractional differential equation. Comput. Math. Appl. 59(2010)1300-1309.

[11] X. Zhang, L. Liu, Y. Wu. The eigenvalue problem for a singular higher order fractional diferential equation involving fractional derivatives. Appl. Math. Comput. 218(2012)8526-8536.

[12] X. Zhang, Y. Han. Existence and uniqueness of positive solutions for higher order nonlocal fractional differential equations.Appl.Math.Lett. 25(2012)555-560.

[13] M.Zima, P. Drygaś. Existence of positive solutions for a kind of periodic boundary value problem at resonance.Bound.ValueProbl.1(2013)1-10

# ON THE $(h,q)$-EULER POLYNOMIALS AND NUMBERS

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ABSTRACT. In this paper, we consider the fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$ which was defined by Kim [T. Kim, On $p$-adic interpolating function for $q$-Euler numbers and its derivatives, J. Math. Anal. Appl. 339(2008) 598-608]. By using this integral, we define the $(h,q)$-Euler polynomials and numbers and give some interesting identities of these polynomials and numbers.

## 1. INTRODUCTION

Let $p$ be a fixed prime number with $p \equiv 1 (\mathrm{mod}\ 2)$. Throughout this work, $\mathbb{Z}_p$, $\mathbb{C}$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will, respectively, denote the ring of $p$-adic integers, the complex field, the field of $p$-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. The $p$-adic norm $|\cdot|_p$ is normalized by $|p|_p = \frac{1}{p}$. When one talks of $q$-extension, $q$ is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, we assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p < 1$. We use the notation

$$[x]_q = \frac{1-q^x}{1-q} \text{ and } [x]_{-q} = \frac{1-(-q)^x}{1+q}, \qquad (\text{see } [1,4,5,11,12,15,21,22,24]). \tag{1}$$

Let $C(\mathbb{Z}_p)$ be the set of continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the $p$-adic $q$-integral on $\mathbb{Z}_p$ is defined by Kim to be

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \qquad (\text{see } [2,3,6-25]). \tag{2}$$

It is well-known that the classical Euler polynomials $E_n(x)$ are defined by means of the following generating function:

$$\int_{\mathbb{Z}_p} e^{t(x+y)} d\mu_{-q}(y) = \frac{2}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \qquad (\text{see } [8-13,15,16,19,22-25]). \tag{3}$$

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For a fixed $h \in \mathbb{C}_p$ with $|1-h|_p < p^{-\frac{1}{1-p}}$, we define the $(h,q)$-Euler polynomials as

$$\mathcal{E}_{n,q}^{(h)}(x) = \int_{\mathbb{Z}_p} [y+x]_q^n h^y d\mu_{-q}(y). \tag{4}$$

When $x=0$, $\mathcal{E}_{n,q}^{(h)} = \mathcal{E}_{n,q}^{(h)}(0)$ are called the $(h,q)$-Euler numbers. When $q=1$, $E_n^{(h)}(x) = \mathcal{E}_{n,1}^{(h)}(x)$ is called the $(h)$-Euler polynomials as follows

$$E_n^{(h)}(x) = \int_{\mathbb{Z}_p} (y+x)^n h^y d\mu_{-q}(y), \tag{5}$$

and $E_n^{(h)} = E_n^{(h)}(0)$ is called the $(h)$-Euler numbers.

In this paper, by using the fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$, we define the $(h,q)$-Euler polynomials and discuss their properties. Furthermore, we give some interesting identities of these polynomials and numbers.

## 2. The $(h,q)$-Euler polynomials

From (4), we observe that

$$\mathcal{E}_{n,q}^{(h)}(x) = \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} q^{lx} (-1)^l \int_{\mathbb{Z}_p} (q^l h)^y d\mu_{-q}(y). \tag{6}$$

We note that

$$\begin{aligned} \int_{\mathbb{Z}_p} (q^l h)^y d\mu_{-q}(y) &= \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} (q^{l+1}h)^y (-1)^y \\ &= \frac{[2]_q}{2} \lim_{N\to\infty} \frac{1+(q^{l+1}h)^{p^N}}{1+q^{l+1}h} \\ &= \frac{[2]_q}{1+q^{l+1}h}. \end{aligned} \tag{7}$$

By (6) and (7), we obtain the following theorem.

**Theorem 2.1.** *For $n \in \mathbb{N} \cup \{0\}$, we have*

$$\mathcal{E}_{n,q}^{(h)}(x) = \frac{[2]_q}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} q^{lx} (-1)^l \frac{1}{1+q^{l+1}h}. \tag{8}$$

Now, we observe that

$$\begin{aligned} & \mathcal{E}_{n,q}^{(h)}(x) \\ = & \frac{[2]_q}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} q^{lx} (-1)^l \sum_{m=0}^{\infty} (-1)^m q^{(l+1)m} h^m \\ = & \frac{[2]_q}{(1-q)^n} \sum_{m=0}^{\infty} (-1)^m h^m q^m \sum_{l=0}^{n} \binom{n}{l} q^{l(x+m)} (-1)^l \\ = & [2]_q \sum_{m=0}^{\infty} (-1)^m h^m q^m \frac{1}{(1-q)^n} (q^{x+m}-1)^n \end{aligned}$$

$$= [2]_q \sum_{m=0}^{\infty} (-1)^m h^m q^m [x+m]_q^n. \tag{9}$$

By (9), we obtain the following theorem.

**Theorem 2.2.** *For $n \in \mathbb{N} \cup \{0\}$, we have*

$$\mathcal{E}_{n,q}^{(h)}(x) = [2]_q \sum_{m=0}^{\infty} (-1)^m h^m q^m [x+m]_q^n. \tag{10}$$

Let us consider the generating function of the $(h,q)$-Euler polynomials as $F_{h,q}(x,t) = \sum_{n=0}^{\infty} \mathcal{E}_{n,q}^{(h)}(x)\frac{t^n}{n!}$, where $t \in \mathbb{C}_p$ with $|1-t|_p < 1$ or $t \in \mathbb{C}$ with $|t| < 1$. Then we see that

$$\begin{aligned} F_{h,q}(x,t) &= \sum_{n=0}^{\infty} \mathcal{E}_{n,q}^{(h)}(x)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left( [2]_q \sum_{m=0}^{\infty} (-1)^m h^m q^m [x+m]_q^n \right) \frac{t^n}{n!} \\ &= [2]_q \sum_{m=0}^{\infty} (-1)^m h^m q^m \sum_{n=0}^{\infty} [x+m]_q^n \frac{t^n}{n!} \\ &= [2]_q \sum_{m=0}^{\infty} (-1)^m h^m q^m e^{[x+m]_q t}. \end{aligned} \tag{11}$$

By (11), we obtain the following theorem.

**Theorem 2.3.** *Let $F_{h,q}(x,t)$ be the generating function of $(h,q)$-Euler polynomials as $F_q(x,t) = \sum_{n=0}^{\infty} \mathcal{E}_{n,q}^{(h)}(x)\frac{t^n}{n!}$. Then we have*

$$F_{h,q}(x,t) = [2]_q \sum_{m=0}^{\infty} (-1)^m h^m q^m e^{[x+m]_q t}. \tag{12}$$

From Theorem 2.3, we note that

$$\begin{aligned} lim_{q\to 1} F_{h.q}(x,t) &= 2\sum_{m=0}^{\infty} (-1)^m h^m e^{(x+m)t} \\ &= 2e^{xt} \sum_{m=0}^{\infty} (-1)^m h^m e^{mt} \\ &= \frac{2}{he^t+1} e^{xt} \end{aligned} \tag{13}$$

and

$$\begin{aligned} &\int_{\mathbb{Z}_p} h^y e^{t(x+y)} d\mu_{-q}(y) \\ &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} h^y (x+y)^n d\mu_{-q}(y) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} E_n^{(h)}(x) \frac{t^n}{n!}, \end{aligned} \tag{14}$$

where $E_n^{(h)}(x)$ are the $(h)$-Euler polynomials. We also observe that if $f_1(x) = f(x+1)$, then we have

$$\begin{aligned} qI_{-q}(f_1) &= \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} (-1)^x f(x+1) q^{x+1} \\ &= \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=1}^{p^N} (-1)^{x-1} f(x) q^x \\ &= -\lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} (-1)^x f(x) q^x \\ &\quad + \lim_{N\to\infty} \frac{[2]_q}{2} \left( (-1)^{p^N-1} q^{p^N} f(p^N) + f(0) \right) \\ &= -\lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} (-1)^{x-1} f(x) q^x + [2]_q f(0) \\ ] &= -I_{-q}(f) + [2]_q f(0). \end{aligned} \tag{15}$$

By (15), we obtain the following integral equation.

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0). \tag{16}$$

We note that if $f(y) = h^y e^{(x+y)t}$, by (5) and (16), then we get

$$\int_{\mathbb{Z}_p} h^y e^{t(x+y)} d\mu_{-q}(y) = \frac{2}{he^t + 1} e^{xt}. \tag{17}$$

From (13), (14) and (17)

**Theorem 2.4.** *Let $E_n^{(h)}(x)$ be $(h)$-Euler polynomials with the generating function $F_{h,1}(x,t) = \sum_{n=0}^{\infty} E_n^{(h)}(x) \frac{t^n}{n!}$. Then we have*

$$F_{h,1}(x,t) = \frac{2}{he^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n^{(h)}(x) \frac{t^n}{n!}, \tag{18}$$

*In particular, when $h = 1$ and $q = 1$, we get*

$$F_{1,1}(x,t) = \frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \tag{19}$$

*where $E_n(x)$ are the classical Euler polynomials.*

From (16), we note that if $f(x) = h^x [x]_q^n$, then we have

$$q \int_{\mathbb{Z}_p} h^{x+1} [x+1]_q^n d\mu_{-q}(x) + \int_{\mathbb{Z}_p} h^x [x]_q^n d\mu_{-q}(x) = [2]_q 0_q^n. \tag{20}$$

By (20), we obtain the following theorem.

**Theorem 2.5.** *For $n \in \mathbb{N} \cup \{0\}$, we have*

$$qh\mathcal{E}_{n,q}^{(h)}(1) + \mathcal{E}_{n,q}^{(h)} = [2]_q \delta_{n,0}, \tag{21}$$

*where $\delta_{n,0}$ is the Kronecker's symbol.*

We note that

$$\begin{aligned}[x+y]_q^n &= ([x]_q + q^x[y]_q)^n \\ &= \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^{lx} [y]_q^l. \end{aligned} \tag{22}$$

By (22), we get

$$\begin{aligned}\mathcal{E}_{n,q}^{(h)}(x) &= \int_{\mathbb{Z}_p} h^y [x+y]_q^n d\mu_{-q}(y) \\ &= \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^{lx} \int_{\mathbb{Z}_p} [y]_q^l d\mu_{-q}(y) \\ &= \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^{lx} \mathcal{E}l, q^{(h)}(x) \\ &= \left(\mathcal{E}_q^{(h)}(x) + [x]_q\right)^n, \end{aligned} \tag{23}$$

with the usual convention about replacing $(\mathcal{E}_q^{(h)}(x))^l = \mathcal{E}_{n,q}^{(h)}(x)$. Thus, by (21) and (23), we obtain the following theorem.

**Theorem 2.6.** *For $n \in \mathbb{N} \cup \{0\}$, we have*

$$qh\left(q\mathcal{E}_q^{(h)}(1) + 1\right)^n + \mathcal{E}_{n,q}^{(h)} = [2]_q \delta_{n,0} \tag{24}$$

*with the usual convention about replacing $(\mathcal{E}_q^{(h)}(x))^l = \mathcal{E}_{n,q}^{(h)}(x)$, where $\delta_{n,0}$ is the Kronecker's symbol.*

## 3. Remarks

In this section, we assume that $q, h \in \mathbb{C}$ with $|q| < 1$ and $|h| < 1$. For $s \in \mathbb{C}$, we define the $(h,q)$-Zeta functions as follows:

$$\zeta_q^{(h)}(s,x) = [2]_q \sum_{n=0}^{\infty} \frac{(-1)^n q^n h^h}{[n+x]_q^s}, \quad x \neq 0, -1, -2, \cdots . \tag{25}$$

By (10) and (25), we obtain the following theorem.

**Theorem 3.1.** *For $m \in \mathbb{N}$, we have*

$$\zeta_q^{(h)}(-m,x) = \mathcal{E}_{m,q}^{(h)}(x). \tag{26}$$

We note that if $d \in \mathbb{N}$ with $d \equiv 1 (\mathrm{mod}\ 2)$, then we have

$$\begin{aligned}&\int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) \\ = &\lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x \end{aligned}$$

$$
\begin{aligned}
&= \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{a=0}^{d-1} \sum_{x=0}^{p^N-1} f(a+dx)(-q)^{a+dx} \\
&= \lim_{N\to\infty} \frac{1+q^d}{1+q^{dp^N}} \cdot \frac{1+q}{1+q^d} \sum_{a=0}^{d-1} \sum_{x=0}^{p^N-1} f(a+dx)(-q)^{a+dx} \\
&= \frac{1}{[d]_{-q}} \lim_{N\to\infty} \frac{1}{[p^N]_{-q^d}} \sum_{a=0}^{d-1} \sum_{x=0}^{p^N-1} f(x)f(a+dx)(-q)^{a+dx} \\
&= \frac{1}{[d]_{-q}} \sum_{a=0}^{d-1} (-q)^a \lim_{N\to\infty} \frac{1}{[p^N]_{-q^d}} \sum_{x=0}^{p^N-1} f(x)f(a+dx)(-q^d)^x \\
&= \frac{1}{[d]_{-q}} \sum_{a=0}^{d-1} (-q)^a \int_{\mathbb{Z}_p} f(a+dx)d\mu_{-q}(x). \qquad (27)
\end{aligned}
$$

By (27), we obtain the following theorem.

**Theorem 3.2.** *Let $d \in \mathbb{N}$ with $d \equiv 1 (\mathrm{mod}\ 2)$. Then, we have*

$$
\int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = \frac{1}{[d]_{-q}} \sum_{a=0}^{d-1} (-q)^a \int_{\mathbb{Z}_p} f(a+dx)d\mu_{-q}(x). \qquad (28)
$$

We note that if $f(y) = h^y[x+y]_q^n$, then we derive

$$
\begin{aligned}
&\int_{\mathbb{Z}_p} h^y[x+y]_q^n d\mu_{-q}(y) \\
&= \frac{1}{[d]_{-q}} \sum_{a=0}^{d-1} (-q)^a \int_{\mathbb{Z}_p} h^{a+dy}[a+dy+x]_q^n d\mu_{-q}(y) \\
&= \frac{1}{[d]_{-q}} \sum_{a=0}^{d-1} (-q)^a h^a \int_{\mathbb{Z}_p} h^{dy}[a+dy+x]_q^n d\mu_{-q}(y) \\
&= \frac{[d]_q^n}{[d]_{-q}} \sum_{a=0}^{d-1} (-q)^a h^a \int_{\mathbb{Z}_p} h^{dy}[\frac{a+x}{d}+y]_{q^d}^n d\mu_{-q}(y) \\
&= \frac{[d]_q^n}{[d]_{-q}} \sum_{a=0}^{d-1} (-q)^a h^a \mathcal{E}_{n,q^d}^{(h)}\left(\frac{a+x}{d}\right), \qquad (29)
\end{aligned}
$$

where $d \in \mathbb{N}$ with $d \equiv 1 (\mathrm{mod}\ 2)$.

**Theorem 3.3.** *For $n \in \mathbb{N} \cup \{0\}$, we have*

$$
\mathcal{E}_{n,q}^{(h)}(x) = \frac{[d]_q^n}{[d]_{-q}} \sum_{a=0}^{d-1} (-q)^a h^a \mathcal{E}_{n,q^d}^{(h)}\left(\frac{a+x}{d}\right). \qquad (30)
$$

## 4. Results and discussion

The results of our works are some interesting identities of the $(h,q)$-Euler polynomials as follows. The first result is to fine the generating function of the $(h,q)$-Euler polynomials in

Theorem 2.3. We note that the $(h,q)$-Euler polynomial is the generalization of the $q$-Euler polynomial and the $(h)$-Euler polynomials (see Theorem 2.4). The second result is to find the recurrence formula of the $(h,q)$-Euler polynomials in Theorem 2.6. The third result is to compare with the $(h,q)$-Euler polynomials and the $(h,q)$-Zeta functions in Theorem 3.1. The fourth result is to find the distribution equation of the $(h,q)$-Euler polynomials in Theorem 3.3.

In the future, we will work to find some interesting identities between the $(h,q)$-Euler polynomials and the $q$-Euler polynomials and to compare with the zeros of the $(h,q)$-Euler polynomials and the zeros of the $q$-Euler polynomials.

## 5. Conclusions

We defined $(h,q)$-Euler polynomials which are $q$-Euler polynomials with some weight function in Eq.(6) and remarked that if $h=1$, $(1,q)$-Euler polynomials are $q$-Euler polynomials. In Theorems 2.1 and 2.2, we found some properties of $(h,q)$-Euler polynomials. In Theorem 2.3, we obtained some specific type of the generating function for $(h,q)$-Euler polynomials and noted that when $h=q$, we saw the more specific type of the generating function of $(h,q)$-Euler polynomials.

Furthermore, we investigated very interesting recursion formula of $(h,q)$-Euler polynomials in Theorem 3.1. In section 3, we defined the $(h,q)$-Zeta functions and gave some relation with the $(h,q)$-Zeta funtion and $(h,q)$-Euler polynomials. Finally, we obtained a distribution identity equation of $(h,q)$-Euler polynomials in Theorem 3.3.

**Competing interests**

The authors declare that they have no competing interests.

**Authors' contributions**

All authors contributed equally to this work. All authors read and approved the final manuscript.

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## References

[1] W. A. Al-Salem, *q-Bernoulli numbers and polynomials*, *Math. Nachr.* **17** (1959) 239–260 (1959).
[2] A. Bayad, J. Chikhi, Non linear recurrences for Apostol-Bernoulli-Euler numbers of higher order,, *Adv. Stud. Contemp. Math (Kyungshang)* **22** (2012), no. 1, 1–6.
[3] A. Bayad, T. Kim, *Identities for Apostol-type Frobenius-Euler polynomials resulting from the study of a nonlinear operator*, *Russ. J. MAth. Phys.* **23** (2016), no. 2, 164–171.
[4] L. Carlitz, *Expansion of q-Bernoulli numbers*, *Duke Math. J.* **25** (1958) 355–364.
[5] L. Carlitz, *q-Bernoulli and Eulerian numbers*, *Trans. Amer. Math. Soc* **76** (1954). 332–350.
[6] L. Carlitz, *q-Bernoulli and polynomials*, *Duke Math. J.* **15** (1948). 987–1000.
[7] D. V. Dolgy, D. J. Kang, T. Kim, B. Lee, *Some new identities on the twisted* $(h,q)$*-Euler numbers and q-Bernstein polynomials*, *J. Comput. Anal. Appl.* **14** (2012), no. 5, 974–984.
[8] L. C. Jang, J. G. Lee, *A note on the Barnes-type q-Euler polynomials*, *Adv. Difference Equ.* **2015**, 2015:250, 7 pp.

[9] D. S. Kim, T. Kim, *Barnes-type Boole polynomials*, *Contrib. Discrete Math.* **11** (2016), no. 1, 7–15.
[10] D. S. Kim, T. Kim, *Some identities of degenerate Euler polynomials arising from p-adic fermionic integral on $\mathbb{Z}_p$*, *Integral Transforms Spec. Funct.* **26** (2015), no. 4, 295–302.
[11] D. S. Kim, T. Kim, *Symmetric identities of higher-order degenerate q-Euler polynomials*, *J. Nonlinear Sci. Appl.* **9** (2016), no. 2, 443–451.
[12] T. Kim, D. S. Kim, H.-I. Kwon, J.-J. Seo, D. V. Dolgy, *Some identities of q-Euler polynomials under the symmetric group of degree n*, *J. Nonlinear Sci. Appl.* **9** (2016), no. 3, 1077–1082.
[13] T. Kim, *q-Euler numbers and polynomials associated with p-adic q-integrals*, *J. Nonlinear Math. Phys.* **14** (2007), no. 1, 15–27.
[14] T. Kim, *Symmetry of power sum polynomials and multivariate fermionic p-adic invariant integral on $\mathbb{Z}_p$*, *Russ. J. Math. Phys.* **16** (2009), no .1, 93–96.
[15] T. Kim, *Some identities on the q-Euler polynomials of higher order and q-Stirling numbers by the fermionic p-adic integral on $\mathbb{Z}_p$*, *Russ. J. Math. Phys.* **16** (2009), no. 4, 484–491.
[16] T. Kim, D. S. Kim, H.-I. Kwon, J.-J. Seo, *Some identities for degenerate Frobenius-Euler numbers arising from nonlinear differential equations*, *Italian J. Pure and Appl. Math.* **36** (2016), 843–850.
[17] T. Kim, D. S. Kim, *Differential equations associated with Catalan-Daehee numbers and their applications*, *Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Mathemáticas* **2016** (2016), 1–11.
[18] T. Kim, D. S. Kim, *A note on nonlinear Changhee differential equations*, *Russ. J. Math. Phys.* **23** (2016), no. 1, 88–92.
[19] T. Kim, D. V. Dolgy, L. C. Jang, H. I. Kwon, *Some identities of degenerate q-Euler polynomials under the symmetry group n*, *J. Nonlinear Sci. Appl.* **9** (2016), no. 6, 4707–4712.
[20] T. Kim, *q-Volkenborn integration*, *Russ. J. Math. Phys.* **9** (2002), no. 3, 288-299.
[21] T. Kim, D. S. Kim, D. V. Dolgy, *Degenerate q-Euler polynomials*, *Adv. Difference Equ.* **2015** 2015:246, 11 pp.
[22] T. Kim, *Some properties on the integral of the product of several Euler polynomials*, *Quaest. Math.* **38** (2015), no. 4, 553–562.
[23] A. Sharma, *q-Bernoulli and Euler numbers of higher order*, *Duke Math. J.* **25** (1958) 343–353.
[24] K. Shiratani, S. Yokoyama, *An application of p-adic convolutions*, *Mem. Fac. Sci. Kyushu Univ. Ser. A* **36** (1982), no. 1, 73–83.
[25] S. L. Sobolev, *Roots of Euler polynomials*, *Theory of cubature formulas and numerical mathematics (Proc. Conf., Novosibirsk, 1978) (Russian)* pp. 125–141, 253,

# HERMITE-HADAMARD TYPE INEQUALITIES FOR $n$-TIMES DIFFERENTIABLE AND $\alpha$-LOGARITHMICALLY PREINVEX FUNCTIONS

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**ABSTRACT:** In the paper, the concept of $\alpha$-logarithmically preinvex function is introduced, and by creating an integral identity involving an $n$-times differentiable function, some new Hermite-Hadamard type inequalities for $\alpha$-logarithmically preinvex functions are established.

**KEY WORDS:** Integral identity, Hermite-Hadamard type inequality, $\alpha$-logarithmically preinvex function.

## 1. Introduction

Throughout this paper, let $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}_+ = (0, \infty)$, $\mathbb{N}$ denote the set of all positive integers, $I$ denote the interval in $\mathbb{R}$, and $A$ denote the set in $\mathbb{R}^n, n \in \mathbb{N}$.

First, let us recall some definitions of various convex functions.

**Definition 1.1.** A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1.1}$$

holds for all $x, y \in I$ and $t \in [0, 1]$. If the inequality (1.1) reverses, then $f$ is said to be concave on $I$.

**Definition 1.2** ([1])**.** A set $A \subseteq \mathbb{R}^n$ is said to be invex with respect to the map $\eta : A \times A \to \mathbb{R}^n$, if for every $x, y \in A$ and $t \in [0, 1]$

$$y + t\eta(x, y) \in A. \tag{1.2}$$

The invex set $A$ is also called a $\eta$-connected set.

It is obvious that every convex set is invex with respect to the map $\eta(x, y) = x - y$, but there exist invex sets which are not convex (see [1], for example).

**Definition 1.3** ([1])**.** Let $A \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : A \times A \to \mathbb{R}^n$. For every $x, y \in A$, the $\eta$-path $P_{xv}$ joining the points $x$ and $v = x + \eta(y, x)$ is defined by

$$P_{xv} = \{v | v = x + t\eta(y, x), t \in [0, 1]\}. \tag{1.3}$$

**Definition 1.4** ([13])**.** Let $A \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : A \times A \to \mathbb{R}^n$. A function $f : A \to \mathbb{R}$ is said to be preinvex with respect to $\eta$, if for every $x, y \in A$ and $t \in [0, 1]$

$$f(y + t\eta(x, y)) \leq tf(x) + (1-t)f(y). \tag{1.4}$$

The function $f$ is said to be preincave if and only if $-f$ is preinvex.

Every convex function is preinvex with respect to the map $\eta(x, y) = x - y$, but not conversely (see [13], for instance).

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**Definition 1.5** ([10]). Let $A \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : A \times A \to \mathbb{R}^n$. The function $f : A \to \mathbb{R}_+$ on the set $A$ is said to be logarithmically preinvex with respect to $\eta$, if for every $x, y \in A$ and $t \in [0, 1]$

$$f(y + t\eta(x, y)) \le \big[f(x)\big]^t \big[f(y)\big]^{1-t}. \tag{1.5}$$

For properties and applications of preinvex and logarithmically preinvex functions, please refer to [8, 9, 14, 16] and closely related references therein.

The most important inequality in the theory of convex functions, the well known Hermite-Hadamard's integral inequality, may be stated as follows. Let $I \subseteq \mathbb{R}$ and $a, b \in I$ with $a < b$. If $f : [a, b] \subseteq I \to \mathbb{R}$ is a convex function on $[a, b]$, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \le \frac{f(a)+f(b)}{2}. \tag{1.6}$$

If $f$ is concave on $[a, b]$, then the inequality (1.6) is reversed.

The inequality (1.6) has been generalized by many mathematicians. Some of them may be recited as follows.

**Theorem 1.1** ([5]). *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I$ and $a, b \in I$ with $a < b$. If $|f'(x)|^q$ for $q \ge 1$ is convex function on $[a, b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{4}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{1/q} \tag{1.7}$$

*and*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{4}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{1/q}. \tag{1.8}$$

**Theorem 1.2** ([6]). *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable on $I$, $a, b \in I$ with $a < b$, and $p > 1$. If $|f'(x)|^{p/(p-1)}$ is convex on $[a, b]$, then*

$$\begin{aligned}\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| &\le \frac{b-a}{16}\left(\frac{4}{p+1}\right)^{1/p} \\ &\times \left\{\left[|f'(a)|^{p/(p-1)}+3|f'(b)|^{p/(p-1)}\right]^{1-1/p} + \left[3|f'(a)|^{p/(p-1)}+|f'(b)|^{p/(p-1)}\right]^{1-1/p}\right\}.\end{aligned} \tag{1.9}$$

**Theorem 1.3** ([9]). *Let $A \subseteq \mathbb{R}$ be an open invex set with respect to $\eta : A \times A \to \mathbb{R}$ and $a, b \in A$ with $\eta(a, b) > 0$ for all $a \ne b$. If $f : A \to \mathbb{R}_+$ is a preinvex function on $A$, then the following inequality holds*

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \le \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,\mathrm{d}x \le \frac{f(a)+f(b)}{2}. \tag{1.10}$$

**Theorem 1.4** ([11]). *Let $A \subseteq \mathbb{R}$ be an open invex set with respect to $\eta : A \times A \to \mathbb{R}$ and $a, b \in A$ with $\eta(a, b) > 0$ for all $a \ne b$. Suppose that $f : A \to \mathbb{R}$ is a differentiable function. If $|f^{(n)}|^q, q > 1, q \in \mathbb{R}$, is a log-preinvex function on $A$, then we have the inequality*

$$\begin{aligned}&\left|f\left(a+\frac{1}{2}\eta(b,a)\right) - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,\mathrm{d}x\right| \\ &\qquad \le \frac{\eta(b,a)\sqrt{|f'(a)|}}{2^{1/p}(p+1)^{1/p}q^{1/q}}\left[\frac{\big(|f'(b)|\big)^{q/2} - \big(|f'(a)|\big)^{q/2}}{\big(\ln\big(|f'(b)|\big) - \ln\big(|f'(a)|\big)\big)}\right]^{1/q},\end{aligned} \tag{1.11}$$

*where $\frac{1}{p} + \frac{1}{q} = 1$.*

**Theorem 1.5** ([7]). *Let $A \subseteq \mathbb{R}$ be an open invex set with respect to $\eta : A \times A \to \mathbb{R}$ and $a, b \in A$ with $\eta(a,b) > 0$ for all $a \neq b$. Suppose $f : A \to \mathbb{R}$ is a function such that $f^{(n)}$ exits on $A$ and $f^{(n)}$ is integrable on $[a, a+\eta(b,a)]$ for $n \in \mathbb{N}, n \geq 1$. If $|f^{(n)}|^q$ is log-preinvex on $A$ for $n \in \mathbb{N}, n \geq 1, q \geq 1$, then we have the following inequality*

$$\left|\sum_{k=0}^{n-1} \frac{\left[(-1)^k+1\right](\eta(b,a))^k}{2^{k+1}(k+1)!} f^{(k)}\left(a+\frac{1}{2}\eta(b,a)\right) - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,\mathrm{d}x\right| \leq \frac{(\eta(b,a))^n\left|f^{(n)}(a)\right|(n!)^{1/q-1}}{2^{(n+1)(q-1)/q}(n+1)^{1-1/q}}\left\{[E_2(n,q)]^{1/q}+[E_3(n,q)]^{1/q}\right\}, \quad (1.12)$$

*where*

$$E_2(n,q) = \frac{(-1)^{n+1}}{q^{n+1}\left[\ln\left(\left|f^{(n)}(b)\right|\right)-\ln\left(\left|f^{(n)}(a)\right|\right)\right]^{n+1}} + \left(\frac{\left|f^{(n)}(b)\right|}{\left|f^{(n)}(a)\right|}\right)^{q/2}\sum_{k=0}^{n}\frac{(-1)^k}{(n-k)!q^{k+1}2^{n-k}\left[\ln\left(\left|f^{(n)}(b)\right|\right)-\ln\left(\left|f^{(n)}(a)\right|\right)\right]^{k+1}} \quad (1.13)$$

*and*

$$E_3(n,q) = \frac{\left|f^{(n)}(b)\right|^q}{q^{n+1}\left[\ln\left(\left|f^{(n)}(b)\right|\right)-\ln\left(\left|f^{(n)}(a)\right|\right)\right]^{n+1}\left|f^{(n)}(a)\right|^q} - \left(\frac{\left|f^{(n)}(b)\right|}{\left|f^{(n)}(a)\right|}\right)^{q/2}\sum_{k=0}^{n}\frac{1}{(n-k)!q^{k+1}2^{n-k}\left[\ln\left(\left|f^{(n)}(b)\right|\right)-\ln\left(\left|f^{(n)}(a)\right|\right)\right]^{k+1}}. \quad (1.14)$$

Recently, some related inequalities for preinvex functions were also obtained in [4, 15, 16, 17].

## 2. New Definitions and lemmas

Now we introduce the concept of $\alpha$-logarithmic preinvex function.

**Definition 2.1.** Let $A \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : A \times A \to \mathbb{R}^n$. The function $f : A \to \mathbb{R}_+$ on the set $A$ is said to be $\alpha$-logarithmically preinvex function with respect to $\eta$, if for every $x, y \in A, t \in [0,1]$ and some $\alpha \in (0,1]$

$$f(y+t\eta(x,y)) \leq \left[f(x)\right]^{t^\alpha}\left[f(y)\right]^{1-t^\alpha}. \quad (2.1)$$

Clearly, when taking $\alpha = 1$ in (2.1), then $f$ becomes the standard logarithmically preinvex convex function on $A$.

In order to obtain our main results, we need the following lemmas.

**Lemma 2.1.** *For $n \in \mathbb{N}$, let $A \subseteq \mathbb{R}$ be an open invex set with respect to $\eta : A \times A \to \mathbb{R}$ and $a, b \in A$ with $\eta(a,b) > 0$ for all $a \neq b$. If $f : A \to \mathbb{R}$ is an $n$-times differentiable function on $A$ and $f^{(n)}$ is integrable on the $\eta$-path $P_{bc}$ for $c = b + \eta(a,b)$, then*

$$\sum_{k=0}^{n-1}\frac{1}{(k+1)!\eta(a,b)}\left[(b+\eta(a,b)-x)^{k+1}-(b-x)^{k+1}\right]f^{(k)}(x) - \frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)} f(x)\,\mathrm{d}x = \frac{(-1)^{n+1}[\eta(a,b)]^n}{n!}\left[\int_0^{\frac{x-b}{\eta(a,b)}} t^n f^{(n)}(b+t\eta(a,b))\,\mathrm{d}t + \int_{\frac{x-b}{\eta(a,b)}}^1 (t-1)^n f^{(n)}(b+t\eta(a,b))\,\mathrm{d}t\right], \quad (2.2)$$

*where $x \in [b, b+\eta(b,a)]$.*

*Proof.* Since $a, b \in A$ and $A$ is an invex set with respect to $\eta$, for every $t \in [0,1]$, it follows that $b + t\eta(a,b) \in A$. When $n = 1$, by integrating by part in the right-hand side of (2.2), it turns out that

$$\begin{aligned}
&\eta(a,b)\left[\int_0^{\frac{x-b}{\eta(a,b)}} tf'(b+t\eta(a,b))\,\mathrm{d}t + \int_{\frac{x-b}{\eta(a,b)}}^1 (t-1)f'(b+t\eta(a,b))\,\mathrm{d}t\right] \\
&= \frac{x-b}{\eta(a,b)}f(x) - \int_0^{\frac{x-b}{\eta(a,b)}} f(b+t\eta(a,b))\,\mathrm{d}t - \frac{x-b-\eta(a,b)}{\eta(a,b)}f(x) - \int_{\frac{x-b}{\eta(a,b)}}^1 f(b+t\eta(a,b))\,\mathrm{d}t \\
&= f(x) - \frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)} f(x)\,\mathrm{d}x.
\end{aligned}$$

Hence, the identity (2.2) holds for $n = 1$.

When $n = m$ and $m \geq 1$, suppose that the identity (2.2) is valid.

When $n = m + 1$, by the hypothesis, we have

$$\begin{aligned}
&\frac{(-1)^{m+2}[\eta(a,b)]^{m+1}}{(m+1)!}\left[\int_0^{\frac{x-b}{\eta(a,b)}} t^{m+1}f^{(m+1)}(b+t\eta(a,b))\,\mathrm{d}t\right. \\
&\left.+\int_{\frac{x-b}{\eta(a,b)}}^1 (t-1)^{m+1}f^{(m+1)}(b+t\eta(a,b))\,\mathrm{d}t\right] \\
&= \frac{(-1)^{m+2}[\eta(a,b)]^m}{(m+1)!}\left[\left(\frac{x-b}{\eta(a,b)}\right)^{m+1} f^{(m)}(x) - (m+1)\int_0^{\frac{x-b}{\eta(a,b)}} t^m f^{(m)}(b+t\eta(a,b))\,\mathrm{d}t\right. \\
&\left.- \left(\frac{x-b-\eta(a,b)}{\eta(a,b)}\right)^{m+1} f^{(m)}(x) - (m+1)\int_{\frac{x-b}{\eta(a,b)}}^1 (t-1)^m f^{(m)}(b+t\eta(a,b))\,\mathrm{d}t\right] \\
&= \frac{(-1)^{m+2}f^{(m)}(x)}{(m+1)!\eta(a,b)}\left[(x-b)^{m+1} - (x-b-\eta(a,b))^{m+1}\right] \\
&+ \frac{(-1)^{m+1}[\eta(a,b)]^m}{m!}\left[\int_0^{\frac{x-b}{\eta(a,b)}} t^m f^{(m)}(b+t\eta(a,b))\,\mathrm{d}t + \int_{\frac{x-b}{\eta(a,b)}}^1 (t-1)^m f^{(m)}(b+t\eta(a,b))\,\mathrm{d}t\right] \\
&= \sum_{k=0}^{m} \frac{1}{(k+1)!\eta(a,b)}\left[(b+\eta(a,b)-x)^{k+1} - (b-x)^{k+1}\right]f^{(k)}(x) - \frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)} f(x)\,\mathrm{d}x.
\end{aligned}$$

Therefore, when $n = m + 1$, the identity (2.2) holds. By induction, the proof of Lemma 2.1 is complete. □

*Remark* 2.1. Under the conditions of Lemma 2.1, taking $x = b + \frac{\eta(a,b)}{2}$, we get

$$\begin{aligned}
&\sum_{k=0}^{n-1} \frac{[\eta(a,b)]^k[1+(-1)^k]}{(k+1)!2^{k+1}} f^{(k)}\left(b + \frac{\eta(a,b)}{2}\right) - \frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)} f(x)\,\mathrm{d}x \\
&\quad = \frac{(-1)^{n+1}[\eta(a,b)]^n}{n!}\left[\int_0^{\frac{1}{2}} t^n f^{(n)}(b+t\eta(a,b))\,\mathrm{d}t + \int_{\frac{1}{2}}^1 (t-1)^n f^{(n)}(b+t\eta(a,b))\,\mathrm{d}t\right], \quad (2.3)
\end{aligned}$$

which may be found in [7].

**Lemma 2.2** ([14])**.** *Let $\mu > 0$ and $x \geq 0$. Then*

$$E(n;\mu,x) \triangleq \int_0^x t^n \mu^t\,\mathrm{d}t = \begin{cases} \frac{(-1)^{n+1}n!}{(\ln\mu)^{n+1}} + n!\mu^x \sum_{k=0}^n \frac{(-1)^k x^{n-k}}{(n-k)!(\ln\mu)^{k+1}}, & \mu \neq 1, \\ \frac{x^{n+1}}{n+1}, & \mu = 1 \end{cases} \quad (2.4)$$

*for* $n \geq 0, n \in \mathbb{N}$.

**Lemma 2.3** ([12]). *Let* $r > 0$, $\mu > 0$ *and* $x \geq 0$. *Then*

$$F(r;\mu,x) \triangleq \int_0^x t^{r-1}\mu^t \,\mathrm{d}t = \begin{cases} x^r\mu^x \sum_{k=1}^{\infty} \frac{(-x\ln\mu)^{k-1}}{(r)_k} < \infty, & \mu \neq 1, \\ \frac{x^r}{r}, & \mu = 1, \end{cases} \tag{2.5}$$

*where* $(r)_k = r(r+1)(r+2)\cdots(r+k-1)$.

**Lemma 2.4** ([3]). *Let* $0 < \phi \leq 1 \leq \psi$ *and* $0 < t, s \leq 1$. *Then*

$$\phi^{t^s} \leq \phi^{st} \tag{2.6}$$

*and*

$$\psi^{t^s} \leq \psi^{st+1-s}. \tag{2.7}$$

.

## 3. Hermite-Hadamard type inequalities

Now we start out to establish some new Hermite-Hadamard type inequalities for $n$-times differentiable and $\alpha$-logarithmically preinvex functions.

**Theorem 3.1.** *For* $n \in \mathbb{N}$ *and* $n \geq 1$, *let* $A \subseteq \mathbb{R}$ *be an open invex set with respect to* $\eta : A \times A \to \mathbb{R}$ *and* $a, b \in A$ *with* $\eta(a,b) > 0$ *for all* $a \neq b$. *Suppose that* $f : A \to \mathbb{R}$ *is an* $n$*-times differentiable function on* $A$ *and* $f^{(n)}$ *is integrable on the* $\eta$*-path* $P_{bc}$ *for* $c = b + \eta(a,b)$. *If* $\left|f^{(n)}\right|^q$ *is* $\alpha$*-logarithmically preinvex function on* $A$ *for* $q \geq 1$, *then for* $\alpha \in (0,1]$ *and* $x \in [b, b+\eta(a,b)]$

$$\begin{aligned} &\left|\sum_{k=0}^{n-1} \frac{1}{(k+1)!\eta(a,b)}\left[(b+\eta(a,b)-x)^{k+1} - (b-x)^{k+1}\right] f^{(k)}(x) - \frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)} f(x)\,\mathrm{d}x\right| \\ &\leq \frac{[\eta(a,b)]^{(n+1-q)/q}}{n!(n+1)^{1-1/q}} \Bigg\{ (x-b)^{(n+1)(q-1)/q} \left|f^{(n)}(a)\right|^{\xi} \left|f^{(n)}(b)\right|^{\zeta} \left[E\left(n;\mu^{q\alpha},\frac{x-b}{\eta(a,b)}\right)\right]^{1/q} \\ &\quad + (b+\eta(a,b)-x)^{(n+1)(q-1)/q} \left|f^{(n)}(a)\right|^{\delta} \left|f^{(n)}(b)\right|^{\sigma} \left[E\left(n;\mu^{-q\alpha},\frac{b+\eta(a,b)-x}{\eta(a,b)}\right)\right]^{1/q} \Bigg\}, \end{aligned} \tag{3.1}$$

*where* $\mu = \frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}$,

$$\{\xi,\zeta\} = \begin{cases} \{0,1\}, & \text{if } 0 < \mu \leq 1, \\ \{1-\alpha,\alpha\}, & \text{if } \mu > 1, \end{cases} \tag{3.2}$$

$$\{\delta,\sigma\} = \begin{cases} \{\alpha,1-\alpha\}, & \text{if } 0 < \mu \leq 1, \\ \{1,0\}, & \text{if } \mu > 1, \end{cases} \tag{3.3}$$

*and* $E(n;\mu,x)$ *is defined in* (2.4).

*Proof.* Since $a, b \in A$ and $A$ is an invex set with respect to $\eta$, for every $t \in [0,1]$, we have $b + t\eta(a,b) \in A$. Using Lemma 2.1, Hölder's inequality and $\alpha$-logarithmic preinvexity of $\left|f^{(n)}\right|^q$, we deduce that

$$\begin{aligned} &\left|\sum_{k=0}^{n-1} \frac{1}{(k+1)!\eta(a,b)}\left[(b+\eta(a,b)-x)^{k+1} - (b-x)^{k+1}\right] f^{(k)}(x) - \frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)} f(x)\,\mathrm{d}x\right| \\ &\leq \frac{[\eta(a,b)]^n}{n!}\left[\int_0^{\frac{x-b}{\eta(a,b)}} t^n \left|f^{(n)}(b+t\eta(a,b))\right|\mathrm{d}t + \int_{\frac{x-b}{\eta(a,b)}}^1 (1-t)^n \left|f^{(n)}(b+t\eta(a,b))\right|\mathrm{d}t\right] \end{aligned}$$

$$
\begin{aligned}
&\le \frac{[\eta(a,b)]^n}{n!}\Bigg\{\bigg[\int_0^{\frac{x-b}{\eta(a,b)}} t^n\,\mathrm{d}t\bigg]^{1-1/q}\bigg[\int_0^{\frac{x-b}{\eta(a,b)}} t^n\big|f^{(n)}(b+t\eta(a,b))\big|^q\,\mathrm{d}t\bigg]^{1/q}\\
&\quad+\bigg[\int_{\frac{x-b}{\eta(a,b)}}^1 (1-t)^n\,\mathrm{d}t\bigg]^{1-1/q}\bigg[\int_{\frac{x-b}{\eta(a,b)}}^1 (1-t)^n\big|f^{(n)}(b+t\eta(a,b))\big|^q\,\mathrm{d}t\bigg]^{1/q}\Bigg\}\\
&\le \frac{[\eta(a,b)]^{(n+1-q)/q}}{n!(n+1)^{1-1/q}}\Bigg\{(x-b)^{(n+1)(q-1)/q}\bigg[\int_0^{\frac{x-b}{\eta(a,b)}} t^n\big|f^{(n)}(a)\big|^{qt^\alpha}\big|f^{(n)}(b)\big|^{q(1-t^\alpha)}\,\mathrm{d}t\bigg]^{1/q}\\
&\quad+(b+\eta(a,b)-x)^{(n+1)(q-1)/q}\bigg[\int_{\frac{x-b}{\eta(a,b)}}^1 (1-t)^n\big|f^{(n)}(a)\big|^{qt^\alpha}\big|f^{(n)}(b)\big|^{q(1-t^\alpha)}\,\mathrm{d}t\bigg]^{1/q}\Bigg\}\\
&= \frac{[\eta(a,b)]^{(n+1-q)/q}\big|f^{(n)}(b)\big|}{n!(n+1)^{1-1/q}}\Bigg\{(x-b)^{(n+1)(q-1)/q}\bigg[\int_0^{\frac{x-b}{\eta(a,b)}} t^n\mu^{qt^\alpha}\,\mathrm{d}t\bigg]^{1/q}\\
&\quad+(b+\eta(a,b)-x)^{(n+1)(q-1)/q}\bigg[\int_{\frac{x-b}{\eta(a,b)}}^1 (1-t)^n\mu^{qt^\alpha}\,\mathrm{d}t\bigg]^{1/q}\Bigg\},
\end{aligned}
\tag{3.4}
$$

where $\mu=\dfrac{\big|f^{(n)}(a)\big|}{\big|f^{(n)}(b)\big|}$.

By Lemma 2.2 and Lemma 2.4, for $0<\mu\le 1$, we give

$$
\int_0^{\frac{x-b}{\eta(a,b)}} t^n\mu^{qt^\alpha}\,\mathrm{d}t\le\int_0^{\frac{x-b}{\eta(a,b)}} t^n\mu^{q\alpha t}\,\mathrm{d}t=E\bigg(n;\mu^{q\alpha},\frac{x-b}{\eta(a,b)}\bigg)
\tag{3.5}
$$

and

$$
\begin{aligned}
&\int_{\frac{x-b}{\eta(a,b)}}^1 (1-t)^n\mu^{qt^\alpha}\,\mathrm{d}t\le\int_{\frac{x-b}{\eta(a,b)}}^1 (1-t)^n\mu^{q\alpha t}\,\mathrm{d}t\\
&=\int_0^{\frac{b+\eta(a,b)-x}{\eta(a,b)}} t^n\mu^{q\alpha(1-t)}\,\mathrm{d}t=\mu^{q\alpha}E\bigg(n;\mu^{-q\alpha},\frac{b+\eta(a,b)-x}{\eta(a,b)}\bigg);
\end{aligned}
\tag{3.6}
$$

for $\mu>1$, we obtain

$$
\int_0^{\frac{x-b}{\eta(a,b)}} t^n\mu^{qt^\alpha}\,\mathrm{d}t\le\int_0^{\frac{x-b}{\eta(a,b)}} t^n\mu^{q(\alpha t+1-\alpha)}\,\mathrm{d}t=\mu^{q(1-\alpha)}E\bigg(n;\mu^{q\alpha},\frac{x-b}{\eta(a,b)}\bigg)
\tag{3.7}
$$

and

$$
\begin{aligned}
&\int_{\frac{x-b}{\eta(a,b)}}^1 (1-t)^n\mu^{qt^\alpha}\,\mathrm{d}t\le\int_{\frac{x-b}{\eta(a,b)}}^1 (1-t)^n\mu^{q(\alpha t+1-\alpha)}\,\mathrm{d}t\\
&=\int_0^{\frac{b+\eta(a,b)-x}{\eta(a,b)}} t^n\mu^{q(\alpha(1-t)+1-\alpha)}\,\mathrm{d}t=\mu^qE\bigg(n;\mu^{-q\alpha},\frac{b+\eta(a,b)-x}{\eta(a,b)}\bigg).
\end{aligned}
\tag{3.8}
$$

Substituting (3.5) to (3.8) into (3.4), we get the inequality (3.1).
Theorem 3.1 is thus proved. □

**Corollary 3.1.1.** *Under the assumptions of Theorem 3.1,*

(1) *if $q=1$, we know*

$$
\begin{aligned}
&\bigg|\sum_{k=0}^{n-1}\frac{1}{(k+1)!\eta(a,b)}\big[(b+\eta(a,b)-x)^{k+1}-(b-x)^{k+1}\big]f^{(k)}(x)-\frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)} f(x)\,\mathrm{d}x\bigg|\\
&\le\frac{[\eta(a,b)]^n}{n!}\Big\{\big|f^{(n)}(a)\big|^\xi\big|f^{(n)}(b)\big|^\zeta E\bigg(n;\mu^\alpha,\frac{x-b}{\eta(a,b)}\bigg)
\end{aligned}
$$

$$+\left|f^{(n)}(a)\right|^{\delta}\left|f^{(n)}(b)\right|^{\sigma}E\left(n;\mu^{-\alpha},\frac{b+\eta(a,b)-x}{\eta(a,b)}\right)\Bigg\}; \tag{3.9}$$

(2) *if* $\alpha=1$, *we get*

$$\begin{aligned}&\left|\sum_{k=0}^{n-1}\frac{1}{(k+1)!\eta(a,b)}\left[(b+\eta(a,b)-x)^{k+1}-(b-x)^{k+1}\right]f^{(k)}(x)-\frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)}f(x)\,\mathrm{d}x\right|\\ &\le\frac{[\eta(a,b)]^{(n+1-q)/q}}{n!(n+1)^{1-1/q}}\Bigg\{(x-b)^{(n+1)(q-1)/q}\left|f^{(n)}(b)\right|\left[E\left(n;\mu^{q},\frac{x-b}{\eta(a,b)}\right)\right]^{1/q}\\ &\quad+(b+\eta(a,b)-x)^{(n+1)(q-1)/q}\left|f^{(n)}(a)\right|\left[E\left(n;\mu^{-q},\frac{b+\eta(a,b)-x}{\eta(a,b)}\right)\right]^{1/q}\Bigg\};\end{aligned} \tag{3.10}$$

(3) *if* $\alpha=1$ *and* $q=1$, *we have*

$$\begin{aligned}&\left|\sum_{k=0}^{n-1}\frac{1}{(k+1)!\eta(a,b)}\left[(b+\eta(a,b)-x)^{k+1}-(b-x)^{k+1}\right]f^{(k)}(x)-\frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)}f(x)\,\mathrm{d}x\right|\\ &\le\frac{[\eta(a,b)]^{n}}{n!}\left\{\left|f^{(n)}(b)\right|E\left(n;\mu,\frac{x-b}{\eta(a,b)}\right)+\left|f^{(n)}(a)\right|E\left(n;\mu,\frac{b+\eta(a,b)-x}{\eta(a,b)}\right)\right\}.\end{aligned} \tag{3.11}$$

**Theorem 3.2.** *For* $n\in\mathbb{N}$ *and* $n\ge1$, *let* $A\subseteq\mathbb{R}$ *be an open invex set with respect to* $\eta:A\times A\to\mathbb{R}$ *and* $a,b\in A$ *with* $\eta(a,b)>0$ *for all* $a\ne b$. *Suppose that* $f:A\to\mathbb{R}$ *is an* $n$*-times differentiable function on* $A$ *and* $f^{(n)}$ *is integrable on the* $\eta$*-path* $P_{bc}$ *for* $c=b+\eta(a,b)$. *If* $\left|f^{(n)}\right|^{q}$ *is* $\alpha$*-logarithmically preinvex function on* $A$ *for* $q>1$, *then for* $\alpha\in(0,1]$, $x\in[b,b+\eta(a,b)]$ *and* $0\le r\le nq$, *we get*

$$\begin{aligned}&\left|\sum_{k=0}^{n-1}\frac{1}{(k+1)!\eta(a,b)}\left[(b+\eta(a,b)-x)^{k+1}-(b-x)^{k+1}\right]f^{(k)}(x)-\frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)}f(x)\,\mathrm{d}x\right|\\ &\le\frac{[\eta(a,b)]^{(r+1-q)/q}}{n!}\left(\frac{q-1}{nq+q-r-1}\right)^{1-1/q}\\ &\quad\times\Bigg\{(x-b)^{n+1-(r+1)/q}\left|f^{(n)}(a)\right|^{\xi}\left|f^{(n)}(b)\right|^{\zeta}\left[F\left(r+1;\mu^{q\alpha},\frac{x-b}{\eta(a,b)}\right)\right]^{1/q}\\ &\quad+(b+\eta(a,b)-x)^{n+1-(r+1)/q}\left|f^{(n)}(a)\right|^{\delta}\left|f^{(n)}(b)\right|^{\sigma}\left[F\left(r+1;\mu^{-q\alpha},\frac{b+\eta(a,b)-x}{\eta(a,b)}\right)\right]^{1/q}\Bigg\},\end{aligned} \tag{3.12}$$

*where* $\mu$, $\{\xi,\zeta\}$ *and* $\{\delta,\sigma\}$ *are given in Theorem 3.1, and* $F(r;\mu,x)$ *is defined by* (2.5).

*Proof.* Since $a,b\in A$ and $A$ is an invex set with respect to $\eta$, for every $t\in[0,1]$, it is easy to see that $b+t\eta(a,b)\in A$. From Lemma 2.1, Hölder's inequality and $\alpha$-logarithmic preinvexity of $\left|f^{(n)}\right|^{q}$, one has that

$$\begin{aligned}&\left|\sum_{k=0}^{n-1}\frac{1}{(k+1)!\eta(a,b)}\left[(b+\eta(a,b)-x)^{k+1}-(b-x)^{k+1}\right]f^{(k)}(x)-\frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)}f(x)\,\mathrm{d}x\right|\\ &\le\frac{[\eta(a,b)]^{n}}{n!}\left[\int_0^{\frac{x-b}{\eta(a,b)}}t^{n}\left|f^{(n)}(b+t\eta(a,b))\right|\mathrm{d}t+\int_{\frac{x-b}{\eta(a,b)}}^{1}(1-t)^{n}\left|f^{(n)}(b+t\eta(a,b))\right|\mathrm{d}t\right]\\ &\le\frac{[\eta(a,b)]^{n}}{n!}\left\{\left[\int_0^{\frac{x-b}{\eta(a,b)}}t^{(nq-r)/(q-1)}\,\mathrm{d}t\right]^{1-1/q}\left[\int_0^{\frac{x-b}{\eta(a,b)}}t^{r}\left|f^{(n)}(b+t\eta(a,b))\right|^{q}\mathrm{d}t\right]^{1/q}\right.\end{aligned}$$

$$
\begin{aligned}
&+\left[\int_{\frac{x-b}{\eta(a,b)}}^{1}(1-t)^{(nq-r)/(q-1)}\,\mathrm{d}t\right]^{1-1/q}\left[\int_{\frac{x-b}{\eta(a,b)}}^{1}(1-t)^{r}\left|f^{(n)}(b+t\eta(a,b))\right|^{q}\,\mathrm{d}t\right]^{1/q}\Bigg\}\\
&\leq\frac{[\eta(a,b)]^{(r+1-q)/q}}{n!}\left(\frac{q-1}{nq+q-r-1}\right)^{1-1/q}\\
&\quad\times\Bigg\{(x-b)^{n+1-(r+1)/q}\left[\int_{0}^{\frac{x-b}{\eta(a,b)}}t^{r}\left|f^{(n)}(a)\right|^{qt^{\alpha}}\left|f^{(n)}(b)\right|^{q(1-t^{\alpha})}\,\mathrm{d}t\right]^{1/q}\\
&\quad+(b+\eta(a,b)-x)^{n+1-(r+1)/q}\left[\int_{\frac{x-b}{\eta(a,b)}}^{1}(1-t)^{r}\left|f^{(n)}(a)\right|^{qt^{\alpha}}\left|f^{(n)}(b)\right|^{q(1-t^{\alpha})}\,\mathrm{d}t\right]^{1/q}\Bigg\}\\
&=\frac{[\eta(a,b)]^{(r+1-q)/q}\left|f^{(n)}(b)\right|}{n!}\left(\frac{q-1}{nq+q-r-1}\right)^{1-1/q}\Bigg\{(x-b)^{n+1-(r+1)/q}\left[\int_{0}^{\frac{x-b}{\eta(a,b)}}t^{r}\mu^{qt^{\alpha}}\,\mathrm{d}t\right]^{1/q}\\
&\quad+(b+\eta(a,b)-x)^{n+1-(r+1)/q}\left[\int_{\frac{x-b}{\eta(a,b)}}^{1}(1-t)^{r}\mu^{qt^{\alpha}}\,\mathrm{d}t\right]^{1/q}\Bigg\},
\end{aligned}
\tag{3.13}
$$

where $\mu=\dfrac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}$.

By Lemma 2.3 and Lemma 2.4, for $0<\mu\leq 1$, we show

$$
\int_{0}^{\frac{x-b}{\eta(a,b)}}t^{r}\mu^{qt^{\alpha}}\,\mathrm{d}t\leq\int_{0}^{\frac{x-b}{\eta(a,b)}}t^{r}\mu^{q\alpha t}\,\mathrm{d}t=F\left(r+1;\mu^{q\alpha},\frac{x-b}{\eta(a,b)}\right)
\tag{3.14}
$$

and

$$
\begin{aligned}
&\int_{\frac{x-b}{\eta(a,b)}}^{1}(1-t)^{r}\mu^{qt^{\alpha}}\,\mathrm{d}t\leq\int_{\frac{x-b}{\eta(a,b)}}^{1}(1-t)^{r}\mu^{q\alpha t}\,\mathrm{d}t\\
&=\int_{0}^{\frac{b+\eta(a,b)-x}{\eta(a,b)}}t^{r}\mu^{q\alpha(1-t)}\,\mathrm{d}t=\mu^{q\alpha}F\left(r+1;\mu^{-q\alpha},\frac{b+\eta(a,b)-x}{\eta(a,b)}\right);
\end{aligned}
\tag{3.15}
$$

for $\mu>1$, we state

$$
\int_{0}^{\frac{x-b}{\eta(a,b)}}t^{r}\mu^{qt^{\alpha}}\,\mathrm{d}t\leq\int_{0}^{\frac{x-b}{\eta(a,b)}}t^{r}\mu^{q(\alpha t+1-\alpha)}\,\mathrm{d}t=\mu^{q(1-\alpha)}F\left(r+1;\mu^{q\alpha},\frac{x-b}{\eta(a,b)}\right)
\tag{3.16}
$$

and

$$
\begin{aligned}
&\int_{\frac{x-b}{\eta(a,b)}}^{1}(1-t)^{r}\mu^{qt^{\alpha}}\,\mathrm{d}t\leq\int_{\frac{x-b}{\eta(a,b)}}^{1}(1-t)^{r}\mu^{q(\alpha t+1-\alpha)}\,\mathrm{d}t\\
&=\int_{0}^{\frac{b+\eta(a,b)-x}{\eta(a,b)}}t^{r}\mu^{q(\alpha(1-t)+1-\alpha)}\,\mathrm{d}t=\mu^{q}F\left(r+1;\mu^{-q\alpha},\frac{b+\eta(a,b)-x}{\eta(a,b)}\right).
\end{aligned}
\tag{3.17}
$$

Substituting (3.14) to (3.17) into (3.13), we get to the inequality (3.12).
The proof of Theorem 3.2 is established. □

**Corollary 3.2.1.** *Under the assumptions of Theorem 3.2,*

(1) *if $\alpha=1$, we write*

$$
\begin{aligned}
&\left|\sum_{k=0}^{n-1}\frac{1}{(k+1)!\eta(a,b)}\left[(b+\eta(a,b)-x)^{k+1}-(b-x)^{k+1}\right]f^{(k)}(x)-\frac{1}{\eta(a,b)}\int_{b}^{b+\eta(a,b)}f(x)\,\mathrm{d}x\right|\\
&\leq\frac{[\eta(a,b)]^{(r+1-q)/q}}{n!}\left(\frac{q-1}{nq+q-r-1}\right)^{1-1/q}
\end{aligned}
$$

$$\times\left\{(x-b)^{n+1-(r+1)/q}\left|f^{(n)}(b)\right|\left[F\left(r+1;\mu^q,\frac{x-b}{\eta(a,b)}\right)\right]^{1/q}\right.$$
$$\left.+(b+\eta(a,b)-x)^{n+1-(r+1)/q}\left|f^{(n)}(a)\right|\left[F\left(r+1;\mu^{-q},\frac{b+\eta(a,b)-x}{\eta(a,b)}\right)\right]^{1/q}\right\}; \tag{3.18}$$

(2) *if* $r=nq$, *we find*

$$\left|\sum_{k=0}^{n-1}\frac{1}{(k+1)!\eta(a,b)}\left[(b+\eta(a,b)-x)^{k+1}-(b-x)^{k+1}\right]f^{(k)}(x)-\frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)}f(x)\,\mathrm{d}x\right|$$
$$\le\frac{[\eta(a,b)]^{(nq+1-q)/q}}{n!}\left\{(x-b)^{1-1/q}\left|f^{(n)}(a)\right|^{\xi}\left|f^{(n)}(b)\right|^{\zeta}\left[F\left(nq+1;\mu^{q\alpha},\frac{x-b}{\eta(a,b)}\right)\right]^{1/q}\right.$$
$$\left.+(b+\eta(a,b)-x)^{1-1/q}\left|f^{(n)}(a)\right|^{\delta}\left|f^{(n)}(b)\right|^{\sigma}\left[F\left(nq+1;\mu^{-q\alpha},\frac{b+\eta(a,b)-x}{\eta(a,b)}\right)\right]^{1/q}\right\}. \tag{3.19}$$

*In particular, if* $r=nq$ *and* $\alpha=1$, *one has*

$$\left|\sum_{k=0}^{n-1}\frac{1}{(k+1)!\eta(a,b)}\left[(b+\eta(a,b)-x)^{k+1}-(b-x)^{k+1}\right]f^{(k)}(x)-\frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)}f(x)\,\mathrm{d}x\right|$$
$$\le\frac{[\eta(a,b)]^{(nq+1-q)/q}}{n!}\left\{(x-b)^{1-1/q}\left|f^{(n)}(b)\right|\left[F\left(nq+1;\mu^{q},\frac{x-b}{\eta(a,b)}\right)\right]^{1/q}\right.$$
$$\left.+(b+\eta(a,b)-x)^{1-1/q}\left|f^{(n)}(a)\right|\left[F\left(nq+1;\mu^{-q},\frac{b+\eta(a,b)-x}{\eta(a,b)}\right)\right]^{1/q}\right\}; \tag{3.20}$$

(3) *if* $r=0$, *we observe*

$$\left|\sum_{k=0}^{n-1}\frac{1}{(k+1)!\eta(a,b)}\left[(b+\eta(a,b)-x)^{k+1}-(b-x)^{k+1}\right]f^{(k)}(x)-\frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)}f(x)\,\mathrm{d}x\right|$$
$$\le\begin{cases}\frac{[\eta(a,b)]^{(1-q)/q}}{n!(q\alpha)^{1/q}}\left(\frac{q-1}{nq+q-1}\right)^{1-1/q}\left\{(x-b)^{n+1-1/q}\left|f^{(n)}(a)\right|^{\xi}\right.\\ \times\left|f^{(n)}(b)\right|^{\zeta-\alpha(x-b)/\eta(a,b)}\left[\frac{\left|f^{(n)}(a)\right|^{(x-b)q\alpha/\eta(a,b)}-\left|f^{(n)}(b)\right|^{(x-b)q\alpha/\eta(a,b)}}{\ln\left|f^{(n)}(a)\right|-\ln\left|f^{(n)}(b)\right|}\right]^{1/q}\\ +(b+\eta(a,b)-x)^{n+1-1/q}\left|f^{(n)}(a)\right|^{\delta-\alpha(b+\eta(a,b)-x)/\eta(a,b)}\left|f^{(n)}(b)\right|^{\sigma}\\ \left.\times\left[\frac{\left|f^{(n)}(b)\right|^{(b+\eta(a,b)-x)q\alpha/\eta(a,b)}-\left|f^{(n)}(a)\right|^{(b+\eta(a,b)-x)q\alpha/\eta(a,b)}}{\ln\left|f^{(n)}(b)\right|-\ln\left|f^{(n)}(a)\right|}\right]^{1/q}\right\}, & \text{if } \mu\ne1,\\ \frac{\left|f^{(n)}(b)\right|}{n!\eta(a,b)}\left(\frac{q-1}{nq+q-1}\right)^{1-1/q}\left[(x-b)^{n+1}+(b+\eta(a,b)-x)^{n+1}\right], & \text{if } \mu=1.\end{cases} \tag{3.21}$$

*Specially, if* $r=0$ *and* $\alpha=1$, *one has*

$$\left|\sum_{k=0}^{n-1}\frac{1}{(k+1)!\eta(a,b)}\left[(b+\eta(a,b)-x)^{k+1}-(b-x)^{k+1}\right]f^{(k)}(x)-\frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)}f(x)\,\mathrm{d}x\right|$$

$$\leq \begin{cases} \frac{[\eta(a,b)]^{(1-q)/q}}{n!q^{1/q}}\left(\frac{q-1}{nq+q-1}\right)^{1-1/q}\Big\{(x-b)^{n+1-1/q}\big|f^{(n)}(b)\big|^{(b+\eta(a,b)-x)/\eta(a,b)} \\ \times\left[\frac{\big|f^{(n)}(a)\big|^{(x-b)q/\eta(a,b)}-\big|f^{(n)}(b)\big|^{(x-b)q/\eta(a,b)}}{\ln\big|f^{(n)}(a)\big|-\ln\big|f^{(n)}(b)\big|}\right]^{1/q} \\ +(b+\eta(a,b)-x)^{n+1-1/q}\big|f^{(n)}(a)\big|^{(x-b)/\eta(a,b)} \\ \times\left[\frac{\big|f^{(n)}(b)\big|^{(b+\eta(a,b)-x)q/\eta(a,b)}-\big|f^{(n)}(a)\big|^{(b+\eta(a,b)-x)q/\eta(a,b)}}{\ln\big|f^{(n)}(b)\big|-\ln\big|f^{(n)}(a)\big|}\right]^{1/q}\Big\}, & \text{if } \mu\neq 1, \\ \frac{\big|f^{(n)}(b)\big|}{n!\eta(a,b)}\left(\frac{q-1}{nq+q-1}\right)^{1-1/q}\big[(x-b)^{n+1}+(b+\eta(a,b)-x)^{n+1}\big], & \text{if } \mu=1. \end{cases} \tag{3.22}$$

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## References


[1] T. Antczak, *Mean value in invexity analysis*, Nonlinear Anal. **60** (2005), no. 8, 1471–1484; Available online at http://dx.doi.org/10.1016/j.na.2004.11.005.

[2] A. Barani, A. G. Ghazanfariand, and S. S. Dragomir, *Hermite-Hadamard inequality for functions whose derivatives absolute values are preinvex*, J. Inequal. Appl. **2012** (2012):247; Available online at http://dx.doi.org/10.1186/1029-242X-2012-247.

[3] R. F. Bai, F. Qi, B. Y. Xi, *Hermite-Hadamard type inequalities for the m- and $(\alpha, m)$-logarithmically convex functions*, Filomat **27:1** (2013), 1–7; Available online at http://dx.doi.org/10.2298/FIL1301001B.

[4] A. Çelik, B. Dal, M. Kisa, *On integral inequalities of hermite-hadamard type for logarithmically H-Preinvex functions via fractional integrals*, International Journal of Pure and Applied Mathematics, **16:1** (2016), 279–289.

[5] S. S. Dragomir and R. P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. Lett. **11** (1998), no. 5, 91–95; Available online at http://dx.doi.org/10.1016/S0893-9659(98)00086-X.

[6] U. S. Kirmaci, *Inequalities for differentiable mappings and applications to special means of real numbers to midpoint formula*, Appl. Math. Comput. **147** (2004), no. 1, 137–146.

[7] M. A. Latif and S. S. Dragomir, *On Hermite-Hadamard type integral inequalities for n-times differentiable log-preinvex functions*, Miskolc Mathematical Notes, **16**, no. 1 (2016), 339–342.

[8] S. R. Mohan and S. K Neogy, *On invex sets and preinvex function*, J. Math. Anal. Appl. **189** (1995), no. 3, 901–908; Available online at http://dx.doi.org/10.1006/jmaa.1995.1057.

[9] M. A. Noor, *Hermite-Hadamard integral inequalities for log-preinvex functions*, J. Math. Anal. Approx. Theory **2** (2007),126–131.

[10] M. A. Noor, *Variational-like inequalities*, Optimization **30** (1994),323–330.

[11] M. Z. Sarikaya, H. Bozkurt, and N. Alp, *On Haermite-Hadamard type integral inequalities for preinvex and log-preinvex functions*, Contemporary Analysis and Applied Mathematics **1** (2013), no. 2, 237–252.

[12] J. Wang, J. Deng and M. Fečkan, *Haermite-Hadamard type inequalities for r-convex functions via Riemann-Liouville fractional integrals*, Ukrainian Math. J. **65** (2013), 193–211.

[13] T. Weir and B. Mond, *Preinvex functions in multiple objective optimization*, Journal of Mathematical Analysis and Applications **136** (1998), 29–38.

[14] S. H. Wang and X. M. Liu, *New Hermite-Hadamard type integral inequalities for n-times differentiable and s-logarithmically preinvex functions*, Abstact and Appllied Analysis, **2014** (2014), Article ID 725987, 11 pages; Available online at http://dx.doi.org/10.1155/2014/725987.

[15] S. H. Wang and F. Qi, *Hermite-Hadamard type integral inequalities for n-times differentiable and preinvex functions*, Journal of Inequalities and Applications **49**(2014); Available online at http://dx.doi.org/10.1186/1029-242X-2014-49.

[16] Y. Wang, S.-H. Wang, and F. Qi, *Simpson type integral inequalities in which the power of the absolute value of the first derivative of the integrand is s-preinvex*, Facta Univ. Ser. Math. Inform. **28** (2013), no. 2, 151–159.

[17] Y. Wang, M. M. Zheng, and F. Qi, *Integral inequalities of Hermite-Hadamard type for functions whose derivatives are $\alpha$-preinvex*, Journal of Inequalities and Applications **2014:97** (2014); Available online at http://dx.doi.org/10.1186/1029-242X-2014-97.

# Existence of solutions for boundary value problems of fractional differential equation in Banach spaces*

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**Abstract**

In this paper, with the help of a new estimation technique for the measure of noncompactness, under more general conditions of growth and noncompactness measure, combining with the Sadovskii's fixed point theorem and Leray-Schauder type fixed point theorem of condensing mapping, we obtain the existence of solutions for the following boundary value problems of fractional differential equation in Banach spaces

$$\begin{cases} -{}^{C}D_{0^+}^{\beta}u(t) = f(t,u(t)), \quad t \in J, \\ u(0) = u(1) = \theta, \end{cases}$$

where $1 < \beta \leq 2$ is a real number, $J = [0,1]$, ${}^{C}D_{0^+}^{\beta}$ is the Caputo fractional derivative of order $\beta$, $f : J \times E \to E$ is continuous, $E$ is a Banach spaces, $\theta$ is the zero element of $E$.

**Keywords:** Banach spaces; fractional differential equation; measure of noncompactness; condensing mapping

**Mathematics Subject Classification(2010):** 34A08; 34B15; 47H08; 49K27

## 1 Introduction

Differential equations of fractional order occur more frequently in difference research areas and engineering, such as physics, chemistry. For the details, one can

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see the monographs [1-5]. Recently, many people pay attention to the boundary value problem for fractional order ordinary differential equation involving Caputo's derivative, see [8-29] and the references therein. However, as far as we know, there are few papers studied the existence of solution to boundary value problems for fractional differential equation involving Caputo's derivative in abstract space.

In 1988, Guo and Lakshmikantham [6] firstly studied the existence of solutions for the following boundary value problem of nonlinear second order differential equation in Banach spaces

$$\begin{cases} -u''(t) = f(t, u(t)), \quad t \in [0, 1], \\ u(0) = u(1) = \theta, \end{cases} \tag{1}$$

where $f : J \times E \to E$ is continuous, $E$ is a Banach spaces, $\theta$ is the zero element of $E$.

In 2008, Li and Guo [7] obtained an existence result of positive solution of BVP (1) by employing a new estimate of noncompactness measure and the fixed point index theory of condensing mapping.

Motivated by the above mentioned works, in this paper, with the help of a new estimation technique of noncompactness measure, under more general conditions of growth and noncompactness measure, combining with the Sadovskii's fixed point theorem and Leray-Schauder type fixed point theorem of condensing mapping, we consider the existence of solutions for the following boundary value problem (BVP) of fractional differential equation in Banach space $E$

$$\begin{cases} -{}^{C}D_{0^+}^{\beta} u(t) = f(t, u(t)), \quad t \in J, \\ u(0) = u(1) = \theta, \end{cases} \tag{2}$$

where $1 < \beta \leq 2$ is real number, $J = [0, 1]$, ${}^{C}D_{0^+}^{\beta}$ is the Caputo fractional derivative of order $\beta$, $f : J \times E \to E$ is continuous, $\theta$ is the zero element of Banach space $E$. It is easy to see that if $\beta = 2$, then BVP (2) will degrade into the problem (1), which was studied in [6] and [7].

# 2 Preliminaries

Throughout this paper, we denote by $C(J, E)$ the Banach space of all continuous $E$-value functions on interval $J$ with the supnorm $\|u\|_C = \sup_{t \in J} \|u(t)\|$.

**Definition 2.1** *The fractional integral of order $q > 0$ with the lower limit zero for a function $g$ is defined as*

$$I_{0^+}^q g(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s) ds, \quad t > 0,$$

*where $\Gamma(\cdot)$ is the gamma function.*

**Definition 2.2** *The Caputo fractional derivative of order $q > 0$ with the lower limit zero for a function $g$ is defined as*

$${}^C D_{0^+}^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s) ds, \quad t > 0, \quad 0 \le n-1 < q < n,$$

*where the function $g(t)$ has absolutely continuous derivatives up to order $n-1$.*

If $g$ is an abstract function with values in $E$, then the integrals which appear in Definitions 2.1 and 2.2 are taken in Bochner's sense.

Next, we recall some properties of the measure of noncompactness that will be used in the proof of our main results. In order to no confusion may occur, we denote by $\alpha(\cdot)$ the Kuratowski measure of noncompactness on both the bounded sets of $E$ and $C(J,E)$. For the details of the definition and properties of the measure of noncompactness, we refer to the monographs [30] and [31]. For any $D \subset C(J,E)$ and $t \in J$, set $D(t) = \{u(t) \mid u \in D\} \subset E$. If $D \subset C(J,E)$ is bounded, then $D(t)$ is bounded in $E$ and $\alpha(D(t)) \le \alpha(D)$.

**Lemma 2.3** ([36]). *Let $E$ be a Banach space, and let $D \subset C(J,E)$ be bounded and equicontinuous. Then $\alpha(D(t))$ is continuous on $J$, and*

$$\max_{t \in J} \alpha(D(t)) = \alpha(D) = \alpha(D(J)).$$

**Lemma 2.4** ([35]). *Let $E$ be a Banach space, and let $D = \{u_n\} \subset C(J,E)$ be a bounded and countable set. Then $\alpha(D(t))$ is Lebesgue integral on $J$, and*

$$\alpha\Big(\Big\{ \int_J u_n(t) dt \mid n \in \mathbb{N} \Big\}\Big) \le 2 \int_J \alpha(D(t)) dt.$$

**Lemma 2.5** ([32,33]). *Let $E$ be a Banach space, and let $D \subset E$ be bounded. Then there exists a countable set $D_0 \subset D$, such that $\alpha(D) \le 2\alpha(D_0)$.*

**Lemma 2.6** ([34]). *Let $u \in C(J, E)$, $\psi \in L(J, \mathbb{R}^+)$. Then*

$$\int_J u(s)\psi(s)ds \in \int_J \psi(s)ds\overline{co}\, u(J),$$

*where $u(J) = \{u(t) \mid t \in J\}$.*

**Lemma 2.7** ([31]). *Let $E$ be a Banach space. Assume that $D \subset E$ is a bounded closed and convex set on $E$, $Q : D \to D$ is condensing. Then $Q$ has at least one fixed point in $D$.*

**Lemma 2.8** ([36]). *Let $E$ be a Banach space. Assume that $Q : D \to D$ is condensing, if $\{x \mid x = \lambda Qx, 0 < \lambda < 1\}$ is a bounded set. Then $Q$ has at least one fixed point in $D$.*

With the help of the existence of solutions for fractional differential equation in real number space, we firstly get the corresponding existence and uniqueness of the fractional linear differential equation in Banach spaces.

**Lemma 2.9.** *Assume that $1 < \beta \leq 2$ and $J = [0, 1]$. Then for any $h \in C(J, E)$, the boundary value problems of fractional linear differential equation in Banach spaces (LBVP)*

$$\begin{cases} -{}^C D^{\beta}_{0^+} u(t) = h(t), \quad t \in J, \\ \\ u(0) = u(1) = \theta, \end{cases} \tag{3}$$

*has a unique solution*

$$u(t) = \int_0^1 G(t, s)h(s)ds := Th(t), \tag{4}$$

*where*

$$G(t, s) = \frac{1}{\Gamma(\beta)} \begin{cases} t(1-s)^{\beta-1} - (t-s)^{\beta-1}, & 0 \leq s \leq t \leq 1, \\ t(1-s)^{\beta-1}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{5}$$

**Proof.** From the expression (4), we first easily know that $u(t)$ is a solution of LBVP (3). Next, we prove $u(t)$ is uniqueness solution.

Suppose $u_1(t), u_2(t) \in C(J, E)$ is solutions of LBVP (3). For any $\varphi \in E^*$, where $E^*$ is conjugate space of $E$. Let $r(t) = \varphi(u_1(t) - u_2(t))$. Then $r(t)$ is a solution

about pure variable $t$ of fractional linear differential equation

$$\begin{cases} -{}^{C}D_{0^+}^{\beta} r(t) = 0, \quad 1 < \beta \leq 2, \quad t \in J, \\ \\ r(0) = r(1) = 0. \end{cases}$$

Evidently, $r(t) \equiv 0$, according to the arbitrary of $\varphi \in E^*$, we know $u_1(t) - u_2(t) \equiv \theta$, also, $u_1(t) \equiv u_2(t)$ on $J$. Thus, $u(t)$ expressed by (4) is a unique solution of LBVP (3). This completes the proof of Lemma 2.9. $\square$

**Lemma 2.10.** *The integral operator $T : C(J, E) \to C(J, E)$ defined by (4) satisfies the following inequality of norm*

$$\|T\| \leq \frac{1}{\Gamma(\beta+1)} \Big( \beta^{\frac{1}{1-\beta}} - \beta^{\frac{\beta}{1-\beta}} \Big).$$

**Proof.** For any $h \in C(J, E)$, by the definition of $T$, we have

$$\|Th(t)\| \leq \| \int_0^1 G(t, s) h(s) ds \|,$$

replace above $G(t, s)$ with the expression of $G(t, s)$ defined by (5), we obtain

$$\|Th(t)\| \leq \Big( \frac{t}{\Gamma(\beta+1)} - \frac{t^{\beta}}{\Gamma(\beta+1)} \Big) \|h\|_C,$$

and since, we easily know

$$\frac{1}{\Gamma(\beta+1)} \Big( t - t^{\beta} \Big) \|h\|_C \leq \frac{1}{\Gamma(\beta+1)} \Big( \beta^{\frac{1}{1-\beta}} - \beta^{\frac{\beta}{1-\beta}} \Big) \|h\|_C$$

Hence

$$\|Th\| \leq \frac{1}{\Gamma(\beta+1)} \Big( \beta^{\frac{1}{1-\beta}} - \beta^{\frac{\beta}{1-\beta}} \Big) \|h\|_C,$$

namely,

$$\|T\| \leq \frac{1}{\Gamma(\beta+1)} \Big( \beta^{\frac{1}{1-\beta}} - \beta^{\frac{\beta}{1-\beta}} \Big).$$

This proof is completed. $\square$

Let us introduce the following assumptions which are used hereafter.

(F1) There exist constants $c_0, c_1 > 0$ satisfying $c_1 < \Big( \frac{1}{\Gamma(\beta+1)} \Big( \beta^{\frac{1}{1-\beta}} - \beta^{\frac{\beta}{1-\beta}} \Big) \Big)^{-1}$, such that for any $t \in J$ and $x \in E$, $\|f(t, x)\| \leq c_1 \|x\| + c_0$;

(F2) There exist constant $L$ satisfying $0 < L < \frac{\left(\frac{1}{\Gamma(\beta+1)}\left(\beta^{\frac{1}{1-\beta}}-\beta^{\frac{\beta}{1-\beta}}\right)\right)^{-1}}{4}$, such that for any $t \in J$ and bounded set $D \subset E$, $\alpha(f(t,D)) \leq L\alpha(D)$;

(F3) There exist constant $L$ satisfying $0 < L < \frac{\left(\frac{1}{\Gamma(\beta+1)}\left(\beta^{\frac{1}{1-\beta}}-\beta^{\frac{\beta}{1-\beta}}\right)\right)^{-1}}{2}$, such that for every bounded set $D \subset E$, $\alpha(f(J,D)) < L\alpha(D)$;

(F4) There exist constant $L$ satisfying $0 < L < \left(\frac{1}{\Gamma(\beta+1)}\left(\beta^{\frac{1}{1-\beta}}-\beta^{\frac{\beta}{1-\beta}}\right)\right)^{-1}$, such that for every bounded set $D \subset E$, $\alpha(f(J,D)) < L\alpha(D)$.

Defined by the following integral operator $Q : C(J,E) \to C(J,E)$

$$Qu(t) = \int_0^1 G(t,s)f(s,u(s))ds. \tag{6}$$

To be convenient for next prove procedure, we firstly introduce the following two Lemmas.

**Lemma 2.11.** *Assume that (F1) and (F2) hold. Then $Q : C(J,E) \to C(J,E)$ is a condensing mapping.*

**Proof.** By (6) we easily know that $Q$ is a continuous operator. Next, we show that $Q$ is a condensing mapping.

When (F1) hold, we see easily that $Q$ maps bounded set in $C(J,E)$ into bounded and equicontinuous set in $C(J,E)$. For every non-relatively compact bounded set $D \subset C(J,E)$, we will prove in the following that $\alpha(Q(D)) < \alpha(D)$.

For every bounded set $D \subset C(J,E)$, by the Lemma 2.5, there exists a countable subset $D_1 = \{u_n\} \subset D$, such that $\alpha(Q(D)) \leq 2\alpha(Q(D_1))$. Since $\alpha(Q(D_1))$ is a bounded and equicontinuous set, by the Lemma 2.3, we have $\alpha(Q(D_1)) = \max_{t\in J}\alpha(Q(D_1(t)))$. By the Lemma 2.4, for any $t \in J$, we can obtain

$$\begin{aligned}
\alpha(Q(D_1(t))) &= \alpha\Big(\Big\{\int_0^1 G(t,s)f(s,u_n(s))ds \mid n \in \mathbb{N}\Big\}\Big) \\
&\leq 2\int_0^1 \alpha\Big(\Big\{G(t,s)f(s,u_n(s))ds \mid n \in \mathbb{N}\Big\}\Big)ds \\
&\leq 2\int_0^1 G(t,s)\alpha\Big(f(s,D_1(s))\Big)ds.
\end{aligned}$$

According to properties of the measure of noncompactness and condition (F2), we have

$$\alpha(f(s, D_1(s)) \leq L\alpha(D_1(s)) \leq L\alpha(D_1) \leq L\alpha(D).$$

Further

$$\alpha(Q(D_1(t))) \leq 2L\alpha(D) \int_0^1 G(t,s)ds.$$

From this inequality, we have

$$\begin{aligned} \alpha(Q(D_1)) &= \max_{t\in J} \alpha(Q(D_1(t))) \\ &\leq 2L\alpha(D) \max_{t\in J} \int_0^1 G(t,s)ds \\ &\leq \frac{2L}{\Gamma(\beta+1)} \Big(\beta^{\frac{1}{1-\beta}} - \beta^{\frac{\beta}{1-\beta}}\Big)\alpha(D). \end{aligned}$$

From the condition (F2), we get that

$$\alpha(Q(D)) \leq 2\alpha(Q(D_1)) \leq \frac{4L}{\Gamma(\beta+1)} \Big(\beta^{\frac{1}{1-\beta}} - \beta^{\frac{\beta}{1-\beta}}\Big)\alpha(D) < \alpha(D).$$

Then, $Q : C(J,E) \to C(J,E)$ is a condensing mapping. This completes the proof of Lemma 2.11. $\square$

**Lemma 2.12.** *Assume that (F1) and (F3) hold. Then $Q : C(J,E) \to C(J,E)$ is a condensing mapping.*

**Proof.** By $Q$ defined by (6) and condition (F1), we easily know that $Q$ maps bounded set in $Q : C(J,E)$ into bounded and equicontinuous set in $Q : C(J,E)$. For every non-relatively compact bounded set $D \subset C(J,E)$, we hope prove that $\alpha(Q(D)) < \alpha(D)$.

For every bounded set $D \subset C(J,E)$, since $Q(D)$ is bounded and equicontinuous set, by the Lemma 2.3, we know $\alpha(Q(D)) = \max_{t\in J} \alpha(Q(D(t)))$.

Denote $B = D(J)$, then $B \subset E$ is bounded and $\alpha(B) \leq 2\alpha(D)$. For any $t \in J, u \in D$, by the Lemma 2.6, we obtain

$$Qu(t) = \int_0^1 G(t,s)f(s,u(s)) \in \int_0^1 G(t,s)ds\overline{\text{co}}\, f(J\times B),$$

$$Q(D)(t) \subset \int_0^1 G(t,s)ds\overline{\text{co}}\, f(J\times B).$$

Combining with properties of the measure of noncompactness and condition (F3), we have

$$\begin{aligned}\alpha(Q(D)(t)) &\leq \int_0^1 G(t,s)ds\alpha\Big(\overline{\text{co}}\, f(J\times B)\Big)\\ &= \int_0^1 G(t,s)ds\alpha\Big(f(J\times B)\Big)\\ &\leq L\int_0^1 G(t,s)ds\alpha(B)\\ &\leq 2L\int_0^1 G(t,s)ds\alpha(D)\\ &\leq \frac{2L}{\Gamma(\beta+1)}\Big(\beta^{\frac{1}{1-\beta}}-\beta^{\frac{\beta}{1-\beta}}\Big)\alpha(D).\end{aligned}$$

From condition (F3), make the maximum value on both sides of this inequality, we know

$$\alpha(Q(D)) = \max_{t\in J}\alpha(Q(D(t))) \leq \frac{2L}{\Gamma(\beta+1)}\Big(\beta^{\frac{1}{1-\beta}}-\beta^{\frac{\beta}{1-\beta}}\Big)\alpha(D) < \alpha(D).$$

Therefore,$Q : C(J,E)\to C(J,E)$ is a condensing mapping. This proof is completed. □

By the definition of $Q$, we know that the solution of BVP (2) is equivalent to the fixed point of $Q$. We will find the fixed point of $Q$ by using the Leray-Schauder fixed point theorem and the Sadovskii's fixed point theorem about condensing mapping.

# 3 Main results

**Theorem 3.1.** *Assume that $f : J\times E\to E$ is continuous and satisfies the conditions (F1) and (F2). Then BVP (2) has at least one solution.*

**Proof.** We firstly prove that $\Omega = \{u\in C(J,E) \mid u = \lambda Qu, 0<\lambda<1\}$ is a bounded set in $C(J,E)$. In fact, for any $u\in\Omega$ satisfies

$$u(t) = \lambda\, Qu(t) = \lambda\int_0^1 G(t,s)f(s,u(s))ds,\ 0<\lambda<1.$$

Combining with condition (F1), we have

$$\begin{aligned}\|u(t)\| &= \lambda\|\int_0^1 G(t,s)f(s,u(s))ds\| \\ &\leq \int_0^1 G(t,s)(c_0+c_1\|u(s)\|)ds \\ &\leq c_0\int_0^1 G(t,s)ds + c_1T_0\|u(t)\|,\end{aligned}$$

where $T\|u(t)\| = \int_0^1 G(t,s)\|u(s)\|ds$ is defined by (3). Since $\|T\| \leq \frac{1}{\Gamma(\beta+1)}\Big(\beta^{\frac{1}{1-\beta}} - \beta^{\frac{\beta}{1-\beta}}\Big)$, we can obtain $\|c_1T\| < \frac{c_1}{\Gamma(\beta+1)}\Big(\beta^{\frac{1}{1-\beta}} - \beta^{\frac{\beta}{1-\beta}}\Big) < 1$, and by the famous perturbation theorem of unit operator, we easily know that $I - c_1T$ exists a bounded inverse operator $(I-c_1T)^{-1}$, and

$$(I-c_1T)^{-1} = \sum_{n=0}^{\infty}(c_1T)^n$$

is positive. Hence

$$\begin{aligned}\|u\|_C &\leq \frac{c_0}{\Gamma(\beta+1)}\Big(\beta^{\frac{1}{1-\beta}} - \beta^{\frac{\beta}{1-\beta}}\Big)(I-c_1T)^{-1}(1) \\ &= \frac{c_0}{\Gamma(\beta+1)}\Big(\beta^{\frac{1}{1-\beta}} - \beta^{\frac{\beta}{1-\beta}}\Big)\sum_{n=0}^{\infty}(c_1T)^n(1).\end{aligned}$$

Since $c_1 < \Big(\frac{1}{\Gamma(\beta+1)}\Big(\beta^{\frac{1}{1-\beta}} - \beta^{\frac{\beta}{1-\beta}}\Big)\Big)^{-1}$, then there exist $\varepsilon > 0$ such that $c_1 + \varepsilon < \Big(\frac{1}{\Gamma(\beta+1)}\Big(\beta^{\frac{1}{1-\beta}} - \beta^{\frac{\beta}{1-\beta}}\Big)\Big)^{-1}$. Namely

$$\|c_1^nT^n\| \leq c_1^n\|T\|^n \leq \Big(\frac{c_1}{c_1+\varepsilon}\Big)^n.$$

By the fact that $\sum_{n=0}^{\infty}\Big(\frac{c_1}{c_1+\varepsilon}\Big)^n$ is convergence, we know that $\sum_{n=0}^{\infty}c_1^n\|T^n\|$ is convergence. Denote $M_0 = \sum_{n=0}^{\infty}c_1^n\|T^n\| < +\infty$. Then $\|u\|_C \leq \frac{c_0}{\Gamma(\beta+1)}\Big(\beta^{\frac{1}{1-\beta}} - \beta^{\frac{\beta}{1-\beta}}\Big)M_0$. Therefore, combined with Lemma 2.8 and Lemma 2.11, we get that the operator $Q$ has a fixed point, which is the solution of BVP (2). This proof is completed. □

Similar with the Proof of Theorem 3.1, we can obtain the following result.

**Theorem 3.2.** *Assume that $f : J \times E \to E$ is continuous and satisfies the conditions (F1) and (F3). Then BVP (2) has at least one solution.*

**Theorem 3.3.** *Assume that $f : J \times E \to E$ is continuous and satisfies the conditions (F1) and (F4). Then BVP (2) has at least one solution.*

**Proof.** Firstly, when (F1) hold, we easily know $Q$ maps bounded set in $C(J, E)$ into bounded and equicontinuous set in $C(J, E)$.

Secondly, let $\Omega_R = \overline{B_C}(\theta, R)$ be a closed ball in $C(J, E)$, where

$$R \geq \frac{c_0\Big(\beta^{\frac{1}{1-\beta}} - \beta^{\frac{\beta}{1-\beta}}\Big)}{\Gamma(\beta+1) - c_1\Big(\beta^{\frac{1}{1-\beta}} - \beta^{\frac{\beta}{1-\beta}}\Big)}.$$

Clearly, $\Omega_R$ is a bounded set in $C(J, E)$. For any $t \in J, u \in \Omega_R$, according to the condition (F1), we have

$$\begin{aligned}
\|(Qu)(t)\| &\leq \int_0^1 G(t,s)\|f(s,u(s))\|ds \\
&\leq \int_0^1 G(t,s)(c_0 + c_1\|u(s)\|)ds \\
&\leq \frac{c_0}{\Gamma(\beta+1)}\Big(\beta^{\frac{1}{1-\beta}} - \beta^{\frac{\beta}{1-\beta}}\Big) + \frac{c_1}{\Gamma(\beta+1)}\Big(\beta^{\frac{1}{1-\beta}} - \beta^{\frac{\beta}{1-\beta}}\Big)R \\
&\leq R,
\end{aligned}$$

then $Q(\Omega_R) \subset \Omega_R$ and it is continuous operator. Let $\Omega_0 = \overline{\text{co}}\, Q(\Omega_R)$, obvious $\Omega_0$ is a bounded closed and convex set in $C(J, E)$ and it is equicontinuous. Since $\Omega_0 \subset \Omega_R$, then we have $Q(\Omega_0) \subset Q(\Omega_R) \subset \Omega_0$ and $Q : \Omega_0 \to \Omega_0$ is a continuous operator.

Finally, we will prove $Q : \Omega_0 \to \Omega_0$ is a condensing operator.

For any non-relatively compact bounded $B$ in $\Omega_0$, we can obtain that both $B$ and $Q(B)$ is a bounded set and equicontinuous set. By the Lemma 2.3, we have

$$\alpha(B) = \max_{t\in J}\alpha(B(t)) = \alpha(B(J)),\ \alpha(Q(B)) = \max_{t\in J}\alpha(Q(B(t))).$$

By the proof of the Lemma 2.12 and the Lemma 2.6, for any $t \in J$, $u \in B$, we have

$$Q(B)(t) \subset \int_0^1 G(t,s)ds\overline{\text{co}}\, f(J \times B(J)).$$

From the properties of the measure of noncompactness and condition (F4), we easily obtain

$$\alpha(Q(B)(t)) \leq \int_0^1 G(t,s)ds\alpha(\overline{\text{co}}\, f(J \times B(J)))$$

$$\begin{aligned} &\leq L\int_0^1 G(t,s)ds\alpha(B(J)) \\ &= L\int_0^1 G(t,s)ds\alpha(B). \end{aligned}$$

Combining with condition (F4), we have

$$\begin{aligned} \alpha(Q(B)(t)) &= \max_{t\in J}\alpha(Q(B(t))) \\ &\leq L\alpha(B)\max_{t\in J}\int_0^1 G(t,s)ds \\ &\leq \frac{L}{\Gamma(\beta+1)}\Big(\beta^{\frac{1}{1-\beta}}-\beta^{\frac{\beta}{1-\beta}}\Big)\alpha(B) \\ &< \alpha(B). \end{aligned}$$

Therefore, $Q:\Omega_0\to\Omega_0$ is a condensing operator. By the Lemma 2.7, we obtain that the $Q$ has at least one fixed point, which is the solution of BVP (2). This completes the proof of Theorem 3.3. □

# References


[1] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Differential Equations, Wiley, New York, 1993.

[2] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.

[3] R. Hilfer, Applications of Fractional Calcus in Physics, World Scientific, Singapore, 2000.

[4] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, in: North-Holland Mathematics Studies, vol. 204, Elsevier Science B.V., Amsterdam, 2006.

[5] S. Heikkila, V. Lakshmikantham, Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations, MaRcel Dekker, Inc.,New York, 1994.

[6] D. Guo, V. Lakshmikantham, Multiple solutions of two-point boundary value problems of ordinary differential equations in Banach spaces, J. Math. Anal. Appl. 129 (1988) 211-222.

[7] Y. Li, C. Guo, Positive solutions to nonlinear second-order boundary value problems in order Banach spaces, J. Lanzhou Univ. Nat. Sci. 44(6) (2008) 120-123. (in Chinese).

[8] Z. Bai, H. L*ü*, Positive solutions for boundary value problem of nonlinear fractional differential equations, J. Math. Anal. Appl. 311 (2005) 495-505.

[9] S. Zhang, Existence of solution for a boundary value problem of fractional order, Acta Math. Sci. Ser. B Engl. Ed. 26 (2006) 220-228.

[10] V. Lakshmikantham, A.S. Vatsala, Basic theory of fractional differential equations, Nonlinear Anal. 69 (2008) 2677-2682.

[11] V. Lakshmikanthama, S. Leela, A Krasnoselskii-Krein-type uniqueness result for fractional differential equations, Nonlinear Anal. 71 (2009) 3421-3424.

[12] X. Su, L. Liu, Existence of solution for boundary value problem of nonlinear fractional differential equation, Appl. Math. J. Chinese. Univ. Ser. B 22 (2007) 291-298.

[13] X. Su, S. Zhang, Solutions to boundary value problems for nonlinear differential equations of fractional order, Electron, J. Differerential Equations, 2009, No. 26. 15pp.

[14] M. Benchohra, S. Hamani, S.K. Ntouyas, Boundary value problems for differential equations with fractional order and nonlocal conditions, Nonlinear Anal. 71 (2009) 2391-2396.

[15] B. Ahmad, S.K. Ntouyas, A. Assolami, Caputo type fractional differential equations with nonlocal Riemann-Liouville integral boundary conditions, J. Appl. Math. Comput. 41 (2013) 339-350.

[16] J. Tariboon, S.K. Ntouyas, W. Sudsutad, Fractional integral problems for fractional differential equations via Caputo derivative, Adv. Difference Equ. 2014, 181, 17pp.

[17] J. Tariboon, S.K. Ntouyas, A. Singubol, Boundary value problems for fractional differential equations with fractional multiterm integral conditions, J. Appl. Math. 2014, Art.ID 806156, 10pp.

[18] B. Ahmad, S.K. Ntouyas, Existence results for a coupled system of Caputo type sequential fractional differential equations with nonlocal integral boundary conditions, Appl. Math. and Comput. 266 (2015) 615-622.

[19] A. Guezane-Lakoud, R. Khaldi, Solvability of a fractional boundary value problem with fractional integral condition, Nonlinear Anal. 75 (2012) 2692-2700.

[20] X. Yang, Z. Wei, W. Dong, Existence of positive solutions for the boundary value problem of nonlinear fractional differential equations, Commun. Nonlinear Sci. Numer. Simul. 17 (2012) 85-92.

[21] Z. Wei, C. Pang, Y. Ding, Positive solutions of singular Caputo fractional differential equations with integral boundary conditions, Commun. Nonlinear Sci. Numer. Simul. 17 (2012) 3148-3160.

[22] Z. Bai, Eigenvalue intervals for a class of fractional boundary value problem, Comput. Math. Appl.64 (2012) 3253-3257.

[23] G. Wang, Monotone iterative technique for boundary value problems of a nonlinear fractional differential equation with deviating arguments, J. Comput. Appl. Math. 236 (2012) 2425-2430.

[24] R. P. Agarwal, B. Ahmad, Existence theorey for anti-periodic boundary value problems of fractional differential equations and inclusions, Comput. Math. Appl. 62 (2011) 1200-1214.

[25] J. Yue, J. Sun, S. Zhang, Existence of positive solution for BVP of nonlinear fractional differential equation, Discrete Dyn. Nat. Soc. 2015, Art.ID 736108, 6pp.

[26] B. Ahmad, S. Sivasundaram, On four-point nonlocal boundary value problems of nonlinear integro-differential equations of fractional order, Appl. Math. Comput. 217 (2010) 480-487.

[27] Z.W. Lv, J. Liang, T.J. Xiao, Solutions to the Cauchy problem for differential equations in Banach spaces with fractional order, Comput. Math. Appl. 62 (2011) 1303-1311.

[28] V. Lakshmikantham1, J.V. Devi, Theory of fractional differential equations in a Banach space, Eur. J. Pure Appl. Math. 1 (2008) 38-45.

[29] R.P. Agarwal, V. Lakshmikantham, J.J. Nieto, On the concept of solutions for fractional differential equations with uncertainly, Nonlinear Anal. 72 (2010) 2859-2862.

[30] J. Banas, K. Goebel, Measure of Noncompactness in Banach Spaces, Lect. Notes Pure Appl. Math., vol. 60, Marcel Pekker, New York, 1980.

[31] K. Deimling, Nonlinear Functional Analysis, Springer-Verlag, New York, 1985.

[32] Y. Li, Existence of solutions of initial value problems for abstract semilinear evolution equations, Acta Math. Sin. 48 (2005) 1089-1094 (in Chinese).

[33] P. Chen, Y. Li, Monotone iterative technique for a class of semilinear evolution equations with nonlocal conditions, Results Math. 63 (2013) 731-744.

[34] Y. Li, X. Wang, Positive periodic solutions for ordinary differential equations in ordered Banach spaces, Xibei Shifan Daxue Xuebao Ziran Kexue Ban 38(1) (2002) 1-5. (in Chinese).

[35] H.P. Heinz, On the behaviour of measure of noncompactness with respect to differentiation and integration of vector-valued functions, Nonlinear Anal. 7 (1983) 1351-1371.

[36] D. Guo, V. Lakshmikantham, Nonlinear Problems in Abstract Cones, Academic Press, New York, 1988.

# On the Higher Order Difference equation

$$x_{n+1} = Ax_n + Bx_{n-l} + Cx_{n-k} + \frac{\gamma x_{n-k}}{Dx_{n-s} + Ex_{n-t}}$$

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## ABSTRACT

The main objective of this paper is to study the global stability of the positive solutions, the boundedness and the periodic character of the difference equation

$$x_{n+1} = Ax_n + Bx_{n-l} + Cx_{n-k} + \frac{\gamma x_{n-k}}{Dx_{n-s} + Ex_{n-t}}, \qquad n = 0,\ 1,\ ...,$$

where the parameters $A,\ B,\ C,\ D, E, \gamma \in (0,\ \infty)$ and the initial conditions $x_{-\sigma}, x_{-\sigma+1}, ..., x_{-1},\ x_0$ are positive real numbers where $\sigma = max\{l,\ k,\ s,\ t\}$. Numerical examples will be given to explicate our results.

**Keywords:** Difference equations, Stability, Global stability, Periodic solutions, Boundedness.

**Mathematics Subject Classification: 39A10**

## 1. INTRODUCTION

Difference equations appeared much earlier than differential equations. But it is only recently that difference equations gained the place they deserve. There is no doubt the interest in difference calculus is related to computers which let effectively apply approximate methods to solve nonlinear difference equations and systems of difference equations [1 - 6]. Particularly, the boundedness, persistence, local asymptotic stability, global character, and the existence of positive periodic solutions can be discussed in many papers [7 - 36].

In [7] El-Dessoky investigated the global stability character and the periodicity of solutions of the recursive sequence

$$x_{n+1} = \frac{ax_{n-l}+bx_{n-k}}{c+dx_{n-l}x_{n-k}}.$$

Xiu-Mei Jia et al. [8] studied the dynamical behavior of rational difference equation,

$$y_{n+1} = \frac{r+py_n+y_{n-k}}{qy_n+y_{n-k}}.$$

Kosmala et al. [9] obtained the periodic and the global stability of solutions of rational difference equation

$$y_{n+1} = \frac{p+y_{n-1}}{qy_n+y_{n-1}}.$$

In [10] Abo-Zeid studied the global asymptotic stability of all solutions of the difference equation

$$x_{n+1} = \frac{Ax_{n-2}}{B+Cx_nx_{n-1}x_{n-2}}.$$

In [11] Devault et al. investigated the dynamics of the difference equation

$$x_{n+1} = p + \frac{x_{n-k}}{x_n}.$$

Elsayed et al. [12] investigated the global stability and the periodicity of solutions of the difference equation

$$x_{n+1} = ax_n + \frac{b+cx_{n-1}}{d+ex_{n-1}}.$$

Zayed et al. [13] obtained some qualitative behavior of the solutions of the difference equation,

$$x_{n+1} = \gamma x_{n-k} + \frac{ax_n+bx_{n-k}}{cx_n-dx_{n-k}}.$$

In [14] El-Dessoky obtained the global stability character, the boundedness and the periodicity of the positive solutions of the difference equation

$$x_{n+1} = ax_n + bx_{n-k} + cx_{n-l} - \frac{dx_{n-s}}{ex_{n-s}-\alpha x_{n-t}}.$$

Our goal is to investigate some qualitative behavior of the positive solutions of the difference equation

$$x_{n+1} = Ax_n + Bx_{n-l} + Cx_{n-k} + \frac{\gamma x_{n-k}}{Dx_{n-s}+Ex_{n-t}}, \qquad n = 0,\ 1,\ ..., \tag{1}$$

where the initial conditions $x_{-\sigma}, x_{-\sigma+1}, ..., x_{-1}, x_0$ are positive real numbers where $\sigma = max\{l,\ k,\ s,\ t\}$ and the coefficients $A,\ B,\ C,\ D, E, \gamma \in (0,\ \infty)$.

Let $J$ be some interval of real numbers and let

$$G : J^{\sigma+1} \to J,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-\sigma}, x_{-\sigma+1}, ..., x_0 \in J$, the difference equation

$$x_{n+1} = G(x_n, x_{n-1}, ..., x_{n-\sigma}), \quad n = 0, 1, ..., \tag{2}$$

has a unique solution $\{x_n\}_{n=-\sigma}^{\infty}$.

DEFINITION 1.1. *The linearized equation of Eq. (2) about the equilibrium $\overline{x}$ is the linear difference equation*

$$y_{n+1} = q_1y_n + q_2y_{n-l} + q_3y_{n-k} + q_4y_{n-s} + q_5y_{n-t}. \tag{3}$$

$q_1 = \frac{\partial G(\overline{x},\ \overline{x},\ \overline{x},\ \overline{x},\ \overline{x})}{\partial x_n},\ q_2 = \frac{\partial G(\overline{x},\ \overline{x},\ \overline{x},\ \overline{x},\ \overline{x})}{\partial x_{n-l}},\ q_3 = \frac{\partial G(\overline{x},\ \overline{x},\ \overline{x},\ \overline{x},\ \overline{x})}{\partial x_{n-k}},\ q_4 = \frac{\partial G(\overline{x},\ \overline{x},\ \overline{x},\ \overline{x},\ \overline{x})}{\partial x_{n-s}},\ q_5 = \frac{\partial G(\overline{x},\ \overline{x},\ \overline{x},\ \overline{x},\ \overline{x})}{\partial x_{n-t}}.$

*The characteristic equation associated with Eq. (3) is*

$$q(\lambda) = q_1 + q_2\lambda^{\sigma-l} + q_3\lambda^{\sigma-k} + q_4\lambda^{\sigma-s} + q_5\lambda^{\sigma-t} = 0, \tag{4}$$

THEOREM 1.2. ***[1]:*** *Assume that $q_1,\ q_2,\ q_3,\ q_4$ and $q_5 \in R$. Then*

$$|q_1| + |q_2| + |q_3| + |q_4| + |q_5| < 1, \tag{4}$$

*is a sufficient condition for the locally asymptotically stability of the Eq. (2).*

THEOREM 1.3. *[4, 5]: Let $h : [\zeta, \eta]^{\sigma+1} \to [\zeta, \eta]$, be a continuous function, where $\sigma$ is a positive integer, and where $[\zeta, \eta]$ is an interval of real numbers. Consider the difference equation*

$$x_{n+1} = h(x_n,\ x_{n-1}, ...,\ x_{n-\sigma}), \qquad n = 0, 1, ... \ . \tag{5}$$

*Suppose that g satisfies the following conditions.*

(1) For each integer $i$ with $1 \le i \le \sigma + 1$; the function $h(z_1,\ z_2,\ ...,\ z_{\sigma+1})$ is weakly monotonic in $z_i$ for fixed $z_1,\ z_2,\ ...,\ z_{i-1},\ z_{i+1},\ ...,\ z_{\sigma+1}$.

(2) If $m,\ M$ is a solution of the system

$$m = h(m_1,\ m_2,\ ...,\ m_{\sigma+1}), \quad M = h(M_1,\ M_2,\ ...,\ M_{\sigma+1}),$$

then $m = M$, where for each $i = 1,\ 2,\ ...,\ \sigma + 1$, we set

$$m_i = \begin{cases} m, & \text{if } h \text{ is non-decreasing in } z_i \\ M, & \text{if } h \text{ is non-increasing in } z_i \end{cases}$$

*and*

$$M_i = \begin{cases} M, & \text{if } h \text{ is non-decreasing in } z_i \\ m, & \text{if } h \text{ is non-increasing in } z_i. \end{cases}$$

Then there exists exactly one equilibrium point $\overline{x}$ of Eq. (5), and every solution of Eq. (5) converges to $\overline{x}$.

## 2. LOCAL STABILITY OF EQ. (1)

Eq. (1) has equilibrium point and is given by

$$\overline{x} = A\overline{x} + B\overline{x} + C\overline{x} + \tfrac{\gamma\overline{x}}{D\overline{x}+E\overline{x}},$$

or

$$\overline{x}(D+E)\left(1-A-B-C\right) = \gamma.$$

If $A + B + c < 1$; then the only positive equilibrium point $\overline{x}$ of Eq. (1) is given by

$$\overline{x} = \tfrac{\gamma}{(D+E)(1-A-B-C)}.$$

THEOREM 2.1. *The equilibrium point $\overline{x}$ of Eq. (1) is locally asymptotically stable if*

$$A + B + C < 1.$$

**Proof:** Suppose that $G : (0,\ \infty)^5 \longrightarrow (0,\ \infty)$ be a continuous function defined by

$$G(u_1,\ u_2,\ u_3,\ u_4,\ u_5) = Au_1 + Bu_2 + Cu_3 + \tfrac{\gamma u_3}{Du_4+Eu_5}. \tag{6}$$

Therefore, it follows that

$$\begin{aligned} \tfrac{\partial G(u_1,\ u_2,\ u_3,\ u_4,\ u_5)}{\partial u_1} &= A, \quad \tfrac{\partial G(u_1,\ u_2,\ u_3,\ u_4,\ u_5)}{\partial u_2} = B, \quad \tfrac{\partial G(u_1,\ u_2,\ u_3,\ u_4,\ u_5)}{\partial u_3} = C + \tfrac{\gamma}{Du_4+Eu_5}, \\ \tfrac{\partial G(u_1,\ u_2,\ u_3,\ u_4,\ u_5)}{\partial u_4} &= -\tfrac{D\gamma u_3}{(Du_4+Eu_5)^2}, \quad \tfrac{\partial G(u_1,\ u_2,\ u_3,\ u_4,\ u_5)}{\partial u_5} = -\tfrac{E\gamma u_3}{(Du_4+Eu_5)^2}. \end{aligned}$$

So, we can write

$$\begin{aligned} \tfrac{\partial G(\overline{x},\ \overline{x},\ \overline{x},\ \overline{x},\ \overline{x})}{\partial u_1} &= A = p_1, \quad \tfrac{\partial G(\overline{x},\ \overline{x},\ \overline{x},\ \overline{x},\ \overline{x})}{\partial u_2} = B = p_2, \quad \tfrac{\partial G(\overline{x},\ \overline{x},\ \overline{x},\ \overline{x},\ \overline{x})}{\partial u_3} = 1 - A - B = p_3, \\ \tfrac{\partial G(\overline{x},\ \overline{x},\ \overline{x},\ \overline{x},\ \overline{x})}{\partial u_4} &= \tfrac{-D\gamma\left(\frac{\gamma}{(D+E)(1-A-B-C)}\right)}{(D+E)^2\left(\frac{\gamma}{(D+E)(1-A-B-C)}\right)^2} = \tfrac{D(A+B+C-1)}{(D+E)} = p_4, \\ \tfrac{\partial G(\overline{x},\ \overline{x},\ \overline{x},\ \overline{x},\ \overline{x})}{\partial u_5} &= \tfrac{-E\gamma\left(\frac{\gamma}{(D+E)(1-A-B-C)}\right)}{(D+E)^2\left(\frac{\gamma}{(D+E)(1-A-B-C)}\right)^2} = \tfrac{E(A+B+C-1)}{(D+E)} = p_5. \end{aligned}$$

Then the linearized equation of Eq. (1) about $\overline{x}$ is

$$y_{n+1} - p_1y_n - p_2y_{n-l} - p_3y_{n-k} - p_4y_{n-s} - p_5y_{n-t} = 0.$$

It follows by Theorem 1 that, Eq. (1) is local asymptotically stable if and only if

$$|p_1| + |p_2| + |p_3| + |p_4| + |p_5| < 1.$$

Thus,

$$\begin{aligned} |p_1| + |p_2| + |p_3| + |p_4| + |p_5| &= |A| + |B| \\ &\quad + |1 - A - B| + \left|\tfrac{D(A+B+C-1)}{(D+E)}\right| + \left|\tfrac{E(A+B+C-1)}{(D+E)}\right| \\ &= 1 + A + B + C - 1 = A + B + C, \end{aligned}$$

for

$$A + B + C < 1.$$

The proof is complete.

THEOREM 2.2. *The equilibrium $\overline{x}$ of Eq. (1) is unstable if $A + B + C > 1$.*

**Example 1.** The solution of Eq. (1) is local stability if $l = 2,\ k = s = 3,\ t = 4,\ A = B = \gamma = 0.1,\ C = 0.2,\ D = 1$ and $E = 0.6$ and the initial conditions $x_{-4} = 0.2$, $x_{-3} = 0.7$, $x_{-2} = 0.3,\ x_{-1} = 2.1$ and $x_0 = 1.1$ (See Fig. 1).

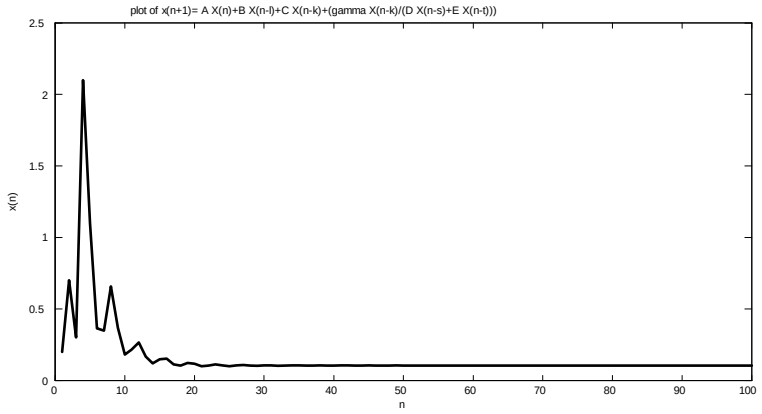



Figure 1. Plot the behavior of the solution of Eq. (1).

**Example 2.** The solution of Eq. (1) is local stability if $l = 4,\ k = t = 2,\ s = 3,\ A = 0.09,\ B = C = 0.01,\ D = 0.3,\ E = 0.5$ and $\gamma = 0.6$ and the initial conditions $x_{-4} = 0.2$, $x_{-3} = 0.7$, $x_{-2} = 0.2,\ x_{-1} = 2.1$ and $x_0 = 1.1$ (See Fig. 2).

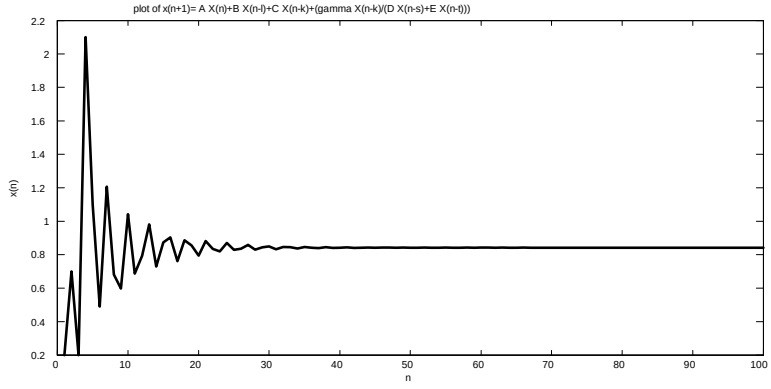



Figure 2. Plot the behavior of the solution of Eq. (1).

**Example 3.** The solution of Eq. (1) is unstable if $l = 4,\ k = t = 2,\ s = 3,\ A = C = 0.2,\ B = \gamma = 0.7,\ D = 0.1$ and $E = 0.5$ and the initial conditions $x_{-4} = 0.2,\ x_{-3} = 0.7,\ x_{-2} = 0.3,\ x_{-1} = 2.1$ and $x_0 = 1.1$ (See Fig. 3).

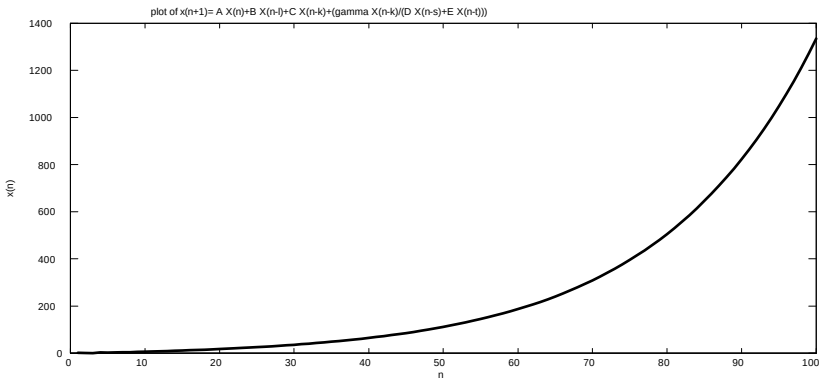



Figure 3. Plot the behavior of the solution of Eq. (1) is unstable.

## 3. GLOBAL ATTRACTIVITY OF EQ. (1)

THEOREM 3.1. *The equilibrium point $\overline{x}$ is a global attractor of Eq. (1) if $A + B + C < 1$ and $\gamma > 1$.*

**Proof:** Suppose that $\zeta$ and $\eta$ are real numbers and assume that $h : [\zeta, \eta]^5 \longrightarrow [\zeta, \eta]$ is a function defined by

$$h(v_1,\ v_2,\ v_3,\ v_4,\ v_5) = Av_1 + Bv_2 + Cv_3 + \frac{\gamma v_3}{Dv_4 + Ev_5}. \tag{7}$$

Then

$$\begin{aligned} \frac{\partial h(v_1,\ v_2,\ v_3,\ v_4,\ v_5)}{\partial v_1} &= A,\ \frac{\partial h(v_1,\ v_2,\ v_3,\ v_4,\ v_5)}{\partial v_2} = B,\ \frac{\partial h(v_1,\ v_2,\ v_3,\ v_4,\ v_5)}{\partial v_3} = C + \frac{\gamma}{Dv_4 + Ev_5}, \\ \frac{\partial h(v_1,\ v_2,\ v_3,\ v_4,\ v_5)}{\partial v_4} &= -\frac{D\gamma v_3}{(Dv_4 + Ev_5)^2},\ \frac{\partial h(v_1,\ v_2,\ v_3,\ v_4,\ v_5)}{\partial v_5} = -\frac{E\gamma v_3}{(Dv_4 + Ev_5)^2}. \end{aligned}$$

First, we can see that the function $h(v_1,\ v_2,\ v_3,\ v_4,\ v_5)$ increasing in $v_1,\ v_2,\ v_3$ and decreasing in $v_4,\ v_5$ for $C + \dfrac{\gamma}{Dv_4 + Ev_5} > 0$.

Let $(m, M)$ be a solution of the system $M = h(M,\ M,\ M,\ m,\ m)$ and $m = h(m,\ m,\ m,\ M,\ M)$. Then from Eq. (1), we see that

$$M = AM + BM + CM + \frac{\gamma M}{Dm + Em} \text{ and } m = Am + Bm + Cm + \frac{\gamma m}{DM + EM},$$

and then

$$M(1 - A - B - C) = \frac{\gamma M}{Dm + Em} \text{ and } m(1 - A - B - C) = \frac{\gamma m}{DM + EM},$$

thus

$$(1 - A - B - C)(D + E)Mm = \gamma M$$

and

$$(1 - A - B - C)(D + E)Mm = \gamma m.$$

Subtracting we obtain

$$\gamma(M - m) = 0,$$

then

$$M = m.$$

It follows by Theorem 2 that $\overline{x}$ is a global attractor of Eq. (1). This completes the proof.

Second, we can see that the function $h(v_1,\ v_2,\ v_3,\ v_4,\ v_5)$ increasing in $v_1,\ v_2$ and decreasing in $v_3,\ v_4,\ v_5$ for $C + \dfrac{\gamma}{Dv_4 + Ev_5} < 0$.

Let $(m, M)$ be a solution of the system $M = h(M,\ M,\ m,\ m,\ m)$ and $m = h(m,\ m,\ M,\ M,\ M)$. Then from Eq. (1), we see that

$$M = AM + BM + Cm + \tfrac{\gamma m}{Dm+Em} \text{ and } m = Am + Bm + CM + \tfrac{\gamma M}{DM+EM},$$

and then

$$M(1 - A - B) - Cm = \tfrac{\gamma}{D+E},$$

and

$$m(1 - A - B) - CM = \tfrac{\gamma}{D+E}.$$

Subtracting we obtain

$$(M - m)(1 - A - B - C) = 0,$$

under the condition $A + B + C < 1$, we see that

$$M = m.$$

It follows by Theorem 2 that $\overline{x}$ is a global attractor of Eq. (1). This completes the proof.

**Example 4.** The solution of Eq. (1) is global stability if $l = 2,\ k = s = 3,\ t = 4,\ A = 0.5,\ B = 0.07,\ C = 0.01,\ D = 2,\ E = 0.6$ and $\gamma = 0.1$ and the initial conditions $x_{-4} = 0.2,\ x_{-3} = 0.7,\ x_{-2} = 0.3,\ x_{-1} = 2.1$ and $x_0 = 1.1$ (See Fig. 4).

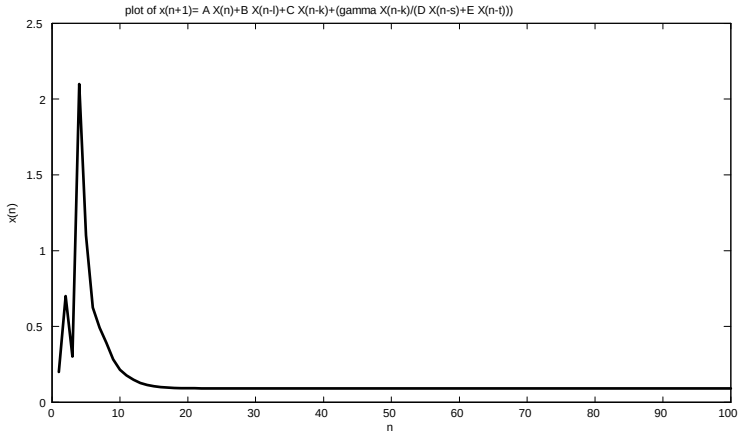



Figure 4. Plot the behavior of the solution of Eq. (1) is global stability.

## 4. EXISTENCE OF PERIODIC SOLUTIONS

THEOREM 4.1. *If $l, k, t$ are an even and $s$ is an odd then Eq. (1) has a prime period two solutions if and only if*

$$(i) \quad (E - D)(A + B + C + 1) + 4D > 0.$$

**Proof:** First Let there exists a prime period two solution

$$...P,\ Q,\ P,\ Q,\ ...,$$

of Eq. (1). We see from Eq. (1) when $l,\ k$ and $t$ are even and $s$ odd that

$$P = AQ + BQ + CQ + \frac{\gamma Q}{DP+EQ} \text{ and } Q = AP + BP + CP + \frac{\gamma P}{DQ+EP}.$$

Therefore,

$$DP^2 + EPQ = D(A+B+C)PQ + E(A+B+C)Q^2 + \gamma Q, \tag{8}$$

and

$$DQ^2 + EPQ = D(A+B+C)PQ + E(A+B+C)P^2 + \gamma P. \tag{9}$$

Subtracting (9) from (8) gives

$$D(P^2-Q^2) + E(A+B+C)(P^2-Q^2) + \gamma(P-Q) = 0,$$

then

$$(P-Q)\left[(D+E(A+B+C))(P+Q)+\gamma\right] = 0$$

Since $P \neq Q$, then

$$P + Q = \frac{-\gamma}{D+E(A+B+C)}. \tag{10}$$

Again, adding (8) and (9) yields

$$D(Q^2+P^2) + 2EPQ = 2D(A+B+C)PQ + E(A+B+C)(Q^2+P^2) + \gamma(Q+P),$$

then

$$2(E-D(A+B+C))PQ = (E(A+B+C)-D)(Q^2+P^2) + \gamma(Q+P). \tag{11}$$

By using (10), (11) and the relation

$$P^2+Q^2 = (P+Q)^2 - 2PQ \quad \text{for all } P,\ Q \in R,$$

we obtain

$$(E(A+B+C)-D)((P+Q)^2-2PQ) + \gamma(Q+P) = 2(E-D(A+B+C))PQ,$$

$$2\left[E-D(A+B+C)+E(A+B+C)-D\right]PQ = (E(A+B+C)-D)(P+Q)^2 + \gamma(Q+P),$$

$$2(E-D)(A+B+C+1)PQ = \left(\frac{-\gamma}{D+E(A+B+C)}\right)^2 (E(A+B+C)-D) + \gamma\left(\frac{-\gamma}{D+E(A+B+C)}\right),$$

$$\begin{aligned} 2(E-D)(A+B+C+1)PQ &= \frac{\gamma^2(E(A+B+C)-D)-\gamma^2(D+E(A+B+C))}{(D+E(A+B+C))^2}, \\ 2(E-D)(A+B+C+1)PQ &= -\frac{2D\gamma^2}{(D+E(A+B+C))^2}. \end{aligned}$$

Then,

$$PQ = -\left(\frac{D\gamma^2}{(D+E(A+B+C))^2}\right)\left(\frac{1}{(E-D)(A+B+C+1)}\right). \tag{12}$$

Now it is obvious from Eq. (10) and Eq. (12) that $P$ and $Q$ are the two distinct roots of the quadratic equation

$$t^2 + \frac{\gamma}{D+E(A+B+C)}t - \left(\frac{D\gamma^2}{(D+E(A+B+C))^2}\right)\left(\frac{1}{(E-D)(A+B+C+1)}\right) = 0,$$

$$(D+E(A+B+C))\,t^2 + \gamma t - \frac{D\gamma^2}{(E-D)(A+B+C+1)(D+E(A+B+C))} = 0, \tag{13}$$

and so

$$\begin{aligned}\left(\frac{\gamma}{D+E(A+B+C)}\right)^2+4\left(\frac{D\gamma^2}{(D+E(A+B+C))^2}\right)\left(\frac{1}{(E-D)(A+B+C+1)}\right) &> 0,\\ 1+\left(\frac{4D}{(E-D)(A+B+C+1)}\right) &> 0,\end{aligned}$$

or

$$(E-D)(A+B+C+1)+4D>0.$$

For $E>D$ then the Inequality (i) holds.

Second suppose that Inequality (i) is true. We will show that Eq. (1) has a prime period two solution.

Suppose that

$$P=\tfrac{\gamma(\zeta-1)}{2(D+E\alpha)} \text{ and } Q=-\tfrac{\gamma(1+\zeta)}{2(D+E\alpha)},$$

where $\zeta=\sqrt{1+\frac{4D}{(E-D)(\alpha+1)}}$and $\alpha=A+B+C$.

We see from the inequality (i) that

$$(E-D)(A+B+C+1)+4D>0,$$

which equivalents to

$$(E-D)(\alpha+1)+4D>0.$$

Therefore $P$ and $Q$ are distinct real numbers.

Set

$$x_{-l}=Q,\ x_{-k}=Q,\ x_{-s}=P,\ x_{-t}=Q,...,\ x_{-3}=P,\ x_{-2}=Q,\ x_{-1}=P,\ x_0=Q.$$

We would like to show that

$$x_1=x_{-1}=P \quad \text{and} \quad x_2=x_0=Q.$$

It follows from Eq. (1) that

$$x_1=AQ+BQ+CQ+\tfrac{\gamma Q}{DP+EQ}=\alpha\left(\tfrac{\gamma(-1-\zeta)}{2(D+E\alpha)}\right)+\frac{\gamma\left(\frac{\gamma(-1-\zeta)}{2(D+E\alpha)}\right)}{D\left(\frac{\gamma(-1+\zeta)}{2(D+E\alpha)}\right)+E\left(\frac{\gamma(-1-\zeta)}{2(D+E\alpha)}\right)}.$$

Dividing the denominator and numerator by $2\left(D+E\alpha\right)$ we get

$$x_1=\alpha\left(\tfrac{\gamma(-1-\zeta)}{2(D+E\alpha)}\right)+\tfrac{\gamma(-1-\zeta)}{D(-1+\zeta)+E(-1-\zeta)}=\alpha\left(\tfrac{\gamma(-1-\zeta)}{2(D+E\alpha)}\right)+\tfrac{\gamma(1+\zeta)}{(D+E)+(E-D)\zeta}.$$

Multiplying the denominator and numerator of the right side by $(D+E)-(D-E)\,\zeta$

$$\begin{aligned}x_1 &= \alpha\left(\tfrac{\gamma(-1-\zeta)}{2(D+E\alpha)}\right)+\tfrac{\gamma(1+\zeta)((D+E)-(E-D)\zeta)}{((D+E)+(E-D)\zeta)((D+E)-(E-D)\zeta)}=\alpha\left(\tfrac{\gamma(-1-\zeta)}{2(D+E\alpha)}\right)+\tfrac{\gamma\left((D+E)-(E-D)\zeta+(D+E)\zeta-(E-D)\zeta^2\right)}{(D+E)^2-(E-D)^2\zeta^2},\\
&= \alpha\left(\tfrac{\gamma(-1-\zeta)}{2(D+E\alpha)}\right)+\tfrac{\gamma\left((D+E)+2D\zeta-(E-D)\zeta^2\right)}{(D+E)^2-(E-D)^2\zeta^2}=\alpha\left(\tfrac{\gamma(-1-\zeta)}{2(D+E\alpha)}\right)+\frac{\gamma\left((D+E)+2D\zeta-(E-D)\left(\frac{(E-D)(\alpha+1)+4D}{(E-D)(\alpha+1)}\right)\right)}{(D+E)^2-(E-D)^2\left(\frac{(E-D)(\alpha+1)+4D}{(E-D)(\alpha+1)}\right)},\\
&= \alpha\left(\tfrac{\gamma(-1-\zeta)}{2(D+E\alpha)}\right)+\frac{\gamma\left((D+E)+2D\zeta-\left(\frac{(E-D)(\alpha+1)+4D}{(\alpha+1)}\right)\right)}{(D+E)^2-(E-D)\left(\frac{(E-D)(\alpha+1)+4D}{(\alpha+1)}\right)}=\alpha\left(\tfrac{\gamma(-1-\zeta)}{2(D+E\alpha)}\right)+\frac{\gamma\left((D+E)+2D\zeta-\left((E-D)+\frac{4D}{(\alpha+1)}\right)\right)}{(D+E)^2-(E-D)\left((E-D)+4\frac{D}{(\alpha+1)}\right)},\\
&= \alpha\left(\tfrac{\gamma(-1-\zeta)}{2(D+E\alpha)}\right)+\frac{\gamma\left(D+E+2D\zeta-E+D-\frac{4D}{(\alpha+1)}\right)}{(D+E)^2-(E-D)^2-\frac{4(E-D)D}{(\alpha+1)}}=\alpha\left(\tfrac{\gamma(-1-\zeta)}{2(D+E\alpha)}\right)+\frac{\gamma D\left(2+2\zeta-\frac{4}{(\alpha+1)}\right)}{4DE-\frac{4(E-D)D}{(\alpha+1)}},\\
&= \alpha\left(\tfrac{\gamma(-1-\zeta)}{2(D+E\alpha)}\right)+\tfrac{\gamma((1+\zeta)(\alpha+1)-2)}{2E(\alpha+1)-2(E-D)}=\gamma\left(\tfrac{\alpha(-1-\zeta)+(1+\zeta)(\alpha+1)-2}{2(D+E\alpha)}\right)=\tfrac{\gamma(\zeta-1)}{2(D+E\alpha)}=P.\end{aligned}$$

Similarly as before, it is easy to show that

$$x_2 = Q.$$

Then by induction we get

$$x_{2n} = Q \quad \text{and} \quad x_{2n+1} = P \quad \text{for all} \quad n \geq -2.$$

Thus Eq. (1) has the prime period two solution

$$...,P,\ Q,\ P,\ Q,...,$$

where $P$ and $Q$ are the distinct roots of the quadratic equation (13) and the proof is complete.

**Example 5.** Figure (5) shows the Eq. (1) has a prime period two solution when $l = 2,\ k = t = 4,\ s = 3,$ $A = 0.01,\ B = 0.02,\ C = 0.03,\ D = 0.2,\ E = 0.3$ and $\gamma = 0.1$ and the initial conditions $x_{-4} = x_{-2} = x_0 = Q$ and $x_{-3} = x_{-1} = P$.

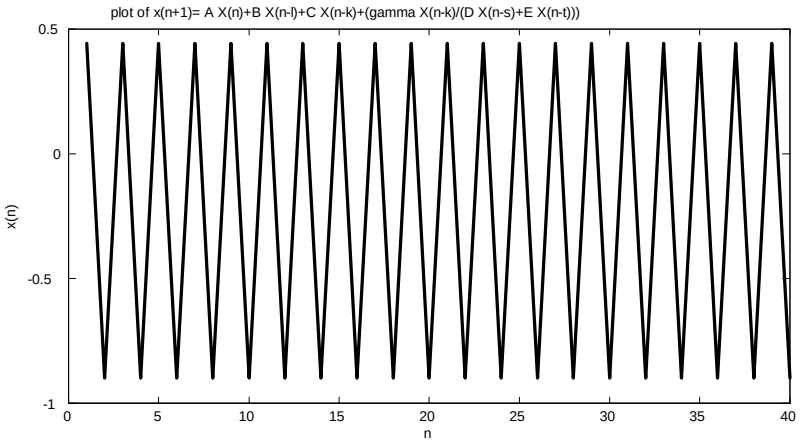



Figure 5. Plot the solution of Eq. (1) has a periodic solution.

THEOREM 4.2. *Eq. (1) has a prime period two solutions if and only if*

$$(D-E)(A+B+C+1)+4E>0\ ,l,k,s-even\ and\ t-odd.$$

THEOREM 4.3. *Eq. (1) has a prime period two solutions if and only if*

$$1-C-3A-3B>0\ \ l,s,t-even\ and\ k-odd.$$

THEOREM 4.4. *Eq. (1) has a prime period two solutions if and only if*

$$A+B+C>3\ l,k-even\ and\ s,t-odd.$$

THEOREM 4.5. *Eq. (1) has a prime period two solutions if and only if*

$$(D-E)\left(A+B+1-C\right)>4D(A+B)\ \ l,s-even\ and\ k,t-odd.$$

THEOREM 4.6. *Eq. (1) has a prime period two solutions if and only if*

$$(E-D)(1+A+B-C)>4E(A+B) \;\; l,t-even \text{ and } k,s-odd.$$

THEOREM 4.7. *Eq. (1) has a prime period two solutions if and only if*

$$(D-E)\,(1+A+C-B)+4E(1-B)>0,\;\; k,s-even \text{ and } l,t-odd.$$

THEOREM 4.8. *Eq. (1) has a prime period two solutions if and only if*

$$(E-D)\,(1+A+C-B)+4D\,(1-B)>0,\;\; k,t-even \text{ and } l,s-odd.$$

THEOREM 4.9. *Eq. (1) has a prime period two solutions if and only if*

$$A+C+3B>3,\;\; k-even \text{ and } l,s,t-odd.$$

THEOREM 4.10. *Eq. (1) has a prime period two solutions if and only if*

$$(D-E)\,(1+A-C-B)>4DA,\;\; s-even \text{ and } l,k,t-odd.$$

THEOREM 4.11. *Eq. (1) has a prime period two solutions if and only if*

$$(E-D)\,(1+A-C-B)>4EA,\;\; t-even \text{ and } l,k,s-odd.$$

THEOREM 4.12. *Eq. (1) has a prime period two solutions if and only if*

$$3A+B+C<1,\;\; s,t-even \text{ and } l,k-odd.$$

THEOREM 4.13. *Eq. (1) has no prime period two solutions if $l,\ k\ ,\ t$ and $s$ are an even when $1+A+B+C\neq 0$.*

**Proof.** Let there exists a prime period two solution $...P,\ Q,\ P,\ Q,\ ...$, of Eq. (1). We see from Eq. (1) when $l,\ k\ ,\ t$ and $s$ are even

$$P=AQ+BQ+CQ+\tfrac{\gamma Q}{DQ+EQ}, \tag{14}$$

and

$$Q=AP+BP+CP+\tfrac{\gamma P}{DP+EP}. \tag{15}$$

Subtracting (14) from (15) gives

$$(1+A+B+C)\,(P-Q)=0,$$

then

$$(P-Q)\,[(D+E)(A+B+C)(P+Q)+\gamma]=0$$

Since $1+A+B+C\neq 0$, then $P=Q$. This is a contradiction. Thus, the proof is completed.

**Example 6.** Figure (6) shows the Eq. (1) has no period two solution when $l = 2, k = s = t = 4$, $A = 0.1$, $B = D = 0.2$, $C = 0.6$, $E = 0.3$ and $\gamma = 0.1$ and the initial conditions $x_{-4} = 0.2$, $x_{-3} = 0.7$, $x_{-2} = 0.3$, $x_{-1} = 2.1$ and $x_0 = 1.1$.

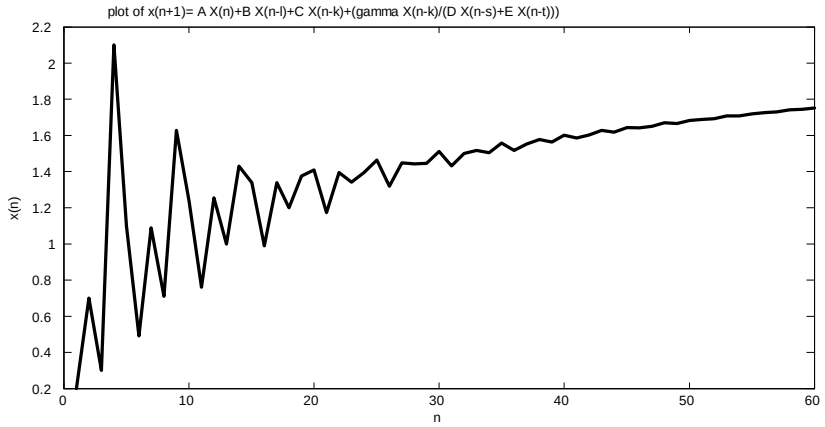



Figure 6. Plot the solution of Eq. (1) has no periodic.

THEOREM 4.14. *Eq. (1) has no prime period two solutions if $l$ is an even and $k$, $s$, $t$ are an odd when $A + B - C + 1 \neq 0$.*

THEOREM 4.15. *Eq. (1) has no a prime period two solutions if $l$, $k$, $s$, $t$ are an odd when $1 - A + B + C \neq 0$.*

THEOREM 4.16. *Eq. (1) has no a prime period two solutions if $k$, $t$, $s$ are an even and $l$ is an odd when $1 + A + C - B \neq 0$.*

## 5. BOUNDEDNESS OF THE SOLUTIONS OF EQ. (1)

THEOREM 5.1. *Let $\{x_n\}$ be a solution of Eq. (1). Then the following statements are true:*
*(i) Suppose $\gamma < D$ and for some $N \geq 0$, the intial conditions*

$$x_{N-\sigma+1},\ ...,\ x_{N-1},\ x_N \in \left[\tfrac{\gamma}{D},\ 1\right],$$

*are valid, then we have the inequality*

$$\tfrac{\gamma}{D}(A + B + C) + \tfrac{\gamma^2}{(D^2+\gamma E)} \leq x_n \leq (A + B + C) + \tfrac{\gamma}{(\gamma+E)}, \tag{16}$$

*for all $n \geq N$.*
*(ii) Suppose $\gamma > D$ and for some $N \geq 0$, the intial conditions*

$$x_{N-\sigma+1},\ ...,\ x_{N-1},\ x_N \in \left[1,\ \tfrac{\gamma}{D}\right],$$

*are valid, then we have the inequality*

$$(A + B + C) + \tfrac{\gamma}{(\gamma+E)} \leq x_n \leq \frac{\gamma}{D}(A + B + C) + \tfrac{\gamma^2}{(D^2+\gamma E)}, \tag{17}$$

*for all $n \geq N$.*

**Proof:** First of all, if for some $N > 0$, $\frac{\gamma}{D} \leq x_N \leq 1$, we have

$$x_{N+1} = Ax_N + Bx_{N-l} + Cx_{N-k} + \tfrac{\gamma x_{N-k}}{Dx_{N-s}+Ex_{N-t}} \leq A + B + C + \tfrac{\gamma x_{N-k}}{Dx_{N-s}+Ex_{N-t}}.$$

But, it is easy to see that $Dx_{N-s} + Ex_{N-t} \geq \gamma + E$, then we get

$$x_{N+1} \leq A + B + C + \frac{\gamma}{\gamma + E}. \tag{18}$$

Similarly, we can show that

$$x_{N+1} \geq \frac{\gamma}{D}(A + B + C) + \frac{\gamma x_{N-k}}{Dx_{N-s} + Ex_{N-t}} \tag{19}$$

But, one can see that $Dx_{N-s} + Ex_{N-t} \leq \frac{D^2 + \gamma E}{D}$, then

$$x_{N+1} \geq \frac{\gamma}{D}(A + B + C) + \frac{\gamma^2}{D^2 + \gamma E} \tag{20}$$

From (18) and (20) we deduce for all $n \geq N$ that the inequality (16) is valid. Hence, the proof of part (i) is completed.

Similarly, if $1 \leq x_N \leq \frac{\gamma}{D}$, then we can prove part (ii) which is omitted here for convenience. Thus, the proof is now completed.

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## REFERENCES

1. V. L. Kocic and G. Ladas, Global Behavior of Nonlinear Difference Equations of Higher Order with Applications, Kluwer Academic Publishers, Dordrecht, 1993.
2. W. G. Kelley, A. C. Peterson, Difference Equations: An Introduction With Applications, Academic Press, New York, 1991
3. R. P. Agarwal, Difference Equations and Inequalities, Marcel Dekker, New York, 1992.
4. E. A. Grove, G. Ladas, Periodicities in Nonlinear Difference Equations, Chapman & Hall/CRC Press, Boca Raton, FL, 2005.
5. M. R. S. Kulenović, G. Ladas, Dynamics of Second Order Rational Difference Equations; With Open Problems and Conjectures, Chapman & Hall/CRC Press, Boca Raton, FL, 2001.
6. S. N. Elaydi, An Introduction to Difference Equations, Springer-Verlag New York Inc., 1996.
7. M. M. El-Dessoky, *Qualitative behavior of rational difference equation of big Order*, Discrete Dyn. Nat. Soc., 2013, (2013), Article ID 495838, 6 pages.
8. Xiu-Mei Jia and Wan-Tong Li, *Boundedness and Global Attractivity of a Higher-Order Nonlinear Difference Equation*, Discrete Dyn. Nat. Soc., 2010, (2010), Article ID 610467, 17 pages.
9. Chang-you Wang, Shu Wang, Zhi-wei Wang, Fei Gong, and Rui-fang Wang, *Asymptotic Stability for a Class of Nonlinear Difference Equations*, Discrete Dyn. Nat. Soc., 2010, (2010), Article ID 791610, 10 pages.
10. R. Abo-Zeid, *On the oscillation of a third order rational difference equation*, J. Egypt. Math. Soc., 23, (2015), 62–66.
11. R. Devault, C., Kent, W. Kosmala, *On the recursive sequence* $x_{n+1} = p + \frac{x_{n-k}}{x_n}$, J. Difference Equ. Appl., 9, (2003), 721–730.
12. E. M. Elsayed, M. M. El-Dessoky, Asim Asiri, *Dynamics and Behavior of a Second Order Rational Difference equation*, J. Comput. Anal. Appl., 16, (2014), 794-807.
13. E. M. E. Zayed, M. A. El-Moneam, *On the Rational Recursive Sequence* $X_{n+1} = \gamma X_{n-k} + \frac{aX_n + bX_{n-k}}{cX_n - dX_{n-k}}$, Bulletin of the Iranian Mathematical Society, 36, (2010), 103-115.
14. M. M. El-Dessoky, *Dynamics and Behavior of the Higher Order Rational Difference equation*, J. Comput. Anal. Appl., 21, (2016), (4), 743-760.

15. R. Abo-Zeid, *Attractivity of two nonlinear third order difference equations*, J. Egypt. Math. Soc., 21, (2013), 241–247.
16. G. Papaschinopoulos, C. J., Schinas, G. Stefanidou, *On the nonautonomous difference equation* $x_{n+1} = A_n + \frac{x_{n-1}^p}{x_n^q}$, Appl. Math. Comput., 217, (2011), 5573-5580.
17. Mehmet Gűműş, *The Periodicity of Positive Solutions of the Nonlinear Difference Equation* $x_{n+1} = \alpha + \frac{x_{n-k}^p}{x_n^p}$, Discrete Dyn. Nat. Soc., 2013, (2013), Article ID 742912, 3 pages.
18. M. A. El-Moneam, *On the Dynamics of the Higher Order Nonlinear Rational Difference Equation*, Math. Sci. Lett., 3, (2014), 121-129.
19. S. Basu, O. Merino, *Global behavior of solutions to two classes of second-order rational difference equations*, Adv. Differ. Equ., 2009, (2009), Article ID 128602, 27 pages.
20. Mehmet Gűműş and Őzkan Őcalan, *Some Notes on the Difference Equation* $x_{n+1} = \alpha + \frac{x_{n-1}}{x_n^k}$, Discrete Dyn. Nat. Soc., 2012, (2012), Article ID 258502, 12 pages.
21. S. Ebru Das., *Dynamics of a nonlinear rational difference equation*, Hacet. J. Math. Stat., 42, (2013), 9-14.
22. Miron B. Bekker, Martin J. Bohner, Hristo D. Voulov, *Asymptotic behavior of solutions of a rational system of difference equations*, J. Nonlinear Sci. Appl., 7, (2014), 379-382.
23. You-Hui Su and Wan-Tong Li, *Global attractivity of a higher order nonlinear difference equation*, J. Difference Equ. Appl., 11, (2005), 947-958.
24. M. M. El-Dessoky, On the dynamics of higher Order difference equations $x_{n+1} = ax_n + \frac{\alpha x_n x_{n-l}}{\beta x_n + \gamma x_{n-k}}$, J. Comput. Anal. Appl., 22, (2017), 1309-1322.
25. Huili Ma, Hui Feng, Jiaofeng Wang, Wandi Ding, *Boundedness and asymptotic behavior of positive solutions for di erence equations of exponential form*, J. Nonlinear Sci. Appl., 8, (2015), 893-899.
26. E. M. Elsayed, M. M. El-Dessoky, Asim Asiri, Dynamics and Behavior of a Second Order Rational Difference equation, J. Comput. Anal. Appl., 16, (2014), 794-807.
27. M. M. El-Dessoky, On the dynamics of a higher Order rational difference equations, J. Egypt. Math. Soc. (to apper).
28. E. M. Elsayed, *Dynamics and behavior of a higher order rational difference equation*, J. Nonlinear Sci. Appl., 9, (2016), 1463-1474.
29. N. Touafek, *On a second order rational difference equation*, Hacet. J. Math. Stat., 41, (2012), 867-874.
30. M. M. El-Dessoky, Qualitative behavior of rational difference equation of big Order, Discrete Dyn Nat Soc, Vol. 2013, (2013), Article ID 495838, 6 pages.
31. E. M. Elsayed and M. M. El-Dessoky, Dynamics and Global Behavior for a Fourth – Order Rational Difference Equation, Hacet. J. Math. Stat., 42(5), (2013), 479-494.
32. E. M. E. Zayed, *Dynamics of the nonlinear rational difference equation* $x_{n+1} = Ax_n + Bx_{n-k} + \frac{px_n + x_{n-k}}{q + x_{n-k}}$, Eur. J. Pure Appl. Math., 3, (2010), 254-268.
33. E. M. Elsayed, M. M. El-Dessoky, E. O. Alzahrani, The Form of The Solution and Dynamics of a Rational Recursive Sequence, J. Comput. Anal. Appl., 17, (2014), 172-186.
34. M. A. El-Moneam, E. M. E. Zayed, On the dynamics of the nonlinear rational difference equation $x_{n+1} = Ax_n + Bx_{n-k} + Cx_{n-l} + \frac{bx_{n-k}}{dx_{n-k} - ex_{n-l}}$, J. Egypt. Math. Soc., 23, (2015), 494–499.
35. A. M. Ahmed, N. A. Eshtewy, Attractivity of the recursive sequence $x_{n+1} = \frac{A - Bx_{n-2}}{C + Dx_{n-1}}$, J. Egypt. Math. Soc., 24, (2016), 392–395.
36. M. M. El-Dessoky and E. O. Alzahrani, On the Difference equation $x_{n+1} = Ax_n + \frac{B \sum_{i=0}^{k} x_{n-i}}{C + D \prod_{i=0}^{k} x_{n-i}}$, J. Comput. Anal. Appl., 23, (2017), 78-91.

# AN ADDITIVE $(\alpha, \beta)$-FUNCTIONAL EQUATION AND LINEAR MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we investigate the additive $(\alpha, \beta)$-functional equation

$$f(x) + \overline{\alpha} f(\alpha y) = \beta^{-1} f(\beta(x + y)) \tag{0.1}$$

for all complex numbers $\alpha$ with $|\alpha| = 1$ and for a fixed nonzero complex number $\beta$.

Using the fixed point method and the direct method, we prove the Hyers-Ulam stability of the additive $(\alpha, \beta)$-functional equation (0.1) in complex Banach spaces.

## 1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [21] concerning the stability of group homomorphisms.

The functional equation $f(x + y) = f(x) + f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [9] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [17] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [8] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. See [5, 7, 13, 14, 15, 18, 19, 20, 22] for more information on functional equations.

We recall a fundamental result in fixed point theory.

**Theorem 1.1.** [2, 6] *Let $(X, d)$ be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

*for all nonnegative integers $n$ or there exists a positive integer $n_0$ such that*

(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \geq n_0$;

(2) *the sequence $\{J^n x\}$ converges to a fixed point $y^*$ of $J$;*

(3) *$y^*$ is the unique fixed point of $J$ in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*

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(4) $d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy)$ *for all* $y \in Y$.

In 1996, Isac and Rassias [10] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [3, 4, 16]).

In Section 2, we solve the additive $(\alpha, \beta)$-functional equation (0.1) in vector spaces and prove the Hyers-Ulam stability of the additive $(\alpha, \beta)$-functional equation (0.1) in Banach spaces by using the fixed point method.

In Section 3, we prove the Hyers-Ulam stability of the additive $(\alpha, \beta)$-functional equation (0.1) in Banach spaces by using the direct method.

Throughout this paper, assume that $X$ is a complex normed space and that $Y$ is a complex Banach space. Let $\beta$ be a fixed nonzero complex number.

## 2. Additive $(\alpha, \beta)$-functional equation (0.1) in complex Banach spaces $I$

We solve the additive $(\alpha, \beta)$-functional equation (0.1) in complex vector spaces.

**Lemma 2.1.** *Let $X$ and $Y$ be complex vector spaces. If a mapping $f : X \to Y$ satisfies*

$$f(x) + \overline{\alpha} f(\alpha y) = \beta^{-1} f(\beta(x + y)) \tag{2.1}$$

*for all $x, y \in X$ and all $\alpha \in \mathbb{T} := \{\mu \in \mathbb{C} \mid |\mu| = 1\}$, then $f : X \to Y$ is $\mathbb{C}$-linear.*

*Proof.* Assume that $f : X \to Y$ satisfies (2.1).

Letting $x = y = 0$ in (2.1), we get $(1 + \overline{\alpha}) f(0) = \beta^{-1} f(0)$ for all $\alpha \in \mathbb{T}$. So $f(0) = 0$.

Letting $\alpha = 1$, $y = -x$ in (2.1), we get $f(x) + f(-x) = 0$ and so $f(-x) = -f(x)$ for all $x \in X$.

Letting $\alpha = 1$, $x = 0$ and replacing $y$ by $x + y$ in (2.1), we get

$$f(x + y) = \beta^{-1} f(\beta(x + y))$$

for all $x, y \in X$. Letting $\alpha = 1$ in (2.1), we get $f(x) + f(y) = \beta^{-1} f(\beta(x + y))$ and so

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in X$.

Letting $y = -x$ in (2.1), we get $f(x) + \overline{\alpha} f(-\alpha x) = 0$ and so $f(\alpha x) = \alpha f(x)$ for all $x \in X$ and all $\alpha \in \mathbb{T}$. By the same reasoning as in the proof of [12, Theorem 2.1], the mapping $f : X \to Y$ is $\mathbb{C}$-linear. □

Using the fixed point method, we prove the Hyers-Ulam stability of the additive $(\alpha, \beta)$-functional equation (2.1) in complex Banach spaces.

**Theorem 2.2.** *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2},\frac{y}{2}\right) \leq \frac{L}{2}\varphi(x,y) \tag{2.2}$$

*for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\left\|f(x) + \overline{\alpha} f(\alpha y) - \beta^{-1} f(\beta(x+y))\right\| \leq \varphi(x,y) \tag{2.3}$$

*for all $x, y \in X$ and all $\alpha \in \mathbb{T}$. Then there exists a unique $\mathbb{C}$-linear mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{L}{2(1-L)}(\varphi(0,2x) + \varphi(x,x)) \tag{2.4}$$

*for all $x \in X$.*

*Proof.* Let $\alpha = 1$.

Letting $y = x$ in (2.3), we get

$$\left\|2f(x) - \beta^{-1} f(2\beta x)\right\| \leq \varphi(x,x) \tag{2.5}$$

for all $x \in X$.

Replacing $y$ by $2x$ and letting $x = 0$ in (2.3), we get

$$\left\|f(2x) - \beta^{-1} f(2\beta x)\right\| \leq \varphi(0,2x) \tag{2.6}$$

for all $x \in X$.

It follows from (2.5) and (2.6) that

$$\|f(2x) - 2f(x)\| \leq \varphi(0,2x) + \varphi(x,x) \tag{2.7}$$

for all $x \in X$.

Consider the set

$$S := \{h : X \to Y, \quad h(0) = 0\}$$

and introduce the generalized metric on $S$:

$$d(g,h) = \inf\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \leq \mu(\varphi(0,2x) + \varphi(x,x)),\ \forall x \in X\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that $(S,d)$ is complete (see [11]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$\|g(x) - h(x)\| \leq \varepsilon(\varphi(0,2x) + \varphi(x,x))$$

for all $x \in X$. Hence

$$\begin{aligned}\|Jg(x) - Jh(x)\| &= \left\|2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right)\right\| \leq 2\varepsilon\left(\varphi(0,x) + \varphi\left(\frac{x}{2},\frac{x}{2}\right)\right) \\ &\leq 2\varepsilon\frac{L}{2}(\varphi(0,2x) + \varphi(x,x)) = L\varepsilon(\varphi(0,2x) + \varphi(x,x))\end{aligned}$$

for all $x \in X$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g,h)$$

for all $g, h \in S$.

It follows from (2.7) that

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| \leq \varphi(0,x) + \varphi\left(\tfrac{x}{2},\tfrac{x}{2}\right) \leq \tfrac{L}{2}(\varphi(0,2x) + \varphi(x,x))$$

for all $x \in X$. So $d(f, Jf) \leq \frac{L}{2}$.

By Theorem 1.1, there exists a mapping $A : X \to Y$ satisfying the following:

(1) $A$ is a fixed point of $J$, i.e.,

$$A(x) = 2A\left(\frac{x}{2}\right) \tag{2.8}$$

for all $x \in X$. The mapping $A$ is a unique fixed point of $J$ in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that $A$ is a unique mapping satisfying (2.8) such that there exists a $\mu \in (0,\infty)$ satisfying

$$\|f(x) - A(x)\| \leq \mu(\varphi(0,2x) + \varphi(x,x))$$

for all $x \in X$;

(2) $d(J^l f, A) \to 0$ as $l \to \infty$. This implies the equality

$$\lim_{l\to\infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in X$;

(3) $d(f,A) \leq \frac{1}{1-L}d(f,Jf)$, which implies

$$\|f(x) - A(x)\| \leq \frac{L}{2(1-L)}(\varphi(0,2x) + \varphi(x,x))$$

for all $x \in X$.

It follows from (2.2) and (2.3) that

$$\begin{aligned}&\left\|A(x) + \overline{\alpha}A(\alpha y) - \beta^{-1}A(\beta(x+y))\right\| \\ &= \lim_{n\to\infty} 2^n\left\|f\left(\frac{x}{2^n}\right) + \overline{\alpha}f\left(\frac{\alpha y}{2^n}\right) - \beta^{-1}f\left(\beta\left(\frac{x+y}{2^n}\right)\right)\right\| \\ &\leq \lim_{n\to\infty} 2^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right) = 0\end{aligned}$$

for all $x, y \in X$ and all $\alpha \in \mathbb{T}$. So

$$A(x) + \overline{\alpha}A(\alpha y) - \beta^{-1}A\left(\beta(x+y)\right) = 0$$

for all $x, y \in X$ and all $\alpha \in \mathbb{T}$. By Lemma 2.1, the mapping $A : X \to Y$ is $\mathbb{C}$-linear. □

**Corollary 2.3.** *Let $r > 1$ and $\theta$ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying*

$$\left\|f(x) + \overline{\alpha}f(\alpha y) - \beta^{-1}f(\beta(x+y))\right\| \leq \theta(\|x\|^r + \|y\|^r) \tag{2.9}$$

*for all $x, y \in X$ and all $\alpha \in \mathbb{T}$. Then there exists a unique $\mathbb{C}$-linear mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{2^r + 2}{2^r - 2}\theta\|x\|^r$$

*for all $x \in X$.*

*Proof.* The proof follows from Theorem 2.2 by taking $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Then we can choose $L = 2^{1-r}$ and we get the desired result. □

**Theorem 2.4.** *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

*for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.3). Then there exists a unique $\mathbb{C}$-linear mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{1}{2(1-L)}(\varphi(0, 2x) + \varphi(x, x))$$

*for all $x \in X$.*

*Proof.* It follows from (2.7) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \leq \frac{1}{2}(\varphi(0, 2x) + \varphi(x, x))$$

for all $x \in X$.

Let $(S, d)$ be the generalized metric space defined in the proof of Theorem 2.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{2}g(2x)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

**Corollary 2.5.** *Let $r < 1$ and $\theta$ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying (2.9). Then there exists a unique $\mathbb{C}$-linear mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{2 + 2^r}{2 - 2^r}\theta\|x\|^r$$

*for all* $x \in X$.

*Proof.* The proof follows from Theorem 2.4 by taking $\varphi(x,y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Then we can choose $L = 2^{r-1}$ and we get desired result. □

## 3. Additive $(\alpha,\beta)$-functional equation (0.1) in complex Banach spaces $II$

In this section, using the direct method, we prove the Hyers-Ulam stability of the additive $(\alpha,\beta)$-functional equation (2.1) in complex Banach spaces.

**Theorem 3.1.** *Let* $\varphi : X^2 \to [0,\infty)$ *be a function and let* $f : X \to Y$ *be a mapping satisfying* $f(0) = 0$ *and*

$$\begin{aligned} \Psi(x,y) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) &< \infty, \\ \left\| f(x) + \overline{\alpha} f(\alpha y) - \beta^{-1} f(\beta(x+y)) \right\| &\leq \varphi(x,y) \end{aligned} \tag{3.1}$$

*for all* $x, y \in X$ *and all* $\alpha \in \mathbb{T}$. *Then there exists a unique* $\mathbb{C}$*-linear mapping* $A : X \to Y$ *such that*

$$\|f(x) - A(x)\| \leq \frac{1}{2}(\Psi(0,2x) + \Psi(x,x)) \tag{3.2}$$

*for all* $x \in X$.

*Proof.* Let $\alpha = 1$.

It follows from (2.7) that

$$\left\| f(x) - 2f\left(\frac{x}{2}\right) \right\| \leq \varphi(0,x) + \varphi\left(\frac{x}{2}, \frac{x}{2}\right)$$

for all $x \in X$. Hence

$$\begin{aligned} \left\| 2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right) \right\| &\leq \sum_{j=l}^{m-1} \left\| 2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right) \right\| \\ &\leq \sum_{j=l}^{m-1} \left( 2^j \varphi\left(0, \frac{x}{2^j}\right) + 2^j \varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right) \right) \end{aligned} \tag{3.3}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (3.3) that the sequence $\{2^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since $Y$ is a Banach space, the sequence $\{2^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.3), we get (3.2).

Now, let $T : X \to Y$ be another additive mapping satisfying (3.2). Then we have

$$\begin{aligned}\|A(x)-T(x)\| &= \left\|2^qA\left(\frac{x}{2^q}\right)-2^qT\left(\frac{x}{2^q}\right)\right\| \\ &\le \left\|2^qA\left(\frac{x}{2^q}\right)-2^qf\left(\frac{x}{2^q}\right)\right\|+\left\|2^qT\left(\frac{x}{2^q}\right)-2^qf\left(\frac{x}{2^q}\right)\right\| \\ &\le 2^q\Psi\left(0,\frac{2x}{2^q}\right)+2^q\Psi\left(\frac{x}{2^q},\frac{x}{2^q}\right),\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of $A$.

The rest of the proof is similar to the proof of Theorem 2.2. □

**Corollary 3.2.** *Let $r > 1$ and $\theta$ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying* (2.9)*. Then there exists a unique $\mathbb{C}$-linear mapping $A : X \to Y$ such that*

$$\|f(x)-A(x)\| \le \frac{2^r+2}{2^r-2}\theta\|x\|^r$$

*for all $x \in X$.*

*Proof.* The proof follows from Theorem 3.1 by taking $\varphi(x,y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. □

**Theorem 3.3.** *Let $\varphi : X^2 \to [0,\infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$,* (3.1) *and*

$$\Psi(x,y) := \sum_{j=0}^{\infty}\frac{1}{2^j}\varphi(2^jx,2^jy) < \infty$$

*for all $x, y \in X$. Then there exists a unique $\mathbb{C}$-linear mapping $A : X \to Y$ such that*

$$\|f(x)-A(x)\| \le \frac{1}{2}(\Psi(0,2x)+\Psi(x,x)) \tag{3.4}$$

*for all $x \in X$.*

*Proof.* It follows from (2.7) that

$$\left\|f(x)-\frac{1}{2}f(2x)\right\| \le (\varphi(0,2x)+\varphi(x,x))$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|\frac{1}{2^l}f(2^lx)-\frac{1}{2^m}f(2^mx)\right\| &\le \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f\left(2^jx\right)-\frac{1}{2^{j+1}}f\left(2^{j+1}x\right)\right\| \qquad (3.5)\\ &\le \sum_{j=l}^{m-1}\left(\frac{1}{2^{j+1}}\varphi(0,2^{j+1}x)+\frac{1}{2^{j+1}}\varphi(2^jx,2^jx)\right)\end{aligned}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (3.5) that the sequence $\{\frac{1}{2^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.5), we get (3.4).

The rest of the proof is similar to the proofs of Theorems 2.2 and 3.1. □

**Corollary 3.4.** *Let $r < 1$ and $\theta$ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (2.9)*. Then there exists a unique $\mathbb{C}$-linear mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{2 + 2^r}{2 - 2^r}\theta\|x\|^r$$

*for all $x \in X$.*

*Proof.* The proof follows from Theorem 3.3 by taking $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. □

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## References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.
[2] L. Cădariu, V. Radu, *Fixed points and the stability of Jensen's functional equation*, J. Inequal. Pure Appl. Math. **4**, no. 1, Art. ID 4 (2003).
[3] L. Cădariu, V. Radu, *On the stability of the Cauchy functional equation: a fixed point approach*, Grazer Math. Ber. **346** (2004), 43–52.
[4] L. Cădariu, V. Radu, *Fixed point methods for the generalized stability of functional equations in a single variable*, Fixed Point Theory Appl. **2008**, Art. ID 749392 (2008).
[5] A. Chahbi, N. Bounader, *On the generalized stability of d'Alembert functional equation*, J. Nonlinear Sci. Appl. **6** (2013), 198–204.
[6] J. Diaz, B. Margolis, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Am. Math. Soc. **74** (1968), 305–309.
[7] G. Z. Eskandani, P. Găvruta, *Hyers-Ulam-Rassias stability of pexiderized Cauchy functional equation in 2-Banach spaces,* J. Nonlinear Sci. Appl. **5** (2012), 459–465.
[8] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–43.
[9] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. U.S.A. **27** (1941), 222–224.
[10] G. Isac, Th. M. Rassias, *Stability of $\psi$-additive mappings: Applications to nonlinear analysis*, Internat. J. Math. Math. Sci. **19** (1996), 219–228.

[11] D. Miheţ, V. Radu, *On the stability of the additive Cauchy functional equation in random normed spaces*, J. Math. Anal. Appl. **343** (2008), 567–572.
[12] C. Park, *Homomorphisms between Poisson $JC^*$-algebras*, Bull. Braz. Math. Soc. **36** (2005), 79–97.
[13] C. Park, *Orthogonal stability of a cubic-quartic functional equation*, J. Nonlinear Sci. Appl. **5** (2012), 28–36.
[14] C. Park, K. Ghasemi, S. G. Ghaleh, S. Jang, *Approximate n-Jordan $*$-homomorphisms in $C^*$-algebras,* J. Comput. Anal. Appl. **15** (2013), 365-368.
[15] C. Park, A. Najati, S. Jang, *Fixed points and fuzzy stability of an additive-quadratic functional equation,* J. Comput. Anal. Appl. **15** (2013), 452–462.
[16] V. Radu, *The fixed point alternative and the stability of functional equations*, Fixed Point Theory **4** (2003), 91–96.
[17] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Am. Math. Soc. **72** (1978), 297–300.
[18] K. Ravi, E. Thandapani, B. V. Senthil Kumar, *Solution and stability of a reciprocal type functional equation in several variables,* J. Nonlinear Sci. Appl. **7** (2014), 18–27.
[19] D. Shin, C. Park, Sh. Farhadabadi, *On the superstability of ternary Jordan $C^*$-homomorphisms,* J. Comput. Anal. Appl. **16** (2014), 964–973.
[20] D. Shin, C. Park, Sh. Farhadabadi, *Stability and superstability of $J^*$-homomorphisms and $J^*$-derivations for a generalized Cauchy-Jensen equation,* J. Comput. Anal. Appl. **17** (2014), 125–134.
[21] S. M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ. New York, 1960.
[22] C. Zaharia, *On the probabilistic stability of the monomial functional equation,* J. Nonlinear Sci. Appl. **6** (2013), 51–59.

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# Riemann-Liouville fractional Hermite-Hadamard inequalities for $h$-preinvex functions

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**Abstract**

In this paper, we first prove two Hermite-Hadamard type inequalities for $h$-preinvex functions via Riemann-Liouville fractional integrals, and then, by introducing an integral identity including the second order derivatives of a given function, we establish some Hermite-Hadamard type inequalities for functions whose second order derivatives are $h$-preinvex via Riemann-Liouville fractional integrals.

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*Key words and phrases*: Hermite-Hadamard's inequality; $h$-preinvex functions; Riemann-Liouville fractional integrals.

## 1 Introduction

In [27], Sarikaya et al. considered Hermite-Hadamard type inequalities involving Riemann-Liouville fractional integrals and established the following interesting inequalities.

**Theorem 1.1** *Let $f : [u, v] \to \mathbb{R}$ be a positive function with $0 \le u < v$ and let $f \in L^1[u, v]$. Suppose $f$ is a convex function on $[u, v]$, then the following inequalities for fractional integrals hold:*

$$f\Big(\frac{u+v}{2}\Big) \le \frac{\Gamma(\alpha+1)}{2(v-u)^{\alpha}}[J^{\alpha}_{u^+}f(v) + J^{\alpha}_{v^-}f(u)] \le \frac{f(u)+f(v)}{2}, \qquad (1.1)$$

*where the symbol $J^{\alpha}_{u^+}f$ and $J^{\alpha}_{v^-}f$ denote respectively the left-sided and right-sided Riemann-Liouville fractional integrals of the order $\alpha \in \mathbb{R}^+$ defined by*

$$J^{\alpha}_{u^+}f(x) = \frac{1}{\Gamma(\alpha)}\int_u^x (x-t)^{\alpha-1}f(t)\mathrm{d}t, \quad u < x$$

*and*

$$J^{\alpha}_{v^-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^v (t-x)^{\alpha-1}f(t)\mathrm{d}t, \quad x < v.$$

*Here, $\Gamma(\alpha)$ is the Gamma function and its definition is*

$$\Gamma(\alpha) = \int_0^{\infty} e^{-\mu}\mu^{\alpha-1}\mathrm{d}\mu.$$

We observe that, for $\alpha = 1$, the inequality (1.1) can be reduced to the following termed Hermite-Hadamard inequality

$$f\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u}\int_u^v f(x)\mathrm{d}x \leq \frac{f(u)+f(v)}{2}, \tag{1.2}$$

where $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex mapping on the interval $I$ of real numbers and $u, v \in I$ with $u < v$.

In recent years, many researchers have studied error estimations with respect to the inequality (1.2); for refinements, counterparts, generalization please refer to [7, 16, 17, 22, 35–37].

We evoke, now, some basic definitions as follows.

**Definition 1.1** *([34]) A set $S \subseteq \mathbb{R}^n$ is said to be invex set with respect to the mapping $\eta : S \times S \to \mathbb{R}^n$, if $x + t\eta(y, x) \in S$ for every $x, y \in S$ and $t \in [0, 1]$. The invex set $S$ is also termed an $\eta$-connected set.*

**Definition 1.2** *([34]) A function $f$ defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex respecting $\eta$, if*

$$f\big(x + t\eta(y, x)\big) \leq (1-t)f(x) + tf(y), \forall x, y \in K, t \in [0, 1]. \tag{1.3}$$

*The function $f$ is said to be preincave if and only if $-f$ is preinvex.*

Very recently, some new generalizations of integral inequalities in connection with the preinvexity were explored by Du, Liao and Li [6], Hussain and Qaisar [8], Latif and Dragomir [14], Li and Du [15], respectively.

**Definition 1.3** *([31]) Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ be a positive function, $h \not\equiv 0$. We say that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is h-convex, or that $f$ belongs to the class $SX(h, I)$, if $f$ is non-negative and for all $x, y \in I$ and $t \in (0, 1)$, one has*

$$f\big(tx + (1-t)y\big) \leq h(t)f(x) + h(1-t)f(y). \tag{1.4}$$

*If inequality (1.4) is reversed, then $f$ is called h-concave, i.e. $f \in SV(h, I)$.*

If $h(t) = t$, then any non-negative convex mapping belongs to $SX(h, I)$ and each non-negative concave mapping belongs to $SV(h, I)$; if $h(t) = \frac{1}{t}$, then $SX(h, I) = Q(I)$; if $h(t) = 1$, then $SX(h, I) \supseteq P(I)$; and if $h(t) = t^s$ for $s \in (0, 1]$, then $SX(h, I) \supseteq K_s^2$.

**Definition 1.4** *([19]) Let $h : [0, 1] \to \mathbb{R}$ be a non-negative function and $h \neq 0$. The function $f$ on the invex set $X$ is said to be $h$-preinvex with respect to $\eta$, if*

$$f\big(x + t\eta(y, x)\big) \leq h(1 - t)f(x) + h(t)f(y) \tag{1.5}$$

*for each $x, y \in X$ and $t \in [0, 1]$ where $f(\cdot) > 0$.*

**Definition 1.5** *([31]) We say that $h : J \subseteq \mathbb{R} \to \mathbb{R}$ is a super-multiplicative function, if for any $x, y \in J$ with $xy \in J$ , one has*

$$h(xy) \geq h(x)h(y).$$

**Definition 1.6** *([1]) We say that $h : J \subseteq \mathbb{R} \to \mathbb{R}$ is a super-additive function, if for every $x, y \in J$ with $x + y \in J$, one has*

$$h(x + y) \geq h(x) + h(y).$$

In order to prove some of our results in the present paper, we need the following Condition C given by Mohan and Neogy in [24].

**Condition C**: Let $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, we say that the mapping $\eta$ satisfies the condition C if for any $x, y \in \mathbb{R}^n$,

$(C_1)$ $\quad \eta\big(x, x + t\eta(y, x)\big) = -t\eta(y, x)$,

$(C_2)$ $\quad \eta\big(y, x + t\eta(y, x)\big) = (1 - t)\eta(y, x)$,

for all $t \in [0, 1]$.

Note that for any $x, y \in \mathbb{R}^n$, $t_1, t_2 \in [0, 1]$ and from Condition C, we can deduce

$$\eta\Big(x + t_2\eta(y, x), x + t_1\eta(y, x)\Big) = (t_2 - t_1)\eta(y, x).$$

Also, if $t_1 = 0$, we have

$$\eta\Big(x + t_2\eta(y, x), x\Big) = t_2\eta(y, x).$$

In [21], Noor et al. proved the following variant of the Fejér-Hermite-Hadamard inequality type under $h$-preinvexity.

**Theorem 1.2** *Let $f : I \subseteq \mathbb{R} \to (0, \infty)$ be an $h$-preinvex function with $\eta(v, u) > 0$, $h(\frac{1}{2}) \neq 0$ and let $w : [u; u + \eta(v, u)] \to \mathbb{R}$ is a non-negative, integrable function and symmetric regarding $u + \frac{1}{2}\eta(v, u)$, then recurring to Condition C, we have*

$$\begin{aligned}\frac{1}{2h(\frac{1}{2})} f\Big(\frac{2u + \eta(v, u)}{2}\Big) \int_u^{u+\eta(v,u)} w(x)\mathrm{d}x &\leq \int_u^{u+\eta(v,u)} f(x)w(x)\mathrm{d}x \\ &\leq \frac{f(u) + f(v)}{2}[h(t) + h(1 - t)] \int_u^{u+\eta(v,u)} w(x)\mathrm{d}x\end{aligned} \tag{1.6}$$

**Corollary 1.1** *In Theorem 1.2, letting* $w(x) = 1$ *or* $\frac{1}{\eta(v,u)}$, *we can obtain*

$$\frac{1}{2h(\frac{1}{2})} f\Big(\frac{2u+\eta(v,u)}{2}\Big) \le \frac{1}{\eta(v,u)} \int_u^{u+\eta(v,u)} f(x)\mathrm{d}x \le \frac{f(u)+f(v)}{2}[h(t)+h(1-t)], \tag{1.7}$$

*specially for* $\eta(v,u) = v-u$, *we can get*

$$\frac{1}{2h(\frac{1}{2})} f\Big(\frac{u+v}{2}\Big) \le \frac{1}{v-u} \int_u^v f(x)\mathrm{d}x \le \frac{f(u)+f(v)}{2}[h(t)+h(1-t)]. \tag{1.8}$$

**Corollary 1.2** *In Theorem 1.2, if* $f$ *is an* $h$*-preincave function and* $w(x) = 1$ *or* $\frac{1}{\eta(v,u)}$, *we can get*

$$\frac{f(u)+f(v)}{2}[h(t)+h(1-t)] \le \frac{1}{\eta(v,u)} \int_u^{u+\eta(v,u)} f(x)\mathrm{d}x \le \frac{1}{2h(\frac{1}{2})} f\Big(\frac{2u+\eta(v,u)}{2}\Big). \tag{1.9}$$

In some of the recent literatures, Riemann-Liouville fractional Hermite-Hadamard type inequalities are applied widely in the field of analysis, and many new results of fractional Hermite-Hadamard type inequalities are gained based on the original Hermite-Hadamard inequalities for functions of different classes. For example, refer to for convex functions [4, 9–11, 25, 27, 28], for $m$-convex functions [32, 38] and $(s,m)$-convex functions [2], for $r$-convex functions [33], for harmonically convex functions [3, 13], for quasi-geometrically convex functions [12], for GA-$s$-convex functions [18], for preinvex functions [23, 26], for $(\alpha,m)$-preinvex functions [5], for $h$-convex functions [20] and references cited therein.

Motivated and inspired by these results and the recent developments in this area, in the present paper, two Hermite-Hadamard's inequalities for $h$-preinvex functions via fractional integrals are firstly established and the obtained results of [30] are generalized. Secondly, a second-order new identity for fractional integrals is found. By virtue of this integral identity, we present the left-sided new Hermite-Hadamard type inequalities for $h$-preinvex functions and $h$-preincave functions via Riemann-Liouville fractional integrals. Some results proved in this paper can be viewed as generalization of several known results of [29, 30].

# 2 Main Results

In this section, we are going to prove our main results.

**Theorem 2.1** *Let* $f : I \subseteq \mathbb{R} \to (0,\infty)$ *be an* $h$*-preinvex function with* $\eta(b,a) > 0$ *and* $f \in L_1[a,b]$. *Then one has the following inequality recurring to Condition*

*C via fractional integrals*

$$\begin{aligned}&\frac{\Gamma(\alpha)}{\eta^{\alpha}(b,a)}\Big[J^{\alpha}_{a^+}f\big(a+\eta(b,a)\big)+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\Big]\\&\leq\big[f(a)+f\big(a+\eta(b,a)\big)\big]\int_0^1 t^{\alpha-1}[h(t)+h(1-t)]\mathrm{d}t\\&\leq\frac{2\big[f(a)+f\big(a+\eta(b,a)\big)\big]}{(\alpha p-p+1)^{\frac{1}{p}}}\bigg(\int_0^1\big(h(t)\big)^q\mathrm{d}t\bigg)^{\frac{1}{q}},\end{aligned}\tag{2.1}$$

*where* $p^{-1}+q^{-1}=1$ *with* $p,q>1$.

*Proof.* Since $f$ is $h$-preinvex, we have

$$f\big(x+(1-t)\eta(y,x)\big)\leq h(t)f(x)+h(1-t)f(y)$$

and

$$f\big(x+t\eta(y,x)\big)\leq h(1-t)f(x)+h(t)f(y).$$

By adding these inequalities we can deduce

$$f\big(x+(1-t)\eta(y,x)\big)+f\big(x+t\eta(y,x)\big)\leq[h(t)+h(1-t)][f(x)+f(y)].\tag{2.2}$$

By employing (2.2) with $x=a$ and $y=a+\eta(b,a)$ we have

$$\begin{aligned}&f\Big(a+(1-t)\eta\big(a+\eta(b,a),a\big)\Big)+f\Big(a+t\eta\big(a+\eta(b,a),a\big)\Big)\\&\leq\big[h(t)+h(1-t)\big]\big[f(a)+f\big(a+\eta(b,a)\big)\big].\end{aligned}\tag{2.3}$$

By making use of Condition C for the left hand side of (2.3), we can get

$$\begin{aligned}&f\Big(a+(1-t)\eta\big(a+\eta(b,a),a\big)\Big)+f\Big(a+t\eta\big(a+\eta(b,a),a\big)\Big)\\&=f\big(a+(1-t)\eta(b,a)\big)+f\big(a+t\eta(b,a)\big).\end{aligned}\tag{2.4}$$

Utilizing (2.4) in (2.3), we have

$$\begin{aligned}&f\Big(a+(1-t)\eta(b,a)\Big)+f\Big(a+t\eta(b,a)\Big)\\&\leq\big[h(t)+h(1-t)\big]\Big[f(a)+f\Big(a+\eta(b,a)\Big)\Big].\end{aligned}\tag{2.5}$$

Then multiplying both sides of (2.5) by $t^{\alpha-1}$ and integrating the resulting inequality with respect to $t$ over $[0,1]$, we obtain

$$\begin{aligned}&\int_0^1 t^{\alpha-1}\Big[f\Big(a+(1-t)\eta(b,a)\Big)+f\Big(a+t\eta(b,a),a\Big)\Big]\mathrm{d}t\\&\leq\int_0^1 t^{\alpha-1}\big[h(t)+h(1-t)\big]\big[f(a)+f\big(a+\eta(b,a)\big)\big]\mathrm{d}t\end{aligned}$$

and

$$\begin{aligned}&\frac{\Gamma(\alpha)}{\eta^{\alpha}(b,a)}\Big[J^{\alpha}_{a^+}f\big(a+\eta(b,a)\big)+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\Big]\\&\leq\big[f(a)+f\big(a+\eta(b,a)\big)\big]\int_0^1 t^{\alpha-1}[h(t)+h(1-t)]\mathrm{d}t\end{aligned}\tag{2.6}$$

and therefore the first inequality of (2.1) is proved.

To prove the second inequality in (2.1), by virtue of Hölder's inequality for the right hand side of (2.6), we get

$$\begin{aligned}\int_0^1 t^{\alpha-1}[h(t)+h(1-t)]\mathrm{d}t&\leq\Big(\int_0^1\big(t^{\alpha-1}\big)^p\mathrm{d}t\Big)^{\frac{1}{p}}\Big(\int_0^1\big(h(t)+h(1-t)\big)^q\mathrm{d}t\Big)^{\frac{1}{q}}\\&=\Big(\frac{1}{\alpha p-p+1}\Big)^{\frac{1}{p}}\Big(\int_0^1\big(h(t)+h(1-t)\big)^q\mathrm{d}t\Big)^{\frac{1}{q}}.\end{aligned}$$

Due to the Minkowski inequality, we may deduce

$$\begin{aligned}&\Big(\frac{1}{\alpha p-p+1}\Big)^{\frac{1}{p}}\Big(\int_0^1\big(h(t)+h(1-t)\big)^q\mathrm{d}t\Big)^{\frac{1}{q}}\\&\leq\Big(\frac{1}{\alpha p-p+1}\Big)^{\frac{1}{p}}\Big[\Big(\int_0^1\big(h(t)\big)^q\mathrm{d}t\Big)^{\frac{1}{q}}+\Big(\int_0^1\big(h(1-t)\big)^q\mathrm{d}t\Big)^{\frac{1}{q}}\Big]\\&=\frac{2}{(\alpha p-p+1)^{\frac{1}{p}}}\Big(\int_0^1\big(h(t)\big)^q\mathrm{d}t\Big)^{\frac{1}{q}},\end{aligned}$$

where the proof is completed.

We point out, now, some special cases of Theorem 2.1.

**Corollary 2.1** *Letting $\eta(b,a)=b-a$, the inequalities (2.1) reduce to Theorem 2.1 given by Tunç in [30, Page 561].*

**Corollary 2.2** *In Theorem 2.1, letting $\alpha=1$, we can deduce*

$$\begin{aligned}\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\mathrm{d}x&\leq\Big[f(a)+f\big(a+\eta(b,a)\big)\Big]\int_0^1 h(t)\mathrm{d}t\\&\leq\Big[f(a)+f\big(a+\eta(b,a)\big)\Big]\Big(\int_0^1\big(h(t)\big)^q\mathrm{d}t\Big)^{\frac{1}{q}}\end{aligned}$$

*for h-preinvex functions.*

**Corollary 2.3** *(1) If we take $h(t)=t$ in Corollary 2.2, we get*

$$\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\mathrm{d}x\leq\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}\leq\frac{f(a)+f\big(a+\eta(b,a)\big)}{(q+1)^{\frac{1}{q}}}$$

*for preinvex functions.*
*(2) If we take $h(t)=1$ in Corollary 2.2, we have*

$$\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\mathrm{d}x \le f(a)+f\big(a+\eta(b,a)\big)$$

*for P-preinvex functions.*
*(3) If we take $h(t)=t^s$ in Corollary 2.2, we also obtain*

$$\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\mathrm{d}x \le \frac{f(a)+f\big(a+\eta(b,a)\big)}{s+1} \le \frac{f(a)+f\big(a+\eta(b,a)\big)}{(sq+1)^{\frac{1}{q}}}$$

*for s-preinvex functions in the second sense with $s\in(0,1]$.*

**Theorem 2.2** *Let $f: I\to(0,\infty)$ be an h-preinvex function with $\eta(b,a)>0$, $h$ be super-additive on $I$ and $f\in L_1[a,b]$, $h\in L_1[0,1]$. Then one has inequality for h-preinnvex functions via fractional integrals*

$$\frac{\Gamma(\alpha)}{\eta^\alpha(b,a)}\Big[J^\alpha_{a^+}f\big(a+\eta(b,a)\big)+J^\alpha_{(a+\eta(b,a))^-}f(a)\Big] \le \frac{h(1)}{\alpha}\big[f(a)+f\big(a+\eta(b,a)\big)\big]. \tag{2.7}$$

*Proof.* Since $f$ is $h$-preinvex and $h$ is super-additive, by virtue of (2.5), we have

$$\begin{aligned} f\big(a+(1-t)\eta(b,a)\big)+f\big(a+t\eta(b,a)\big) &\le \big[h(t)+h(1-t)\big]\big[f(a)+f\big(a+\eta(b,a)\big)\big] \\ &\le h(1)\big[f(a)+f\big(a+\eta(b,a)\big)\big]. \end{aligned} \tag{2.8}$$

Then multiplying both sides of (2.8) by $t^{\alpha-1}$ and integrating the resulting inequality respecting $t$ over $[0,1]$, we get

$$\begin{aligned} &\int_0^1 t^{\alpha-1}\Big[f\big(a+(1-t)\eta(b,a)\big)+f\big(a+t\eta(b,a)\big)\Big]\mathrm{d}t \\ &\le \int_0^1 t^{\alpha-1}h(1)\Big[f(a)+f\big(a+\eta(b,a)\big)\Big]\mathrm{d}t \end{aligned}$$

and

$$\begin{aligned} &\frac{\Gamma(\alpha)}{\eta^\alpha(b,a)}\Big[J^\alpha_{a^+}f\big(a+\eta(b,a)\big)+J^\alpha_{(a+\eta(b,a))^-}f(a)\Big] \\ &\le h(1)\Big[f(a)+f\big(a+\eta(b,a)\big)\Big]\int_0^1 t^{\alpha-1}\mathrm{d}t, \end{aligned} \tag{2.9}$$

which completes the proof.

As special cases, we provide the following results for the Theorem 2.2.

**Corollary 2.4** *Letting $\eta(b,a)=b-a$, the inequality* (2.7) *reduces to Theorem* 2.4 *proven by Tunç in [30, Page 562].*

**Corollary 2.5** *If we take $\alpha = 1$ with $\eta(b,a) = b - a$ in Theorem 2.2, then the inequality (2.7) becomes a special version of right hand side of (1.8).*

We prove, next, some Hermite-Hadamard type inequalities for mappings whose derivatives are differentiable $h$-preinvex via fractional integrals. To do this, we present the following lemma.

**Lemma 2.1** *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \to \mathbb{R}$ and $a, b \in A$ with $\eta(b,a) > 0$. Suppose that $f : A \to \mathbb{R}$ be a twice differentiable function on $A$. If $f'' \in L[a, a+\eta(b,a)]$ , then the following identity for Riemann-Liouville fractional integrals with $\alpha > 0$ holds:*

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\Big[J^{\alpha}_{a^+}f(a+\eta(b,a)) + J^{\alpha}_{(a+\eta(b,a))^-}f(a)\Big] - f\Big(\frac{2a+\eta(b,a)}{2}\Big)\\ =&\frac{\eta^2(b,a)}{2}\bigg[\int_0^{\frac{1}{2}}\big(t-\lambda(t)\big)f''\big(a+t\eta(b,a)\big)\mathrm{d}t + \int_{\frac{1}{2}}^1\big((1-t)-\lambda(t)\big)f''\big(a+t\eta(b,a)\big)\mathrm{d}t\bigg],\end{aligned} \tag{2.10}$$

*where*

$$\lambda(t) = \frac{1 - t^{\alpha+1} - (1-t)^{\alpha+1}}{\alpha+1}. \tag{2.11}$$

*Proof.* Set

$$\begin{aligned}I_1 &= \frac{\eta^2(b,a)}{2}\int_0^{\frac{1}{2}} tf''\big(a+t\eta(b,a)\big)\mathrm{d}t,\\ I_2 &= \frac{\eta^2(b,a)}{2}\int_{\frac{1}{2}}^1 (1-t)f''\big(a+t\eta(b,a)\big)\mathrm{d}t,\end{aligned}$$

and

$$I_3 = \frac{\eta^2(b,a)}{2}\int_0^1 \frac{1-t^{\alpha+1}-(1-t)^{\alpha+1}}{\alpha+1}f''\big(a+t\eta(b,a)\big)\mathrm{d}t.$$

Since $a, b \in A$ and $A$ is an invex set regarding $\eta$, for every $t \in [0,1]$, we have $a + t\eta(b,a) \in A$. Integrating by part yields that

$$\begin{aligned}I_1 &= \frac{\eta^2(b,a)}{2}\bigg[\frac{1}{\eta(b,a)}tf'\big(a+t\eta(b,a)\big)\Big|_0^{\frac{1}{2}} - \frac{1}{\eta(b,a)}\int_0^{\frac{1}{2}} f'\big(a+t\eta(b,a)\big)\mathrm{d}t\bigg]\\ &= \frac{\eta(b,a)}{4}f'\big(\frac{2a+\eta(b,a)}{2}\big) - \frac{1}{2}f\big(a+t\eta(b,a)\big)\Big|_0^{\frac{1}{2}}\\ &= \frac{\eta(b,a)}{4}f'\big(\frac{2a+\eta(b,a)}{2}\big) - \frac{1}{2}\Big[f\Big(\frac{2a+\eta(b,a)}{2}\Big) - f(a)\Big].\end{aligned}$$

Analogously we also have

$$I_2 = -\frac{\eta(b,a)}{4}f'\big(\frac{2a+\eta(b,a)}{2}\big) + \frac{1}{2}\Big[f(a+\eta(b,a)) - f\Big(\frac{2a+\eta(b,a)}{2}\Big)\Big].$$

On the other hand, integrating by part, it yields that

$$\begin{aligned} I_3 =& \frac{\eta^2(b,a)}{2}\Bigg[\frac{1-t^{\alpha+1}-(1-t)^{\alpha+1}}{(\alpha+1)\eta(b,a)}f'\big(a+t\eta(b,a)\big)\Big|_0^1 \\ & -\int_0^1 \frac{-(\alpha+1)t^{\alpha}+(\alpha+1)(1-t)^{\alpha}}{(\alpha+1)\eta(b,a)}f'\big(a+t\eta(b,a)\big)\mathrm{d}t\Bigg] \\ =& \frac{\eta^2(b,a)}{2}\Bigg[\frac{t^{\alpha}-(1-t)^{\alpha}}{\eta^2(b,a)}f\big(a+t\eta(b,a)\big)\Big|_0^1 \\ & -\int_0^1 \frac{\alpha t^{\alpha-1}+\alpha(1-t)^{\alpha-1}}{\eta^2(b,a)}f\big(a+t\eta(b,a)\big)\mathrm{d}t\Bigg] \\ =& \frac{f(a)+f\big(a+t\eta(b,a)\big)}{2}-\frac{\alpha}{2}\Bigg[\int_0^1\Big(t^{\alpha-1}+(1-t)^{\alpha-1}\Big)f\big(a+t\eta(b,a)\big)\mathrm{d}t\Bigg]. \end{aligned} \tag{2.12}$$

Let $u=a+t\eta(b,a)$ and using the reduction formula $\Gamma(\alpha+1)=\alpha\Gamma(\alpha)(\alpha>0)$ for Euler gamma function, we get

$$\frac{\alpha}{2}\int_0^1 t^{\alpha-1}f\big(a+t\eta(b,a)\big)\mathrm{d}t=\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}J^{\alpha}_{(a+\eta(b,a))^-}f(a) \tag{2.13}$$

and similarly we also have

$$\frac{\alpha}{2}\int_0^1 (1-t)^{\alpha-1}f\big(a+t\eta(b,a)\big)\mathrm{d}t=\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}J^{\alpha}_{a^+}f\big(a+t\eta(b,a)\big). \tag{2.14}$$

Applying (2.13) and (2.14) to (2.12), we obtain

$$I_3=\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\Big[J^{\alpha}_{a^+}f\big(a+\eta(b,a)\big)+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\Big].$$

From $I_1$, $I_2$ and $I_3$, it follows that

$$I_1+I_2-I_3=\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\Big[J^{\alpha}_{a^+}f(a+\eta(b,a))+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\Big]-f\Big(\frac{2a+\eta(b,a)}{2}\Big),$$

which is the desired result. The proof of Lemma 2.1 is completed.

**Corollary 2.6** *Letting $\eta(b,a)=b-a$, the formula (2.10) reduces to lemma 2.1 given by Zhang and Wang in [38, page 4]. Clearly, the obtained Lemma 2.1 in the present paper is an extension of a result proved by Zhang et al. in [38].*

With the aid of Lemma 2.1, let us begin with our next results involving fractional integral inequalities for $h$-preinvex functions.

**Theorem 2.3** *Let $h: J\subseteq\mathbb{R}\to\mathbb{R}$ ($[0,1]\subseteq J$) be a non-negative function and $h(t)\geq t$ for $0\leq t\leq 1$. Suppose that $f:[a,a+\eta(b,a)]\subseteq[0,\infty)\to\mathbb{R}$ be a twice differentiable mapping on $(a,a+\eta(b,a))$ with $\eta(b,a)>0$ such that $f''$*

$\in L_1[a, a+\eta(b,a)]$. *If* $|f''|$ *is h-preinvex on* $[a, a+\eta(b,a)]$, *then the following inequality for fractional integrals with* $\alpha > 0$ *hold:*

$$\begin{aligned}&\Big|\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\Big[J^{\alpha}_{a^+}f\big(a+\eta(b,a)\big)+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\Big]-f\Big(\frac{2a+\eta(b,a)}{2}\Big)\Big|\\&\leq\frac{\eta^2(b,a)}{2}\Big[\int_0^{\frac{1}{2}}\Big(t-\lambda(t)\Big)h(t)\mathrm{d}t+\int_{\frac{1}{2}}^1\Big((1-t)-\lambda(t)\Big)h(t)\mathrm{d}t\Big]\Big(|f''(a)|+|f''(b)|\Big),\end{aligned}\tag{2.15}$$

*where* $\lambda(t)$ *is defined by (2.11).*

*Proof.* From Lemma 2.1 and properties of absolute value, we have

$$\begin{aligned}&\Big|\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\Big[J^{\alpha}_{a^+}f\big(a+\eta(b,a)\big)+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\Big]-f\Big(\frac{2a+\eta(b,a)}{2}\Big)\Big|\\&\leq\frac{\eta^2(b,a)}{2}\Big[\int_0^{\frac{1}{2}}\big|t-\lambda(t)\big|\big|f''\big(a+t\eta(b,a)\big)\big|\mathrm{d}t+\int_{\frac{1}{2}}^1\big|(1-t)-\lambda(t)\big|\big|f''\big(a+t\eta(b,a)\big)\big|\mathrm{d}t\Big]\\&=\frac{\eta^2(b,a)}{2}\Big[\int_0^{\frac{1}{2}}t\big|f''\big(a+t\eta(b,a)\big)\big|\mathrm{d}t+\int_{\frac{1}{2}}^1(1-t)\big|f''\big(a+t\eta(b,a)\big)\big|\mathrm{d}t\\&\quad-\int_0^1\lambda(t)|f''\big(a+t\eta(b,a)\big)|\mathrm{d}t\Big].\end{aligned}\tag{2.16}$$

According to the $h$-preinvexity of $|f''|$ on $[a, a+\eta(b,a)]$, we have

$$\begin{aligned}&\Big|\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\Big[J^{\alpha}_{a^+}f\big(a+\eta(b,a)\big)+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\Big]-f\Big(\frac{2a+\eta(b,a)}{2}\Big)\Big|\\&\leq\frac{\eta^2(b,a)}{2}\Big\{\int_0^{\frac{1}{2}}t\Big[h(1-t)|f''(a)|+h(t)|f''(b)|\Big]\mathrm{d}t+\int_{\frac{1}{2}}^1(1-t)\Big[h(1-t)|f''(a)|+h(t)|f''(b)|\Big]\mathrm{d}t\\&\quad-\int_0^1\lambda(t)\Big[h(1-t)|f''(a)|+h(t)|f''(b)|\Big]\mathrm{d}t\Big\}\\&=\frac{\eta^2(b,a)}{2}\Big\{|f''(a)|\Big[\int_0^{\frac{1}{2}}th(1-t)\mathrm{d}t+\int_{\frac{1}{2}}^1(1-t)h(1-t)\mathrm{d}t-\int_0^1\lambda(t)h(1-t)\mathrm{d}t\Big]\\&\quad+|f''(b)|\Big[\int_0^{\frac{1}{2}}th(t)\mathrm{d}t+\int_{\frac{1}{2}}^1(1-t)h(t)\mathrm{d}t-\int_0^1\lambda(t)h(t)\mathrm{d}t\Big]\Big\}\\&=\frac{\eta^2(b,a)}{2}\Big[\int_0^{\frac{1}{2}}\Big(t-\lambda(t)\Big)h(t)\mathrm{d}t+\int_{\frac{1}{2}}^1\Big((1-t)-\lambda(t)\Big)h(t)\mathrm{d}t\Big]\Big(|f''(a)|+|f''(b)|\Big),\end{aligned}\tag{2.17}$$

where we use the following fact that

$$\int_0^{\frac{1}{2}}th(1-t)\mathrm{d}t=\int_{\frac{1}{2}}^1(1-t)h(t)\mathrm{d}t,$$

$$\int_0^{\frac{1}{2}} th(t)\mathrm{d}t = \int_{\frac{1}{2}}^1 (1-t)h(1-t)\mathrm{d}t$$

and

$$\int_0^1 \lambda(t)h(1-t)\mathrm{d}t = \int_0^1 \lambda(t)h(t)\mathrm{d}t.$$

Hence, we obtain the required inequality (2.15). This completes the proof of Theorem 2.3.

Let us discuss some special cases of Theorem 2.3.

**Corollary 2.7** *Under conditions of Theorem 2.3, if we choose $h(t) = t$, then (2.15) becomes the following inequality for preinvex functions*

$$\Big|\frac{\Gamma(\alpha+1)}{2\eta^\alpha(b,a)}\Big[J^\alpha_{a^+}f\big(a+\eta(b,a)\big) + J^\alpha_{(a+\eta(b,a))^-}f(a)\Big] - f\Big(\frac{2a+\eta(b,a)}{2}\Big)\Big|$$
$$\leq \frac{\eta^2(b,a)}{16}\Big[\frac{\alpha^2-\alpha+2}{(\alpha+1)(\alpha+2)}\Big]\Big[|f''(a)|+|f''(b)|\Big],$$

*specially for $\alpha = 1$ and $\eta(b,a) = b-a$, we get the following inequality for convex functions*

$$\Big|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - f\Big(\frac{a+b}{2}\Big)\Big| \leq \frac{(b-a)^2}{24}\Big[\frac{|f''(a)|+|f''(b)|}{2}\Big],$$

*which is is the same as the inequality established by Sarikaya, Saglam, and Yildirim in [29, Theorem 3].*
*Also, in Theorem 2.3, letting $h(t) = 1$, then we get*

$$\Big|\frac{\Gamma(\alpha+1)}{2\eta^\alpha(b,a)}\Big[J^\alpha_{a^+}f\big(a+\eta(b,a)\big) + J^\alpha_{(a+\eta(b,a))^-}f(a)\Big] - f\Big(\frac{2a+\eta(b,a)}{2}\Big)\Big|$$
$$\leq \frac{\eta^2(b,a)}{2}\Big\{\Big[|f''(a)|+|f''(b)|\Big]\Big(\frac{1}{4} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\Big)\Big\}$$

*for $P$-preinvex functions.*

**Corollary 2.8** *In given conditions of Theorem 2.3, if we take $h(t) = t^s$, then (2.15) reduces to the following inequality for $s$-preinvex functions*

$$\Big|\frac{\Gamma(\alpha+1)}{2\eta^\alpha(b,a)}\Big[J^\alpha_{a^+}f\big(a+\eta(b,a)\big) + J^\alpha_{(a+\eta(b,a))^-}f(a)\Big] - f\Big(\frac{2a+\eta(b,a)}{2}\Big)\Big|$$
$$\leq \frac{\eta^2(b,a)}{2}\Big\{\Big[|f''(a)|+|f''(b)|\Big]\Big[\frac{2^{s+1}-1}{2^{s+1}(s+1)(s+2)} - \frac{1}{(s+1)(s+\alpha+2)} + \frac{\beta(s+1,\alpha+2)}{\alpha+1}\Big]\Big\}.$$

**Corollary 2.9** *In Theorem 2.3, if $|f''(x)| \leq M$, then (2.15) becomes the following inequality*

$$\Big|\frac{\Gamma(\alpha+1)}{2\eta^\alpha(b,a)}\Big[J^\alpha_{a^+}f\big(a+\eta(b,a)\big) + J^\alpha_{(a+\eta(b,a))^-}f(a)\Big] - f\Big(\frac{2a+\eta(b,a)}{2}\Big)\Big|$$
$$\leq M\eta^2(b,a)\Big[\int_0^{\frac{1}{2}} th(t)\mathrm{d}t + \int_{\frac{1}{2}}^1 (1-t)h(t)\mathrm{d}t - \int_0^1 \lambda(t)h(t)\mathrm{d}t\Big],$$

*where $\lambda(t)$ is defined by (2.11).*

**Theorem 2.4** *Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ ($[0,1] \subseteq J$) be a non-negative and $h(t) \geq t$ for $0 \leq t \leq 1$. Suppose $f : [a, a+\eta(b,a)] \subseteq [0,\infty) \to \mathbb{R}$ be a twice differentiable mapping on $(a, a+\eta(b,a))$ with $\eta(b,a) > 0$ such that $h^q$, $f'' \in L_1[a, a+\eta(b,a)]$ and $p^{-1} + q^{-1} = 1$ with $p, q > 1$. If $|f''|$ is h-preinvex on $[a, a+\eta(b,a)]$, then the following inequality for fractional integrals with $\alpha > 0$ hold:*

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\Big[J^{\alpha}_{a^+}f(a+\eta(b,a)) + J^{\alpha}_{(a+\eta(b,a))^-}f(a)\Big] - f\Big(\frac{2a+\eta(b,a)}{2}\Big)\right| \\
\leq &\frac{\eta^2(b,a)}{2}\Big[|f''(a)| + |f''(b)|\Big]\Bigg\{\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\Big(\frac{1}{2}\Big)^{1+\frac{1}{p}}\Bigg[\Big(\int_0^{\frac{1}{2}} h^q(t)\mathrm{d}t\Big)^{\frac{1}{q}} + \Big(\int_{\frac{1}{2}}^1 h^q(t)\mathrm{d}t\Big)^{\frac{1}{q}}\Bigg] \\
&+ \frac{1}{\alpha+1}\Big(\frac{p\alpha+p-1}{p\alpha+p+1}\Big)^{\frac{1}{p}}\Big(\int_0^1 h^q(t)\mathrm{d}t\Big)^{\frac{1}{q}}\Bigg\}.
\end{aligned} \tag{2.18}$$

*Proof.* Continuing from inequality (2.17) in the proof of Theorem 2.3, using properties of absolute value again, recurring to definition of $\lambda(t)$ and Hölder's inequality, we have

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\Big[J^{\alpha}_{a^+}f(a+\eta(b,a)) + J^{\alpha}_{(a+\eta(b,a))^-}f(a)\Big] - f\Big(\frac{2a+\eta(b,a)}{2}\Big)\right| \\
\leq &\frac{\eta^2(b,a)}{2}\Big[|f''(a)| + |f''(b)|\Big] \\
&\times\Bigg\{\Big(\int_0^{\frac{1}{2}} t^p\mathrm{d}t\Big)^{\frac{1}{p}}\Big(\int_0^{\frac{1}{2}} h^q(t)\mathrm{d}t\Big)^{\frac{1}{q}} + \Big(\int_{\frac{1}{2}}^1 (1-t)^p\mathrm{d}t\Big)^{\frac{1}{p}}\Big(\int_{\frac{1}{2}}^1 h^q(t)\mathrm{d}t\Big)^{\frac{1}{q}} \\
&+ \frac{1}{\alpha+1}\Big(\int_0^1 \big(1-t^{\alpha+1}-(1-t)^{\alpha+1}\big)^p\mathrm{d}t\Big)^{\frac{1}{p}}\Big(\int_0^1 h^q(t)\mathrm{d}t\Big)^{\frac{1}{q}}\Bigg\} \\
\leq &\frac{\eta^2(b,a)}{2}\Big[|f''(a)| + |f''(b)|\Big] \\
&\times\Bigg\{\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\Big(\frac{1}{2}\Big)^{1+\frac{1}{p}}\Bigg[\Big(\int_0^{\frac{1}{2}} h^q(t)\mathrm{d}t\Big)^{\frac{1}{q}} + \Big(\int_{\frac{1}{2}}^1 h^q(t)\mathrm{d}t\Big)^{\frac{1}{q}}\Bigg] \\
&+ \frac{1}{\alpha+1}\Big(\frac{p\alpha+p-1}{p\alpha+p+1}\Big)^{\frac{1}{p}}\Big(\int_0^1 h^q(t)\mathrm{d}t\Big)^{\frac{1}{q}}\Bigg\}.
\end{aligned}$$

To prove the second inequality above, we use the following fact that

$$\int_0^{\frac{1}{2}} t^p\mathrm{d}t = \int_{\frac{1}{2}}^1 (1-t)^p\mathrm{d}t = \frac{1}{2^{p+1}(p+1)} \tag{2.19}$$

and

$$\int_0^1 \Big(1-t^{\alpha+1}-(1-t)^{\alpha+1}\Big)^p\mathrm{d}t \leq \frac{p\alpha+p-1}{p\alpha+p+1}, \tag{2.20}$$

where we use the following inequality

$$\big(1-t^{\alpha+1}-(1-t)^{\alpha+1}\big)^p \le 1-t^{p(\alpha+1)}-(1-t)^{p(\alpha+1)}$$

for any $t\in[0,1]$, which follows from

$$(A-B)^p \le A^p - B^p$$

for any $A>B\ge 0$ and $p>1$.
Hence, we can get the desired result (2.18).

We give, now, some special cases of Theorem 2.4.

**Corollary 2.10** *In Theorem 2.4, if we take $p=q=2$, then we get*

$$\begin{aligned}
&\Big|\frac{\Gamma(\alpha+1)}{2\eta^\alpha(b,a)}\Big[J^\alpha_{a^+}f(a+\eta(b,a))+J^\alpha_{(a+\eta(b,a))^-}f(a)\Big]-f\Big(\frac{2a+\eta(b,a)}{2}\Big)\Big|\\
\le&\frac{\eta^2(b,a)}{2}\Big[|f''(a)|+|f''(b)|\Big]\Bigg\{\frac{1}{2\sqrt{6}}\Big[\Big(\int_0^{\frac12}h(t^2)\mathrm{d}t\Big)^{\frac12}+\Big(\int_{\frac12}^1 h(t^2)\mathrm{d}t\Big)^{\frac12}\Big]\\
&+\frac{1}{\alpha+1}\Big(\frac{2\alpha+1}{2\alpha+3}\Big)^{\frac12}\Big(\int_0^1 h(t^2)\mathrm{d}t\Big)^{\frac12}\Bigg\},
\end{aligned}$$

*where $h$ is super-multiplicative.*

**Corollary 2.11** *Under assumptions of Theorem 2.4, letting $h(t)=t$, then the inequality (2.18) becomes the following inequality for the preinvex function*

$$\begin{aligned}
&\Big|\frac{\Gamma(\alpha+1)}{2\eta^\alpha(b,a)}\Big[J^\alpha_{a^+}f(a+\eta(b,a))+J^\alpha_{(a+\eta(b,a))^-}f(a)\Big]-f\Big(\frac{2a+\eta(b,a)}{2}\Big)\Big|\\
\le&\frac{\eta^2(b,a)}{2}\Big(\frac{1}{q+1}\Big)^{\frac1q}\Big[|f''(a)|+|f''(b)|\Big]\\
&\times\Bigg\{\Big(\frac{1}{p+1}\Big)^{\frac1p}\Big(\frac12\Big)^{1+\frac1p}\Big[\Big(\frac12\Big)^{1+\frac1q}+\Big(1-\frac{1}{2^{q+1}}\Big)^{\frac1q}\Big]+\frac{1}{\alpha+1}\Big(\frac{p\alpha+p-1}{p\alpha+p+1}\Big)^{\frac1p}\Bigg\},
\end{aligned}$$

*specially for $\alpha=1$, we get the following inequality for preinvex functions*

$$\begin{aligned}
&\Big|\frac{1}{2\eta(b,a)}\int_a^{a+\eta(b,a)}f(t)\mathrm{d}t-f\Big(\frac{2a+\eta(b,a)}{2}\Big)\Big|\\
\le&\frac{\eta^2(b,a)}{4}\Big(\frac{1}{q+1}\Big)^{\frac1q}\Big[|f''(a)|+|f''(b)|\Big]\\
&\times\Bigg\{\Big(\frac{1}{2p+2}\Big)^{\frac1p}\Big[\Big(\frac12\Big)^{1+\frac1q}+\Big(1-\frac{1}{2^{q+1}}\Big)^{\frac1q}\Big]+\Big(\frac{2p-1}{2p+1}\Big)^{\frac1p}\Bigg\}.
\end{aligned}$$

*In Theorem 2.4, if we choose $h(t)=1$, then we get*

$$\left|\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\Big[J^{\alpha}_{a^+}f(a+\eta(b,a))+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\Big]-f\Big(\frac{2a+\eta(b,a)}{2}\Big)\right|$$
$$\leq\frac{\eta^2(b,a)}{2}\Big[|f''(a)|+|f''(b)|\Big]\left\{\frac{1}{2}\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}+\frac{1}{\alpha+1}\Big(\frac{p\alpha+p-1}{p\alpha+p+1}\Big)^{\frac{1}{p}}\right\},$$

*for $P$-preinvex functions.*

**Corollary 2.12** *In Theorem 2.4, if we take $h(t)=t^s$, then the inequality (2.18) becomes the following inequality for $s$-preinvex functions*

$$\left|\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\Big[J^{\alpha}_{a^+}f(a+\eta(b,a))+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\Big]-f\Big(\frac{2a+\eta(b,a)}{2}\Big)\right|$$
$$\leq\frac{\eta^2(b,a)}{2}\Big(\frac{1}{sq+1}\Big)^{\frac{1}{q}}\Big[|f''(a)|+|f''(b)|\Big]$$
$$\times\left\{\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\Big(\frac{1}{2}\Big)^{1+\frac{1}{p}}\left[\Big(\frac{1}{2}\Big)^{s+\frac{1}{q}}+\Big(1-\frac{1}{2^{sq+1}}\Big)^{\frac{1}{q}}\right]+\frac{1}{\alpha+1}\Big(\frac{p\alpha+p-1}{p\alpha+p+1}\Big)^{\frac{1}{p}}\right\}.$$

**Theorem 2.5** *Let $h:J\subseteq\mathbb{R}\to\mathbb{R}$ ($[0,1]\subseteq J$) be a non-negative and super-multiplicative function, $h(t)\geq t$ for $0\leq t\leq 1$. Assume that $f:[a,a+\eta(b,a)]\subseteq[0,\infty)\to\mathbb{R}$ be a twice differentiable mapping on $(a,a+\eta(b,a))$ with $\eta(b,a)>0$ such that $f''\in L_1[a,a+\eta(b,a)]$. If $|f''|^q$ is $h$-preinvex on $[a,a+\eta(b,a)]$, $q\geq 1$ and $|f''(x)|\leq M$, $x\in[a,a+\eta(b,a)]$, then the following inequalities for fractional integrals with $\alpha>0$ hold:*

$$\left|\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\Big[J^{\alpha}_{a^+}f(a+\eta(b,a))+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\Big]-f\big(\frac{2a+\eta(b,a)}{2}\big)\right|$$
$$\leq\frac{M\eta^2(b,a)}{2}\left\{\Big(\frac{1}{8}\Big)^{1-\frac{1}{q}}\Big[\Big(\int_0^{\frac{1}{2}}t[h(t)+h(1-t)]\mathrm{d}t\Big)^{\frac{1}{q}}+\Big(\int_{\frac{1}{2}}^1(1-t)[h(t)+h(1-t)]\mathrm{d}t\Big)^{\frac{1}{q}}\Big]\right.$$
$$\left.+\frac{2^{q\alpha}-1}{2^{q\alpha}(\alpha+1)}\Big(\int_0^1\big[h(t)+h(1-t)\big]\mathrm{d}t\Big)^{\frac{1}{q}}\right\}\tag{2.21}$$
$$\leq\frac{M\eta^2(b,a)}{2}\left\{\Big(\frac{1}{8}\Big)^{1-\frac{1}{q}}\Big[\Big(\int_0^{\frac{1}{2}}[h(t^2)+h(t-t^2)]\mathrm{d}t\Big)^{\frac{1}{q}}+\Big(\int_{\frac{1}{2}}^1\Big[h(t-t^2)+h\Big((1-t)^2\Big)\Big]\mathrm{d}t\Big)^{\frac{1}{q}}\Big]\right.$$
$$\left.+\frac{2^{q\alpha}-1}{2^{q\alpha}(\alpha+1)}\Big(\int_0^1\big[h(t)+h(1-t)\big]\mathrm{d}t\Big)^{\frac{1}{q}}\right\}.\tag{2.22}$$

*Proof.* Continuing from inequality (2.16) in the proof of Theorem 2.3, using properties of absolute value again, recurring to definition of $\lambda(t)$ and power

mean inequality for $q \geq 1$, we have

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\Big[J^{\alpha}_{a^+}f(a+\eta(b,a)) + J^{\alpha}_{(a+\eta(b,a))^-}f(a)\Big] - f\big(\frac{2a+\eta(b,a)}{2}\big)\right| \\
\leq &\frac{\eta^2(b,a)}{2}\bigg\{\Big(\int_0^{\frac{1}{2}} t\mathrm{d}t\Big)^{1-\frac{1}{q}}\Big(\int_0^{\frac{1}{2}} t\big|f''\big(a+t\eta(b,a)\big)\big|^q\mathrm{d}t\Big)^{\frac{1}{q}} \\
&+\Big(\int_{\frac{1}{2}}^1 (1-t)\mathrm{d}t\Big)^{1-\frac{1}{q}}\Big(\int_{\frac{1}{2}}^1 (1-t)\big|f''\big(a+t\eta(b,a)\big)\big|^q\mathrm{d}t\Big)^{\frac{1}{q}} \\
&+\frac{1}{\alpha+1}\Big(\int_0^1 1\mathrm{d}t\Big)^{1-\frac{1}{q}}\bigg[\int_0^1 \big(1-t^{\alpha+1}-(1-t)^{\alpha+1}\big)^q\big|f''\big(a+t\eta(b,a)\big)\big|^q\mathrm{d}t\bigg]^{\frac{1}{q}}\bigg\}.
\end{aligned}$$

According to the $h$-preinvexity of $|f''|^q$ and $|f''| \leq M$, we get

$$\begin{aligned}
\int_0^{\frac{1}{2}} t\big|f''\big(a+t\eta(b,a)\big)\big|^q\mathrm{d}t &\leq \int_0^{\frac{1}{2}} t\Big[h(1-t)|f''(a)|^q + h(t)|f''(b)|^q\Big]\mathrm{d}t \\
&\leq M^q\int_0^{\frac{1}{2}} t[h(t)+h(1-t)]\mathrm{d}t.
\end{aligned}$$

Similarly, we also have

$$\int_{\frac{1}{2}}^1 (1-t)\big|f''\big(a+t\eta(b,a)\big)\big|^q\mathrm{d}t \leq M^q\int_{\frac{1}{2}}^1 (1-t)[h(t)+h(1-t)]\mathrm{d}t$$

and

$$\begin{aligned}
&\int_0^1 \Big(1-t^{\alpha+1}-(1-t)^{\alpha+1}\Big)^q\big|f''\big(a+t\eta(b,a)\big)\big|^q\mathrm{d}t \\
&\leq \Big(1-\frac{1}{2^{q\alpha}}\Big)M^q\int_0^1 [h(t)+h(1-t)]\mathrm{d}t,
\end{aligned}$$

where we use the fact that

$$\big(1-(1-t)^{\alpha+1}-t^{\alpha+1}\big)^q \leq 1-\big[(1-t)^{\alpha+1}+t^{\alpha+1}\big]^q \leq 1-(2^{-\alpha})^q = 1-\frac{1}{2^{q\alpha}}$$

for any $t \in [0,1]$ with $q \geq 1$.
Also

$$\int_0^{\frac{1}{2}} t\mathrm{d}t = \int_{\frac{1}{2}}^1 (1-t)\mathrm{d}t = \frac{1}{8}.$$

Using these results, we see that the inequality (2.21) is proved. To prove (2.22), and using the additional properties of $h$ in hypothetical conditions, we further have

$$\int_0^{\frac{1}{2}} t[h(t)+h(1-t)]\mathrm{d}t \leq \int_0^{\frac{1}{2}} [h(t^2)+h(t-t^2)]\mathrm{d}t$$

and

$$\int_{\frac{1}{2}}^{1}(1-t)[h(t)+h(1-t)]\mathrm{d}t \leq \int_{\frac{1}{2}}^{1}\Big[h(t-t^2)+h\Big((1-t)^2\Big)\Big]\mathrm{d}t.$$

Hence, the proof of (2.22) is completed.

Elementary calculation yields the following result.

**Corollary 2.13** *In Theorem 2.5, if we choose $h(t)=t$, we can obtain*

$$\begin{aligned}&\Big|\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\Big[J^{\alpha}_{a^+}f(a+\eta(b,a))+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\Big]-f\big(\frac{2a+\eta(b,a)}{2}\big)\Big|\\&\leq\frac{M\eta^2(b,a)}{2}\Big(\frac{1}{4}+\frac{2^{q\alpha}-1}{2^{q\alpha}(\alpha+1)}\Big),\end{aligned}$$

*specially for $h(t)=1$, we get*

$$\begin{aligned}&\Big|\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\Big[J^{\alpha}_{a^+}f(a+\eta(b,a))+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\Big]-f\big(\frac{2a+\eta(b,a)}{2}\big)\Big|\\&\leq\frac{M\eta^2(b,a)}{2^{1-\frac{1}{q}}}\Big(\frac{1}{4}+\frac{2^{q\alpha}-1}{2^{q\alpha}(\alpha+1)}\Big).\end{aligned}$$

**Theorem 2.6** *Let $h:J\subseteq\mathbb{R}\to\mathbb{R}$ ($[0,1]\subseteq J$) be a non-negative and super-additive function, $h(t)\geq t$ for $0\leq t\leq 1$. Suppose that $f:[a,a+\eta(b,a)]\subseteq[0,\infty)\to\mathbb{R}$ be a twice differentiable mapping on $(a,a+\eta(b,a))$ with $\eta(b,a)>0$ such that $f''\in L_1[a,a+\eta(b,a)]$. If $|f''|^q$ is h-preinvex on $[a,a+\eta(b,a)]$, $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $|f''(x)|\leq M$, $x\in[a,a+\eta(b,a)]$, then the following inequalities for fractional integrals with $\alpha>0$ hold:*

$$\begin{aligned}&\Big|\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\Big[J^{\alpha}_{a^+}f(a+\eta(b,a))+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\Big]-f\Big(\frac{2a+\eta(b,a)}{2}\Big)\Big|\\&\leq\frac{M\eta^2(b,a)}{2}\Big\{\Big(\frac{1}{2}\Big)^{1+\frac{1}{p}}\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\Big[\Big(\int_0^{\frac{1}{2}}[h(t)+h(1-t)]\mathrm{d}t\Big)^{\frac{1}{q}}+\Big(\int_{\frac{1}{2}}^{1}[h(t)+h(1-t)]\mathrm{d}t\Big)^{\frac{1}{q}}\Big]\\&\quad+\frac{1}{\alpha+1}\Big(\frac{p\alpha+p-1}{p\alpha+p+1}\Big)^{\frac{1}{p}}\Big(\int_0^1[h(t)+h(1-t)]\mathrm{d}t\Big)^{\frac{1}{q}}\Big\}\qquad(2.23)\\&\leq\frac{M\eta^2(b,a)}{2}\Big[\frac{1}{2}\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}h^{\frac{1}{q}}(1)+\frac{1}{\alpha+1}\Big(\frac{p\alpha+p-1}{p\alpha+p+1}\Big)^{\frac{1}{p}}h^{\frac{1}{q}}(1)\Big].\qquad(2.24)\end{aligned}$$

*Proof.* Continuing from inequality (2.16) in the proof of Theorem 2.3, using properties of absolute value again, recurring to definition of $\lambda(t)$ and Hölder's

inequality for $q>1$, we have

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\Big[J^{\alpha}_{a^+}f(a+\eta(b,a))+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\Big]-f\Big(\frac{2a+\eta(b,a)}{2}\Big)\right|\\
\leq&\frac{\eta^2(b,a)}{2}\bigg\{\Big(\int_0^{\frac{1}{2}}t^p\mathrm{d}t\Big)^{\frac{1}{p}}\Big[\int_0^{\frac{1}{2}}\big|f''\big(a+t\eta(b,a)\big)\big|^q\mathrm{d}t\Big]^{\frac{1}{q}}\\
&+\Big(\int_{\frac{1}{2}}^1(1-t)^p\mathrm{d}t\Big)^{\frac{1}{p}}\Big[\int_{\frac{1}{2}}^1\big|f''\big(a+t\eta(b,a)\big)\big|^q\mathrm{d}t\Big]^{\frac{1}{q}}\\
&+\frac{1}{\alpha+1}\Big[\int_0^1\Big(1-t^{\alpha+1}-(1-t)^{\alpha+1}\Big)^p\mathrm{d}t\Big]^{\frac{1}{p}}\Big[\int_0^1\big|f''\big(a+t\eta(b,a)\big)\big|^q\mathrm{d}t\Big]^{\frac{1}{q}}\bigg\}.
\end{aligned}$$

According to the $h$-preinvexity of $|f''|^q$ and $|f''|\leq M$, we can get

$$\begin{aligned}
\int_0^{\frac{1}{2}}\big|f''\big(a+t\eta(b,a)\big)\big|^q\mathrm{d}t&\leq|f''(a)|^q\int_0^{\frac{1}{2}}h(1-t)\mathrm{d}t+|f''(b)|^q\int_0^{\frac{1}{2}}h(t)\mathrm{d}t\\
&\leq M^q\int_0^{\frac{1}{2}}[h(t)+h(1-t)]\mathrm{d}t.
\end{aligned}$$

Similarly we also have

$$\int_{\frac{1}{2}}^1\big|f''\big(a+t\eta(b,a)\big)\big|^q\mathrm{d}t\leq M^q\int_{\frac{1}{2}}^1[h(t)+h(1-t)]\mathrm{d}t$$

and

$$\int_0^1\big|f''\big(a+t\eta(b,a)\big)\big|^q\mathrm{d}t\leq M^q\int_0^1[h(t)+h(1-t)]\mathrm{d}t.$$

By virtue of the above results and the fact (2.19) and the inequality (2.20), we complete the proof of (2.23).
Using the supper-additive property of $h$ in the assumptions, we further have

$$h(t)+h(1-t)\leq h(1).$$

Hence, the proof of (2.24) is completed.

Finally we shall obtain estimate of Riemann-Liouville fractional Hermite-Hadamard inequality for for $h$-preincave functions.

**Theorem 2.7** *Let $h:J\subseteq\mathbb{R}\to\mathbb{R}$ ($[0,1]\subseteq J$) be a non-negative function, $h(t)\geq t$ for $0\leq t\leq 1$ and $f:[a,a+\eta(b,a)]\subseteq[0,\infty)\to\mathbb{R}$ be a twice differentiable mapping on $(a,a+\eta(b,a))$ with $\eta(b,a)>0$ such that $f''\in L_1[a,a+\eta(b,a)]$. If $|f''|^q$ is $h$-preincave on $[a,a+\eta(b,a)]$, $p,q>1$, $p^{-1}+q^{-1}=1$, then the following inequality for fractional integrals with $\alpha>0$ hold:*

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\Big[J^{\alpha}_{a^+}f(a+\eta(b,a))+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\Big]-f\Big(\frac{2a+\eta(b,a)}{2}\Big)\right|\\
\leq&\frac{\eta^2(b,a)}{2}\Big[\Big(\frac{1}{2p+2}\Big)^{\frac{1}{p}}+\frac{1}{\alpha+1}\Big(\frac{p\alpha+p-1}{p\alpha+p+1}\Big)^{\frac{1}{p}}\Big]\Big[\frac{1}{2h(\frac{1}{2})}\Big]^{\frac{1}{q}}\Big|f''\Big(\frac{2a+\eta(b,a)}{2}\Big)\Big|
\end{aligned}$$

*Proof.* Continuing from inequality (2.16) in the proof of Theorem 2.3, using properties of absolute value again, recurring to definition of $\lambda(t)$ and Hölder's inequality, we have

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J^{\alpha}_{a^+}f(a+\eta(b,a))+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\right]-f\Big(\frac{2a+\eta(b,a)}{2}\Big)\right| \\
\leq&\frac{\eta^2(b,a)}{2}\Bigg\{\Big(\int_0^{\frac{1}{2}}t^p\mathrm{d}t\Big)^{\frac{1}{p}}\Big[\int_0^{\frac{1}{2}}\big|f''\big(a+t\eta(b,a)\big)\big|^q\mathrm{d}t\Big]^{\frac{1}{q}} \\
&+\Big(\int_{\frac{1}{2}}^1(1-t)^p\mathrm{d}t\Big)^{\frac{1}{p}}\Big[\int_{\frac{1}{2}}^1\big|f''\big(a+t\eta(b,a)\big)\big|^q\mathrm{d}t\Big]^{\frac{1}{q}} \\
&+\frac{1}{\alpha+1}\Big[\int_0^1\Big(1-t^{\alpha+1}-(1-t)^{\alpha+1}\Big)^p\mathrm{d}t\Big]^{\frac{1}{p}}\Big[\int_0^1\big|f''\big(a+t\eta(b,a)\big)\big|^q\mathrm{d}t\Big]^{\frac{1}{q}}\Bigg\} \\
\leq&\frac{\eta^2(b,a)}{2}\Bigg\{\Big(\frac{1}{2^{p+1}(p+1)}\Big)^{\frac{1}{p}}\Big[\Big(\int_0^{\frac{1}{2}}\big|f''\big(a+t\eta(b,a)\big)\big|^q\mathrm{d}t\Big)^{\frac{1}{q}}+\Big(\int_{\frac{1}{2}}^1\big|f''\big(a+t\eta(b,a)\big)\big|^q\mathrm{d}t\Big)^{\frac{1}{q}}\Big] \\
&+\frac{1}{\alpha+1}\Big(\frac{p\alpha+p-1}{p\alpha+p+1}\Big)^{\frac{1}{p}}\Big[\int_0^1\big|f''\big(a+t\eta(b,a)\big)\big|^q\mathrm{d}t\Big]^{\frac{1}{q}}\Bigg\}.
\end{aligned}$$

To prove the second inequality above, we here use the fact (2.19) and the inequality(2.20) again.

Also, $|f''|^q$ is $h$-preincave on $[a,a+\eta(b,a)]$, by inequalities (1.9) we have

$$\int_0^{\frac{1}{2}}\big|f''\big(a+t\eta(b,a)\big)\big|^q\mathrm{d}t\leq\int_0^1\big|f''\big(a+t\eta(b,a)\big)\big|^q\mathrm{d}t\leq\frac{1}{2h(\frac{1}{2})}\Big|f''\Big(\frac{2a+\eta(b,a)}{2}\Big)\Big|^q.$$

Similarly, we also have

$$\int_{\frac{1}{2}}^1\big|f''\big(a+t\eta(b,a)\big)\big|^q\mathrm{d}t\leq\frac{1}{2h(\frac{1}{2})}\Big|f''\Big(\frac{2a+\eta(b,a)}{2}\Big)\Big|^q$$

and

$$\int_0^1\big|f''\big(a+t\eta(b,a)\big)\big|^q\mathrm{d}t\leq\frac{1}{2h(\frac{1}{2})}\Big|f''\Big(\frac{2a+\eta(b,a)}{2}\Big)\Big|^q.$$

Therefore, we can get

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J^{\alpha}_{a^+}f(a+\eta(b,a))+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\right]-f\Big(\frac{2a+\eta(b,a)}{2}\Big)\right| \\
\leq&\frac{\eta^2(b,a)}{2}\Big[\Big(\frac{1}{2p+2}\Big)^{\frac{1}{p}}+\frac{1}{\alpha+1}\Big(\frac{p\alpha+p-1}{p\alpha+p+1}\Big)^{\frac{1}{p}}\Big]\Big[\frac{1}{2h(\frac{1}{2})}\Big]^{\frac{1}{q}}\Big|f''\Big(\frac{2a+\eta(b,a)}{2}\Big)\Big|.
\end{aligned}$$

Direct computation provides the following corollary.

**Corollary 2.14** *In given conditions of Theorem 2.7, if we take $h(t)=t$, we obtain the following inequality for the preincave functions*

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J^{\alpha}_{a^+}f(a+\eta(b,a))+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\right]-f\Big(\frac{2a+\eta(b,a)}{2}\Big)\right| \\
\leq&\frac{\eta^2(b,a)}{2}\Big[\Big(\frac{1}{2p+2}\Big)^{\frac{1}{p}}+\frac{1}{\alpha+1}\Big(\frac{p\alpha+p-1}{p\alpha+p+1}\Big)^{\frac{1}{p}}\Big]\Big|f''\Big(\frac{2a+\eta(b,a)}{2}\Big)\Big|.
\end{aligned}$$

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## References

[1] H. Alzer, A superadditive property of Hadamard's gamma function, *Abh. Math. Semin. Univ. Hambg.*, **79**, no. 1, 11-23, (2009).

[2] G. A. Anastassiou, Generalised fractional Hermite-Hadamard inequalities involving $m$-convexity and $(s,m)$–convexity, *Facta Universitatis (Niš) Ser. Math. Inform.*, **28**, 107-126, (2013).

[3] F. Chen, Extensions of the Hermite-Hadamard inequality for harmonically convex functions via fractional integrals, *Appl. Math. Comput.*, **268**, 121-128, (2015).

[4] S. S. Dragomir, M. I. Bhatti, M. Iqbal, and M. Muddassar, Some new Hermite-Hadamard's type fractional integral inequalities, *J. Comput. Anal. Appl.*, **18**, 655-661, (2015).

[5] T. S, Du, J. G. Liao, L. Z. Chen, and M. U. Awan, Properties and Riemann-Liouville fractional Hermite-Hadamard inequalities for the generalized $(\alpha,m)$-preinvex functions, *J. Inequal. Appl.*, **2016**, Article Number 306, 24 pages, (2016).

[6] T. -S. Du, J. -G. Liao, and Y. -J. Li, Properties and integral inequalities of Hadamard-Simpson type for the generalized $(s,m)$-preinvex functions, *J. Nonlinear Sci. Appl.*, **9**, 3112-3126, (2016).

[7] T. Du, Y. Li, and Z. Yang, A generalization of Simpson's inequality via differentiable mapping using extended $(s,m)$-convex functions, *Appl. Math. Comput.*, **293**, 358-369, (2017).

[8] S. Hussain and S. Qaisar, More results on Hermite-Hadamard type inequality through $(\alpha,m)$-preinvexity, *J. Appl. Anal. Comput.*, **6**, 293-305, (2016).

[9] S. -R. Hwang, K. -L. Tseng, and K. -C. Hsu, New inequalities for fractional integrals and their applications, *Turkish J. Math.*, **40**, 471-486, (2016).

[10] S. -R. Hwang, S. -Y. Yeh, and K. -L. Tseng, Refinements and similar extensions of Hermite-Hadamard inequality for fractional integrals and their applications, *Appl. Math. Comput.*, **249**, 103-113, (2014).

[11] M. Iqbal, M. I. Bhatti, and K. Nazeer, Generalization of inequalities analogous to Hermite-Hadamard inequality via fractional integrals, *Bull. Korean Math. Soc.*, **52**, 707-716, (2015).

[12] İ. İşcan, New general integral inequalities for quasi-geometrically convex functions via fractional integrals, *J. Inequal. Appl.*, **2013**, Article Number 491, 15 pages, (2013).

[13] İ. İşcan and S. Wu, Hermite-Hadamard type inequalities for harmonically convex functions via fractional integrals, *Appl. Math. Comput.*, **238**, 237-244, (2014).

[14] M. A. Latif and S. S. Dragomir, Generalization of Hermite-Hadamard type inequalities for $n$-times differentiable functions which are $s$-preinvex in the second sense with applications, *Hacet. J. Math. Stat.*, **44**, 839-853, (2015).

[15] Y. J. Li and T. S. Du, A generalization of Simpson type inequality via differentiable functions using extended $(s,m)_\phi$-preinvex functions, *J. Comput. Anal. Appl.*, **22**, 613-632, (2017).

[16] Y. J. Li and T. S. Du, Some Simpson type integral inequalities for functions whose third derivatives are $(\alpha,m)$-GA-convex functions, *J. Egyptian Math. Soc.*, **24**, no. 2, 175-180, (2016).

[17] Y. J. Li, T. S. Du, and B. Yu, Some new integral inequalities of Hadamard-Simpson type for extended $(s,m)$-preinvex functions, *Ital. J. Pure Appl. Math.*, **36**, 583-600, (2016).

[18] Y. M. Liao, J. H. Deng, and J. R. Wang, Riemann-Liouville fractional Hermite-Hadamard inequalities. Part II: for twice differentiable geometric-arithmetically $s$-convex functions, *J. Inequal. Appl.*, **2013**, Article Number 517, 13 pages, (2013).

[19] M. Matłoka, Inequalities for $h$-preinvex functions, *Appl. Math. Comput.*, **234**, 52-57, (2014).

[20] M. Matłoka, Some inequalities of Hadamard type for mappings whose second derivatives are $h$-convex via fractional integrals, *J. Fract. Calc. Appl.*, **6**, no. 1, 110-119, (2015).

[21] M. A. Noor, K. I. Noor, M. U. Awan, and J. Li, On Hermite-Hadamard inequalities for $h$-preinvex functions, *Filomat*, **28**, no. 7, 1463-1474, (2014).

[22] M. A. Noor, K. I. Noor, M. U. Awan, and S. Costache, Some integral inequalities for harmonically $h$-convex functions, *Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys.*, **77**, no. 1, 5-16, (2015).

[23] M. A. Noor, K. I. Noor, M. V. Mihai, and M. U. Awan, Fractional Hermite-Hadamard inequalities for some classes of differentiable preinvex functions, *Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys.*, **78**, no. 3, 163-174, (2016).

[24] S. R. Mohan and S. K. Neogy, On invex sets and preinvex functions, *J. Math. Anal. Appl.*, **189**, 901-908, (1995).

[25] M. E. Özdemir, S. S. Dragomir, and Ç. Yildiz, The Hadamard inequality for convex function via fractional integrals, *Acta Math. Sci. Ser. B Engl. Ed.*, **33B**, 1293-1299, (2013).

[26] S. Qaisar, M. Iqbal, and M. Muddassar, New Hermite-Hadamard's inequalities for preinvex functions via fractional integrals, *J. Comput. Anal. Appl.*, **20**, 1318-1328, (2016).

[27] M. Z. Sarikaya, E. Set, H. Yaldiz, and N. Başak, Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities, *Math. Comput. Modelling*, **57**, 2403-2407, (2013).

[28] M. Z. Sarikaya, H. Filiz, and M. E. Kiris, On some generalized integral inequalities for Riemann-Liouville fractional integrals, *Filomat*, **29**, 1307-1314, (2015).

[29] M. Z. Sarikaya, A. Saglam, and H. Yildirim, New inequalities of Hermite-Hadamard type for functions whose second derivatives absolute values are convex and quasi-convex, https://arxiv.org/abs/1005.0451v1, (2010).

[30] M. Tunç, On new inequalities for $h$-convex functions via Riemann-Liouville fractional integration, *Filomat*, **27**, no. 4, 559-565, (2013).

[31] S. Varošanec, On $h$-convexity, *J. Math. Anal. Appl.*, **326**, no. 1, 303-311, (2007).

[32] J. R. Wang, X. Z. Li, M. Fečkan, and Y. Zhou, Hermite-Hadamard-type inequalities for Riemann-Liouville fractional integrals via two kinds of convexity, *Appl. Anal.*, **92**, no. 11, 2241-2253, (2013).

[33] J. Wang, J. Deng, and M. Fečkan, Hermite-Hadamard-type inequalities for $r$-convex functions based on the use of Riemann-Liouville frantional integrals, *Ukrainian Math. J.*, **65**, 193-211, (2013).

[34] T. Weir and B. Mond, Pre-invex functions in multiple objective optimization, *J. Math. Anal. Appl.*, **136**, 29-38, (1988).

[35] S. Wu, On the weighted generalization of the Hermite-Hadamard inequality and its applications, *Rocky Mountain J. Math.*, **39**, 1741-1749, (2009).

[36] S. Wu, L. Debnath, Inequalities for convex sequences and their applications, *Comput. Math. Appl.*, **54** , 525-534, (2007).

[37] S. -H. Wu, B. Sroysang, J. -S. Xie, and Y. -M. Chu, Parametrized inequality of Hermite-Hadamard type for functions whose third derivative absolute values are quasi-convex, *SpringerPlus*, **4** , Article Number 831, 9 pages, (2015).

[38] Y. R. Zhang and J. R. Wang, On some new Hermite-Hadamard inequalities involving Riemann-Liouville fractional integrals, *J. Inequal. Appl.*, **2013**, Article Number 220, 27 pages, (2013).

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# SOME HERMITE–HADAMARD AND SIMPSON TYPE INEQUALITIES FOR CONVEX FUNCTIONS VIA FRACTIONAL INTEGRALS WITH APPLICATIONS

MUHAMMAD IQBAL, MUSTAFA HABIB, NASIR SIDDIQUI,
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Abstract. In this paper, a new general identity for Riemann−Liouville fractional integrals is established. Then by making use of the established identity, we establish some new inequalities of the Simpson and the Hermite−Hadamard type for functions whose absolute values of derivatives are convex. Our results have some relationships with the results, proved in [3, 6, 10], and the analysis used in the proofs is simple.

## 1. Introduction

**Definition 1.** *Let $I \subset \mathbb{R}$ be an interval. The function $f : I \to \mathbb{R}$ is said to be convex on $I$, if for all $a, b \in I$ with $a \leq b$ and $\lambda \in [0, 1]$, satisfies the inequality*

$$f(\lambda a + (1-\lambda) b) \leq \lambda f(a) + (1-\lambda) f(b)$$

The inequalities discovered by Hermite and Hadamard for convex functions are very important in the literature (see, e.g.,[[12], p. 137], ).These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval $I$ of real numbers and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Both inequalities hold in the reversed direction for $f$ to be concave.

Hadamard's inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found; see, for example, ([3],[5]−[6], [9]−[10], [12]) and the references cited therein.

In [10], a variant of Hermite−Hadamard type inequalities was obtained, which follows as:

**Theorem 1.** *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on $I^\circ$ and let $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex function on $[a, b]$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{4}\left(\frac{|f'(a)| + |f'(b)|}{2}\right) \tag{2}$$

In [6] authors proved the following version of Hermite−Hadamard type inequalities:

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2000 *Mathematics Subject Classification.* 26D15, 26A51, 26A33.
*Key words and phrases.* Hermite−Hadamard's Inequality, Simpson's Inequality, Convex Functions, Riemann−Liouville fractional integral.

**Theorem 2.** *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on $I^\circ$ and let $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex function on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left(\frac{|f'(a)|+|f'(b)|}{2}\right) \tag{3}$$

The Simpson's inequality is very important and well known in the literature. For recent refinements, counterparts, generalizations and new Simpson's type inequalities, see ([7],[13], [16], [17]).

In [16], Sarikaya et al. obtained inequality for differentiable convex mappings which is connected with Simpson's inequality, is as follow:

**Theorem 3.** *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f' \in L[a, b]$ where $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex function on $[a, b]$, then the following inequality holds:*

$$\left|\frac{1}{3}\left\{2f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right\} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{5(b-a)}{36}\left(\frac{|f'(a)|+|f'(b)|}{2}\right) \tag{4}$$

In [3], the authors generalize some inequalities related to Hermite−Hadamard and Simpsons inequality for functions whose derivatives in absolute value are convex functions as:

**Theorem 4.** *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on $I^\circ$ and let $a, b \in I^\circ$ with $a < b$. If $0 \leq \lambda \leq 1$ and $|f'|^{\frac{1}{q}}$ for $q \geq 1$ is a convex on $[a, b]$, then the following inequality holds:*

$$\begin{aligned}
&\left|(1-\lambda)f\left(\frac{a+b}{2}\right)+\lambda\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\quad\leq \frac{b-a}{8}\left(\frac{1}{6}\right)^{\frac{1}{q}}(1-2\lambda+2\lambda^2)^{1-\frac{1}{q}} \\
&\quad\times\left[\left(\left\{2-3\lambda+2\lambda^3\right\}|f'(a)|^q+\left\{4-9\lambda+12\lambda^2-2\lambda^3\right\}|f'(b)|^q\right)^{\frac{1}{q}}\right. \\
&\qquad\left.+\left(\left\{4-9\lambda+12\lambda^2-2\lambda^3\right\}|f'(a)|^q+\left\{2-3\lambda+2\lambda^3\right\}|f'(b)|^q\right)^{\frac{1}{q}}\right]
\end{aligned} \tag{5}$$

**Remark 1.** *On letting $\lambda =, 0, 1, \frac{1}{3}$ with $q = 1$ , inequality (5) reduces to inequalities (2), (3) and (4), respectively..*

It is well known that the integral inequalities play an important role in nonlinear analysis. In the recent years, these inequalities have been improved and generalized in a number of ways and a large number of research papers have been written on these inequalities, (see, [1]−[2], [4], [10], [15]) and the references therein.

In recent paper, [10] Sarikaya et. al. proved a variant of Hermite−Hadamard's inequalities in fractional integral forms as follows:

**Theorem 5.** *Let $f : [a, b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L[a, b]$. If $f$ is convex function on $[a, b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a^+}f(b)+J^\alpha_{b^-}f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{6}$$

**Remark 2.** *For $\alpha = 1$, inequality (6) reduces to inequality (1).*

In the following, we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper.

**Definition 2.** *Let $f \in L[a,b]$, the Reimann−Liouville integrals $J^{\alpha}_{a^+}$ and $J^{\alpha}_{b^-}$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J^{\alpha}_{a^+} = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \qquad x > \alpha$$

*and*

$$J^{\alpha}_{b^-} = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \qquad x < \alpha$$

*respectively. Here, $\Gamma(\alpha) = \int_0^{\infty} e^{-t}u^{\alpha-1}du$ is the Gamma function and $J^0_{a^+}f(x) = J^0_{b^-}f(x) = f(x)$.*

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral. Properties concerning this operator can be found in ([8] , [11] , [14]).

The aim of this paper is to establish some new Hermite−Hadamard and Simpson type inequalities in the form of fractional integrals for functions whose absolute values of derivatives are convex. we derive a general integral identity via Riemann−Liouville fractional integrals.

## 2. Main Results

In order to prove our main results we need the following integral identity:

**Lemma 1.** *Let $I \subset \mathbb{R}$ be an open interval, $a, b \in I$ with $a < b$ and $f : [a,b] \to \mathbb{R}$ be a differentiable function such that $f'$ is integrable and $0 < \alpha \leq 1$ on $(a,b)$ with $a < b$,then the following identity for Riemann−Liouville fractional integrals holds:*

$$\left(1 - \frac{2}{2^{\alpha}}\lambda\right) f\left(\frac{a+b}{2}\right) + \lambda\frac{f(a)+f(b)}{2^{\alpha}} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a^+}f(b) + J^{\alpha}_{b^-}f(a)\right]$$
$$= \frac{b-a}{2^{\alpha+2}}\sum_{n=1}^{4} I_n$$

*where*

$$I_1 = \int_0^1 \left[(1-t)^{\alpha} - \lambda\right] f'\left(ta + (1-t)\tfrac{a+b}{2}\right) dt,$$
$$I_2 = \int_0^1 \left[\lambda - (1-t)^{\alpha}\right] f'\left(tb + (1-t)\tfrac{a+b}{2}\right) dt,$$
$$I_3 = \int_0^1 \left[2^{\alpha} - \lambda - (2-t)^{\alpha}\right] f'\left(t\tfrac{a+b}{2} + (1-t)a\right) dt,$$
$$I_4 = \int_0^1 \left[\lambda - 2^{\alpha} + (2-t)^{\alpha}\right] f'\left(t\tfrac{a+b}{2} + (1-t)b\right) dt.$$

*Proof.* Integrating by parts and substituting $u = ta + (1-t)\frac{a+b}{2}$

$$I_1 = \int_0^1 \left[(1-t)^{\alpha} - \lambda\right] f'\left(ta + (1-t)\frac{a+b}{2}\right) dt$$
$$= \frac{2\left[(1-t)^{\alpha} - \lambda\right] f'\left(ta + (1-t)\frac{a+b}{2}\right) dt}{a-b}\Bigg|_0^1$$
$$+ \frac{2\alpha}{a-b}\int_0^1 (1-t)^{\alpha} f'\left(ta + (1-t)\frac{a+b}{2}\right) dt$$
$$= \frac{2\lambda}{b-a}f(a) + \frac{2(1-\lambda)}{b-a}f\left(\frac{a+b}{2}\right) - \frac{2^{\alpha+1}\alpha}{(b-a)^{\alpha+1}}\int_a^{\frac{a+b}{2}} (u-a)^{\alpha-1} f(u)du$$

Analogously

$$I_2 = \frac{2\lambda}{b-a}f(b) + \frac{2(1-\lambda)}{b-a}f\left(\frac{a+b}{2}\right) - \frac{2^{\alpha+1}\alpha}{(b-a)^{\alpha+1}}\int_{\frac{a+b}{2}}^{b}(b-u)^{\alpha-1}f(u)du$$

$$I_3 = \frac{2\lambda}{b-a}f(a) + \frac{2(2^{\alpha}-1-\lambda)}{b-a}f\left(\frac{a+b}{2}\right) - \frac{2^{\alpha+1}\alpha}{(b-a)^{\alpha+1}}\int_{a}^{\frac{a+b}{2}}(b-u)^{\alpha-1}f(u)du$$

$$I_4 = \frac{2\lambda}{b-a}f(b) + \frac{2(2^{\alpha}-1-\lambda)}{b-a}f\left(\frac{a+b}{2}\right) - \frac{2^{\alpha+1}\alpha}{(b-a)^{\alpha+1}}\int_{\frac{a+b}{2}}^{b}(b-u)^{\alpha-1}f(u)du$$

Multiplying above equalities by $\frac{b-a}{2^{\alpha+2}}$, then adding, to get required identity. □

**Theorem 6.** *Let $I \subset \mathbb{R}$ be an open interval, $a, b \in I$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ be a differentiable function such that $f'$ is integrable and $0 < \alpha \leq 1$ on $(a, b)$ with $a < b$ and $0 \leq \lambda \leq 1$. If $|f'|$ is a convex on $[a, b]$, then the following inequality for Riemann−Liouville fractional integrals holds:*

$$\left|\left(1-\frac{2}{2^{\alpha}}\lambda\right)f\left(\frac{a+b}{2}\right) + \lambda\frac{f(a)+f(b)}{2^{\alpha}} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a^+}f(b) + J^{\alpha}_{b^-}f(a)\right]\right|$$
$$\leq \frac{b-a}{2^{\alpha+2}}\left[\frac{2}{\alpha+1}\left\{1+2^{\alpha}-(1-\zeta)^{\alpha+1}-(2-\xi)^{\alpha+1}\right\}\right.$$
$$\left.+(1-2\zeta)\lambda + (2^{\alpha}-\lambda)(1-2\xi)\right]\left(|f'(a)| + |f'(b)|\right) \quad (7)$$

*where $\zeta = 1 - \lambda^{\frac{1}{\alpha}}$, and $\xi = 2 - (2^{\alpha} - \lambda)^{\frac{1}{\alpha}}$.*

*Proof.* Let $\zeta = 1 - \lambda^{\frac{1}{\alpha}}$, and $\xi = 2 - (2^{\alpha} - \lambda)^{\frac{1}{\alpha}}$ then

$$A = \int_0^1 |(1-t)^{\alpha} - \lambda|\, dt = \frac{1-2(1-\zeta)^{\alpha+1}}{\alpha+1} + (1-2\zeta)\lambda$$

$$B = \int_0^1 |2^{\alpha} - (2-t)^{\alpha} - \lambda|\ dt = \frac{1+2^{\alpha+1}-2(2-\xi)^{\alpha+1}}{\alpha+1} + (2^{\alpha}-\lambda)(1-2\xi)$$

$$C = \int_0^1 |(1-t)^{\alpha} - \lambda|\, t\ dt = \frac{1-2(1-\zeta)^{\alpha+2}}{(\alpha+1)(\alpha+2)} - \frac{2\zeta(1-\zeta)^{\alpha+1}}{\alpha+1} + \frac{\lambda}{2}(1-2\zeta^2)$$

$$D = \int_0^1 |2^{\alpha} - (2-t)^{\alpha} - \lambda|\, t\ dt$$
$$= \frac{1+2^{\alpha+2}-2(2-\xi)^{\alpha+2}}{(\alpha+1)(\alpha+2)} + \frac{1-2\xi(2-\xi)^{\alpha+1}}{\alpha+1} + (2^{\alpha}-\lambda)(\frac{1}{2}-\xi^2)$$

By using the properties of modulus on Lemma 1 and convexity of $|f'|$, we have

$$\left|\left(1-\frac{2}{2^{\alpha}}\lambda\right)f\left(\frac{a+b}{2}\right) + \lambda\frac{f(a)+f(b)}{2^{\alpha}} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a^+}f(b) + J^{\alpha}_{b^-}f(a)\right]\right|$$
$$\leq \frac{b-a}{2^{\alpha+2}}\sum_{n=1}^{4}|I_n| \quad (8)$$

$$
\begin{aligned}
|I_1| &\le \int_0^1 |(1-t)^\alpha - \lambda| \left| f'\left(ta + (1-t)\frac{a+b}{2}\right)\right| dt \\
&= \int_0^1 |(1-t)^\alpha - \lambda| \left| f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\right| dt \\
&\le \int_0^1 |(1-t)^\alpha - \lambda| \left\{\frac{1+t}{2}\,|f'(a)| + \frac{1-t}{2}\,|f'(b)|\right\} dt \\
&= \frac{|f'(a)| + |f'(b)|}{2}\int_0^1 |(1-t)^\alpha - \lambda|\ dt + \frac{|f'(a)| - |f'(b)|}{2}\int_0^1 |(1-t)^\alpha - \lambda|\, t\ dt \\
&= \frac{A}{2}\{|f'(a)| + |f'(b)|\} + \frac{C}{2}\{|f'(a)| - |f'(b)|\}
\end{aligned}
$$

Analogously

$$
\begin{aligned}
|I_2| &\le \frac{A}{2}\{|f'(a)| + |f'(b)|\} - \frac{C}{2}\{|f'(a)| - |f'(b)|\} \\
|I_3| &\le B|f'(a)| - \frac{D}{2}\{|f'(a)| - |f'(b)|\} \\
|I_4| &\le B|f'(b)| + \frac{D}{2}\{|f'(a)| - |f'(b)|\}
\end{aligned}
$$

To get desired result, substituting the above inequalities into the inequality (8). □

**Corollary 1.** *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on $I^\circ$ and let $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex function on $[a, b]$, then the following inequality holds:*

$$
\begin{aligned}
&\left|(1-\lambda)\, f\left(\frac{a+b}{2}\right) + \lambda\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\qquad\qquad \le \frac{b-a}{8}\left[2\lambda^2 - 2\lambda + 1\right]\left(|f'(a)| + |f'(b)|\right) \quad (9)
\end{aligned}
$$

*Proof.* Setting $\alpha = 1$ in Theorem 5, we get the required result. □

**Remark 3.** *For setting $\lambda = 0$, inequality (9) reduces to inequality (2).*
*For setting $\lambda = 1$, inequality (9) reduces to inequality (3).*

**Theorem 7.** *Let $I \subset \mathbb{R}$ be an open interval, $a, b \in I$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ be a differentiable function such that $f'$ is integrable and $0 < \alpha \le 1$ on $(a, b)$ with $a < b$ and $0 \le \lambda \le 1$. If $|f'|$ is a convex on $[a, b]$, then the following inequality for Riemann−Liouville fractional integrals holds:*

$$
\begin{aligned}
&\left|\left(1 - \frac{2}{2^\alpha}\lambda\right) f\left(\frac{a+b}{2}\right) + \lambda\frac{f(a)+f(b)}{2^\alpha} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a^+}f(b) + J^\alpha_{b^-}f(a)\right]\right| \\
&\qquad\qquad \le \frac{b-a}{2^{\alpha+1}}\left[\mu\left|f'\left(\frac{a+b}{2}\right)\right| + \nu\left(\frac{|f'(a)| + |f'(b)|}{2}\right)\right]
\end{aligned}
$$

*where*

$$
\begin{aligned}
\mu &= \tfrac{2^{\alpha+2}+2(1-\zeta)^{\alpha+2}-2(2-\xi)^{\alpha+2}}{(\alpha+1)(\alpha+2)} + \tfrac{2-2(1-\zeta)^{\alpha+2}-2\xi(2-\xi)^{\alpha+1}}{\alpha+1} \\
&\quad +(2^\alpha - \lambda)(\tfrac{1}{2} - \xi^2) - \tfrac{\lambda}{2}(1-2\zeta^2) + (1-2\zeta)\lambda \\
\nu &= \tfrac{2(2-\xi)^{\alpha+2}-2(1-\zeta)^{\alpha+2}-2^{\alpha+2}}{(\alpha+1)(\alpha+2)} + \tfrac{2^{\alpha+1}-2\zeta(1-\zeta)^{\alpha+1}-2(1-\xi)(2-\xi)^{\alpha+1}}{\alpha+1} \\
&\quad -(2^\alpha - \lambda)(\tfrac{1}{2} - \xi^2) + \tfrac{\lambda}{2}(1-2\zeta^2) + (2^\alpha - \lambda)(1-2\xi).
\end{aligned}
$$

*Proof.* By using the properties of modulus on $I_1$ and convexity of $|f'|$, we have

$$\begin{aligned}|I_1| &\leq \int_0^1 |(1-t)^\alpha - \lambda| \left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right| dt \\ &\leq \int_0^1 |(1-t)^\alpha - \lambda| \left\{t\,|f'(a)| + (1-t)\left|f'\left(\frac{a+b}{2}\right)\right|\right\} dt \\ &\leq (A-C)\left|f'\left(\frac{a+b}{2}\right)\right| + C|f'(a)|\end{aligned}$$

Analogously

$$\begin{aligned}|I_2| &\leq (A-C)\left|f'\left(\frac{a+b}{2}\right)\right| + C|f'(b)| \\ |I_3| &\leq D\left|f'\left(\frac{a+b}{2}\right)\right| + (B-D)|f'(a)| \\ |I_4| &\leq D\left|f'\left(\frac{a+b}{2}\right)\right| + (B-D)|f'(b)|\end{aligned}$$

Substituting the above inequalities into the following inequality

$$\left|\left(1-\frac{2}{2^\alpha}\lambda\right) f\left(\frac{a+b}{2}\right) + \lambda\frac{f(a)+f(b)}{2^\alpha} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a^+}f(b) + J^\alpha_{b^-}f(a)\right]\right| \leq \frac{b-a}{2^{\alpha+2}}\sum_{n=1}^{4}|I_n|$$

which completes the proof. □

**Corollary 2.** *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on $I^\circ$ and let $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex function on $[a,b]$, then the following inequality holds:*

$$\left|(1-\lambda) f\left(\frac{a+b}{2}\right) + \lambda\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{12}\left[\left(2-3\lambda+2\lambda^3\right)\left|f'\left(\frac{a+b}{2}\right)\right| + (1-3\lambda+6\lambda^2-2\lambda^3)\left(\frac{|f'(a)|+|f'(b)|}{2}\right)\right] \quad (10)$$

*Proof.* Setting $\alpha = 1$ in Theorem 6, we get the required result. □

The corresponding version for powers of the absolute value of the derivative is incorporated in the following theorem.

**Theorem 8.** *Let $I \subset \mathbb{R}$ be an open interval, $a, b \in I$ with $a < b$ and $f : [a,b] \to \mathbb{R}$ be a differentiable function such that $f'$ is integrable and $0 < \alpha \leq 1$ on $(a,b)$ with $a < b$ and $0 \leq \lambda \leq 1$. If $|f'|^{\frac{1}{q}}$ for $q \geq 1$ is a convex on $[a,b]$, then the following inequality for Riemann–Liouville fractional integrals holds:*

$$\left|\left(1-\frac{2}{2^\alpha}\lambda\right) f\left(\frac{a+b}{2}\right) + \lambda\frac{f(a)+f(b)}{2^\alpha} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a^+}f(b) + J^\alpha_{b^-}f(a)\right]\right| \leq \frac{b-a}{2^{\alpha+2}}\times \left[A\left\{\left(\frac{A-C}{2A}|f'(a)|^q + \frac{A+C}{2A}|f'(b)|^q\right)^{\frac{1}{q}} + \left(\frac{A+C}{2A}|f'(a)|^q + \frac{A-C}{2A}|f'(b)|^q\right)^{\frac{1}{q}}\right\}\right. \quad (11)$$
$$\left. + B\left\{\left(\frac{2B-D}{2B}|f'(a)|^q + \frac{D}{2B}|f'(b)|^q\right)^{\frac{1}{q}} + \left(\frac{D}{2B}|f'(a)|^q + \frac{2B-D}{2B}|f'(b)|^q\right)^{\frac{1}{q}}\right\}\right].$$

*where*

$$\begin{aligned}
A &= \frac{1-2(1-\zeta)^{\alpha+1}}{\alpha+1}+(1-2\zeta)\lambda \\
B &= \frac{1+2^{\alpha+1}-2(2-\xi)^{\alpha+1}}{\alpha+1}+(2^{\alpha}-\lambda)(1-2\xi) \\
C &= \frac{1-2(1-\zeta)^{\alpha+2}}{(\alpha+1)(\alpha+2)}-\frac{2\zeta(1-\zeta)^{\alpha+1}}{\alpha+1}+\frac{\lambda}{2}(1-2\zeta^2) \\
D &= \frac{1+2^{\alpha+2}-2(2-\xi)^{\alpha+2}}{(\alpha+1)(\alpha+2)}+\frac{1-2\xi(2-\xi)^{\alpha+1}}{\alpha+1}+(2^{\alpha}-\lambda)(\frac{1}{2}-\xi^2)
\end{aligned}$$

*Proof.* Using the well-known power-mean integral inequality , we have

$$\begin{aligned}
&\left|\left(1-\frac{2}{2^{\alpha}}\lambda\right)f\left(\frac{a+b}{2}\right)+\lambda\frac{f(a)+f(b)}{2^{\alpha}}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a^+}f(b)+J^{\alpha}_{b^-}f(a)\right]\right| \\
&\leq\frac{b-a}{2^{\alpha+2}}\Bigg[\left(\int_0^1\left|(1-t)^{\alpha}-\lambda\right|dt\right)^{1-\frac{1}{q}}\left\{\left(\int_0^1\left|(1-t)^{\alpha}-\lambda\right|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\qquad\left.+\left(\int_0^1\left|(1-t)^{\alpha}-\lambda\right|\left|f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right\}+ \\
&\left(\int_0^1\left|2^{\alpha}-(2-t)^{\alpha}-\lambda\right|\ dt\right)^{1-\frac{1}{q}}\left\{\left(\int_0^1\left|2^{\alpha}-(2-t)^{\alpha}-\lambda\right|\left|f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\qquad\left.+\left(\int_0^1\left|2^{\alpha}-(2-t)^{\alpha}-\lambda\right|\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right\}\Bigg].
\end{aligned}$$

Which completes the proof. □

**Remark 4.** *For $\alpha=1$, inequality (11) reduces to inequality (5).*

**Theorem 9.** *Let $I\subset\mathbb{R}$ be an open interval, $a,b\in I$ with $a<b$ and $f:[a,b]\to\mathbb{R}$ be a differentiable function such that $f'$ is integrable and $0<\alpha\leq 1$ on $(a,b)$ with $a<b$ and $0\leq\lambda\leq 1$. If $|f'|^{\frac{1}{q}}$ for $q\geq 1$ is a convex on $[a,b]$, then the following inequality for Riemann−Liouville fractional integrals holds:*

$$\begin{aligned}
&\left|\left(1-\frac{2}{2^{\alpha}}\lambda\right)f\left(\frac{a+b}{2}\right)+\lambda\frac{f(a)+f(b)}{2^{\alpha}}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a^+}f(b)+J^{\alpha}_{b^-}f(a)\right]\right|\leq\frac{b-a}{2^{\alpha+1}}\times \\
&\left[A\left\{\left(\frac{C}{A}|f'(a)|^q+\left(1-\frac{C}{A}\right)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(\frac{C}{A}|f'(b)|^q+\left(1-\frac{C}{A}\right)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right\}\right. \\
&\left.+B\left\{\left(\left(1-\frac{D}{B}\right)|f'(a)|^q+\frac{D}{B}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(\left(1-\frac{D}{B}\right)|f'(b)|^q+\frac{D}{B}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right\}\right].
\end{aligned}$$

*where*

$$\begin{aligned} A &= \frac{1-2(1-\zeta)^{\alpha+1}}{\alpha+1}+(1-2\zeta)\lambda \\ B &= \frac{1+2^{\alpha+1}-2(2-\xi)^{\alpha+1}}{\alpha+1}+(2^{\alpha}-\lambda)(1-2\xi) \\ C &= \frac{1-2(1-\zeta)^{\alpha+2}}{(\alpha+1)(\alpha+2)}-\frac{2\zeta(1-\zeta)^{\alpha+1}}{\alpha+1}+\frac{\lambda}{2}(1-2\zeta^2) \\ D &= \frac{1+2^{\alpha+2}-2(2-\xi)^{\alpha+2}}{(\alpha+1)(\alpha+2)}+\frac{1-2\xi(2-\xi)^{\alpha+1}}{\alpha+1}+(2^{\alpha}-\lambda)(\frac{1}{2}-\xi^2) \end{aligned}$$

**Corollary 3.** *Let $f: I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on $I^{\circ}$ and let $a,b \in I^{\circ}$ with $a<b$. If $0 \le \lambda \le 1$ and $|f'|^{\frac{1}{q}}$ for $q \ge 1$ is a convex on $[a,b]$, then the following inequality holds:*

$$\begin{aligned} &\left|(1-\lambda)f\left(\frac{a+b}{2}\right)+\lambda\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\quad \le \frac{b-a}{3}\left(\frac{1}{3}\right)^{\frac{1}{q}}(1-2\lambda+2\lambda^2)^{1-\frac{1}{q}} \\ &\times\left[\left(\left\{1-3\lambda+6\lambda^2-2\lambda^3\right\}|f'(a)|^q+\left\{2-3\lambda+2\lambda^3\right\}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right. \\ &\left.+\left(\left\{1-3\lambda+6\lambda^2-2\lambda^3\right\}|f'(b)|^q+\left\{2-3\lambda+2\lambda^3\right\}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right] \quad (12) \end{aligned}$$

*Proof.* Setting $\alpha=1$ in Theorem 9, we get the required result. □

## 3. Applications To Quadrature Formulae

In this section, some particular inequalities which generalize some classical results such as: trapezoid inequality, Simpsons inequality, midpoint inequality and others, are pointed out.

**Proposition 1.** *(Midpoint Inequality). Under the assumptions Corollary 2 with $\lambda=0$ in inequality (10), then the following inequality holds,*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{24}\left(|f'(a)|+4\left|f'\left(\frac{a+b}{2}\right)\right|+|f'(b)|\right)$$

**Proposition 2.** *(Midpoint Inequality). Under the assumptions Corollary 3 with $\lambda=0$ in inequality (12), then the following inequality holds,*

$$\begin{aligned} \left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{8}&\left[\left(\frac{1}{3}|f'(a)|^q+\frac{2}{3}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right. \\ &\left.+\left(\frac{1}{3}|f'(b)|^q+\frac{2}{3}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right] \end{aligned}$$

**Proposition 3.** *(Trapezoid Inequality). Under the assumptions Corollary* 2 *with* $\lambda = 1$ *in inequality* (10)*, then the following inequality holds,*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{12}\left(|f'(a)| + \left|f'\left(\frac{a+b}{2}\right)\right| + |f'(b)|\right)$$

**Proposition 4.** *(Trapezoid Inequality). Under the assumptions Corollary* 3 *with* $\lambda = 1$ *in inequality* (12)*, then the following inequality holds,*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{8}\left[\left(\frac{2}{3}|f'(a)|^q + \frac{1}{3}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}} + \left(\frac{2}{3}|f'(b)|^q + \frac{1}{3}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right]$$

**Proposition 5.** *Under the assumptions Corollary* 2 *with* $\lambda = \frac{1}{2}$ *in inequality* (10)*, then the following inequality holds,*

$$\left|\frac{1}{2}\left\{f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right\} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{32}\left(|f'(a)| + 2\left|f'\left(\frac{a+b}{2}\right)\right| + |f'(b)|\right)$$

**Proposition 6.** *Under the assumptions Corollary* 3 *with* $\lambda = \frac{1}{2}$ *in inequality* (12)*, then the following inequality holds,*

$$\left|\frac{1}{2}\left\{f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right\} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{16}\times \left[\left(\frac{1}{2}|f'(a)|^q + \frac{1}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}} + \left(\frac{1}{2}|f'(b)|^q + \frac{1}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right]$$

**Proposition 7.** *(Simpson Inequality). Under the assumptions Corollary* 2 *with* $\lambda = \frac{1}{3}$ *in inequality* (10)*, then the following inequality holds,*

$$\left|\frac{1}{3}\left\{2f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right\} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{324}\left(8|f'(a)| + 29\left|f'\left(\frac{a+b}{2}\right)\right| + 8|f'(b)|\right)$$

**Proposition 8.** *(Simpson Inequality). Under the assumptions Corollary* 3 *with* $\lambda = \frac{1}{3}$ *in inequality* (12)*, then the following inequality holds,*

$$\left|\frac{1}{3}\left\{2f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right\} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{5(b-a)}{72}\times \left[\left(\frac{16}{45}|f'(a)|^q + \frac{29}{45}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}} + \left(\frac{16}{45}|f'(b)|^q + \frac{29}{45}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right]$$

## References


[1] G. Anastassiou, M.R. Hooshmandasl, A. Ghasemi, F. Moftakharzadeh, Montogomery identities for fractional integrals and related fractional inequalities, J. Ineq. Pure Appl. Math. 10 (4) Art. 97,2009.

[2] S. Belarbi, Z. Dahmani, On some new fractional integral inequalities, J. Ineq. Pure Appl. Math. 10 (3) Art. 86, 2009.

[3] Bo-Yan Xi and Feng Qi, Some Integral Inequalities of Hermite-Hadamard Type for Convex Functions with Applications to Means, Journal of Function Spaces and Applications, Vol. 2012, Art. ID 980438, doi:10.1155/2012/980438.

[4] Z. Dahmani, New inequalities in fractional integrals, International Journal of Nonlinear Scinece, 9(4), 493-497, 2010.

[5] S. S. Dragomir, M. I. Bhatti, M. Iqbal, and M. Muddassar, Some new fractional Integral Hermite–Hadamard type inequalities, Journal of Computational Analysis and Application, 18(4), 655−661, 2015.

[6] S. S. Dragomir and R. P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, Applied Mathematics Letters, 11 ( 5), 9195, 1998.

[7] S.S. Dragomir, R.P. Agarwal, P. Cerone, On Simpsons inequality and applications, J. Inequal. Appl. 5, 533−579, 2000.

[8] R. Gorenflo, F. Mainardi, Fractional calculus: integral and differential equations of fractional order, Springer Verlag, Wien (1997), 223−276.

[9] M. Iqbal, M. I. Bhatti, and K. Nazeer, Generaliation of inequalities analogous to Hermite−Hadamard inequality via fractional integrals, Bull. Korean Math. Soc. 52( 3), 707−716, 2015.

[10] U. S. Kirmaci, Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, Applied Mathematics and Computation, 147 (1), 137146, 2004.

[11] S. Miller, B. Ross, An introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley & Sons, USA, 1993, 2.

[12] J.E. Pečarić, F. Proschan, Y.L. Tong, Convex Functions, Partial Orderings and Statistical Applications, Academic Press, Boston, 1992.

[13] J. Pečarić and S. Varošanec, A note on Simpson.s inequality for functions of bounded variation, Tamkang J. Math., 31(3), 239−242, 2000.

[14] I. Podlubni, Fractional Differential Equations, Academic Press, San Diego, 1999.

[15] M.Z. Sarikaya, N. Aktan, On the generalization of some integral inequalities and their applications, Mathematical and computer Modelling, 54 (9), 2175 −2182, 2011.

[16] M.Z. Sarikaya, E. Set, M.E. Özdemir, On new inequalities of Simpsons type for convex functions, RGMIA Res. Rep. Coll. 13 (2) Article2, 2010.

[17] Y. Shi and Z. Liu, Some sharp Simpson type inequalities and applications, Appl. Math. E-Notes, 9, 205−215, 2009.



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# The differentiability for fuzzy $n$-cell mappings and the KKT optimality conditions for a class of fuzzy constrained minimization problem *

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**Abstract** In this paper, the concept of generalized difference for fuzzy $n$-cell numbers is presented, and we use the generalized difference to introduce and study the differentiability for fuzzy $n$-cell mappings. Next, the convexity for fuzzy $n$-cell mappings is studied, which is based on the concept of the partial ordering of fuzzy $n$-cell numbers proposed in this study. Finally, using the differentiability and the convexity for fuzzy $n$-cell mappings, we obtain the KKT optimality conditions for a class of fuzzy constrained minimization problem.

***Keywords:*** Fuzzy $n$-cell numbers; fuzzy $n$-cell mappings; differentiability; fuzzy optimization.

## 1. Introduction

Since the concept and operations of fuzzy set were introduced by Zadeh [1], enormous researchers have been dedicated on development of various aspects of the theory and applications of fuzzy sets. The occurrence of randomness and imprecision in the real world is inevitable owing to some unexpected situations. Therefore, imposing the uncertainty upon the conventional optimization problems is an interesting research topic.

The theory and methods of mathematical programming are important components of optimization. The importance of the derivative of a function in the study of mathematical programming is well-known. Given that our interest is in fuzzy objective mappings, it is necessary to introduce a concept of derivative for fuzzy mappings. Toward this end, in the fuzzy analysis, there are a variety of notions of derivative for fuzzy mappings. The concept of fuzzy derivative first introduced by Chang and Zadeh [2] in 1972. Since then, numerous definitions of the differentiability for fuzzy mappings have been presented. In 1983, Puri and Ralescu [3] defined the derivative and $G$-derivative for fuzzy mappings from an open subset of a normal space into $n$-dimension fuzzy number space $E^n$ by using embedding theorem (which shows how to isometrically embed $E^n$ into a Banach space as a closed convex cone of vertex zero) and Hukuhara difference. In 1987, Kaleva [4] discussed the $G$-derivative, obtained a sufficient condition of the $H$-differentiability for fuzzy mappings from $[a, b]$ into $E^n$ and a necessary condition for the $H$-differentiability of fuzzy mapping from $[a, b]$ into $E^1$. In 2003, Wang and Wu [5] put forward the concepts of directional derivative, differential and sub-differential for fuzzy mappings from $R^n$ into $E^1$ by using Hukuhara difference. However, the usual Hukuhara difference between two fuzzy numbers exists only under very restrictive conditions [4] and the $H$-difference of two fuzzy numbers does not always exist [6, 7]. The $g$-difference between two fuzzy numbers proposed in [7] overcomes these shortcomings of the above discussed concepts and the $g$-difference of two fuzzy numbers always exists. Based on the $g$-difference for two fuzzy numbers, Bede [8] introduced and studied new generalized differentiability concepts for fuzzy valued functions in 2013, in particular, a new very general fuzzy differentiability concept was defined and studied, the so-called $g$-derivative, and it was

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shown that the $g$-derivative is the most general among all similar definitions.

Motivated both by [8] and the importance of the concept of differentiability for fuzzy optimization, the paper focuses on the concept of differentiability for fuzzy mappings, which is based on the generalized difference of fuzzy $n$-cell numbers presented in this paper. The Karush-Kuhn-Tucker optimality conditions play an important role in the area of optimization theory and have been studied for over a century. We extend the concept of convexity for real-valued functions to fuzzy $n$-cell mappings based on the partial ordering $\preceq_c$ introduced in this paper, and then establish the KKT optimality conditions for an optimization problem with a fuzzy $n$-cell objective mapping.

The remainder of the paper is organised as follows: First of all, we give the preliminary terminology used in the present paper. And then, the generalized difference of fuzzy $n$-cell numbers is introduced. We use the generalized difference of fuzzy $n$-cell numbers to study differentiability for fuzzy $n$-cell mappings, and convexity for fuzzy $n$-cell mappings based on the partial ordering $\preceq_c$ is discussed in Section 4. At last, using the convexity and differentiability for fuzzy $n$-cell mappings, section 5 deals with the Karush-Kuhn-Tucker optimality conditions for a class of constrained fuzzy minimization problem.

## 2. Preliminaries

Throughout this paper, $F(R^n)$ denotes the set of all fuzzy subsets on $n$-dimensional Euclidean space $R^n$. A fuzzy set $\widetilde{u}$ on $R^n$ is a mapping $\widetilde{u}: R^n \to [0,1]$. For each fuzzy set $\widetilde{u}$, we denote its $r$-level set as $[\widetilde{u}]^r = \{x \in R^n : \widetilde{u}(x) \geq r\}$ for any $r \in (0,1]$. The support of $\widetilde{u}$ we denote by supp$\widetilde{u}$ where supp$\widetilde{u} = \{x \in R^n : \widetilde{u}(x) > 0\}$. The closure of supp$\widetilde{u}$ defines the 0-level of $\widetilde{u}$, i.e. $[\widetilde{u}]^0 = cl(\text{supp}\widetilde{u})$. Here $cl(M)$ denotes the closure of set $M$. Fuzzy set $\widetilde{u} \in F(R^n)$ is called a fuzzy number if
(1) $\widetilde{u}$ is a normal fuzzy set, i.e. there exists an $x_0 \in R^n$ such that $\widetilde{u}(x_0) = 1$,
(2) $\widetilde{u}$ is a convex fuzzy set, i.e. $\widetilde{u}(\lambda x + (1-\lambda)y) \geq \min\{\widetilde{u}(x), \widetilde{u}(y)\}$ for any $x, y \in R^n$ and $\lambda \in [0,1]$,
(3) $\widetilde{u}$ is upper semi-continuous,
(4) $[\widetilde{u}]^0 = cl(\text{supp}\widetilde{u}) = cl(\bigcup_{r\in(0,1]}[\widetilde{u}]^r)$ is compact.
We will denote $E^n$ the set of fuzzy numbers [9, 10, 11, 12].

It is clear that any $u \in R^n$ can be regarded as a fuzzy number $\widetilde{u}$ defined by

$$\widetilde{u}(x) = \begin{cases} 1, & x = u, \\ 0, & \text{otherwise.} \end{cases}$$

In particular, the fuzzy number $\widetilde{0}$ is defined as $\widetilde{0}(x) = 1$ if $x = 0$, and $\widetilde{0}(x) = 0$ otherwise.

**Definition 2.1.** [13] If $\widetilde{u} \in E^n$, and $[\widetilde{u}]^r$ is a cell, i.e., for any $r \in [0,1]$,

$$[\widetilde{u}]^r = \prod_{i=1}^{n}[u_i^-(r), u_i^+(r)] = [u_1^-(r), u_1^+(r)] \times [u_2^-(r), u_2^+(r)] \times \cdots \times [u_n^-(r), u_n^+(r)],$$

where $u_i^-(r), u_i^+(r) \in R$ with $u_i^-(r) \leq u_i^+(r)$ $(i = 1, 2, \cdots, n)$, then we call $\widetilde{u}$ a fuzzy $n$-cell number. Denote the collection of all fuzzy $n$-cell numbers by $L(E^n)$.

For any $r \in [0,1]$, $l_i[\widetilde{u}]^r = u_i^+(r) - u_i^-(r)$ $(i = 1, 2, \cdots, n)$ is called the $r$-level length of a fuzzy $n$-cell number $\widetilde{u}$ with respect to the ith component.

**Theorem 2.1.** [13] (Representation theorem). If $\widetilde{u} \in L(E^n)$, then for $i = 1, 2, \cdots, n$, $u_i^-(r), u_i^+(r)$ are real-valued functions on $[0,1]$, and satisfy
(1) $u_i^-(r)$ are non-decreasing, left continuous at $r \in (0,1]$ and right continuous at $r = 0$,
(2) $u_i^+(r)$ are non-increasing, left continuous at $r \in (0,1]$ and right continuous at $r = 0$,
(3) $u_i^-(r) \leq u_i^+(r)$ (it is equivalent to $u_i^-(1) \leq u_i^+(1)$).

Conversely if $a_i(r), b_i(r)$ $(i = 1, 2, \cdots, n)$ are real-valued functions on $[0,1]$ which satisfy conditions (1)-(3), then there exists a unique $\widetilde{u} \in L(E^n)$ such that $[\widetilde{u}]^r = \prod_{i=1}^{n}[a_i(r), b_i(r)]$ for any $r \in [0,1]$.

**Theorem 2.2.** [13] Let $\widetilde{u}, \widetilde{v} \in L(E^n)$ and $k \in R$. Then for any $r \in [0,1]$,
(1) $[\widetilde{u} + \widetilde{v}]^r = [\widetilde{u}]^r + [\widetilde{v}]^r = \prod_{i=1}^{n}[u_i^-(r) + v_i^-(r), u_i^+(r) + v_i^+(r)]$,

(2) $[k\widetilde{u}]^r = k[\widetilde{u}]^r = \begin{cases} \prod_{i=1}^n [ku_i^-(r), ku_i^+(r)], & k \geq 0, \\ \prod_{i=1}^n [ku_i^+(r), ku_i^-(r)], & k < 0, \end{cases}$

(3) $[\widetilde{u}\widetilde{v}]^r = \prod_{i=1}^n [\min\{u_i^-(r)v_i^-(r), u_i^-(r)v_i^+(r), u_i^+(r)v_i^-(r), u_i^+(r)v_i^+(r)\},$

$$\max\{u_i^-(r)v_i^-(r), u_i^-(r)v_i^+(r), u_i^+(r)v_i^-(r), u_i^+(r)v_i^+(r)\}].$$

Given $\widetilde{u}, \widetilde{v} \in L(E^n)$, the distance $D : L(E^n) \times L(E^n) \to [0, +\infty)$ between $\widetilde{u}$ and $\widetilde{v}$ is defined by the equation

$$D(\widetilde{u}, \widetilde{v}) = \sup_{r\in[0,1]} d([\widetilde{u}]^r, [\widetilde{v}]^r) = \sup_{r\in[0,1]} \max_{1\leq i\leq n} \{| u_i^-(r) - v_i^-(r) |, | u_i^+(r) - v_i^+(r) |\}.$$

Then $(L(E^n), D)$ is a complete metric space, and satisfies $D(\widetilde{u}+\widetilde{w}, \widetilde{v}+\widetilde{w}) = D(\widetilde{u}, \widetilde{v})$, $D(k\widetilde{u}, k\widetilde{v}) = |k|D(\widetilde{u}, \widetilde{v})$ for any $\widetilde{u}, \widetilde{v}, \widetilde{w} \in L(E^n)$, $k \in R$.

In recent years, several authors have discussed different ordering relation of fuzzy numbers [14]. To the best of our knowledge, very few investigations have been appeared to study ordering relation of fuzzy $n$-cell numbers. For this reason, an ordering $\preceq_c$ of fuzzy $n$-cell numbers will be introduced.

**Definition 2.2.** Let $\tau : L(E^n) \to R^n$ be a vector-valued function defined by

$$\begin{aligned} \tau(\widetilde{u}) &= (2\int_0^1 r\frac{\int\cdots\int_{[\widetilde{u}]^r} x_1dx_1dx_2\cdots dx_n}{\int\cdots\int_{[\widetilde{u}]^r} 1dx_1dx_2\cdots dx_n}dr, 2\int_0^1 r\frac{\int\cdots\int_{[\widetilde{u}]^r} x_2dx_1dx_2\cdots dx_n}{\int\cdots\int_{[\widetilde{u}]^r} 1dx_1dx_2\cdots dx_n}dr, \cdots, 2\int_0^1 r\frac{\int\cdots\int_{[\widetilde{u}]^r} x_ndx_1dx_2\cdots dx_n}{\int\cdots\int_{[\widetilde{u}]^r} 1dx_1dx_2\cdots dx_n}dr) \\ &= (\int_0^1 r(\widetilde{u}_1^+(r) + u_1^-(r))dr, \int_0^1 r(\widetilde{u}_2^+(r) + u_2^-(r))dr, \cdots, \int_0^1 r(\widetilde{u}_n^+(r) + u_n^-(r))dr), \end{aligned}$$

where $\int_0^1 r\frac{\int\cdots\int_{[\widetilde{u}]^r} x_idx_1dx_2\cdots dx_n}{\int\cdots\int_{[\widetilde{u}]^r} 1dx_1dx_2\cdots dx_n}dr$ $(i = 1, 2, \cdots, n)$ are the Lebesgue integral of $r\frac{\int\cdots\int_{[\widetilde{u}]^r} x_idx_1dx_2\cdots dx_n}{\int\cdots\int_{[\widetilde{u}]^r} 1dx_1dx_2\cdots dx_n}$ $(i = 1, 2, \cdots, n)$ on $[0, 1]$. The vector-valued function $\tau$ is called a ranking value function defined on $L(E^n)$.

In this case $\tau(\widetilde{u})$ represents a centroid of the fuzzy $n$-cell number $\widetilde{u}$. From the ranking value function $\tau(\widetilde{u})$, we consider the following ordering relation $\preceq_c$ on $L(E^n)$.

**Definition 2.3.** Let $\widetilde{u}, \widetilde{v} \in L(E^n)$, $C \subseteq R^n$ be a closed convex cone with $0 \in C$ and $C \neq R^n$. We say that $\widetilde{u} \preceq_c \widetilde{v}$ ($\widetilde{u}$ precedes $\widetilde{v}$) if $\tau(\widetilde{v}) \in \tau(\widetilde{u}) + C$ $(\tau(\widetilde{v}) - \tau(\widetilde{u}) \in C)$.

Obviously the order relation $\preceq_c$ is reflexive and transitive, and $\preceq_c$ is a partially ordered relation on $L(E^n)$. For $\widetilde{u}, \widetilde{v} \in L(E^n)$, if either $\widetilde{u} \preceq_c \widetilde{v}$ or $\widetilde{v} \preceq_c \widetilde{u}$, then we say that $\widetilde{u}$ and $\widetilde{v}$ are comparable, otherwise non-comparable. If $\widetilde{u}, \widetilde{v} \in E^1$, $C = [0, +\infty) \subseteq R$, then Definition 2.3 coincides with Definition 2.5 from [14].

We say that $\widetilde{u} \prec_c \widetilde{v}$ if $\widetilde{u} \preceq_c \widetilde{v}$ and $\tau(\widetilde{u}) \neq \tau(\widetilde{v})$. Sometimes we may write $\widetilde{v} \succeq_c \widetilde{u}$ (resp. $\widetilde{v} \succ_c \widetilde{u}$) instead of $\widetilde{u} \preceq_c \widetilde{v}$ (resp. $\widetilde{u} \prec_c \widetilde{v}$).

**Remark 2.1.** Let $\widetilde{u}, \widetilde{v} \in L(E^n)$, $k_1, k_2 \in R$. According to Theorem 2.2 and Definition 2.2, it is easy to verify that $\tau(k_1\widetilde{u} + k_2\widetilde{v}) = k_1\tau(\widetilde{u}) + k_2\tau(\widetilde{v})$.

**Theorem 2.3.** Let $\widetilde{u}_1, \widetilde{u}_2, \widetilde{v}_1, \widetilde{v}_2, \in L(E^n)$, $k_1, k_2 \in [0, +\infty]$, $C \subseteq R^n$ be a closed convex cone with $0 \in C$ and $C \neq R^n$. If $\widetilde{u}_1 \preceq_c \widetilde{v}_1$ and $\widetilde{u}_2 \preceq_c \widetilde{v}_2$, then $k_1\widetilde{u}_1 + k_2\widetilde{u}_2 \preceq_c k_1\widetilde{v}_1 + k_2\widetilde{v}_2$.

The proof is similar to the proof of Theorem 2.3 in [15]

## 3. Generalized difference for fuzzy $n$-cell numbers

**Definition 3.1.** [16] Let $\widetilde{u}, \widetilde{v} \in L(E^n)$. The generalized difference ($g$-difference for short) of $\widetilde{u}$ and $\widetilde{v}$ is given by its level sets as

$$[\widetilde{u} \ominus_g \widetilde{v}]^r = \prod_{i=1}^n [\inf_{\beta\geq r} \min\{u_i^-(\beta) - v_i^-(\beta), u_i^+(\beta) - v_i^+(\beta)\}, \sup_{\beta\geq r} \max\{u_i^-(\beta) - v_i^-(\beta), u_i^+(\beta) - v_i^+(\beta)\}],$$

where $\beta \in [r, 1]$.

**Remark 3.1.** If $\widetilde{u}, \widetilde{v} \in E^1$, we have

$$[\widetilde{u} \ominus_g \widetilde{v}]^r = [\inf_{\beta\geq r} \min\{u^-(\beta) - v^-(\beta), u^+(\beta) - v^+(\beta)\}, \sup_{\beta\geq r} \max\{u^-(\beta) - v^-(\beta), u^+(\beta) - v^+(\beta)\}],$$

which coincides with Definition 7 of reference [8].

According to Proposition 13 in [7], we have the following conclusion.

**Theorem 3.1.** Let $\widetilde{u}, \widetilde{v} \in L(E^n)$. If $l_i[\widetilde{u}]^r \leq l_i[\widetilde{v}]^r$ or $l_i[\widetilde{u}]^r \geq l_i[\widetilde{v}]^r$ for any $r \in [0,1]$ and $i = 1, 2, \cdots, n$, then the $g$-difference $\widetilde{u} \ominus_g \widetilde{v}$ exists and $\widetilde{u} \ominus_g \widetilde{v} \in L(E^n)$.

From now on, throughout this paper, we will assume that the $g$-difference $\widetilde{u} \ominus_g \widetilde{v}$ for any fuzzy $n$-cell numbers $\widetilde{u}$ and $\widetilde{v}$ exists.

**Theorem 3.2.** For any $\widetilde{u}, \widetilde{v}, \widetilde{w}, \in L(E^n)$, we have

(1) $\widetilde{u} \ominus_g \widetilde{u} = \widetilde{0}$, $\widetilde{u} \ominus_g \widetilde{0} = \widetilde{u}$, $\widetilde{0} \ominus_g \widetilde{u} = -\widetilde{u}$,

(2) $\widetilde{u} \ominus_g \widetilde{v} = -(\widetilde{v} \ominus_g \widetilde{u})$,

(3) $k(\widetilde{u} \ominus_g \widetilde{v}) = k\widetilde{u} \ominus_g k\widetilde{v}$, for any $k \in R$,

(4) $k_1\widetilde{u} \ominus_g k_2\widetilde{u} = (k_1 - k_2)\widetilde{u}$, for any $k_1, k_2 \in R$ and $k_1 \cdot k_2 \geq 0$,

(5) $\widetilde{u} \ominus_g (-\widetilde{v}) = \widetilde{v} \ominus_g (-\widetilde{u})$, $(-\widetilde{u}) \ominus_g \widetilde{v} = (-\widetilde{v}) \ominus_g \widetilde{u}$,

(6) $(\widetilde{u} + \widetilde{v}) \ominus_g \widetilde{v} = \widetilde{u}$,

(7) $\widetilde{0} \ominus_g (\widetilde{u} \ominus_g \widetilde{v}) = \widetilde{v} \ominus_g \widetilde{u} = (-\widetilde{u}) \ominus_g (-\widetilde{v})$,

(8) $\widetilde{u} \ominus_g \widetilde{v} = \widetilde{v} \ominus_g \widetilde{u} = \widetilde{w}$ if and only if $\widetilde{w} = -\widetilde{w}$.

**Proof.** The proof of (1), (3) are immediate.

(2) According to Definition 3.1, for any $r \in [0,1]$, we have

$$\begin{aligned}
& -[\widetilde{v} \ominus_g \widetilde{u}]^r \\
= & -\prod_{i=1}^n [\inf_{\beta \geq r} \min\{v_i^-(\beta) - u_i^-(\beta), v_i^+(\beta) - u_i^+(\beta)\}, \ \sup_{\beta \geq r} \max\{v_i^-(\beta) - u_i^-(\beta), v_i^+(\beta) - u_i^+(\beta)\}] \\
= & \prod_{i=1}^n [-\sup_{\beta \geq r} \max\{v_i^-(\beta) - u_i^-(\beta), v_i^+(\beta) - u_i^+(\beta)\}, \\
& \quad -\inf_{\beta \geq r} \min\{v_i^-(\beta) - u_i^-(\beta), v_i^+(\beta) - u_i^+(\beta)\}] \\
= & \prod_{i=1}^n [-\sup_{\beta \geq r}(-\min\{u_i^-(\beta) - v_i^-(\beta), u_i^+(\beta) - v_i^+(\beta)\}), \\
& \quad -\inf_{\beta \geq r}(-\max\{u_i^-(\beta) - v_i^-(\beta), u_i^+(\beta) - v_i^+(\beta)\})] \\
= & \prod_{i=1}^n [\inf_{\beta \geq r} \min\{u_i^-(\beta) - v_i^-(\beta), u_i^+(\beta) - v_i^+(\beta)\}, \ \sup_{\beta \geq r} \max\{u_i^-(\beta) - v_i^-(\beta), u_i^+(\beta) - v_i^+(\beta)\}] \\
= & [\widetilde{u} \ominus_g \widetilde{v}]^r.
\end{aligned}$$

It follows from Theorem 2.2 that $\widetilde{u} \ominus_g \widetilde{v} = -(\widetilde{v} \ominus_g \widetilde{u})$.

(4) For any $r \in [0,1]$, it follows from Definition 3.1 that

$$\begin{aligned}
& [k_1\widetilde{u} \ominus_g k_2\widetilde{u}]^r \\
= & \prod_{i=1}^n [\inf_{\beta \geq r} \min\{(k_1 - k_2)u_i^-(\beta), (k_1 - k_2)u_i^+(\beta)\}, \ \sup_{\beta \geq r} \max\{(k_1 - k_2)u_i^-(\beta), (k_1 - k_2)u_i^+(\beta)\}].
\end{aligned}$$

If $k_1 - k_2 \geq 0$, for any $r \in [0,1]$, it is obvious that

$$[k_1\widetilde{u} \ominus_g k_2\widetilde{u}]^r = \prod_{i=1}^n [(k_1 - k_2)u_i^-(r), (k_1 - k_2)u_i^+(r)] = [(k_1 - k_2)\widetilde{u}]^r.$$

On the other hand, if $k_1 - k_2 < 0$, for any $r \in [0,1]$, we have from Theorem 2.2 that

$$\begin{aligned}
& [k_1\widetilde{u} \ominus_g k_2\widetilde{u}]^r \\
= & \prod_{i=1}^n [\inf_{\beta \geq r} \min\{(k_1 - k_2)u_i^-(\beta), (k_1 - k_2)u_i^+(\beta)\}, \ \sup_{\beta \geq r} \max\{(k_1 - k_2)u_i^-(\beta), (k_1 - k_2)u_i^+(\beta)\}] \\
= & \prod_{i=1}^n [(k_1 - k_2) \sup_{\beta \geq r} \max\{u_i^-(\beta), u_i^+(\beta)\}, \ (k_1 - k_2) \inf_{\beta \geq r} \min\{u_i^-(\beta), u_i^+(\beta)\}] \\
= & \prod_{i=1}^n [(k_1 - k_2)u_i^+(r), (k_1 - k_2)u_i^-(r)] \\
= & [(k_1 - k_2)\widetilde{u}]^r.
\end{aligned}$$

Then $k_1\widetilde{u} \ominus_g k_2\widetilde{u} = (k_1 - k_2)\widetilde{u}$.

(5) According to Definition 3.1 and Theorem 2.2, for any $r \in [0,1]$, we have

$$\begin{aligned}
&[\widetilde{u} \ominus_g (-\widetilde{v})]^r \\
= \ & \prod_{i=1}^n [\inf_{\beta \geq r} \min\{u_i^-(\beta) + v_i^+(\beta), u_i^+(\beta) + v_i^-(\beta)\}, \ \sup_{\beta \geq r} \max\{u_i^-(\beta) + v_i^+(\beta), u_i^+(\beta) + v_i^-(\beta)\}] \\
= \ & \prod_{i=1}^n [\inf_{\beta \geq r} \min\{v_i^-(\beta) + u_i^+(\beta), v_i^+(\beta) + u_i^-(\beta)\}, \ \sup_{\beta \geq r} \max\{v_i^-(\beta) + u_i^+(\beta), v_i^+(\beta) + u_i^-(\beta)\}] \\
= \ & [\widetilde{v} \ominus_g (-\widetilde{u})]^r.
\end{aligned}$$

Then $\widetilde{u} \ominus_g (-\widetilde{v}) = \widetilde{v} \ominus_g (-\widetilde{u})$. It follows from (3) that $(-\widetilde{u}) \ominus_g \widetilde{v} = (-\widetilde{v}) \ominus_g \widetilde{u}$.

(6) For any $r \in [0,1]$, we have from Theorem 2.2 that

$$\begin{aligned}
[(\widetilde{u} + \widetilde{v}) \ominus_g \widetilde{v}]^r &= \prod_{i=1}^n [\inf_{\beta \geq r} \min\{(u_i^-(\beta) + v_i^-(\beta)) - v_i^-(\beta), (u_i^+(\beta) + v_i^+(\beta)) - v_i^+(\beta)\}, \\
&\qquad \sup_{\beta \geq r} \max\{(u_i^-(\beta) + v_i^-(\beta)) - v_i^-(\beta), (u_i^+(\beta) + v_i^+(\beta)) - v_i^+(\beta)\}] \\
&= \prod_{i=1}^n [\inf_{\beta \geq r} \min\{u_i^-(\beta), u_i^+(\beta)\}, \ \sup_{\beta \geq r} \max\{u_i^-(\beta), u_i^+(\beta)\}] \\
&= \prod_{i=1}^n [\inf_{\beta \geq r} u_i^-(\beta), \ \sup_{\beta \geq r} u_i^+(\beta)] \\
&= \prod_{i=1}^n [u_i^-(r), \ u_i^+(r)] \\
&= [\widetilde{u}]^r.
\end{aligned}$$

Then $(\widetilde{u} + \widetilde{v}) \ominus_g \widetilde{v} = \widetilde{u}$.

(7) It follows from (1), (2) and (3) that the proof of (7) is immediate.

(8) We have from (2) that the proof of (8) is immediate.

For any $\widetilde{u}, \widetilde{v} \in L(E^n)$, using the method with that Bede proved Proposition 15 in [8], we can show that $D(\widetilde{u}, \widetilde{v}) = D(\widetilde{u} \ominus_g \widetilde{v}, \widetilde{0})$.

## 4. The differentiability and convexity for fuzzy $n$-cell mappings

In this work, let $M$ be a convex set of $m$-dimensional Euclidean space $R^m$. We consider mapping $\widetilde{F}$ from $M$ into $L(E^n)$, such a mapping is called a fuzzy $n$-cell mapping. For the sake of brevity, $\widetilde{F}$ is called a fuzzy mapping. For any $r \in [0,1]$, we denote $[\widetilde{F}(t)]^r$ by $F_r(t) = \prod_{i=1}^n [F_i^-(r,t), F_i^+(r,t)]$.

Let $\widetilde{F} : M \to L(E^n)$ be a fuzzy mapping and $\widetilde{u} \in L(E^n)$. For $t_0 \in \text{int}M$, we write $\lim_{t \to t_0} \widetilde{F}(t) = \widetilde{u}$, if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that, for $0 < \|t - t_0\| < \delta$, we have $D(\widetilde{F}(t), \widetilde{u}) < \varepsilon$.

We say that $\widetilde{F}$ is continuous at $t_0 \in \text{int}M$ if $\lim_{t \to t_0} \widetilde{F}(t) = \widetilde{F}(t_0)$.

**Theorem 4.1.** Let $\widetilde{F} : M \to L(E^n)$ be a fuzzy mapping such that $\widetilde{F}(t) = f(t) \cdot \widetilde{u}$, where $f(t) : M \to R$ be a real-valued function on $M$, $\widetilde{u} \in L(E^n)$ and $\widetilde{u} \neq 0$. If $f$ is continuous at $t_0$, then $\widetilde{F}$ is continuous at $t_0$ and

$$\lim_{t \to t_0} \widetilde{F}(t) = \widetilde{u} \cdot \lim_{t \to t_0} f(t).$$

**Proof.** Assume that $f$ is continuous at $t_0$. Then for every $\varepsilon > 0$, there exists a $\delta > 0$ such that, for $0 < \|t - t_0\| < \delta$, we have $|f(t) - f(t_0)| < \frac{\varepsilon}{D(\widetilde{u}, \widetilde{0})}$. According to the sign-preserving theorem of limit, we have

$$\begin{aligned}
D(\widetilde{F}(t), \widetilde{F}(t_0)) &= D(f(t) \cdot \widetilde{u}, f(t_0) \cdot \widetilde{u}) \\
&= \sup_{r \in [0,1]} \max_{1 \leq i \leq n} \{|f(t) - f(t_0)| \cdot |u_i^-(r)|, |f(t) - f(t_0)| \cdot |u_i^+(r)|\} \\
&= |f(t) - f(t_0)| \sup_{r \in [0,1]} \max_{1 \leq i \leq n} \{|u_i^-(r)|, |u_i^+(r)|\} \\
&= |f(t) - f(t_0)| \cdot D(\widetilde{u}, \widetilde{0}) \\
&< \varepsilon,
\end{aligned}$$

which implies that $\widetilde{F}$ is continuous at $t_0$.

**Definition 4.1.** [15] Let $\widetilde{F}: M \to L(E^n)$, $t_0 = (t_1^0, t_2^0, \cdots, t_m^0) \in \text{int}M$ and $t = (t_1, t_2, \cdots, t_m) \in \text{int}M$. If $g$-difference $\widetilde{F}(t) \ominus_g \widetilde{F}(t_0)$ exists and there exist $\widetilde{u}_j \in L(E^n)$ $(j = 1, 2, \cdots, m)$, such that

$$\lim_{t \to t_0} \frac{D(\widetilde{F}(t) \ominus_g \widetilde{F}(t_0), \sum_{j=1}^m \widetilde{u}_j(t_j - t_j^0))}{d(t, t_0)} = 0,$$

then we say that $\widetilde{F}$ is differentiable at $t_0$ and the fuzzy vector $(\widetilde{u}_1,\ \widetilde{u}_2, \cdots, \widetilde{u}_m)$ is the gradient of $\widetilde{F}$ at $t_0$, denoted by $\nabla\widetilde{F}(t_0)$, i.e., $\nabla\widetilde{F}(t_0) = (\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m)$.

**Remark 4.1.** Let $\widetilde{F}: M \to L(E^n)$, $t_0 = (t_1^0, \cdots, t_j^0, \cdots, t_m^0) \in \text{int}M$ and $h \in R$ with $t = (t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) \in \text{int}M$. Then the gradient $\nabla\widetilde{F}(t_0)$ exists at $t_0$ if and only if and $\widetilde{F}(t) \ominus_g \widetilde{F}(t_0)$ exists and there are $\widetilde{u}_j \in L(E^n)$ $(j = 1, 2, \cdots, m)$, such that

$$\widetilde{u}_j = \lim_{h \to 0} \frac{\widetilde{F}(t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) \ominus_g \widetilde{F}(t_1^0, \cdots, t_j^0, \cdots, t_m^0)}{h}.$$

Here the limit is taken in the metric space $(L(E^n), D)$.

**Theorem 4.2.** Let $\widetilde{F}: M \to L(E^n)$ be a fuzzy mapping such that $\widetilde{F}(t) = f(t) \cdot \widetilde{u}$, where $f(t): M \to R$ be a continuous function on $M$, $\widetilde{u} \in L(E^n)$ and $\widetilde{u} \neq 0$. If $\widetilde{F}(t) \ominus_g \widetilde{F}(t_0)$ exists, then the gradient $\nabla\widetilde{F}(t_0) = (\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m)$ of $\widetilde{F}$ exists at $t_0$ if and only if the gradient $\nabla f = (\frac{\partial f}{\partial t_1}, \frac{\partial f}{\partial t_2}, \cdots, \frac{\partial f}{\partial t_m})$ of $f$ exists at $t_0$ and

$$\widetilde{u}\frac{\partial f}{\partial t_j}|_{t=t_0} = \widetilde{u}_j \ (j = 1, 2, \cdots, m).$$

**Proof.** It follows from the sign-preserving theorem of limit and Theorem 3.2 that

$$\widetilde{F}(t) \ominus_g \widetilde{F}(t_0) = (f(t) - f(t_0)) \cdot \widetilde{u}.$$

Only if: Taking $h \in R$, such that $t = (t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) \in \text{int}M$, then

$$\begin{aligned} & \widetilde{u} \lim_{h\to 0} \tfrac{f(t_1^0, \cdots, t_j^0+h, \cdots, t_m^0) - f(t_1^0, \cdots,\ t_j^0, \cdots,\ t_m^0)}{h} \\ = & \lim_{h\to 0} \tfrac{(f(t_1^0, \cdots, t_j^0+h, \cdots, t_m^0) - f(t_1^0, \cdots,\ t_j^0, \cdots,\ t_m^0)) \cdot \widetilde{u}}{h} \\ = & \lim_{h\to 0} \tfrac{\widetilde{F}(t_1^0, \cdots, t_j^0+h, \cdots, t_m^0) \ominus_g \widetilde{F}(t_1^0, \cdots,\ t_j^0, \cdots,\ t_m^0)}{h} \\ = & \widetilde{u}_j, \end{aligned}$$

which implies that the gradient $\nabla f = (\frac{\partial f}{\partial t_1}, \frac{\partial f}{\partial t_2}, \cdots, \frac{\partial f}{\partial t_m})$ of $f$ exists at $t_0$ and $\widetilde{u}\frac{\partial f}{\partial t_j}|_{t=t_0} = \widetilde{u}_j$ $(j = 1, 2, \cdots, m)$.

If: Taking $h \in R$, such that $t = (t_1^0, \cdots, t_j^0 + h, \cdots, t_m^0) \in \text{int}M$, then

$$\begin{aligned} & \lim_{h\to 0} \tfrac{\widetilde{F}(t_1^0, \cdots, t_j^0+h, \cdots, t_m^0) \ominus_g \widetilde{F}(t_1^0, \cdots,\ t_j^0, \cdots,\ t_m^0)}{h} \\ = & \lim_{h\to 0} \tfrac{(f(t_1^0, \cdots, t_j^0+h, \cdots, t_m^0) - f(t_1^0, \cdots, t_j^0+h, \cdots, t_m^0)) \cdot \widetilde{u}}{h} \\ = & \tfrac{\partial f}{\partial t_j}|_{t=t_0} \cdot \widetilde{u}, \end{aligned}$$

which implies that the gradient $\nabla\widetilde{F}(t_0) = (\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_m)$ of $\widetilde{F}$ exists at $t_0$ and $\widetilde{u}_j = \widetilde{u}\frac{\partial f}{\partial t_j}|_{t=t_0}$ $(j = 1, 2, \cdots, m)$.

**Theorem 4.3.** Let $\widetilde{F}: M \to L(E^n)$ be a fuzzy mapping such that $\widetilde{F}(t) = f(t) \cdot \widetilde{u}$, where $f(t): M \to R$ be a continuous function on $M$, $\widetilde{u} \in L(E^n)$ and $\widetilde{u} \neq 0$. If $\widetilde{F}(t) \ominus_g \widetilde{F}(t_0)$ exists and $f$ is differentiable at $t_0$, then $\widetilde{F}$ is differentiable at $t_0$.

**Proof.** It follows from the sign-preserving theorem of limit and Theorem 3.2 that $\widetilde{F}(t) \ominus_g \widetilde{F}(t_0) = (f(t) - f(t_0)) \cdot \widetilde{u}$. According to Theorem 4.2, we have

$$\begin{aligned} & D(\widetilde{F}(t) \ominus_g \widetilde{F}(t_0), \textstyle\sum_{j=1}^m \widetilde{u}_j(t_j - t_j^0)) \\ = & D(\widetilde{u}(f(t) - f(t_0)), \widetilde{u} \textstyle\sum_{j=1}^m \frac{\partial f}{\partial t_j}|_{t=t_0}(t_j - t_j^0)) \\ = & D(\widetilde{u}, \widetilde{0}) \cdot |f(t) - f(t_0) - \textstyle\sum_{j=1}^m \frac{\partial f}{\partial t_j}|_{t=t_0}(t_j - t_j^0)|, \end{aligned}$$

where, $\widetilde{u}_j = \lim_{h\to 0} \frac{\widetilde{F}(t_1^0,\cdots,t_j^0+h,\cdots,t_m^0)\ominus_g \widetilde{F}(t_1^0,\cdots,t_j^0,\cdots,t_m^0)}{h}$. This completes the proof.

Convexity plays a key role in mathematical economics, engineering, management science, and optimization theory. Therefore, research into convexity is one of the most important aspects of mathematical programming. Next, using the partial ordering relation $\preceq_c$ we will extend the concept of convexity for real-valued functions to fuzzy mappings.

**Definition 4.2.** Let $\widetilde{F} : M \to L(E^n)$ be a fuzzy mapping. $\widetilde{F}$ is said to be convex (c.) on $M$ if

$$\widetilde{F}(\lambda t_1 + (1-\lambda)t_2) \preceq_c \lambda\widetilde{F}(t_1) + (1-\lambda)\widetilde{F}(t_2)$$

for any $t_1, t_2 \in M$ and $\lambda \in [0,1]$.

**Remark 4.2.** If $\widetilde{F}, \widetilde{G} : M \to L(E^n)$ are convex fuzzy mappings and $\alpha, \beta \geq 0$, then $\alpha\widetilde{F} + \beta\widetilde{G}$ is a convex fuzzy mapping on $M$.

**Theorem 4.4.** Let $\widetilde{F} : M \to L(E^n)$ be a fuzzy mapping such that $\widetilde{F}(t) = f(t)\cdot\widetilde{u}$, where $f(t) : M \to R$ be a real-valued function on $M$, $\widetilde{u} \in L(E^n)$ and $\widetilde{u} \succeq_c \widetilde{0}$. $\widetilde{F}$ is convex on $M$ if and only if $f$ is convex on $M$.

**Proof.** If: Let $C = R^{n+} \subseteq R^n$, where $R^{n+} = \{(x_1, x_2, \cdots, x_n) \in R^n : x_1 \geq 0, x_2 \geq 0, \cdots, x_n \geq 0\}$. Assume that the real-valued function $f$ is convex on $M$, then for any $t_1, t_2 \in M$ and $\lambda \in [0,1]$, we have

$$f(\lambda t_1 + (1-\lambda)t_2) \leq \lambda f(t_1) + (1-\lambda)f(t_2).$$

According to $\widetilde{u} \succeq_c \widetilde{0}$, we obtain

$$\tau(\widetilde{u})(\lambda f(t_1) + (1-\lambda)f(t_2)) \in \tau(\widetilde{u})f(\lambda t_1 + (1-\lambda)t_2) + C.$$

For any $t_1, t_2 \in M$ and $\lambda \in [0,1]$, it follows from Remark 2.1 that

$$\tau(\lambda\widetilde{u}f(t_1) + (1-\lambda)\widetilde{u}f(t_2)) \in \tau(\widetilde{u}f(\lambda t_1 + (1-\lambda)t_2)) + C,$$

i.e., $\tau(\lambda\widetilde{F}(t_1) + (1-\lambda)\widetilde{F}(t_2)) \in \tau(\widetilde{F}(\lambda t_1 + (1-\lambda)t_2)) + C$, which implies that

$$\widetilde{F}(\lambda t_1 + (1-\lambda)t_2) \preceq_c \lambda\widetilde{F}(t_1) + (1-\lambda)\widetilde{F}(t_2).$$

Therefore, $\widetilde{F}$ is convex on $M$.

Only if: Let $\widetilde{F}$ be convex on $M$. For any $t_1, t_2 \in M$ and $\lambda \in [0,1]$,

$$\widetilde{F}(\lambda t_1 + (1-\lambda)t_2) \preceq_c \lambda\widetilde{F}(t_1) + (1-\lambda)\widetilde{F}(t_2),$$

i.e., $\tau(\lambda\widetilde{F}(t_1) + (1-\lambda)\widetilde{F}(t_2)) \in \tau(\widetilde{F}(\lambda t_1 + (1-\lambda)t_2)) + C$, which implies that

$$\tau(\lambda\widetilde{u}f(t_1) + (1-\lambda)\widetilde{u}f(t_2)) \in \tau(\widetilde{u}f(\lambda t_1 + (1-\lambda)t_2)) + C.$$

According to Remark 2.1, we have

$$\tau(\widetilde{u})(\lambda f(t_1) + (1-\lambda)f(t_2)) \in \tau(\widetilde{u})f(\lambda t_1 + (1-\lambda)t_2) + C.$$

For any $t_1, t_2 \in M$ and $\lambda \in [0,1]$, we have from $\widetilde{u} \succeq_c \widetilde{0}$ that

$$f(\lambda t_1 + (1-\lambda)t_2) \leq \lambda f(t_1) + (1-\lambda)f(t_2).$$

Therefore, $f$ is convex on $M$.

**Theorem 4.5.** Let $\widetilde{F} : M \to L(E^n)$ be a fuzzy mapping such that $\widetilde{F}(t) = \widetilde{u}_0 + \widetilde{u}_1 f_1(t) + \widetilde{u}_2 f_2(t) + \cdots + \widetilde{u}_l f_l(t)$, where $f_k : M \to R$ be real-valued functions on $M$, $\widetilde{u}_k \in L(E^n)$ and $\widetilde{u}_k \succeq_C \widetilde{0}$ $(k = 0, 1, 2, \cdots, l)$. If $f_k$ $(k = 1, 2, \cdots, l)$ are convex on $M$, then $\widetilde{F}$ is convex on $M$.

**Proof.** Let $f_k$ $(k = 1, 2, \cdots, l)$ be convex fuzzy-number-valued functions. Then we have

$$f_k(\lambda t_1 + (1-\lambda)t_2) \leq \lambda f_k(t_1) + (1-\lambda)f_k(t_2)$$

for any $t_1, t_2 \in M$ and $\lambda \in [0,1]$. It follows from $\widetilde{u}_k \succeq_C \widetilde{0}$ and Remark 2.1 that

$$\widetilde{u}_k \cdot f_k(\lambda t_1 + (1-\lambda)t_2) \preceq_C \lambda \widetilde{u}_k \cdot f_k(t_1) + (1-\lambda)\widetilde{u}_k \cdot f_k(t_2) \ \ (k = 0, 1, 2, \cdots, l),$$

for any $t_1, t_2 \in M$ and $\lambda \in [0,1]$. According to Theorem 2.3, we have

$$\widetilde{u}_0 + \sum_{k=1}^{l} \widetilde{u}_k \cdot f_k(\lambda t_1 + (1-\lambda)t_2) \preceq_C \lambda(\widetilde{u}_0 + \sum_{k=1}^{l} \widetilde{u}_k \cdot f_k(t_1)) + (1-\lambda)(\widetilde{u}_0 + \sum_{k=1}^{l} \widetilde{u}_k \cdot f_k(t_2)),$$

which implies that

$$\widetilde{F}(\lambda t_1 + (1-\lambda)t_2) \preceq_c \lambda F(t_1) + (1-\lambda)F(t_2),$$

for any $t_1, t_2 \in M$ and $\lambda \in [0,1]$. This completes the proof.

## 5. Convex fuzzy programming

It is well known that convexity plays an important role in the aspect of optimality conditions for mathematical programming. Based on the new concept of convexity for fuzzy mappings, the KKT optimality conditions for a class of fuzzy optimization problems are established.

Let $\widetilde{F}(t), \widetilde{g}_1(t), \widetilde{g}_2(t), \cdots, \widetilde{g}_m(t)$ be fuzzy mappings on $M$. We consider the following fuzzy constrained minimization problem (FCMP1):

Minimize $\widetilde{F}(t)$

Subject to $\widetilde{g}_k(t) \preceq_c \widetilde{u}_k \ (k = 1, 2, \cdots, l)$,

where $\widetilde{u}_k \in L(E^n)$. In the following arguments the feasible set $T = \{t \in \text{int}M : \widetilde{g}_k(t) \preceq_c \widetilde{u}_k, k = 1, 2, \cdots, l\}$ of fuzzy constrained minimization problem (FCMP1) is assumed to be a compact convex set. If $t_0 \in \text{int}M$ and for no $t \in \text{int}M$ such that $\widetilde{F}(t) \prec_c \widetilde{F}(t_0)$, then $t_0$ is called an optimal solution or a global optimal solution to the problem FCMP1, $\widetilde{F}(t_0)$ is called the optimal objective value of $\widetilde{F}$.

Let $C = R^{n+} \subseteq R^n$. We have from Definition 2.2 and Definition 2.3 that the problem FCMP1 can be written as following constrained minimization problem (FCMP2):

Minimize $\widetilde{F}(t)$

Subject to $\int_0^1 r[(\widetilde{g}_k(t))_1^+(r) + (\widetilde{g}_k(t))_1^-(r)]dr \le \int_0^1 r[(\widetilde{u}_k)_1^+(r) + (\widetilde{u}_k)_1^-(r)]dr, \ k = 1, 2, \cdots, l,$

$\int_0^1 r[(\widetilde{g}_k(t))_2^+(r) + (\widetilde{g}_k(t))_2^-(r)]dr \le \int_0^1 r[(\widetilde{u}_k)_2^+(r) + (\widetilde{u}_k)_2^-(r)]dr, \ k = 1, 2, \cdots, l,$

$\cdots$

$\int_0^1 r[(\widetilde{g}_k(t))_n^+(r) + (\widetilde{g}_k(t))_n^-(r)]dr \le \int_0^1 r[(\widetilde{u}_k)_n^+(r) + (\widetilde{u}_k)_n^-(r)]dr, \ k = 1, 2, \cdots, l.$

Let $G_s(t) = \int_0^1 r[(\widetilde{g}_k(t))_i^+(r) + (\widetilde{g}_k(t))_i^-(r)]dr - \int_0^1 r[(\widetilde{u}_k)_i^+(r) + (\widetilde{u}_k)_i^-(r)]dr$ $(i = 1, 2, \cdots, n, \ k = 1, 2, \cdots, l,$ $s = i \times k = 1, 2, \cdots, n \times l)$, denoted $p = n \times l$. The problem FCMP2 can be written as follows (FCMP3):

Minimize $\widetilde{F}(t)$

Subject to $G_s(t) \le 0, \ s = 1, 2, \cdots, p,$

where $G_s : M \to R$ are real-valued functions. It is obvious that the feasible sets of problems (FCMP1) and (FCMP3) are the same.

In the rest of this article, $G_s$ $(s = 1, 2, \cdots, p)$ are assumed to be convex functions on $M$, and continuously differentiable at $t_0 \in T$.

**Theorem 5.1.** Suppose that the fuzzy objective function $\widetilde{F}(t) = f(t) \cdot \widetilde{u}$, where $f(t) : M \to R$, $\widetilde{u} \in L(E^n)$ and $\widetilde{u} \succeq_c \widetilde{0}$. let $f$ be convex on $M$ and continuously differentiable at $t_0 \in \text{int}M$. If there exist Lagrange multipliers $\mu_s \ge 0$ $(s = 1, 2, \cdots, p)$, such that

(1) $\nabla f(t_0) + \sum_{s=1}^{p} \mu_s \cdot \nabla G_s(t_0) = \mathbf{0}$,

(2) $\mu_s \cdot G_s(t_0) = 0$ $(s = 1, 2, \cdots, p)$,

then $t_0$ is an optimal solution of FCMP3.

**Proof.** Since $f$ is convex on $M$ and continuously differentiable at $t_0 \in \text{int}M$. We consider the following constrained optimization problem

Minimize $f(t)$

Subject to $G_s(t) \leq 0$ $(s = 1, 2, \cdots, p)$.

It is obvious that conditions (1) and (2) are the KKT optimality conditions for this optimization problem. Therefore, we conclude that $t_0$ is an optimal solution of the real-valued objective function $f$, i.e.,

$$f(t_0) \leq f(t), \tag{5.1}$$

for any $t \in \text{int}M$.

Suppose that $t_0$ is not a solution of problem FCMP3. Then, there exists a $\bar{t} \in \text{int}M$ such that

$$\widetilde{F}(\bar{t}) \prec_c \widetilde{F}(t_0).$$

Let $C = R^{n+} \subseteq R^n$, we have

$$\tau(f(t_0) \cdot \widetilde{u}) \in \tau(f(\bar{t}) \cdot \widetilde{u}) + C.$$

Thus

$$f(t_0) \cdot \tau(\widetilde{u}) \in f(\bar{t}) \cdot \tau(\widetilde{u}) + C,$$

and

$$f(t_0) \cdot \tau(\widetilde{u}) \neq f(\bar{t}) \cdot \tau(\widetilde{u}).$$

It follows from $\widetilde{u} \succeq_c \widetilde{0}$ that $f(\bar{t}) < f(t_0)$. This is a contradiction with (5.1), then $t_0$ is an optimal solution of FCMP3.

**Example 5.1.** Let

$$\widetilde{u}(x_1, x_2) = \begin{cases} 20x_2 - 18, & 0.9 \leq x_2 \leq 0.95,\ 200x_2 - 90 \leq x_1 \leq 290 - 200x_2, \\ -0.1x_1 + 11, & 100 \leq x_1 \leq 110,\ 1.45 - 0.005x_1 \leq x_2 \leq 0.45 + 0.005x_1, \\ -20x_2 + 20, & 0.95 \leq x_2 \leq 1,\ 290 - 200x_2 \leq x_1 \leq 200x_2 - 90, \\ 0.1x_1 - 9, & 90 \leq x_1 \leq 100,\ 0.45 + 0.005x_1 \leq x_2 \leq 1.45 - 0.005x_1, \\ 0, & \text{otherwise}, \end{cases}$$

$C = R^{2+} \subseteq R^2$. Then for any $r \in [0, 1]$,

$$[\widetilde{u}]^r = [90 + 10r, 100 - 10r] \times [0.9 + 0.05r, 1 - 0.05r].$$

According to Definition 2.3, $\tau(\widetilde{u}) = (95, 0.95)$, thus $\widetilde{u} \succeq_c \widetilde{0}$. Suppose that the fuzzy objective function $\widetilde{F} : [1, +\infty) \to L(E^2)$ and $\widetilde{F}(t) = f(t) \cdot \widetilde{u}$, where $f(t) = e^t - t^2$. We consider the following fuzzy constrained minimization problem:

Minimize $\widetilde{F}(t)$,

Subject to $G_1(t) = t^2 - 25 \leq 0, \quad G_2(t) = -t + 1 \leq 0.$

It is obvious that $G_s(t)$ $(s = 1, 2)$ are convex functions on $[1, +\infty)$, and continuously differentiable at $t_0 = 1$.

Since $f$ is a convex function on $[1, +\infty)$, and continuously differentiable at $t_0 = 1$. On the other hand, the condition (1) and (2) from Theorem 5.1 are satisfied for $\mu_1 = 0$, and $\mu_2 = e - 2$. Therefore, $t_0 = 1$ is an optimal solution of FCMP3.

**Theorem 5.2.** Suppose that the fuzzy objective function $\widetilde{F}(t) = \widetilde{u}_1 f_1(t) + \widetilde{u}_2 f_2(t) + \cdots + \widetilde{u}_l f_l(t)$, where $f_k(t) : M \to R$, $\widetilde{u}_k \in L(E^n)$ and $\widetilde{u}_k \succeq_c \widetilde{0}$ $(k = 1, 2, \cdots, l)$. Let real-valued functions $f_k$ are convex on $M$ and continuously differentiable at $t_0 \in \text{int}M$. If there exist Lagrange multipliers $\mu_s^k \geq 0$ $(s = 1, 2, \cdots, p,\ k = 1, 2, \cdots, l)$, such that

(1) $\nabla f_k(t_0) + \sum_{s=1}^{p} \mu_s^k \cdot \nabla G_s(t_0) = \mathbf{0}$ $(k = 1, 2, \cdots, l)$,

(2) $\mu_s^k \cdot G_s(t_0) = 0$ $(s = 1, 2, \cdots, p,\ k = 1, 2, \cdots, l)$,

then $t_0$ is an optimal solution of FCMP3.

**Proof.** Since the real-valued functions $f_k(t)$ $(k = 1, 2, \cdots, l)$ are convex on $M$ and continuously differentiable at $t_0 \in \text{int}M$. We consider the following constrained optimization problem

Minimize $f_k(t)$ $(k = 1, 2, \cdots, l)$,

Subject to $\quad G_s(t) \leq 0 \ (s = 1, 2, \cdots, p)$.

It is obvious that conditions (1) and (2) are the KKT optimality conditions for this optimization problem. Therefore, we conclude that $t_0$ is an optimal solution of the real-valued objective functions $f_k$, i.e.,

$$f_k(t_0) \leq f_k(t) \ (k = 1, 2, \cdots, l).$$

for any $t \in \text{int}M$. Let $C = R^{n+} \subseteq R^n$. We have from $\widetilde{u}_k \succeq_c \widetilde{0}$ that

$$\tau(\widetilde{u}_k) f_k(t) \in \tau(\widetilde{u}_k) f_k(t_0) + C \ (k = 1, 2, \cdots, l).$$

Thus,

$$\sum_{k=1}^{l} \tau(\widetilde{u}_k) f_k(t) \in \sum_{k=1}^{l} \tau(\widetilde{u}_k) f_k(t_0) + C.$$

According to Remark 2.1, we obtain

$$\tau(\sum_{k=1}^{l} u_k f_k(t)) \in \tau(\sum_{k=1}^{l} u_k f_k(t_0)) + C,$$

i.e., $\widetilde{F}(t) \in \widetilde{F}(t_0) + C$. Therefore, $\widetilde{F}(t_0) \preceq_c \widetilde{F}(t)$, which implies that $t_0$ is an optimal solution of FCMP3.

### 6. Conclusion

The objective of this paper is to introduce the differentiability concept for fuzzy mappings and an ordering relation on the fuzzy $n$-cell number space is considered. We have used the differentiability for fuzzy mappings to obtain the KKT optimality conditions for fuzzy constrained minimization problem based on the ordering relation $\preceq_c$ .

Future research includes studying other types of optimality conditions for fuzzy constrained minimization problem. One alternative is to define the concept of invex function using differentiability and the ordering relation $\preceq_c$ for fuzzy mappings.

## References

[1] L.A. Zadeh, Fuzzy sets, Inform Control. 8 (1965) 338-353.

[2] S.L. Chang, L.A. Zadeh, On Fuzzy Mapping and Control, IEEE T Syst Man Cy-S. 2 (1972) 30-34.

[3] M.L. Puri, D.A. Ralescu, Differentials of Fuzzy Functions, J. Math. Anal. Appl. 91 (1983) 552-558.

[4] O. Kaleva, Fuzzy differential equations, Fuzzy Sets Syst. 24 (1987) 301-317.

[5] G.X. Wang, C.X. Wu, Directional derivatives and subdifferential of convex fuzzy mappings and application in convex fuzzy programming, Fuzzy Sets Syst. 138 (2003) 559-591.

[6] L. Stefanini, A generalization of Hukuhara difference, in: D. Dubois, M.A. Lubiano, H. Prade, M.A. Gil, P. Grzegorzewski, O. Hryniewicz (Eds.), Soft Methods for Handling Variability and Imprecision, in: Series on Advances in Soft Computing, Springer, 2008.

[7] L. Stefanini, A generalization of Hukuhara difference and division for interval and fuzzy arithmetic, Fuzzy Sets Syst. 161 (2010) 1564-1584.

[8] B. Bede, L. Stefanini, Generalized differentiability of fuzzy-valued functions, Fuzzy Sets Syst. 230 (2013) 119-141.

[9] P. Diamond, P. Kloeden, Characterization of compact subsets of fuzzy sets, Fuzzy Sets Syst. 29 (1989) 341-348.

[10] M. Ma, On embedding problems of fuzzy number spaces: Part 5, Fuzzy Sets Syst. 55 (1993) 313-318.

[11] C.V. Negoita, D.A. Ralescu, Application of Fuzzy Sets to Systems Analysis, Wiley, NewYork, 1975.

[12] C.X. Wu, M. Ma, J.X. Fang, Structure Theory of fuzzy Analysis, Guizhou Scientific Publication, 1994 (in Chinese).

[13] G.X. Wang, C.X. Wu, Fuzzy $n$-cell numbers and the differential of fuzzy $n$-cell number value mappings, Fuzzy Sets Syst. 130 (2002) 367-381.

[14] R. Goetschel, W. Voxman, Elementary fuzzy calculus, Fuzzy Sets Syst. 18 (1986) 31-43.

[15] S.X. Hai, Z.T. Gong, The gradient and subgradient for fuzzy mappings based on generalized difference of $n$-cell fuzzy-numbers, J Comput Anal Appl. (submit)

[16] S.X. Hai, Z.T. Gong, H.X. Li, Generalized differentiability for $n$-dimensional fuzzy-number-valued functions and fuzzy optimization, Inform Sci. 374 (2016) 151 – 163.

[17] Z.T. Gong, S.X. Hai, Convexity of $n$-dimensional fuzzy number-valued functions and its applications, Fuzzy Sets Syst. 295 (2016) 19 – 36.

# Monotone iterative technique for fractional partial differential equations with impulses *

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**Abstract**

In this article, we use a monotone iterative technique based on the presence of lower and upper solutions to discuss the existence of mild solutions for the initial value problem of the impulsive time fractional order partial differential equation of Volterra type in an ordered Banach space $E$

$$\begin{cases} {D_0}^q u(t) + Au(t) = f(t, u(t), Tu(t)), \quad t \in J,\ t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k)), \quad k = 1, 2, \cdots, m, \\ u(0) = x_0, \end{cases}$$

where ${D_0}^q$ is the Caputo fractional derivative of order $q$, $0 < q < 1$, $A : D(A) \subset E \to E$ is a closed linear operator and $-A$ is a generator of equicontinuous $C_0$-semigroup, $f \in C(J \times E \times E, E)$, $J = [0, a]$, $a > 0$ is a constant, $T$ is a Volterra integral operator, $0 < t_1 < t_2 < \cdots < t_m < a$, $I_k \in C(E, E)$, $k = 1, 2, \cdots, m$ and $x_0 \in E$. Under wide monotone conditions and the noncompactness measure condition of nonlinearity $f$, we obtain the existence of extremal mild solutions and unique mild solution between lower and upper solutions. The results obtained generalize the recent conclusions on this topic. An example is also given to illustrate that our results are valuable.

**Key Words:** Impulsive fractional order integro-differential evolution equation; lower and upper solution; equicontinuous semigroup; measure of noncompactness; monotone iterative technique

**MR(2010) Subject Classification:** 34A37; 34G20; 35R11; 45J05

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# 1 Introduction

Fractional order models are found to be more adequate than integer order models in some real world problems. Fractional order derivatives describe the property of memory and heredity of materials, and this is the major advantage of fractional order derivatives compared with integer order derivatives. In recent years, fractional order differential calculus has attracted many physicists, mathematicians and engineers, and notable contributions have been made to both theory and applications of fractional differential equations. It has been found that the differential equations involving fractional order derivatives in time are more realistic to describe many phenomena in practical cases than those of integer order in time. For instance, fractional calculus concepts have been used in the modelling of neurons [1], viscoelastic materials [2]. Other examples from fractional order differential equations can be found in [3-7] and the references therein.

One of the branches of fractional differential equations and dynamics is the theory of time fractional order evolution equations. Since time fractional order semilinear evolution equations are abstract formulations for many problems arising in engineering and physics, time fractional evolution equations have attracted increasing attention in recent years, see [8-16] and the references therein.

In this article, we use a monotone iterative technique based on the presence of lower and upper solutions to discuss the existence of mild solutions for the initial value problem (IVP) of impulsive time fractional order partial differential equation of Volterra type in an ordered Banach space $E$

$$\begin{cases} D_0{}^q u(t) + Au(t) = f(t, u(t), Tu(t)), \quad t \in J,\ t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k)), \quad k = 1, 2, \cdots, m, \\ u(0) = x_0, \end{cases} \tag{1.1}$$

where $D_0{}^q$ is the Caputo fractional derivative of order $q$, $0 < q < 1$, $A : D(A) \subset E \to E$ is a closed linear operator, $-A$ generates a $C_0$-semigroup $S(t)(t \geq 0)$ in $E$, $J = [0, a]$, $a > 0$ is a constant, $f \in C(J \times E \times E, E)$, $x_0 \in E$, $0 < t_1 < t_2 < \cdots < t_m < a$, $I_k \in C(E, E)$, $k = 1, 2, \cdots, m$, and

$$Tu\,(t) := \int_0^t K(t, s)u(s)ds \tag{1.2}$$

is a Volterra integral operator with integral kernel $K \in C(\triangle, \mathbb{R}^+)$, $\triangle = \{(t, s) \in \mathbb{R}^2 \mid 0 \leq s \leq t \leq a\}$, $\Delta u|_{t=t_k}$ stands the jump of $u(t)$ at $t = t_k$, i.e., $\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-)$, where $u(t_k^+)$ and $u(t_k^-)$ represent the right and left limits of $u(t)$ at $t = t_k$, respectively.

The monotone iterative technique based on lower and upper solutions is an effective and

flexible method, which yields monotone sequences of lower and upper approximate solutions that converge to the minimal and maximal solutions between the lower and upper solutions. In 1982, Du and Lakshmikantham [17] established a monotone iterative method for an initial value problem of ordinary differential equation in an ordered banach space. Later, Li [18,19], Chen and Li [20,21] developed the monotone iterative method for the abstract evolution equations in abstract space.

The theory of impulsive differential equations is a new and important branch of differential equation theory, which has an extensive physical background and realistic mathematical model, and hence has been emerging as an important area of investigation in recent years, see [22]. Correspondingly, the existence of solutions to impulsive integro-differential equations in Banach spaces has also been studied by several authors, see for example [23,24] and the references therein. But all of the results mentioned above are for the differential equations of integer order. To the best of the author's knowledge, no results yet exist for the initial value problem of the impulsive time fractional order integro-differential evolution equation (1.1) by using the monotone iterative technique. The purpose of this paper is to establish the monotone iterative method for IVP (1.1) in an ordered Banach space $E$. Under the positivity assumption for the $C_0$ semigroup $S(t)$ and some monotone conditions combined with the noncompactness measure condition of nonlinearity $f$, we obtain the results on the existence and uniqueness of mild solutions for IVP(1.1).

## 2 Preliminaries

Let $(E, \|\cdot\|)$ be an ordered Banach space with the partial order " $\leq$ ". Then the positive cone $P = \{x \in E \mid x \geq \theta\}$ is normal with normal constant $N$. Denote $C(J, E)$ the Banach space of all $E$-value continuous functions with the supremum norm $\|u\|_C = \sup_{t\in J} \|u(t)\|$. Clearly, $C(J, E)$ is also an ordered Banach space, which partial order " $\leq$ " is reduced by the positive cone $P_C = \{u \in C(J, E) \mid u(t) \geq \theta, t \in J\}$. Set

$$PC(J, E) = \{\, u : J \to E \mid u(t) \text{ is continuous at } t \neq t_k,$$
$$\text{left continuous at } t = t_k, \text{ and } u(t_k^+) \text{ exists, } k = 1, 2, \cdots, m \,\}.$$

Evidently, $PC(J, E)$ is also an ordered Banach space with the supremum norm $\|u\|_{PC} = \sup_{t\in J} \|u(t)\|$, its partial order " $\leq$ " is reduced by the positive function cone $P_{PC} = \{u \in PC(J, E) \mid u(t) \geq \theta, t \in J\}$. $P_{PC}$ is also a normal with the same normal constant $N$. For $v, w \in PC(J, E)$ with $v \leq w$, we use $[v, w]$ to denote the order interval $\{u \in PC(J, E) \mid v \leq u \leq w\}$, and for every $t \in J$, we use $[v(t), w(t)]$ to represent the order interval $\{x \in E \mid v(t) \leq x \leq w(t)\}$ in $E$. Let $J' := J\backslash\{t_1, t_2, \cdots, t_m\}$. We denote by $PC^1(J, E) = \{u \in PC(J, E) \cap C^1(J', E) \mid u'(t_k^+)$ and $u'(t_k^-)$ exist$\}$. Set $L(J, E)$ be the Banach space of all $E$-valued Bochner integrable functions defined on $J$ with the norm $\|u\|_1 = \int_0^t \|u(t)\|dt$.

Let $0 < q < 1$. The Caputo fractional order derivative of order $q$ with the lower limit 0 for a function $g \in C^1(J)$ is defined as

$$D_0{}^q f(t) = \frac{1}{\Gamma(1-q)} \int_0^t \frac{g'(s)}{(t-s)^q}\, ds, \qquad t > 0, \tag{2.1}$$

where $\Gamma(\cdot)$ is the Gamma function. See [7].

If $g$ is an abstract function with values in $E$, the definition of its Caputo fractional order derivative is same. In that case then the integrals appeared in (2.1) is taken in Bochner's sense.

Let $A : D(A) \subset E \to E$ be a closed linear operator and $E_1$ denote the Banach space $D(A)$ with the graphic norm $\|x\|_1 = \|x\| + \|Ax\|$. We assume that $-A$ generates a $C_0$-semigroup $S(t)$ $(t \geq 0)$ of linear bounded operators in $E$. Denote by $\mathscr{L}(E)$ the Banach space of all linear bounded operators in $E$. By the exponential boundedness of $C_0$-semigroup, there exist $M > 0$ and $\omega \in \mathbb{R}$ such that

$$\|S(t)\|_{\mathscr{L}(E)} \leq M\, e^{\omega t}, \qquad t \geq 0. \tag{2.2}$$

If $\omega = 0$, we call $S(t)$ a uniformly bounded semigroup. Let $h \in L(J, E)$ and consider the initial value problem of the linear time fractional order evolution equation (LIVP)

$$\begin{cases} D_0{}^q u(t) + Au(t) = h(t), & t \in J, \\ u(0) = x_0. \end{cases} \tag{2.3}$$

By [11], we have the following existence and uniqueness result

**Lemma 2.1** *Assume that the $C_0$-semigroup $S(t)$ $(t \geq 0)$ generated by $-A$ is a uniformly bounded and analytic semigroup. If $h \in C(J, E)$ is uniformly Hölder continuous on $J$, then the linear initial value problem (2.3) has a unique solution expressed by*

$$u(t) = U(t)x_0 + \int_0^t (t-s)^{q-1} V(t-s)h(s)ds \tag{2.4}$$

*where $U(t),\ V(t) : [0, \infty) \to \mathscr{L}(E)$ are strongly continuous functions of linear bounded operator value given by*

$$\begin{aligned} U(t)x &= \int_0^\infty \zeta_q(\vartheta) S(t^q\vartheta)x d\vartheta, \qquad x \in E,\ \ t \geq 0, \\ V(t)x &= q\int_0^\infty \vartheta\zeta_q(\vartheta) S(t^q\vartheta)x d\vartheta, \quad x \in E,\ \ t \geq 0, \end{aligned} \tag{2.5}$$

*where*

$$\zeta_q(\vartheta) = \frac{1}{q}\, \vartheta^{-1-(1/q)} \rho_q(\vartheta^{-1/q}) \tag{2.6}$$

*is a probability density function on* $(0, +\infty)$, *in which*

$$\rho_q(\vartheta) = \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^{n-1}\vartheta^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q), \quad \vartheta \in (0, +\infty).$$

By [11], the probability density function $\zeta_q(\vartheta)$ given by (2.6) satisfies the following condition

$$\int_0^{\infty}\zeta_q(\vartheta)d\vartheta = 1, \qquad \int_0^{\infty}\vartheta\zeta_q(\vartheta)d\vartheta = \frac{1}{\Gamma(1+q)}. \tag{2.7}$$

When $S(t)(t \geq 0)$ is a uniformly bounded $C_0$-semigroup, we can also define two operator value functions $U(t)$ and $V(t)$ by (2.5). From [11, 12], we have the following result

**Lemma 2.2** *Assume that* $S(t)(t \geq 0)$ *is a uniformly bounded* $C_0$*-semigroup. Then operators* $U(t)$ *and* $V(t)$ *defined by (2.5) have the following properties:*

(1). For evrey $t \geq 0$, $U(t)$ and $V(t)$ are linear bounded operators, and

$$\|U(t)x\| \leq M\|x\|, \qquad \|V(t)x\| \leq \frac{M}{\Gamma(q)}\|x\|, \qquad x \in E. \tag{2.8}$$

(2). $U(t)$ and $V(t)$ are strongly continuous on $[0, +\infty)$.

(3). When $S(t)(t \geq 0)$ is equicontinuous, $U(t)$ and $V(t)$ are continuous in $[0, +\infty)$ by the operator norm.

In this case, by means of Lemma 2.2 (2), the function $u$ gived by (2.5) belongs to $C(J, E)$, we call it a mild solution of the linear fractional order evolution equation (2.3). That is:

**Definition 2.1** *Let* $S(t)(t \geq 0)$ *be a uniformly bounded* $C_0$*-semigroup and* $h \in L(J, E)$. *By the mild solution of the LIVP (2.3), we mean that the function* $u \in C(J, E)$ *satisfying the integral equation*

$$u(t) = U(t)x_0 + \int_0^t (t-s)^{q-1}V(t-s)h(s)ds.$$

Let $h \in PC(J, E)$, $y_k \in E$, $k = 1, 2, \cdots, m$. We consider the initial value problem of the linear impulsive time fractional order evolution equation (LIVP)

$$\begin{cases} D_0{}^q u(t) + Au(t) = h(t), & t \in J, \quad t \neq t_k, \\ \Delta u|_{t=t_k} = y_k, & k = 1, 2, \cdots, m, \\ u(0) = x_0 \in E. \end{cases} \tag{2.9}$$

Let $J_1 = [0, t_1]$, $J_k = (t_{k-1}, t_k]$, $k = 2, 3, \cdots, m+1$, where $t_{m+1} = a$. Using Definition 2.1, from $J_1$ to $J_{m+1}$ interval by interval, we can easily obtain the following result.

**Lemma 2.3** *For every $h \in PC(J,E)$ and $y_k \in E$, $k = 1, 2, \cdots, m$, the LIVP (2.9) has a unique mild solution $u \in PC(J,E)$ given by*

$$u(t) = \begin{cases} U(t)x_0 + \int_0^t (t-s)^{\alpha-1} V(t)(t-s)h(s)ds, \quad t \in J_1, \\ U(t-t_1)(u(t_1)+y_1) + \int_{t_1}^t (t-s)^{q-1} V(t-s)h(s)ds, \ \ t \in J_2, \\ \quad \cdots \cdots \cdots \\ U(t-t_m)(u(t_m)+y_m) + \int_{t_m}^t (t-s)^{q-1} V(t-s)h(s)ds, \ t \in J_{m+1}. \end{cases} \tag{2.10}$$

**Remark 2.1** *Note that the operator value functions $U(t)$ and $V(t)$ do not possess the properties of semigroup. The mild solution of the LIVP (2.9) can be expressed only by using piecewise function.*

We consider the nonlinear impulsive time fractional order evolution equation (1.1). By Lemma 2.3, a function $u \in PC(J,E)$ is called a mild solution of IVP (1.1) if $u$ satisfies the piecewise integral equation

$$u(t) = \begin{cases} U(t)x_0 + \int_0^t (t-s)^{\alpha-1} V(t)(t-s)f(s,u(s),Tu(s))ds, \quad t \in J_1, \\ U(t-t_1)(u(t_1)+I_1(u(t_1))) + \int_{t_1}^t (t-s)^{q-1} V(t-s)f(s,u(s),Tu(s))ds, \ \ t \in J_2, \\ \quad \cdots \cdots \cdots \\ U(t-t_m)(u(t_m)+I_m(u(t_m))) + \int_{t_m}^t (t-s)^{q-1} V(t-s)f(s,u(s),Tu(s))ds, \ t \in J_{m+1}. \end{cases}$$

We will use the monotone iterative method based on lower and upper solutions to discuss the existence of extremal mild solutions for IVP (1.1). Next, we introduce the concepts of lower and upper solutions for IVP (1.1).

**Definition 2.2** *If a function $v_0 \in PC^1(J,E) \cap PC(J,E_1)$ and satisfies inequalities*

$$\begin{cases} {D_0}^q v_0(t) + Av_0(t) \le f(t, v_0(t), Tv_0(t)), \quad t \in J, \ t \ne t_k, \\ \Delta v_0|_{t=t_k} \le I_k(v_0(t_k)), \quad k = 1, 2, \cdots, m, \\ v_0(0) \le x_0, \end{cases} \tag{2.11}$$

*we called it a lower solution of IVP (1.1). If all the inequalities of (2.11) are inverse, we call it an upper solution of IVP (1.1).*

Our discussion needs that the $S(t)(t \geq 0)$ is a positive $C_0$-semigroup, that is, $S(t)x \geq \theta$ for any $x \geq \theta$ and $t \geq 0$. For more details of the properties of the positive $C_0$-semigroup, see [19,25]. Clearly, by the (2.5) we obtain that:

**Lemma 2.4** *If $S(t)(t \geq 0)$ is a uniformly bounded positive $C_0$-semigroup in $E$, then $U(t)$ and $V(t)$ are positive operators in $E$ for every $t \in [0, +\infty)$.*

Let $C \geq 0$ is a constant, I denote the identity operator in $E$. It is easy to see that $-(A+C\mathrm{I})$ generates a $C_0$-semigroup $S_1(t) = e^{-Ct}S(t)\,(t \geq 0)$ in $E$, and $S_1(t)$ is a positive $C_0$ semigroup if $S(t)$ is a positive $C_0$-semigroup. If $C \geq \omega$, then by (2.2), $S_1(t)$ is a a uniformly bounded $C_0$-semigroup, for more details please see [26]. Hence we can define the corresponding operator value functions $U_1(t)$ and $V_1(t)$ as follows

$$\begin{aligned} U_1(t)x &= \int_0^\infty \zeta_q(\vartheta)S_1(t^q\vartheta)xd\vartheta, \qquad x \in E,\ \ t \geq 0, \\ V_1(t)x &= q\int_0^\infty \vartheta\zeta_q(\vartheta)S_1(t^q\vartheta)xd\vartheta, \quad x \in E,\ \ t \geq 0. \end{aligned} \tag{2.12}$$

$U_1(t)$ and $V_1(t)$ have completely same properties with $U(t)$ and $V(t)$. If the semigroup $S(t)$ is not uniformly bounded, we choose $C \geq \omega$ such that $S_1(t)$ uniformly bounded. In this case, the mild solution of IVP (1.1) can be expressed by $U_1(t)$ and $V_1(t)$.

Next, we recall some properties about the measure of noncompactness that will be used in the proof of our main results. Let $\alpha(\cdot)$ denote the Kuratowski measure of noncompactness of the bounded set. For the details of the definitions and properties of the measure of noncompactness, see [27]. For $\forall\ B \subset PC(J, E)$ and $t \in J$, set $B(t) = \{u(t) : u \in B\} \subset E$. If $B$ is bounded in $PC(J, E)$, then $B(t)$ is bounded in $E$ and $\alpha(B(t)) \leq \alpha(B)$.

**Lemma 2.5** [27] *Let $B \subset C(J, E)$ be bounded and equicontinuous. Then $\alpha(B(t))$ is continuous on $J$, and*

$$\alpha\Big(\Big\{\int_J u(t) \mid u \in B\Big\}\Big) \leq \int_J \alpha(B(t))dt.$$

**Lemma 2.6** [28] *Assume that $B = \{u_n\} \subset PC(J, E)$ is a countable set and there exists a function $m \in L^1(J, \mathbb{R}^+)$ such that for every $n \in \mathbb{N}$*

$$\|u_n(t)\| \leq m(t), \qquad \text{a.e. } t \in J.$$

*Then $\alpha(B(t))$ is Lebesgue integral on $J$, and*

$$\alpha\Big(\Big\{\int_J u_n(t)dt\Big\}\Big) \leq 2\int_J \alpha(B(t))dt.$$

Our discussion also need the following generalized Gronwall inequality which can be fund in [29].

**Lemma 2.7** *Let $C_0 \geq 0$ be a constant and $a \in L(J)$ be a nonnegative function. If $\varphi \in L(J)$ is nonnegative and satisfies*

$$\varphi \leq a(t) + C_0 \int_0^t (t-s)^{q-1}\varphi(s)ds, \quad t \in J,$$

*then*

$$\varphi(t) \leq a(t) + \int_0^t \Big[\sum_{n=1}^{\infty} \frac{(C_0\Gamma(q))^n}{\Gamma(nq)}(t-s)^{nq-1}a(s)\Big]ds, \quad t \in J.$$

# 3 Main Results

In this section, we use the monotone iterative method based on lower and upper solutions to discuss the existence of mild solution for IVP (1.1). We assume that the operator $A : D(A) \subset E \to E$ satisfies

(H0) $A : D(A) \subset E \to E$ be a closed linear operator, $-A$ generates a positive and equicontinuous $C_0$-semigroup $S(t)\,(t \geq 0)$.

Our main results as follows:

**Theorem 3.1** *Let $E$ be an ordered Banach space, and let the positive cone $P$ be normal. Assume that $A : D(A) \subset E \to E$ satisfies the assumption (H0), $f \in C(J \times E \times E, E)$, $I_k \in C(E, E)$, $k = 1, 2, \cdots, m$, and IVP (1.1) has a lower solution $v_0$ and an upper solution $w_0$ with $v_0 \leq w_0$. If the following conditions are satisfied:*

(H1) There exists a constant $C > 0$ such that

$$f(t, x_2, y_2) - f(t, x_1, y_1) \geq -C(x_2 - x_1),$$

for $\forall\, t \in J$, $v_0(t) \leq x_1 \leq x_2 \leq w_0(t)$, and $Tv_0(t) \leq y_1 \leq y_2 \leq Tw_0(t)$.

(H2) $I_k(x)$ is increasing on the order interval $[v_0(t), w_0(t)]$ for $t \in J$, $k = 1, 2, \cdots, m$.

(H3) There exists a constant $L > 0$ such that

$$\alpha(\{f(t, x_n, y_n)\}) \leq L(\alpha(\{x_n\}) + \alpha(\{y_n\})),$$

for $\forall\, t \in J$, and increasing or decreasing monotonic sequences $\{x_n\} \subset [v_0(t), w_0(t)]$ and $\{y_n\} \subset [Tv_0(t), Tw_0(t)]$,

*then IVP (1.1) has minimal and maximal mild solutions between $v_0$ and $w_0$, and they can be obtained by a monotone iterative procedure starting from $v_0$ and $w_0$, respectively.*

**Proof.** Without losing the generality, in the assumption (H1) we may assume that $C \geq \omega$, where $\omega$ is the growth exponent of $S(t)$ given by (2.2). Then the $C_0$-semigroup

$$S_1(t) = e^{-Ct}S(t), \qquad t \geq 0 \tag{3.1}$$

generated by $-(A + C\mathrm{I})$ is uniformly bounded, positive and equicontinuous. Let $U_1(t)(t \geq 0)$ and $V_1(t)(t \geq 0)$ be the operator value function defined by (2.12), then they have the properties in Lemma 2.2, specially they satisfy

$$\|U_1(t)\,x\| \leq M\|x\|, \qquad \|V_1(t)\,x\| \leq \frac{M}{\Gamma(q)}\,\|x\|, \qquad t \geq 0, \;\; x \in E. \tag{3.2}$$

For every $u \in PC(J,\, E)$, set

$$G(u)(t) = f(t, u(t), Tu(t)) + Cu(t), \qquad t \in J. \tag{3.3}$$

Then $G : PC(J,\, E) \to PC(J,\, E)$ is a continuous mapping. We rewrite the equation (1.1) to the form of

$$\begin{cases} D_0{}^q u(t) + (A + C\mathrm{I})\,u(t) = G(u)\,(t), \quad t \in J,\ t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k)), \quad k = 1, 2, \cdots, m, \\ u(0) = x_0, \end{cases} \tag{3.4}$$

then by Lemma 2.3, the mild solution of this equation, equivalently IVP (1.1), which means that $u \in PC(J,\, E)$ satisfies the piecewise integral equation

$$u(t) = \begin{cases} U_1(t)x_0 + \displaystyle\int_0^t (t-s)^{\alpha-1} V_1(t)(t-s)\,G(u)(s)\,ds, \quad t \in J_1, \\ U_1(t-t_1)(u(t_1) + I_1(u(t_1))) + \displaystyle\int_{t_1}^t (t-s)^{q-1} V_1(t-s)\,G(u)(s)\,ds, \quad t \in J_2, \\ \quad \cdots\ \cdots\ \cdots \\ U_1(t-t_m)(u(t_m) + I_m(u(t_m))) + \displaystyle\int_{t_m}^t (t-s)^{q-1} V_1(t-s)G(u)(s)ds,\ t \in J_{m+1}. \end{cases}$$

We define the mapping $Q : [v_0,\, w_0] \to PC(J,\, E)$ by

$$Qu(t) = \begin{cases} U_1(t)x_0 + \int_0^t (t-s)^{\alpha-1} V_1(t)(t-s)\, G(u)(s)\, ds, \ \ t \in J_1, \\ U_1(t-t_1)(u(t_1) + I_1(u(t_1))) + \int_{t_1}^t (t-s)^{q-1} V_1(t-s)\, G(u)(s)\, ds, \ \ t \in J_2, \\ \quad \cdots \cdots \cdots \\ U_1(t-t_m)(u(t_m) + I_m(u(t_m))) + \int_{t_m}^t (t-s)^{q-1} V_1(t-s) G(u)(s) ds, \ t \in J_{m+1}, \end{cases} \tag{3.5}$$

then the mild solution of IVP (1.1) is equivalent to the fixed point of $Q$. Clearly, $Q : [v_0,\, w_0] \to PC(J,\, E)$ is continuous. Since operators $U_1(t)$ and $V_1(t)$ are positive, by the assumptions (H1) and (H2), $Q$ is increasing in $[v_0,\, w_0]$. We use monotone iterative method of increasing operator to find the fixed point of $Q$.

Firstly, we show that $v_0 \le Qv_0$ and $Qw_0 \le w_0$.

Let $h(t) = D_0{}^q v_0(t) + Av_0(t) + Cv_0(t)$. Then $h \in PC(J, E)$ and $v_0$ is the unique mild solution of the the linear impulsive time fractional evolution equation (LIVP)

$$\begin{cases} D_0{}^q u(t) + (A + C\mathrm{I})\, u(t) = h(t), \qquad t \in J,\ t \ne t_k, \\ \Delta u|_{t=t_k} = \Delta v_0|_{t=t_k}, \qquad k = 1, 2, \cdots, m, \\ u(0) = v_0(0) \in E. \end{cases} \tag{3.6}$$

By the definition of lower solution

$$v_0(0) \le x_0, \qquad h(t) \le G(v_0)(t), \quad t \in J',$$

$$\Delta v_0|_{t=t_k} \le I_k(v_0(t_k)), \quad k = 1,\, 2,\, \cdots,\, m.$$

Hence by Lemma 2.3 and the positivity of operators $U_1(t)$ and $V_1(t)$, we have

$$v_0(t) \ = \ \begin{cases} U_1(t)v_0(0) + \int_0^t (t-s)^{\alpha-1} V_1(t)(t-s)h(s)ds, \ t \in J_1, \\ U_1(t-t_1)(v_0(t_1) + \Delta v_0|_{t=t_1}) + \int_{t_1}^t (t-s)^{q-1} V_1(t-s)h(s)ds, \quad t \in J_2, \\ \quad \cdots \cdots \cdots \\ U_1(t-t_m)(v_0(t_m) + \Delta v_0|_{t=t_m}) + \int_{t_m}^t (t-s)^{q-1} V_1(t-s)h(s)ds, \ t \in J_{m+1} \end{cases}$$

$$\leq \begin{cases} U_1(t)v_0(0) + \int_0^t (t-s)^{\alpha-1} V_1(t)(t-s) G(v_0)(s)\, ds,\ t \in J_1, \\ U_1(t-t_1)(v_0(t_1) + I_1(v_0(t_1))) + \int_{t_1}^t (t-s)^{q-1} V_1(t-s) G(v_0)(s)\, ds,\ t \in J_2, \\ \quad \cdots \cdots \cdots \\ U_1(t-t_m)(v_0(t_m) + I_m(v_0(t_m))) + \int_{t_m}^t (t-s)^{q-1} V_1(t-s) G(v_0)(s)\, ds,\ t \in J_{m+1} \end{cases}$$

$$= Q\, v_0(t).$$

This means that $v_0 \leq Qv_0$. Using a similar method, we can prove that $Qw_0 \leq w_0$. Combining these facts and the increasing property of $Q$ in $[v_0, w_0]$, we see that $Q$ maps $[v_0,\, w_0]$ into itself. Hence, $Q : [v_0,\, w_0] \to [v_0,\, w_0]$ is a continuously increasing operator.

Secondly, we prove that the image set $Q([v_0, w_0])$ is equicontinuous in every interval $J_k$, $k = 1,\, 2, \cdots,\, m+1$.

For $\forall\ u \in [v_0,\, w_0]$, by the assumptions (H1) and (H2), we have

$$G(v_0) \leq G(u)(t) \leq G(w_0), \quad t \in J,$$

and

$$v_0(t_k) + I_k(v_0(t_k)) \leq u(t_k) + I_k(u(t_k)) \leq w_0(t_k) + I_k(w_0(t_k)),\ \ k = 1, 2, \cdots, m.$$

Hence by the normality of the cone $P$, there exist positive constant $M^*$ and $L_k$, $k = 1, 2, \cdots, m$, such that

$$\begin{aligned} \|G(u)(t)\| &\leq M^*, \quad t \in J, \\ \|u(t_k) + I_k(u(t_k))\| &\leq L_k, \quad k = 1, 2, \cdots, m. \end{aligned} \tag{3.7}$$

Consider the case of $J_1$. Let $t^{'}, t^{''} \in J_1$ and $0 < t^{'} < t^{''}$. We show that $\|Qu(t^{''}) - Qu(t^{'})\| \to 0$ independently of $u$ as $t^{''} - t^{'} \to 0$. By the definition of $Q$, we have

$$\begin{aligned} Qu\,(t^{''}) - Qu\,(t^{'}) &= U_1(t^{''})x_0 - U_1(t^{'})x_0 \\ &\quad + \int_{t^{'}}^{t^{''}} (t^{''} - s)^{q-1} V_1(t^{''} - s) G(u)(s) ds \\ &\quad + \int_0^{t^{'}} [(t^{''} - s)^{q-1} - (t^{'} - s)^{q-1}] V_1(t^{''} - s) G(u)(s) ds \\ &\quad + \int_0^{t^{'}} (t^{'} - s)^{q-1} [V_1(t^{''} - s) - V_1(t^{'} - s)] G(u)(s) ds \end{aligned}$$

$$= \quad S_{11} + S_{12} + S_{13} + S_{14},$$

where

$$\begin{aligned} S_{11} &= U_1(t^{''})x_0 - U_1(t^{'})x_0, \\ S_{12} &= \int_{t'}^{t''} (t^{''} - s)^{q-1} V_1(t^{''} - s) G(u)(s) ds, \\ S_{13} &= \int_0^{t'} [(t^{''} - s)^{q-1} - (t^{'} - s)^{q-1}] V_1(t^{''} - s) G(u)(s) ds, \\ S_{14} &= \int_0^{t'} (t^{'} - s)^{q-1} [V_1(t^{''} - s) - V_1(t^{'} - s)] G(u)(s) ds. \end{aligned}$$

Since

$$\|Qu\,(t^{''}) - Qu\,(t^{'})\| \leq \|S_{11}\| + \|S_{12}\| + \|S_{13}\| + \|S_{14}\|,$$

we only need to check that $\|S_{1i}\| \to 0$ independently of $u \in [v_0,\, w_0]$ as $t^{''} - t^{'} \to 0$, $i = 1,\, 2,\, 3,\, 4$. For $S_{11}$, by Lemma 2.2 (2), $U_1(t)x_0$ is continuous on $J$, hence it is uniformly continuous on $J$ and we have

$$\|S_{11}\| = \|U_1(t^{''})\,x_0 - U_1(t^{'})\,x_0\| \to 0 \quad (t^{''} - t^{'} \to 0).$$

For $S_{12}$ and $S_{13}$, by (3.2) and (3.7) we have

$$\begin{aligned} \|S_{12}\| &\leq \int_{t'}^{t''} (t^{''} - s)^{q-1} \|V_1(t^{''} - s) G(u)(s)\| ds \\ &\leq \frac{MM^*}{q\,\Gamma(q)} (t^{''} - t^{'})^q \to 0 \quad (t^{''} - t^{'} \to 0). \end{aligned}$$

$$\begin{aligned} \|S_{13}\| &\leq \int_0^{t'} [(t^{'} - s)^{q-1} - (t^{''} - s)^{q-1}]\, \|V_1(t^{''} - s) G(u)(s)\| ds \\ &\leq \frac{MM^*}{\Gamma(q)} \int_0^{t'} [(t^{'} - s)^{q-1} - (t^{''} - s)^{q-1}] ds \\ &= \frac{MM^*}{q\,\Gamma(q)} [(t^{'})^q - (t^{''})^q + (t^{''} - t^{'})^q] \\ &\leq \frac{MM^*}{q\,\Gamma(q)} (t^{''} - t^{'})^q \to 0 \quad (t^{''} - t^{'} \to 0). \end{aligned}$$

For $S_{14}$, using (3.2), (3.7), Lemma 2.2(3) and the Lebesgue bounded convergence theorem of integration, we have

$$\|S_{14}\| \leq \int_0^{t'} (t^{'} - s)^{q-1} \|V_1(t^{''} - s) - V_1(t^{'} - s)\| \cdot \|G(u)(s)\|\, ds.$$

$$\begin{aligned}
&\leq M^* \int_0^{t^{'}} (t^{'}-s)^{q-1} \|V_1(t^{''}-s) - V_1(t^{'}-s)\|\, ds \\
&= M^* \int_0^{t^{'}} r^{q-1} \|V_1(t^{''}-t^{'}+r) - V_1(r)\|\, dr \\
&\leq M^* \int_0^{a} r^{q-1} \|V_1(t^{''}-t^{'}+r) - V_1(r)\|\, dr \to 0 \quad (t^{''}-t^{'}\to 0).
\end{aligned}$$

As a result, $\|Q(u)(t^{''}) - Q(u)(t^{'})\|$ tends to 0 independently of $u \in [v_0, w_0]$ as $t^{''}-t^{'}\to 0$, which means that $Q([v_0, w_0])$ is equicontinuous in the interval $J_1$.

Consider the case of $J_2$. For $t^{'}, t^{''} \in J_2$ with $t^{'} < t^{''}$, we have

$$\begin{aligned}
Qu\,(t^{''}) - Qu\,(t^{'}) &= (\,U_1(t^{''}-t_1) - U_1(t^{'}-t_1)\,)\,(u(t_1) + I_1(u(t_1))\,) \\
&\quad + \int_{t^{'}}^{t^{''}} (t^{''}-s)^{q-1} V_1(t^{''}-s) G(u)(s)\, ds \\
&\quad + \int_{t_1}^{t^{'}} [(t^{''}-s)^{q-1} - (t^{'}-s)^{q-1}] V_1(t^{''}-s) G(u)(s)\, ds \\
&\quad + \int_{t_1}^{t^{'}} (t^{'}-s)^{q-1} [V_1(t^{''}-s) - V_1(t^{'}-s)] G(u)(s)\, ds \\
&= S_{21} + S_{22} + S_{23} + S_{24},
\end{aligned}$$

where

$$\begin{aligned}
S_{21} &= (\,U_1(t^{''}-t_1) - U_1(t^{'}-t_1)\,)\,(u(t_1) + I_1(u(t_1))\,), \\
S_{22} &= \int_{t^{'}}^{t^{''}} (t^{''}-s)^{q-1} V_1(t^{''}-s) G(u)(s)\, ds, \\
S_{23} &= \int_{t_1}^{t^{'}} [(t^{''}-s)^{q-1} - (t^{'}-s)^{q-1}] V_1(t^{''}-s) G(u)(s)\, ds, \\
S_{24} &= \int_{t_1}^{t^{'}} (t^{'}-s)^{q-1} [V_1(t^{''}-s) - V_1(t^{'}-s)] G(u)(s)\, ds.
\end{aligned}$$

It is obvious that

$$\|Qu\,(t^{''}) - Qu\,(t^{'})\| \leq \|S_{21}\| + \|S_{22}\| + \|S_{23}\| + \|S_{24}\|.$$

Therefore, we only need to check that $\|S_{2i}\| \to 0$ independently of $u \in [v_0,\, w_0]$ as $t^{''}-t^{'}\to 0$, $i = 1,\, 2,\, 3,\, 4$. For $S_{21}$, by Lemma 2.2 (3) and (3.7), we have that

$$\|S_{21}\| = \|(\,U_1(t^{''}-t_1) - U_1(t^{'}-t_1)\,)\,(u(t_1) + I_1(u(t_1))\,)\|$$

$$\begin{aligned} &\leq \|U_1(t^{''}-t_1)-U_1(t^{'}-t_1)\|\cdot\|u(t_1)+I_1(u(t_1))\| \\ &\leq L_1\|U_1(t^{''}-t_1)-U_1(t^{'}-t_1)\|\to 0 \quad (t^{''}-t^{'}\to 0). \end{aligned}$$

For $S_{22}$, similarly to $S_{12}$, we have

$$\|S_{22}\|\leq \frac{MM^*}{q\,\Gamma(q)}\,(t^{''}-t^{'})^q\to 0 \quad (t^{''}-t^{'}\to 0).$$

For $S_{23}$, by (3.2) and (3.7) we have

$$\begin{aligned} \|S_{23}\| &\leq \int_{t_1}^{t^{'}}[(t^{'}-s)^{q-1}-(t^{''}-s)^{q-1}]\,\|V_1(t^{''}-s)G(u)(s)\|ds \\ &\leq \frac{MM^*}{\Gamma(q)}\int_{t_1}^{t^{'}}[(t^{'}-s)^{q-1}-(t^{''}-s)^{q-1}]ds \\ &= \frac{MM^*}{q\,\Gamma(q)}\,[(t^{'}-t_1)^q-(t^{''}-t_1)^q+(t^{''}-t^{'})^q] \\ &\leq \frac{MM^*}{q\,\Gamma(q)}\,(t^{''}-t^{'})^q\to 0 \quad (t^{''}-t^{'}\to 0). \end{aligned}$$

For $S_{24}$, by (3.7) and lemma 2.2(3), we have

$$\begin{aligned} \|S_{24}\| &\leq \int_{t_1}^{t^{'}}(t^{'}-s)^{q-1}\|V_1(t^{''}-s)-V_1(t^{'}-s)\|\cdot\|G(u)(s)\|\,ds. \\ &\leq M^*\int_{t_1}^{t^{'}}(t^{'}-s)^{q-1}\|V_1(t^{''}-s)-V_1(t^{'}-s)\|\,ds \\ &= M^*\int_0^{t_1-t^{'}}r^{q-1}\|V_1(t^{''}-t^{'}+r)-V_1(r)\|\,dr \\ &\leq M^*\int_0^{a}r^{q-1}\|V_1(t^{''}-t^{'}+r)-V_1(r)\|\,dr\to 0 \quad (t^{''}-t^{'}\to 0). \end{aligned}$$

Consequently, $\|Qu\,(t^{''})-Qu\,(t^{'})\|$ tends to 0 independently of $u\in[v_0,w_0]$ as $t^{''}-t^{'}\to 0$. This means that $Q([v_0,w_0])$ is equicontinuous in the interval $J_2$.

Continuing such a process interval by interval up to $J_{m+1}$, we can prove that $Q([v_0,\,w_0]$ is equicontinuous in every interval $J_k$, $k=1,\,2,\,\cdots,\,m+1$.

Now, we define two sequences $\{v_n\}$ and $\{w_n\}$ in $[v_0,w_0]$ by the iterative schemes

$$v_n=Q\,v_{n-1},\quad w_n=Q\,w_{n-1},\qquad n=1,\,2,\,\cdots. \tag{3.8}$$

Then from the monotonicity of $Q$, one can easy to prove that

$$v_0 \le v_1 \le \cdots v_n \le \cdots \le w_n \le \cdots \le w_1 \le w_0. \tag{3.9}$$

We prove that $\{v_n\}$ and $\{w_n\}$ are uniformly convergent in $J$.

For convenience, let $B = \{v_n \mid n \in \mathbb{N}\}$ and $B_0 = \{v_{n-1} \mid n \in \mathbb{N}\}$. Since $B = Q(B_0) \subset Q([v_0, w_0])$, so that $B$ is equicontinuous in every interval $J_k$, $k = 2, 3, \cdots, m$. From $B_0 = B \cup \{v_0\}$ it is follows that $\alpha(B_0(t)) = \alpha(B(t))$ for $t \in J$. Denote

$$\varphi(t) = \alpha(B(t)) = \alpha(B_0(t)), \quad t \in J. \tag{3.10}$$

By Lemma 2.5, $\varphi \in PC(J, \mathbb{R}^+)$. We from $J_1$ to $J_{m+1}$ interval by interval show that $\varphi(t) \equiv 0$ in $J$.

For every $t \in J$, by Lemma 2.6 we get that

$$\begin{aligned}
\alpha(T(B_0)(t)) &= \alpha\Big(\Big\{ \int_0^t K(t,s) v_{n-1}(s) ds \;\Big|\; n \in \mathbb{N}\Big\}\Big) \\
&\le 2\int_0^t \alpha(\{K(t,s) v_{n-1}(s) \mid n \in \mathbb{N}\}) ds \\
&= 2\int_0^t K(t,s)\, \alpha(\{v_{n-1}(s) \mid n \in \mathbb{N}\}) ds \\
&\le 2K_0 \int_0^t \varphi(s) ds,
\end{aligned}$$

where $K_0 = \max_{(t,s)\in\Delta} K(t,s)$. Therefore

$$\begin{aligned}
\int_0^t (t-s)^{q-1} \alpha(T(B_0)(s))\, ds &\le 2K_0 \int_0^t (t-s)^{q-1} \Big[ \int_0^s \varphi(r) dr \Big] ds \\
&= \frac{2K_0}{q} \int_0^t (t-r)^q \varphi(r) dr, \\
&\le \frac{2aK_0}{q} \int_0^t (t-s)^{q-1} \varphi(s) ds, \quad t \in J.
\end{aligned} \tag{3.11}$$

For $\forall\, t \in J_1$, by (3.5), using Lemma 2.6, the assumption (H3), (3.2) and (3.11), we have

$$\begin{aligned}
\varphi(t) &= \alpha(B(t)) = \alpha(\, Q(B_0)(t)\,) \\
&= \alpha\Big(\Big\{ U_1(t)x_0 + \int_0^t (t-s)^{q-1} V_1(t-s) G(v_{n-1})(s)\, ds \Big\}\Big) \\
&= \alpha\Big(\Big\{ \int_0^t (t-s)^{q-1} V_1(t-s) G(v_{n-1})(s)\, ds \Big\}\Big)
\end{aligned}$$

$$
\begin{aligned}
&\leq 2\int_0^t \alpha\left(\left\{(t-s)^{q-1}V_1(t-s)\,G(v_{n-1})(s)\,\right\}\right) ds \\
&\leq \frac{2M}{\Gamma(q)}\int_0^t (t-s)^{q-1}\alpha\left(\{\,G(v_{n-1})(s)\;ds\,\}\right) ds \\
&= \frac{2M}{\Gamma(q)}\int_0^t (t-s)^{q-1}\alpha\left(\{\,f(s,v_{n-1},Tv_{n-1}(s))+Cv_{n-1}(s)\,\}\right) ds \\
&\leq \frac{2M}{\Gamma(q)}\int_0^t (t-s)^{q-1}\left[L(\,\alpha(B_0(s))+\alpha(\,T(B_0)(s))\,)+C\alpha(B_0(s))\,\right] ds \\
&= \frac{2M}{\Gamma(q)}(L+C)\int_0^t (t-s)^{q-1}\varphi(s)\,ds+\frac{2M}{\Gamma(q)}\int_0^t (t-s)^{q-1}\alpha(T(B_0)(s))\,ds \\
&\leq \frac{2M}{\Gamma(q)}\Big(L+C+\frac{2aK_0L}{q}\Big)\int_0^t (t-s)^{q-1}\varphi(s)ds.
\end{aligned}
$$

Hence by Lemma 2.7, $\varphi(t)\equiv 0$ in $J_1$. In particular, $\alpha(B(t_1))=\alpha(B_0(t_1))=\varphi(t_1)=0$, this means that $B(t_1)$ and $B_0(t_1)$ are precompact in $E$. Hence, from the continuity of $I_1$ we obtain that $I_1(B_0(t_1))$ is precompact in $E$, and $\alpha(I_1(B_0(t_1)))=0$.

For $\forall\, t\in J_2$, since

$$
\begin{aligned}
\alpha(\{U_1(t-t_1)[v_{n-1}(t_1)+I_1(v_{n-1}(t_1))]\}) &\leq \alpha(U_1(t-t_1)(B_0(t_1)+I_1(B_0(t_1)))\,) \\
&\leq M(\alpha(B_0(t_1))+\alpha(I_1(B_0(t_1)))\,) \;=\; 0,
\end{aligned}
$$

using (3.5) and a similar argument above, we have

$$
\begin{aligned}
\varphi(t) &= \alpha(B(t))=\alpha(\,Q(B_0)(t)\,) \\
&= \alpha\Big(\Big\{U_1(t-t_1)[v_{n-1}(t_1)+I_1(v_{n-1}(t_1))]+\int_{t_1}^t (t-s)^{q-1}V_1(t-s)G(v_{n-1})(s)\,ds\Big\}\Big) \\
&\leq \alpha(\{U_1(t-t_1)[v_{n-1}(t_1)+I_1(v_{n-1}(t_1))]\}) \\
&\quad +\alpha\Big(\Big\{\int_{t_1}^t (t-s)^{q-1}V_1(t-s)G(v_{n-1})(s)\,ds\Big\}\Big) \\
&= \alpha\Big(\Big\{\int_0^t (t-s)^{q-1}V_1(t-s)G(v_{n-1})(s)\,ds\;\Big\}\Big) \\
&\leq 2\int_{t_1}^t \alpha\left(\left\{(t-s)^{q-1}V_1(t-s)\,G(v_{n-1})(s)\,\right\}\right) ds \\
&\leq \frac{2M}{\Gamma(q)}\int_{t_1}^t (t-s)^{q-1}\alpha\left(\{\,G(v_{n-1})(s)\,\}\right) ds
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2M}{\Gamma(q)}\int_{t_1}^{t}(t-s)^{q-1}\alpha\left(\{\, f(s,v_{n-1},Tv_{n-1}(s))+Cv_{n-1}(s)\,\}\right)ds\\
&\le \frac{2M}{\Gamma(q)}\int_{t_1}^{t}(t-s)^{q-1}[\,L(\alpha(B_0(s))+\alpha(T(B_0)(s)))+C\alpha(B_0(s))\,]\,ds\\
&= \frac{2M}{\Gamma(q)}(L+C)\int_{t_1}^{t}(t-s)^{q-1}\varphi(s)\,ds+\frac{2ML}{\Gamma(q)}\int_{t_1}^{t}(t-s)^{q-1}\alpha(T(B_0)(s))\,ds\\
&\le \frac{2M}{\Gamma(q)}(L+C)\int_{0}^{t}(t-s)^{q-1}\varphi(s)\,ds+\frac{2ML}{\Gamma(q)}\int_{0}^{t}(t-s)^{q-1}\alpha(T(B_0)(s))\,ds\\
&\le \frac{2M}{\Gamma(q)}\Big(L+C+\frac{2aK_0L}{q}\Big)\int_{0}^{t}(t-s)^{q-1}\varphi(s)ds.
\end{aligned}
$$

Again by Lemma 2.7, $\varphi(t)\equiv 0$ on $J_2$, from which we obtain that $\alpha(B_0(t_2))=0$, and therefore $\alpha(I_2(B_0(t_2)))=0$. Continuing such a process interval by interval up to $J_{m+1}$, we can prove that $\varphi\equiv 0$ in every $J_k$.

Therefore, for every $J_k$, $\{v_n\}$ is equicontinuous on $J_k$ and $\{v_n(t)\}$ is precompact in $E$ for every $t\in J_k$. By the Arzela-Ascoli theorem, $\{v_n\}$ has a subsequence which is uniformly convergent in $J_k$. Combining this with the monotonicity (3.9), we easily prove that $\{v_n\}$ itself is uniformly convergent in $J_k$, $k=1,\,2,\,\cdots,\,m+1$. Consequently, $\{v_n(t)\}$ is uniformly convergent over the whole of $J$.

Using a similar argument to that for $\{v_n(t)\}$, we can prove that $\{w_n(t)\}$ is also uniformly convergent on $J$. Hence, $\{v_n\}$ and $\{w_n\}$ are convergent in the Banach space $PC(J,E)$. Set

$$\underline{u}=\lim_{n\to\infty}v_n \quad\text{and}\quad \overline{u}=\lim_{n\to\infty}w_n \quad\text{ in }\ PC(J,\,E). \tag{3.12}$$

Letting $n\to\infty$ in (3.8) and (3.9), we see that $v_0\le\underline{u}\le\overline{u}\le w_0$ and

$$\underline{u}=Q\,\underline{u} \quad\text{and}\quad \overline{u}=Q\,\overline{u}. \tag{3.13}$$

By the monotonicity of $Q$, it is easy to prove that $\underline{u}$ and $\overline{u}$ are the minimal and maximal fixed points of $Q$ in $[v_0,\,w_0]$, and therefore, they are the minimal and maximal mild solutions of IVP (1.1) in $[v_0,\,w_0]$, respectively.

This completes the proof of Theorem 3.1. □

In Theorem 3.1, if $E$ is weakly sequentially complete, the condition (H3) holds automatically. In fact, when $E$ is an ordered and weakly sequentially complete Banach space, by Theorem 2.2 in paper [30], we know that any monotonic and order-bounded sequence is precompact. Let $\{x_n\}$ and $\{y_n\}$ be two increasing or decreasing sequences in condition (H3), then by condition (H1), $\{\,f(t,\,x_n,\,y_n)+Cx_n\}$ is monotonic and order-bounded sequence. By the property of measure of

noncompactness, we have

$$\alpha(\{\, f(t,\, x_n,\, y_n)\}) \le \alpha(\{\, f(t,\, x_n,\, y_n) + Cx_n\}) + C\,\alpha(\{x_n\}) = 0.$$

Hence, condition (H3) holds for any $L > 0$. From Theorem 3.1, we obtain that

**Corollary 3.1** *Let $E$ be an ordered and weakly sequentially complete Banach space, and let the positive cone $P$ be normal. Assume that $A : D(A) \subset E \to E$ satisfies the assumption (H0), $f \in C(J \times E \times E,\, E)$, and $I_k \in C(E, E)$, $k = 1, 2, \cdots, m$. If IVP (1.1) has a lower solution $v_0$ and an upper solution $w_0$ with $v_0 \le w_0$ and the assumptions (H1) and (H2) are satisfied, then IVP (1.1) has minimal and maximal mild solutions between $v_0$ and $w_0$, which can be obtained by a monotone iterative procedure starting from $v_0$ and $w_0$, respectively.*

Now we discuss the uniqueness of the mild solution for IVP (1.1) in $[v_0,\, w_0]$. In theorem 3.1, if replacing the assumption (H3) by the condition:

(H4) There exist positive constants $C_1$ and $C_2$ such that

$$f(t,\, x_2,\, y_2) - f(t,\, x_1,\, y_1) \le C_1(x_2 - x_1) + C_2(y_2 - y_1),$$

for $\forall\, t \in J$, and $v_0(t) \le x_1 \le x_2 \le w_0(t),\ Tv_0(t) \le y_1 \le y_2 \le Tw_0(t)$,

we have the following uniqueness result.

**Theorem 3.2** *Let $E$ be an ordered Banach space, and let the positive cone $P$ be normal. Assume that $A : D(A) \subset E \to E$ satisfies the assumption (H0), $f \in C(J \times E \times E,\, E)$, and $I_k \in C(E, E)$, $k = 1, 2, \cdots, m$. If IVP(1.1) has a lower solution $v_0$ and an upper solution $w_0$ with $v_0 \le w_0$ such that conditions (H1), (H2) and (H4) hold, then IVP (1.1) has a unique mild solution between $v_0$ and $w_0$, which can be obtained by a monotone iterative procedure starting from $v_0$ or $w_0$.*

**Proof.** We firstly prove that (H1) and (H4) can deduce (H3). For $t \in J$, let $\{x_n\} \subset [v_0(t),\, w_0(t)]$ and $\{y_n\} \subset [Tv_0(t),\, Tw_0(t)]$ be two increasing sequences. For $m, n \in \mathbb{N}$ with $m > n$, by (H1) and (H4),

$$\begin{aligned} \theta &\le (\, f(t,\, x_m,\, y_m) - f(t,\, x_n,\, y_n)\,) + C(x_m - x_n) \\ &\le (C + C_1)(x_m - x_n) + C_2(y_m - y_n). \end{aligned}$$

By this and the normality of cone $P$, we have

$$\|f(t,\, x_m,\, y_m) - f(t,\, x_n,\, y_n)\|$$

$$\leq N\,\|(C+C_1)(x_m-x_n)+C_2(y_m-y_n)\|+C\,\|x_m-x_n\|$$

$$\leq (C+NC+NC_1)\|x_m-x_n\|+NC_2\|y_m-y_n\|.$$

From this and the definition of the measure of noncompactness, it follows that

$$\begin{aligned}\alpha(\{\,f(t,\,x_n,\,y_n)\}) &\leq (C+NC+NC_1)\,\alpha(\{x_n\})+NC_2\,\alpha(\{y_n\})\\ &\leq L\,(\alpha(\{x_n\})+\alpha(\{y_n\})),\end{aligned}$$

where $L=C+NC+NC_1+NC_2$. If $\{x_n\}$ and $\{y_n\}$ are two decreasing sequences, the above inequality is also valid. Hence (H3) holds.

Therefore, by Theorem 3.1, IVP (1.1) has a minimal mild solution $\underline{u}$ and a maximal mild solution $\overline{u}$ in $[v_0,\,w_0]$. Let $\{v_n\}$ and $\{w_n\}$ be the sequences defined by the iterative scheme (3.8). Then by the proof of Theorem 3.1, we know that (3.9), (3.12) and (3.13) are valid. We show that $\underline{u}(t)\equiv\overline{u}(t)$ on $J$. Set

$$\psi(t)=\|\overline{u}(t)-\underline{u}(t)\|,\qquad t\in J,\tag{3.14}$$

then $\psi\in PC(J,\,\mathbb{R}^+)$. We need to show that $\psi(t)\equiv 0$ on $J$. We from $J_1$ to $J_{m+1}$ interval by interval show that $\psi(t)\equiv 0$ on $J$.

For every $t\in J_1$, using (3.5), (3.8), (3.9) and assumption (H1) and (H4), we obtain that

$$\begin{aligned}\theta &\leq \overline{u}(t)-\underline{u}(t)=Q\overline{u}\,(t)-Q\underline{u}\,(t)\\ &= \int_0^t(t-s)^{q-1}V_1(t-s)\,(\,G(\overline{u})(s)-G(\underline{u})(s)\,)\,ds\\ &\leq \int_0^t(t-s)^{q-1}V_1(t-s)\big((C+C_1)(\overline{u}(s)-\underline{u}(s))+C_2(T\overline{u}(s)-T\underline{u}(s))\big)\,ds.\end{aligned}$$

Hence, by the the normality of cone $P$ and (3.2), we have

$$\begin{aligned}\psi(t) &= \|\overline{u}(t)-\underline{u}(t)\|\\ &\leq N\Big\|\int_0^t(t-s)^{q-1}V_1(t-s)\big((C+C_1)(\overline{u}(s)-\underline{u}(s))+C_2(T\overline{u}(s)-T\underline{u}(s))\big)\,ds\Big\|\\ &\leq \frac{MN}{\Gamma(q)}\Big[(C+C_1)\int_0^t(t-s)^{q-1}\psi(s)\,ds+C_2\int_0^t(t-s)^{q-1}\|T\overline{u}(s)-T\underline{u}(s)\|\,ds\Big]\\ &\leq \frac{MN}{\Gamma(q)}\Big[(C+C_1)\int_0^t(t-s)^{q-1}\psi(s)\,ds+C_2K_0\int_0^t(t-s)^{q-1}\int_0^s\psi(r)drds\Big]\\ &\leq \frac{MN}{\Gamma(q)}\Big[(C+C_1)\int_0^t(t-s)^{q-1}\psi(s)\,ds+\frac{C_2K_0}{q}\int_0^t(t-r)^{q}\psi(r)dr\Big]\end{aligned}$$

$$\begin{aligned} &\leq \frac{MN}{\Gamma(q)}\Big[(C+C_1)\int_0^t (t-s)^{q-1}\psi(s)\,ds + \frac{aC_2K_0}{q}\int_0^t (t-s)^{q-1}\psi(s)ds\Big] \\ &= \frac{MN}{\Gamma(q)}\Big(C+C_1+\frac{aC_2K_0}{q}\Big)\int_0^t (t-s)^{q-1}\psi(s)\,ds. \end{aligned}$$

So we obtain that $\psi(t)\equiv 0$ on $J_1$ by Lemma 2.7.

For $t\in J_2$, noting that $\underline{u}(t_1)=\overline{u}(t_1)$ and $I_1(\overline{u}(t_1))=I_1(\underline{u}(t_1))$, using (3.5) and the same argument as above for $t\in J_1$, by (3.5) we can prove that

$$\begin{aligned} \psi(t) &\leq \frac{NM}{\Gamma(q)}\Big(C+C_1+\frac{aC_2K_0}{q}\Big)\int_{t_1}^t (t-s)^{q-1}\psi(s)ds \\ &\leq \frac{NM}{\Gamma(q)}\Big(C+C_1+\frac{aC_2K_0}{q}\Big)\int_0^t (t-s)^{q-1}\psi(s)ds, \qquad t\in J_2. \end{aligned}$$

Again by Lemma 2.7, we have $\psi(t)\equiv 0$ on $J_2$.

Continuing such a process interval by interval up to $J_{m+1}$, we obtain that $\psi(t)\equiv 0$ over the whole of $J$. Hence, $\underline{u}(t)\equiv\overline{u}(t)$ on $J$ and $\widetilde{u}:=\underline{u}=\overline{u}$ is a unique mild solution of IVP (1.1) in $[v_0,\, w_0]$, which can be obtained by the monotone iterative procedure (3.8) starting from $v_0$ or $w_0$.

This completes the proof of Theorem 3.2. □

**Remark 3.1** *Since the condition (H4) can be more easily verified than (H3), the applications of Theorem 3.2 are convenient.*

# 4 Application

In order to illustrate the applicability of our main results, we consider the initial-boundary value problem of time fractional order parabolic partial differential equation with impulses and integral term

$$\begin{cases} \partial_t{}^q u - \nabla^2 u = g(t,\, x,\, u(t,x),\, Tu(t,x)), \quad (t,x)\in J\times\Omega, \quad t\neq t_k, \\ \Delta u|_{t=t_k} = c_k\, u(t_k,x), \quad x\in\Omega; \quad k=1,\,2,\,\cdots,\,m, \\ u|_{\partial\Omega}=0, \\ u(0,x)=\varphi_0(x), \quad x\in\Omega, \end{cases} \tag{4.1}$$

where $\partial_t{}^q$ is the Caputo fractional order partial derivative of order $q$, $0<q<1$, $\nabla^2$ is the Laplace operator, $J=[0,a]$, $a>0$, $0<t_1<t_2<\cdots<t_m<a$, $\Omega\subset\mathbb{R}^N$ is a bounded domain with a sufficiently smooth boundary $\partial\Omega$, $g\in C(J\times\overline{\Omega}\times\mathbb{R}\times\mathbb{R})$ and satisfies the growth condition

$$|g(t,\,x,\,\xi,\,\eta)|\leq b_0+b_1|\xi|+b_2|\eta|, \qquad (t,\,x,\,\xi,\,\eta)\in J\times\overline{\Omega}\times\mathbb{R}\times\mathbb{R}, \tag{4.2}$$

with positive constants $b_0,\ b_1,\ b_2$, and

$$Tu(t,x) := \int_0^t K(t,s)u(s,x)ds \tag{4.3}$$

is a Volterra integral operator with integral kernel $K \in C(\triangle, \mathbb{R}^+)$, $\triangle := \{(t,s) \in \mathbb{R}^2 \mid 0 \le s \le t \le a\}$; $c_1,\ c_2,\ \cdots,\ c_m$ are positive constants, and the initial value function $\varphi_0 \in L^2(\Omega)$.

Let $E = L^2(\Omega)$, $P = \{u \in L^2(\Omega) \mid u(x) \ge 0,\ a.e.\ x \in \Omega\}$. Then $E$ is a Banach space, $P$ is a normal cone of $E$. We define an operator $A$ in $L^2(\Omega)$ by

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega), \quad Au = -\nabla^2 u, \tag{4.4}$$

From [26, Chapter 7, Theorem 3.2], we know that $-A$ generates a positive and analytic semigroup $S(t)\ (t \ge 0)$ in $E$. Let

$$f(t,\ v,\ w) := g(t,\ x,\ v(x),\ w(x)), \qquad t \in J, \quad v,\ w \in E. \tag{4.5}$$

Then by the condition (4.2), $f : J \times E \times E \to E$ is continuous. Let $I_k = c_k\,\mathrm{I}$, $k = 1,\ 2,\ \cdots,\ m$, where I is the identity operator in $L^2(\Omega)$. For the fuction $u : J \times \Omega \to \mathbb{R}$, let $u(t) = u(t,\cdot)$. Then the equation (4.1) be transformed into the following abstract form of IVP (1.1) in $L^2(\Omega)$

$$\begin{cases} D_0{}^q u(t) + Au(t) = f(t, u(t), Tu(t)), \quad t \in J, \quad t \ne t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k)), \quad k = 1, 2, \cdots, m, \\ u(0) = \varphi_0. \end{cases} \tag{4.6}$$

Let $\lambda_1$ be the first eigenvalue of $A$. It is well known that $\lambda_1 > 0$ and it has a unique positive eigenfunction $\phi_1 \in C^2(\Omega) \cap C_0(\overline{\Omega})$ satisfied $\max_{x \in \overline{\Omega}} \phi_1(x) = 1$. Let

$$\mu(t) = 1 + \textstyle\sum_{t > t_k} c_k, \qquad t \in J, \tag{4.7}$$

then $\mu \in PC(J)$ and $\Delta\mu|_{t=t_k} = c_k$, $k = 1,\ 2,\ \cdots,\ m$. In order to solve the problem (4.1), we make the following assumptions:

(A1) $\varphi_0 \in L^2(\Omega)$ and $0 \le \varphi_0(x) \le \phi_1(x)$ for $a.e.\ x \in \Omega$.

(A2) $g(t,\ x,\ 0,\ 0) \ge 0$ and $g(t,\ x,\ \mu(t)\phi_1(x),\ \phi_1(x)\,T\mu(t)) \le \lambda_1\,\mu(t)\,\phi_1(x)$ for every $(x,\ t) \in J \times \overline{\Omega}$.

(A3) In $J \times \overline{\Omega} \times \mathbb{R} \times \mathbb{R}$, $g(t,\ x,\ \xi,\ \eta)$ has continuous partial derivative $g_\xi{}'(t,\ x,\ \xi,\ \eta)$ and $g_\eta{}'(t,\ x,\ \xi,\ \eta)$.

**Theorem 4.1** *If the assumptions (A1)-(A3) are satisfied, then the equation (4.1) has a unique $L^2$-mild solution between 0 and $\mu(t)\phi_1(x)$.*

**Proof.** Consider IVP (4.6). From Definition 2.2 and the assumptions (A1) and (A2) we see that $v_0(t) \equiv 0$ is a lower solution of IVP (4.6) and $w_0(t) = \mu(t)\phi_1$ is an upper solution of IVP (4.6). From (A3) it is easy to verify that $f$ satisfies the assumption (H1) and (H4). Clearly, for $I_k = c_k\mathrm{I}$, $k = 1, 2, \cdots, m$, (H2) holds. Therefore, by Theorem 3.2, IVP (4.6) has a unique mild solution between $v_0$ and $w_0$, that is, the equation (4.1) has a unique $L^2$-mild solution between 0 and $\mu(t)\phi_1(x)$. □

# References


[1] B. Lundstrom, M. Higgs, W. Spain, A. Fairhall, Fractional by neocortial pyramidal neurons. Nat Neurosci. 11(2008), 1335-1342.

[2] Y. Rossikhin, M. Shitikova, Application of fractional derivatives to the analysis of damped vibrationsof viscoelaticsingle mass system. Acta Mech. 120(1997), 109-125.

[3] X. Zhang, L. Liu, Y. Wu, The uniqueness of positive solution for a fractional order model of turbulent flow in a porous medium, Appl. Math. Lett. 37(2014), 26-33.

[4] X. Zhang, Positive solutions for a class of singular fractional differential equation with infinite points boundary value conditions, Appl. Math. Lett. 39(2015), 22-27.

[5] P. Chen, Y. Li, Nonlocal problem for fractional evolution equations of mixed type with the measure of noncompactness, Abst. Appl. Anal. Vol. 2013, Article ID 784816, 12 pages.

[6] Z. Ouyang, Existence and uniqueness of the solutions for a class of nonlinear fractional order partial differential equations with delay, Comput. Math. Appl. 61(2011), 860-870.

[7] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, in: North-Holland Mathematics Studies, vol. 204, Elsevier Science B.V., Amsterdam, 2006.

[8] C. Lizama, Solutions of two-term time fractional order differential equations with nonlocal initial conditions, Electron. J. Qual. Theo. Diff. Equat. 82(2012), 1-9.

[9] C. Lizama, An operator theoretical approach to a class of fractional order differential equations. Appl. Math. Lett. 24(2)(2011), 184-190.

[10] J. Wang, Y. Zhou, W. Wei, Optimal feedback control for semilinear fractional evolution equations in Banach spaces, Syst. Contr. Lett. 61(2012), 472-476.

[11] M.M. EI-Borai, Some probability densities and funddamental solutions of fractional evolution equations, Chaos, Solitons and Fractals, 14(2002), 433-440.

[12] M.M. El-Borai, K.E. El-Nadi, E.G. El-Akabawy, On some fractional evolution equations, Comput. Math. Appl. 59(2010), 1352-1355.

[13] P. Chen, Y. Li, Existence of mild solutions for fractional evolution equations with mixed monotone nonlocal conditions, Z. Angew. Math. Phys. 65(2014), 711-728.

[14] P. Chen, Y. Li, Q. Li, Existence of mild solutions for fractional evolution equations with nonlocal initial conditions, Ann. Polon. Math. 110(2014), 13-24.

[15] R. Wang, T.J. Xiao, J. Liang, A note on the fractional Cauchy problems with nonlocal conditions, Appl. Math. Lette. 24(2011), 1435-1442.

[16] K. Balachandran, J.J. Trujillo, The nonlocal Cauchy problem for nonlinear fractional integrodifferential equations in Banach spaces, Nonlinear Anal. 72(2010), 4587-4593.

[17] S. Du, V. Lakshmikantham, Monotone iterative technique for differential equtions in Banach spaces, J. Math. Anal. Appl. 87(1982), 454-459.

[18] Y. Li, Existence of solutions to initial value problems for abstract semilinear evolution equations, Acta Mathematica Sinica, 48(2005), 1089-1094.

[19] Y. Li, The positive solutions of abstract semilinear evolution equations and their applications, Acta Mathematica Sinica, 39(1996), 666-672.

[20] P. Chen, Y. Li, Monotone iterative technique for a class of semilinear evolution equations with nonlocal conditions, Results Math. 63(2013), 731-744.

[21] P. Chen, Y. Li, Mixed monotone iterative technique for a class of semilinear impulsive evolution equations in Banach spaces, Nonlinear Anal. 74(2011), 3578-3588.

[22] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, Theory of Impulsive Differential Equations, World Scientific,Singapore, 1989.

[23] J. Liang, J. H. Liu and T.J. Xiao, Nonlocal impulsive problems for nonlinear differential equations in Banach spaces, Math. Comput. Model. 49(2009), 798-804.

[24] P. Chen, Y. Li, H. Yang, Perturbation method for nonlocal impulsive evolution equations, Nonlinear Anal. Hybrid Syst. (8)2013, 22-30.

[25] J. Banasiak, L. Arlotti, Perturbations of Positive Semigroups with Applications, Springer-verlag, London, 2006.

[26] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-verlag, Berlin, 1983.

[27] K. Deimling, Nonlinear Functional Analysis, Springer-Verlag, New York, 1985.

[28] H.P. Heinz, On the behaviour of measure of noncompactness with respect to differentiation and integration of rector-valued functions, Nonlinear Anal. 7(1983), 1351-1371.

[29] H. Ye, J. Gao, Y. Ding, A generalized Gronwall inequality and its application to a fractional differential equation, J. Math. Anal. Appl. 328(2007), 1075-1081.

[30] Y. Du, Fixed pionts of increasing operators in Banach spaces and applications, Appl. Anal. 38 (1990),1-20.

# EVP, MINIMAX THEOREMS AND EXISTENCE OF NONCONVEX EQUILIBRIA IN COMPLETE G-METRIC SPACES

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ABSTRACT. We prove generalized EVP (Ekeland's variational principle) and generalized Takahashi's nonconvex minimization theorem by Ω-distance on G-metric spaces. As a result of last theorems, we get generalized flower petal theorem.

## 1. INTRODUCTION

EVP, studied first one in 1972, has found a multitude of applications in different fields of analysis. It has also served to provide simple and elegant proofs of known results. The best references for those are by Ekeland himself: his survey article [2], his book with Aubin [1] and [4].

## 2. EVP

**Definition 2.1.** [3] Let $X \neq \varnothing$. The function $G : X \times X \times X \to [0, \infty)$ is said to be G-metric when

(i) $G(u, v, w) = 0$ if $u = v = w$ (coincidence),
(ii) $G(u, u, v) > 0$ for all $u, v \in X$, where $u \neq v$,
(iii) $G(u, u, w) \leq G(u, v, w)$ for all $u, v, w \in X$, with $w \neq v$,
(iv) $G(u, v, w) = G(P\{u, v, w\})$, where $p$ is a permutation of $u, v, w$ (symmetry),
(v) $G(u, v, w) \leq G(u, a, a) + G(a, v, w)$ for all $u, v, w, a \in X$ (rectangle inequality).

In this paper, $\varphi : (-\infty, \infty) \to (0, \infty)$ is a nondecreasing function. We say the function $h : X \to (-\infty, \infty]$ is lower semicontinuous from above (shortly lsca) at $w_0 \in X$ when for every sequence $\{w_n\}$ in $X$ with $w_n \to w_0$ and $h(w_1) \geq h(w_2) \geq \cdots \geq h(w_n) \geq \cdots$, we have

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$h(w_0) \leq \lim_{n\to\infty} h(w_n)$. We say $h$ is lsca on $X$ when $h$ is lsca at every point of $X$. We say $h$ is proper if $h \neq \infty$.

**Definition 2.2.** [3] Let $(X, G)$ be a G-metric space.

(1) A sequence $\{u_n\}$ in $X$ is a G-Cauchy sequence when, for every $\varepsilon > 0$, there exists a positive integer $n_0$ such that if $m, n, l \geq n_0$, then $G(u_m, u_n, u_l) < \varepsilon$.

(2) A sequence $\{u_n\}$ in $X$ is G-convergent to a point $u \in X$ when for every $\varepsilon > 0$, there exists a positive integer $n_0$ such that for all $m, n \geq n_0$ we have $G(u_m, u_n, u) < \varepsilon$.

**Definition 2.3.** [5] Let $(X, G)$ be a G-metric space. A function $\Omega : X \times X \times X \to [0, \infty)$ is said to be an $\Omega$-distance on $X$ when

(a) $\Omega(u, v, w) \leq \Omega(u, a, a) + \Omega(a, v, w)$ for all $u, v, w \in X$;

(b) For any $u \in X$, $\Omega(u, ., .) : X \to [0, \infty)$ is lower semicontinuous;

(c) For each $\varepsilon > 0$, there exists a $\delta > 0$ such that $\Omega(u, a, a) \leq \delta$ and $\Omega(a, v, w) \leq \delta$ imply $G(u, v, w) \leq \varepsilon$.

**Example 2.4.** [5] Let $(X, d)$ be a metric space and $G : X^3 \to [0, \infty)$ defined by $G(u, v, w) = \max\{d(u, v), d(u, w), d(v, w)\}$ for all $u, v, w \in X$. Then $\Omega = G$ is an $\Omega$-distance on $X$.

**Lemma 2.5.** [5] *Let $(X, G)$ be a G-metric space and $\Omega$ an $\Omega$-distance on $X$. Let also $\{u_n\}$, $\{v_n\}$ be sequences in $X$, $\{\alpha_n\}$ and $\{\beta_n\}$ sequences in $[0, \infty)$ converging to zero and let $u, v, w, a \in X$. Then we have*

(1) *if $\Omega(v, u_n, u_n) \leq \alpha_n$ and $\Omega(u_n, v, w) \leq \beta_n$ for $n \in \mathbb{N}$, then $G(v, v, w) < \varepsilon$ and hence $w = v$;*
(2) *if $\Omega(v_n, u_n, u_n) \leq \alpha_n$ and $\Omega(u_n, u_m, w) \leq \beta_n$ for any $m > n \in \mathbb{N}$, Then $G(v_n, v_m, w) \to 0$ and hence $v_n \to w$;*
(3) *if $\Omega(u_n, u_m, u_l) \leq \alpha_n$ for any $l, m, n \in \mathbb{N}$ with $n \leq m \leq l$, then $\{u_n\}$ is a G-Cauchy sequence;*
(4) *if $\Omega(u_n, a, a) \leq \alpha_n$ for any $n \in \mathbb{N}$, then $\{u_n\}$ is a G-Cauchy sequence.*

**Lemma 2.6.** *Let $\Omega$ be an $\Omega$-distance on $X \times X \times X$. If $\{u_n\}$ is a sequence in $X$ with $\limsup_{n\to\infty}\{\Omega(u_n, u_m, u_l) : n \leq m \leq l\} = 0$, then $\{u_n\}$ is a G-Cauchy sequence in $X$.*

*Proof.* Suppose $\alpha_n = \sup\{\Omega(u_n, u_m, u_l)\}$. We have $\lim_{n\to\infty} \alpha_n = 0$. By Lemma 2.5, we obtain that $\{u_n\}$ is a G-Cauchy sequence in $X$. □

**Lemma 2.7.** *Let $f : X \to (-\infty, \infty]$ be a function and $\Omega$ a $\Omega$-distance on $X \times X \times X$. We define the set $P(u)$ by*

$$P(u) = \{v \in X : v \neq u,\ \Omega(u, v, v) \leq \varphi(f(u))(f(u) - f(v))\}.$$

*If $v \in P(u)$, then we have $f(v) \leq f(u)$ and $P(v) \subseteq P(u)$.*

*Proof.* Let $v \in P(u)$. Then $v \neq u$ and $\Omega(u, v, v) \leq \varphi(f(u))(f(u)-f(v))$. Since $\Omega(u, v, v) \geq 0$ for any $u, v \in X$ and $\varphi$ is nondecreasing in $(0, \infty)$, we have $f(u) \geq f(v)$. If $P(v) = \emptyset$, then $P(v) \subseteq P(u)$. If $P(v) \neq \emptyset$, then let $w \in P(v)$. We have $w \neq v$ and $\Omega(v, w, w) \leq \varphi(f(v))(f(v) - f(w))$. Then, we have $f(v) \geq f(w)$. Also we have

$$\Omega(u, w, w) \leq \Omega(u, v, v) + \Omega(v, w, w) \leq \varphi\big(f(u)\big)\big(f(u) - f(w)\big).$$

We claim that $w \neq u$. Let $w = u$; then $\Omega(u, w, w) = 0$. On the other hand

$$\Omega(u, v, v) \leq \varphi\big(f(u)\big)\big(f(u) - f(v)\big) \leq \varphi\big(f(u)\big)\big(f(u) - f(w)\big) = 0.$$

Then $\Omega(u, v, v) = 0$; for each $\varepsilon > 0$, we have $\Omega(u, w, w) = 0 < \delta$, $\Omega(w, v, v) = 0 < \delta \Longrightarrow G(w, v, v) < \varepsilon$. Then $G(w, v, v) = 0$ and $w = v$, which is a contradiction. Therefore $w \in P(u)$ and hence $P(v) \subseteq P(u)$. □

**Proposition 2.8.** *Let $f : X \to (-\infty, \infty]$ be a proper lsca and bounded from below function. Let also $\Omega$ be an $\Omega$-distance on $X \times X \times X$. For each $u \in X$, let*

$$P(u) = \big\{v \in X : v \neq u,\ \Omega(u, v, v) \leq \varphi\big(f(u)\big)\big(f(u) - f(v)\big)\big\}.$$

*If $\{u_n\}$ is a sequence in $X$ such that $P(u_n)$ is a nonempty set and $u_{n+1} \in P(u_n)$ for all $n \in \mathbb{N}$, then there exists $u_0 \in X$ such that $u_n \to u_0$ and $u_0 \in \bigcap_{n=1}^{\infty} P(u_n)$.*

*Also, if $f(u_{n+1}) \leq \inf_{w \in P(u_n)} f(w) + 1/n$ for all $n \in \mathbb{N}$, then $\bigcap_{n=1}^{\infty} P(u_n)$ has only one point.*

*Proof.* First we show that $\{u_n\}$ is a Cauchy sequence. Whereas $u_{n+1} \in P(u_n)$, by Lemma 2.7, $f(u_n) \geq f(u_{n+1})$ for all $n \in \mathbb{N}$, so $\{f(u_n)\}$ is nonincreasing. On the other hand, $f$ is bounded from below; then $r = \lim_{n\to\infty} f(u_n)$, so $f(u_n) \geq r$ for all $n \in \mathbb{N}$.

We show that $\limsup_{n\to\infty}\{\Omega(u_n, u_m, u_m) : m > n\} = 0$. We have

$$\begin{aligned}\Omega(u_n, u_m, u_m) &\leq \Omega(u_n, u_{n+1}, u_{n+1}) + \Omega(u_{n+1}, u_m, u_m)\\ &\leq \Omega(u_n, u_{n+1}, u_{n+1}) + \Omega(u_{n+1}, u_{n+2}, u_{n+2}) + \Omega(u_{n+2}, u_m, u_m)\end{aligned}$$

Therefore $\Omega(u_n,u_m,u_m)\leq\sum_{j=n}^{m-1}\Omega(u_j,u_{j+1},u_{j+1})\leq\varphi\big(f(u_1)\big)\big(f(u_n)-r\big)$, for all $m,n\in\mathbb{N}$ with $m>n$. Put $\alpha_n=\varphi\big(f(u_1)\big)\big(f(u_n)-r\big)$ for all $n\in\mathbb{N}$. We have $\sup\{\Omega(u_n,u_m,u_m):m>n\}\leq\alpha_n$ for all $n\in\mathbb{N}$. So $\lim_{n\to\infty}f(u_n)=r$. We have $\lim_{n\to\infty}\alpha_n=0$ and

$$\lim_{n\to\infty}\sup\{\Omega(u_n,u_m,u_m):m>n\}=0.$$

By Lemma 2.6, $\{u_n\}$ is a G-Cauchy sequence, and $X$ is a G-complete metric space, so there exists $u_0\in X$ such that $u_n\to u_0$. We show that $u_0\in\bigcap_{n=1}^{\infty}P(u_n)$. Since $f$ is lsca, then $f(u_0)\leq\lim_{n\to\infty}f(u_n)=r\leq f(u_k)$. Suppose that $n\in\mathbb{N}$ is fixed for all $m\in\mathbb{N}$ with $m>n$. We have $\Omega(u_n,u_m,u_m)\leq\sum_{j=n}^{m-1}\Omega(u_j,u_{j+1},u_{j+1})\leq\varphi\big(f(u_n)\big)\big((f(u_n)-f(u_0)\big)$. Since $\Omega(u,.,.):X\to(0,\infty)$ is lower semi continuous, then

$$\Omega(u_n,u_0,u_0)\leq\varphi\big(f(u_n)\big)\big(f(u_n)-f(u_0)\big).\tag{2.1}$$

Also $u_0\neq u_n$ for all $n\in\mathbb{N}$. Suppose contrary, that there exists $j\in\mathbb{N}$ such that $u_0=u_j$. Since $\Omega(u_j,u_{j+1},u_{j+1})\leq\varphi\big(f(u_j)\big)\big(f(u_j)-f(u_{j+1})\big)\leq\varphi\big(f(u_j)\big)\big(f(u_j)-f(u_0)\big)=0$, so we would have $\Omega(u_j,u_{j+1},u_{j+1})=0$. Similarly, we would have $\Omega(u_{j+1},u_{j+2},u_{j+2})=0$. Now, for $\varepsilon>0$, we would have $\Omega(u_j,u_{j+1},u_{j+1})=0<\delta$ and $\Omega(u_{j+1},u_{j+2},u_{j+2})=0<\delta$. Then $G(u_j,u_{j+2},u_{j+2})<\varepsilon$, and by Definition 2.2, we would have $u_j=u_{j+2}$, which is contradiction.

Since $u_{j+1}\in P(u_j)$, then $P(u_{j+1})\subseteq P(u_j)$ and $u_{j+2}\in P(u_{j+1})$, so $u_{j+2}\in P(u_j)$ and therefore $u_{j+2}\neq u_j$. We conclude that $u_0\neq u_n$ for all $n\in\mathbb{N}$. By (2.1) we have $u_0\in\bigcap_{n=1}^{\infty}P(u_n)$, thus $\bigcap_{n=1}^{\infty}P(u_n)\neq\emptyset$. Now we assume that $f(u_{n+1})\leq\inf_{w\in P(u_n)}f(w)+1/n$ for all $n\in\mathbb{N}$. We show that $\bigcap_{n=1}^{\infty}P(u_n)=\{u_0\}$. Let $t\in\bigcap_{n=1}^{\infty}P(u_n)$; then

$$\begin{aligned}\Omega(u_n,t,t)&\leq\varphi\big(f(u_n)\big)\big(f(u_n)-f(t)\big)\\&\leq\varphi\big(f(u_1)\big)\big(f(u_n)-\inf_{w\in S(u_n)}f(w)\big)\\&\leq\varphi\big(f(u_1)\big)\big(f(u_n)-f(u_{n+1})+1/n\big).\end{aligned}$$

Let $\beta_n=\varphi\big(f(u_1)\big)\big(f(u_n)-f(u_{n+1})+1/n\big)$, for all $n\in\mathbb{N}$. Then $\lim_{n\to\infty}\beta_n=0$, thus $\lim_{n\to\infty}\Omega(u_n,t,t)=0$; also $\{u_m\}$ is G-Cauchy. Then $\lim_{n\to\infty}\Omega(u_m,u_m,u_n)=0$, and we obtain $u_n\to t$. By uniqueness, we have $t=u_0$. Then $\bigcap_{n=1}^{\infty}P(u_n)=\{u_0\}$. □

**Theorem 2.9** (Generalized EVP)**.** *Let $f:X\to(-\infty,\infty]$ be a proper lsca and bounded from below function. Let also $\Omega$ be an $\Omega$-distance on $X\times X\times X$. Then there exists $t\in X$ such that*

$$\Omega(t,u,u)>\varphi\big(f(t)\big)\big(f(t)-f(u)\big)\ \textit{for all}\ u\in X\ \textit{with}\ u\neq t.$$

*Proof.* Suppose contrary, that for each $u \in X$, there exists $v \in X$ with $v \neq u$ such that $\Omega(u,v,v) \leq \varphi\big(f(u)\big)\big(f(u)-f(v)\big)$. That would mean that $P(u) \neq \emptyset$ for each $u \in X$. Since $f$ is proper, there would exist $u \in X$ such that $f(u) \neq \infty$. We define a sequence $\{u_n\}$ as follows: let $u_1 = x$, and choose $u_2 \in P(u_1)$ such that $f(u_2) \leq \inf_{u\in P(u_1)} f(u) + 1$. Suppose $u_n \in X$ is so defined, and choose $u_{n+1} \in P(u_n)$ such that $f(u_{n+1}) \leq \inf_{u\in P(u_n)} f(u) + 1/n$. By Proposition 2.8, there would exist $u_0 \in X$ such that $\bigcap_{n=1}^{\infty} P(u_n) = \{u_0\}$. By Lemma 2.7, $P(u_0) \subseteq \bigcap_{n=1}^{\infty} P(u_n) = \{u_0\}$, so $P(u_0) = \{u_0\}$, which is a contradiction. Therefore, there exists $t \in X$ such that

$$\Omega(t,u,u) > \varphi\big(f(t)\big)\big(f(t)-f(u)\big) \text{ for all } u \in X \text{ with } u \neq t. \qquad \square$$

**Theorem 2.10** (Generalized Caristi's common fixed point theorem for a family of multivalued maps)**.** *Let $f : X \to (-\infty,\infty]$ be a proper lsca and bounded from below function. Let also $\Omega$ be an $\Omega$-distance on $X \times X \times X$. Let $J$ be any index set and for each $j \in J$, suppose $P_j : X \to 2^X$ is a multivalued map with nonempty values such that for each $u \in X$, there exists $v = v(u,j) \in P_j(u)$ with*

$$\Omega(u,v,v) \leq \varphi\big(f(u)\big)\big(f(u)-f(v)\big). \tag{2.2}$$

*Then there exists $t \in X$ such that $t \in \bigcap_{j\in J} P_j(t)$, and $\Omega(t,t,t) = 0$.*

*Proof.* By Theorem 2.9, there exists $t \in X$ such that $\Omega(t,u,u) > \varphi\big(f(t)\big)\big(f(t)-f(u)\big)$ for all $u \in X$ with $u \neq t$. Now we show that $t \in \bigcap_{j\in J} P_j(t)$ and $\Omega(t,t,t) = 0$. According to the assumption, there exists $w(t,j) \in P_j(t)$ such that $\Omega(t,w,w) \leq \varphi\big(f(t)\big)\big(f(t)-f(w(t,j))\big)$. We claim that $w(t,j) = t$, for all $j \in J$. If, on the contrary, $w(t,j_0) \neq t$ for some $j_0 \in J$, then

$$\Omega(t,w,w) \leq \varphi\big(f(t)\big)\big(f(t)-f(w)\big) < \Omega(t,w,w).$$

which is a contradiction. Therefore $t = w(t,j) \in P_j(t)$ for all $j \in P$.

Since $\Omega(t,t,t) \leq \varphi\big(f(t)\big)\big(f(t)-f(t)\big) = 0$, we obtain $\Omega(t,t,t) = 0$. $\square$

**Corollary 2.11** (Generalized Caristi's common fixed point theorem for a family of single-valued maps)**.** *Let $f : X \to (-\infty,\infty]$ be a proper lsca and bounded from below function. Let also $\Omega$ be an $\Omega$-distance on $X \times X \times X$. Let $J$ be any index set and for each $j \in J$, let $g_j : X \to X$ be a single-valued map so that*

$$\Omega\big(u,g_j(u),g_j(u)\big) \leq \varphi\big(f(u)\big)\big(f(u)-f(g_j(u))\big) \tag{2.3}$$

*is established for each $u \in X$. Then there exists $t \in X$ such that $g_j(t) = t$ for each $j \in J$ and $\Omega(t,t,t) = 0$.*

*Proof.* Let $P_j : X \to X$ and $P_j(x) = \{g_j(u)\}$, for all $u \in X$ and $j \in J$. Then by Theorem 2.10, $g_j(t) = t$ for each $j \in J$ and $\Omega(t, t, t) = 0$. □

**Remark 2.12.** (a) Corollary 2.11 implies Theorem 2.10.

Suppose that for each $u \in X$, there exists $v(u, j) \in P_j(u)$ such that

$$\Omega\big(u, v(u, j), v(u, j))\big) \leq \varphi\big(f(u)\big)\big(f(u) - f(v(u, j))\big)$$

for each $j \in J$, and let $g_j(u) = v(u, j)$. Then $g_j$ is single-valued map and

$$\Omega\big(u, g_j(u), g_j(u))\big) \leq \varphi\big(f(u)\big)\big(f(u) - f(g_j(u))\big)$$

for all $u \in X$. By Corollary 2.11, there exists $t \in X$ such that $t = g_j(t) \in P_j(t)$ for each $j \in J$ and $\Omega(t, t, t) = 0$

(b) Theorem 2.10 implies Theorem 2.9.

Suppose contrary, that for each $u \in X$, there exists $v \in X$ with $v \neq u$ such that

$$\Omega(u, v, v) \leq \varphi\big(f(u)\big)\big(f(u) - f(v)\big).$$

Define $P : X \to 2^X \setminus \{\emptyset\}$ by

$$P(u) = \{v \in X : v \neq u, \Omega(u, v, v) \leq \varphi\big(f(u)\big)\big(f(u) - f(v)\big)\}.$$

By Theorem 2.10, $P$ has a fixed point $t \in X$; this means $t \in P(t)$. This is a contradiction, because $t \notin P(t)$.

**Theorem 2.13** (Nonconvex maximal element theorem for a family of multivalued maps)**.** *Let $f : X \to (-\infty, \infty]$ be a proper lsca and bounded from below function. Let also $\Omega$ be an $\Omega$-distance on $X \times X \times X$ and $J$ be any index set. For each $j \in J$, let $P_j : X \to 2^X$ be a multivalued map. Suppose that for each $(u, j) \in X \times J$ with $P_j(u) \neq \emptyset$, there exists $v = v(u, j) \in X$ with $v \neq u$ such that* (2.2) *holds. Then there exists $t \in X$ such that $P_j(t) = \emptyset$ for each $j \in J$.*

*Proof.* By Theorem 2.9, there exists $t \in X$, such that $\Omega(t, u, u) > \varphi\big(f(t)\big)\big(f(t) - f(u)\big)$ for all $u \in X$ with $u \neq t$. We prove that $P_j(t) = \emptyset$ for each $j \in J$. Indeed, if $P_{j_0}(t) \neq \emptyset$, for some $j_0 \in J$, according to the assumption, there would exist $w = w(t, j_0) \in X$ with $w \neq t$ such that $\Omega(t, w, w) \leq \varphi\big(f(t)\big)\big(f(t) - f(w)\big)$. Also $\Omega(t, w, w) > \varphi\big(f(t)\big)\big(f(t) - f(w)\big)$, which is a contradiction. □

**Remark 2.14.** Theorem 2.13 implies Theorem 2.9.

Suppose contrary, that for each $u \in X$, there exists $v \in X$ with $v \neq u$ such that

$$\Omega(u,v,v) \leq \varphi\big(f(u)\big)\big(f(u)-f(v)\big).$$

For each $u \in X$, we define

$$P(u) = \{v \in X : v \neq u,\ \Omega(u,v,v) \leq \varphi\big(f(u)\big)\big(f(u)-f(v)\big)\}.$$

Then $P(u) \neq \emptyset$ for all $u \in X$. But by Theorem 2.13, there would exist $t \in X$ such that $P(t) = \emptyset$, which is a contradiction.

## 3. Nonconvex optimization and minimax theorems

**Theorem 3.1** (Generalized Takahashi's nonconvex minimization theorem)**.** *Let $f : X \to (-\infty, \infty]$ be a proper lsca and bounded from below function. Also, let $\Omega$ be an $\Omega$-distance on $X \times X \times X$. Suppose that for any $u \in X$ with $f(u) > \inf_{w\in X} f(w)$ there exists $v \in X$ with $v \neq u$ such that* (2.2) *holds. Then there exists $t \in X$ such that $f(t) = \inf_{w\in X} f(w)$.*

*Proof.* By Theorem 2.9, there exists $t \in X$ such that $\Omega(t,u,u) > \varphi\big(f(t)\big)\big(f(t)-f(u)\big)$, for all $u \in X$, $u \neq t$. Now we prove that $f(t) = \inf_{w\in X} f(w)$. On the contrary, let $f(t) > \inf_{w\in X} f(w)$. According to the assumption, there would exist $v = v(t) \in X$, with $v \neq t$ such that $\Omega(t,v,v) \leq \varphi\big(f(t)\big)\big(f(t)-f(v)\big)$. Then we would have

$$\Omega(t,v,v) \leq \varphi\big(f(t)\big)\big(f(t)-f(v)\big) < \Omega(t,v,v)$$

which is a contradiction. □

**Remark 3.2.** Using Theorem 3.1, we can infer Theorem 2.9.

If we could not, then for each $u \in X$, there would exist $v \in X$ with $v \neq u$ such that $\Omega(u,v,v) \leq \varphi\big(f(u)\big)\big(f(u)-f(v)\big)$. By Theorem 3.1, there would exist $t \in X$ such that $f(t) = \inf_{w\in X} f(w)$. According to the assumption, there would exist $z \in X$ with $z \neq u$, such that $\Omega(t,z,z) \leq \varphi\big(f(t)\big)\big(f(t)-f(z)\big) \leq 0$. Then $\Omega(t,z,z) = 0$ and $f(t) = f(z) = \inf_{w\in X} f(w)$. There would exist $w \in X$ with $w \neq z$ such that $\Omega(z,w,w) \leq \varphi\big(f(z)\big)\big(f(z)-f(w)\big) \leq 0$. Then we would have $\Omega(z,w,w) = 0$ and $f(t) = f(z) = f(w) = \inf_{u\in X} f(u)$. Since $\Omega(t,w,w) \leq \Omega(t,z,z) + \Omega(z,w,w)$, then $\Omega(t,w,w) = 0$. For $\varepsilon > 0$ we would have $\Omega(t,z,z) = 0 < \delta$, $\Omega(z,w,w) = 0 < \delta$; then $G(t,w,w) < \varepsilon$, that is, $t = w$. Also for $\varepsilon > 0$ we would have $\Omega(z,t,t) = 0 < \delta$, $\Omega(t,w,w) = 0 < \delta$; then $G(z,w,w) < \varepsilon$ that is, $z = w$, which is a contradiction.

**Theorem 3.3** (Nonconvex minimax theorem)**.** *Let $G : X \times X \to (-\infty, \infty]$ be a proper lsca and bounded from below function in the first argument. Suppose that for each $u \in X$ with $\{x \in X : G(u,x) > \inf_{a\in X} G(a,x)\} \neq \emptyset$, there exists $v = v(u) \in X$ with $v \neq u$ such that*

$$\Omega(u,v,v) \leq \varphi\big(G(u,t)\big)\big(G(u,t) - G(v,t)\big) \tag{3.1}$$

*for all $t \in \{x \in X : G(u,x) > \inf_{a\in X} G(a,x)\}$.*

*Then $\inf_{u\in X} \sup_{v\in X} G(u,v) = \sup_{v\in X} \inf_{u\in X} G(u,v)$.*

*Proof.* By Theorem 3.1, for every $v \in X$, there exists $u(v) \in X$ such that $G\big(u(v),v\big) = \inf_{u\in X} G(u,v)$. Then, $\sup_{v\in X} G\big(u(v),v\big) = \sup_{v\in X} \inf_{u\in X} G(u,v)$.

Replacing $u(v)$ by an arbitrary $u \in X$, we obtain

$$\inf_{u\in X} \sup_{v\in X} G(u,v) = \sup_{v\in X} \inf_{u\in X} G(u,v). \qquad \square$$

**Theorem 3.4** (Nonconvex equilibrium theorem)**.** *Let $G$ and $\varphi$ be the same as in Theorem* 3.3*. Let for each $u \in X$ with $\{x \in X : G(u,x) < 0\} \neq \emptyset$, there exist $v = v(u) \in X$ with $v \neq u$ such that* (3.1) *holds for all $t \in X$. Then there exists $y \in X$ such that $G(y,v) \geqslant 0$ for all $v \in X$.*

*Proof.* By Theorem 2.9, for each $w \in X$, there exists $y(w) \in X$ such that $\Omega(y(w),u,u) > \varphi\big(G(y(w),w)\big)\big(G(y(w),w) - G(u,w)\big)$ for all $u \in X$ with $u \neq y(w)$. We show that there exists $y \in X$ such that $G(y,v) \geq 0$ for all $v \in X$. Suppose contrary, that for each $u \in X$ there exists $v \in X$ such that $G(u,v) < 0$. Then for each $u \in X$, $\{x \in X : G(u,x) < 0\} \neq \emptyset$. According to the assumption, there would exist $v = v(y(w))$, $y \neq y(w)$ such that $\Omega(y(w),v,v) \leq \varphi\big(G(y(w),w)\big)\big(G(y(w),w) - G(v,w)\big)$, which is a contradiction. $\square$

**Example 3.5.** Let $X = [0,1]$ and $G(u,v,w) = \max\{|u-v|, |u-w|, |v-w|\}$. Then $(X,G)$ is a complete G-metric space. Suppose that $a, b$ are positive real numbers with $a \geqslant b$. Let $H : X \times X \to R$ with $H(u,v) = au - bv$. Therefore, the function $u \mapsto H(u,v)$ is proper, lower semicontinuous and bounded from below, and $H(1,v) \geq 0$ for every $v \in X$. Also $H(u,v) \geq 0$ for every $u \in [\frac{b}{a}, 1]$ and every $v \in X$. In fact, for every $u \in [0, \frac{b}{a}]$, $H(u,v) = au - bv < 0$ when $v \in [\frac{a}{b}u, 1]$. Then set $\{x \in X : H(u,x) < 0\} \neq \emptyset$ for every $u \in [0, \frac{b}{a}]$. Let $u, v \in X$, $u \geq v$; we have $u - v = \frac{1}{a}\{(au - bx) - (av - bx)\}$, for every $x \in X$. Define $\varphi : [0,\infty) \to [0,\infty)$ by $\varphi(t) = \frac{1}{a}$. Then $G(u,v,v) \leq \varphi\big(H(u,x)\big)\big(H(u,x) - H(v,x)\big)$, for every $u \geqslant v$, and $u, v, x \in X$. By Theorem 3.4, there exists $y \in X$ such that $H(y,v) \geqslant 0$ for every $v \in X$.

**Theorem 3.6.** *Let $\Omega, \varphi$ be the same as in Theorem 2.9. For each $j \in J$, let $P_j : X \to X$ be multivalued maps with nonempty values, $g_j, h_j : X \times X \to \mathbb{R}$ be functions and $\{a_j\}$ and $\{b_j\}$ families of real numbers. Suppose that:*

(i) *For each $(u, j) \in X \times J$, there exists $v = v(u, j) \in P_j(u)$ such that $g_j(u, v) \geq a_j$ and $\Omega(u, v, v) \leq \varphi\big(f(u)\big)\big(f(u) - f(v)\big)$;*
(ii) *For each $(x, j) \in X \times J$, there exists $w = w(x, j) \in P_j(x)$ such that $h_j(x, w) \leq bj$ and $\Omega(x, w, w) \leq \varphi\big(f(x)\big)\big(f(x) - f(w)\big)$.*

*Then there exists $u_0 \in P_j(u_o)$ such that $g_j(u_0, u_0) \geq a_j$ and $h_j(u_0, u_0) \leq b_j$ for all $j \in J$ and $\Omega(u_0, u_0, u_0) = 0$.*

*Proof.* By Theorem 2.9, there exists $t \in X$ such that $\Omega(t, u, u) > \varphi\big(f(t)\big)\big(f(t) - f(u)\big)$, for all $u \in X$ with $u \neq t$. For each $j \in J$, by (i) there exists $w_1 = w_1(t, j) \in P_j(t)$ such that $g_j(t, w_1) \geq a_j$ and $\Omega(t, , w_1, w_1) \leq \varphi\big(f(t)\big)\big(f(t) - f(w_1)\big)$. Also according to (ii), there exists $w_2 = w_2(t, j) \in P_j(t)$ such that $h_j(t, w_2) \leq bj$ and $\Omega(t, , w_2, w_2) \leq \varphi\big(f(t)\big)\big(f(t) - f(w_2)\big)$. If $w_1 \neq t$, then $\Omega(t, , w_1, w_1) \leq \varphi\big(f(t)\big)\big(f(t) - f(w_1)\big) < \Omega(t, w_1, w_1)$, which is a contradiction. Therefore $w_1 = t$. Similarly, we have $w_2 = t$. Since $\Omega(t, , t, t) \leq \varphi\big(f(t)\big)\big(f(t) - f(t)\big) = 0$, hence $\Omega(t, t, t) = 0$. □

**Remark 3.7.** (a) In Theorem 3.6, put $g_j = h_j = F_j$ and $a_j = b_j = c_j$; then there exists $u_0 \in P_j(u_0)$ such that $F_j(u_0, u_0) = c_j$ for all $j \in J$ and $\Omega(u_0, u_0, u_0) = 0$.

(b) In (a), put $P_j(u) = X$ for all $u \in X$; then there exists $u_0 \in X$ such that $F_j(u_0, u_0) = cj$ for all $j \in J$ and $\Omega(u_0, u_0, u_0) = 0$.

**Remark 3.8.** From Theorem 3.5, we can infer Theorem 2.9.

Suppose contrary, that for each $u \in X$, there exists $v \in X$ with $v \neq u$ such that

$$\Omega(u, v, v) \leq \varphi\big(f(u)\big)\big(f(u) - f(v)\big).$$

Define $P : X \to X \setminus \{\emptyset\}$ by $P(u) = \{v \in X : v \neq u\}$ and a function $F : X \times X \to \mathbb{R}$ by $F(u, v) = \chi_{P(u)}(v)$, where $\chi_A$ is the characteristic function for an arbitrary set $A$. We would have $v \in P(u) \iff F(u, v) = 1$. Then for each $u \in X$, there would exist $v \in X$ such that $F(u, v) = 1$ and $\Omega(t, , u, u) \leq \varphi\big(f(t)\big)\big(f(t) - f(u)\big)$. According to Remark 3.7(a) with $c = 1$, there would exist $u_0 \in X$ such that $F(u_0, u_0) = 1$ and $\Omega(u_0, u_0, u_0) = 0$. Then $u_0 \in P(u_0)$. This is a contradiction.

## 4. Applications

Let $(X, G)$ be a $G$-metric space and $a, b \in X$. Suppose that $\kappa : X \to (0, \infty)$ is a function and $\Omega$ a $\Omega$-distance on $X$. Define

$$\Omega_\varepsilon(a, b, \kappa) = \{u \in X : \varepsilon\Omega(a, u, u) \leq \kappa(a)\big(\Omega(b, a, a) - \Omega(b, u, u)\big)\}$$

such that $\varepsilon \in (0, \infty)$ and $a, b \in X$.

**Lemma 4.1.** *Let $\Omega, f,$and $\varphi$ be the same as in Theorem 2.9. Let $\varepsilon > 0$ and $\Omega$ be an $\Omega$-distance on $X$. Suppose that there exists $x \in X$ such that $f(x) < \infty$ and $\Omega(x, x, x) = 0$. Then there exists $t \in X$ such that*

(i) $\varepsilon\Omega(x, t, t) \leq \varphi\big(f(x)\big)\big(f(x) - f(t)\big)$;
(ii) $\Omega(t, u, u) > \varphi\big(f(t)\big)\big(f(t) - f(u)\big)$ *for all $u \in X$ with $u \neq t$.*

*Proof.* Let $x \in X$, $f(x) < +\infty$ and $\Omega(x, x, x) = 0$. Put

$$V = \{u \in X : \varepsilon\Omega(x, u, u) \leq \varphi\big(f(x)\big)\big(f(x) - f(u)\big)\}.$$

The space $(V, G)$ is a nonempty complete G-metric space. By Theorem 2.9, there exists $t \in V$ such that $\varepsilon\Omega(t, u, u) > \varphi\big(f(t)\big)\big(f(t) - f(u)\big)$ for all $u \in V$ with $u \neq t$. For any $u \in X \setminus V$, since

$$\begin{aligned}\varepsilon[\Omega(x, t, t) + \Omega(t, u, u)] &\geq \varepsilon\Omega(x, u, u) > \varphi\big(f(x)\big)\big(f(x) - f(u)\big) \\ &\geq \varepsilon\Omega(x, t, t) + \varphi\big(f(t)\big)\big(f(t) - f(u)\big),\end{aligned}$$

we have $\varepsilon\Omega(t, u, u) > \varphi\big(f(t)\big)\big(f(t) - f(u)\big)$ for all $u \in X \setminus V$. Then $\varepsilon\Omega(t, u, u) > \varphi\big(f(t)\big)\big(f(t) - f(t)\big)$ for all $u \in X$ with $u \neq t$. □

**Theorem 4.2** (Generalized flower petal theorem)**.** *Suppose that $N$ is a proper complete subset of a $G$-metric space $X$ and $a \in N$. Let $\Omega$ be an $\Omega$-distance on $X$ with $\Omega(a, a, a) = 0$. Let $b \in X \setminus N$, $\Omega(b, N, N) = \inf_{u \in N} \Omega(b, u, u) \geq r$, $\Omega(b, a, a) = s > 0$, and let there exist a function $\kappa : X \to (0, \infty)$ satisfying $\kappa(u) = \varphi\big(\Omega(b, u, u)\big)$ for some nondecreasing function $\varphi : (-\infty, \infty] \to (0, \infty)$. Then for each $\varepsilon > 0$, there exists $t \in N \cap \Omega_\varepsilon(a, b, \kappa)$ such that $\Omega_\varepsilon(t, b, \kappa) \cap (N \setminus \{t\}) = \emptyset$ and $\Omega(a, t, t) \leq \varepsilon^{-1}\kappa(a)(s - r)$.*

*Proof.* The space $(N, G)$ is a complete G-metric space. Consider the function $f : N \to (-\infty, \infty]$ defined by $f(u) = \Omega(b, u, u)$. Since $f(a) = \Omega(b, a, a) = s < \infty$ and $\Omega(b, N, N) = \inf_{u \in N} \Omega(b, u, u) \geq r$, $f$ is a proper lower semicontinuous and bounded from below function. By Lemma 4.2, there exists $t \in N$ such that

(i) $\varepsilon\Omega(a,t,t) \leq \kappa(a)\big(f(a)-f(t)\big)$;

(ii) $\varepsilon\Omega(t,u,u) > \kappa(t)\big(f(t)-f(t)\big)$ for all $u \in N$ with $u \neq t$.

Applying (i), we get $t \in N \bigcap \Omega_\varepsilon(a,b,\kappa)$. Applying (i) again, we get $\Omega(a,t,t) \leq \varepsilon^{-1}\kappa(a)\big(\Omega(b,a,a)-\Omega(b,t,t)\big) \leq \varepsilon^{-1}\kappa(a)(s-r)$. By (ii), we obtain $\varepsilon\Omega(t,u,u) > \kappa(t)\big(\Omega(b,t,t)-\Omega(b,u,u)\big)$ for all $u \in N$ with $u \neq t$. Therefore $u \notin \Omega_\varepsilon(t,b,\kappa)$ for all $u \in N \setminus \{t\}$ or $\Omega_\varepsilon(t,b,\kappa) \cap (N \setminus \{t\}) = \emptyset$. □

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## References

[1] J.-P. Aubin, I. Ekeland, *Applied Nonlinear Analysis*, Wiley (1984).

[2] I. Ekeland, *Remarques sur les problmes variationnels, I*, C. R. Acad. Sci. Paris Sér. A–B, **275** (1972), 1057–1059.

[3] Z. Mustafa, B. Sims, *A new approach to generalized metric spaces*, J. Nonlinear Convex Anal., **7** (2006), 289–392.

[4] S. Plubtieng, Th. Seangwattana, *The Borwein-Preiss variational principle for nonconvex countable systems of equilibrium problems*, J. Nonlinear Sci. Appl. **9** (2016), no. 5, 2224–2232.

[5] R. Saadati, S. M. Vaezpour, P. Vetro, B. E. Rhoades, *Fixed point theorems in generalized partially ordered G-metric spaces*, Math. Comput. Modelling, **52** (2010), 797–801.

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# GEOMETRIC PROPERTIES OF BESSEL FUNCTIONS FOR THE CLASSES OF JANOWSKI STARLIKE AND CONVEX FUNCTIONS

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Abstract. Applications of Bessel differential equations have attracted the univalent function theorists in recent years. In the present investigation, we establish certain sufficient conditions for Bessel function to be in the class of Janowski starlike and Janowski convex functions. Further, certain sufficient condition for an integral operator defined using Bessel function to be in the class of Janowski starlike and Janowski convex functions are determined.

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## 1. Introduction

Let $\mathcal{A}$ be the class of functions $f$ normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions which are also univalent in $\mathbb{U}$. A function $f \in \mathcal{A}$ is said to be starlike of order $\alpha$ $(0 \leq \alpha < 1)$, if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{U}).$$

This function class is denoted by $\mathcal{S}^*(\alpha)$. We also write $\mathcal{S}^*(0) =: \mathcal{S}^*$, where $\mathcal{S}^*$ denotes the class of functions $f \in \mathcal{A}$ that are starlike in $\mathbb{U}$ with respect to the origin. A function $f \in \mathcal{A}$ is said to be convex of order $\alpha$ $(0 \leq \alpha < 1)$ if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathbb{U}).$$

This class is denoted by $\mathcal{K}(\alpha)$. Further, $\mathcal{K} = \mathcal{K}(0)$, the well-known standard class of convex functions. It is an established fact that

$$f \in \mathcal{K}(\alpha) \Longleftrightarrow zf' \in \mathcal{S}^*(\alpha).$$

There has been a continuous interest shown on the Geometric and other related properties of Bessel functions (like hypergeometric functions) after many papers have been published by Baricz [2](see also the other works of Baricz) in recent times. One such problem of Baricz [3] was to find conditions on the triplet $p$, $b$ and $c$ such that the function $u_{p,b,c}$ is starlike and convex of order $\alpha$. In earlier investigations, finding conditions on the parameters for which the Gaussian hypergeometric function belong to the various classes of functions have been discussed in detail by Shanmugam [20], Sivasubramanian *et al.* [21] and Sivasubramanian and Sokol [22] (See also [6, 7, 10, 11, 12, 13, 16, 17] ).

Let us consider the following second-order linear homogenous differential equation (see, for details, [3]):

$$z^2\omega''(z) + bz\omega'(z) + [cz^2 - p^2 + (1-b)p]\omega(z) = 0 \quad (b, c, p \in \mathbb{C}). \tag{1.2}$$

The function $\omega_{p,b,c}(z)$, which is called the generalized Bessel function of the first kind of order $p$, is defined as a particular solution of (1.2). Further, the function $\omega_{p,b,c}(z)$ has the familiar representation

$$\omega_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-c)^n}{n!\Gamma\left(p + n + \frac{b+1}{2}\right)} \left(\frac{z}{2}\right)^{2n+p} \quad (z \in \mathbb{C}), \tag{1.3}$$

where $\Gamma$ stands for the Euler gamma function. The series (1.3) permits the study of Bessel, modified Bessel and spherical Bessel functions all together. Solutions of (1.2) are referred as the generalized Bessel function of order $p$. The particular solution given by (1.3) is called the generalized Bessel function of the first kind order of $p$. Although the series defined above is convergent everywhere, the function $\omega_{p,b,c}$ is generally not univalent in $\mathbb{U}$. By ratio test, the radius of convergence for the series in (1.3) is infinity and hence $\omega_{p,b,c}(z)$ converges everywhere for all $b, c, p \in \mathbb{C}$ and for all $z \in \mathbb{U}$.

It is worth mentioning that, in particular, for $b = c = 1$ in (1.3), we obtain the familiar Bessel function of the first kind of order $p$ defined by

$$J_p(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(p+n+1)} \left(\frac{z}{2}\right)^{2n+p} \quad (z \in \mathbb{C}). \tag{1.4}$$

Further, for the choices $c = 1$ and $b = 2$ in (1.3), we obtain the familiar spherical Bessel function of the first kind of order $p$ defined by

$$S_p(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(p+n+3/2)} \left(\frac{z}{2}\right)^{2n+p} \quad (z \in \mathbb{C}). \tag{1.5}$$

For the choices of $b = 1$ and $c = -1$ in (1.3), we obtain the modified Bessel function of the first kind of order $p$ defined by

$$I_p(z) = \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(p+n+1)} \left(\frac{z}{2}\right)^{2n+p} \quad (z \in \mathbb{C}). \tag{1.6}$$

From (1.3), it is clear that $\omega(0) = 0$. Therefore, it follows from (1.3) that

$$\omega_{p,b,c}(z) = \left[2^p \Gamma\left(p + \frac{b+1}{2}\right)\right]^{-1} z^p \sum_{n=0}^{\infty} \frac{(-c/4)^n \ \Gamma\left(p + (b+1)/2\right)}{n!\Gamma\left(p+n+(b+1)/2\right)} z^{2n} \quad (z \in \mathbb{C}). \tag{1.7}$$

Let us set

$$u_{p,b,c}(z) = \sum_{n=0}^{\infty} b_n z^n,$$

where

$$b_n = \frac{(-c/4)^n \ \Gamma\left(p + (b+1)/2\right)}{n!\Gamma\left(p+n+(b+1)/2\right)}.$$

Hence, (1.7) becomes

$$\omega_{p,b,c}(z) = \left[2^p \Gamma\left(p + \frac{b+1}{2}\right)\right]^{-1} z^p u_{p,b,c}(z^2). \tag{1.8}$$

By using the well-known Pochhammer symbol (or the shifted factorial) $(\lambda)_\mu$ defined, for $\lambda, \mu \in \mathbb{C}$ and in terms of the Euler $\Gamma$ function, by

$$(\lambda)_\mu := \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\mu = 0; \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\mu = n \in \mathbb{N}; \lambda \in \mathbb{C}). \end{cases}$$

In view of the fact that $(0)_0 := 1$, the series representation for the function $u_{p,b,c}$ is given by

$$u_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-c/4)^n z^n}{(\kappa)_n (n)!} \quad \left(\kappa := p + (b+1)/2 \notin \mathbb{Z}_0^-\right) \tag{1.9}$$

and therefore,

$$z u_{p,b,c}(z) = z + \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1} z^n}{(\kappa)_{n-1} (n-1)!} \quad \left(\kappa := p + (b+1)/2 \notin \mathbb{Z}_0^-\right) \tag{1.10}$$

where $\mathbb{N} := \{1, 2, ...\}$ and $\mathbb{Z}_0^- := \{0, -1, -2, ...\}$. The function $u_{p,b,c}$ is called the generalized and normalized Bessel function of the first kind of order $p$. We note that by the ratio test, the radius of

convergence of the series $u_{p,b,c}$ is infinity. Moreover, the function $u_{p,b,c}$ is analytic in $\mathbb{C}$ and satisfies the differential equation $4z^2u''(z)+4\kappa zu'(z)+czu(z)=0$. Also, if $b,p,c\in\mathbb{C}$ and $\kappa\notin\mathbb{Z}_0^-$, then the function $u_{p,b,c}$ satisfies the recursive relation

$$4ku'_{p,b,c}(z)=-cu_{p+1,b,c}(z)\quad(z\in\mathbb{C}). \tag{1.11}$$

Further, for $z=1$, we denote $u_{p,b,c}(z)$ simply by $u_p(1)$. For $f\in\mathcal{A}$, we define the operator $I_{p,b,c}f(z)$ by

$$I_{p,b,c}\ f(z)=zu_{p,b,c}(z)*f(z)=z+\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(\kappa)_{n-1}}\frac{a_n\ z^n}{(n-1)!}, \tag{1.12}$$

where $\kappa=p+(b+1)/2\notin\mathbb{Z}_0^-$. In fact, the function $I_{p,b,c}f(z)$ given by (1.12) is an elementary transform of the generalized hypergeometric function. Thus, it is easy to see that

$$I_{p,b,c}f(z)=z{}_0F_1(\kappa;-c/4z)*f(z).$$

For the special choices of $b=c=1$ in (1.12), $I_{p,b,c}f$ reduces to $J_p:\mathcal{A}\to\mathcal{A}$ related with Bessel function, defined by

$$J_p\ f(z)=zu_{p,1,1}(z)*f(z)=z+\sum_{n=2}^{\infty}\frac{(-1/4)^{n-1}a_n}{(p+1)_{n-1}}\frac{z^n}{(n-1)!}. \tag{1.13}$$

For the special choices of $b=1$ and $c=-1$ in (1.12), $I_{p,b,c}f$ reduces to $M_p:\mathcal{A}\to\mathcal{A}$ related with modified Bessel function, defined by

$$M_pf(z)=zu_{p,1,-1}(z)*f(z)=z+\sum_{n=2}^{\infty}\frac{a_n}{(4)^{n-1}(p+1)_{n-1}}\frac{z^n}{(n-1)!} \tag{1.14}$$

where $*$ denotes the usual Hadamard product or convolution of power series.

If $f$ and $g$ are analytic in $\mathbb{U}$, then we say that the function $f$ is *subordinate* to $g$, if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with $w(0)=0$ and $|w(z)|<1$ $(z\in\mathbb{U})$, such that $f(z)=g(w(z))$ $(z\in\mathbb{U})$. We denote this subordination by $f\prec g$ or $f(z)\prec g(z)$ $(z\in\mathbb{U})$.

For $-1\le F<E\le 1$, let

$$\mathcal{S}^*[E,F]=\left\{f\in\mathcal{A}:\frac{zf'(z)}{f(z)}\prec\frac{1+Ez}{1+Fz}\quad(z\in\mathbb{U})\right\}$$

and

$$\mathcal{K}[E,F]=\left\{f\in\mathcal{A}:1+\frac{zf''(z)}{f'(z)}\prec\frac{1+Ez}{1+Fz}\quad(z\in\mathbb{U})\right\}.$$

It is fairly straightforward to see that $\mathcal{S}^*[1,-1]$ is the familiar class of starlike functions $\mathcal{S}^*$, $\mathcal{S}^*[1-2\gamma,-1]\,(0\le\gamma<1)$ is the class of starlike functions of order $\gamma$ and also the class $\mathcal{S}^*[\lambda,0]$ is denoted by $\mathcal{S}^*_\lambda$. Further, $\mathcal{K}[1,-1]$ is the familiar class of convex functions $\mathcal{K}$, $\mathcal{K}[1-2\gamma,-1]\,(0\le\gamma<1)$ is the class of convex functions of order $\gamma$ and also the class $\mathcal{K}[\lambda,0]$ is denoted by $\mathcal{K}_\lambda$. These two classes have been investigated in several works, for example, see [18, 19].

The connection between the Janowski starlike, Janowski convex functions and the Bessel functions is not considered so far. In the present paper, we obtain mapping properties between various subclasses of $\mathcal{S}$ motivated by the works of Anbudurai and Parvatham [1] (see also [5, 10, 11, 12, 16, 17, 18, 24]).

## 2. Sufficient conditions for Bessel functions to be in $\mathcal{S}^*[E,F]$ and $\mathcal{K}[E,F]$ involving Jack's Lemma

In the present section, we determine certain sufficient conditions involving Jack's Lemma for $u_{p,b,c}$ and $z\,u_{p,b,c}$ to be in the class of Janowski starlike and Janowski convex functions. To prove the main theorems we need the following lemma.

**Lemma 2.1.** [9] *Let $\omega$ be regular in the unit disk $\mathbb{U}$ with $\omega(0)=0$. If $|\omega(z)|$ attains a maximum value on the circle $|z|=r\,(0\le r<1)$ at a point $z$, then $z_1\omega'(z_1)=m\omega(z_1)$ where $m$ is real and $m\ge 1$.*

**Lemma 2.2.** [18] *Let a function $f$ of the form (1.1) satisfy*

$$\left|\frac{zf'(z)}{f(z)}-1\right|^{1-\alpha}\ \left|\frac{zf''(z)}{f'(z)}\right|^{\alpha}<\frac{(E-F)(2+E+E^2)^{\alpha}}{(1+|F|)(1+E)^{2\alpha}} \tag{2.1}$$

*for fixed constants $E$, $F$ and $\alpha$ such that $-1\le F<E\le 1$, $\alpha\ge 0$ and $z\in\mathbb{U}$. Then $f\in\mathcal{S}^*[E,F]$.*

**Theorem 2.1.** *Let $f\in\mathcal{A}$. If*

$$\left|(I_{p,b,c}f(z))'-1\right|^{1-\beta}\ \left|\frac{z(I_{p,b,c}f(z))''}{(I_{p,b,c}f(z))'}\right|^{\beta}<\frac{1}{2^{\beta}}\quad(\beta\ge 0), \tag{2.2}$$

*then $I_{p,b,c}f$ is univalent in $\mathbb{U}$.*

*Proof.* We know that

$$I_{p,b,c}f(z)=z+\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!}\,a^n\,z^n$$

in $\mathcal{A}$. Define $\omega$ by $\omega(z)=(I_{p,b,c}f(z))'-1$ for $z\in\mathbb{U}$. Then it follows that $\omega$ is analytic in $\mathbb{U}$ with $\omega(0)=0$. In view of (2.2), we have

$$|\omega(z)|^{1-\beta}\ \left|\frac{z\omega'(z)}{1+\omega(z)}\right|^{\beta}=|\omega(z)|\left|\frac{z\omega'(z)}{\omega(z)}\frac{1}{1+\omega(z)}\right|^{\beta}<\frac{1}{2^{\beta}}. \tag{2.3}$$

Suppose that there exists a point $z_1\in\mathbb{U}$ such that $\max\limits_{|z|\le|z_1|}|\omega(z)|=|\omega(z_1)|=1$. Then, by Lemma 3.1, we can put

$$\frac{z_1\omega'(z_1)}{\omega(z_1)}=m\ge 1.$$

Therefore, we obtain

$$|\omega(z_1)|\left|\frac{z_1\omega'(z_1)}{\omega(z_1)}\frac{1}{1+\omega(z)}\right|^{\beta}\ge\left(\frac{m}{2}\right)^{\beta}\ge\frac{1}{2^{\beta}}$$

which contradicts the condition (2.3). This shows that $|\omega(z)|=|(I_{p,b,c}f(z))'-1|<1$ which implies that $\Re(I_{p,b,c}f(z))'>0$ for $z\in\mathbb{U}$. Therefore, by the Noshiro-Warschawski theorem [8], $I_{p,b,c}f$ is univalent in $\mathbb{U}$. □

**Theorem 2.2.** *Let $f\in\mathcal{A}$, $c\in\mathbb{C}$ and $\kappa>0$. If $u_{p,b,c}$ defined by (1.9) satisfies the inequality*

$$\left|\frac{zu'_{p,b,c}(z)}{u_{p,b,c}(z)}\right|<\frac{E-F}{1+|F|}, \tag{2.4}$$

*where $-1\le F<E<1$, $-1\le F\le 0$ and $z\in\mathbb{U}$, then $zu_{p,b,c}\in\mathcal{S}^*[E,F]$.*

*Proof.* Let us define a function $F$ by

$$F(z) = zu_{p,b,c}(z) \quad (z \in \mathbb{U}).$$

In view of (2.4), we have

$$\left|\frac{zF'(z)}{F(z)} - 1\right| < \frac{E-F}{1+|F|}. \tag{2.5}$$

An application of Lemma 2.2 with $\alpha = 0$ proves Theorem 2.2. □

**Theorem 2.3.** *Let $f \in \mathcal{A}$, $c \in \mathbb{C}$ and $\kappa > 0$. If $u_{p,b,c}$ defined by (1.9) satisfies the inequality*

$$\left|\frac{zu''_{p,b,c}(z)}{u'_{p,b,c}(z)}\right| < \frac{(E-F)(2+E+E^2)}{(1+|F|)(1+E)^2}, \tag{2.6}$$

*where $-1 \leq F < E < 1$ and $-1 \leq F \leq 0$, then $u_{p,b,c}$ is starlike of order $(E+F)/2F$ and type $|F|$ with respect to* 1.

*Proof.* Let $h : \mathbb{U} \to \mathbb{C}$ be defined by

$$h(z) = \frac{u_{p,b,c}(z) - b_0}{b_1}.$$

Then $h \in \mathcal{A}$ and $h$ satisfies

$$\begin{aligned}\left|\frac{zh''(z)}{h'(z)}\right| &= \left|\frac{zu''_{p,b,c}(z)}{u'_{p,b,c}(z)}\right| \\ &< \frac{(E-F)(2+E+E^2)}{(1+|F|)(1+E)^2}.\end{aligned}$$

An application of Lemma 2.2 with $\alpha = 1$ implies that $u_{p,b,c}$ is starlike of order $(E+F)/2F$ and type $|F|$ with respect to 1 as the value of $b_0 = 1$. □

**Theorem 2.4.** *Let $f \in \mathcal{A}$, $c \in \mathbb{C}$ and $\kappa > 0$. If $u_{p,b,c}$ defined by (1.9) satisfies the inequality*

$$\left|\frac{zu'_{p+1,b,c}(z)}{u_{p+1,b,c}(z)}\right| < \frac{E-F}{1+|F|}, \tag{2.7}$$

*where $-1 \leq F < E < 1$, $-1 \leq F \leq 0$ and $c \neq 0$, then $u_{p,b,c}(z) \in \mathcal{K}[E,F]$.*

*Proof.* By virtue of Theorem 2.2, $zu_{p+1,b,c} \in \mathcal{S}^*[E,F]$. In view of (1.11), $zu'_{p,b,c}(z) = b_1 u_{p+1,b,c}(z)$, where $b_1 = -c/4\kappa \neq 0$. Therefore, we have $zu'_{p,b,c} \in \mathcal{S}^*[E,F]$, which implies $u_{p,b,c} \in \mathcal{K}[E,F]$. □

**Remark 2.1.** *Note that, the conclusions of Theorem 2.2, Theorem 2.3 and Theorem 2.4 hold in the disk $|z| < 4/|c|$ where $0 < |c| < 4$ which is larger than the unit disk. By applying as in Theorems 2,3, and 5 of Owa and Srivastava* [15] *to the function $F(z) = {}_0F_1(\kappa, z)$ and using the transformation $F(z) = u_{p,b,c}(-4z/c)$ and replacing $z$ by $-cz/4$, we obtain that Theorem 2.2, Theorem 2.3 and Theorem 2.4 hold in the disk $|z| < 4/|c|$.*

**Theorem 2.5.** *Let $c \in \mathbb{C}$, $-1 \leq F < E < 1$, $-1 \leq F \leq 0$ and $\kappa > 0$. If the Bessel's inequality*

$$(1-F)\frac{|c|}{4\kappa}u_{p+1,b,|c|}(1) + (E-F)u_{p,b,|c|}(1) \leq 2(E-F) \tag{2.8}$$

*is satisfied, then $zu_{p,b,c} \in \mathcal{S}^*[E,F]$.*

*Proof.* A special case of Theorem 3 in [1] gives a sufficient condition for a function $f \in \mathcal{S}^*[E,F]$ and is given by

$$\sum_{n=2}^{\infty} [n(1-F)-(1-E)]\,|A_n| \leq E-F,$$

where

$$A_n = \frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!}.$$

To prove the theorem, we need to show that

$$\begin{aligned} T: &= \sum_{n=2}^{\infty} [n(1-F)-(1-E)]\,|A_n| \\ &= \sum_{n=2}^{\infty} [n(1-F)-(1-E)] \left| \frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!} \right| \\ &= (1-F)\sum_{n=2}^{\infty} n \left| \frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!} \right| - (1-E)\sum_{n=2}^{\infty} \left| \frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!} \right| \\ &\leq (1-F)\sum_{n=2}^{\infty} \frac{(|c|/4)^{n-1}}{(\kappa)_{n-1}(n-2)!} + (E-F)\sum_{n=2}^{\infty} \frac{(|c|/4)^{n-1}}{(\kappa)_{n-1}(n-1)!} \\ &= (1-F)\frac{|c|}{4\kappa} u_{p+1,b,|c|}(1) + (E-F)\left(u_{p,b,|c|}(1)-1\right), \end{aligned}$$

which is bounded above by $E-F$ if (2.8) is satisfied. □

For the choices of $E=\lambda$ and $F=0$, we get the following corollary.

**Corollary 2.1.** *Let $c \in \mathbb{C}$, $-1 \leq F < E < 1$, $-1 \leq F \leq 0$ and $\kappa > 0$. If the Bessel's inequality*

$$\frac{|c|}{4\kappa}\, u_{p+1,b,|c|}(1) + \lambda u_{p,b,c}(1) \leq 2\lambda \tag{2.9}$$

*is satisfied, then $zu_{p,b,|c|} \in \mathcal{S}^*_{\lambda}$.*

**Theorem 2.6.** *Let $c \in \mathbb{C}$ and $\kappa > 0$. If the Bessel's inequality*

$$(1-F)\frac{(|c|/4)^2}{\kappa(\kappa+1)}\, u_{p+2,b,|c|}(1) + (2+E-3F)\frac{|c|}{4\kappa}\, u_{p+1,b,|c|}(1) + (E-F)\, u_{p,b,|c|}(1) \leq 2(E-F) \tag{2.10}$$

*is satisfied, then the operator $zu_{p,b,c} \in \mathcal{K}[E,F]$.*

*Proof.* By an analogous similar result [1] mentioned as in the earlier theorem, a sufficient condition for $f \in \mathcal{K}[E,F]$ is that

$$\sum_{n=2}^{\infty} n\,[n(1-F)-(1-E)]\,|A_n| \leq E-F,$$

where

$$A_n = \frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!}.$$

Then we have to show that

$$T_1 := \sum_{n=2}^{\infty} n\left[n(1-F)-(1-E)\right]|A_n| \leq E-F. \tag{2.11}$$

Writing $n = n-1+1$, and proceeding with the calculation as in the previous theorem, we get

$$\begin{aligned} T_1 &= \sum_{n=2}^{\infty} n\left(n(1-F)-(1-E)\right)\left|\frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!}\right| \\ &\leq \sum_{n=2}^{\infty}\left(n(1-F)-(1-E)\right)\frac{(|c|/4)^{n-1}}{(\kappa)_{n-1}(n-2)!} + \sum_{n=2}^{\infty}\left(n(1-F)-(1-E)\right)\frac{(|c|/4)^{n-1}}{(\kappa)_{n-1}(n-1)!}. \end{aligned}$$

Breaking the above inequality into two parts and simplifying, we observe that the summation is bounded above by $E-F$ if (2.10) is satisfied. □

For the choices of $E=\lambda$ and $F=0$, we get the following corollary.

**Corollary 2.2.** *Let $c \in \mathbb{C}$ and $\kappa > 0$. If the Bessel's inequality*

$$\frac{(|c|/4)^2}{\kappa(\kappa+1)}u_{p+2,b,|c|}(1) + (2+\lambda)\frac{|c|}{4\kappa}u_{p+1,b,|c|}(1) + \lambda u_{p,b,|c|}(1) \leq 2\lambda \tag{2.12}$$

*is satisfied, then the operator $zu_{p,b,c} \in \mathcal{K}_\lambda$.*

**Remark 2.2.** *For the choices of $E = 1-2\alpha$ $(0 \leq \alpha < 1)$ and $F=-1$, each of the above theorems reduces to the results obtained by Baricz* [3].

## 3. Inclusion properties involving the class of Janowski starlike and convex functions

Let a function $f \in \mathcal{A}$ is said to be in the class $R^\tau(A,B)$ if

$$\left|\frac{f'(z)-1}{\tau(A-B)-B(f'(z)-1)}\right| < 1 \quad (-1 \leq B < A \leq 1; \tau \in \mathbb{C}\setminus\{0\}; z \in \mathrm{U}). \tag{3.1}$$

Clearly, a function $f$ belongs to $R^\tau(A,B)$ if and only if there exists a function $w$ regular in $\mathbb{U}$ satisfying $w(0)=0$ and $|w(z)|<1 \quad z \in \mathbb{U}$ such that

$$1+\frac{1}{\tau}(f'(z)-1) = \frac{1+Aw(z)}{1+Bw(z)} \ \ (z \in \mathbb{U}).$$

The class $R^\tau(A,B)$ was introduced by Dixit and Pal [6]. For $\tau=1$, $A=\beta$, $B=-\beta$, $(0<\beta\leq 1)$, $R^\tau(A,B)$ reduces to the class of functions $f \in \mathcal{A}$ satisfying the inequality

$$\left|\frac{f'(z)-1}{f'(z)+1}\right| < \beta \ \ (z \in \mathbb{U};\ \ 0<\beta\leq 1),$$

which was studied by Caplinger and Cauchy [4] and Padmanaban [14].

Now we aim at investigating various mapping and inclusion properties involving the class of Janowski starlike and Janowski convex functions. To prove the main results we need the following lemmas.

**Lemma 3.1.** [6] *Let a function $f$ of the form (1.1) be in $R^{\tau}(A,B)$. Then*

$$|a_n| \leq \frac{(A-B)|\tau|}{n}.$$

*The result is sharp for the function*

$$f(z) = \int_0^z \left(1 + \frac{(A-B)|\tau|z^{n-1}}{1+Bz^{n-1}}\right) dz \quad (n \geq 2;\ z \in \mathbb{U})$$

**Lemma 3.2.** [6] *Let a function $f$ of the form (1.1) satisfy the inequality*

$$\sum_{n=2}^{\infty}(1+|B|)n|a_n| \leq (A-B)|\tau| \ \ (-1 \leq B < A \leq 1;\ \tau \in \mathbb{C}).$$

*Then $f \in R^{\tau}(A,B)$. The result is sharp for the function*

$$f(z) = z + \frac{(A-B)\tau}{(1+|B|)n}z^n \quad (n \geq 2;\ z \in \mathbb{U}).$$

**Theorem 3.1.** *Let $c \in \mathbb{C}$, $\kappa > 0$. Suppose that $f \in R^{\tau}(A,B)$. If the Bessel's inequality*

$$2u_{p,b,|c|}(1) - 4\frac{(\kappa-1)}{c}\left(u_{p-1,b,|c|}(1)-1\right) \leq \frac{1}{(1+B)} + 1 \tag{3.2}$$

*is satisfied, then $zu_{p,b,c}(z^2) * f(z) \in R^{\tau}(A,B)$*

*Proof.* Suppose that $f \in R^{\tau}(A,B)$. We note that

$$zu_{p,b,c}(z^2) = z + \sum_{n=2}^{\infty} \frac{(\frac{-c}{4})^{n-1}}{(\kappa)_{n-1}(n-1)!}\ z^{2n-1}.$$

By Lemma 3.2, it is enough to show that

$$\sum_{n=2}^{\infty}(1+|B|)\ (2n-1)\ \left|\frac{(\frac{-c}{4})^{n-1}}{(\kappa)_{n-1}(n-1)!}\ a_n\right| \leq (A-B)|\tau|.$$

Then by a similar proof as in the earlier theorem, we get

$$(A-B)\ |\tau|\ (1+|B|)\ \left[2u_{p,b,|c|}(1) - 4\frac{(\kappa-1)}{|c|}\left(u_{p-1,b,|c|}(1)-1\right) - 1\right] \leq (A-B)\ |\tau|,$$

which completes the proof of Theorem 3.1. □

**Theorem 3.2.** *Let $c \in \mathbb{C}$ and $\kappa > 0$. Suppose that $f \in R^{\tau}(A,B)$ and satisfy the condition*

$$u_{p,b,|c|}(1) \leq \frac{1}{1+|B|} + 1. \tag{3.3}$$

*Then $I_{p,b,c}\ f \in R^{\tau}(A,B)$.*

*Proof.* Let $f$ be of the form (1.1) belong to the class $R^{\tau}(A,B)$. By Lemma 3.2, it suffices to show that

$$\sum_{n=2}^{\infty} n(1+|B|)|A_n| \leq (A-B)|\tau|, \tag{3.4}$$

where

$$A_n = \frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!}\, a_n.$$

By virtue of Lemma 3.1 and making use of the fact that $|-c/4|^n \leq (|c|/4)^n$, we obtain

$$\begin{aligned}\sum_{n=2}^{\infty} n(1+|B|) \left|\frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!}\, a_n\right| &\leq (1+|B|)\,|\tau|\,(A-B)\sum_{n=2}^{\infty}\frac{(|c|/4)^{n-1}}{(\kappa)_{n-1}(n-1)!} \\ &= (1+|B|)\,|\tau|\,(A-B)\,[u_{p,b,|c|}(1)-1],\end{aligned}$$

which is bounded above by $(A-B)|\tau|$ in view of (3.3). This completes the proof of Theorem 3.2. □

**Theorem 3.3.** *Let $c \in \mathbb{C}$ and $\kappa > 0$. Suppose that $f \in R^{\tau}(A,B)$. If the Bessel's inequality*

$$(1-F)u_{p,b,|c|}(1) - (1-E)\frac{4(\kappa-1)}{|c|}u_{p-1,b,|c|}(1) \leq \frac{E-F}{(A-B)|\tau|} + E - F - \frac{(1-E)4(\kappa-1)}{|c|} \tag{3.5}$$

*is satisfied, then the operator $I_{p,b,c}\, f \in \mathcal{S}^*[E,F]$.*

*Proof.* Let $f$ be of the form (1.1) belong to the class $R^{\tau}(A,B)$. A special case of Theorem 3 [1] gives a sufficient condition that

$$\sum_{n=2}^{\infty}[n(1-F)-(1-E)]\,|A_n| \leq E-F,$$

where

$$A_n = \frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!}\, a_n.$$

Then we have to show that

$$T := \sum_{n=2}^{\infty}[n(1-F)-(1-E)]\,|A_n| \leq E-F. \tag{3.6}$$

Since, $f \in R^{\tau}(A,B)$, in virtue of Lemma 3.1,

$$\begin{aligned} T &\leq \sum_{n=2}^{\infty}[n(1-F)-(1-E)]\left|\frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!}\right|\frac{(A-B)|\tau|}{n} \\ &= (A-B)|\tau|\left[(1-F)\sum_{n=2}^{\infty}\left|\frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!}\right| - (1-E)\sum_{n=2}^{\infty}\left|\frac{(-c4)^{n-1}}{(\kappa)_{n-1}(n!)}\right|\right] \\ &\leq (A-B)|\tau|\left[(1-F)\sum_{n=2}^{\infty}\frac{(|c|/4)^{n-1}}{(\kappa)_{n-1}(n-1)!} - (1-E)\sum_{n=2}^{\infty}\frac{(|c|/4)^{n-1}}{(\kappa)_{n-1}(n!)}\right] \\ &= (A-B)|\tau|\left[(1-F)(u_{p,b,|c|}(1)-1) - (1-E)\frac{4(\kappa-1)}{|c|}\left(u_{p-1,b,|c|}(1) - 1 - \frac{|c|/4}{(\kappa-1)}\right)\right], \end{aligned}$$

which is bounded above by $E-F$ if (3.5) is satisfied. □

For the choices of $E=\lambda$ and $F=0$, we get the following corollary.

**Corollary 3.1.** *Let $c \in \mathbb{C}$, $\kappa > 0$ and $\lambda \in [0,1]$. Suppose that $f \in R^{\tau}(A,B)$ and satisfy the condition*

$$u_{p,b,|c|}(1) \; + \; (\lambda - 1)\, \frac{4(\kappa - 1)}{c}\, [u_{p-1,b,|c|}(1) - 1] \; \leq \; \left( \frac{1}{(A-B)\,|\tau|} + 1 \right) \lambda.$$

*Then the operator $I_{p,b,c} f \in S^{*}_{\lambda}$.*

**Theorem 3.4.** *Let $c \in \mathbb{C}$ and $\kappa > 0$. Suppose that $f \in R^{\tau}(A,B)$. If the Bessel's inequality*

$$(1-F)\frac{|c|}{4\kappa} u_{p+1,b,|c|}(1) + (E-F) u_{p,b,|c|}(1) \leq (E-F) \left( \frac{1}{(A-B)|\tau|} + 1 \right) \tag{3.7}$$

*is satisfied, then the operator $I_{p,b,c} f \in \mathcal{K}\,[E,F]$.*

*Proof.* Let $f$ be of the form (1.1) belong to the class $R^{\tau}(A,B)$. We need to show (see [1]) that

$$\sum_{n=2}^{\infty} n\,[(n(1-F) - (1-E))]\,|A_n| \leq E - F,$$

where

$$A_n = \frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!}\; a_n.$$

Since, $f \in R^{\tau}(A,B)$, in virtue of Lemma 3.1,

$$\begin{aligned}
T: \;&=\; \sum_{n=2}^{\infty} n\,[n(1-F) - (1-E)]\,|A_n| \leq E - F \\
T \;&\leq\; \sum_{n=2}^{\infty} (n(1-F) - (1-E)) \left| \frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!} \right| (A-B)|\tau| \\
&=\; (A-B)|\tau| \left[ (1-F) \sum_{n=2}^{\infty} n \left| \frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!} \right| - (1-E) \sum_{n=2}^{\infty} \left| \frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n-1)!} \right| \right] \\
&\leq\; (A-B)|\tau| \left[ (1-F) \sum_{n=2}^{\infty} \frac{(|c|/4)^{n-1}}{(\kappa)_{n-1}(n-2)!} + (E-F) \sum_{n=2}^{\infty} \frac{(|c|/4)^{n-1}}{(\kappa)_{n-1}(n-1)!} \right] \\
&=\; (A-B)|\tau| \left[ (1-F) \frac{|c|}{4\kappa} u_{p+1,b,|c|}(1) + (E-F) \left( u_{p,b,|c|}(1) - 1 \right) \right],
\end{aligned}$$

which is bounded above by $E-F$ if (3.7) is satisfied. This completes the proof of Theorem 3.4. □

For the choices of $E = \lambda$ and $F = 0$, we get the following corollary.

**Corollary 3.2.** *Let $c \in \mathbb{C}$ and $\kappa > 0$. Suppose that $f \in R^{\tau}(A,B)$. If the Bessel's inequality*

$$\frac{|c|}{4\kappa} u_{p+1,b,|c|}(1) + \lambda\, u_{p,b,|c|}(1) \leq \lambda \left( \frac{1}{(A-B)|\tau|} + 1 \right) \tag{3.8}$$

*is satisfied, then the operator $I_{p,b,c} f \in \mathcal{K}_{\lambda}$.*

## 4. Sufficient conditions for Bessel's integral operator to be in the class of $\mathcal{S}^*[E,F]$ and $\mathcal{K}[E,F]$

As in the work of Baricz [3], one can look at other linear operators acting on $u_{p,b,c}$ to obtain similar results. In this section, we make use of this idea in the case of a particular integral operator. We continue our earlier work that was done in the earlier section. That is, we determine sufficient conditions for the integral operator $g$ defined by (4.1) to be in the class of Janowski starlike and Janowski convex functions as follows:

$$\begin{aligned} g(z) &= \int_0^z u_p(t)dt \\ &= z+\sum_{n=2}^{\infty}\frac{b_{n-1}}{n}z^n \\ &= z+\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n!)}z^n. \end{aligned} \tag{4.1}$$

**Theorem 4.1.** *Let $c\in\mathbb{C}$ and $\kappa>0$. Further, let $-1\le F<E<1$ and $-1\le F\le 0$. If the Bessel's inequality*

$$(1-F)u_{p,b,|c|}(1)-(1-E)\frac{4(\kappa-1)}{|c|}u_{p-1,b,|c|}(1)\le 2(E-F)-(1-E)\frac{4(\kappa-1)}{|c|} \tag{4.2}$$

*is satisfied, then the function $g\in\mathcal{S}^*[E,F]$ where $g$ is defined by (4.1).*

*Proof.* To prove the theorem, we have to show that

$$T:=\sum_{n=2}^{\infty}\left[n(1-F)-(1-E)\right]|B_n|\le E-F, \tag{4.3}$$

where

$$B_n=\frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n)!}.$$

Then

$$\begin{aligned} T &= \sum_{n=2}^{\infty}\left[n(1-F)-(1-E)\right]\left|\frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n)!}\right| \\ &\le (1-F)\sum_{n=2}^{\infty}\frac{(|c|/4)^{n-1}}{(\kappa)_{n-1}(n-1)!}-(1-E)\sum_{n=2}^{\infty}\frac{(|c|/4)^{n-1}}{(\kappa)_{n-1}(n)!} \\ &= (1-F)\left(u_{p,b,|c|}(1)-1\right)-(1-E)\left[\frac{4(\kappa-1)}{|c|}\left(u_{p-1,b,|c|}(1)-1-\frac{|c|/4}{(\kappa-1)}\right)\right], \end{aligned}$$

which is bounded above by $E-F$ if (4.2) is satisfied. □

For the choices of $E=\lambda$ and $F=0$, we get the following corollary.

**Corollary 4.1.** *Let $c\in\mathbb{C}$ and $\kappa>0$. Further, let $\lambda\ge 0$. If the Bessel's inequality*

$$u_{p,b,|c|}(1)-(1-\lambda)\frac{4(\kappa-1)}{|c|}u_{p-1,b,|c|}(1)\le 2\lambda+(1-\lambda)\frac{4(\kappa-1)}{|c|} \tag{4.4}$$

*is satisfied, then the function $g \in \mathcal{S}_\lambda^*$ where $g$ is defined by (4.1).*

**Theorem 4.2.** *Let $c \in \mathbb{C}$ and $\kappa > 0$. Further, let $-1 \leq F < E < 1$ and $-1 \leq F \leq 0$ and $g$ be defined as in (4.1). If the Bessel's inequality*

$$(1-F)\frac{|c|}{4\kappa}u_{p+1,b,|c|}(1) + (E-F)u_{p,b,|c|}(1) \leq 2(E-F) \tag{4.5}$$

*is satisfied, then the integral operator $g \in \mathcal{K}[E,F]$.*

*Proof.* We have to show that

$$T_4 := \sum_{n=2}^{\infty} n\left(n(1-F)-(1-E)\right)|B_n| \leq E-F, \tag{4.6}$$

where

$$B_n = \frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n)!}.$$

Then

$$\begin{aligned} T_4 &= \sum_{n=2}^{\infty} n\left[n(1-F)-(1-E)\right]\left|\frac{(-c/4)^{n-1}}{(\kappa)_{n-1}(n)!}\right| \\ &\leq \sum_{n=2}^{\infty}\left[n(1-F)-(1-E)\right]\frac{(|-c/4|)^{n-1}}{(\kappa)_{n-1}(n-1)!} \\ &= (1-F)\sum_{n=2}^{\infty}\frac{(|c|/4)^{n-1}}{(\kappa)_{n-1}(n-2)!} + (E-F)\sum_{n=2}^{\infty}\frac{(|c|/4)^{n-1}}{(\kappa)_{n-1}(n-1)!} \\ &= (1-F)\frac{|c|}{4\kappa}u_{p+1,b,|c|}(1) + (E-F)\left(u_{p,b,|c|}(1)-1\right), \end{aligned}$$

which is bounded above by $E-F$ if (4.5) is satisfied. □

For the choices of $E = \lambda$ and $F = 0$, we get the following corollary.

**Corollary 4.2.** *Let $c \in \mathbb{C}$ and $\kappa > 0$. Further, let $-1 \leq F < E < 1$ and $-1 \leq F \leq 0$ and $g$ be defined as in (4.1). If the Bessel's inequality*

$$\frac{|c|}{4\kappa}u_{p+1,b,|c|}(1) + \lambda u_{p,b,|c|}(1) \leq 2\lambda \tag{4.7}$$

*is satisfied, then $g \in \mathcal{K}_\lambda$.*

## 5. Consequences and observations

Since the study generalized Bessel function permits the study of Bessel, modifed Bessel and spherical Bessel functions all together, each of these Theorems can also be stated for the Bessel, modified Bessel and spherical Bessel functions for special choices of the parameters $b$ and $c$. However, we leave all these results for the interested readers.

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## References

[1] M. Anbu Durai and R. Parvatham, Convolutions with Hypergeometric functions, Bull. Malay. Math. Soc. (Second Series) **23** (2000), 153–161.

[2] Á. Baricz, Geometric properties of generalized Bessel function, publ. Math. Debrecan **73** (2008), 155–178.

[3] Á. Baricz, Generalized Bessel functions of the first kind, PhD thesis, Babes-Bolyai University, Cluj-Napoca, 2008.

[4] T. R. Caplinger and W. M. Causey, A class of univalent functions, Proc. Amer. Math. Soc. **39** (1973), 357–361.

[5] E. Deniz, H. Orhan and H.M. Srivastava, Some sufficient conditions for univalence of certain families of integral operators involving generalized Bessel functions, Taiwansese J. Math. **15**(2) (2011), 883–917.

[6] K. K. Dixit And S. K. Pal, On a class of univalent functions related to complex order, Indian J. Pure Appl. Math. **26**(9)(1995), 889–896.

[7] A. Gangadharan, T. N. Shanmugam and H. M. Srivastava, Generalized hypergeometric functions associated with k-uniformly convex functions, Comput. Math. Appl. **44** (2002), 1515–1526.

[8] A. W. Goodman, On uniformly starlike functions, J. Math. Anal. Appl. **155** (1991), 364–370.

[9] I. S. Jack, Functions starlike and convex of order $\alpha$, J. London Math. Soc. **3** (1971), 469–474.

[10] T. Janani and G. Murugusundaramoorthy, Inclusion results on subclasses of starlike and convex functions associated with struve functions, Ital. J. Pure Appl. Math. **32** (2014), 467–476.

[11] S. R. Mondal and A. Swaminathan, Geometric properties of generalized Bessel functions, Bull. Malay. Math. Soc. **35**(1) (2012), 179–194.

[12] G. Murugusundaramoorthy, K. Vijaya and M. Kasturi, A note on subclasses of starlike and convex functions associated with Bessel functions, J. Nonlinear Funct. Anal. 2014, 1–11.

[13] G. Murugusundaramoorthy and N. Magesh, On certain subclasses of analytic functions associated with hypergeometric functions, Appl. Math. Lett. **24** (2011), 494–500.

[14] K. S. Padmanabhan, On a certain class of functions whose derivatives have a positive real part in the unit disc, Ann. Polon. Math. **23** (1970/71), 73–81.

[15] S. Owa and H. M. Srivastava, Univalent and starlike generalized hypergeometric functions, Canad. J. Math. **39** (1987), 1057-1077 .

[16] S. Porwal and K. K. Dixit, An application of generalized Bessel functions on certain analytic functions, Acta Univ. M. Belii. Ser. Math. 2013, 51–57,.

[17] S. Porwal and K. K. Dixit, An application of certain convolution operator involving hypergeometric functions, J. Rajasthan Acad. Phys. Sci. **9**(2) (2010), 173–186.

[18] R. K. Raina, On Univalent and starlike Wright's hypergeometric function, Rend. Sem. Mat. Univ. Padova **95** (1996), 11–22.

[19] T. N. Shanmugam, Convolution and differential subordination, Internat. J. Math. Math. Sci. **12** (1989), 333–340

[20] T. N. Shanmugam, Hypergeometric functions in the geometric function theory, Appl. Math. Comput. **187** (2007), 433–444.

[21] S. Sivasubramanian, T. Rosy and K. Muthunagai, Certain sufficient conditions for a subclass of analytic functions involving Hohlov operator, Comput. Math. Appl. **62** (2011), 4479–4485.

[22] S. Sivasubramanian and J. Sokół, Hypergeomtric transforms in certain classes of functions, Math. Comp. Modelling, **54** (2011), 3076–3082.

[23] A. Swaminathan, Certain suffienct conditions on Gaussian hypergeometric functions, J. Inequal. Pure Appl. Math. **5**(4), Art.83 (2004), 10 pp.

[24] N. Yagmur and H. Orhan, Starlikeness and convexity of generalized Struve functions, Abstr. Appl. Anal. 2013 (2013), Article ID 954513, 6 pp.

# ON THE JENSEN-TYPE INEQUALITY FOR THE $\bar{g}$-INTEGRAL.

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Abstract. We consider the pseudo-integral with respect to a $\sigma-\oplus$-measure of set-valued functions which was defined by Grbíc et al. Román-Flores et al.(2007) proved the Jensen type inequality for fuzzy integral with respect to a fuzzy measure. In this paper, we prove the Jensen type inequality for the $\bar{g}$-integral with respect to a $\sigma-\oplus_g$-measure under some sufficient conditions.

## 1. Introduction

Benvenuti-Mesiar [2], Deschrijver [3], J. Fang [4], Mesiar-Pap [9], Ralescu-Adams [10], and Wu-Wang-Ma [12] provided the properties and applications of the generalized fuzzy integral which is a generalization of fuzzy integrals.

The integrals of set-valued functions was introduced by Aumanm [1], and Jang [6,7] and Zhang-Guo [13] investigated some properties of the generalized fuzzy integral of set-valued functions. Not long ago, authors in [8,11] proved the Jensen type inequality for the fuzzy integral and for the generalized Sugeno integral. We consider the pseudo-integral with respect to a $\sigma-\oplus$-measure of set-valued functions which was defined by Grbíc et al [5]. Román-Flores et al. [11] proved the Jensen type inequality for fuzzy integral with respect to a fuzzy measure. In this paper, we prove the Jensen type inequality for $\bar{g}$-integral with respect to a $\sigma-\oplus_g$-measure under some sufficient conditions.

## 2. Jensen type inequality for the $g$-integral

Let $[a,b]$ be a closed (in some cases can be considered semiclosed) subinterval of $\overline{\mathbb{R}} = [-\infty,\infty]$ and let $\preccurlyeq$ be a total order on $[a,b]$. We introduce a semiring which is a structure $([a,b],\oplus,\odot)$ as follows.

*Key words and phrases.* Sugeno integral, $\sigma-\oplus$-measure, $\bar{g}$-integral, Jensen inequality.

**Definition 2.1.** ([2,7,9]) (1) A function $\oplus : [a,b] \times [a,b] \longrightarrow [a,b]$ is called a pseudo-addition if it is commutative, non-decreasing with respect to $\preccurlyeq$, associative, and with a zero (natural) element denoted by $\mathbf{0}$, that is, for each $x \in [a,b]$, $\mathbf{0} \oplus x = x$ holds (usually $\mathbf{0}$ is either $a$ or $b$).

(2) A function $\odot : [a,b] \times [a,b] \longrightarrow [a,b]$ is called a pseudo-multiplication if it is commutative, positively non-decreasing, that is, $x \preccurlyeq y$ implies $x \odot z \preccurlyeq y \odot z$ for all $z \in [a,b]_+ = \{x | x \in [a,b], \mathbf{0} \preccurlyeq x\}$, associative and there exists a unit element $\mathbf{1} \in [a,b]$, that is, for each $x \in [a,b], \mathbf{1} \odot x = x$.

(3) The structure $([a,b], \oplus, \odot)$ is called a semiring if $\mathbf{0} \odot x = \mathbf{0}$ and $\odot$ is a distributive pseudo-multiplication with respect to $\oplus$, that is, $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$.

(4) A set function $\mu : \Sigma \longrightarrow [a,b]$ is a $\sigma - \oplus$-measure if it satisfies the following two conditions:

(i) $\mu(\emptyset) = \mathbf{0}$ (if $\oplus$ is not idempotent);

(ii) $\mu(\cup_{i=1}^{\infty} A_i) = \oplus_{i=1}^{\infty} \mu(A_i)$ holds for any sequence $(A_i)_{i \in \mathbb{N}}$ of disjoint sets from $\Sigma$.

We note that for a real interval $[a,b] = [0,\infty]$, a pseudo-addition $\oplus$ and a pseudo-multiplication $\odot$ are generated by a strictly monotone bijective function $g : [0,\infty] \longrightarrow [0,\infty]$, that is, pseudo-operations are given by

$$x \oplus_g y = g^{-1}(g(x) + g(y)) \text{ and } x \odot_g y = g^{-1}(g(x)g(y)).$$

Now, the pseudo-integral, known as the $g$-integral, of some measurable function $f : X \longrightarrow [0,\infty]$ is

$$(g) \int_X f d\mu = \int_X^{\oplus_g} f \odot_g d\mu = g^{-1}\left( \int_X (g \circ f)\ d(g \circ \mu) \right) \tag{1}$$

where $g \circ \mu$ is the Lebesgue measure and the integral on the right-hand side of $(A)$ is the Lebesgue integral (see [7,9]). Let $(X, \Sigma, \mu)$ be a $\sigma - \oplus$-measure space. Grbíc et al. [5] defined the pseudo-integral of an interval-valued function $F$ on $A \in \sum$ as follows;

$$\int_A^{\oplus} F \odot d\mu = \{ \int_A^{\oplus} f \odot d\mu | f \in S(F) \} \tag{2}$$

where $\mu$ is a $\sigma - \oplus$-measure and $S(F)$ is the set of all selections of $F$. Let $L^1(\eta)$ be the set of all Lebesgue integrable functions on the Lebesgue space $([0,\infty), \sum, \eta)$ and $f \in L^1_{\oplus}(\mu)$ if and only if $g \circ f \in L^1(g \circ \mu)$. We introduce the definition of $g$-integrable boundedness of a set-valued function $F$ as follows:

**Definition 2.2.** ([5]) Let $g$ be a strictly monotone bijective function. A set-valued function $F$ is $g$ -integrable bounded if there is a function $h \in L^1_{\oplus}(\mu)$ such that

(i) $\oplus_{\alpha \in F(x)} \alpha \preccurlyeq h(x)$, for the idempotent pseudo-addition,

(ii) $\sup_{\alpha \in F(x)} \alpha \preccurlyeq h(x)$, for the pseudo-addition given by an increasing generator $g$,

(iii) $\inf_{\alpha \in F(x)} \alpha \preccurlyeq h(x)$, for the pseudo-addition given by a decreasing generator $g$.

From Proposition 11 in [5], we note that if $F$ is a pseudo-integrable bounded set-valued function, then $F$ is pseudo-integrable, that is, $\int_X^{\oplus} F \odot d\mu \neq \emptyset$.

**Theorem 2.1.** *(Theorem 2.4 [5]) Let $F$ be a pseudo-integrable bounded interval-valued function with border functions $f_l$ and $f_r$. Then we have*

$$\int_X^{\oplus} F \odot d\mu = \left[\int_X^{\oplus} f_l \odot d\mu, \int_X^{\oplus} f_r \odot d\mu\right], \tag{3}$$

Now, we obtain the following Jensen type inequality for the $g$-integral with respect to a $\sigma - \oplus_g$-measure.

**Theorem 2.2.** *Let $g$ be a decreasing function and $(X, \sum, g\circ\mu)$ be the Lebesgue measure space and $f \in L^1_{\oplus}\mu$ with $(g)\int_X f d\mu = m$. If $\Phi : [o, \infty) \longrightarrow [0, \infty)$ is strictly increasing function such that $\Phi(x) \leq x$, for every $x \in [0, m]$ and $\Phi(f) \in L^1_{\oplus}(\mu)$, then we have*

$$\Phi\left((g)\int_X f d\mu\right) \leq (g)\int_X \Phi(f) d\mu. \tag{4}$$

*Proof.* Since $\Phi(f) \leq f$ and $g$ is decrasing,

$$g \circ \Phi(f) \geq g \circ f. \tag{5}$$

By (5) and monotonicity of the Lebesgue integral with respect to $g \circ \mu$, we have

$$\int_X g \circ \Phi(f)\ dg \circ \mu \geq \int_X g \circ f\ dg \circ \mu. \tag{6}$$

Since $g^{-1}$ is decreasing, by (6), we have

$$g^{-1}\int_X g \circ \Phi(f)\ dg \circ \mu \leq g^{-1}\int_X g \circ f\ dg \circ \mu. \tag{7}$$

By (7),

$$\begin{aligned}
\Phi\left((g)\int_X f d\mu\right) &= \Phi\left(g^{-1}\int_X g \circ f\ dg \circ \mu\right) \\
&\leq g^{-1}\int_X g \circ f\ dg \circ \mu \\
&\leq g^{-1}\int_X g \circ \Phi(f)\ dg \circ \mu \\
&= (g)\int_X \Phi(f) d\mu.
\end{aligned}$$

□

**Theorem 2.3.** *Let $g$ be an increasing function and $(X, \sum, g \circ \mu)$ be the Lebesgue measure space and $f \in L^1_{\oplus}(\mu)$ with $(g)\int_X f d\mu = m$. If $\Phi : [0, \infty) \longrightarrow [0, \infty)$ is strictly increasing function such that $\Phi(x) \geq x$, for every $x \in [0, m]$, and $\Phi(f) \in L^1_{\oplus}(\mu)$, then*

$$\Phi\left((g)\int_X f d\mu\right) \geq (g)\int_X \Phi(f) d\mu. \tag{8}$$

*Proof.* Since $\Phi(f) \leq f$ and $g$ is increasing,

$$g \circ \Phi(f) \leq g \circ f. \tag{9}$$

By (9) and monotonicity of the Lebesgue integral with respect to $g \circ \mu$, we have

$$\int_X g \circ \Phi(f)\ dg \circ \mu \leq \int_X g \circ f\ dg \circ \mu. \tag{10}$$

Since $g^{-1}$ is increasing, by (10), we have

$$g^{-1} \int_X g \circ \Phi(f)\ dg \circ \mu \leq g^{-1} \int_X g \circ f\ dg \circ \mu. \tag{11}$$

By (11),

$$\begin{aligned} \Phi\left((g)\int_X f d\mu\right) &= \Phi\left(g^{-1}\int_X g \circ f\ dg \circ \mu\right) \\ &\geq g^{-1}\int_X (g \circ f)\ dg \circ \mu \\ &\geq g^{-1}\int_X g \circ \Phi(f)\ dg \circ \mu \\ &= (g)\int_X \Phi(f) d\mu. \end{aligned}$$

□

## 3. Jensen type inequality for the $\bar{g}$-integral

Let $I([0,\infty])$ be the set of all bounded closed intervals in $[0,\infty]$ as follows :

$$I([0,\infty]) = \{\bar{a} = [a_l, a_r] | a_l, a_r \in [0,\infty] \text{ and } a_l \leq a_r\}$$

For these intervals, we define the order, the strictly order, and strong strictly order of intervals as follows:

**Definition 3.1.** ([5]) If $\bar{a} = [a_l, a_r], \bar{b} = [b_l, b_r] \in I([0,\infty])$, then we define order ($\leq$), strictly order ($<$), and strong strictly order ($\prec_s$) as follows :

(a) $\bar{a} \leq \bar{b}$ if and only if $a_l \leq b_l$ and $a_r \leq b_r$,
(b) $\bar{a} < \bar{b}$ if and only if $\bar{a} \leq \bar{b}$ and $\bar{a} \neq \bar{b}$,
(c) $\bar{a} \prec_s \bar{b}$ if and only if $a_l < b_l$ and $a_r < b_r$.

**Definition 3.2.** A mapping $\overline{\Phi} = [\Phi_l, \Phi_r] : [0,\infty) \times [0,\infty) \longrightarrow [0,\infty) \times [0,\infty)$ by $\overline{\Phi}(x,y) = [\Phi_l(x), \Phi_r(y)]$ is called a strictly increasing function if for all $\bar{x} = [x_l, x_r], \bar{y} = [y_l, y_r] \in [0,\infty) \times [0,\infty)$,

$$\bar{x} \prec_s \bar{y} \Rightarrow \overline{\Phi}(\bar{x}) \prec_s \overline{\Phi}(y).$$

From Definition 2.2, we directly obtain the following theorem.

**Theorem 3.1.** *Let* $\overline{\Phi} = [\Phi_l, \Phi_r] : [0,\infty)\times[0,\infty) \longrightarrow [0,\infty)\times[0,\infty)$ *by* $\overline{\Phi}(x,y) = [\Phi_l(x), \Phi_r(y)]$ *be a mapping. Then* $\overline{\Phi} = [\Phi_l, \Phi_r]$ *is an strictly increasing function if and only if* $\Phi_l$ *and* $\Phi_r$ *are strictly increasing functions.*

We remark that if $f \in L^1_{\oplus}(\nu) = L^1(g\circ\mu)$, then $f$ is a $(g)$-integrable function and $g^{-1}\int_X (g\circ g)\, dg\circ\mu$ is finite. By Theorem 2.1, and (1), we obtain the following theorem.

**Theorem 3.2.** *Let* $F$ *be a pseudo-inequality bounded interval-valued function with border functions* $f_l$ *and* $f_r$*. If* $\bar{g} = [g_l, g_r]$ *is a monotone function and* $g_l \circ f_l \in L^1(g_l\circ\mu)$ *and* $g_r \circ f_r \in L^1(g_r\circ\mu)$*. Then we have*

$$(\bar{g})\int_X \bar{f} d\mu = \left[(g_l)\int_X f_l d\mu, (g_r)\int_X f_r d\mu\right]. \tag{12}$$

*Proof.* By (1), we have

$$\begin{aligned}(g_l)\int_X f_l d\mu &= g_l^{-1}\left(\int_X (g_l\circ f)\, dg_l\circ\mu\right)\\ &= \int_X^{\oplus_{g_l}} f\odot_{g_l} d\mu.\end{aligned} \tag{13}$$

and

$$\begin{aligned}(g_r)\int_X f_r d\mu &= g_r^{-1}\left(\int_X (g_r\circ f)\, dg_r\circ\mu\right)\\ &= \int_X^{\oplus_{g_r}} f\odot_{g_r} d\mu.\end{aligned} \tag{14}$$

By (13) and (14), and Theorem 2.1, we have

$$\begin{aligned}(\bar{g})\int_X \bar{f} d\mu &= \int_X^{\oplus_{\bar{g}}} F\odot_{\bar{g}} d\mu\\ &= \left[\int_X^{\oplus_{g_l}} f_l\odot_{g_l} d\mu, \int_X^{\oplus_{g_r}} f_r\odot_{g_r} d\mu\right]\\ &= \left[(g_l)\int_X f_l d\mu, (g_r)\int_X f_r d\mu\right].\end{aligned}$$

□

Finally, we obtain the following Jensen inequality for the $\bar{g}$-integral with respect to a .

**Theorem 3.3.** *Let* $\bar{g}$ *be a decreasing function and* $(X, \sum, g_s\circ\mu)$ *be the Lebesgue measure space for* $s = l, r$ *and* $g_l\circ f_l \in L^1(g_l\circ\mu)$ *and* $g_r\circ f_r \in L^1(g_r\circ\mu)$ *with* $(g_l)\int_X f_l d\mu = m_l$ *and* $(g_r)\int_X f_r d\mu = m_r$*. If* $\overline{\Phi} = [\Phi_l, \Phi_r] : [0,\infty)\times[0,\infty) \longrightarrow [0,\infty)\times[0,\infty)$ *by* $\overline{\Phi}(x,y) = [\Phi_l(x), \Phi_r(y)]$ *is strictly increasing function such that* $\overline{\Phi}(x,y) \le (x,y)$ *for every* $(x,y) \in [0,m_l]\times[0,m_r]$ *and* $\Phi_l(f_l) \in L^1(g_l\circ\mu)$ *and* $\Phi_r(f_r) \in L^1(g_r\circ\mu)$*, then we have*

$$\overline{\Phi}\left((\bar{g})\int_X F d\mu\right) \le (\bar{g})\int_X \overline{\Phi}(F) d\mu. \tag{15}$$

*Proof.* By Theorem 3.1 we have the following two inequalities :

$$(\bar{g})\int_X F d\mu = \left[(g_l)\int_X f_l d\mu, (g_r)\int_X f_r d\mu\right] \tag{16}$$

and

$$(\bar{g})\int_X \overline{\Phi}(F)d\mu = \left[(g_l)\int_X \Phi_l(f_l)d\mu, (g_r)\int_X \Phi_r(f_r)d\mu\right]. \tag{17}$$

By Theorem 2.2, we have

$$\Phi_s\left((g_s)\int_X f_s d\mu\right) \leq (g_s)\int_X \Phi_s(f_x)d\mu, \tag{18}$$

for $s = l, r$. By (16), (17) and (18), we obtain the following result :

$$\begin{aligned}
\Phi\left((\bar{g}\int_X F d\mu\right) &= \left[\Phi_l\left((g_l)\int_X f_l d\mu\right), \Phi_r\left((g_r)\int_X f_r d\mu\right)\right] \\
&\leq \left[(g_l)\int_X \Phi_l(f_l)d\mu, (g_r)\int_X \Phi_r(f_r)d\mu\right] \\
&= (\bar{g})\int_X \left[\Phi_l(f_l), \Phi_r(f_r)\right] d\mu \\
&= (\bar{g})\int_X \overline{\Phi}(F)d\mu.
\end{aligned}$$

□

**Theorem 3.4.** *Let $\bar{g}$ be an increasing function and $(X, \sum, g_s \circ \mu)$ be the Legesgue measure space and $g_l \circ f_l \in L^1(g_l \circ \mu)$ and $g_r \circ f_r \in L^1(g_r \circ \mu)$ with $(g_l)\int_X f_l d\mu = m_l$ and $(g_r)\int_X f_r d\mu = m_r$.*

*If $\overline{\Phi} = [\Phi_l, \Phi_r] : [0,\infty) \times [0,\infty) \longrightarrow [0,\infty) \times [0,\infty)$ by $\overline{\Phi}(x,y) = [\Phi_l(x), \Phi_r(y)]$ for all $(x,y) \in [0,\infty) \times [0,\infty)$ is strictly increasing such that $\overline{\Phi}(x,y) \geq (x,y)$ for every $(x,y) \in [0, m_l] \times [0, m_r]$ and $\Phi_l(f_l) \in L^1(g_l \circ \mu)$ and $\Phi_r(f_r) \in L^1(g_r \circ \mu)$,*

$$\Phi_l\left((\bar{g})\int_X F d\mu\right) \geq (\bar{g})\int_X \overline{\Phi}(F)d\mu. \tag{19}$$

*Proof.* By using Theorem 2.3,we have

$$\Phi_s\left((g_s)\int_X f_s d\mu\right) \geq (g_s)\int_X \Phi_s(f_s)d\mu \tag{20}$$

for $s = l, r$. By (16), (17) and (20), we obtain the following result:

$$\begin{aligned}
\Phi\left((\bar{g})\int_X F d\mu\right) &= \left[\Phi_l\left((g_l)\int_X f_l d\mu\right), \Phi_r\left((g_r)\int_X f_r d\mu\right)\right] \\
&\geq \left[(g_l)\int_X \Phi_l(f_l)d\mu, (g_r)\int_X \Phi_r(f_r)d\mu\right] \\
&= (\bar{g})\int_X \left[\Phi_l(f_l), \Phi_r(f_r)\right] d\mu \\
&= (\bar{g})\int_X \overline{\Phi}(F)d\mu.
\end{aligned}$$

□

## References

[1] R. J. Aumann, *Integrals of set-valued functions*, *J. Math. Anal. Appl.* **12** (1965) 1-12.
[2] P. Benvenuti, R. Mesiar, *Pseudo-arithmetical operations as a basis for the general measure and integration theory*, *Inf. Sci.* **160** (2004) 1-11.
[3] G. Deschrijver, *Generalized arithmetic operators and their relationship to t-norms in interval- valued fuzzy set theory*, *Fuzzy Sets Syst.* **160** (2009) 3080-3102.
[4] J. Fang, *On the convergence theorems of generalized fuzzy integral sequence*, *Fuzzy Sets Syst.* **124** (2001) 117-123.
[5] T. Grbíc. I Štajner-Papuga, M. Štrboja, *An approach to pseudo-integration of set-valued functions*, *Inform. Sci.* **181** (2011) 2278-2292.
[6] L. C. Jang, *A note on the interval-valued generalized fuzzy integral by means of an interval-representable pseudo-multiplication and their convergence properties*, *Fuzzy Sets Syst.* **222** (2013) 45-57.
[7] L. C. Jang, *A note on the pseudo Stolarsky type inequality for the $\bar{g}$-integral*, *Appl. Math. Comp.* **269** (2015) 809-815.
[8] M. Kaluszka, A. Okolewski, M. Boczek, *On the Jensen type inequality for generalized Sugeno integral*, *Inf. Sci.* **266** (2014) 140-147.
[9] R. Mesiar, E. Pap, *Idemponet integral as limit of g-integrals*, *Fuzzy Sets Syst.* **102** (1999) 385-392.
[10] D. R. Ralescu, G. Adams, *The fuzzy integral*, *J. Math. Anal. Appl.* **75** (1980) 562-570.
[11] H. Román-Flores, A. Flores-Franulic, Y. Chalco Cano, *A Jensen type inequality for fuzzy integrals*, *Inf. Sci.* **177** (2007) 3192-3201.
[12] C. Wu, S. Wang, M. Ma, *Generalized fuzzy integrals : Part 1. Fundamental concept*, *Fuzzy Sets Syst.* **70** (1995) 75-87.
[13] D. Zhang, C. Guo, *Generalized fuzzy integrals of set-valued functions*, *Fuzzy Sets Syst.* **76** (1995) 365-373.

# SOME NEW HERMITE-HADAMARD TYPE INEQUALITIES FOR OPERATOR $m$-CONVEX AND $(\alpha, m)$-CONVEX FUNCTIONS ON THE CO-ORDINATES

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Abstract. In this paper, operator $m$-convex and $(\alpha, m)$-convex function on the co-ordinates are defined, and some new integral inequalities of Hermite-Hadamard type for operator $m$-convex and $(\alpha, m)$-convex on the co-ordinates are established.

## 1. Introduction

Throughout this paper, we adopt the notations: $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}_0 = [0, \infty)$.

The following inequality holds for any convex function $f$ defined on $\mathbb{R}$ and $a, b \in \mathbb{R}$ with $a < b$

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if $f$ is concave on $[a, b]$. The inequality (1.1) is well known in the literature as Hermite-Hadamard's inequality. We note that the Hermite-Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality.

The concept of $m$-convexity was first introduced by G. Toader in [19] (see also [2]) and it is defined as follows:

**Definition 1.1** ([19])**.** The function $f : [0, b] \to \mathbb{R}$, $b > 0$ is said to be $m$-convex, where $m \in [0, 1]$, if for every $x, y \in [0, b]$ and $t \in [0, 1]$, we have

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y). \tag{1.2}$$

The class of $(\alpha, m)$-convex functions was also first introduced in [16] and it is defined as follows:

**Definition 1.2** ([16])**.** The function $f : [0, b] \to \mathbb{R}$, $b > 0$ is said to be $(\alpha, m)$-convex, where $(\alpha, m) \in [0, 1]^2$, if we have

$$f(tx + m(1-t)y) \leq t^\alpha f(x) + m(1-t^\alpha)f(y) \tag{1.3}$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$.

Also, the $m$-convex and $(\alpha, m)$-convex functions on the co-ordinates defined in a rectangle from the plane were introduced as follows.

**Definition 1.3** ([17])**.** Let $\triangle := [0, b] \times [0, d]$ be the bidimensional interval in $\mathbb{R}_0^2$ with $b > 0$ and $d > 0$. For some $m \in [0, 1]$, the function $f : \triangle \to \mathbb{R}$ is said to be $m$-convex if the following inequality

$$f(\lambda x + (1-\lambda)z, \lambda y + m(1-\lambda)w) \leq \lambda f(x, y) + m(1-\lambda)f(z, w) \tag{1.4}$$

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holds for all $(x,y),(z,w)\in\triangle$ and $\lambda\in(0,1)$.

**Definition 1.4** ([17]). For some $m\in[0,1]$, a function $f:\triangle:=[0,b]\times[0,d]\subseteq\mathbb{R}_0^2\to\mathbb{R}$ which is $m$-convex on $\triangle$ will be called $m$-convex on the co-ordinates with $b>0$ and $d>0$ if the partial mappings

$$f_y:[a,b]\to\mathbb{R},f_y(u):=f(u,y)$$

and

$$f_x:[c,d]\to\mathbb{R},f_x(v):=f(x,v)$$

are $m$-convex for all $y\in[c,d]$ and $x\in[a,b]$.

**Definition 1.5** ([17]). Let $\triangle:=[0,b]\times[0,d]$ be the bidimensional interval in $\mathbb{R}_0^2$ with $b>0$ and $d>0$. For some $(\alpha,m)\in[0,1]^2$, the function $f:\triangle\to\mathbb{R}$ is said to be $(\alpha,m)$-convex if the following inequality

$$f(\lambda x+(1-\lambda)z,\lambda y+m(1-\lambda)w)\leq\lambda^\alpha f(x,y)+m(1-\lambda^\alpha)f(z,w)\tag{1.5}$$

holds for all $(x,y),(z,w)\in\triangle$ and $\lambda\in(0,1)$.

**Definition 1.6** ([17]). For some $(\alpha,m)\in[0,1^2]$, a function $f:\triangle:=[0,b]\times[0,d]\subseteq\mathbb{R}_0^2\to\mathbb{R}$ which is $(\alpha,m)$-convex on $\triangle$ will be called $(\alpha,m)$-convex on the co-ordinates with $b>0$ and $d>0$ if the partial mappings

$$f_y:[a,b]\to\mathbb{R},f_y(u):=f(u,y)$$

and

$$f_x:[c,d]\to\mathbb{R},f_x(v):=f(x,v)$$

are $(\alpha,m)$-convex for all $y\in[c,d]$ and $x\in[a,b]$.

In recent years several extensions and generalizations have been considered for classical convexity. A significant generalization of convex functions is that of operator functions introduced by S. S. Dragomir in [6].

We review the operator order in $B(H)$ and the continuous functional calculus for a bounded self-adjoint operator. For self-adjoint operators $A,B\in B(H)$, we write $A\leq B$ if $\langle Ax,x\rangle\leq\langle Bx,x\rangle$ for every vector $x\in H$, we call it the operator order.

Let $A$ be a bounded self-adjoint linear operator on a complex Hilbert space $(H;\langle.,.\rangle)$. The Gelfand map establishes a $*$-isometrically isomorphism $\Phi$ between the set $C(Sp(A))$ of all continuous complex-valued functions defined on the spectrum of $A$, denoted $Sp(A)$, and the $C^*$-algebra $C^*(A)$ generated by $A$ and the identity operator $1_H$ on $H$ as follows (see for instance [8], p.3). For any $f,g\in C(Sp(A))$ and any $\alpha,\beta\in\mathbb{C}$, we have

$(i)$ $\Phi(\alpha f+\beta g)=\alpha\Phi(f)+\beta\Phi(g)$;

$(ii)$ $\Phi(fg)=\Phi(f)\Phi(g)$ and $\Phi(f^*)=\Phi(f)^*$;

$(iii)$ $\|\Phi(f)\|=\|f\|:=\sup_{t\in Sp(A)}|f|$;

$(iv)$ $\Phi(f_0)=1_H$ and $\Phi(f_1)=A$, where $f_0(t)=1$ and $f_1(t)=t$ for $t\in Sp(A)$.

With this notation, we define

$$f(A):=\Phi(f)\quad\text{for all}\quad f\in C(Sp(A))\tag{1.6}$$

and we call it the continuous functional calculus for a bounded self-adjoint operator $A$.

A real valued continuous function $f$ on an interval $I\subseteq\mathbb{R}$ is said to be operator convex (operator concave) if the operator inequality

$$f((1-\lambda)A+\lambda B)\leq(\geq)(1-\lambda)f(A)+\lambda f(B)\tag{1.7}$$

holds in the operator order in $B(H)$, for all $\lambda \in [0,1]$ and for every bounded self-adjoint operators $A$ and $B$ in $B(H)$ whose spectra are contained in $I$.

In [20], Wang defined operator $m$-convex and $(\alpha, m)$-convex functions in the following way:.

**Definition 1.7.** Let $[0,b] \subseteq \mathbb{R}_0$ with $b > 0$ and $K$ be a convex set of $B(H)^+$. A continuous function $f : [0,b] \to \mathbb{R}$ is said to be operator $m$-convex on $[0,b]$ for operators in $K$, if

$$f(tA + m(1-t)B) \leq tf(A) + m(1-t)f(B) \tag{1.8}$$

in the operator order in $B(H)$, for all $t \in [0,1]$ and every positive operators $A$ and $B$ in $K$ whose spectra are contained in $[0,b]$ and for some fixed $m \in [0,1]$.

**Definition 1.8.** Let $[0,b] \subseteq \mathbb{R}_0$ with $b > 0$ and $K$ be a convex set of $B(H)^+$. A continuous function $f : [0,b] \to \mathbb{R}$ is said to be operator $(\alpha, m)$-convex on $[0,b]$ for operators in $K$, if

$$f(tA + m(1-t)B) \leq t^\alpha f(A) + m(1-t^\alpha)f(B) \tag{1.9}$$

in the operator order in $B(H)$, for all $t \in [0,1]$ and every positive operators $A$ and $B$ in $K$ whose spectra are contained in $[0,b]$ and for some fixed $(\alpha, m) \in [0,1]^2$.

Also, author proved the following inequalities in [20]:

**Theorem 1.1** ([20]). *Let the continuous function $f : \mathbb{R}_0 \to \mathbb{R}$ be operator $(\alpha, m)$-convex for operators in $K \subseteq B(H)^+$ with $(\alpha, m) \in (0,1]^2$. Then for all positive operator $A, B \in K$ with spectra in $\mathbb{R}_0$, the following inequality holds:*

$$\int_0^1 f(tA + (1-t)B)\,\mathrm{d}t \leq min\left\{ \frac{f(A) + \alpha m f\left(\frac{B}{m}\right)}{\alpha + 1}, \frac{f(B) + \alpha m f\left(\frac{A}{m}\right)}{\alpha + 1} \right\}. \tag{1.10}$$

**Theorem 1.2** ([20]). *Let the continuous function $f : \mathbb{R}_0 \to \mathbb{R}$ be operator $(\alpha, m)$-convex for operators in $K \subseteq B(H)^+$ with $(\alpha, m) \in (0,1]^2$. Then for all positive operator $A, B \in K$ with spectra in $\mathbb{R}_0$, the following inequalities hold:*

$$\begin{aligned} f\left(\frac{A+B}{2}\right) \leq & \frac{1}{2^\alpha} \int_0^1 \left[ f(tA + (1-t)B) + m(2^\alpha - 1) f\left( \frac{(1-t)A + tB}{m} \right) \right] \mathrm{d}t \\ \leq & \frac{1}{2^{\alpha+1}(\alpha+1)} \left\{ f(A) + f(B) + m(\alpha + 2^\alpha - 1) \left[ f\left(\frac{A}{m}\right) + f\left(\frac{B}{m}\right) \right] \right. \\ & \left. + \alpha m^2 (2^\alpha - 1) \left[ f\left(\frac{A}{m^2}\right) + f\left(\frac{B}{m^2}\right) \right] \right\}. \end{aligned} \tag{1.11}$$

**Theorem 1.3** ([20]). *Let the continuous function $f : \mathbb{R}_0 \to \mathbb{R}$ be operator $(\alpha, m)$-convex for operators in $K \subseteq B(H)^+$ with $(\alpha, m) \in (0,1]^2$. Then for all positive operator $A, B \in K$ with spectra in $\mathbb{R}_0$, the following inequality holds:*

$$\int_0^1 f(tA + (1-t)B)\,\mathrm{d}t \leq \frac{f(A) + f(B) + \alpha m \left[ f\left(\frac{A}{m}\right) + f\left(\frac{B}{m}\right) \right]}{2(\alpha + 1)}. \tag{1.12}$$

**Theorem 1.4** ([20]). *Let the continuous function $f : \mathbb{R}_0 \to \mathbb{R}$ be operator $(\alpha, m)$-convex for operators in $K \subseteq B(H)^+$ with $(\alpha, m) \in (0,1]^2$. Then for all positive operator $A, B \in K$ with spectra in $\mathbb{R}_0$, the following inequality holds:*

$$\int_0^1 \left[ f(tA + m(1-t)B) + f(tB + m(1-t)A) \right] \mathrm{d}t \leq \frac{(1 + m\alpha)[f(A) + f(B)]}{\alpha + 1}. \tag{1.13}$$

**Theorem 1.5** ([20]). *Let the continuous function $f : \mathbb{R}_0 \to \mathbb{R}$ be operator $(\alpha, m)$-convex for operators in $K \subseteq B(H)^+$ with $(\alpha, m) \in (0, 1]^2$. Then for all positive operator $A, B \in K$ with spectra in $\mathbb{R}_0$, the following inequalities hold:*

$$\begin{aligned} & f\left(\frac{2-m}{2}B + \frac{m}{2}(mA)\right) \\ \leq & \frac{1}{2^\alpha}\int_0^1 \left[f\big(t(2-m)B + (1-t)m^2A\big) + m(2^\alpha - 1)f\left(\frac{(1-t)(2-m)B + tm^2A}{m}\right)\right] \mathrm{d}t \\ \leq & \frac{1}{2^\alpha(\alpha+1)}\Big[f((2-m)B) + m\big(\alpha + 2^\alpha - 1\big)f(mA) \\ & + m^2\alpha(2^\alpha - 1)f\left(\frac{(2-m)B}{m^2}\right)\Big]. \end{aligned} \tag{1.14}$$

For recent results related to Hermite-Hadamard type inequalities are given in [1], [4], [5], [7], [8], [9], [10], [13], [14], and plenty of references therein.

The main purpose of this paper is to establish some new Hadamard type inequalities for operator $m$)-convex and $(\alpha, m)$-convex functions on the co-ordinates.

## 2. operator co-ordinated $m$-convex and $(\alpha, m)$-convex functions

Let $I_1, I_2$ be real intervals and let $f : I_1 \times I_2 \to \mathbb{R}$ be a Borel measurable and essentially bounded function. Let $X = (X_1, X_2)$ be a 2-tuple of bounded self-adjoint operators on Hilbert spaces $H_1, H_2$ such that the spectrum of $X_i$ is contained in $I_i$ for $i = 1, 2$. We say that such a 2-tuple is in the domain of $f$. If

$$X_i = \int_{I_i} \lambda_i E_i(d\lambda_i), i = 1, 2$$

is the spectral decomposition of $X_i$ where $E_i$ is a bounded positive measure on $I_i$, we define

$$f(X) = \int_{I_1 \times I_2} f(\lambda_1, \lambda_2) E_1(d\lambda_1) \otimes E_2(d\lambda_2)$$

as a bounded self-adjoint operator on the tensor product $H_1 \otimes H_2$. If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction have the property that

$$f(X_1, X_2) = f_1(X_1) \otimes f_2(X_2),$$

whenever $f$ can be separated as a product $f(t_1, t_2) = f_1(t_1)f_2(t_2)$ of 2 functions each depending on only one variable.

With above functional calculus, we say that a function $f : I_1 \times I_2 \to \mathbb{R}$ is operator convex if $f$ is continuous and the operator inequality

$$f(tX + (1-t)Y) \leq tf(X) + (1-t)f(Y) \tag{2.1}$$

holds for all 2-tuples of self-adjoint operators $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$ in the domain of $f$ acting on any Hilbert spaces $H_1, H_2$ and for all $t \in [0, 1]$.

In [21], Hermite-Hadamard type inequality for the co-ordinated operator convex functions is given.

**Theorem 2.1.** *Suppose that a continuous function $f : I_1 \times I_2 \subseteq \mathbb{R}^2 \to \mathbb{R}$ is operator convex on the co-ordinates for all 2-tuples of self-adjoint operators in the domain of $f$ acting on any Hilbert spaces $H_1, H_2$. Then we have the inequalities*

$$f\left(\frac{A+C}{2}, \frac{B+D}{2}\right)$$

$$
\begin{aligned}
&\leq \frac{1}{2}\left[\int_0^1 f\left(tA+(1-t)C, \frac{B+D}{2}\right)\mathrm{d}t + \int_0^1 f\left(\frac{A+C}{2}, \lambda B+(1-\lambda)D\right)\mathrm{d}\lambda\right] \\
&\leq \int_0^1\int_0^1 f(tA+(1-t)C, \lambda B+(1-\lambda)D)\,\mathrm{d}t\,\mathrm{d}\lambda \\
&\leq \frac{1}{4}\left[\int_0^1 f(tA+(1-t)C, B)\,\mathrm{d}t + \int_0^1 f(tA+(1-t)C, D)\,\mathrm{d}t\right. \\
&\quad \left. + \int_0^1 f(A, \lambda B+(1-\lambda)D)\,\mathrm{d}\lambda + \int_0^1 f(C, \lambda B+(1-\lambda)D)\,\mathrm{d}\lambda\right] \\
&\leq \frac{f(A,B)+f(A,D)+f(C,B)+f(C,D)}{4}, \qquad (2.2)
\end{aligned}
$$

*where* $(A,B),(C,D)\in B(H_1)\otimes B(H_2)$ *with spectra in* $I_1\times I_2$.

For some fundamental results on operator convex and operator monotone functions of several variables, see [11], [12], [15], and the references therein

Now we give the concepts of operator $m$-convex and $(\alpha,m)$-convex functions on the co-ordinates.

**Definition 2.1.** A continuous function $f:[0,b]\times[0,d]\subseteq\mathbb{R}_0^2\to\mathbb{R}$ is said to be operator $m$-convex with $b>0$ and $d>0$ for some fixed $m\in[0,1]$ if the operator inequality

$$f(tX_1+(1-t)Y_1, tX_2+m(1-t)Y_2)\leq tf(X_1,X_2)+m(1-t)f(Y_1,Y_2) \qquad (2.3)$$

holds for all 2-tuples of self-adjoint operators $X=(X_1,X_2)$ and $Y=(Y_1,Y_2)$ in the domain of $f$ acting on any Hilbert spaces $H_1,H_2$ and for all $t\in(0,1)$.

**Definition 2.2.** A continuous function $f:[0,b]\times[0,d]\subseteq\mathbb{R}_0^2\to\mathbb{R}$ which is operator $m$-convex on $[0,b]\times[0,d]$ with $b>0$ and $d>0$ is said to be operator $m$-convex on the co-ordinates for some fixed $m\in[0,1]$ if the partial mapping

$$f_{X_2}: I_1\to\mathbb{R}, f_{X_2}(u):=f(u,X_2)$$

and

$$f_{X_1}: I_2\to\mathbb{R}, f_{X_1}(v):=f(X_1,v)$$

are operator $m$-convex for all operators $X_2\in B(H_2)$ and $X_1\in B(H_1)$ whose spectra are contained in $[0,d]$ and $[0,b]$, respectively.

**Definition 2.3.** A continuous function $f:[0,b]\times[0,d]\subseteq\mathbb{R}_0^2\to\mathbb{R}$ is said to be operator $(\alpha,m)$-convex with $b>0$ and $d>0$ for some fixed $(\alpha,m)\in[0,1]^2$ if the operator inequality

$$f(tX_1+(1-t)Y_1, tX_2+m(1-t)Y_2)\leq t^\alpha f(X_1,X_2)+m(1-t^\alpha)f(Y_1,Y_2) \qquad (2.4)$$

holds for all 2-tuples of self-adjoint operators $X=(X_1,X_2)$ and $Y=(Y_1,Y_2)$ in the domain of $f$ acting on any Hilbert spaces $H_1,H_2$ and for all $t\in(0,1)$.

**Definition 2.4.** A continuous function $f:[0,b]\times[0,d]\subseteq\mathbb{R}_0^2\to\mathbb{R}$ which is operator $(\alpha,m)$-convex on $[0,b]\times[0,d]$ with $b>0$ and $d>0$ is said to be operator $(\alpha,m)$-convex on the co-ordinates for some fixed $(\alpha,m)\in[0,1]^2$ if the partial mapping

$$f_{X_2}: I_1\to\mathbb{R}, f_{X_2}(u):=f(u,X_2)$$

and

$$f_{X_1}: I_2\to\mathbb{R}, f_{X_1}(v):=f(X_1,v)$$

are operator $(\alpha,m)$-convex for all operators $X_2\in B(H_2)$ and $X_1\in B(H_1)$ whose spectra are contained in $[0,d]$ and $[0,b]$, respectively.

*Remark* 2.1. It can be easily seen that for $(\alpha, m) \in \{(1,1),(1,m)\}$ one obtains the classes of operator convex and operator $m$-convex functions of two variables, respectively.

The following lemmas hold:

**Lemma 2.1.** *For $b > 0$, $d > 0$, and some fixed $m \in [0,1]$, every operator $m$-convex mapping $f : [0,b] \times [0,d] \subseteq \mathbb{R}_0^2 \to \mathbb{R}$ is operator $m$-convex on the co-ordinates, but the converse is not generally true.*

*Proof.* Suppose that $f$ is operator $m$-convex mapping on $[0,b] \times [0,d]$. Consider $f_{X_1} : [0,d] \to \mathbb{R}, f_{X_1}(v) := f(X_1, v)$. Then for all $t \in (0,1)$ and operators $A, C \in B(H_2)$ with spectra in $[0,d]$, one has

$$\begin{aligned} f_{X_1}(tA + m(1-t)C) &= f(tX_1 + (1-t)X_1, tA + m(1-t)C) \\ &\leq tf(X_1, A) + m(1-t)f(X_1, C) = tf_{X_1}(A) + m(1-t)f_{X_1}(C), \end{aligned}$$

where $X_1 \in B(H_1)$ with spectra in $[0,b]$. It shows the operator $m$-convexity of $f_{X_1}$.

The fact that $f_{X_2} : [0,b] \to \mathbb{R}, f_{X_2}(u) := f(u, X_2)$ is also operator $m$-convex on $[0,b]$ for all operators $X_2 \in B(H_2)$ with spectra in $[0,d]$ goes likewise and we shall omit the details.

In [21], authors gave a mapping $f : [0,1]^2 \to \mathbb{R}_0$ defined by $f(r_1, r_2) = r_1 \times r_2$ which is operator convex on the co-ordinates but is not operator convex. We consider the same function with $m = 1$ to prove that the operator $m$-convexity on the co-ordinates does not imply the operator $m$-convexity.

The Lemma 2.1 is thus proved. □

Similarly, we state the following elementary results without proof.

**Lemma 2.2.** *For $b > 0$, $d > 0$, and some fixed $(\alpha, m) \in [0,1]^2$, every operator $(\alpha, m)$-convex mapping $f : [0,b] \times [0,d] \subseteq \mathbb{R}_0^2 \to \mathbb{R}$ is operator $(\alpha, m)$-convex on the co-ordinates, but the converse is not generally true.*

## 3. Hermite-Hadamard type inequalities for operator $m$-convex and $(\alpha, m)$-convex functions on the co-ordinates

We will now point out some new inequalities of the Hermite-Hadamard type.

**Theorem 3.1.** *Let some fixed $(\alpha, m) \in (0,1]^2$ and a continuous function $f : \mathbb{R}_0^2 \to \mathbb{R}$ be operator $(\alpha, m)$-convex on the co-ordinates for all 2-tuples of positive self-adjoint operators in the domain of $f$ acting on any Hilbert spaces $H_1, H_2$. Then one has*

$$\int_0^1 \int_0^1 f(tA + (1-t)C, \lambda B + (1-\lambda)D)\, \mathrm{d}\lambda\, \mathrm{d}t \leq \frac{min\{v_1, v_2\} + min\{v_3, v_4\}}{2(\alpha+1)}, \tag{3.1}$$

*where $(A,B),(C,D) \in B(H_1) \times B(H_2)$ with spectra in $\mathbb{R}_0^2$, and*

$$\begin{aligned} v_1 &= \int_0^1 f(tA + (1-t)C, B)\, \mathrm{d}t + \alpha m \int_0^1 f\left(tA + (1-t)C, \frac{D}{m}\right) \mathrm{d}t, \\ v_2 &= \int_0^1 f(tA + (1-t)C, D)\, \mathrm{d}t + \alpha m \int_0^1 f\left(tA + (1-t)C, \frac{B}{m}\right) \mathrm{d}t, \\ v_3 &= \int_0^1 f(A, \lambda B + (1-\lambda)D)\, \mathrm{d}\lambda + \alpha m \int_0^1 f\left(\frac{C}{m}, \lambda B + (1-\lambda)D\right) \mathrm{d}\lambda, \\ v_4 &= \int_0^1 f(C, \lambda B + (1-\lambda)D)\, \mathrm{d}\lambda + \alpha m \int_0^1 f\left(\frac{A}{m}, \lambda B + (1-\lambda)D\right) \mathrm{d}\lambda. \end{aligned} \tag{3.2}$$

*Proof.* Since the spectrum of $tA+(1-t)C$ and $\lambda B+(1-\lambda)D$ are contained in $\mathbb{R}_0$, and $f$ is continuous, the operator valued integrals $\int_0^1 f(tA+(1-t)C)\,\mathrm{d}t$, $\int_0^1 f(\lambda B+(1-\lambda)D)\,\mathrm{d}\lambda$ and $\int_0^1\int_0^1 f(tA+(1-t)C,\lambda B+(1-\lambda)D)\,\mathrm{d}t\,\mathrm{d}\lambda$ exist.

From the operator co-ordinated $(\alpha,m)$-convexity of $f$ and the inequality (1.10) it is easy to see that

$$\begin{aligned}&\int_0^1 f(tA+(1-t)C,\lambda B+(1-\lambda)D)\,\mathrm{d}t\\ \leq& min\Bigg\{\frac{f(A,\lambda B+(1-\lambda)D))+\alpha m f\big(\frac{C}{m},\lambda B+(1-\lambda)D)\big)}{\alpha+1},\\ &\frac{f(C,\lambda B+(1-\lambda)D))+\alpha m f\big(\frac{A}{m},\lambda B+(1-\lambda)D)\big)}{\alpha+1}\Bigg\}.\end{aligned}$$

Integrating this inequality on $[0,1]$ over $\lambda$, we deduce

$$\begin{aligned}&\int_0^1\int_0^1 f(tA+(1-t)C,\lambda B+(1-\lambda)D)\,\mathrm{d}t\,\mathrm{d}\lambda\\ \leq&\frac{1}{\alpha+1}min\Bigg\{\int_0^1 f(A,\lambda B+(1-\lambda)D)\,\mathrm{d}\lambda+\alpha m\int_0^1 f\bigg(\frac{C}{m},\lambda B+(1-\lambda)D\bigg)\,\mathrm{d}\lambda,\\ &\int_0^1 f(C,\lambda B+(1-\lambda)D)\,\mathrm{d}\lambda+\alpha m\int_0^1 f\bigg(\frac{A}{m},\lambda B+(1-\lambda)D\bigg)\,\mathrm{d}\lambda\Bigg\}.\end{aligned}\tag{3.3}$$

By a similar argument we get

$$\begin{aligned}&\int_0^1\int_0^1 f(tA+(1-t)C,\lambda B+(1-\lambda)D)\,\mathrm{d}\lambda\,\mathrm{d}t\\ \leq&\frac{1}{\alpha+1}min\Bigg\{\int_0^1 f(tA+(1-t)C,B)\,\mathrm{d}t+\alpha m\int_0^1 f\bigg(tA+(1-t)C,\frac{D}{m}\bigg)\,\mathrm{d}t,\\ &\int_0^1 f(tA+(1-t)C,D)\,\mathrm{d}t+\alpha m\int_0^1 f\bigg(tA+(1-t)C,\frac{B}{m}\bigg)\,\mathrm{d}t\Bigg\}.\end{aligned}\tag{3.4}$$

Summing the inequalities (3.3) and (3.4) and dividing by 2, we get the inequality (3.1).

The proof thus is complete. □

**Corollary 3.1.1.** *Under the assumptions of Theorem 3.1, choosing $\alpha=1$, we get the inequality for operator m-convex:*

$$\int_0^1\int_0^1 f(tA+(1-t)C,\lambda B+(1-\lambda)D)\,\mathrm{d}\lambda\,\mathrm{d}t\leq\frac{min\{u_1,u_2\}+min\{u_3,u_4\}}{4},\tag{3.5}$$

*where*

$$\begin{aligned}u_1&=\int_0^1 f(tA+(1-t)C,B)\,\mathrm{d}t+m\int_0^1 f\bigg(tA+(1-t)C,\frac{D}{m}\bigg)\,\mathrm{d}t,\\ u_2&=\int_0^1 f(tA+(1-t)C,D)\,\mathrm{d}t+m\int_0^1 f\bigg(tA+(1-t)C,\frac{B}{m}\bigg)\,\mathrm{d}t,\\ u_3&=\int_0^1 f(A,\lambda B+(1-\lambda)D)\,\mathrm{d}\lambda+m\int_0^1 f\bigg(\frac{C}{m},\lambda B+(1-\lambda)D\bigg)\,\mathrm{d}\lambda,\end{aligned}$$

$$u_4 = \int_0^1 f(C, \lambda B + (1-\lambda)D)\,\mathrm{d}\lambda + m\int_0^1 f\Big(\frac{A}{m}, \lambda B + (1-\lambda)D\Big)\,\mathrm{d}\lambda. \tag{3.6}$$

*Furthermore, for $\alpha, m = 1$ we have*

$$\begin{aligned}
&\int_0^1\int_0^1 f(tA + (1-t)C, \lambda B + (1-\lambda)D)\,\mathrm{d}\lambda\,\mathrm{d}t \\
\le&\frac{1}{4}\Big[\int_0^1 f(tA + (1-t)C, B)\,\mathrm{d}t + \int_0^1 f(tA + (1-t)C, D)\,\mathrm{d}t \\
&+ \int_0^1 f(A, \lambda B + (1-\lambda)D)\,\mathrm{d}\lambda + \int_0^1 f(C, \lambda B + (1-\lambda)D)\,\mathrm{d}\lambda\Big].
\end{aligned} \tag{3.7}$$

**Theorem 3.2.** *Let some fixed $(\alpha, m) \in (0,1]^2$ and a continuous function $f : \mathbb{R}_0^2 \to \mathbb{R}$ be operator $(\alpha, m)$-convex on the co-ordinates for all 2-tuples of positive self-adjoint operators in the domain of $f$ acting on any Hilbert spaces $H_1, H_2$. Then one has*

$$\begin{aligned}
&f\Big(\frac{A+C}{2}, \frac{B+D}{2}\Big) \\
\le&\frac{1}{2^{\alpha+1}}\Big\{\int_0^1\Big[f\Big(\frac{A+C}{2}, \lambda B + (1-\lambda)D\Big) + m(2^\alpha - 1)f\Big(\frac{A+C}{2}, \frac{(1-\lambda)B + \lambda D}{m}\Big)\Big]\,\mathrm{d}\lambda \\
&+ \int_0^1\Big[f\Big(tA + (1-t)C, \frac{B+D}{2}\Big) + m(2^\alpha - 1)f\Big(\frac{(1-t)A + tC}{m}, \frac{B+D}{2}\Big)\Big]\,\mathrm{d}t\Big\} \\
\le&\frac{1}{2^{\alpha+2}(\alpha+1)}\Big\{f\Big(\frac{A+C}{2}, B\Big) + f\Big(\frac{A+C}{2}, D\Big) + f\Big(A, \frac{B+D}{2}\Big) + f\Big(C, \frac{B+D}{2}\Big) \\
&+ m(\alpha + 2^\alpha - 1)\Big[f\Big(\frac{A+C}{2}, \frac{B}{m}\Big) + f\Big(\frac{A+C}{2}, \frac{D}{m}\Big) + f\Big(\frac{A}{m}, \frac{B+D}{2}\Big) + f\Big(\frac{C}{m}, \frac{B+D}{2}\Big)\Big] \\
&+ \alpha m^2(2^\alpha - 1)\Big[f\Big(\frac{A+C}{2}, \frac{B}{m^2}\Big) + f\Big(\frac{A+C}{2}, \frac{D}{m^2}\Big) \\
&+ f\Big(\frac{A}{m^2}, \frac{B+D}{2}\Big) + f\Big(\frac{C}{m^2}, \frac{B+D}{2}\Big)\Big]\Big\},
\end{aligned} \tag{3.8}$$

*where $(A, B), (C, D) \in B(H_1) \times B(H_2)$ with spectra in $\mathbb{R}_0^2$.*

*Proof.* By operator co-ordinated $(\alpha, m)$-convexity of $f$ and and the inequality (1.11), we can give

$$\begin{aligned}
&f\Big(\frac{A+C}{2}, \frac{B+D}{2}\Big) \\
\le&\frac{1}{2^\alpha}\int_0^1\Big[f\Big(tA + (1-t)C, \frac{B+D}{2}\Big) + m(2^\alpha - 1)f\Big(\frac{(1-t)A + tC}{m}, \frac{B+D}{2}\Big)\Big]\,\mathrm{d}t \\
\le&\frac{1}{2^{\alpha+1}(\alpha+1)}\Big\{f\Big(A, \frac{B+D}{2}\Big) + f\Big(C, \frac{B+D}{2}\Big) \\
&+ m(\alpha + 2^\alpha - 1)\Big[f\Big(\frac{A}{m}, \frac{B+D}{2}\Big) + f\Big(\frac{C}{m}, \frac{B+D}{2}\Big)\Big] \\
&+ \alpha m^2(2^\alpha - 1)\Big[f\Big(\frac{A}{m^2}, \frac{B+D}{2}\Big) + f\Big(\frac{C}{m^2}, \frac{B+D}{2}\Big)\Big]\Big\}
\end{aligned} \tag{3.9}$$

and

$$f\Big(\frac{A+C}{2}, \frac{B+D}{2}\Big)$$

$$
\begin{aligned}
&\leq \frac{1}{2^{\alpha}} \int_0^1 \left[ f\left( \frac{A+C}{2}, \lambda B + (1-\lambda) D \right) + m(2^{\alpha}-1) f\left( \frac{A+C}{2}, \frac{(1-\lambda)B + \lambda D}{m} \right) \right] \mathrm{d}\lambda \\
&\leq \frac{1}{2^{\alpha+1}(\alpha+1)} \left\{ f\left( \frac{A+C}{2}, B \right) + f\left( \frac{A+C}{2}, D \right) \right. \\
&\quad + m(\alpha + 2^{\alpha} - 1) \left[ f\left( \frac{A+C}{2}, \frac{B}{m} \right) + f\left( \frac{A+C}{2}, \frac{D}{m} \right) \right] \\
&\quad \left. + \alpha m^2 (2^{\alpha}-1) \left[ f\left( \frac{A+C}{2}, \frac{B}{m^2} \right) + f\left( \frac{A+C}{2}, \frac{D}{m^2} \right) \right] \right\} \qquad (3.10)
\end{aligned}
$$

Summing the inequalities (3.9) and (3.10) and dividing by 2, we get the inequality (3.8). The proof is completed. □

**Corollary 3.2.1.** *Under the assumptions of Theorem 3.2, choosing $\alpha = 1$, we get the inequality for operator m-convex:*

$$
\begin{aligned}
&f\left( \frac{A+C}{2}, \frac{B+D}{2} \right) \\
&\leq \frac{1}{4} \left\{ \int_0^1 \left[ f\left( \frac{A+C}{2}, \lambda B + (1-\lambda) D \right) + m f\left( \frac{A+C}{2}, \frac{(1-\lambda)B + \lambda D}{m} \right) \right] \mathrm{d}\lambda \right. \\
&\quad \left. + \int_0^1 \left[ f\left( tA + (1-t)C, \frac{B+D}{2} \right) + m f\left( \frac{(1-t)A + tC}{m}, \frac{B+D}{2} \right) \right] \mathrm{d}t \right\} \\
&\leq \frac{1}{16} \left\{ f\left( \frac{A+C}{2}, B \right) + f\left( \frac{A+C}{2}, D \right) + f\left( A, \frac{B+D}{2} \right) + f\left( C, \frac{B+D}{2} \right) \right. \\
&\quad + 2m \left[ f\left( \frac{A+C}{2}, \frac{B}{m} \right) + f\left( \frac{A+C}{2}, \frac{D}{m} \right) + f\left( \frac{A}{m}, \frac{B+D}{2} \right) + f\left( \frac{C}{m}, \frac{B+D}{2} \right) \right] \\
&\quad \left. + m^2 \left[ f\left( \frac{A+C}{2}, \frac{B}{m^2} \right) + f\left( \frac{A+C}{2}, \frac{D}{m^2} \right) + f\left( \frac{A}{m^2}, \frac{B+D}{2} \right) + f\left( \frac{C}{m^2}, \frac{B+D}{2} \right) \right] \right\}. \qquad (3.11)
\end{aligned}
$$

*Furthermore, for $\alpha, m = 1$ we have*

$$
\begin{aligned}
&f\left( \frac{A+C}{2}, \frac{B+D}{2} \right) \\
&\leq \frac{1}{2} \left[ \int_0^1 f\left( \frac{A+C}{2}, \lambda B + (1-\lambda) D \right) \mathrm{d}\lambda + \int_0^1 f\left( tA + (1-t)C, \frac{B+D}{2} \right) \mathrm{d}t \right] \\
&\leq \frac{1}{4} \left[ f\left( \frac{A+C}{2}, B \right) + f\left( \frac{A+C}{2}, D \right) + f\left( A, \frac{B+D}{2} \right) + f\left( C, \frac{B+D}{2} \right) \right]. \qquad (3.12)
\end{aligned}
$$

**Theorem 3.3.** *Let some fixed $(\alpha, m) \in (0,1]^2$ and a continuous function $f : \mathbb{R}_0^2 \to \mathbb{R}$ be operator $(\alpha, m)$-convex on the co-ordinates for all 2-tuples of positive self-adjoint operators in the domain of $f$ acting on any Hilbert spaces $H_1, H_2$. Then one has*

$$
\begin{aligned}
&\int_0^1 \int_0^1 f(tA + (1-t)C, \lambda B + (1-\lambda) D) \, \mathrm{d}\lambda \, \mathrm{d}t \\
&\leq \frac{1}{4(\alpha+1)} \left\{ \int_0^1 f(tA + (1-t)C, B) \, \mathrm{d}t + \int_0^1 f(tA + (1-t)C, D) \, \mathrm{d}t \right. \\
&\quad + \alpha m \left[ \int_0^1 f\left( tA + (1-t)C, \frac{B}{m} \right) \mathrm{d}t + \int_0^1 f\left( tA + (1-t)C, \frac{D}{m} \right) \mathrm{d}t \right] \\
&\quad \left. + \int_0^1 f(A, \lambda B + (1-\lambda) D) \, \mathrm{d}\lambda + \int_0^1 f(C, \lambda B + (1-\lambda) D) \, \mathrm{d}\lambda \right.
\end{aligned}
$$

$$+\alpha m\Big[\int_0^1 f\Big(\frac{A}{m},\lambda B+(1-\lambda)D\Big)\,\mathrm{d}\lambda+\int_0^1 f\Big(\frac{C}{m},\lambda B+(1-\lambda)D\Big)\,\mathrm{d}\lambda\Big]\Big\},\tag{3.13}$$

*where* $(A,B),(C,D)\in B(H_1)\times B(H_2)$ *with spectra in* $\mathbb{R}_0^2$.

*Proof.* Using operator co-ordinated $(\alpha,m)$-convexity of $f$ and the inequality (1.12), we can write

$$\begin{aligned}&\int_0^1 f(tA+(1-t)C,\lambda B+(1-\lambda)D)\,\mathrm{d}t\\ \le&\frac{1}{2(\alpha+1)}\Big\{f(A,\lambda B+(1-\lambda)D)+f(C,\lambda B+(1-\lambda)D)\\ &+\alpha m\Big[f\Big(\frac{A}{m},\lambda B+(1-\lambda)D\Big)+f\Big(\frac{C}{m},\lambda B+(1-\lambda)D\Big)\Big]\Big\}.\end{aligned}$$

Integrating this inequality on $[0,1]$ over $\lambda$, we deduce

$$\begin{aligned}&\int_0^1\int_0^1 f(tA+(1-t)C,\lambda B+(1-\lambda)D)\,\mathrm{d}t\,\mathrm{d}\lambda\\ \le&\frac{1}{2(\alpha+1)}\Big\{\int_0^1 f(A,\lambda B+(1-\lambda)D)\,\mathrm{d}\lambda+\int_0^1 f(C,\lambda B+(1-\lambda)D)\,\mathrm{d}\lambda\\ &+\alpha m\Big[\int_0^1 f\Big(\frac{A}{m},\lambda B+(1-\lambda)D\Big)\,\mathrm{d}\lambda+\int_0^1 f\Big(\frac{C}{m},\lambda B+(1-\lambda)D\Big)\,\mathrm{d}\lambda\Big]\Big\}.\end{aligned}\tag{3.14}$$

By a similar argument we get

$$\begin{aligned}&\int_0^1\int_0^1 f(tA+(1-t)C,\lambda B+(1-\lambda)D)\,\mathrm{d}\lambda\,\mathrm{d}t\\ \le&\frac{1}{2(\alpha+1)}\Big\{\int_0^1 f(tA+(1-t)C,B)\,\mathrm{d}t+\int_0^1 f(tA+(1-t)C,D)\,\mathrm{d}t\\ &+\alpha m\Big[\int_0^1 f\Big(tA+(1-t)C,\frac{B}{m}\Big)\,\mathrm{d}t+\int_0^1 f\Big(tA+(1-t)C,\frac{D}{m}\Big)\,\mathrm{d}t\Big]\Big\}.\end{aligned}\tag{3.15}$$

Summing the inequalities (3.14) and (3.15) and dividing by 2, we get the inequality (3.13).

The proof thus is complete. □

**Corollary 3.3.1.** *Under the assumptions of Theorem 3.3, choosing* $\alpha=1$*, we get the inequality for operator m-convex:*

$$\begin{aligned}&\int_0^1\int_0^1 f(tA+(1-t)C,\lambda B+(1-\lambda)D)\,\mathrm{d}\lambda\,\mathrm{d}t\\ \le&\frac{1}{8}\Big\{\int_0^1 f(tA+(1-t)C,B)\,\mathrm{d}t+\int_0^1 f(tA+(1-t)C,D)\,\mathrm{d}t+m\Big[\int_0^1 f\Big(tA+(1-t)C,\frac{B}{m}\Big)\,\mathrm{d}t\\ &+\int_0^1 f\Big(tA+(1-t)C,\frac{D}{m}\Big)\,\mathrm{d}t\Big]+\int_0^1 f(A,\lambda B+(1-\lambda)D)\,\mathrm{d}\lambda+\int_0^1 f(C,\lambda B+(1-\lambda)D)\,\mathrm{d}\lambda\\ &+m\Big[\int_0^1 f\Big(\frac{A}{m},\lambda B+(1-\lambda)D\Big)\,\mathrm{d}\lambda+\int_0^1 f\Big(\frac{C}{m},\lambda B+(1-\lambda)D\Big)\,\mathrm{d}\lambda\Big]\Big\}.\end{aligned}\tag{3.16}$$

*Furthermore, for* $\alpha,m=1$ *we have*

$$\int_0^1\int_0^1 f(tA+(1-t)C,\lambda B+(1-\lambda)D)\,\mathrm{d}\lambda\,\mathrm{d}t$$

$$\leq \frac{1}{4}\Big[\int_0^1 f(tA+(1-t)C,B)\,\mathrm{d}t+\int_0^1 f(tA+(1-t)C,D)\,\mathrm{d}t + \int_0^1 f(A,\lambda B+(1-\lambda)D)\,\mathrm{d}\lambda+\int_0^1 f(C,\lambda B+(1-\lambda)D)\,\mathrm{d}\lambda\Big]. \tag{3.17}$$

**Theorem 3.4.** *Let some fixed $(\alpha,m)\in(0,1]^2$ and a continuous function $f:\mathbb{R}_0^2\to\mathbb{R}$ be operator $(\alpha,m)$-convex on the co-ordinates for all 2-tuples of positive self-adjoint operators in the domain of $f$ acting on any Hilbert spaces $H_1,H_2$. Then one has*

$$\int_0^1\int_0^1 \big[f(tA+m(1-t)C,\lambda B+m(1-\lambda)D)+f(tC+m(1-t)A,\lambda B+m(1-\lambda)D)\big]\,\mathrm{d}t\,\mathrm{d}\lambda$$
$$\leq \frac{1+m\alpha}{2(\alpha+1)}\Big[\int_0^1 f(tA+m(1-t)C,B)\,\mathrm{d}t+\int_0^1 f(tA+m(1-t)C,D)\,\mathrm{d}t + \int_0^1 f(A,\lambda B+m(1-\lambda)D)\,\mathrm{d}\lambda+\int_0^1 f(C,\lambda B+m(1-\lambda)D)\,\mathrm{d}\lambda\Big], \tag{3.18}$$

*where $(A,B),(C,D)\in B(H_1)\times B(H_2)$ with spectra in $\mathbb{R}_0^2$.*

*Proof.* Using operator co-ordinated $(\alpha,m)$-convexity of $f$ and the inequality (1.13), we can write

$$\int_0^1 \big[f(tA+m(1-t)C,\lambda B+m(1-\lambda)D)+f(tC+m(1-t)A,\lambda B+m(1-\lambda)D)\big]\,\mathrm{d}t$$
$$\leq \frac{(1+m\alpha)[f(A,\lambda B+m(1-\lambda)D)+f(C,\lambda B+m(1-\lambda)D]}{\alpha+1}.$$

Integrating this inequality on $[0,1]$ over $\lambda$, we deduce

$$\int_0^1\int_0^1 \big[f(tA+m(1-t)C,\lambda B+m(1-\lambda)D)+f(tC+m(1-t)A,\lambda B+m(1-\lambda)D)\big]\,\mathrm{d}t\,\mathrm{d}\lambda$$
$$\leq \frac{1+m\alpha}{\alpha+1}\Big[\int_0^1 f(A,\lambda B+m(1-\lambda)D)\,\mathrm{d}\lambda+\int_0^1 f(C,\lambda B+m(1-\lambda)D)\,\mathrm{d}\lambda\Big]. \tag{3.19}$$

By a similar argument we get

$$\int_0^1\int_0^1 \big[f(tA+m(1-t)C,\lambda B+m(1-\lambda)D)+f(tC+m(1-t)A,\lambda B+m(1-\lambda)D)\big]\,\mathrm{d}t\,\mathrm{d}\lambda$$
$$\leq \frac{1+m\alpha}{\alpha+1}\Big[\int_0^1 f(tA+m(1-t)C,B)\,\mathrm{d}t+\int_0^1 f(tA+m(1-t)C,D)\,\mathrm{d}t\Big]. \tag{3.20}$$

Summing the inequalities (3.19) and (3.20) and dividing by 2, we get the inequality (3.18).

The proof thus is complete. □

**Corollary 3.4.1.** *Under the assumptions of Theorem 3.3, choosing $\alpha=1$, we get the inequality for operator $m$-convex:*

$$\int_0^1\int_0^1 \big[f(tA+m(1-t)C,\lambda B+m(1-\lambda)D)+f(tC+m(1-t)A,\lambda B+m(1-\lambda)D)\big]\,\mathrm{d}t\,\mathrm{d}\lambda$$
$$\leq \frac{1+m}{4}\Big[\int_0^1 f(tA+m(1-t)C,B)\,\mathrm{d}t+\int_0^1 f(tA+m(1-t)C,D)\,\mathrm{d}t + \int_0^1 f(A,\lambda B+m(1-\lambda)D)\,\mathrm{d}\lambda+\int_0^1 f(C,\lambda B+m(1-\lambda)D)\,\mathrm{d}\lambda\Big]. \tag{3.21}$$

*Furthermore, for $\alpha, m = 1$ we have*

$$\begin{aligned}
&\int_0^1\int_0^1 f(tA+(1-t)C,\lambda B+(1-\lambda)D)\,\mathrm{d}\lambda\,\mathrm{d}t\\
\le&\frac{1}{4}\bigg[\int_0^1 f(tA+(1-t)C,B)\,\mathrm{d}t+\int_0^1 f(tA+(1-t)C,D)\,\mathrm{d}t\\
&+\int_0^1 f(A,\lambda B+(1-\lambda)D)\,\mathrm{d}\lambda+\int_0^1 f(C,\lambda B+(1-\lambda)D)\,\mathrm{d}\lambda\bigg].
\end{aligned}\tag{3.22}$$

**Theorem 3.5.** *Let some fixed $(\alpha, m) \in (0, 1]^2$ and a continuous function $f : \mathbb{R}_0^2 \to \mathbb{R}$ be operator $(\alpha, m)$-convex on the co-ordinates for all 2-tuples of positive self-adjoint operators in the domain of $f$ acting on any Hilbert spaces $H_1, H_2$. Then one has*

$$\begin{aligned}
&f\bigg(\frac{2-m}{2}C+\frac{m}{2}(mA),\frac{2-m}{2}D+\frac{m}{2}(mB)\bigg)\\
\le&\frac{1}{2^{\alpha+1}}\bigg[\int_0^1 f\bigg(\frac{2-m}{2}C+\frac{m}{2}(mA),\lambda(2-m)D+(1-\lambda)m^2B\bigg)\,\mathrm{d}\lambda\\
&+m(2^\alpha-1)\int_0^1 f\bigg(\frac{2-m}{2}C+\frac{m}{2}(mA),\frac{(1-\lambda)(2-m)D+\lambda m^2B}{m}\bigg)\,\mathrm{d}\lambda\\
&+\int_0^1 f\bigg(t(2-m)C+(1-t)m^2A,\frac{2-m}{2}D+\frac{m}{2}(mB)\bigg)\,\mathrm{d}t\\
&+m(2^\alpha-1)\int_0^1 f\bigg(\frac{(1-t)(2-m)C+tm^2A}{m},\frac{2-m}{2}D+\frac{m}{2}(mB)\bigg)\,\mathrm{d}t\bigg]\\
\le&\frac{1}{2^{\alpha+2}(\alpha+1)}\bigg[f\bigg(\frac{2-m}{2}C+\frac{m}{2}(mA),(2-m)D\bigg)+m\big(\alpha+2^\alpha-1\big)f\bigg(\frac{2-m}{2}C+\frac{m}{2}(mA),mB\bigg)\\
&+m^2\alpha(2^\alpha-1)f\bigg(\frac{2-m}{2}C+\frac{m}{2}(mA),\frac{(2-m)D}{m^2}\bigg)+f\bigg((2-m)C,\frac{2-m}{2}D+\frac{m}{2}(mB)\bigg)\\
&+m\big(\alpha+2^\alpha-1\big)f\bigg(mA,\frac{2-m}{2}D+\frac{m}{2}(mB)\bigg)\\
&+m^2\alpha(2^\alpha-1)f\bigg(\frac{(2-m)C}{m^2},\frac{2-m}{2}D+\frac{m}{2}(mB)\bigg)\bigg],
\end{aligned}\tag{3.23}$$

*where $(A, B), (C, D) \in B(H_1) \times B(H_2)$ with spectra in $\mathbb{R}_0^2$.*

*Proof.* From operator co-ordinated $(\alpha, m)$-convexity of $f$ and the inequality (1.14), we can deduce

$$\begin{aligned}
&f\bigg(\frac{2-m}{2}C+\frac{m}{2}(mA),\frac{2-m}{2}D+\frac{m}{2}(mB)\bigg)\\
\le&\frac{1}{2^\alpha}\int_0^1\bigg[f\bigg(t(2-m)C+(1-t)m^2A,\frac{2-m}{2}D+\frac{m}{2}(mB)\bigg)\\
&+m(2^\alpha-1)f\bigg(\frac{(1-t)(2-m)C+tm^2A}{m},\frac{2-m}{2}D+\frac{m}{2}(mB)\bigg)\bigg]\,\mathrm{d}t\\
\le&\frac{1}{2^{\alpha+1}(\alpha+1)}\bigg\{f\bigg((2-m)C,\frac{2-m}{2}D+\frac{m}{2}(mB)\bigg)+m\big(\alpha+2^\alpha-1\big)f\bigg(mA,\frac{2-m}{2}D+\frac{m}{2}(mB)\bigg)\\
&+m^2\alpha(2^\alpha-1)f\bigg(\frac{(2-m)C}{m^2},\frac{2-m}{2}D+\frac{m}{2}(mB)\bigg)\bigg\}
\end{aligned}\tag{3.24}$$

and

$$
\begin{aligned}
&f\bigg(\frac{2-m}{2}C+\frac{m}{2}(mA),\frac{2-m}{2}D+\frac{m}{2}(mB)\bigg)\\
\leq&\frac{1}{2^{\alpha}}\int_0^1\bigg[f\bigg(\frac{2-m}{2}C+\frac{m}{2}(mA),\lambda(2-m)D+(1-\lambda)m^2B\bigg)\\
&+m(2^{\alpha}-1)f\bigg(\frac{2-m}{2}C+\frac{m}{2}(mA),\frac{(1-\lambda)(2-m)D+\lambda m^2B}{m}\bigg)\bigg]\,\mathrm{d}\lambda\\
\leq&\frac{1}{2^{\alpha+1}(\alpha+1)}\bigg\{f\bigg(\frac{2-m}{2}C+\frac{m}{2}(mA),(2-m)D\bigg)+m\big(\alpha+2^{\alpha}-1\big)f\bigg(\frac{2-m}{2}C+\frac{m}{2}(mA),mB\bigg)\\
&+m^2\alpha(2^{\alpha}-1)f\bigg(\frac{2-m}{2}C+\frac{m}{2}(mA),\frac{(2-m)D}{m^2}\bigg)\bigg\}. \qquad (3.25)
\end{aligned}
$$

Summing the inequalities (3.24) and (3.25) and dividing by 2, we get the inequality (3.23).

The proof is completed. □

**Corollary 3.5.1.** *Under the assumptions of Theorem 3.5, choosing $\alpha=1$, we get the inequality for operator m-convex:*

$$
\begin{aligned}
&f\bigg(\frac{2-m}{2}C+\frac{m}{2}(mA),\frac{2-m}{2}D+\frac{m}{2}(mB)\bigg)\\
\leq&\frac{1}{4}\bigg[\int_0^1 f\bigg(\frac{2-m}{2}C+\frac{m}{2}(mA),\lambda(2-m)D+(1-\lambda)m^2B\bigg)\,\mathrm{d}\lambda\\
&+m\int_0^1 f\bigg(\frac{2-m}{2}C+\frac{m}{2}(mA),\frac{(1-\lambda)(2-m)D+\lambda m^2B}{m}\bigg)\,\mathrm{d}\lambda\\
&+\int_0^1 f\bigg(t(2-m)C+(1-t)m^2A,\frac{2-m}{2}D+\frac{m}{2}(mB)\bigg)\,\mathrm{d}t\\
&+m\int_0^1 f\bigg(\frac{(1-t)(2-m)C+tm^2A}{m},\frac{2-m}{2}D+\frac{m}{2}(mB)\bigg)\,\mathrm{d}t\bigg]\\
\leq&\frac{1}{16}\bigg[f\bigg(\frac{2-m}{2}C+\frac{m}{2}(mA),(2-m)D\bigg)+2mf\bigg(\frac{2-m}{2}C+\frac{m}{2}(mA),mB\bigg)\\
&+m^2f\bigg(\frac{2-m}{2}C+\frac{m}{2}(mA),\frac{(2-m)D}{m^2}\bigg)+f\bigg((2-m)C,\frac{2-m}{2}D+\frac{m}{2}(mB)\bigg)\\
&+2mf\bigg(mA,\frac{2-m}{2}D+\frac{m}{2}(mB)\bigg)\\
&+m^2f\bigg(\frac{(2-m)C}{m^2},\frac{2-m}{2}D+\frac{m}{2}(mB)\bigg)\bigg], \qquad (3.26)
\end{aligned}
$$

*Furthermore, for $\alpha,m=1$ we have*

$$
\begin{aligned}
&f\bigg(\frac{A+C}{2},\frac{B+D}{2}\bigg)\\
\leq&\frac{1}{2}\bigg[\int_0^1 f\bigg(\frac{A+C}{2},\lambda B+(1-\lambda)D\bigg)\,\mathrm{d}\lambda+\int_0^1 f\bigg(tA+(1-t)C,\frac{B+D}{2}\bigg)\,\mathrm{d}t\bigg]\\
\leq&\frac{1}{4}\bigg[f\bigg(\frac{A+C}{2},B\bigg)+f\bigg(\frac{A+C}{2},D\bigg)+f\bigg(A,\frac{B+D}{2}\bigg)+f\bigg(C,\frac{B+D}{2}\bigg)\bigg]. \qquad (3.27)
\end{aligned}
$$

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## References

[1] M. Alomari and M. Darus, *The Hadamard's inequality for s-convex function of 2-variables on the co-ordinates*, Int. Journal of Math. Analysis, **2:13** (2008), 629–638.

[2] S. S. Dragomir and G. Toader, *Some inequalities for m-convex functions*, Studia Univ. Babes-Bolyai, Mathematica, **38:1** (1993), 21–28.

[3] S. S. Dragomir, *On some new inequalities of Hermite-Hadamard type for m-convex functions*, Tamkang J. Math., **3:1** (2002), 45–55.

[4] S. S. Dragomir, *An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products*, J. Inequal. Pure Appl. Math., **3:2** (2002), 775–788; Available online at `http://jipam.vu.edu.cn`.

[5] S. S. Dragomir, *An inequality improving the second Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products*, J. Inequal. Pure Appl. Math., **3:3** (2002), Article 35; Available online at `http://jipam.vu.edu.cn`.

[6] S. S. Dragomir, *Hermite-Hadamard's type inequalities for operator convex functions*, Applied Mathematics and Computation, **218** (2011), 766–772; Available online at `http://www.elsevier.com/locate/amc`.

[7] S. S. Dragomir and S. Fitzpatrick, *The Hadamards inequality for s-convex functions in the second sense*, Demonstratio Math., **32:4** (1999), 687–696.

[8] T. Furuta, J. M. Hot, J. Pečarić, and Y. Seo, *Mond-Pečarić method in operator inequalities*, Inequalities for Bounded Selfadjoint Operators on a Hilbert Space. Element, Zagreb, 2005.

[9] A. G. Ghazanfari,*Some new Hermite-Hadamard type inequalities for two operator convex functions*; Available online at `http://arXiv:1207.0928v1[math.FA]4Jul2012`.

[10] A. G. Ghazanfari,*The Hermite-Hadamard type inequalities for operator s-convex functions*, Journal of Advanced Research in Pure Mathematics, **6:3** (2014), 52–61; Available online at `http://dx.doi.org/10.5373/jarpm.1876.110613`.

[11] F. Hansen, *Operator convex functions of several variables*, Publ. RIMS, Kyoto Univ., **33** (1997), 443–463; Available online at `http://dx.doi.org/10.2977/prims/1195145324`.

[12] F. Hansen, *Operator monotone functions of several variables*; Available online at `http://arXiv:math/0205147v1[math.OA]14May2002`.

[13] H. Hudzik and L. Maligranda, *Some remarks on s-convex functions*, Aequationes Math., **48** (1994), 100–111.

[14] E. Kikianty, *Hermite-Hadamard inequality in the geometry of Banach spaces*, PhD thesis, Victoria University, 2010.

[15] A. Koráyi, *On some classes of analytic functions of several variables*, Transactions of the American Mathematical Society, **101:3** (1961), 520–554; Available online at `http://www.jstor.org/stable/1993476`.

[16] V. G. A. Mihesan, *A generalization of the convexity* , Seminar on Functional Equations, Approx. and Convex., Cluj-Napoca, Romania (1993).

[17] M. E. Ozdemir, E. Set, and M. Z. Sardari, *Some new Hadamard type inequalities for co-ordinated m-convex and* $(\alpha, m)$ *-convex functions*, Hacettepe Journal of Mathematics and Statistics, **40** (2011), 219–229.

[18] E. Set, M. Z. Sardari, M. E. Ozdemir, and J. Rooin, *On generalizations of the Hadamard inequality for* $(\alpha, m)$*-convex functions*, RGMIA Res. Rep. Coll., **12:4**, Article 4 (2009).

[19] G. Toader,*Some generalizations of the convexity*, Proc. Colloq. Approx. Opt. ClujNapoca, (1984), 329–338;

[20] S. H. Wang, *New integral inequalities of Hermite-Hadamard type for operator* $m-$*convex and* $(\alpha; m)-$*convex functions*, J. Computational analysis and applications, **22:4** (2017), 744–753, in press.

[21] S. H. Wang, *Hermite-Hadamard type inequalities for operator convex functions on the co-ordinates*, J. Nonlinear Sci. Appl., (2016), submit.

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# ADDITIVE-QUADRATIC $\rho$-FUNCTIONAL INEQUALITIES IN FUZZY BANACH SPACES: A FIXED POINT APPROACH

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Abstract. Let

$$M_1f(x,y) := \frac{3}{4}f(x+y)-\frac{1}{4}f(-x-y)+\frac{1}{4}f(x-y)+\frac{1}{4}f(y-x)-f(x)-f(y),$$
$$M_2f(x,y) := 2f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)+f\left(\frac{y-x}{2}\right)-f(x)-f(y).$$

Using the fixed point method, we prove the Hyers-Ulam stability of the additive-quadratic $\rho$-functional inequalities

$$N\left(M_1f(x,y)-\rho M_2f(x,y),t\right)\geq\frac{t}{t+\varphi(x,y)} \tag{0.1}$$

and

$$N\left(M_2f(x,y)-\rho M_1f(x,y),t\right)\geq\frac{t}{t+\varphi(x,y)} \tag{0.2}$$

in fuzzy Banach spaces, where $\rho$ is a fixed real number with $\rho\neq 1$.

## 1. Introduction and preliminaries

Katsaras [19] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [15, 21, 48]. In particular, Bag and Samanta [3], following Cheng and Mordeson [11], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [20]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [4].

We use the definition of fuzzy normed spaces given in [3, 25, 26] to investigate the Hyers-Ulam stability of additive $\rho$-functional inequalities in fuzzy Banach spaces.

**Definition 1.1.** [3, 25, 26, 27] Let $X$ be a real vector space. A function $N: X\times\mathbb{R}\to[0,1]$ is called a *fuzzy norm* on $X$ if for all $x,y\in X$ and all $s,t\in\mathbb{R}$,

($N_1$) $N(x,t)=0$ for $t\leq 0$;
($N_2$) $x=0$ if and only if $N(x,t)=1$ for all $t>0$;
($N_3$) $N(cx,t)=N(x,\frac{t}{|c|})$ if $c\neq 0$;
($N_4$) $N(x+y,s+t)\geq\min\{N(x,s),N(y,t)\}$;
($N_5$) $N(x,\cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty}N(x,t)=1$.
($N_6$) for $x\neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

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The pair $(X, N)$ is called a *fuzzy normed vector space*.

The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [25, 28].

**Definition 1.2.** [3, 28, 26, 27] Let $(X, N)$ be a fuzzy normed vector space. A sequence $\{x_n\}$ in $X$ is said to *be convergent* or *converge* if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, $x$ is called the *limit* of the sequence $\{x_n\}$ and we denote it by $N\text{-}\lim_{n\to\infty} x_n = x$.

**Definition 1.3.** [3, 28, 26, 27] Let $(X, N)$ be a fuzzy normed vector space. A sequence $\{x_n\}$ in $X$ is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces $X$ and $Y$ is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to $x_0$ in $X$, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on $X$ (see [4]).

The stability problem of functional equations originated from a question of Ulam [47] concerning the stability of group homomorphisms.

The functional equation $f(x+y) = f(x)+f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [17] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [39] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [16] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The functional equation $f(x+y)+f(x-y) = 2f(x)+2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional equation was proved by Skof [46] for mappings $f : E_1 \to E_2$, where $E_1$ is a normed space and $E_2$ is a Banach space. Cholewa [12] noticed that the theorem of Skof is still true if the relevant domain $E_1$ is replaced by an Abelian group. The stability problems of various functional equations have been extensively investigated by a number of authors (see [1, 5, 9, 10, 14, 22, 24, 29, 34, 35, 36, 40, 41, 42, 43, 44, 45, 49, 50]).

We recall a fundamental result in fixed point theory.

**Theorem 1.4.** [6, 13] *Let $(X, d)$ be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1}x) = \infty$$

*for all nonnegative integers $n$ or there exists a positive integer $n_0$ such that*

(1) $d(J^n x, J^{n+1}x) < \infty, \qquad \forall n \geq n_0$;

(2) *the sequence $\{J^n x\}$ converges to a fixed point $y^*$ of $J$;*

(3) $y^*$ *is the unique fixed point of* $J$ *in the set* $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;
(4) $d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy)$ *for all* $y \in Y$.

In 1996, G. Isac and Th.M. Rassias [18] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [7, 8, 30, 31, 38]).

Park [32, 33] defined additive $\rho$-functional inequalities and proved the Hyers-Ulam stability of the additive $\rho$-functional inequalities in Banach spaces and non-Archimedean Banach spaces.

In Section 2, we prove the Hyers-Ulam stability of the additive-quadratic $\rho$-functional inequality (0.1) in fuzzy Banach spaces by using the fixed point method.

In Section 3, we prove the Hyers-Ulam stability of the additive-quadratic $\rho$-functional inequality (0.2) in fuzzy Banach spaces by using the fixed point method.

Throughout this paper, assume that $X$ is a real vector space and $(Y, N)$ is a fuzzy Banach space. Let $\rho$ be a real number with $\rho \neq 1$.

## 2. Additive-quadratic $\rho$-functional inequality (0.1)

In this section, we prove the Hyers-Ulam stability of the additive-quadratic $\rho$-functional inequality (0.1) in fuzzy Banach spaces.

**Theorem 2.1.** *Let* $\varphi : X^2 \to [0, \infty)$ *be a function such that there exists an* $L < 1$ *with*

$$\varphi(x, y) \leq \frac{L}{4}\varphi(2x, 2y) \leq \frac{L}{2}\varphi(2x, 2y) \tag{2.1}$$

*for all* $x, y \in X$.
(i) *Let* $f : X \to Y$ *be an odd mapping satisfying*

$$N(M_1 f(x, y) - \rho M_2 f(x, y), t) \geq \frac{t}{t + \varphi(x, y)} \tag{2.2}$$

*for all* $x, y \in X$ *and all* $t > 0$. *Then* $A(x) := N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ *exists for each* $x \in X$ *and defines an additive mapping* $A : X \to Y$ *such that*

$$N(f(x) - A(x), t) \geq \frac{(2 - 2L)t}{(2 - 2L)t + L\varphi(x, x)} \tag{2.3}$$

*for all* $x \in X$ *and all* $t > 0$.
(ii) *Let* $f : X \to Y$ *be an even mapping satisfying* $f(0) = 0$ *and* (2.2). *Then* $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$ *exists for each* $x \in X$ *and defines a quadratic mapping* $Q : X \to Y$ *such that*

$$N(f(x) - Q(x), t) \geq \frac{(2 - 2L)t}{(2 - 2L)t + L\varphi(x, x)} \tag{2.4}$$

*for all* $x \in X$ *and all* $t > 0$.

*Proof.* (i) Letting $y = x$ in (2.2), we get

$$N(f(2x) - 2f(x), t) \geq \frac{t}{t + \varphi(x, x)} \tag{2.5}$$

and so

$$N\left(f(x)-2f\left(\frac{x}{2}\right),t\right)\geq\frac{t}{t+\varphi\left(\frac{x}{2},\frac{x}{2}\right)} \tag{2.6}$$

for all $x\in X$.

Consider the set

$$S:=\{g:X\to Y\}$$

and introduce the generalized metric on $S$:

$$d(g,h)=\inf\left\{\mu\in\mathbb{R}_+ : N(g(x)-h(x),\mu t)\geq\frac{t}{t+\varphi(x,x)},\ \forall x\in X,\forall t>0\right\},$$

where, as usual, $\inf\phi=+\infty$. It is easy to show that $(S,d)$ is complete (see [23, Lemma 2.1]).

Now we consider the linear mapping $J:S\to S$ such that

$$Jg(x):=2g\left(\frac{x}{2}\right)$$

for all $x\in X$.

Let $g,h\in S$ be given such that $d(g,h)=\varepsilon$. Then

$$N(g(x)-h(x),\varepsilon t)\geq\frac{t}{t+\varphi(x,x)}$$

for all $x\in X$ and all $t>0$. Hence

$$\begin{aligned}N(Jg(x)-Jh(x),L\varepsilon t) &= N\left(2g\left(\frac{x}{2}\right)-2h\left(\frac{x}{2}\right),L\varepsilon t\right)=N\left(g\left(\frac{x}{2}\right)-h\left(\frac{x}{2}\right),\frac{L}{2}\varepsilon t\right)\\ &\geq \frac{\frac{Lt}{2}}{\frac{Lt}{2}+\varphi\left(\frac{x}{2},\frac{x}{2}\right)}\geq\frac{\frac{Lt}{2}}{\frac{Lt}{2}+\frac{L}{2}\varphi(x,x)}=\frac{t}{t+\varphi(x,x)}\end{aligned}$$

for all $x\in X$ and all $t>0$. So $d(g,h)=\varepsilon$ implies that $d(Jg,Jh)\leq L\varepsilon$. This means that

$$d(Jg,Jh)\leq Ld(g,h)$$

for all $g,h\in S$.

It follows from (2.6) that $N\left(f(x)-2f\left(\frac{x}{2}\right),\frac{L}{2}t\right)\geq\frac{t}{t+\varphi(x,x)}$ for all $x\in X$ and all $t>0$. So $d(f,Jf)\leq\frac{L}{2}$.

By Theorem 1.4, there exists a mapping $A:X\to Y$ satisfying the following:

(1) $A$ is a fixed point of $J$, i.e.,

$$A\left(\frac{x}{2}\right)=\frac{1}{2}A(x) \tag{2.7}$$

for all $x\in X$. Since $f:X\to Y$ is odd, $A:X\to Y$ is an odd mapping. The mapping $A$ is a unique fixed point of $J$ in the set

$$M=\{g\in S : d(f,g)<\infty\}.$$

This implies that $A$ is a unique mapping satisfying (2.7) such that there exists a $\mu\in(0,\infty)$ satisfying

$$N(f(x)-A(x),\mu t)\geq\frac{t}{t+\varphi(x,x)}$$

for all $x\in X$;

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in X$;

(3) $d(f, A) \leq \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, A) \leq \frac{L}{2-2L}.$$

This implies that the inequality (2.3) holds.

By (2.2),

$$N\left(2^n\left(f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right), 2^n t\right) \geq \frac{t}{t + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)}$$

and so

$$N\left(4^n\left(f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right), t\right) \geq \frac{\frac{t}{2^n}}{\frac{t}{2^n} + \frac{L^n}{2^n}\varphi(x,y)}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{\frac{t}{2^n}}{\frac{t}{2^n} + \frac{L^n}{2^n}\varphi(x,y)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$N\left(A(x+y) - A(x) - A(y), t\right) = 1$$

for all $x, y \in X$ and all $t > 0$. So the mapping $A : X \to Y$ is additive.

(ii) Letting $y = x$ in (2.2), we get

$$N\left(\frac{1}{2}f(2x) - 2f(x), t\right) \geq \frac{t}{t + \varphi(x, x)} \tag{2.8}$$

and so

$$N\left(f(x) - 4f\left(\frac{x}{2}\right), t\right) \geq \frac{\frac{t}{2}}{\frac{t}{2} + \varphi\left(\frac{x}{2}, \frac{x}{2}\right)} = \frac{t}{t + 2\varphi\left(\frac{x}{2}, \frac{x}{2}\right)} \tag{2.9}$$

for all $x \in X$.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t + \varphi(x, x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(4g\left(\frac{x}{2}\right) - 4h\left(\frac{x}{2}\right), L\varepsilon t\right) = N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{4}\varepsilon t\right) \\ &\geq \frac{\frac{Lt}{4}}{\frac{Lt}{4} + \varphi\left(\frac{x}{2}, \frac{x}{2}\right)} \geq \frac{\frac{Lt}{4}}{\frac{Lt}{4} + \frac{L}{4}\varphi(x, x)} = \frac{t}{t + \varphi(x, x)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq L d(g, h)$$

for all $g, h \in S$.

It follows from (2.9) that $N\left(f(x) - 4f\left(\frac{x}{2}\right), \frac{L}{2}t\right) \geq \frac{t}{t+\varphi(x,x)}$ for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq \frac{L}{2}$.

By Theorem 1.4, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) $Q$ is a fixed point of $J$, i.e.,

$$Q\left(\frac{x}{2}\right) = \frac{1}{4}Q(x) \tag{2.10}$$

for all $x \in X$. Since $f : X \to Y$ is even, $Q : X \to Y$ is a even mapping. The mapping $Q$ is a unique fixed point of $J$ in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that $Q$ is a unique mapping satisfying (2.10) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(f(x) - Q(x), \mu t) \geq \frac{t}{t + \varphi(x, x)}$$

for all $x \in X$;

(2) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right) = Q(x)$$

for all $x \in X$;

(3) $d(f, Q) \leq \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, Q) \leq \frac{L}{2 - 2L}.$$

This implies that the inequality (2.4) holds.

By (2.2),

$$N\left(4^n\left(\frac{1}{2}f\left(\frac{x+y}{2^n}\right) + \frac{1}{2}f\left(\frac{x-y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right), 4^n t\right) \geq \frac{t}{t + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)}$$

and so

$$N\left(4^n\left(\frac{1}{2}f\left(\frac{x+y}{2^n}\right) + \frac{1}{2}f\left(\frac{x-y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right), t\right) \geq \frac{\frac{t}{4^n}}{\frac{t}{4^n} + \frac{L^n}{4^n}\varphi(x, y)}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{\frac{t}{4^n}}{\frac{t}{4^n} + \frac{L^n}{4^n}\varphi(x,y)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$N\left(\frac{1}{2}Q(x+y) + \frac{1}{2}Q(x-y) - Q(x) - Q(y), t\right) = 1$$

for all $x, y \in X$ and all $t > 0$. So the mapping $Q : X \to Y$ is quadratic. □

**Corollary 2.2.** *Let $\theta \geq 0$ and let $p$ be a real number with $p > 2$. Let $X$ be a normed vector space with norm $\|\cdot\|$.*

(i) *Let $f : X \to Y$ be an odd mapping satisfying*

$$N\left(M_1 f(x, y) - \rho M_2 f(x, y), t\right) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \tag{2.11}$$

*for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$N\left(f(x) - A(x), t\right) \geq \frac{(2^p - 2)t}{(2^p - 2)t + 2\theta\|x\|^p}$$

*for all $x \in X$ and all $t > 0$.*

(ii) *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.11). Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N\left(f(x) - Q(x), t\right) \geq \frac{(2^p - 4)t}{(2^p - 4)t + 4\theta\|x\|^p}$$

*for all $x \in X$ and all $t > 0$.*

*Proof.* The proof follows from Theorem 2.1 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Choosing $L = 2^{1-p}$ for an odd mapping case and $L = 2^{2-p}$ for an even mapping case, then we obtain the desired results. □

**Theorem 2.3.** *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \leq 4L\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \tag{2.12}$$

*for all $x, y \in X$..*

(i) *Let $f : X \to Y$ be an odd mapping satisfying (2.2). Then $A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n} f\left(2^n x\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$N\left(f(x) - A(x), t\right) \geq \frac{(2 - 2L)t}{(2 - 2L)t + \varphi(x, x)}$$

*for all $x \in X$ and all $t > 0$.*

(ii) *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.2). Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f\left(2^n x\right)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N\left(f(x) - Q(x), t\right) \geq \frac{(2 - 2L)t}{(2 - 2L)t + \varphi(x, x)}$$

*for all $x \in X$ and all $t > 0$.*

*Proof.* Let $(S, d)$ be the generalized metric space defined in the proof of Theorem 2.1.

(i) It follows from (2.5) that

$$N\left(f(x) - \frac{1}{2}f(2x), \frac{1}{2}t\right) \geq \frac{t}{t + \varphi(x, x)}$$

for all $x \in X$ and all $t > 0$.

(ii) It follows from (2.8) that

$$N\left(f(x) - \frac{1}{4}f(2x), \frac{1}{2}t\right) \geq \frac{t}{t + \varphi(x, x)}$$

for all $x \in X$ and all $t > 0$.

The rest of the proof is similar to the proof of Theorem 2.1. □

**Corollary 2.4.** *Let $\theta \geq 0$ and let $p$ be a real number with $0 < p < 1$. Let $X$ be a normed vector space with norm $\|\cdot\|$.*
*(i) Let $f : X \to Y$ be an odd mapping satisfying (2.11). Then $A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$N\left(f(x) - A(x), t\right) \geq \frac{(2-2^p)t}{(2-2^p)t + 2\theta\|x\|^p}$$

*for all $x \in X$ and all $t > 0$.*
*(ii) Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.11). Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N\left(f(x) - Q(x), t\right) \geq \frac{(4-2^p)t}{(4-2^p)t + 4\theta\|x\|^p}$$

*for all $x \in X$ and all $t > 0$.*

*Proof.* The proof follows from Theorem 2.3 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Choosing $L = 2^{p-1}$ for an odd mapping case and $L = 2^{p-2}$ for an even mapping case, then we obtain the desired results. □

## 3. Additive-quadratic $\rho$-functional inequality (0.2)

In this section, we prove the Hyers-Ulam stability of the additive-quadratic $\rho$-functional inequality (0.2) in fuzzy Banach spaces.

**Theorem 3.1.** *Let $\varphi : X^2 \to [0, \infty)$ be a function satisfying (2.1).*
*(i) Let $f : X \to Y$ be an odd mapping satisfying*

$$N\left(M_2 f(x, y) - \rho M_1 f(x, y), t\right) \geq \frac{t}{t + \varphi(x, y)} \tag{3.1}$$

*for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$N\left(f(x) - A(x), t\right) \geq \frac{(1-L)t}{(1-L)t + \varphi(x, x)}$$

*for all $x \in X$ and all $t > 0$.*
*(ii) Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.1). Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N\left(f(x) - Q(x), t\right) \geq \frac{(1-L)t}{(1-L)t + \varphi(x, x)}$$

*for all $x \in X$ and all $t > 0$.*

*Proof.* (i) Letting $y = 0$ in (3.1), we get

$$N\left(f(x) - 2f\left(\frac{x}{2}\right), t\right) = N\left(2f\left(\frac{x}{2}\right) - f(x), t\right) \geq \frac{t}{t + \varphi(x, 0)} \tag{3.2}$$

for all $x \in X$.

Consider the set

$$S := \{g : X \to Y\}$$

and introduce the generalized metric on $S$:

$$d(g,h) = \inf\left\{\mu \in \mathbb{R}_+ : N(g(x) - h(x), \mu t) \geq \frac{t}{t+\varphi(x,0)},\ \forall x \in X, \forall t > 0\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that $(S,d)$ is complete (see [23, Lemma 2.1]).

The rest of the proof is similar to the proof of Theorem 2.1 (i).

(ii) Letting $y = 0$ in (3.1), we get

$$N\left(f(x) - 4f\left(\frac{x}{2}\right), t\right) = N\left(4f\left(\frac{x}{2}\right) - f(x), t\right) \geq \frac{t}{t+\varphi(x,0)} \tag{3.3}$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.1 (ii). □

**Corollary 3.2.** *Let $\theta \geq 0$ and let $p$ be a real number with $p > 2$. Let $X$ be a normed vector space with norm $\|\cdot\|$.*
(i) *Let $f : X \to Y$ be an odd mapping satisfying*

$$N\left(M_2 f(x,y) - \rho M_1 f(x,y), t\right) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \tag{3.4}$$

*for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$N\left(f(x) - A(x), t\right) \geq \frac{(2^p - 2)t}{(2^p - 2)t + 2^p\theta\|x\|^p}$$

*for all $x \in X$ and all $t > 0$.*
(ii) *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.4). Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N\left(f(x) - Q(x), t\right) \geq \frac{(2^p - 4)t}{(2^p - 4)t + 2^p\theta\|x\|^p}$$

*for all $x \in X$ and all $t > 0$.*

*Proof.* The proof follows from Theorem 3.1 by taking $\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Choosing $L = 2^{1-p}$ for an odd mapping case and $L = 2^{2-p}$ for an even mapping case, then we obtain the desired results. □

**Theorem 3.3.** *Let $\varphi : X^2 \to [0,\infty)$ be a function satisfying (2.12).*
(i) *Let $f : X \to Y$ be an odd mapping satisfying (3.1). Then $A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n} f\left(2^n x\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$N\left(f(x) - A(x), t\right) \geq \frac{(1-L)t}{(1-L)t + L\varphi(x,x)}$$

*for all $x \in X$ and all $t > 0$.*

(ii) *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.1). Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N(f(x) - Q(x), t) \geq \frac{(1-L)t}{(1-L)t + L\varphi(x,x)}$$

*for all $x \in X$ and all $t > 0$.*

*Proof.* Let $(S, d)$ be the generalized metric space defined in the proof of Theorem 3.1.

(i) It follows from (3.2) that

$$N\left(f(x) - \frac{1}{2}f(2x), \frac{t}{2}\right) \geq \frac{t}{t + \varphi(2x, 0)}$$

and so

$$N\left(f(x) - \frac{1}{2}f(2x), Lt\right) \geq \frac{2Lt}{2Lt + \varphi(2x, 0)} = \frac{t}{t + \varphi(x, 0)}$$

for all $x \in X$ and all $t > 0$.

(ii) It follows from (3.3) that

$$N\left(f(x) - \frac{1}{4}f(2x), \frac{t}{4}\right) \geq \frac{t}{t + \varphi(2x, 0)}$$

and so

$$N\left(f(x) - \frac{1}{4}f(2x), Lt\right) \geq \frac{4Lt}{4Lt + \varphi(2x, 0)} = \frac{t}{t + \varphi(x, 0)}$$

for all $x \in X$ and all $t > 0$.

The rest of the proof is similar to the proof of Theorem 2.1. □

**Corollary 3.4.** *Let $\theta \geq 0$ and let $p$ be a real number with $0 < p < 1$. Let $X$ be a normed vector space with norm $\|\cdot\|$.*

(i) *Let $f : X \to Y$ be an odd mapping satisfying (3.4). Then $A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$N(f(x) - A(x), t) \geq \frac{(2-2^p)t}{(2-2^p)t + 2^p\theta\|x\|^p}$$

*for all $x \in X$.*

(ii) *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.4). Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N(f(x) - Q(x), t) \geq \frac{(4-2^p)t}{(4-2^p)t + 2^p\theta\|x\|^p}$$

*for all $x \in X$.*

*Proof.* The proof follows from Theorem 3.3 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Choosing $L = 2^{p-1}$ for an odd mapping case and $L = 2^{p-2}$ for an even mapping case, then we obtain the desired results. □

## References


[1] M. Adam, *On the stability of some quadratic functional equation*, J. Nonlinear Sci. Appl. **4** (2011), 50–59.
[2] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.
[3] T. Bag, S. K. Samanta, *Finite dimensional fuzzy normed linear spaces*, J. Fuzzy Math. **11** (2003), 687–705.
[4] T. Bag, S. K. Samanta, *Fuzzy bounded linear operators*, Fuzzy Sets and Systems **151** (2005), 513–547.
[5] L. Cădariu, L. Găvruta, P. Găvruta, *On the stability of an affine functional equation,* J. Nonlinear Sci. Appl. **6** (2013), 60–67.
[6] L. Cădariu, V. Radu, *Fixed points and the stability of Jensen's functional equation*, J. Inequal. Pure Appl. Math. **4**, no. 1, Art. ID 4 (2003).
[7] L. Cădariu, V. Radu, *On the stability of the Cauchy functional equation: a fixed point approach*, Grazer Math. Ber. **346** (2004), 43–52.
[8] L. Cădariu, V. Radu, *Fixed point methods for the generalized stability of functional equations in a single variable*, Fixed Point Theory Appl. **2008**, Art. ID 749392 (2008).
[9] A. Chahbi, N. Bounader, *On the generalized stability of d'Alembert functional equation*, J. Nonlinear Sci. Appl. **6** (2013), 198–204.
[10] I. Chang, Y. Lee, *Additive and quadratic type functional equation and its fuzzy stability*, Results Math. **63** (2013), 717–730.
[11] S. C. Cheng, J. M. Mordeson, *Fuzzy linear operators and fuzzy normed linear spaces*, Bull. Calcutta Math. Soc. **86** (1994), 429–436.
[12] P. W. Cholewa, *Remarks on the stability of functional equations,* Aequationes Math. **27** (1984), 76–86.
[13] J. Diaz, B. Margolis, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc. **74** (1968), 305–309.
[14] G. Z. Eskandani, P. Găvruta, *Hyers-Ulam-Rassias stability of pexiderized Cauchy functional equation in 2-Banach spaces,* J. Nonlinear Sci. Appl. **5** (2012), 459–465.
[15] C. Felbin, *Finite dimensional fuzzy normed linear spaces*, Fuzzy Sets and Systems **48** (1992), 239–248.
[16] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.
[17] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A. **27** (1941), 222–224.
[18] G. Isac, Th. M. Rassias, *Stability of $\psi$-additive mappings: Appications to nonlinear analysis*, Internat. J. Math. Math. Sci. **19** (1996), 219–228.
[19] A. K. Katsaras, *Fuzzy topological vector spaces II*, Fuzzy Sets and Systems **12** (1984), 143–154.
[20] I. Kramosil, J. Michalek, *Fuzzy metric and statistical metric spaces*, Kybernetica **11** (1975), 326–334.
[21] S. V. Krishna, K. K. M. Sarma, *Separation of fuzzy normed linear spaces*, Fuzzy Sets and Systems **63** (1994), 207–217.
[22] G. Lu, Y. Wang, P. Ye *n-Jordan $*$-derivations on induced fuzzy $C^*$-algebras*, J. Comput. Anal. Appl. **20** (2016), 266–276.
[23] D. Miheţ, V. Radu, *On the stability of the additive Cauchy functional equation in random normed spaces*, J. Math. Anal. Appl. **343** (2008), 567–572.
[24] D. Mihet, R. Saadati, *On the stability of the additive Cauchy functional equation in random normed spaces*, Appl. Math. Lett. **24** (2011), 2005–2009.
[25] A. K. Mirmostafaee, M. Mirzavaziri, M. S. Moslehian, *Fuzzy stability of the Jensen functional equation*, Fuzzy Sets and Systems **159** (2008), 730–738.
[26] A. K. Mirmostafaee, M. S. Moslehian, *Fuzzy versions of Hyers-Ulam-Rassias theorem*, Fuzzy Sets and Systems **159** (2008), 720–729.
[27] A. K. Mirmostafaee, M. S. Moslehian, *Fuzzy approximately cubic mappings*, Inform. Sci. **178** (2008), 3791–3798.
[28] M. Mirzavaziri and M.S. Moslehian, *A fixed point approach to stability of a quadratic equation*, Bull. Braz. Math. Soc. **37** (2006), 361–376.
[29] E. Movahednia, S. M. S. M. Mosadegh, C. Park, D. Shin, *Stability of a lattice preserving functional equation on Riesz space: fixed point alternative*, J. Comput. Anal. Appl. **21** (2016), 83–89.
[30] C. Park, *Fixed points and Hyers-Ulam-Rassias stability of Cauchy-Jensen functional equations in Banach algebras*, Fixed Point Theory Appl. **2007**, Art. ID 50175 (2007).

[31] C. Park, *Generalized Hyers-Ulam-Rassias stability of quadratic functional equations: a fixed point approach*, Fixed Point Theory Appl. **2008**, Art. ID 493751 (2008).
[32] C. Park, *Additive $\rho$-functional inequalities and equations,* J. Math. Inequal. **9** (2015), 17–26.
[33] C. Park, *Additive $\rho$-functional inequalities in non-Archimedean normed spaces,* J. Math. Inequal. **9** (2015), 397–407.
[34] C. Park, *Stability of ternary quadratic derivation on ternary Banach algebras: revisited,* J. Comput. Anal. Appl. **20** (2016), 21–23.
[35] C. Park, K. Ghasemi, S. G. Ghaleh, S. Jang, *Approximate n-Jordan $*$-homomorphisms in $C^*$-algebras,* J. Comput. Anal. Appl. **15** (2013), 365–368.
[36] C. Park, A. Najati, S. Jang, *Fixed points and fuzzy stability of an additive-quadratic functional equation,* J. Comput. Anal. Appl. **15** (2013), 452–462.
[37] W. Park, J. Bae, *Approximate quadratic forms on restricted domains*, J. Comput. Anal. Appl. **20** (2016), 388–410.
[38] V. Radu, *The fixed point alternative and the stability of functional equations*, Fixed Point Theory **4** (2003), 91–96.
[39] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.
[40] K. Ravi, E. Thandapani, B. V. Senthil Kumar, *Solution and stability of a reciprocal type functional equation in several variables,* J. Nonlinear Sci. Appl. **7** (2014), 18–27.
[41] S. Schin, D. Ki, J. Chang, M. Kim, *Random stability of quadratic functional equations: a fixed point approach*, J. Nonlinear Sci. Appl. **4** (2011), 37–49.
[42] S. Shagholi, M. Bavand Savadkouhi, M. Eshaghi Gordji, *Nearly ternary cubic homomorphism in ternary Fréchet algebras,* J. Comput. Anal. Appl. **13** (2011), 1106–1114.
[43] S. Shagholi, M. Eshaghi Gordji, M. Bavand Savadkouhi, *Stability of ternary quadratic derivation on ternary Banach algebras,* J. Comput. Anal. Appl. **13** (2011), 1097–1105.
[44] D. Shin, C. Park, S. Farhadabadi, *On the superstability of ternary Jordan $C^*$-homomorphisms,* J. Comput. Anal. Appl. **16** (2014), 964–973.
[45] D. Shin, C. Park, S. Farhadabadi, *Stability and superstability of $J^*$-homomorphisms and $J^*$-derivations for a generalized Cauchy-Jensen equation,* J. Comput. Anal. Appl. **17** (2014), 125–134.
[46] F. Skof, *Propriet locali e approssimazione di operatori*, Rend. Sem. Mat. Fis. Milano **53** (1983), 113–129.
[47] S. M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ. New York, 1960.
[48] J. Z. Xiao, X. H. Zhu, *Fuzzy normed spaces of operators and its completeness*, Fuzzy Sets and Systems **133** (2003), 389–399.
[49] C. Zaharia, *On the probabilistic stability of the monomial functional equation,* J. Nonlinear Sci. Appl. **6** (2013), 51–59.
[50] S. Zolfaghari, *Approximation of mixed type functional equations in p-Banach spaces*, J. Nonlinear Sci. Appl. **3** (2010), 110–122.

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# Optimal control of a special predator-prey system with functional response and toxicant

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**Abstract**

This paper is devoted to the optimal harvesting problem for a diffusive population dynamics with functional response in a polluted environment . $C_0$-semigroup theory is used to obtain the existence and uniqueness of the positive strong solution for the controlled system. The first order necessary optimality condition is derived by means the technique of tangent-normal cones and adjoint system of the state. The second-order necessary and sufficient optimality conditions are established by making use of the second order Fréchet derivative of the associated Lagrange function.

**Keywords**: Optimal harvesting; optimal conditions; functional response; toxicant

## 1 Introduction

The optimal control problems of population dynamics have been widely studied, such as N.C. Apreutesei [1] studied for a Lotka-Volterra system of three differential equations, some necessary conditions of optimality were founded in order to maximize the total number of individuals. W.Ko [2-3] considered a diffusive two-competing-prey and one-predator system with functional response (Beddington-DeAngelis and ratio-dependent), showed the properties for the positive steady-state solutions of the corresponding elliptic system with Robin boundary. Then N.C. Apreutesei [4] studied for a reaction-diffusion system as follows

$$\begin{cases} \dfrac{\partial y_1}{\partial t} = \alpha_1 \Delta y_1 + y_1 g_1(y_1) + u_1 y_1 - y_1 y_2 f(y_1), \\ \dfrac{\partial y_2}{\partial t} = \alpha_2 \Delta y_2 - a y_2 + b y_1 y_2 f(y_1) + c y_2 y_3 h(y_3), \\ \dfrac{\partial y_3}{\partial t} = \alpha_3 \Delta y_3 + y_3 g_3(y_3) + u_3 y_3 - y_3 y_2 h(y_3), \\ \dfrac{\partial y_i}{\partial v}(t,x) = 0, \ on \ \Sigma = [0,T] \times \partial\Omega, \ i = 1,2,3, \\ y_i(0,x) = y_i^0(x), \ x \in \ \Omega, \ i = 1,2,3. \end{cases} \tag{1.1}$$

the author considered the general functional response $y_i f(y_i)$, which contains the classical various Holling type, the existence of an optimal solution and first and second order optimality conditions

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were proved. E.Casas [5] investigated an abstract formulation for optimization problems in some $L^p$ spaces, devoted to reduce the classical gap between the necessary and sufficient conditions for optimization problems in Banach spaces. Other models from population dynamics and optimal control problems can be found in [6-9]. However, those papers were not take into account toxicant factor. Among the practical problems, it determines the real rate of the biological individual and the behavior of individual. To this end, Luo [10-12] first formulated a new age-dependent toxicant population model in an environment with small toxicant capacity, effectively bridge the research between age-structure and polluted environment. Inspired his works, this paper propose a more realistic models with toxicant-population in a small content of the environment.

The aim of this paper is to seek the maximum of the following functional, which gives the profit from harvesting less the cost of harvesting:

$$J(u,\nu)=\sum_{i=1}^{3}\int_0^T\int_\Omega\left[K_iu_i(t,x)y_i(t,x)-\frac{1}{2}C_iu_i^2(t,x)\right]dxdt-\frac{1}{2}\int_0^T C_4[\nu(t)]^2dt. \tag{OH}$$

where $K_i$ are selling price factors, positive constants $C_i$ and $C_4$ represents the cost factors of harvesting and the cost factor of administering pollution of environment, respectively; $u=(u_1,u_2,u_3)$ are the proportions of the populations to be harvested, $\nu(t)$ is the exogenous toxicant input rate the moment $t$, and the state $y=(y_1,y_2,y_3)$ is the solution of the following system corresponding to $(u_1,u_2,u_3)$:

$$\begin{cases}\dfrac{\partial y_1}{\partial t}=\alpha_1\Delta y_1+y_1[g_1(y_1)-r_1c_{10}]-y_1y_2f(y_1)-u_1y_1,\\ \dfrac{\partial y_2}{\partial t}=\alpha_2\Delta y_2-(a-r_2c_{20})y_2+by_1y_2f(y_1)+cy_2y_3h(y_3)-u_2y_2,\\ \dfrac{\partial y_3}{\partial t}=\alpha_3\Delta y_3+y_3[g_3(y_3)-r_3c_{30}]-y_3y_2h(y_3)-u_3y_3,\\ \dfrac{dc_{i0}}{dt}=kc_e(t)-gc_{i0}(t)-mc_{i0}(t),i=1,2,3,\\ \dfrac{dc_e}{dt}=-k_1c_e(t)[y_1(t)+y_2(t)+y_3(t)]+g_1\displaystyle\sum_{i=1}^{3}c_{i0}(t)y_i(t)-hc_e(t)+\nu(t)\end{cases} \tag{1.2}$$

for $(t,x)\in Q$, subject to some Neumann boundary conditions

$$\frac{\partial y_i}{\partial\nu}(t,x)=0,\ on\ \Sigma=[0,T]\times\partial\Omega,\ i=1,2,3$$

and to the initial conditions

$$y_i(0,x)=y_i^0(x),\ x\in\ \Omega,\ i=1,2,3.$$

which descried a diffusive one-predator and two-competing-prey system in a spatially inhomogeneous environment, where $Q=(0,T)\times\Omega$, $\Omega$ is a bounded domain in $\mathbb{R}^d(d\geq1)$ with the boundary $\partial\Omega$ of class $C^{2+\sigma}(\sigma>0)$, we denote by $y_i(t,x)$ the density of individuals of $i$th population at the moment $t$ and in the location $x\in\overline{\Omega}$. $c_0(t)$ is the concentration of the toxicant in an organism at the moment $t$, $c_e(t)$ is the concentration of the toxicant in the environment at the moment $t$. The function $u_i$ is the harvesting rate of population $y_i$, and the coefficients $\alpha_1,\alpha_2,\alpha_3,a,b,c$ are all positive constants. For the simplicity, we have assumed that $f$ and $h$ depend only on $y_1$ and on $y_3$ respectively, but the reasoning and the main results remain true also in the case when $f$ and $h$ depend on $y_2$ too, parameter $a$ is the per capita death rate of the predator.

The admissible control set is defined as

$$\mathcal{U}_{ad} = \big\{(u,\nu) \in [L^2(Q)]^3 \times L^\infty(0,T) |\ 0 \le u_i(t,x) \le 1,\ a.e\ in\ Q,\ 0 \le \nu(t) \le h\ a.e\ in\ (0,T)\big\}.$$

Throughout this paper, we always assume that:
($H_1$) $g_1, g_3$ are continuous and bounded on $(0,\infty)$;
($H_2$) $f, h$ are continuous and positive on $(0,\infty)$ and bounded on bounded sets;
($H_3$) $y_i^0 \in H^2(\Omega), y_i^0 > 0$ on $\Omega$ and $\partial y_i^0/\partial\nu = 0$ *a.e. on* $\partial\Omega,\ i = 1,2,3$;
($H_4$) $\nu(\cdot) \in L^2[0,T], 0 \le \nu(t) \le \nu_1 < +\infty$;
($H_5$) $0 \le c_{i0}(0) \le 1, 0 \le c_e(0) \le 1$;
($H_6$) $g \le k \le g + m, \nu \le h$.

The paper is organized as follows: In section 2, we use results from the semigroup theory and some well-known existence theorems from [13-14] to derive the global existence and uniqueness of a positive strong solution of the controlled system (1.2), Section 3 is devoted to first order necessary optimality conditions for (OH). Necessary and sufficient second order optimality conditions are given in Section 4.

## 2 Basic properties of the solution

This section concerns the most important properties of the dynamics system with diffusion. Existence, uniqueness and positivity of the solution will be proved. Thus formally, system (1.1) can be written as an infinite dimensional Cauchy problem of the form

$$\begin{cases} \dfrac{dy}{dt}(t) = Ay(t) + F(t, y(t)), \quad t \in [0,T], \\ y(0) = y_0, \end{cases} \tag{2.1}$$

where $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ is the infinitesimal generator of a $C_0$-semigroup of contractions $\{S(t)\}_{t\ge 0}$ on the Banach space $\mathbb{X}$, if $\mathbb{X}$ is a Hilbert space, $A$ is called dissipative if $(Ax, x) \le 0, \forall x \in D(A)$, and $F : [0,T] \times \mathbb{X} \to \mathbb{X}$ is measurable in $t$ and Lipschtiz in $x \in \mathbb{X}$ uniformly with respect to $t$.

We shall employ a general existence result which we use in the sequel (Proposition 1.2, p.175,[14]).

**Theorem 2.1** For each $y_0 \in \mathbb{X}$, the initial value problem (2.1) has a unique mild solution $y \in C([0,T];\mathbb{X})$, and

$$y(t) = S(t)y_0 + \int_0^t S(t-s)F(s, y(s))ds, \quad t \in [0,T].$$

In addition, if $\mathbb{X}$ is a Hilbert space, A is self-adjoint and dissipative on $\mathbb{X}$, and $y_0 \in D(A)$, then the mild solution is in fact a strong solution and $y \in W^{1,2}([\delta,T];X), \forall \delta \in [0,T]$.

Thus, we work in the Hilbert space $H = (L^2(\Omega))^3$, where the operator $A : D(A) \subset H \to H$,

$$A = \begin{pmatrix} \alpha_1\Delta & 0 & 0 \\ 0 & \alpha_2\Delta & 0 \\ 0 & 0 & \alpha_3\Delta \end{pmatrix}, \qquad F(t, y(t)) = \begin{pmatrix} F_1(t, y(t)) \\ F_2(t, y(t)) \\ F_3(t, y(t)) \end{pmatrix},$$

for $y = (y_1, y_2, y_3) \in D(A), D(A) = \Big\{y = (y_1, y_2, y_3) \in (H^2(\Omega))^3, \dfrac{\partial y_i}{\partial \nu} = 0\ on\ \partial\Omega, i = 1,2,3\Big\}$, $y^0 = (y_1^0, y_2^0, y_3^0)$ is the initial value of $y$, and $F = (F_1, F_2, F_3)$ is the nonlinear term in (2.1), that is

$$\begin{cases} F_1(t, y(t)) = y_1 g_1(y_1) - y_1 y_2 f(y_1) - u_1 y_1, \\ F_2(t, y(t)) = -a y_2 + b y_1 y_2 f(y_1) + c y_2 y_3 h(y_3) - u_2 y_2, \\ F_3(t, y(t)) = y_3 g_3(y_3) - y_3 y_2 h(y_3) - u_3 y_3, \end{cases} \tag{2.2}$$

**Theorem 2.2** Suppose that $y^0 = (y_1^0, y_2^0, y_3^0) \in D(A)$, and $y_i^0 > 0, i = 1, 2, 3$. Then for each $u \in \mathcal{U}_{ad}$, the system (2.1) has a unique nonnegative solution $(y(t,x), c_0(t), c_e(t))$, such that

(i) $(y_i(t,x), c_{i0}(t), c_e(t)) \in L^\infty(Q) \cap L^2(0,T; H^2(\Omega) \cap L^\infty(0,T; H^1(\Omega))) \times L^\infty(0,T) \times L^\infty(0,T)$,

(ii) $0 \le c_{i0}(t) \le 1, 0 \le c_e(t) \le 1, \forall\ t \in (0,T)$.

**Proof** Since $F$ is not satisfy Lipschtiz conditions, we cannot apply the theorem 2.1 directly for our problem, usually we use a truncation procedure for $F$, consider the truncated initial value problem

$$\begin{cases} \dfrac{\partial y^N}{\partial t}(t,x) = Ay^N(t) + F^N(t, y(t)), \quad t \in [0,T], \\ y^N(0) = y_0, \end{cases} \tag{2.3}$$

where $F^N = (F_1^N, F_2^N, F_3^N)$ is obtained from $F = (F_1, F_2, F_3)$, a fixed large number $N > 0$. If $|y_i| \le N$, then $y_i$ in $F(t, y_1, y_2, y_3)$ remains unchanged, if $y_i > N$, then $y_i$ from (2.2) is replaced by $N$, if $y_i < -N$, then $y_i$ from (2.2) is replaced by $-N$. Thus function $F^N$ becomes Lipschitz continuous with respect to $t$, according theorem 2.1, the problem (2.3) admits a unique strong solution $y^N = (y_1^N, y_2^N, y_3^N) \in W^{1,2}([\delta, T]; H) \cap L^2(0,T; D(A)), \forall \delta \in [0,T]$.

To begin with, we shall that $y \in L^2(0,T; H^2(\Omega) \cap L^\infty(0,T; H^1(\Omega))$. On one hand, from theorem 2.1 we know $y \in L^2(0,T; H^2(\Omega)$, on the other hand, from (2.3) we derive that

$$\int_Q \Big|\frac{\partial y_1^N}{\partial t}\Big|^2 dsdx - 2\alpha_1 \int_Q \frac{\partial y_1^N}{\partial t} \Delta y_1^N dsdx + \alpha^2 \int_Q |\Delta y_1^N|^2 dsdx = \int_Q |F_1(t, y(t))|^2 dsdx,$$

Using the regularity of $y_1^N$ and the Green's formula, we have

$$\int_Q \Big|\frac{\partial y_1^N}{\partial t}\Big|^2 dsdx + 2\alpha_1 \int_\Omega |\nabla y_1^N|^2 dx + \alpha^2 \int_Q |\Delta y_1^N|^2 dsdx = \int_Q |F_1(t, y(t))|^2 dsdx + 2\alpha_1 \int_\Omega |\nabla y_1^0|^2 dx.$$

Since $y_1^N \in W^{1,2}(0,T; H)$ and $y_1^0 \in H^2(\Omega)$, by the Lipschitz property of $F_1^N$ we deduce that

$$2\alpha_1 \int_\Omega |\nabla y_1^N|^2 dx \le \int_Q |y^N| dxds + 2\alpha_1 \int_\Omega |\nabla y_1^0|^2 dx < +\infty.$$

Thus, we have $y_1 \in L^\infty(0,T; H^1(\Omega))$, analogously $y_2, y_3$ are proved.

Further more, it remains to prove that $y^N \in L^\infty(Q)$, $(c_{i0}, c_e) \in L^\infty(0,T)$. Indeed, consider the following auxiliary initial value problems

$$\begin{cases} \dfrac{\partial \rho_1^N}{\partial t}(t,x) = \Delta \rho_1^N(t) + F_1^N(t, y(t)) - M_N, \quad t \in [0,T], \\ \rho_1^N(0) = y_1^0 - \|y_1^0\|_{L^\infty(\Omega)} \end{cases} \tag{2.4}$$

and

$$\begin{cases} \dfrac{\partial \omega_1^N}{\partial t}(t,x) = \Delta \omega_1^N(t) + F_1^N(t, y(t)) + M_N, \quad t \in [0,T], \\ \omega_1^N(0) = y_1^0 + \|y_1^0\|_{L^\infty(\Omega)}, \end{cases} \tag{2.5}$$

where $M_N = \max\Big\{\|F_i^N(\cdot, y(t))\|_{L^\infty(Q)}, \|y_i^0\|_{L^\infty(Q)}, i = 1, 2, 3\Big\}$.

By theorem 2.1 the function $\rho_1^N$ and $\omega_1^N$ in $C([0,T]; \mathbb{X})$ is a mild solution to problem (2.4) and (2.5), the solution of these can be written as

$$\rho_1(t) = S(t)(y_1^0 - \|y_1^0\|_{L^\infty(\Omega)}) + \int_0^t S(t-s)(F_1^N(s, y_1, y_2, y_3) - M_N) ds,$$

$$\omega(t) = S(t)(y_1^0 + \|y_1^0\|_{L^\infty(\Omega)}) + \int_0^t S(t-s)(F_1^N(s, y_1, y_2, y_3) + M_N)ds.$$

Remark that their solutions are

$$\rho_1(t,x) = y_1^N(t,x) - M_N t - \|y_1^0\|_{L^\infty(\Omega)}, \quad \omega_1(t,x) = y_1^N(t,x) + M_N t + \|y_1^0\|_{L^\infty(\Omega)},$$

since $|F_1^N(t, y^N)| \leq M_N$, from the comparison principle of linear parabolic equation, we deduce that $\rho_1^N(0) \leq 0,\ \omega_1^N(0) \geq 0$, that is

$$|y_1^N(t,x)| \leq M_N t + \|y_1^0\|_{L^\infty(\Omega)},$$

and in the same manner to prove that $y_2^N, y_3^N$ hold for $(t,x) \in Q$. Therefor, $y_i^N \in L^\infty(Q)$. To prove $(c_{i0}, c_e) \in L^\infty(0,T)$, we define $G : \mathbb{X} \to \mathbb{X}$, from (1.2) we can deduce that

$$G_i(t) = c_{i0}(t) = c_{i0}(0)\exp\{-(g+m)t\} + k\int_0^t c_e(s)\exp\big\{(s-t)(g+m)\big\}ds, i = 1,2,3, \tag{2.6}$$

$$\begin{aligned} G_4(t) = c_e(t) = c_e(0)\exp\Big\{ -\int_0^t \big[\sum_{i=1}^3 k_1 y_i(\tau) + h\big]d\tau\Big\} \\ + \int_0^t \Big[g_1\big(\sum_{i=1}^3 c_{i0}(s)y_i(s)\big) + \nu(s)\Big]\exp\Big\{\int_t^s \big[\sum_{i=1}^3 k_1 y_i(\tau) + h\big]d\tau\Big\}ds, \end{aligned} \tag{2.7}$$

if the hypothesis $(\mathrm{H}_6)$ hold, it is clear that $0 \leq c_{i0}(t) \leq 1, 0 \leq c_e(t) \leq 1$ and $(c_{i0}, c_e) \in L^\infty(0,T),\ i = 1,2,3.$ the specific process can refer to [15].

Moreover, we shall that $y_i^N$ are positive on $Q$, to this end, let $y_i^N = (y_i^N)^+ - (y_i^N)^-$, where

$$(y_i^N)^+(t,x) = \sup\{y_i^N(t,x), 0\}, \quad (y_i^N)^-(t,x) = -\inf\{y_i^N(t,x), 0\},\ i = 1,2,3. \tag{2.8}$$

Multiplying the first equation from (2.1) by $y_1^N$ we have

$$\frac{1}{2}\frac{\partial}{\partial t}|(y_1^N)^-|^2 = \alpha_1(y_1^N)^-\Delta(y_1^N)^- + |(y_1^N)^-|^2\big[g_1(y_1^N) - y_2^N f(y_1^N) - u_1\big]. \tag{2.9}$$

Integrating (2.9) on $\Omega$ and using Greens formula we get

$$\frac{1}{2}\int_\Omega \frac{\partial}{\partial t}|(y_1^N)^-|^2 dx = -\alpha_1\int_\Omega |\nabla(y_1^N)^-|^2 dx + \int_\Omega |(y_1^N)^-|^2\big[g_1(y_1^N) - y_2^N f(y_1^N) - u_1\big]dx.$$

By integrating over $[0,t]$, for $t \in [0,T]$, and take into consideration of the uniformly boundedness of $y_i^N$, it is not difficult to see that there exists a constant $C^N > 0$ depending on $N$ such that

$$\frac{1}{2}\int_\Omega |(y_1^N)^-|^2 dx + \alpha_1\int_0^t\int_\Omega |\nabla(y_1^N)^-|^2 dxds \leq C^N\int_0^t\int_\Omega |y_1^N(s)|^2 dxds.$$

Gronwalls inequality lead to

$$\int_\Omega |(y_1^N)^-|^2 dx \leq 0, \quad \forall t \in [0,T],$$

that is $(y_1^N)^- = 0$, by the definition of (2.8) we conclude that $y_1^N(t,x) > 0$, analogously we get $y_2^N(t,x) > 0$ and $y_3^N(t,x) > 0$.

In addition, we prove the uniqueness of the solution. For any $(y^1, c_0^1, c_e^1)$ and $(y^2, c_0^2, c_e^2)$ are two solutions of problem (1.2), where $y^1 = (y_1^1, y_2^1, y_3^1),\ c_0^1 = (c_{10}^1, c_{20}^1, c_{30}^1),\ y^2 = (y_1^2, y_2^2, y_3^2),\ c_0^2 = (c_{10}^2, c_{20}^2, c_{30}^2)$, we denote by $\varphi = y^1 - y^2$, then $\varphi$ is the solution of

$$\begin{cases} \dfrac{\partial \varphi_1}{\partial t} = \alpha_1 \Delta \varphi_1 + F_1(t, y_1^1, y_2^1, y_3^1) - F_1(t, y_1^2, y_2^2, y_3^2), \\ \dfrac{\partial \varphi_2}{\partial t} = \alpha_2 \Delta \varphi_2 + F_2(t, y_1^1, y_2^1, y_3^1) - F_2(t, y_1^2, y_2^2, y_3^2), \\ \dfrac{\partial \varphi_3}{\partial t} = \alpha_3 \Delta \varphi_3 + F_3(t, y_1^1, y_2^1, y_3^1) - F_3(t, y_1^2, y_2^2, y_3^2), \\ \dfrac{\partial \varphi_1}{\partial \nu} = \dfrac{\partial \varphi_2}{\partial \nu} = \dfrac{\partial \varphi_3}{\partial \nu} = 0,\ on\ \Sigma, \\ \varphi_1(0,x) = \varphi_2(0,x) = \varphi_3(0,x) = 0. \end{cases} \tag{2.10}$$

Suppose $g_1, g_3, f, h \in C^1[0,\infty)$, $g_1, g_3$ are bounded and $f, h$ are positive and have at most polynomial growth, then from (2.2) we obtain

$$|F_i(t, y_1^1, y_2^1, y_3^1) - F_i(t, y_1^2, y_2^2, y_3^2)| \le c(|\varphi_1| + |\varphi_2| + |\varphi_3|),$$

where $c$ is a positive constant. Multiplying (2.10) by $\varphi_1, \varphi_2, \varphi_3$ respectively, and integrating on $\Omega_T = \Omega \times (0,t)$ we get

$$\begin{aligned} \frac{1}{2}\sum_{i=1}^{3}\int_\Omega |\varphi_i(t)|^2 dx + \sum_{i=1}^{3}\int_{\Omega_T} \alpha_i |\nabla \varphi_i|^2 ds dx &= \sum_{i=1}^{3}\int_{\Omega_T} \varphi_i (F_i(t, y_1^1, y_2^1, y_3^1) - F_i(t, y_1^2, y_2^2, y_3^2)) ds dx \\ &\le C\int_0^t \int_\Omega (|\varphi_1(s)|^2 + |\varphi_2(s)|^2 + |\varphi_3(s)|^2) ds dx. \end{aligned} \tag{2.11}$$

From (2.11) and Gronwall's lemma we have

$$\int_\Omega (|\varphi_1(s)|^2 + |\varphi_2(s)|^2 + |\varphi_3(s)|^2) \le 0,$$

which yields that $\varphi_1 = \varphi_2 = \varphi_3 = 0$, thus we have proved the uniqueness of the $y_i$. However, we can follow by (2.6) and (2.7)

$$|c_0^1(t) - c_0^2(t)| = \sum_{i=1}^{3} |c_{i0}^1(t) - c_{i0}^2(t)| \le 3k \int_0^t |c_e^1(s) - c_e^2(s)| ds,\ i = 1,2,3. \tag{2.12}$$

$$|c_e^1(t) - c_e^2(t)| \le M_1 \sum_{i=1}^{3} \int_0^t |c_{i0}^1(s) - c_{i0}^2(s)| ds, \tag{2.13}$$

where $M_1$ is constant. We define an equivalent norm in $\mathbb{X}$ as follows:

$$\|(c_{i0}, c_e)\|_* = Ess \sup_{t\in(0,T)} e^{-\lambda t} \Big\{ \sum_{i=1}^{3} |c_{i0}(t)| + |c_e(t)| \Big\},$$

by (2.11) and (2.12) we obtain

$$\begin{aligned}\|G(x^1)-G(x)^2\|_* &= \|G_i(x^1)-G_i(x^2), G_4(x^1)-G_4(x^2)\|_* \\ &\le M_2 Ess \sup_{t\in(0,T)} e^{-\lambda t}\int_0^t \Big\{\sum_{i=1}^{3}|c_{i0}^1(s)-c_{i0}^2(s)|+|c_e^1(s)-c_e^2(s)|\Big\}ds \\ &\le M_2 Ess \sup_{t\in(0,T)} e^{-\lambda t}\int_0^t e^{\lambda s}\Big\{e^{-\lambda s}[\sum_{i=1}^{3}|c_{i0}^1(s)-c_{i0}^2(s)|+|c_e^1(s)-c_e^2(s)|]\Big\}ds \\ &\le M_2\|x^1-x^2\|_* Ess \sup_{t\in(0,T)}\{e^{-\lambda t}\int_0^t e^{\lambda s}ds\} \\ &\le \frac{M_2}{\lambda}\|x^1-x^2\|_*,\end{aligned}$$

where $M_2$ is constant, choose $\lambda > M_2$ yields that $G$ is a strict contraction on $(\mathbb{X}, \|\cdot\|_*)$ and consequently has a unique fixed point.

Thus, the system (1.2) has a unique solution $(y_i, c_{i0}, c_e)$. the proof is completed.

# 3 Necessary optimality conditions

In this section, we find some necessary optimality conditions in order to maximize the profit from harvesting less the cost of harvesting.

**Theorem 3.1** If $(u^*, \nu^*)$ is an optimal control and $(y^*, c_{i0}^*, c_e^*)$ is the corresponding optimal state, then

$$\begin{aligned} u_i^*(t,x) &= \mathcal{L}_i\Big(\frac{(K_i-q_i)y_i^*}{C_i}\Big),\ i=1,2,3,\ a.e.\ in\ Q, \\ \nu^*(t) &= \mathcal{L}_4\Big(\frac{q_7(t)}{C_4}\Big),\ a.e.\ in\ (0,T),\end{aligned} \tag{3.1}$$

where

$$\mathcal{L}_j(x)=\begin{cases}0 & x<0\\ x & 0\le x\le H_j\\ H_j & x>H_j\end{cases},\quad j=1,2,3,4.$$

and $q=(q_1,q_2,\dots,q_7)$ is the solution of following adjoint system corresponding to $(u^*,\nu^*)$.

$$\begin{cases}
\frac{\partial q_1}{\partial t} = -\alpha_1\Delta q_1 + [g_1(y_1^*) - r_1c_{10}^* + y_1^*g_1'(y_1^*) - u_1^* - y_2^*f(y_1^*) - y_1^*y_2^*f'(y_1^*)]q_1 \\
\qquad - [by_2^*f(y_1^*) + by_1^*y_2^*f'(y_1^*)]q_2 + [k_1c_e^* - g_1c_{10}^*]q_7 + K_1u_1^*, \\
\frac{\partial q_2}{\partial t} = -\alpha_2\Delta q_2 + y_1^*f(y_1^*)q_1 - [-a - r_2c_{20}^* + by_1^*f(y_1^*) + cy_3^*h(y_3^*)]q_2 \\
\qquad + y_3^*h(y_3^*)q_3 + [k_1c_e^* - g_1c_{20}^*]q_7 + K_2u_2^*, \\
\frac{\partial q_3}{\partial t} = -\alpha_3\Delta q_3 - [g_3(y_3^*) - r_3c_{30}^* + y_3^*g_3'(y_1^*) - u_3^* - y_2^*h(y_3^*) - y_3^*y_2^*h'(y_3^*)]q_3 \\
\qquad - cy_2^*[h(y_3^*) + y_3^*h'(y_3^*)]q_2 + [k_1c_e^* - g_1c_{30}^*]q_7 + K_3u_3^*, \\
\frac{\partial q_j}{\partial t} = (g+m)q_j - g_1y_i^*q_7,\ j=i+3,\ i=1,2,3, \\
\frac{\partial q_7}{\partial t} = -k\sum_{j=4}^{6}q_j + k_1\sum_{i=1}^{3}y_iq_7 + hq_7, \\
q_i(T,x) = 0,\quad x\in\Omega,
\end{cases} \tag{3.2}$$

$$\frac{\partial q_i}{\partial \nu} = 0 \ \ a.e.\ on\ \Sigma,\ i = 1, 2, 3.$$

**Proof** Existence and uniqueness of the solution $q$ to system (3.2) follows by theorem 2.2. Denote by $\mathcal{N}_{\mathcal{U}_{ad}}(u^*, \nu^*)$ the normal cone at $\mathcal{U}_{ad}$ in $(u^*, \nu^*)$,

$$\mathcal{N}_{\mathcal{U}_{ad}}(u^*, \nu^*) = \{v_1 \in L^2(Q),\ v_2 \in L^2(0, T) \text{ satisfying the following formula}\},$$

$$\begin{cases} v_1(t,x) \le 0, & \text{when } u(t,x) = 0, \\ v_1(t,x) = 0, & \text{when } 0 \le u(t,x) \le 1, \\ v_1(t,x) \ge 0, & \text{when } u(t,x) = 1, \end{cases} \quad \begin{cases} v_2(t) \le 0, & \text{when } \nu(t) = 0, \\ v_2(t) = 0, & \text{when } 0 \le \nu(t) \le h, \\ v_2(t) \ge 0, & \text{when } \nu(t) = h, \end{cases}$$

for any given $(\vartheta_1, \vartheta_2) \in \mathcal{T}_{\mathcal{U}_{ad}}(u^*, \nu^*)$ $\vartheta_1 = (\vartheta_{11}, \vartheta_{21}, \vartheta_{31})$, as $\varepsilon > 0$ small enough, $(u^* + \varepsilon\vartheta_1, \nu^* + \varepsilon\vartheta_2) \in \mathcal{U}_{ad}$, we get

$$J(u^* + \varepsilon\vartheta_1, \nu^* + \varepsilon\vartheta_2) \le J(u^*, \nu^*). \tag{3.3}$$

Substituting (2.1) into (3.3) gives that

$$\begin{aligned} &\sum_{i=1}^{3} \int_0^T \int_\Omega K_i(u_i^* + \varepsilon\vartheta_{i1}) y_i^\varepsilon dxdt - \frac{1}{2} \sum_{i=1}^{3} \int_0^T \int_\Omega C_i(u_i^* + \varepsilon\vartheta_{i1})^2 dxdt - \frac{1}{2} \int_0^T C_4(\nu^* + \varepsilon\vartheta_2)^2 dt \\ &\qquad \le \sum_{i=1}^{3} \int_0^T \int_\Omega K_i u_i^* y_i^* dxdt - \frac{1}{2} \sum_{i=1}^{3} \int_0^T \int_\Omega C_i [u_i^*]^2 dxdt - \frac{1}{2} \int_0^T C_4 [\nu^*]^2 dt, \end{aligned}$$

that is

$$\sum_{i=1}^{3} \int_0^T \int_\Omega K_i u_i^* z_i^* dxdt + \sum_{i=1}^{3} \int_0^T \int_\Omega (K_i y_i^* - C_i u_i^*) \vartheta_{i1} dxdt - \int_0^T \nu^* \vartheta_2 dt \le 0, \tag{3.4}$$

where

$$z_i(t,x) = \lim_{\varepsilon \to 0^+} \frac{y_i^\varepsilon(t,x) - y_i^*(t,x)}{\varepsilon}, z_{i+3}(t) = \lim_{\varepsilon \to 0^+} \frac{c_{i0}^\varepsilon(t) - c_{i0}^*(t)}{\varepsilon}, z_7(t) = \lim_{\varepsilon \to 0^+} \frac{c_e^\varepsilon(t) - c_e^*(t)}{\varepsilon}, i = 1, 2, 3,$$

$(y^\varepsilon, c_0^\varepsilon, c_e^\varepsilon)$ is the state corresponding to $(u^* + \varepsilon\vartheta_1, \nu^* + \varepsilon\vartheta_2)$, it follows from the state system (1.2) that $z = (z_1, z_2, \ldots, z_7)$ is the solution of

$$\begin{cases} \dfrac{\partial z_1}{\partial t} = \alpha_1 \Delta z_1 + z_1[g_1(y_1^*) + y_1^* g_1'(y_1^*) - u_1^* - y_2^* f(y_1^*) - y_1^* y_2^* f'(y_1^*)] \\ \qquad\qquad - y_1^* z_2 f(y_1^*) - \vartheta_{11} y_1^*, \\ \dfrac{\partial z_2}{\partial t} = \alpha_2 \Delta z_2 + + b z_1[y_2^* f(y_1^*) + y_1^* y_2^* f'(y_1^*)] + z_2[-a - u_2^* + b y_1^* f(y_1^*) + c y_3^* h(y_3^*)] \\ \qquad\qquad + c z_3[y_2^* h(y_3^*) + y_3^* y_2^* h'(y_3^*)] - \vartheta_{21} y_2^*, \\ \dfrac{\partial z_3}{\partial t} = \alpha_3 \Delta z_3 + z_3[g_3(y_3^*) + y_3^* g_3'(y_1^*) - u_3^* - y_2^* h(y_3^*) - y_3^* y_2^* h'(y_3^*)] \\ \qquad\qquad - z_2 y_3^* h(y_3^*) - \vartheta_{31} y_3^*, \\ \dfrac{\partial z_j}{\partial t} = k z_7(t) - g z_j(t) - m z_j(t),\ j = i + 3,\ i = 1, 2, 3, \\ \dfrac{\partial z_7}{\partial t} = -k_1 c_e^*(t) \sum\limits_{i=1}^{3} z_i + g_1 \sum\limits_{i=1}^{3} c_{i0}(t) z_i(t) + g_1 \sum\limits_{i=1}^{3} y_i^* z_j(t) - [k_1 \sum\limits_{i=1}^{3} y_i^*(t) + h] z_7(t) + \vartheta_2(t), \end{cases} \tag{3.5}$$

for all $(t,x)\in Q$, subject to the boundary and initial conditions

$$\frac{\partial q_i}{\partial \nu}=0 \ \ a.e.\ on\ \Sigma,\ i=1,2,3.$$

$$z_i(0,x)=z_j(0)=0,\ x\in\Omega, j=i+3,\ i=1,2,3.$$

Multiplying the (3.5) by $q_1,q_2,\ldots,q_7$ respectively, integrating on $Q$ and $(0,T)$ and using (3.2) yield

$$\sum_{i=1}^{3}\int_0^T\int_\Omega K_iu_i^*(t,x)z_i(t,x)dxdt=-\sum_{i=1}^{3}\int_0^T\int_\Omega y_i^*(t,x)q_i(t,x)\vartheta_{1i}dxdt+\int_0^T\vartheta_2(t)q_7(t)dt. \quad (3.6)$$

Substituting (3.6) into (3.4) we obtain that

$$\sum_{i=1}^{3}\int_0^T\int_\Omega[(K_i-q_i)y_i^*-C_iu_i^*]\vartheta_{1i}dxdt+\int_0^T(-C_4\nu^*+q_7)\vartheta_2dt\le 0. \quad (3.7)$$

By using the concept of normal cone $\mathcal{U}_{ad}$ at $(u^*,\nu^*)$ [16], we get

$$\Big((K_i-q_i)y_i^*-C_iu_i^*,-C_4\nu^*+q_7\Big)\in\mathcal{N}_{\mathcal{U}_{ad}}(u^*,\nu^*),$$

the proof is completed by the characteristics properties of the normal vector [14].

# 4 Second order optimality conditions

In this section, we discuss the second order sufficient conditions for the controlled system, since the second order optimality conditions can be solved by using the second order Fréchet derivative of the associated Lagrange function, so we introduce the Lagrange function firstly,

$$\mathcal{L}(y,u,\nu,q)=J(u,\nu)-\int_Q q(y_t-Ay-F)^{\mathbf{T}}dtdx-\int_\Sigma q\Big(\frac{\partial y}{\partial\nu}\Big)^{\mathbf{T}}dtdx, \quad (4.1)$$

here the upper index $\mathbf{T}$ is the transposed of any matrix and $\frac{\partial y}{\partial\nu}=\Big(\frac{\partial y_1}{\partial\nu},\frac{\partial y_2}{\partial\nu},\frac{\partial y_3}{\partial\nu}\Big)$, we employ the method from [17-18], let $X=(y,c_0,c_e)$, $U=(u,v)$, $Q=(q_1,q_2,\cdots,q_7)$, then (4.1) can be written in detail as

$$\begin{aligned}\mathcal{L}(X,U,Q)=&\int_0^T\int_\Omega\big[K_1u_1y_1+K_2u_2y_2+K_3u_3y_3-\frac{1}{2}(C_1u_1^2+C_2u_2^2+C_3u_3^2)\big]dxdt-\frac{1}{2}\int_0^TC_4\nu^2dt\\&+\int_Q\Big(\frac{\partial q_1}{\partial t}y_1+\frac{\partial q_2}{\partial t}y_2+\frac{\partial q_3}{\partial t}y_3\Big)dtdx+\int_Q(\alpha_1y_1\Delta q_1+\alpha_2y_2\Delta q_2+\alpha_3y_3\Delta q_3)dtdx\\&+\int_Q[y_1g_1(y_1)q_1+y_3g_3(y_3)q_3-u_1y_1q_1-u_2y_2q_2-u_3y_3q_3+y_1y_2f(y_1)(bq_2-q_1)\\&-ay_2q_2+y_2y_3h(y_3)(cq_2-q_3)]dtdx+\int_0^T\sum_{j=4}^{6}q_j[kc_e-(g+m)c_{i0}]dt\\&-\int_0^Tq_7[k_1c_e(y_1+y_2+y_3)+g_1(c_{10}y_1+c_{20}y_2+c_{30}y_3)-hc_e+\nu]dt,\end{aligned} \quad (4.2)$$

we introduce the Hamiltonian function secondly

$$\begin{aligned}\mathcal{H}(X,U,Q) =&K_1u_1y_1 + K_2u_2y_2 + K_3u_3y_3 + y_1g_1(y_1)q_1 + y_3g_3(y_3)q_3 - u_1y_1q_1 - u_2y_2q_2 - u_3y_3q_3\\ &+ y_1y_2f(y_1)(bq_2 - q_1) - ay_2q_2 + y_2y_3h(y_3)(cq_2 - q_3) + \sum_{j=4}^{6} q_j[kc_e - (g+m)c_{i0}]\\ &- q_7[k_1c_e(y_1+y_2+y_3) + g_1(c_{10}y_1 + c_{20}y_2 + c_{30}y_3) - hc_e + \nu],\end{aligned}$$

Assume that $g_1, g_3, f$ and $h$ are functions of class $C^2$. If $\bar{u}, \bar{\nu}$ is an admissible control and $y, q$ are the corresponding state and adjoint state, then the associated Hessian matrix at $(y, u, q)$ is

$$D^2\mathcal{H}(y,u,\nu,q) = \begin{pmatrix} H_{11} & H_{12} & 0 & H_{14} & 0 & 0 & 0\\ H_{21} & 0 & H_{23} & 0 & H_{25} & 0 & 0\\ 0 & H_{32} & H_{33} & 0 & 0 & H_{36} & 0\\ H_{41} & 0 & 0 & -C_1 & 0 & 0 & 0\\ 0 & H_{52} & 0 & 0 & -C_2 & 0 & 0\\ 0 & 0 & H_{63} & 0 & 0 & -C_3 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & -C_4 \end{pmatrix},$$

and

$$\begin{cases} H_{11} = \bar{q}_1[2g_1'(\bar{y}_1) + \bar{y}_1g_1''(\bar{y}_1)] - \bar{y}_2(\bar{q}_1 - b\bar{q}_2)[2f'(\bar{y}_1) + \bar{y}_1f''(\bar{y}_1)],\\ H_{12} = -(\bar{q}_1 - b\bar{q}_2)[f(\bar{y}_1) + \bar{y}_1f'(\bar{y}_1)],\\ H_{23} = -(\bar{q}_3 - c\bar{q}_2)[h(\bar{y}_3) + \bar{y}_3f'(\bar{y}_3)],\\ H_{33} = \bar{q}_3[2g_3'(\bar{y}_3) + \bar{y}_3g_3''(\bar{y}_3)] - \bar{y}_2(\bar{q}_3 - c\bar{q}_2)[2h'(\bar{y}_3) + \bar{y}_3h''(\bar{y}_3)]\\ H_{14} = H_{41} = K_1 - \bar{q}_1, H_{25} = H_{52} = K_2 - \bar{q}_2, H_{36} = H_{63} = K_3 - \bar{q}_3. \end{cases}$$

Then, we have

$$\mathcal{L}''(\bar{y},\bar{u},\bar{\nu},\bar{q})[(y,u,\nu),(y,u,\nu)] = \int_Q (y,u,\nu)D^2\mathcal{H}(y,u,q)(y,u,\nu)^{\mathbf{T}}dtdx,$$

that is

$$\begin{aligned}&\mathcal{L}''(\bar{y},\bar{u},\bar{\nu},\bar{q})[(y,u,\nu),(y,u,\nu)]\\ &= \int_Q (y_1)^2[\bar{q}_1(2g_1'(\bar{y}_1) + \bar{y}_1g_1''(\bar{y}_1)) - \bar{y}_2(\bar{q}_1 - b\bar{q}_2)(2f'(\bar{y}_1) + \bar{y}_1f''(\bar{y}_1))]dtdx\\ &+ \int_Q (y_3)^2[\bar{q}_3(2g_3'(\bar{y}_3) + \bar{y}_3g_3''(\bar{y}_3)) - \bar{y}_2(\bar{q}_3 - c\bar{q}_2)(2h'(\bar{y}_3) + \bar{y}_3h''(\bar{y}_3))]dtdx\\ &+ \int_Q 2[-y_1y_2(\bar{q}_1 - b\bar{q}_2)(f(\bar{y}_1) + \bar{y}_1f'(\bar{y}_1)) - y_2y_3(\bar{q}_3 - c\bar{q}_2)(h(\bar{y}_3) + \bar{y}_3f'(\bar{y}_3))]dtdx\\ &+ \int_Q [2u_1y_1(K_1 - \bar{q}_1) + 2u_2y_2(K_2 - \bar{q}_2) + 2u_3y_3(K_3 - \bar{q}_3)]dtdx\\ &- \int_Q (C_1u_1^2 + C_2u_2^2 + C_3u_3^2)dtdx - \int_0^T C_4\nu^2dt.\end{aligned}$$

Now we can formulate the second order optimality conditions for our problem.

**Theorem 4.1** (i) (Second order necessary optimality conditions.) Under the hypotheses of Theorem 3.1, if $(u^*, \nu^*)$ is an optimal pair and $q$ is the corresponding adjoint variable, then

$$\mathcal{L}''(y^*,u^*,\nu^*,q)[(y,u,\nu),(y,u,\nu)] \le 0, \quad \forall\ (u,\nu) \in \mathcal{N}_{\mathcal{U}_{ad}}(u^*,\nu^*).$$

(ii) (Second order sufficient optimality conditions.) $\forall\ (u^*, \nu^*) \in \mathcal{U}_{ad}$, together with its corresponding state $y^*$ and adjoint state $q$, if $(y^*, u^*, \nu^*, q)$ satisfies the first order necessary condition (3.1) and the condition

$$\mathcal{L}''(y^*, u^*, \nu^*, q)[(y, u, \nu), (y, u, \nu)] < \kappa(\|v_1\|^2_{L^2(Q)} + \|v_2\|^2_{L^2(0,T)}),\quad \forall\ (v_1, v_2) \in \mathcal{N}_{\mathcal{U}_{ad}},$$

for some $\kappa > 0$, then $(y^*, u^*, \nu^*)$ is an optimal local solution of the controlled system (1.2).

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# References

[1] N.C. Apreutesei, Necessary optimality conditions for a Lotka-Volterra three species system, Math. Model. Nat. Phenom. 1 (2006) 123-135.

[2] W. Ko, K. Ryu, Analysis of diffusive two-competing-prey and one-predator systems with Beddington-DeAngelis functional response, Nonlinear Anal. 71 (9) (2009) 4185-4202.

[3] W. Ko, I. Ahn, A diffusive one-prey and two-competing-predator system with a ratio-dependent functional response: II stationary pattern formation, J. Math. Anal. Appl. 397 (1) (2013) 29-45.

[4] N. Apreutesei, G. Dimitriu, R. Strugariu, An optimal control problem for a two-prey and one-predator model with diffusion, Comput. Math. Appl. 67 (2014) 2127-2143.

[5] E. Casas, M. Mateos, F. Tröltzsch, Necessary and sufficient optimality conditions for optimization problems in function spaces and applications to control theory, ESAIM: Proceedings. 13 (2003).

[6] N. Apreutesei, G. Dimitriu, On a prey-predator reaction-diffusion system with Holling type III functional response, J. Comput. Appl. Math. 235 (2) (2010) 366-379.

[7] N.C. Apreutesei, An optimal control problem for a pest, predator, and plant system, Nonlinear Anal.RWA. 13 (3) (2012) 1391-1400.

[8] W. Chen, M. Wang, Qualitative analysis of predator-prey models with Beddington-DeAngelis functional response and diffusion, Math. Comput. Modelling 42 (2005) 31-44.

[9] Y. Du, Y. Lou, Qualitative behavior of positive solutions of a predatorCprey model: effects of saturation, Proc. Roy. Soc. Edinburgh Sect. A 131 (2001) 321-349.

[10] Z.X. Luo, Z.R. He, Optimal control for age-dependent population hybrid system in a polluted environment, Appl. Math. Comput. 228 (2014) 68-76.

[11] Z.X. Luo, X.L. Fan, Optimal control for an age-dependent competitive species model in a polluted environment. Comput. Math. Appl. 228 (2014) 91-101.

[12] Z.X. Luo, Optimal control for an age-dependent predator-prey system in a polluted environment, J. Appl. Math. Comput. 44 (1-2) (2014) 491-509.

[13] I. Vrabie, $C_0$-Semigroups and Applications, in: Math. Stud., vol. 191, North-Holland, 2003.

[14] V. Barbu, Mathematical Methods in Optimization of Differential Systems, Kluwer Academic Publishers, Dordrecht, 1994.

[15] Z.E. Ma, G.R. Cui , W.D. Wang, Persistence and extinction of a population in a polluted environment, Math. Biosci. 101 (1) (1990) 75-97.

[16] V. Barbu, M. Iannelli, Optimal control of population dynamics. Optim. Theory. Appl. 102 (1) ( 1999) 1-14.

[17] E. Casas, F. Tröltzsch, Second order necessary and sufficient optimality conditions for optimization problems and applications to control theory, SIAM J. Optim. 13 (2) (2002) 406-431.

[18] J.-P. Raymond, F. Tröltzsch, Second order sufficient optimality conditions for nonlinear parabolic control problems with state constraints, Discrete and continuous dynamical systems, 6 (2) (2000) 431-450.

# Existence results for new extended vector variational-like inequality

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**Abstract**

In this paper, we establish and study some new existence theorems for a new extended vector variational-like inequality in a Banach space. The results are proved by using the new definition of $g - f - \eta - \phi - \mu-$quasimonotone of Stampacchia and of Minty type mappings. The obtained results in this article can be viewed as some new and generalized forms which can be applied to several problems.

*Keywords:* New extended vector variational-like inequality; Existence result; C-convex; KKM-mapping; $g - f - \eta - \phi - \mu-$quasimonotonicity; $g - f - \eta - \phi - \mu-$pseudomonotoncity.

## 1. Introduction

In 1980, Giannessi introduced a generalization of variational inequality is the vector variational inequality (for short, VVI) in a finite-dimensional Euclidean space, see [8]. For the past years, vector variational inequalities and their generalizations have been studied and applied in various directions. The vector variational-like inequalities is a generalized form of a vector variational inequalities related to the class of $\eta$-connected sets which is much more general than the class of convex sets. It well Known that monotonicity plays an important role to proving existence of solutions of vector variational inequalities and vector variational-like inequalities. Some important generalizations of monotonicity, such as quasimonotonicity, proper quasimonotonicity, pseu-domonotonicity, dense pseudomonotonicity, semimonotonicity, have been introduced and considered to study various variational inequalities and other related problems. In [9] Ahmad and Irfan obtained existence results for extended vector variational-like inequality and equilibrium problems by using $g$-$h$-$\eta$-quasimonotone of Stampacchia and Minty types.

In this paper, we introduce a new definition for a new extended vector variational-like inequality and we define a new and general form of definitions for quasimonotone of Stampacchia and Minty type mappings. We have some ideas to establish some sufficient conditions to guarantee the existence of solutions. The new problems can be viewed as some unified forms of the previous problem, that is, extended vector variational-like inequalities considered and studied by Ahmad and Irfan [9].

Let $X$ and $Y$ be two real Banach spaces, $K \subset X$ be a nonempty, closed and convex subset, $C \subset Y$ be a pointed, closed and convex cone in $Y$ such that $intC \neq \emptyset$ where $intC$ denote the

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interior of $C$. Then for $x, y \in Y$ , a partial order $\geq_C$ in $Y$ is defined as

$$x \geq_C y \Leftrightarrow x - y \in C.$$

Let $L(X,Y)$ be the space of all continuous linear mappings from $X$ to $Y$. Let $T_1, T_2, \cdots, T_N : K \to L(X,Y), g, f : K \to K, \eta : K \times K \to X$ and $\phi, \mu : K \times K \to Y$ are mappings.
We consider the following *new extended vector variational-like inequalities*:

$$(NEVVLI-I)\begin{cases} Find \;\; x \in K such\ that, \\ \left\langle \sum_{i=1}^{N} T_i(x), \eta(g(y), g(x)) \right\rangle + \phi\left(f(y), f(x)\right) - \mu\left(f(x), f(y)\right) \geq_C 0, \\ \forall y \in K. \end{cases}$$

and

$$(NEVVLI-II)\begin{cases} Find \;\; x \in K such\ that, \\ \left\langle \sum_{i=1}^{N} T_i(y), \eta(g(x), g(y)) \right\rangle + \phi\left(f(x), f(y)\right) - \mu\left(f(y), f(x)\right) \leq_C 0, \\ \forall y \in K. \end{cases}$$

*Special cases:*

(i) If $T_3, T_4, \cdots, T_N \equiv 0$, $T_1 = S, T_2 = T$, $\phi = h$, $\mu \equiv 0$ and $f = g$ then (NEVVLI-I) and (NEVVLI-II) reduces to the following *extended vector variational-like inequalities* considered and studied by Ahmad and Irfan [9]

$$(EVVLI-I)\begin{cases} Find \;\; x \in K such\ that, \\ \langle S(x) + T(x), \eta(g(y), g(x))\rangle + h(g(y), g(x)) \geq_C 0, \\ \forall y \in K, \end{cases}$$

and

$$(EVVLI-II)\begin{cases} Find \;\; x \in K such\ that, \\ \langle S(y) + T(y), \eta(g(x), g(y))\rangle + h(g(x), g(y)) \leq_C 0, \\ \forall y \in K, \end{cases}$$

(ii) If $T_2, T_3, \cdots, T_N \equiv 0, T_1 = T$, $\phi = h$, $\mu \equiv 0$ and $f = g = I$ then (NEVVLI-I) and (NEVVLI-II) reduces to the following *vector variational-like inequalities* considered and studied by Ahmad [1]

$$(VVLI-I)\begin{cases} Find \;\; x \in K such\ that, \\ \langle T(x), \eta(y, x)\rangle + h(y, x) \geq_C 0, \quad \forall y \in K, \end{cases}$$

and

$$(VVLI-I)\begin{cases} Find \;\; x \in K such\ that, \\ \langle T(y), \eta(x, y)\rangle + h(x, y) \leq_C 0, \quad \forall y \in K, \end{cases}$$

(iii) If $T_2, T_3, \cdots, T_N \equiv 0, T_1 = T$, $\phi \equiv 0$, $\mu \equiv 0$ and $g = I$ then (NEVVLI-I) and (NEVVLI-II) reduces to the following vector variational-like inequalities considered and studied by Zhao and Xia [12]

$$(VVLI-I)\begin{cases} Find \;\; x \in K such\ that, \\ \langle T(x), \eta(y, x)\rangle \geq_C 0, \quad \forall y \in K, \end{cases}$$

and

$$(VVLI-I)\begin{cases} Find \;\; x\in K such\ that, \\ \langle T(y),\eta(x,y)\rangle \leq_C 0, \;\; \forall y\in K, \end{cases}$$

The following concepts and results are needed for the results.

**Definition 1.1.** A mapping $f:K\to Y$ is said to be *hemicontinuous* if, for any fixed $x,y\in K$, the mapping $t\mapsto f(x+t(y-x))$ is continuous at $0^+$.

**Definition 1.2.** Let $C:K\to 2^Y$ be a set-valued mapping, $h:K\times K\to Y$ and $g:K\times K\to X$ are the single-valued mappings. Then

(i) $h(\cdot,v)$ is said to be *C-convex* in the first argument if

$$h(tu_1+(1-t)u_2,v)\in th(u_1,v)+(1-t)h(u_2,v)-C,\forall u_1,u_2\in K,\;\; t\in[0,1],$$

(ii) If $K$ is an affine set, the $\eta(x,y)$ is said to be *affine* with respect to $u$ if for any given $v\in K$

$$\eta(tu_1+(1-t)u_2,v)=t\eta(u_1,v)+(1-t)\eta(u_2,v),\forall u_1,u_2\in K,\;\; t\in\mathbb{R},$$

with $u=(tu_1+(1-t)u_2)\in K$.

**Definition 1.3.** Let $T_1,T_2,\cdots,T_N:K\to L(X,Y),g,f:K\to K,\eta:K\times K\to X$ and $\phi,\mu:K\times K\to Y$ are mappings. Then $T_1,T_2,\cdots,T_N$ are said to be $g$-$f$-$\eta$-$\phi$-$\mu$-pseudomonotone if for any $x,y\in K$,

$$\left\langle\sum_{i=1}^{N}T_i(x),\eta(g(y),g(x))\right\rangle+\phi\left(f(y),f(x)\right)-\mu\left(f(x),f(y)\right)\geq_C 0,$$
$$\Rightarrow\left\langle\sum_{i=1}^{N}T_i(y),\eta(g(x),g(y))\right\rangle+\phi\left(f(x),f(y)\right)-\mu\left(f(y),f(x)\right)\leq_C 0.$$

**Example 1.4.** Let $X=\mathbb{R},K=\mathbb{R}_+,Y=\mathbb{R}^2,C=\mathbb{R}^2_+$ and

$$T_1(x)=\begin{pmatrix}1\\1\end{pmatrix},T_2(x)=\begin{pmatrix}2^x\\2^{-x}\end{pmatrix},T_3(x)=\begin{pmatrix}3^x\\3^{-x}\end{pmatrix},\cdots,T_N(x)=\begin{pmatrix}N^x\\N^{-x}\end{pmatrix}$$
$$g(x)=3x,f(x)=2x,\eta(y,x)=4y-5x,\phi(y,x)=\begin{pmatrix}8y-12x\\6y^2-5xy-11x^2\end{pmatrix},$$
$$\mu(x,y)=\begin{pmatrix}3x-4y\\4x^2-2xy-6y^2\end{pmatrix},\forall x,y\in K.$$

Thus

$$\eta(g(y),g(x))=\eta(3y,3x)$$
$$=4(3y)-5(3x)$$
$$=12y-15x,$$

$$\begin{aligned}
&\phi(f(y),f(x)) = \phi(2y,2x)\\
&= \begin{pmatrix} 8(2y)-12(2x) \\ 6(2y)^2-5(2x)(2y)-11(2x)^2 \end{pmatrix}\\
&= \begin{pmatrix} 16y-24x \\ 24y^2-20xy-44x^2 \end{pmatrix},
\end{aligned}$$

and

$$\begin{aligned}
&\mu(f(x),f(y)) = \mu(2y,2x)\\
&= \begin{pmatrix} 3(2x)-4(2y) \\ 4(2x)^2-2(2x)(2y)-2(2y)^2 \end{pmatrix}\\
&= \begin{pmatrix} 6x-8y \\ 16x^2-8xy-24y^2 \end{pmatrix}.
\end{aligned}$$

Then $\forall x, y \in K$

$$\begin{aligned}
&\left\langle \sum_{i=1}^{N} T_i(x), \eta(g(y),g(x)) \right\rangle + \phi\left(f(y),f(x)\right) - \mu\left(f(x),f(y)\right)\\
&= \begin{pmatrix} 1+2^x+3^x+\cdots+N^x \\ 1+2^{-x}+3^{-x}+\cdots+N^{-x} \end{pmatrix}(12y-15x) + \begin{pmatrix} 16y-24x \\ 24x^2-20xy-44y^2 \end{pmatrix} - \begin{pmatrix} 6x-8y \\ 16y^2-8xy-24x^2 \end{pmatrix}\\
&= \begin{pmatrix} 1+2^x+3^x+\cdots+N^x \\ 1+2^{-x}+3^{-x}+\cdots+N^{-x} \end{pmatrix}(12y-15x) + \begin{pmatrix} 16y-24x-6x+8y \\ 24y^2-20xy-44x^2-16x^2+8xy+24y^2 \end{pmatrix}\\
&= \begin{pmatrix} 1+2^x+3^x+\cdots+N^x \\ 1+2^{-x}+3^{-x}+\cdots+N^{-x} \end{pmatrix}(12y-15x) + \begin{pmatrix} 24y-30x \\ 48y^2-12xy-60x^2 \end{pmatrix}\\
&= \begin{pmatrix} 1+2^x+3^x+\cdots+N^x \\ 1+2^{-x}+3^{-x}+\cdots+N^{-x} \end{pmatrix}(12y-15x) + \begin{pmatrix} 2(12y-15x) \\ 2(12y-15x)(y+x) \end{pmatrix}\\
&= \begin{pmatrix} 1+2^x+3^x+\cdots+N^x \\ 1+2^{-x}+3^{-x}+\cdots+N^{-x} \end{pmatrix}(12y-15x) + \begin{pmatrix} 2 \\ 4(y+x) \end{pmatrix}(12y-15x)\\
&= (12y-15x)\left[\begin{pmatrix} 1+2^x+3^x+\cdots+N^x \\ 1+2^{-x}+3^{-x}+\cdots+N^{-x} \end{pmatrix} + \begin{pmatrix} 2 \\ 4(y+x) \end{pmatrix}\right]\\
&= (12y-15x)\begin{pmatrix} 1+2^x+3^x+\cdots+N^x+2 \\ 1+2^{-x}+3^{-x}+\cdots+N^{-x}+4y+4x \end{pmatrix} \geq_C 0,
\end{aligned}$$

implies that $12y \geq 15x$. Thus, $12x \leq 15x \leq 12y \leq 15y$. Therefore, $12x - 15y \leq 0$.

So it follows that

$$\begin{aligned}
&\left\langle \sum_{i=1}^{N} T_i(y), \eta(g(x), g(y)) \right\rangle + \phi\left(f(x), f(y)\right) - \mu\left(f(y), f(x)\right) \\
&= \begin{pmatrix} 1+2^y+3^y+\cdots+N^y \\ 1+2^{-y}+3^{-y}+\cdots+N^{-y} \end{pmatrix} (12x-15y) + \begin{pmatrix} 16x-24y \\ 24y^2-20xy-44x^2 \end{pmatrix} - \begin{pmatrix} 6y-8x \\ 16x^2-8xy-24y^2 \end{pmatrix} \\
&= \begin{pmatrix} 1+2^y+3^y+\cdots+N^y \\ 1+2^{-y}+3^{-y}+\cdots+N^{-y} \end{pmatrix} (12x-15y) + \begin{pmatrix} 16x-24y-6y+8x \\ 24x^2-20xy-44y^2-16y^2+8xy+24x^2 \end{pmatrix} \\
&= \begin{pmatrix} 1+2^y+3^y+\cdots+N^y \\ 1+2^{-y}+3^{-y}+\cdots+N^{-y} \end{pmatrix} (12x-15y) + \begin{pmatrix} 24x-30y \\ 48x^2-12xy-60y^2 \end{pmatrix} \\
&= \begin{pmatrix} 1+2^y+3^y+\cdots+N^y \\ 1+2^{-y}+3^{-y}+\cdots+N^{-y} \end{pmatrix} (12x-15y) + \begin{pmatrix} 2(12x-15y) \\ 2(12x-15y)(x+y) \end{pmatrix} \\
&= \begin{pmatrix} 1+2^y+3^y+\cdots+N^y \\ 1+2^{-y}+3^{-y}+\cdots+N^{-y} \end{pmatrix} (12x-15y) + \begin{pmatrix} 2 \\ 4(x+y) \end{pmatrix} (12x-15y) \\
&= (12x-15y)\left[ \begin{pmatrix} 1+2^y+3^y+\cdots+N^y \\ 1+2^{-y}+3^{-y}+\cdots+N^{-y} \end{pmatrix} + \begin{pmatrix} 2 \\ 4(x+y) \end{pmatrix} \right] \\
&= (12x-15y) \begin{pmatrix} 1+2^y+3^y+\cdots+N^y+2 \\ 1+2^{-y}+3^{-y}+\cdots+N^{-y}+4x+4y \end{pmatrix} \leq_C 0.
\end{aligned}$$

$\Rightarrow T_1, T_2, \cdots, T_N$ are $g$-$f$-$\eta$-$\phi$-$\mu$-pseudomonotone.

**Definition 1.5.** A multi-valued operator $S : X \to 2^{X^*}$ is called quasimonotone if for all $x, y \in X$ the following implications hold:

$$\exists x^* \in S(x) : \langle x^*, y-x\rangle > 0 \Rightarrow \exists y^* \in S(y) : \langle y^*, y-x\rangle \geq 0.$$

**Definition 1.6.** A multi-valued operator $S : X \to 2^{X^*}$ is called *properly quasimonotone* if for all $x_1, x_2, ..., x_n \in X$ and every $y \in Conv\{x_1, x_2, ..., x_n\}$ there exist $i$ such that

$$\forall x_i^* \in S(x_i) : \langle x_i^*, y-x_i\rangle \geq 0.$$

**Definition 1.7.** A mapping $T : K \to L(X, Y)$ is said to be *properly quasimonotone of Stampacchia type* if for all $n \in N$ for all vectors $v_1, v_2, ..., v_n \in K$ and scalars $\lambda_i \geq 0, i = 1, 2, ..., n$ with $\sum_{i=1}^n \lambda_i = 1$ and $u := \sum_{i=1}^n \lambda_i v_i$, $\langle Tu, v_i - u\rangle \geq_C 0$ holds for some $i$. $T$ is said to be *properly quasimonotone of Minty type* if for all vectors $v_1, v_2, ..., v_n \in K$ and scalars $\lambda_i \geq 0, i = 1, 2, ..., n$ with $\sum_{i=1}^n \lambda_i = 1$ and $u := \sum_{i=1}^n \lambda_i v_i$, $\langle Tv_i, v_i - u\rangle \leq_C 0$ holds for some $i$.

**Definition 1.8.** A mapping $T : K \to L(X, Y)$ is said to be *properly $g$-$\eta$-quasimonotone of Stampacchia type* if for all $n \in N$ for all vectors $v_1, v_2, ..., v_n \in K$ and scalars $\lambda_i \geq 0, i = 1, 2, ..., n$ with $\sum_{i=1}^n \lambda_i = 1$ and $u := \sum_{i=1}^n \lambda_i v_i$, $\langle Tu, \eta(g(v_i), g(u))\rangle \geq_C 0$ holds for some $i$. $T$ is said to be *properly $g$-$\eta$-quasimonotone of Minty type* if for all vectors $v_1, v_2, ..., v_n \in K$ and scalars $\lambda_i \geq 0, i = 1, 2, ..., n$ with $\sum_{i=1}^n \lambda_i = 1$, and $u := \sum_{i=1}^n \lambda_i v_i$, $\langle Tv_i, \eta(g(v_i), g(u))\rangle \leq_C 0$ holds for some $i$.

**Definition 1.9.** A mapping $T : K \to L(X, Y)$ is said to be *properly $g$-$h$-$\eta$-quasimonotone of Stampacchia type* if for all $n \in N$ for all vectors $v_1, v_2, ..., v_n \in K$ and scalars $\lambda_i \geq 0, i = 1, 2, ..., n$ with $\sum_{i=1}^n \lambda_i = 1$ and $u := \sum_{i=1}^n \lambda_i v_i$, $\langle Tu, \eta(g(v_i), g(u))\rangle + h(g(v_i), g(u)) \geq_C 0$ holds for some $i$. $T$ is said to be *properly $g$-$h$-$\eta$-quasimonotone of Minty type* if for all vectors $v_1, v_2, ..., v_n \in K$ and scalars $\lambda_i \geq 0, i = 1, 2, ..., n$ with $\sum_{i=1}^n \lambda_i = 1$, and $u := \sum_{i=1}^n \lambda_i v_i$, $\langle Tv_i, \eta(g(u), g(v_i))\rangle + h(g(u), g(v_i)) \leq_C 0$ holds for some $i$.

**Definition 1.10.** A mapping $T : K \to L(X, Y)$ is said to be *properly g-f-η-ϕ-μ-quasimonotone of Stampacchia type* if for all $n \in N$ for all vectors $v_1, v_2, ..., v_n \in K$ and scalars $\lambda_i \geq 0, i = 1, 2, ..., n$ with $\sum_{i=1}^{n} \lambda_i = 1$ and $u := \sum_{i=1}^{n} \lambda_i v_i$, $\langle T(u), \eta(g(v_i), g(u))\rangle + \phi(f(v_i), f(u)) - \mu(f(u), f(v_i)) \geq_C 0$ holds for some $i$. $T$ is said to be *properly g-f-η-ϕ-μ-quasimonotone of Minty type* if for all vectors $v_1, v_2, ..., v_n \in K$ and scalars $\lambda_i \geq 0, i = 1, 2, ..., n$ with $\sum_{i=1}^{n} \lambda_i = 1$, and $u := \sum_{i=1}^{n} \lambda_i v_i$, $\langle T(v_i), \eta(g(u), g(v_i))\rangle + \phi(f(u), f(v_i)) - \mu(f(v_i), f(u)) \leq_C 0$ holds for some $i$.

**Example 1.11.** Let $X = \mathbb{R}, K = \mathbb{R}_+, Y = \mathbb{R}^2, C = \mathbb{R}^2_+$ and

$$T_1(x) = \begin{pmatrix} 1 \\ x \end{pmatrix}, T_2(x) = \begin{pmatrix} 2 \\ x^2 \end{pmatrix}, T_3(x) = \begin{pmatrix} 3 \\ x^3 \end{pmatrix}, \cdots, T_N(x) = \begin{pmatrix} N \\ x^N \end{pmatrix}$$

$$g(x) = 2x, f(x) = 3x, \eta(y, x) = 7y - 5x, \phi(y, x) = \begin{pmatrix} 5y + 3x \\ 5y^2 + 3x^2 \end{pmatrix},$$

$$\mu(x, y) = \begin{pmatrix} 2x + 3y \\ 2x^2 + 3y^2 \end{pmatrix}, \forall x, y \in K.$$

Thus

$$\begin{aligned} &\eta(g(y), g(x)) = \eta(2y, 2x) \\ &= 7(2y) - 5(2x) \\ &= 14y - 10x, \end{aligned}$$

$$\begin{aligned} &\phi(f(y), f(x)) = \phi(3y, 3x) \\ &= \begin{pmatrix} 5(3y) + 3(3x) \\ 5(3y)^2 + 3(3x)^2 \end{pmatrix} \\ &= \begin{pmatrix} 15y + 9x \\ 45y^2 + 27x^2 \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} &\mu(f(x), f(y)) = \mu(3x, 3y) \\ &= \begin{pmatrix} 2(3x) + 3(3y) \\ 2(3x)^2 + 3(3y)^2 \end{pmatrix} \\ &= \begin{pmatrix} 6x + 9y \\ 18x^2 + 27y^2 \end{pmatrix}. \end{aligned}$$

We claim that $T_1, T_2, \cdots, T_N$ are properly *g-f-η-ϕ-μ*-quasimonotone of Stampacchia type. Suppose to the contrary that there exists $x_i \in K, t_i \geq 0, i = 1, 2, ..., n$ with $\sum_{i=1}^{n} t_i = 1$ such that

$$\left\langle \sum_{i=1}^{N} T_i(x), \eta(g(x_i), g(x)) \right\rangle + \phi\left(f(x_i), f(x)\right) - \mu\left(f(x), f(x_i)\right) \not\geq_C 0, i = 1, 2, ..., n$$

where $x_i = \sum_{i=1}^{n} \lambda_i x_i$, it follows that

$$\begin{aligned}
&\left\langle \sum_{i=1}^{N} T_i(x), \eta(g(x_i), g(x)) \right\rangle + \phi\left(f(x_i), f(x)\right) - \mu\left(f(x), f(x_i)\right) \\
&= \begin{pmatrix} 1+2+3+\cdots+N \\ x+x^2+x^3+\cdots+x^N \end{pmatrix} (14x_i - 10x) + \begin{pmatrix} 15x_i + 9x \\ 45{x_i}^2 + 27x^2 \end{pmatrix} - \begin{pmatrix} 6x + 9x_i \\ 18x^2 + 27{x_i}^2 \end{pmatrix} \\
&= \begin{pmatrix} (1+2+3+\cdots+N)(14x_i - 10x) \\ (x+x^2+x^3+\cdots+x^N)(14x_i - 10x) \end{pmatrix} + \begin{pmatrix} 6x_i + 3x \\ 18x_i + 9x \end{pmatrix} \not\geq_C 0 \\
&= \begin{pmatrix} (1+2+3+\cdots+N)(14x_i - 10x) + 6x_i + 3x \\ (x+x^2+x^3+\cdots+x^N)(14x_i - 10x) + 18x_i + 9x \end{pmatrix} \not\geq_C 0
\end{aligned}$$

$i = 1, 2, ..., n,$

which is a contradiction, since

$$(1+2+3+\cdots+N)(14x_i - 10x) + 6x_i + 3x \geq_C 0,$$

and

$$(x+x^2+x^3+\cdots+x^N)(14x_i - 10x) + 18x_i + 9x \geq_C 0,$$

for atleast one $i$. Thus $T_1, T_2, \cdots, T_N$ are properly $g$-$f$-$\eta$-$\phi$-$\mu$-quasimonotone of Stampacchia type.

**Lemma 1.12.** *Let $T_1, T_2, \cdots, T_N : K \to L(X,Y), \eta : K \times K \to X, \phi, \mu : K \times K \to Y$ and $g : K \to K$ be mappings. If $T_1, T_2, \cdots, T_N$ are $g$-$f$-$\eta$-$\phi$-$\mu$-pseudomonotone and properly $g$-$f$-$\eta$-$\phi$-$\mu$-quasimonotone of Stampacchia type, then $T_1, T_2, \cdots, T_N$ are properly $g$-$f$-$\eta$-$\phi$-$\mu$-quasimonotone of Minty type.*

*Proof.* The fact directly follows from Definitions 1.3 and 1.9. □

**Definition 1.13.** Let $D$ be a nonempty subset of a topological Hausdorff space $E$. A mapping $G : D \to 2^E$ (the family of nonempty subset of $E$) is said to be a *KKM mapping* if for any finite subset $\{x_1, x_2, ..., x_n\} \subset D$,
we have $Co\{x_1, ..., x_n\} \subset \bigcup_{i=1}^{n} G(x_i)$. where Co denotes the convex hull operator.

**Lemma 1.14** ([6])**.** *Let $D$ be a nonempty subset of a topological Hausdorff vector space $E$ and $G : D \to 2^E$ be a KKM mapping. If $G(x)$ is closed for any $x \in D$, and compact for some $x \in D$, then $\bigcap\limits_{x \in D} G(x) \neq \emptyset$.*

**Lemma 1.15.** *Let $Y$ be a topological vector space with a pointed, closed and convex cone $C$ such that $intC \neq \emptyset$. If $u, v \in Y$ and $u \notin C$ and $v \in -C$, then $tv + (1-t)u \notin C, \quad \forall t \in (0,1)$.*

*Proof.* Assume that $u, v \in Y$ and $u \notin C$ and $v \in -C$. We must to show that $tv + (1-t)u \notin C$ $\forall t \in (0,1)$. Suppose to the contrary that there exists some $t \in (0,1)$ such that $tv + (1-t)u \in C$. Since $C$ is cone and $v \in -C$, we have $-tv \in C$. Thus $tv + (1-t)u + (-tv) \in C + C \subset C$ and hence $(1-t)u \in C$. By $(1-t) > 0$ and $C$ is cone, it follows that $\frac{1}{(1-t)}(1-t)u \in C$. So $u \in C$. Which is a contradiction. Hence $tv + (1-t)u \notin C, \ \forall t \in (0,1)$. This completes the proof. □

## 2. Existence results

In this section, we establish some existence results for (NEVVLI-I) and (NEVVLI-II) by using Lemma 1.14.

**Lemma 2.1.** *Let $T_1, T_2, \cdots, T_N : K \to L(X,Y), \eta : K \times K \to X, \phi, \mu : K \times K \to Y$ be mappings and $g, f : K \to K$ is affine mapping satisfying the following conditions:*

(a) *$T_1, T_2, \cdots, T_N$ are $g$-$f$-$\eta$-$\phi$-$\mu$-pseudomonotone;*

(b) *for any fixed $x \in X$, the mapping $y \mapsto \left\langle \sum_{i=1}^{N} T_i(y), \eta(g(x), g(y)) \right\rangle$ is hemicontinuous and $\phi(f(x), f(y))$ and $\mu(f(y), f(x))$ are continuous with $\{z_t\} \to x_0 \in K, z_t \in K$;*

(c) *$\phi(\cdot, f(y))$ is $C$-convex in the first variable and $\phi(f(x), f(x)) = 0, \forall x \in K$;*

(d) *$\mu(f(y), \cdot)$ is $C$-concave in the second variable and $\mu(f(x), f(x)) = 0, \forall x \in K$;*

(e) *$\eta(\cdot, g(y))$ is affine in the first variable and $\eta(g(x), g(x)) = 0, \forall x \in K$.*

*Then for any $x_0 \in K$, the following statements are equivalent*

$$(I)\ \left\langle \sum_{i=1}^{N} T_i(x_0), \eta(g(x), g(x_0)) \right\rangle + \phi\left(f(x), f(x_0)\right) - \mu\left(f(x_0), f(x)\right) \geq_C 0,$$

$$(II)\ \left\langle \sum_{i=1}^{N} T_i(x), \eta(g(x_0), g(x)) \right\rangle + \phi\left(f(x_0), f(x)\right) - \mu\left(f(x), f(x_0)\right) \leq_C 0.$$

*Proof.* $T_1, T_2, \cdots, T_N$ are $g$-$f$-$\eta$-$\phi$-$\mu$-pseudomonotone, it follows that $(I) \Rightarrow (II)$.
$(II) \Rightarrow (I)$. Suppose that $(II)$ holds for any $x_0 \in K$

$$\left\langle \sum_{i=1}^{N} T_i(y), \eta(g(x), g(y)) \right\rangle + \phi\left(f(x), f(y)\right) - \mu\left(f(y), f(x)\right) \leq_C 0.$$

For arbitrary $z \in K$, letting $z_t = (1-t)x_0 + tz, \ \ t \in (0,1)$, we have $z_t \in K$ by convexity of $K$. Hence we have

$$\left\langle \sum_{i=1}^{N} T_i(z_t), \eta(g(x_0), g(z_t)) \right\rangle + \phi\left(f(x_0), f(z_t)\right) - \mu\left(f(z_t), f(x_0)\right) \leq_C 0.$$

Now we show that

$$\left\langle \sum_{i=1}^{N} T_i(z_t), \eta(g(z), g(z_t)) \right\rangle + \phi\left(f(z), f(z_t)\right) - \mu\left(f(z_t), f(z)\right) \geq_C 0.$$

Suppose to the contrary that there exists some $t \in (0,1)$ such that

$$\left\langle \sum_{i=1}^{N} T_i(z_t), \eta(g(z), g(z_t)) \right\rangle + \phi\left(f(z), f(z_t)\right) - \mu\left(f(z_t), f(z)\right) \not\geq_C 0.$$

As $C$ is a convex cone and in veiw of (c), (d) and (e) we get

$$
\begin{aligned}
0 &= \left\langle \sum_{i=1}^{N} T_i(z_t), \eta(g(z_t), g(z_t)) \right\rangle + \phi\left(f(z_t), f(z_t)\right) - \mu\left(f(z_t), f(z_t)\right) \\
&= \left\langle \sum_{i=1}^{N} T_i(z_t), \eta(g((1-t)x_0 + tz), g(z_t)) \right\rangle + \phi\left(f((1-t)x_0 + tz), f(z_t)\right) \\
&\quad - \mu\left(f(z_t), f((1-t)x_0 + tz)\right) \\
&= \left\langle \sum_{i=1}^{N} T_i(z_t), \eta((1-t)g(x_0) + tg(z), g(z_t)) \right\rangle + \phi\left((1-t)f(x_0) + tf(z), f(z_t)\right) \\
&\quad - \mu\left(f(z_t), (1-t)f(x_0) + tf(z)\right) \\
&= \left\langle \sum_{i=1}^{N} T_i(z_t), (1-t)\eta(g(x_0)g(z_t)) + t\eta(g(z), g(z_t)) \right\rangle + \phi\left((1-t)f(x_0) + tf(z), f(z_t)\right) \\
&\quad - \mu\left(f(z_t), (1-t)f(x_0) + tf(z)\right) \\
&\leq_C \left\langle \sum_{i=1}^{N} T_i(z_t), (1-t)\eta(g(x_0)g(z_t)) + t\eta(g(z), g(z_t)) \right\rangle + (1-t)\phi\left(f(x_0), f(z_t)\right) \\
&\quad + t\phi(f(z), f(z_t)) - \left[(1-t)\mu\left(f(z_t), f(x_0)\right) + t\mu(f(z_t), f(z))\right] \\
&= \left\langle \sum_{i=1}^{N} T_i(z_t), (1-t)\eta(g(x_0)g(z_t)) \right\rangle + \left\langle \sum_{i=1}^{N} T_i(z_t), t\eta(g(z), g(z_t)) \right\rangle \\
&\quad + (1-t)\phi\left(f(x_0), f(z_t)\right) + t\phi(f(z), f(z_t)) - (1-t)\mu\left(f(z_t), f(x_0)\right) - t\mu(f(z_t), f(z)) \\
&= (1-t)\left\langle \sum_{i=1}^{N} T_i(z_t), \eta(g(x_0)g(z_t)) \right\rangle + t\left\langle \sum_{i=1}^{N} T_i(z_t), \eta(g(z), g(z_t)) \right\rangle \\
&\quad + (1-t)\phi\left(f(x_0), f(z_t)\right) + t\phi(f(z), f(z_t)) - (1-t)\mu\left(f(z_t), f(x_0)\right) - t\mu(f(z_t), f(z)) \\
&= t\left\{ \left\langle \sum_{i=1}^{N} T_i(z_t), \eta(g(z), g(z_t)) + \phi(f(z), f(z_t)) - t\mu(f(z_t), f(z)) \right\rangle \right\} \\
&\quad + (1-t)\left\{ \left\langle \sum_{i=1}^{N} T_i(z_t), \eta(g(x_0)g(z_t)) + \phi\left(f(x_0), f(z_t)\right) - \mu\left(f(z_t), f(x_0)\right) \right\rangle \right\} \\
&\in t\left\{ \left\langle \sum_{i=1}^{N} T_i(z_t), \eta(g(z), g(z_t)) + \phi(f(z), f(z_t)) - t\mu(f(z_t), f(z)) \right\rangle \right\} \\
&\quad + (1-t)\left\{ \left\langle \sum_{i=1}^{N} T_i(z_t), \eta(g(x_0)g(z_t)) + \phi\left(f(x_0), f(z_t)\right) - \mu\left(f(z_t), f(x_0)\right) \right\rangle \right\} - C,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
& t\left\{ \left\langle \sum_{i=1}^{N} T_i(z_t), \eta(g(z), g(z_t)) + \phi(f(z), f(z_t)) - t\mu(f(z_t), f(z)) \right\rangle \right\} \\
& + (1-t)\left\{ \left\langle \sum_{i=1}^{N} T_i(z_t), \eta(g(x_0)g(z_t)) + \phi\left(f(x_0), f(z_t)\right) - \mu\left(f(z_t), f(x_0)\right) \right\rangle \right\} \\
& \in C.
\end{aligned}
$$

Which is a contradiction. Hence

$$\left\langle \sum_{i=1}^{N} T_i(z_t), \eta(g(z), g(z_t)) \right\rangle + \phi\left(f(z), f(z_t)\right) - \mu\left(f(z_t), f(z)\right) \geq_C 0.$$

Condition $(b)$ implies that

$$\left\langle \sum_{i=1}^{N} T_i(x_0), \eta(g(z), g(x_0)) \right\rangle + \phi\left(f(z), f(x_0)\right) - \mu\left(f(x_0), f(z)\right) \geq_C 0, \forall x \in K.$$

This completes the proof. □

**Theorem 2.2.** *Let $X$ and $Y$ be two real Banach spaces and $K \subset X$ a nonempty, compact and convex set. Let $T_1, T_2, \cdots, T_N : K \to L(X,Y), \eta : K \times K \to X, \phi, \mu : K \times K \to Y$ and $g, f : K \to K$ are the mappings satisfying the following conditions:*

(i) *for any fixed $y \in X$, the mapping $x \mapsto \left\langle \sum_{i=1}^{N} T_i(x), \eta(g(y), g(x)) \right\rangle$, $\phi(f(y), f(x))$ and $\mu(f(x), f(y))$ are continuous;*

(ii) *$T_1, T_2, \cdots, T_N$ are properly $g$-$f$-$\eta$-$\phi$-$\mu$-quasimonotone of Stampacchia type;*

(iii) *for all $x \in K, \eta(g(x), g(x)) = 0$ and $\phi(f(x), f(x)) = 0 = \mu(f(x), f(x))$.*

*Then there exists $x \in K$ such that*

$$\left\langle \sum_{i=1}^{N} T_i(x), \eta(g(y), g(x)) \right\rangle + \phi\left(f(y), f(x)\right) - \mu\left(f(x), f(y)\right) \geq_C 0, \quad \forall y \in K.$$

*Proof.* Define a multivalued mapping $M_1 : K \to 2^K$ by

$$M_1(z) = \left\{ \left\langle \sum_{i=1}^{N} T_i(x), \eta(g(z), g(x)) \right\rangle + \phi\left(f(z), f(x)\right) - \mu\left(f(x), f(z)\right) \geq_C 0 \right\}, \quad \forall z \in K,$$

then $M_1(z)$ is nonempty for each $z \in K$. We claim that $M_1$ is a KKM mapping. In fact if it is not the case then there exists $\{x_1, x_2, ..., x_n\} \subset K$, $x = \sum_{i=1}^{n} t_i x_i$ with $t_i > 0, i = 1, 2, ..., n$ and $\sum_{i=1}^{n} t_i = 1$ such that $x \notin \bigcup_{i=1}^{m} M_1(x_i)$.

This implies that

$$\left\langle \sum_{i=1}^{N} T_i(x), \eta(g(x_i), g(x)) \right\rangle + \phi\left(f(x_i), f(x)\right) - \mu\left(f(x), f(x_i)\right) \geq_C 0.$$

This contradicts condition (ii). Therefore $M_1$ is a KKM mapping; Now we prove that for any $z \in K, M_1(z)$ is closed.

In veiw of (i), let there exists a net $\{x_n\} \subset M_1(z)$ such that $x_n \to x \in K$. Because

$$\left\langle \sum_{i=1}^{N} T_i(x_n), \eta(g(z), g(x_n)) \right\rangle + \phi\left(f(z), f(x_n)\right) - \mu\left(f(x_n), f(z)\right) \geq_C 0, \quad \forall n,$$

we have

$$\left\langle \sum_{i=1}^{N} T_i(x), \eta(g(z), g(x)) \right\rangle + \phi\left(f(z), f(x)\right) - \mu\left(f(x), f(z)\right) \geq_C 0.$$

Hence $x \in M_1(z)$ and so $M_1(z)$ is closed. It follows from the compactness of K and closedness of $M_1(z) \subset K$, that $M_1(z)$ is compact. Thus by Lemma 1.14, we have $\bigcap_{z\in K} M_1(z) \neq \emptyset$. Hence there exist $x \in K$ such that

$$\left\langle \sum_{i=1}^{N} T_i(x), \eta(g(y), g(x)) \right\rangle + \phi\left(f(y), f(x)\right) - \mu\left(f(x), f(y)\right) \geq_C 0 \quad \forall y \in K.$$

This completes the proof. □

**Theorem 2.3.** *Let $K$ be a nonempty, bounded, closed and convex subset a real reflexive Banach space $X$ and $Y$ a real Banach space. Let $T_1, T_2, \cdots, T_N : K \to L(X,Y), \eta : K \times K \to X, \phi, \mu : K \times K \to Y$ and $g, f : K \to K$ are the mappings satisfying the following conditions:*

(i) *for any fixed $y \in X$, the mapping $\left\langle \sum_{i=1}^{N} T_i(y), \eta(g(\cdot), g(y)) \right\rangle$, $\phi(f(\cdot), f(y))$ and $\mu(f(y), f(\cdot))$ are lower semicontinuous;*

(ii) *$T_1, T_2, \cdots, T_N$ are properly g-f-$\eta$-$\phi$-$\mu$-quasimonotone of Minty type;*

(iii) *forall $x \in K, \eta(g(x), g(x)) = 0$ and $\phi(f(x), f(x)) = 0 = \mu(f(x), f(x))$.*

*Then there exists $x \in K$ such that*

$$\left\langle \sum_{i=1}^{N} T_i(y), \eta(g(x), g(y)) \right\rangle + \phi\left(f(x), f(y)\right) - \mu\left(f(y), f(x)\right) \leq_C 0, \quad \forall y \in K.$$

*Proof.* Define a multivalued mapping $M_2 : K \to 2^K$ by

$$M_2(z) = \left\{ x \in K : \left\langle \sum_{i=1}^{N} T_i(y), \eta(g(x), g(y)) \right\rangle + \phi\left(f(x), f(y)\right) - \mu\left(f(y), f(x)\right) \leq_C 0 \right\}, \quad \forall z \in K.$$

then $M_2(z)$ is nonempty for each $z \in K$. We claim that $M_2$ is not KKM mapping, then there exists $\{x_1, x_2, ..., x_n\} \subset K$, $x = \sum_{i=1}^{n} t_i x_i$ with $t_i > 0, i = 1, 2, ..., n$ and $\sum_{i=1}^{n} t_i = 1$ such that $x \notin \bigcup_{i=1}^{m} M_2(x_i)$.

This implies that

$$\left\langle \sum_{i=1}^{N} T_i(x_i), \eta(g(x), g(x_i)) \right\rangle + \phi\left(f(x), f(x_i)\right) - \mu\left(f(x_i), f(x)\right) \not\leq_C 0, \quad i = 1, 2, ..., n.$$

This contradicts condition (ii). Therefore $M_2$ is a KKM mapping. Since $K$ is bounded, $M_2(z)$ is bounded. From (ii), we have $M_2(z)$ is convex. Next, we will show that $M_2(z)$ closed.

In veiw of (i), let there exists a net $\{x_n\} \subset M_2(z)$ such that $x_n \to x \in K$. Because

$$\left\langle \sum_{i=1}^{N} T_i(y), \eta(g(x_n), g(y)) \right\rangle + \phi\left(f(x_n), f(y)\right) - \mu\left(f(y), f(x_n)\right) \leq_C 0, \quad \forall n,$$

we have

$$\left\langle \sum_{i=1}^{N} T_i(y), \eta(g(x), g(y)) \right\rangle + \phi\left(f(x), f(y)\right) - \mu\left(f(y), f(x)\right) \leq_C 0.$$

Hence $x \in M_2(z)$ and so $M_2(z)$ is closed.

Since $X$ is reflexive, $M_2(z)$ is weakly compact for all $z \in K$. It follows from Lemma 1.14, that $\bigcap_{z\in K} M_2(z) \neq \emptyset$. Hence there exist $x \in K$ such that

$$\left\langle \sum_{i=1}^{N} T_i(y), \eta(g(x), g(y)) \right\rangle + \phi\left(f(x), f(y)\right) - \mu\left(f(y), f(x)\right) \leq_C 0, \quad \forall y \in K.$$

This completes the proof. □

It is useful to mention that the result of Theorem 2.2 can be viewed as an improvement of the following corollary.

**Corollary 2.4** ([9, Theorem 2.1]). *Let $X$ and $Y$ be two real Banach spaces and $K \subset X$ a nonempty, compact and convex set. Let $S, T : K \to L(X, Y), \eta : K \times K \to X, h : K \times K \to Y$ and $g : K \to K$ are the mappings satisfying the following conditions:*

*(a) for any fixed $y \in X$, the mapping $x \mapsto \langle S(y) + T(y), \eta(g(x), g(y))\rangle$ and $h(g(x), g(y))$ are continuous;*

*(b) $S$ and $T$ are properly $g$-$h$-$\eta$-quasimonotone of Stampacchia type;*

*(c) for all $x \in K, \eta(g(x), g(x)) = 0 = h(g(x), g(x))$.*

*Then there exists $x \in K$ such that*

$$\langle S(x) + T(x), \eta(g(y), g(x))\rangle + h(g(y), g(x)) \geq_C 0, \quad \forall y \in K.$$

*Proof.* By taking $T_3, T_4, \cdots, T_N \equiv 0$, $T_1 = S, T_2 = T$, $\phi = h$, $\mu \equiv 0$ and $f = g$ in Theorem 2.2, we obtain the desired results. □

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## References

[1] R. Ahmad, Existence results for vector variational-like inequalities, *Thai J. Math*, 9(3), 553-561(2011).

[2] R. Ahmad, S.S. Irfan, On generalized nonlinear variational-like inequality problem, *Appl. Math. Lett*, 19, 294-297(2006).

[3] Q.H. Ansari, J.C. Yao, Iterative schemes for solving mixed variational-like inequalities, *J. Optim. Theory Appl*, 108, 527-541(2001).

[4] Q.H. Ansari, J.C. Yao, On nondifferentiable and nonconvex vector optimization problems, *J. Optim. Theory Appl*, 106 (3), 487-500(2000).

[5] A. Deniilidis, N. Hadjisavvas, Characterization of nonsmooth semistrictly quasiconvex and strictly quasiconvex functions, *J. Optm. Theory Appl*, 102, 525-536(1999).

[6] K. Fan, Some properties of convex sets related to fixed point theorems, *Math. Anal*, 266, 519-547(1984).

[7] A.P. Farajzadeh, B.S. Lee, Vector variational-like inequality problem and vector optimization problem, *Appl. Math. Lett*, 23, 48-52(2010).

[8] F. Giannessi, Theorems of alternative quadratic programs and complementarity problems, in: R. Cottle, F. Giannessi, J.L. Lions (Eds.), Variational Inequalities and Complementarity Problems, John Wiley and Sons, 1980.

[9] S.S. Irfan and R. Ahmad, Existence results for extended vector variational-like inequality. *Journal of the Egyptian Mathematical Society*, 23, 144-148(2015).

[10] S.S. Irfan, R. Ahmad, Generalized multivalued vector variational inequalities, *J. Glob. Optim*, 46 (1), 25-30(2010).

[11] S.K. Mishraa, S.Y. Wang, Vector variational-like inequalities and non-smooth vector optimization problems, *Nonlinear Anal*, 64, 1939-1945(2006).

[12] Y. Zhao, Z. Xia, Existence results for system of variational-like inequalities, *Nonlinear Anal. Real World Appl*, 8, 1370-1378(2007).

# Existence of solutions for a new semi-linear evolution equations with impulses *

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**Abstract**

By using monotone iterative technique and operator semigroup theorem, we consider the existence of mild solutions for a class of nonlocal semi-linear evolution equation with not instantaneous impulses in ordered Banach spaces. Finally, an example is given to show the existence results.

**Key Words:** evolution equation; not instantaneous impulses; operator semigroup; upper and lower solutions; monotone iterative technique; mild solutions
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# 1 Introduction

The impulsive differential equations are used to describe mathematical models of many real processes and phenomena studied by physical, chemical, biological, population dynamics, industrial robotics, economics, engineering and so on, see [1]. Applied impulsive mathematical models have become an active research subject in nonlinear science and have attracted more attention in many fields , see[2-4] and references therein.

For more details on differential equations with "abrupt and instantaneous" impulses, one can see for instance the monographs[5-7] and the references therein. By means of monotone iterative method coupled with lower and upper solutions, some sufficient

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conditions for the existence of solutions of impulsive integro-differential equations were established in [8]. Recently, the existence results to impulsive differential equations with nonlocal conditions was studied in [9-15]. Moreover, Chen, Li and Yang[16] used the perturbation method and monotone iterative technique in the presence of lower and upper solutions to discuss the existence of mild solutions for the nonlocal impulsive evolution equation in ordered Banach spaces.

However, it seems that the models with instantaneous impulses could not explain the certain dynamics of evolution processes in pharmacotherapy. For example, one considers the hemodynamic equilibrium of a person, the introduction of the drugs in the bloodstream and the consequent absorption for the body are gradual and continuous process. Hernandez and O'Regan[17] and Pierri et al.[18] initially studied on Cauchy problems for a new type first order evolution equations with not instantaneous impulses of the form:

$$\begin{cases} u'(t) = Au(t) + f(t, u(t)), \quad t \in (s_i, t_{i+1}], i = 0, 1, 2, \cdots, m, \\ u(t) = h_i(t, u(t)), \quad t \in (t_i, s_i], \quad i = 1, 2, \cdots, m. \\ u(0) = u_0. \end{cases}$$

Wang and Li[19], Yu et al.[20] considered periodic boundary value problems for nonlinear evolution equations with non instantaneous impulses. Wang et al.[21] discussed a class of new fractional differential equations with not instantaneous impulses.

However, to the best of our knowledge, the existence mild solutions for nonlocal evolution equations with not instantaneous impulses by means of monotone iterative technique has not been investigated yet. Motivated by this consideration, in this paper, we discuss the existence of mild solutions for the nonlocal evolution equation with not instantaneous impulses in an ordered Banach space $X$

$$\begin{cases} u'(t) + Au(t) = f(t, u(t)), \quad t \in (s_i, t_{i+1}], i = 0, 1, 2, \cdots, m, \\ u(t) = h_i(t, u(t)), \quad t \in (t_i, s_i], \quad i = 1, 2, \cdots, m, \\ u(0) = g(u), \end{cases} \tag{1.1}$$

where $A : D(A) \subset X \to X$ is a closed linear operator and $-A$ generates a $C_0$-semigroup $T(t)(t \geq 0)$ in $X$; $0 = s_0 < t_1 \leq s_1 < t_2 \leq s_2 < t_3 \leq s_3 < \cdots < t_{m-1} \leq s_{m-1} < t_m \leq s_m < t_{m+1} = a$ are pre-fixed numbers, $J = [0, a]$, $a > 0$ is a constant; $f \in C([0, a], X)$. $h_i \in C([t_i, s_i] \times X, X)$ for all $i = 1, 2, \cdots, m$.

## 2 Preliminaries

Throughout this paper, Let $X$ be a Banach space, $A : D(A) \subset X \to X$ be a closed linear operator and $-A$ generate a $C_0$-semigroup $T(t)(t \geq 0)$ in $X$. Denote

$$M \equiv \sup_{t \in J} \|T(t)\|,$$

which is a finite number. For more details of the properties of the operator semigroups and positive $C_0$-semigroup, we refer to the monographs[22, 23] and [24].

Let $X$ be an ordered Banach space with the norm $\|\cdot\|$ and partial order " $\leq$ ", whose positive cone $K = \{x \in X | x \geq \theta\}$ is normal with normal constant $N$. Let $C(J, X)$ with the norm $\|u\|_C = \max_{t \in J} \|u(t)\|$, then $C(J, X)$ is an ordered Banach space induced by the convex cone $K_C = \{u \in C(J, X) \mid u(t) \geq 0, t \in J\}$, and $K_C$ is also a normal cone.

Let $J' = J \setminus \{t_1, t_2, \cdots, t_m\}$, $J'' = J \setminus \{0, t_1, t_2, \cdots, t_m\}$. Evidently, $PC(J, X) = \{u : J \to X \mid u(t)$ is continuous in $J'$, and left continuous at $t_k$, and $u(t_k^+)$ exists, $k = 1, 2, \cdots, m\}$. $PC(J, X)$ is a Banach space with the norm $\|\cdot\|_{PC} = \sup_{t \in J} \|u(t)\|$. Evidently, $PC(J, X)$ is also an order Banach space with the partial order " $\leq$ " induced by the positive cone $K_{PC} = \{u \in PC(J, X) | u(t) \geq \theta, t \in J\}$. $K_{PC}$ is normal with the same normal constant $N$. For $v, w \in PC(J, X)$ with $v \leq w$, we use $[v, w]$ to denote the order interval $\{u \in PC(J, X) \mid v \leq u \leq w\}$ in $PC(J, X)$, and $[v(t), w(t)]$ to denote the order interval $\{u \in X \mid v(t) \leq u(t) \leq w(t), t \in J\}$ in $X$. We use $X_1$ to denote the Banach space $D(A)$ with the graph norm $\|\cdot\|_1 = \|\cdot\| + \|A\cdot\|$. For more details and definitions of the partial and cone, we refer to the monographs [25, 26].

**Definition** 2.1. If functions $v_0 \in PC(J, X) \cap C^1(J'', X) \cap C(J', X_1)$ satisfy

$$\begin{cases} v_0'(t) + Av_0(t) \leq f(t, v_0(t)), \quad t \in (s_i, t_{i+1}], i = 0, 1, 2, \cdots, m, \\ v_0(t) \leq h_i(t, v_0(t)), \quad t \in (t_i, s_i], \quad i = 1, 2, \cdots, m. \\ v_0(0) \leq g(v_0), \end{cases} \tag{2.1}$$

we call $v_0$ a lower solution of problem (1.1); if all the inequalities of (2.1) are inverse, we call it an upper solution of problem (1.1).

Next, we recall some properties of measure of noncompactness that will be used in the proof of our main results. Let $\alpha(\cdot)$ denotes the Kuratowski measure of noncompactness of the bounded set.For the details of the definition and properties of the measure of noncompactness, see [25]. The following lemmas are needed in our arguments.

**Lemma 2.3.**([27]) *Let $B \subset C(J, X)$ be bounded and equicontinuous. Then $\alpha(B(t))$ is continuous on $J$, and*

$$\alpha(B) = \max_{t \in J} \alpha(B(t)) = \alpha(B(J)).$$

**Lemma 2.4.** ([28]) *Let $B = \{u_n\} \subset C(J, X)(n = 1, 2, \cdots)$ be a bounded and countable set. Then $\alpha(B(t))$ is Lebesgue integral on $J$, and*

$$\alpha\Big(\Big\{\int_J u_n(t)dt \mid n \in \mathbb{N}\Big\}\Big) \leq 2\int_J \alpha(B(t))dt. \tag{2.2}$$

# 3 Linear nonlocal problem

Let $I = [t_0, t]$, $t_0 \geq 0$. It is well-known ([22])that for any $x_0 \in D(A)$ and $h \in C^1(I, X)$, the initial value problem of linear evolution equation

$$\begin{cases} u'(t) + Au(t) = h(t), \quad t \in I, \\ u(t_0) = x_0, \end{cases} \tag{3.1}$$

has a unique classical solution $u \in C^1(I, X) \cap C(I, X_1)$ expressed by

$$u(t) = T(t - t_0)x_0 + \int_{t_0}^{t} T(t - s)h(s)ds, t \in I \tag{3.2}$$

If $x_0 \in X$ and $h \in C(I, X)$, the function $u$ given by (3.2) belongs to $C(I, X)$, which is known as a mild solution of IVP(3.1).

To prove our main results, for any $h \in PC(J, X)$ and $y_i \in PC(J, X), i = 1, 2, \cdots, m$, we consider the linear nonlocal evolution equation with not instantaneous impulses in $X$

$$\begin{cases} u'(t) + Au(t) = h(t), \quad t \in (s_i, t_{i+1}], i = 0, 1, 2, \cdots, m, \\ u(t) = y_i(t), \quad t \in (t_i, s_i], \quad i = 1, 2, \cdots, m. \\ u(0) = g(u). \end{cases} \tag{3.3}$$

**Theorem 3.1.** *Let $X$ be a Banach space, $A : D(A) \subset X \to X$ be a closed linear operator and $-A$ generate a $C_0$-semigroup $T(t)(t \geq 0)$ in $X$. For any $h \in PC(J, X)$, $y_i \in PC(J, X), i = 1, 2, \cdots, m$, $g : PC(J, X) \to X$, problem(3.3) has a unique mild solution $u \in PC(J, X)$ given by*

$$\begin{cases} u(t) = T(t)g(u) + \int_0^t T(t - \tau)h(\tau)d\tau, \quad t \in [0, t_1]; \\ u(t) = y_i(t), \quad t \in (t_i, s_i], \quad i = 1, 2, \cdots, m; \\ u(t) = T(t - s_i)y_i(s_i) + \int_{s_i}^t T(t - \tau)h(\tau)d\tau, \quad t \in (s_i, t_{i+1}], \quad i = 1, 2, \cdots, m. \end{cases} \tag{3.4}$$

**Proof** Let $t \in [0, t_1]$, problem(3.3) is equivalent to the linear nonlocal evolution equation without impulse

$$\begin{cases} u'(t) + Au(t) = h(t), \quad t \in [0, t_1], \\ u(0) = g(u). \end{cases} \tag{3.5}$$

Then (3.5) has a unique mild solution $u \in C([0, t_1], X)$ given by

$$u(t) = T(t)g(u) + \int_0^t T(t-\tau)h(\tau)d\tau.$$

Let $t \in (t_i, s_i]$, then $u(t) = y_i(t), \quad i = 1, 2, \cdots, m.$

Let $t \in (s_i, t_{i+1}]$, problem(3.3) is equivalent to the initial value problem of linear evolution equation

$$\begin{cases} u'(t) + Au(t) = h(t), \quad t \in (s_i, t_{i+1}], , i = 1, 2, \cdots, m, \\ u(s_i) = y_i(s_i). \end{cases} \tag{3.6}$$

Then (3.6) has a unique mild solution $u \in C([s_i, t_{i+1}], X)$ given by

$$u(t) = T(t - s_i)y_i(s_i) + \int_{s_i}^t T(t-\tau)h(\tau)d\tau.$$

Inversely, we can verify directly that the function $u \in PC(J, X)$ defined by(3.4) is a mild solution of problem(3.3). Hence problem(3.3) has a unique mild solution $u \in PC(J, X)$ given by (3.4). This completes the proof.

**Remark 3.2.** *In Theorem 3.1, let $X$ be an ordered Banach space,$-A$ generate a positive $C_0$-semigroup $T(t)(t \geq 0)$ in $X$. For any $h \geq \theta$,$g \geq \theta$ and $y_i \geq \theta, i = 1, 2, \cdots, m$, then the mild solution of problem(3.3) is a positive solution.*

# 4 The main results

Now, we are in a position to state and prove our main results of this section.

**Theorem 4.1.** *Let $X$ be an ordered Banach space, whose positive cone $K$ is normal, and $N_0$ be the normal constant. Let $A : D(A) \subset X \to X$ be a closed linear operator and $-A$ generate a compact and positive $C_0$-semigroup $T(t)(t \geq 0)$ in $X$. $f \in C(J \times X, X)$. Assume that problem(1.1) has lower and upper solutions $v_0$ and $w_0$ with $v_0(t) \leq w_0(t)(t \in J)$. Suppose that the following conditions are satisfied:*

*(H1) There exists a constant $C \geq 0$ such that*

$$f(t, x_2) - f(t, x_1) \geq -C(x_2 - x_1), \quad t \in J,$$

*for any* $t \in J$, *and* $v_0(t) \le x_1 \le x_2 \le w_0(t)$.

*(H2) The impulsive functions* $h_i (i = 1, 2, \cdots, m)$ *are satisfy the conditions*

$$h_i(t, x_2) \ge h_i(t, x_1),\ i = 1, 2, \cdots, m,$$

*for* $\forall t \in J, v_0(t) \le x_1 \le x_2 \le w_0(t)$.

*(H3) The nonlocal function* $g(u)$ *is increasing in* $u$ *for* $u \in [v_0, w_0]$.

*(H4)* $h_i \in C(J \times X, X)(i = 1, 2, \cdots, m)$ *are compact operators.*

*(H5)* $g : PC(J, X) \to X$ *is compact operator.*

*Then the problem (1.1) has minimal and maximal mild solutions* $\underline{u}$ *and* $\overline{u}$ *between* $v_0$ *and* $w_0$, *which can be obtained by monotone iterative sequences starting from* $v_0$ *and* $w_0$.

**Proof** It is easy to see that $-(A + CI)$ generates a positive compact semigroup $S(t) = e^{-Ct}T(t)$. Define $D = [v_0, w_0]$. Let $\overline{M} = \sup_{t \in J} \|S(t)\|$, we define an operator $Q : D \to PC(J, X)$ by

$$(Qu)(t) = \begin{cases} S(t)g(u) + \int_0^t S(t-\tau)(f(\tau, u(\tau)) + Cu(\tau))d\tau, \quad t \in [0, t_1]; \\ h_i(t, u(t)), \quad t \in (t_i, s_i], \quad i = 1, 2, \cdots, m; \\ S(t - s_i)h_i(s_i, u(s_i)) + \int_{s_i}^t S(t-\tau)(f(\tau, u(\tau)) + Cu(\tau))d\tau, \\ t \in (s_i, t_{i+1}], \quad i = 1, 2, \cdots, m. \end{cases} \tag{4.1}$$

Since $f$, $h_i$ and $g$ are continuous, so $Q : D \to PC(J, X)$ is continuous. Clearly, from Theorem 3.1, the mild solutions of problem (1.1) are equivalent to the fixed point of operator $Q$.

(i) We show $Q : D \to PC(J, X)$ is an increasing operator.

For $\forall x_1, x_2 \in D$ and $x_1 \le x_2$, from the assumptions (H1) and (H2), we have

$$f(t, x_1(t)) + Cx_1(t) \le f(t, x_2(t)) + Cx_2(t), t \in J. \tag{4.2}$$

and

$$h_i(t, x_1(t)) \le h_i(t, x_2(t)), i = 1, 2, \cdots, m. \tag{4.3}$$

Combining the positive of $C_0$-semigroup $S(t)$ with (4.2), (4.2) and (H3), we have

$$\begin{aligned} & S(t)g(x_1) + \int_0^t S(t-\tau)(f(\tau, x_1(\tau)) + Cx_1(\tau))d\tau \\ \le\ & S(t)g(x_2) + \int_0^t S(t-\tau)(f(\tau, x_2(\tau)) + Cx_2(\tau))d\tau, \quad t \in [0, t_1]; \\ & h_i(t, x_1(t)) \le h_i(t, x_2(t)), t \in (t_i, s_i], \quad i = 1, 2, \cdots, m; \end{aligned}$$

$$
\begin{aligned}
& S(t-s_i)h_i(s_i,x_1(s_i))+\int_{s_i}^{t}S(t-\tau)(f(\tau,x_1(\tau))+Cx_1(\tau))d\tau \\
\leq\ & S(t-s_i)h_i(s_i,x_2(s_i))+\int_{s_i}^{t}S(t-\tau)(f(\tau,x_2(\tau))+Cx_2(\tau))d\tau, \\
& t\in(s_i,t_{i+1}],\quad i=1,2,\cdots,m.
\end{aligned}
$$

Namely, $Q:D\to PC(J,X)$ is an increasing operator.

(ii) We show $v_0\leq Q(v_0)$, $Q(w_0)\leq w_0$.

Let

$$
\begin{cases}
v_0'(t)+Av_0(t)+Cv_0(t)=\bar{f}(t) & t\in(s_i,t_{i+1}],i=0,1,2,\cdots,m,\\
v_0(t)=\bar{h}_i(t), & t\in(t_i,s_i],\quad i=1,2,\cdots,m.\\
v_0(0)=\bar{g}(v_0), &
\end{cases}
\tag{4.4}
$$

by the definition of $v_0$, we have

$$
\begin{cases}
\bar{f}(t)\leq f(t,v_0(t))+Cv_0(t), & t\in(s_i,t_{i+1}],i=0,1,2,\cdots,m,\\
\bar{h}_i(t)\leq h_i(t,v_0(t)), & t\in(t_i,s_i],\quad i=1,2,\cdots,m.\\
\bar{g}(v_0)\leq g(v_0), &
\end{cases}
\tag{4.5}
$$

By Theorem 3.1, (4.5)and (4.6), we have

$$
v_0(t)=\begin{cases}
S(t)\bar{g}(v_0)+\int_0^t S(t-\tau)\bar{f}(\tau)d\tau, & t\in[0,t_1];\\
\bar{h}_i(t), & t\in(t_i,s_i],\quad i=1,2,\cdots,m;\\
S(t-s_i)\bar{h}_i(s_i)+\int_{s_i}^t S(t-\tau)\bar{f}(\tau)d\tau, & \\
t\in(s_i,t_{i+1}], & i=1,2,\cdots,m.
\end{cases}
\tag{4.6}
$$

and

$$
\begin{aligned}
& S(t)\bar{g}(v_0)+\int_0^t S(t-\tau)\bar{f}(\tau)d\tau \\
\leq\ & S(t)g(v_0)+\int_0^t S(t-\tau)(f(\tau,v_0(\tau))+Cv_0(\tau))d\tau, \quad t\in[0,t_1];\\
& \bar{h}_i(t)\leq h_i(t,v_0(t)), \quad t\in(t_i,s_i],\quad i=1,2,\cdots,m;\\
& S(t-s_i)\bar{h}_i(s_i)+\int_{s_i}^t S(t-\tau)\bar{f}(\tau)d\tau \\
\leq\ & S(t-s_i)h_i(s_i,v_0(s_i))+\int_{s_i}^t S(t-\tau)(f(\tau,v_0(\tau))+Cv_0(\tau))d\tau,\\
& t\in(s_i,t_{i+1}],\quad i=1,2,\cdots,m.
\end{aligned}
$$

Namely, $v_0(t) \leq Q(v_0)(t)$. Similarly, it can be shown that $Q(w_0)(t) \leq w_0(t)$. Therefore, $Q : [v_0, w_0] \to [v_0, w_0]$ is a continuously increasing operator.

(iii) We prove that the operator $Q$ has fixed points on $[v_0, w_0]$.

Now, we define two sequences $\{v_n\}$ and $\{w_n\}$ by the iterative scheme

$$v_n = Q(v_{n-1}), \quad w_n = Q(w_{n-1}), \quad n = 1, 2, \cdots \tag{4.7}$$

Then from the monotonicity of operator $Q$ it follows that

$$v_0 \leq v_1 \leq v_2 \leq \cdots \leq v_n \leq \cdots \leq w_n \leq \cdots \leq w_2 \leq w_1 \leq w_0. \tag{4.8}$$

Next, we prove that $\{v_n\}$ and $\{w_n\}$ are convergent in $J$. Let $B = \{v_n \mid n \in \mathbb{N}\}$, $B_0 = \{v_{n-1} \mid n \in \mathbb{N}\}$, then $B_0 = \{v_0\} \cup B$ and $B = Q(B_0)$.

For any $v_{n-1} \in B_0$, let

$$\begin{aligned}
(Q_1 v_{n-1})(t) = &\quad S(t)g(u) + \int_0^t S(t-\tau)(f(\tau, v_{n-1}(\tau)) + Cv_{n-1}(\tau))d\tau, \quad t \in [0, t_1];\\
(Q_2 v_{n-1})(t) = &\quad h_i(t, v_{n-1}(t)), \quad t \in (t_i, s_i], \quad i = 1, 2, \cdots, m;\\
(Q_3 v_{n-1})(t) = &\quad S(t-s_i)h_i(s_i, v_{n-1}(s_i)) + \int_{s_i}^t S(t-\tau)(f(\tau, v_{n-1}(\tau)) + Cv_{n-1}(\tau))d\tau,\\
&\quad t \in (s_i, t_{i+1}], \quad i = 1, 2, \cdots, m.
\end{aligned}$$

For $0 < t \leq a$, by the assumption (H1), we know that

$$f(t, v_0(t)) + Cv_0(t) \leq f(t, v_{n-1}(t)) + Cv_{n-1}(t) \leq f(t, w_0(t)) + Cw_0(t).$$

Since $f(t, v_0(t))$ and $f(t, w_0(t))$ are continuous in compact set $[0, a]$, so their image sets are compact sets in $X$, namely image sets are bounded. Combining this fact with the normality of cone $K$ in $X$, we have $\exists M_1 > 0$, $\forall v_{n-1} \in B_0$,

$$\begin{aligned}
&\|f(t, v_{n-1}(t)) + Cv_{n-1}(t)\|\\
\leq &\quad \|f(t, v_0(t)) + Cv_0(t)\| + N_0\|f(t, w_0(t)) + Cw_0(t) - f(t, v_0(t)) - Cv_0(t)\| \qquad (4.9)\\
\leq &\quad M_1.
\end{aligned}$$

Case 1. For interval $[0, t_1]$ and any $0 < \epsilon < t_1$, let

$$(W_1 v_{n-1})(t) := \int_0^t S(t-\tau)(f(\tau, v_{n-1}(\tau)) + Cv_{n-1}(\tau))d\tau$$

and

$$(W_1^\epsilon v_{n-1})(t) := \int_0^{t-\epsilon} S(t-\tau)(f(\tau, v_{n-1}(\tau)) + Cv_{n-1}(\tau))d\tau,$$

then

$$\begin{aligned}
& \|(W_1 v_{n-1})(t)) - (W_1^{\epsilon} v_{n-1})(t)\| \\
= \; & \| \int_0^t S(t-\tau)(f(\tau, v_{n-1}(\tau)) + C v_{n-1}(\tau)) d\tau \\
& - \int_0^{t-\epsilon} S(t-s)\tau)(f(\tau, v_{n-1}(\tau)) + C v_{n-1}(\tau)) d\tau \| \\
\leq \; & \int_{t-\epsilon}^t \|S(t-\tau)\| \|f(\tau, v_{n-1}(\tau)) + C v_{n-1}(\tau)\| d\tau \\
\leq \; & \overline{M} M_1 \epsilon.
\end{aligned}$$

Therefore, $Y_1(t) \triangleq \{(W_1 v_{n-1})(t) \mid v_{n-1} \in B_0\}$ is precompact in $X$ by using the total boundedness.

On the other hand, by the assumption (H5), $\{S(t)g(v_{n-1}) \mid v_{n-1} \in B_0\}$ is precompact in $X$ due to the compactness of $S(t)$. Therefore, $\{(Q_1 v_{n-1})(t) \mid v_{n-1} \in B_0\}$ is precompact in $X$ for $t \in [0, t_1]$.

Case 2. For $t \in (t_i, s_i], i = 1, 2, \cdots, m$, the set $\{(Q_2 v_{n-1})(t) \mid v_{n-1} \in B_0\}$ is precompact in $X$ by the assumption (H4).

Case 3. For $t \in (s_i, t_{i+1}], i = 1, 2, \cdots, m$, similar to the case 1, $\{(Q_3 v_{n-1})(t) \mid v_{n-1} \in B_0\}$ is precompact in $X$ by (4.9)and the assumption (H4).

Hence, $\{v_n(t)\} = \{Q(v_{n-1})(t) \mid v_{n-1} \in B_0\}$ is precompact in $X$ for $t \in J$, combining this fact with the monotonicity of $\{v_n\}$, we easily prove that $\{v_n(t)\}$ is convergent. Let $\{v_n(t)\} \to \underline{u}(t)$ in $t \in J$. Similarly, we prove that $\{w_n(t)\} \to \overline{u}(t)$ in $t \in J$.

Evidently $\{v_n(t)\}, \{w_n(t)\} \in PC(J, X)$, so $\underline{u}(t)$ and $\overline{u}(t)$ is bounded integrable in $J$). Since for any $t \in J$, $v_n(t) = Q(v_{n-1})(t)$, $w_n(t) = Q(w_{n-1})(t)$, letting $n \to \infty$, by the Lebesgue dominated convergence theorem, we have $\underline{u}(t) = Q(\underline{u})(t)$, $\overline{u}(t) = Q(\overline{u})(t)$ and $\underline{u}(t), \overline{u}(t) \in PC(J, X)$. Combining this with monotonicity (4.8), we have $v_0(t) \leq \underline{u}(t) \leq \overline{u}(t) \leq w_0(t)$.

Next, we prove that $\underline{u}(t)$ and $\overline{u}(t)$ are the minimal and maximal fixed points of $Q$ in $[v_0, w_0]$, respectively. In fact, for any $u^* \in [v_0, w_0], Q(u^*) = u^*$, we have $v_0 \leq u^* \leq w_0$ and $v_1 = Q(v_0) \leq Q(u^*) = u^* \leq Q(w_0) = w_1$. Continuing such progress, we get $v_n \leq u^* \leq w_n$. Letting $n \to \infty$, we get $\underline{u}(t) \leq u^* \leq \overline{u}(t)$. Therefor, $\underline{u}(t)$ and $\overline{u}(t)$ are the minimal and maximal mild solutions of the problem (1.1) between $v_0$ and $w_0$, which can be obtained by monotone iterative sequences starting from $v_0$ and $w_0$, respectively.□

From the Theorem4.1, we obtain the following result.

**Theorem4.2.** *Let $X$ be an ordered Banach space, whose positive cone $K$ is normal, and $N_0$ be the normal constant. Let $A : D(A) \subset X \to X$ be a closed linear operator*

*and $-A$ generate a compact and positive $C_0$-semigroup $T(t)(t \geq 0)$ in $X$. $f \in C(J \times X, X)$. Assume that problem(1.1) has lower and upper solutions $v_0$ and $w_0$ with $v_0(t) \leq w_0(t)(t \in J)$. Suppose that conditions (H1), (H2), (H3) and the following conditions are satisfied:*

*(H6) $\{h_i(\cdot, x_n)\}(i = 1, 2, \cdots, m)$ are precompact in $X$, for any increasing or decreasing monotonic sequence $\{x_n\} \subset [v_0, w_0]$.*

*(H7) $\{g(x_n)\}$ is a precompact set in $X$, for any increasing or decreasing monotonic sequence $\{x_n\} \subset [v_0, w_0]$.*

*Then problem (1.1) has minimal and maximal mild solutions $\underline{u}$ and $\overline{u}$ between $v_0$ and $w_0$, which can be obtained by monotone iterative sequences starting from $v_0$ and $w_0$.*

Next, we discuss the existence of the mild solutions for problem (1.1) under the function $g$ is continuous in PC(J,X) and noncompactness measure conditions.

**Theorem 4.3.** *Let $X$ be an ordered Banach space, whose positive cone $K$ is normal, and $N_0$ be the normal constant. Let $A : D(A) \subset X \to X$ be a closed linear operator and $-A$ generate an equicontinuous and positive $C_0$-semigroup $T(t)(t \geq 0)$ in $X$. $f \in C(J \times X, X)$. $h_i \in C(J \times X, X)(i = 1, 2, \cdots, m)$. $g : PC(J, X) \to X$ be a continuous function. Assume that problem(1.1) has lower and upper solutions $v_0$ and $w_0$ with $v_0(t) \leq w_0(t)(t \in J)$. Suppose that conditions (H1), (H2) and (H3) hold, and satisfy:*

*(H8) There exist a constant $L > 0$ such that*

$$\alpha(\{f(t, x_n)\}) \leq L\alpha(\{x_n\}),$$

*for all $t \in J$, and increasing or decreasing sequence $\{x_n\} \subset [v_0(t), w_0(t)]$.*

*(H9) There exist constants $0 < L_i < 1(i = 1, 2, \cdots, m)$ such that*

$$\alpha(\{h_i(t, x_n)\}) \leq L_i\alpha(\{x_n\}), (i = 1, 2, \cdots, m),$$

*for all $t \in J$, and increasing or decreasing sequence $\{x_n\} \subset [v_0(t), w_0(t)]$.*

*(H10) There exist a constant $L^{'} > 0$ such that*

$$\alpha(\{g(x_n)\}) \leq L^{'}\alpha(\{x_n\}),$$

*for all $t \in J$, and increasing or decreasing sequence $\{x_n\} \subset [v_0(t), w_0(t)]$.*

*(H11) $\overline{M}[L_i + L^{'} + 2(L + C)a] < 1(i = 1, 2, \cdots, m)$.*

*Then the problem (1.1) has minimal and maximal mild solutions $\underline{u}$ and $\overline{u}$ between $v_0$ and $w_0$, which can be obtained by monotone iterative sequences starting from $v_0$ and $w_0$.*

**Proof** From Theorem 4.1, we know that $Q : [v_0, w_0] \to [v_0, w_0]$ is continuous. Furthermore, if conditions (H1), (H2) and (H3) are satisfied, the iterative sequences $\{v_n\}$ and

$\{w_n\}$ defined by (4.7) satisfying (4.8). Therefore, for any $t \in J$, $\{v_n(t)\}$ and $\{w_n(t)\}$ are monotone and order-bounded sequences in $X$.

Next, we prove that $\{v_n\}$ and $\{w_n\}$ are convergent in $J$. Since $T(t)(t \geq 0)$ is an equicontinuous $C_0$-semigroup, so $S(t)(t \geq 0)$ also is an equicontinuous $C_0$-semigroup.

Let $B = \{v_n \mid n \in \mathbb{N}\}$ and $B_0 = \{v_{n-1} \mid n \in \mathbb{N}\}$, by (4.8) and the normality of the positive cone $K$, then $B$ and $B_0$ are bounded.

(i) We prove that $Q(B_0)$ is equicontinuous in $PC(J, X)$.

Combining (H2) and (H3) with the normality of cone $K$ in $X$, we have $\exists M_2 > 0, M_3 > 0$, $\forall v_{n-1} \in B_0$,

$$\|g(v_{n-1})\| \leq \|g(v_0)\| + N_0\|g(w_0) - g(v_0)\| \leq M_2. \tag{4.10}$$

$$\|h_i(t, v_{n-1}(t))\| \leq \|h_i(t, v(t))\| + N_0\|h_i(t, w_0(t)) - g(h_i(t, v_0(t))\| \leq M_3. \tag{4.11}$$

Case 1. For $\forall t', t'' \in [0, t_1]$ and $t' < t''$, by (4.9) and (4.10) we have that

$$\begin{aligned}
& \|(Qv_{n-1})(t'') - (Qv_{n-1})(t')\| \\
= \; & \|S(t'')g(v_{n-1}) + \int_0^{t''} S(t'' - \tau)(f(\tau, v_{n-1}(\tau)) + Cv_{n-1}(\tau))d\tau \\
& -S(t')g(v_{n-1}) - \int_0^{t'} S(t' - \tau)(f(\tau, v_{n-1}(\tau)) + Cv_{n-1}(\tau))d\tau\| \\
\leq \; & \|S(t'') - S(t')\|\|g(v_{n-1})\| \\
& + \int_0^{t'} \|S(t'' - \tau) - S(t' - \tau)\|\|f(\tau, v_{n-1}(\tau)) + Cv_{n-1}(\tau)\|d\tau \\
& + \int_{t'}^{t''} \|S(t'' - \tau)\|\|f(\tau, v_{n-1}(\tau)) + Cv_{n-1}(\tau)\|d\tau \\
\leq \; & M_2\|S(t')\|\|S(t'' - t') - I\| + M_1 \int_0^{t'} \|S(t'' - \tau) - S(t' - \tau)\|d\tau + \overline{M}M_1(t'' - t') \\
\leq \; & M_2\overline{M}\|S(t'' - t') - I\| + M_1 \int_0^{t'} \|S(t'' - t' + \tau) - S(\tau)\|d\tau + \overline{M}M_1(t'' - t') \\
\to \; & 0(t'' - t' \to 0).
\end{aligned}$$

Case 2. For $\forall t', t'' \in (t_i, s_i](i = 1, 2, \cdots, m)$ and $t' < t''$, we have that

$$\|(Qv_{n-1})(t'') - (Qv_{n-1})(t')\| = \|h_i(t'', v_{n-1}(t'')) - h_i(t', v_{n-1}(t'))\| \to 0(t'' - t' \to 0).$$

Case 3. For $\forall t', t'' \in (s_i, t_{i+1}](i = 1, 2, \cdots, m)$ and $t' < t''$, by (4.9) and (4.11) we have that

$$\begin{aligned}
& \|(Qv_{n-1})(t'') - (Qv_{n-1})(t')\| \\
= \; & \|S(t'')h_i(s_i, v_{n-1}(s_i)) + \int_{s_i}^{t''} S(t''-\tau)(f(\tau, v_{n-1}(\tau)) + Cv_{n-1}(\tau))d\tau \\
& -S(t')h_i(s_i, v_{n-1}(s_i)) - \int_{s_i}^{t'} S(t'-\tau)(f(\tau, v_{n-1}(\tau)) + Cv_{n-1}(\tau))d\tau\| \\
\leq \; & \|S(t'') - S(t')\|\|h_i(s_i, v_{n-1}(s_i))\| \\
& + \int_{s_i}^{t'} \|S(t''-\tau) - S(t'-\tau)\|\|f(\tau, v_{n-1}(\tau)) + Cv_{n-1}(\tau)\|d\tau \\
& + \int_{t'}^{t''} \|S(t''-\tau)\|\|f(\tau, v_{n-1}(\tau)) + Cv_{n-1}(\tau)\|d\tau \\
\leq \; & M_3\|S(t')\|\|S(t''-t') - I\| + M_1\int_{s_i}^{t'} \|S(t''-\tau) - S(t'-\tau)\|d\tau + \overline{M}M_1(t''-t') \\
\leq \; & M_3\overline{M}\|S(t''-t') - I\| + M_1\int_0^{t'-s_i} \|S(t''-t'+\tau) - S(\tau)\|d\tau + \overline{M}M_1(t''-t') \\
\to \; & 0(t''-t' \to 0).
\end{aligned}$$

Therefore, $Q(B_0)$ is equicontinuous in $PC(J, X)$.

(ii) We prove that $\alpha(B(t)) = 0$ for $t \in J$.

It follows from $B_0 = \{v_0\} \cup B$ that $\alpha(B(t)) = \alpha(B_0(t))$ for $t \in J$.

Case 1. For $t \in [0, t_1]$, by Lemma 2.3 and Lemma 2.4, we have that

$$\begin{aligned}
& \alpha(B(t)) = \alpha((QB_0)(t)) \\
= \; & \alpha\Big(\Big\{S(t)g(v_{n-1}) + \int_0^t S(t-\tau)(f(\tau, v_{n-1}(\tau)) + Cv_{n-1}(\tau))d\tau \mid n \in \mathbb{N}\Big\}\Big) \\
\leq \; & \alpha(\{S(t)g(v_{n-1}) \mid n \in \mathbb{N}\}) \\
& + \alpha\Big(\Big\{\int_0^t S(t-\tau)(f(\tau, v_{n-1}(\tau)) + Cv_{n-1}(\tau))d\tau \mid n \in \mathbb{N}\Big\}\Big) \\
\leq \; & \overline{M}\alpha(\{g(v_{n-1}) \mid n \in \mathbb{N}\}) + 2\int_0^t \alpha(\{S(t-\tau)(f(\tau, v_{n-1}(\tau)) + Cv_{n-1}(\tau))d\tau \mid n \in \mathbb{N}\}) \\
\leq \; & \overline{M}L^{'}\alpha(B_0) + 2\overline{M}\int_0^t (L + C)\alpha(B_0(\tau))d\tau
\end{aligned}$$

$$\leq \overline{M}L^{'}\max_{t\in J}\alpha(B(t)) + 2\overline{M}(L+C)a\max_{t\in J}\alpha(B(t))$$

$$\leq \overline{M}[L^{'} + 2(L+C)a]\max_{t\in J}\alpha(B(t)).$$

Case 2. For $t \in (t_i, s_i], i = 1, 2, \cdots, m$, by (H9) we have

$$\alpha(B(t)) = \alpha((QB_0)(t)) = \alpha\Big(\Big\{h_i(t, v_{n-1}(t)) \mid n \in \mathbb{N}\Big\}\Big)$$

$$\leq L_i\alpha(B_0(t)) \leq L_i\max_{t\in J}\alpha(B(t)) < \max_{t\in J}\alpha(B(t)).$$

Case 3. For $t \in (s_i, t_{i+1}](i = 1, 2, \cdots, m)$, we have

$$\alpha(B(t)) = \alpha((QB_0)(t))$$

$$= \alpha\Big(\Big\{S(t-s_i)h_i(s_i, v_{n-1}(s_i)) + \int_{s_i}^{t} S(t-\tau)(f(\tau, v_{n-1}(\tau)) + Cv_{n-1}(\tau))d\tau \mid n \in \mathbb{N}\Big\}\Big)$$

$$\leq \alpha(\{S(t-s_i)h_i(s_i, v_{n-1}(s_i)) \mid n \in \mathbb{N}\})$$

$$+\alpha\Big(\Big\{\int_{s_i}^{t} S(t-\tau)(f(\tau, v_{n-1}(\tau)) + Cv_{n-1}(\tau))d\tau \mid n \in \mathbb{N}\Big\}\Big)$$

$$\leq \overline{M}\alpha(\{h_i(s_i, v_{n-1}(s_i)) \mid n \in \mathbb{N}\})$$

$$+2\int_{s_i}^{t}\alpha(\{S(t-\tau)(f(\tau, v_{n-1}(\tau)) + Cv_{n-1}(\tau))d\tau \mid n \in \mathbb{N}\})$$

$$\leq \overline{M}L_i\alpha(B_0) + 2\overline{M}\int_{s_i}^{t}(L+C)\alpha(B_0(\tau))d\tau$$

$$\leq \overline{M}L_i\max_{t\in J}\alpha(B(t)) + 2\overline{M}(L+C)a\max_{t\in J}\alpha(B(t))$$

$$\leq \overline{M}[L_i + 2(L+C)a]\max_{t\in J}\alpha(B(t)).$$

By (H11), we have $\alpha(B(t)) < \max_{t\in J}\alpha(B(t))$, then $\alpha(B(t)) = 0$ in $t \in J$. Therefore, $\{v_n(t)\}$ is precompact in $X$ for $t \in J$, combining this fact with the monotonicity of $\{v_n\}$, we easily prove that $\{v_n(t)\}$ is convergent. Let $\{v_n(t)\} \to \underline{u}(t)$ in $t \in J$. The same idea can be used to prove that $\{w_n(t)\} \to \overline{u}(t)$ in $t \in J$. Similar to the proof of Theorem 4.1, we know that $\underline{u}(t)$ and $\overline{u}(t)$ are the problem(1.1) between $v_0$ and $w_0$, which can be obtained by monotone iterative sequences starting from $v_0$ and $w_0$, respectively. This completes the proof of Theorem 4.3. □

**Remark 4.4.** Analytic semigroup and differentiable semigroup are equicontinuous semigroup ([22]). In the application of partial differential equations, such as parabolic

and strongly damped wave equations, the corresponding solution semigroup are analytic semigroup. Therefore, Theorem 4.3. has extensive applicability.

we discuss the existence of the mild solutions for problem (1.1) under the positive cone is regular.

**Theorem 4.5.** *Let $X$ be an ordered Banach space, whose positive cone $K$ is regular. Let $A : D(A) \subset X \to X$ be a closed linear operator and $-A$ generate a positive $C_0$-semigroup $T(t)(t \geq 0)$ in $X$. $f \in C(J \times X, X)$. $h_i \in C(J \times X, X)(i = 1, 2, \cdots, m)$. $g : PC(J, X) \to X$ be a continuous function. Assume that problem(1.1) has lower and upper solutions $v_0$ and $w_0$ with $v_0(t) \leq w_0(t)(t \in J)$. Suppose that conditions (H1), (H2) and (H3) are satisfied.*

*Then the problem (1.1) has minimal and maximal mild solutions $\underline{u}$ and $\overline{u}$ between $v_0$ and $w_0$, which can be obtained by monotone iterative sequences starting from $v_0$ and $w_0$.*

**Proof** From Theorem 4.1 we know that $Q : [v_0, w_0] \to [v_0, w_0]$ is a continuously increasing operator. Similarly, the two sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ are defined in $[v_0, w_0]$ by the iterative scheme (4.7). By conditions (H1), (H2) and (H3), then $\{v_n(t)\}$ and $\{w_n(t)\}$ are ordered-monotonic and ordered-bounded sequences in $X$.

Using the regularity of the cone $K$, any ordered-monotonic and ordered-bounded sequence in $X$ is convergent. Similar to the proof of Theorem 4.1, we know that $\underline{u}(t)$ and $\overline{u}(t)$ are the problem(1.1) between $v_0$ and $w_0$, which can be obtained by monotone iterative sequences starting from $v_0$ and $w_0$, respectively. This completes the proof of Theorem 4.5. □

**Corollary 4.6.** *Let $X$ be an ordered and weakly sequentially complete Banach space, whose positive cone $K$ is normal, and $N_0$ be the normal constant. Let $A : D(A) \subset X \to X$ be a closed linear operator and $-A$ generate a positive $C_0$-semigroup $T(t)(t \geq 0)$ in $X$. $f \in C(J \times X, X)$. $h_i \in C(J \times X, X)(i = 1, 2, \cdots, m)$. $g : PC(J, X) \to X$ be a continuous function. Assume that problem(1.1) has lower and upper solutions $v_0$ and $w_0$ with $v_0(t) \leq w_0(t)(t \in J)$. Suppose that conditions (H1), (H2) and (H3) are satisfied. Then the problem (1.1) has minimal and maximal mild solutions $\underline{u}$ and $\overline{u}$ between $v_0$ and $w_0$, which can be obtained by monotone iterative sequences starting from $v_0$ and $w_0$.*

**Proof** In an ordered and weakly sequentially complete Banach space, the normal cone $K$ is regular. Then the proof is complete. □

Next, we discuss the existence of mild solutions of problem (1.1), when we don't assume the lower and upper solutions of problem (1.1) be exist.

**Theorem 4.7.** *Let $X$ be an ordered Banach space, whose positive cone $K$ is normal, and $N_0$ be the normal constant. Let $A : D(A) \subset X \to X$ be a closed linear operator and*

*$-A$ generate a positive and compact $C_0$-semigroup $T(t)(t \geq 0)$ in $X$. $f \in C(J \times X, X)$. $h_i \in C(J \times X, X)(i = 1, 2, \cdots, m)$. $g : PC(J, X) \to X$ be a continuous function. Suppose that conditions (H1)–(H5) hold and the following condition is satisfied:*

*(H12) $\exists b \geq 0$, $h \in PC(J, X), h \geq \theta$, $y_i(s_i) \in D(A), y_i \geq \theta, i = 1, 2, \cdots, m$ and $g(u) \in D(A), g(u) \geq \theta$ for any $u \in PC(J, X)$, such that*

$$f(t, u) \leq bu + h(t), \quad h_i(t, u) \leq y_i(t), \quad u \geq 0;$$

$$bu - h(t) \leq f(t, u), \quad -y_i(t) \leq h_i(t, u), \quad u \leq 0.$$

*Then the problem (1.1) has minimal and maximal mild solutions , which can be obtained by monotone iterative procedure.*

**Proof** For $h(t) \geq \theta, y_i(t) \geq \theta$, we consider the linear nonlocal evolution equation with not instantaneous impulses in $X$

$$\begin{cases} u'(t) + Au(t) - bu(t) = h(t), \quad t \in (s_i, t_{i+1}], i = 0, 1, 2, \cdots, m, \\ u(t) = y_i(t), \quad t \in (t_i, s_i], \quad i = 1, 2, \cdots, m. \\ u(0) = g(u), \end{cases} \tag{4.12}$$

Since $-(A - bI)$ generate a positive $C_0$-semigroup $S(t) = e^{bt}T(t)(t \geq 0)$ in $X$. By Theorem 3.1 and assumption (H12), we know that the problem (4.12) has a unique positive solution $u^* \geq \theta$. Let $v_0 = -u^*, w_0 = u^*$, by the conditions (H1)–(H3) and (H12), we get

$$\begin{cases} v_0'(t) + Av_0(t) = bv_0(t) - h(t) \leq f(t, v_0(t)), \quad t \in (s_i, t_{i+1}], i = 0, 1, 2, \cdots, m, \\ v_0(t) = -y_i(t) \leq h_i(t, v_0(t)), \quad t \in (t_i, s_i], \quad i = 1, 2, \cdots, m. \\ v_0(0) = -g(-v_0) \leq g(v_0), \end{cases} \tag{4.13}$$

and

$$\begin{cases} w_0'(t) + Aw_0(t) = bw_0(t) + h(t) \geq f(t, w_0(t)), \quad t \in (s_i, t_{i+1}], i = 0, 1, 2, \cdots, m, \\ w_0(t) = y_i(t) \geq h_i(t, w_0(t)), \quad t \in (t_i, s_i], \quad i = 1, 2, \cdots, m. \\ w_0(0) \geq g(w_0), \end{cases} \tag{4.14}$$

So, we inferred that $v_0$ and $w_0$ are a lower solution and an upper solution of the problem (1.1), respectively. Therefore by Theorem 4.1., the conclusion holds. Then the proof is complete.□

Meanwhile, we can obtain the following results from Theorem 4.2, 4.3, 4.5 and Corollary 4.6, respectively.

**Corollary 4.8.** *Let $X$ be an ordered Banach space, whose positive cone $K$ is normal, and $N_0$ be the normal constant. Let $A : D(A) \subset X \to X$ be a closed linear operator and $-A$ generate a positive and compact $C_0$-semigroup $T(t)(t \geq 0)$ in $X$. $f \in C(J \times X, X)$. $g, h_i(i = 1, 2, \cdots, m)$ are continuous and map a monotonic set into a precompact set and conditions (H1)–(H3) and (H12) hold, then the problem (1.1) has minimal and maximal mild solutions , which can be obtained by monotone iterative procedure.*

**Corollary 4.9.** *Let $X$ be an ordered Banach space, whose positive cone $K$ is normal, and $N_0$ be the normal constant. Let $A : D(A) \subset X \to X$ be a closed linear operator and $-A$ generate a positive $C_0$-semigroup $T(t)(t \geq 0)$ in $X$. $f \in C(J \times X, X)$. $g, h_i(i = 1, 2, \cdots, m)$ are continuous and for any monotonic sequence $\{x_n\}$ satisfy conditions (H8)–(H11) and conditions (H1)–(H3) as well as (H12) hold, then the problem (1.1) has minimal and maximal mild solutions , which can be obtained by monotone iterative procedure.*

**Corollary 4.10.** *Let $X$ be an ordered Banach space, whose positive cone $K$ is regular. Let $A : D(A) \subset X \to X$ be a closed linear operator and $-A$ generate a positive $C_0$-semigroup $T(t)(t \geq 0)$ in $X$. $f \in C(J \times X, X)$. $g, h_i(i = 1, 2, \cdots, m)$ are continuous and conditions (H1)–(H3) as well as (H12) hold, then the problem (1.1) has minimal and maximal mild solutions , which can be obtained by monotone iterative procedure.*

**Corollary 4.11.** *Let $X$ be an ordered and weakly sequentially complete Banach space, whose positive cone $K$ is normal, and $N_0$ be the normal constant. Let $A : D(A) \subset X \to X$ be a closed linear operator and $-A$ generate a positive $C_0$-semigroup $T(t)(t \geq 0)$ in $X$. $f \in C(J \times X, X)$. $g, h_i(i = 1, 2, \cdots, m)$ are continuous and conditions (H1)–(H3) as well as (H12) hold, then the problem (1.1) has minimal and maximal mild solutions, which can be obtained by monotone iterative procedure.*

# 5 Application

In this section, we present one example, which indicates how our abstract results can be applied to concrete problems.

**Example 5.1.** Consider the following nonlocal parabolic partial differential equa-

tion with not instantaneous impulses:

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) + A(x,D)u(x,t) = f(x,t,u(x,t)), \quad x \in \Omega, \\ t \in J, \quad t \in (s_i, t_{i+1}], i = 0,1,2,\cdots,m, \\ u(x,t) = h_i(x,t,u(x,t)), \quad x \in \Omega, \quad t \in (t_i, s_i], \quad i = 1,2,\cdots,m, \\ Bu = 0, \quad (x,t) \in \partial\Omega \times J, \\ u(x,0) = g(u), \quad x \in \Omega, \end{cases} \tag{5.1}$$

where $J = [0,a]$, $a > 0$ is a constant, $0 = s_0 < t_1 \le s_1 < t_2 \le s_2 < t_3 \le s_3 < \cdots < t_{m-1} \le s_{m-1} < t_m \le s_m < t_{m+1} = a$ are pre-fixed numbers, integer $n \ge 1, \Omega \subset \mathbb{R}^n$ is a bounded domain with a sufficiently smooth boundary $\partial\Omega$,

$$A(x,D) = -\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial y_j} + \sum_{i=1}^{n} a_i(x)\frac{\partial}{\partial x_i} + a_0(x)$$

is a strongly elliptic operator of second order, coefficient functions $a_{ij}(x), a_i(x)$ and $a_0(x)$ are Hölder continuous in $\Omega$, $Bu = b_0(x)u + \delta\frac{\partial u}{\partial n}$ is a regular boundary operator on $\partial\Omega$, $f : \overline{\Omega} \times J \times \mathbb{R} \to \mathbb{R}$ is continuous, $h_i : \overline{\Omega} \times J \times \mathbb{R} \to \mathbb{R}$ are also continuous, $i = 1,2,\cdots,m$, $g$ is a continuous function.

Let $X = L^p(\Omega)$ with $p > n+2$, $K = \{u \in L^p(\Omega) \mid u(x) \ge 0 \ \ a.e. \ \ x \in \Omega\}$, and define the operator $A$ as follows:

$$D(A) = \{u \in W^{2,p}(\Omega) \mid Bu = 0\}, \ Au = A(x,D)u.$$

We know that $X$ is a Banach space, $K$ is a regular cone of $X$, and $-A$ generates a positive and analytic $C_0$-semigroup $T(t)(t \ge 0)$ in $X$ (see [22]). Define $u(t) = u(\cdot,t), f(t,u(t)) = f(\cdot,t,u(\cdot,t)), h_i(t,u(t)) = h_i(\cdot,t,u(\cdot,t))$, then system (5.1) can be reformulated as problem (1.1) in $X$. We assume that the following conditions hold:

(i) Let $f(x,t,0) \ge 0, h_i(x,t,0) \ge 0, g(0) \ge 0, x \in \Omega$.

(ii) There exist $w = w(x,t) \in PC(\Omega\times J)\cap C^{2,1}(\Omega\times J'')$, and $w(x,t) \ge 0, x \in \Omega, t \in J$ such that

$$\begin{cases} \frac{\partial}{\partial t}w(x,t) + A(x,D)w(x,t) \ge f(x,t,w(x,t)), \quad x \in \Omega, \\ t \in J, \quad t \in (s_i, t_{i+1}], i = 0,1,2,\cdots,m, \\ w(x,t) \ge h_i(x,t,w(x,t)), \quad x \in \Omega, \quad t \in (t_i, s_i], \quad i = 1,2,\cdots,m, \\ Bw = 0, \quad (x,t) \in \partial\Omega \times J, \\ w(x,0) \ge g(w), \quad x \in \Omega, \end{cases}$$

(iii) The partial derivative $f'_u(x,t,u)$ is continuous on any bounded domain.

(iv) For any $u_1, u_2 \in [0, w(x,t)]$ with $u_1 \le u_2$, for any $x \in \Omega, i = 1,2,\cdots,m$,we have

$$h_i(x,t,u_1(x,t)) \le h_i(x,t,u_2(x,t)), g(u_1) \le g(u_1)$$

**Theorem 5.2.** If assumptions (i), (ii), (iii) and (iv) are satisfied, then the impulsive parabolic partial differential equation (5.1) has minimal and maximal mild solutions between 0 and $w(x,t)$, which can be obtained by a monotone iterative procedure starting from 0 and $w(x,t)$, respectively.

**Proof** From assumptions (i) and(ii) we know that 0 and $w(x,t)$ are lower and upper solutions of problem (5.1), respectively. (iii) implies that condition (H1) is satisfied. (iv) implies that conditions (H2) and (H2) are satisfied. So, by Theorem 4.5., we have the result. Then the proof is complete.□

# 6 Conclusions

In this paper, we consider the existence of mild solutions for the new nonlocal evolution equation with impulses. We initially use the monotone iterative technique to the problem under new impulsive conditions. Hence the results are new.

# References

[1] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, Theory of Impulsive Differential Equations, World Scientific, Singapore, 1989.

[2] M. Frigon, D. O'Regan, Existence results for first-order impulsive differential equations, J. Math. Anal. Appl. 193(1995). 96-113.

[3] M. Frigon, D. O'Regan, First order impulsive initial and periodic problems with variable moments, J. Math. Anal. Appl. 233(1999). 730-739.

[4] N.U.d, K.L. Teo, S.H. Hou, Nonlinear impulsive systems on infinite dimensional spaces, Nonlinear Anal.: Theory, Methods & Applications 54 (2003). 907-925.

[5] J. Liang, J.H. Liu, T.J. Xiao, Nonlocal impulsive problems for nonlinear differential equations in Banach spaces, Math. Comput. Modelling 49(2009).798-804.

[6] S.M. Afonso, E.M. Bonotto, M. Federson, $\check{S}$. Schwabik, Discontinuous local semiflows for Kurzweil equations leading to LaSalle's invariance principle for differential systems with impulses at variable times, J. Differential Equations 250(2011). 2969-3001.

[7] S.M. Afonso, E.M. Bonotto, L. Gimenes, Boundedness of solutions of retarded functional differential equations with variable impulses via generalized ordinary differential equations, Math. Nachr. 285(2012). 545-561.

[8] Y.X. Li, Z.Liu, Monotone iterative technique for addressing impulsive integro-differential equations in Banach spaces, Nonlinear Anal. 66(2007). 83-92.

[9] Z. Fan, G. Li, Existence results for semilinear differential equations with nonlocal and impulsive conditions, J. Funct. Anal. 258(2010). 1709-1727.

[10] S. Ji, G. Li, Existence results for impulsive differential inclusions with nonlocal conditions, Comput. Math. Appl. 62 (2011). 1908-1915.

[11] X. Fu, Y. Cao, Existence for neutral impulsive differential inclusions with nonlocal conditions, Nonlinear Anal. 68(2008). 3707-3718.

[12] Y.K. Chang, A. Anguraj, K. Karthikeyan, Existence for impulsive neutral integrodifferential inclusions with nonlocal initial conditions via fractional operators, Nonlinear Anal. 71(2009). 4377-4386.

[13] Y.K. Chang , V. Kavitha , M. Mallika Arjunan, Existence results for impulsive neutral differential and integrodifferential equations with nonlocal conditions via fractional operators, Nonlinear Analysis: Hybrid Systems 4(2010). 32-43

[14] N. Abada, M. Benchohra, H. Hammouche, Existence and controllability results for nondensely defined impulsive semilinear functional differential inclusions, J. Differ. Equations 246(2009). 3834-3863.

[15] T. Cardinali, P. Rubbioni, Impulsive mild solutions for semilinear differential inclusions with nonlocal conditions in Banach spaces, Nonlinear Anal.75(2012). 871-879.

[16] P.Y. Chen, Y.X. Li, H. Yang, Perturbation method for nonlocal impulsive evolution equations, Nonlinear Analysis: Hybrid Systems 8(2013). 22-30.

[17] E. Hernandez, D. O'Regan, On a new class of abstract impulsive differential equations, Proc. Am. Math. Soc. 141(2013). 1641-1649.

[18] M. Pierri, D. O'Regan, V. Rolnik, Existence of solutions for semi-linear abstract differential equations with not instantaneous impulses, Appl. Math. Comput. 219(2013). 6743-6749.

[19] J.R. Wang, X.Z. Li, Periodic BVP for integer/fractional order nonlinear differential equations with non-instantaneous impulses, J. Appl. Math. Comput. 46(2014). 321-334.

[20] X.L. Yu, J.R. Wang, Periodic boundary value problems for nonlinear impulsive evolution equations on Banach spaces, Commun. Nonlinear Sci. Numer. Simul. 22(2015). 980-989.

[21] J.R. Wang, Y. Zhou, Z. Lin, On a new class of impulsive fractional differential equations, Appl. Math. Comput. 242(2014). 649-657.

[22] A. Pazy, Semigroup of linear operators and applications to partial differential equations, Springer-Verlag, Berlin, 1983.

[23] J. Banasiak, L. Arlotti, Perturbations of Positive Semigroups with Applications, SpringerCverlag, London, 2006.

[24] Y.X. Li, The positive solutions of abstract semilinear evolution equations and their applications, Acta Math. Sinica 39(5)(1996). 666-672 (in Chinese).

[25] K. Deimling, Nonlinear Functional Analysis, Springer-Verlag, New York, 1985.

[26] D.Guo, V. Lakshmikantham, Nonlinear Problems in Abstract Cones, Academic Press, New York, 1988.

[27] D.J. Guo, J.X. Sun, Ordinary Differential Equations in Abstract Spaces, Shandong Science and Technology, Jinan, 1989.

[28] H. P. Heinz; On the behaviour of measure of noncompactness with respect to differentiation and integration of rector-valued functions, Nonlinear Anal. 7(1983). 1351-1371.

# Eigenvalue for a system of Caputo fractional differential equations *

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**Abstract:** In this article, we study the existence of positive solutions for a system of nonlinear differential equations of mixed Caputo fractional orders

$$\begin{cases} {}^cD_{0+}^{\alpha}u(t)+\lambda f(t,u(t),v(t))=0, & 0<t<1,\\ {}^cD_{0+}^{\beta}v(t)+\mu g(t,u(t),v(t))=0, & 0<t<1,\\ u(0)=u'(0)=u''(1)=u'''(0)=0,\\ v(0)=v'(0)=v''(1)=v'''(0)=0, \end{cases}$$

where $3<\alpha,\beta\leq 4$ are real numbers, ${}^cD_{0+}^{\alpha}$,${}^cD_{0+}^{\beta}$ are the Caputo fractional derivatives, and $f,g:[0,1]\times[0,+\infty)\times[0,+\infty)\to[0,+\infty)$ are given continuous functions. By using Krasnoselskii's fixed point theorem, some sufficient conditions for the existence of positive solutions and the eigenvalue intervals on which there exists a positive solution are obtained.

**Keywords**: Fractional order differential equation, Positive solution, Existence, Krasnoselskii's fixed point theorem, Eigenvalue.

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## 1 Introduction

Fractional differential equations describe many phenomena in various fields of engineering and scientific disciplines such as physics, biophysics, chemistry, economics, control theory, see [4, 8]. Recently, fractional differential equations have been of great interest, there are a large number of papers dealing with the existence of positive solutions of nonlinear fractional differential equations by the use of techniques of nonlinear analysis (such as upper and lower solution method, Leray-Schauder theory, etc.), see [1, 2, 5, 7, 9, 10]. In this paper, we consider the system of Caputo fractional differential equations

$$\begin{cases} {}^cD_{0+}^{\alpha}u(t)+\lambda f(t,u(t),v(t))=0, & 0<t<1,\\ {}^cD_{0+}^{\beta}v(t)+\mu g(t,u(t),v(t))=0, & 0<t<1,\\ u(0)=u'(0)=u''(1)=u'''(0)=0,\\ v(0)=v'(0)=v''(1)=v'''(0)=0, \end{cases} \tag{1.1}$$

where $3<\alpha,\beta\leq 4$ are real numbers, ${}^cD_{0+}^{\alpha}$,${}^cD_{0+}^{\beta}$ are the Caputo fractional derivatives, and $f,g:[0,1]\times[0,+\infty)\times[0,+\infty)\to[0,+\infty)$ are given continuous functions. By using Krasnoselskii's fixed point theorem, some sufficient conditions for the existence of positive solutions and the eigenvalue intervals on which there exists a positive solution are obtained.

This paper is organized as follows. In Section 2, we present some basic definitions and properties from the fractional calculus theory. In Section 3, based on the Krasnoselskii's fixed point theorem, we prove two existence theorems of the positive solutions for BVP (1.1). In section 4, an example is presented to illustrate the main results.

## 2 Preliminaries

Let us start with the necessary definitions which are used throughout this paper.

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**Definition 2.1 ([2]).** The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, +\infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \mathrm{d}s, \qquad t > 0,$$

provided the right-hand side is pointwise defined on $(0, +\infty)$.

**Definition 2.2 ([3, 8]).**For a function $f : (0, +\infty) \to \mathbb{R}$, the Caputo derivative of fractional order is defined as

$${}^{c}D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}} \mathrm{d}s, \quad t > 0,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the number $\alpha$.

**Lemma 2.1 ([3, 9]).** Let $\alpha > 0$, then fractional differential equation ${}^{c}D_{0+}^{\alpha} u(t) = 0$ has solutions

$$u(t) = C_1 + C_2 t + \cdots + C_n t^{n-1}, C_i \in \mathbb{R}, i = 1, 2, \cdots, n, n = [\alpha] + 1.$$

**Lemma 2.2 ([3, 9]).** Let $\alpha > 0$, then

$$I_{0+}^{\alpha}\, {}^{c}D_{0+}^{\alpha} u(t) = u(t) + C_1 + C_2 t + \cdots + C_n t^{n-1}, C_i \in \mathbb{R}, i = 1, 2, \cdots, n, n = [\alpha] + 1.$$

In the following, we present Green's function of BVP (1.1).

**Lemma 2.3 .** Let $h_1 \in C[0, 1]$ and $3 < \alpha \leq 4$, the unique solution of problem

$${}^{c}D_{0+}^{\alpha} u(t) + h_1(t) = 0, 0 < t < 1, \tag{2.1}$$

$$u(0) = u'(0) = u''(1) = u'''(0) = 0, \tag{2.2}$$

is
$$u(t) = \int_0^1 G_1(t, s) h_1(s) \mathrm{d}s,$$

where

$$G_1(t,s) = \begin{cases} \dfrac{(\alpha-1)(\alpha-2)t^2(1-s)^{\alpha-3} - 2(t-s)^{\alpha-1}}{2\Gamma(\alpha)}, 0 \leq s \leq t \leq 1, \\ \dfrac{t^2(1-s)^{\alpha-3}}{2\Gamma(\alpha-2)}, \qquad\qquad 0 \leq t \leq s \leq 1. \end{cases} \tag{2.3}$$

Here $G_1(t, s)$ is called the Green's function of BVP (2.1) and (2.2).

**Proof.** We may apply Lemma 2.2 to reduce (2.1) to an equivalent integral equation

$$u(t) = -I_{0+}^{\alpha} h_1(t) + C_1 + C_2 t + C_3 t^2 + C_4 t^3,$$

for some $C_1, C_2, C_3, C_4 \in \mathbb{R}$. Consequently, the general solution of (2.1) is

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h_1(s) \mathrm{d}s + C_1 + C_2 t + C_3 t^2 + C_4 t^3.$$

By (2.2), there are $C_1 = C_2 = C_4 = 0$, and $C_3 = \dfrac{1}{2\Gamma(\alpha-2)} \displaystyle\int_0^1 (1-s)^{\alpha-3} h_1(s) \mathrm{d}s$.
Therefore, the unique solution of problem (2.1) and (2.2) is

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h_1(s) \mathrm{d}s + \frac{1}{2\Gamma(\alpha-2)} \int_0^1 t^2 (1-s)^{\alpha-3} h_1(s) \mathrm{d}s \\ &= \int_0^t \left[ \frac{t^2(1-s)^{\alpha-3}}{2\Gamma(\alpha-2)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \right] h_1(s) \mathrm{d}s + \int_t^1 \frac{t^2(1-s)^{\alpha-3}}{2\Gamma(\alpha-2)} h_1(s) \mathrm{d}s \\ &= \int_0^t \frac{(\alpha-1)(\alpha-2)t^2(1-s)^{\alpha-3} - 2(t-s)^{\alpha-1}}{2\Gamma(\alpha)} h_1(s) \mathrm{d}s + \int_t^1 \frac{t^2(1-s)^{\alpha-3}}{2\Gamma(\alpha-2)} h_1(s) \mathrm{d}s \\ &= \int_0^1 G_1(t, s) h_1(s) \mathrm{d}s. \end{aligned}$$

The proof is finished.

**Lemma 2.4 .** The function $G_1(t,s)$ defined by (2.3) possesses the following properties:

(1) $G_1(t,s) > 0$, for $t,s \in (0,1)$;

(2) $\min\limits_{\frac{1}{4}\le t\le\frac{3}{4}} G_1(t,s) \ge \dfrac{1}{16}\max\limits_{0\le t\le 1} G_1(t,s) = \dfrac{1}{16}G_1(1,s)$, for $s \in (0,1)$.

**Proof.** Let $g_1(t,s) = \dfrac{(\alpha-1)(\alpha-2)t^2(1-s)^{\alpha-3} - 2(t-s)^{\alpha-1}}{2\Gamma(\alpha)}$, $\quad g_2(t,s) = \dfrac{t^2(1-s)^{\alpha-3}}{2\Gamma(\alpha-2)}$.

(1) Since $3 < \alpha \le 4, 0 < s \le t < 1$, so

$$(\alpha-1)(\alpha-2)t^2(1-s)^{\alpha-3} > 2t^2(1-s)^{\alpha-3} > 2t^2(t-s)^{\alpha-3} \ge 2(t-s)^{\alpha-1},$$

therefore, $g_1(t,s) > 0$, obviously, $g_2(t,s) > 0$, thus $G_1(t,s) > 0$, for $t,s \in (0,1)$.

(2) Since

$$\frac{\partial g_1(t,s)}{\partial t} = \frac{(\alpha-1)(\alpha-2)t(1-s)^{\alpha-3} - (\alpha-1)(t-s)^{\alpha-2}}{\Gamma(\alpha)} > 0, \qquad \frac{\partial g_2(t,s)}{\partial t} = \frac{t(1-s)^{\alpha-3}}{\Gamma(\alpha-2)} > 0,$$

so $G_1(t,s)$ is monotone increasing function for $t$.

Thus,

$$0 \le G_1(t,s) \le \max_{0\le t\le 1} G_1(t,s) = G_1(1,s), \qquad t,s \in [0,1].$$

Noticing that

$$\min_{\frac{1}{4}\le t\le\frac{3}{4}} G_1(t,s) = G_1(\frac{1}{4},s) = \begin{cases} \dfrac{(\alpha-1)(\alpha-2)(1-s)^{\alpha-3} - 32\left(\frac{1}{4}-s\right)^{\alpha-1}}{32\Gamma(\alpha)}, & s \in (0,\frac{1}{4}], \\ \dfrac{(1-s)^{\alpha-3}}{32\Gamma(\alpha-2)}, & s \in [\frac{1}{4},1). \end{cases}$$

$$\max_{0\le t\le 1} G_1(t,s) = G_1(1,s) = \frac{(\alpha-1)(\alpha-2)(1-s)^{\alpha-3} - 2(1-s)^{\alpha-1}}{2\Gamma(\alpha)}, \qquad s \in (0,1).$$

Next we proof

$$\min_{\frac{1}{4}\le t\le\frac{3}{4}} G_1(t,s) \ge \frac{1}{16}\max_{0\le t\le 1} G_1(t,s) = \frac{1}{16}G_1(1,s).$$

When $0 < s \le \frac{1}{4}$, since $3 < \alpha \le 4$, we have

$$(\frac{1}{4}-s)^{\alpha-1} = (\frac{1}{4})^{\alpha-1}(1-4s)^{\alpha-1} \le (\frac{1}{4})^2(1-4s)^{\alpha-1} < \frac{1}{16}(1-s)^{\alpha-1},$$

so $$\min_{\frac{1}{4}\le t\le\frac{3}{4}} G_1(t,s) \ge \frac{1}{16}\max_{0\le t\le 1} G_1(t,s) = \frac{1}{16}G_1(1,s).$$

When $\frac{1}{4} \le s < 1$, we obtain

$$\min_{\frac{1}{4}\le t\le\frac{3}{4}} G_1(t,s) = \frac{(1-s)^{\alpha-3}}{32\Gamma(\alpha-2)} = \frac{(\alpha-1)(\alpha-2)(1-s)^{\alpha-3}}{32\Gamma(\alpha)},$$

$$\frac{1}{16}\max_{0\le t\le 1} G_1(t,s) = \frac{1}{16}G_1(1,s) = \frac{(\alpha-1)(\alpha-2)(1-s)^{\alpha-3}}{32\Gamma(\alpha)} - \frac{(1-s)^{\alpha-1}}{16\Gamma(\alpha)}.$$

Obvious that,

$$\min_{\frac{1}{4}\le t\le\frac{3}{4}} G_1(t,s) \ge \frac{1}{16}\max_{0\le t\le 1} G_1(t,s) = \frac{1}{16}G_1(1,s), \qquad s \in (0,1).$$

The proof is finished.

**Lemma 2.5 .** If the function $f \in C([0,1]\times[0,+\infty)\times[0,+\infty),[0,+\infty))$, then the unique solution of BVP (1.1) satisfied

$$\min_{\frac{1}{4}\le t\le\frac{3}{4}} u(t) \ge \frac{1}{16}\|u\|.$$

**Proof.** From lemma 2.3, we known

$$u(t)=\int_0^1 G_1(t,s)\,f(s,u(s),v(s))\,\mathrm{d}s\le\int_0^1 \max_{0\le t\le 1} G_1(t,s)\,f(s,u(s),v(s))\,\mathrm{d}s,$$

and

$$\|u\|=\max_{0\le t\le 1}|u(t)|=\max_{0\le t\le 1}\int_0^1 G_1(t,s)\,f(s,u(s),v(s))\,\mathrm{d}s\le\int_0^1 \max_{0\le t\le 1} G_1(t,s)\,f(s,u(s),v(s))\,\mathrm{d}s.$$

From lemma 2.4, we have

$$\begin{aligned}\min_{\frac{1}{4}\le t\le\frac{3}{4}} u(t)&=\min_{\frac{1}{4}\le t\le\frac{3}{4}}\int_0^1 G_1(t,s)\,f(s,u(s),v(s))\,\mathrm{d}s\\&\ge\frac{1}{16}\int_0^1\max_{0\le t\le 1} G_1(t,s)\,f(s,u(s),v(s))\,\mathrm{d}s\ge\frac{1}{16}\max_{0\le t\le 1}\int_0^1 G_1(t,s)\,f(s,u(s),v(s))\,\mathrm{d}s=\frac{1}{16}\|u\|.\end{aligned}$$

The proof is finished.

Similarly, we can obtain $G_2(t,s)$ if $\alpha$ is replaced by $\beta$,

$$G_2(t,s)=\begin{cases}\dfrac{(\beta-1)(\beta-2)t^2(1-s)^{\beta-3}-2(t-s)^{\beta-1}}{2\Gamma(\beta)}, & 0\le s\le t\le 1,\\ \dfrac{t^2(1-s)^{\beta-3}}{2\Gamma(\beta-2)}, & 0\le t\le s\le 1.\end{cases}\tag{2.4}$$

The function $G_2(t,s)$ defined by (2.4) have the same properties with $G_1(t,s)$, so

$$\min_{\frac{1}{4}\le t\le\frac{3}{4}} G_2(t,s)\ge\frac{1}{16}\max_{0\le t\le 1} G_2(t,s)=\frac{1}{16}G_2(1,s),s\in(0,1).$$

**Lemma 2.6 ([6]).** Let $E$ be a Banach space, and let $P\subset E$ be a cone in $E$. Assume $\Omega_1,\Omega_2$ be two open subsets of $E$ with $\theta\in\Omega_1\subset\overline{\Omega}_1\subset\Omega_2$, and let $T:P\to P$ be a completely continuous operator such that either

(i)$\|Tw\|\le\|w\|,w\in P\cap\partial\Omega_1,\|Tw\|\ge\|w\|,w\in P\cap\partial\Omega_2$, or

(ii)$\|Tw\|\ge\|w\|,w\in P\cap\partial\Omega_1,\|Tw\|\le\|w\|,w\in P\cap\partial\Omega_2$

holds. Then $T$ has a fixed point in $P\cap\overline{\Omega}_2\backslash\Omega_1$.

# 3 Main results and proof

In this section, we establish the existence of positive solutions for BVP (1.1). For convenience, we introduce the following notations

$$\begin{aligned}f_0&=\liminf_{u+v\to 0^+}\min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]}\frac{f(t,u,v)}{u+v}, & g_0&=\liminf_{u+v\to 0^+}\min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]}\frac{g(t,u,v)}{u+v},\\ f^0&=\limsup_{u+v\to 0^+}\max_{t\in[0,1]}\frac{f(t,u,v)}{u+v}, & g^0&=\limsup_{u+v\to 0^+}\max_{t\in[0,1]}\frac{g(t,u,v)}{u+v},\\ f_\infty&=\liminf_{u+v\to\infty}\min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]}\frac{f(t,u,v)}{u+v}, & g_\infty&=\liminf_{u+v\to\infty}\min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]}\frac{g(t,u,v)}{u+v},\\ f^\infty&=\limsup_{u+v\to\infty}\max_{t\in[0,1]}\frac{f(t,u,v)}{u+v}, & g^\infty&=\limsup_{u+v\to\infty}\max_{t\in[0,1]}\frac{g(t,u,v)}{u+v}.\end{aligned}$$

By using the Green's functions $G_i(t,s)\,(i=1,2)$, from Section 2, the problem (1.1) can be written equivalently as the following nonlinear system of integral equations

$$\begin{cases}u(t)=\lambda\displaystyle\int_0^1 G_1(t,s)f(s,u(s),v(s))\mathrm{d}s,0\le t\le 1,\\ v(t)=\mu\displaystyle\int_0^1 G_2(t,s)g(s,u(s),v(s))\mathrm{d}s,0\le t\le 1.\end{cases}$$

We consider the Banach space $X = C[0,1]$ with the norm $\|u\| = \max_{0\le t\le 1} |u(t)|$, and the Banach space $Y = X \times X$ with the norm $\|(u,v)\|_Y = \|u\| + \|v\|$.

We define the cone $P \subset Y$ by

$$P = \left\{ (u,v) \in Y | u(t) \ge 0, v(t) \ge 0, \min_{\frac{1}{4}\le t\le \frac{3}{4}} (u(t)+v(t)) \ge \frac{1}{16} \|(u,v)\|_Y, t \in [0,1] \right\}.$$

For $\lambda, \mu > 0$, we define the operators $T_1, T_2 : Y \to X$ and $T : Y \to Y$ respectively by

$$\begin{cases} T_1(u,v)(t) = \lambda \int_0^1 G_1(t,s) f(s,u(s),v(s)) \mathrm{d}s, 0 \le t \le 1, \\ T_2(u,v)(t) = \mu \int_0^1 G_2(t,s) g(s,u(s),v(s)) \mathrm{d}s, 0 \le t \le 1, \end{cases}$$

and $T(u,v) = (T_1(u,v), T_2(u,v)), (u,v) \in Y$. Thus, the solutions of BVP (1.1) are the fixed points of the operator $T$.

**Lemma 3.1.** $T : P \to P$ is a completely continuous operator.

**Proof.** Let $(u,v) \in P$ be an arbitrary element. From the definition $T_1(u,v)$ and Lemma 2.4, we get

$$\|T_1(u,v)\| = \max_{0\le t\le 1} |T_1(u,v)(t)| \le \int_0^1 \max_{0\le t\le 1} G_1(t,s) f(s,u(s),v(s)) \mathrm{d}s,$$

$$\begin{aligned} \min_{\frac{1}{4}\le t\le \frac{3}{4}} T_1(u,v)(t) &= \min_{\frac{1}{4}\le t\le \frac{3}{4}} \int_0^1 G_1(t,s) f(s,u(s),v(s)) \mathrm{d}s \ge \frac{1}{16} \int_0^1 \max_{0\le t\le 1} G_1(t,s) f(s,u(s),v(s)) \mathrm{d}s \\ &\ge \frac{1}{16} \max_{0\le t\le 1} \int_0^1 G_1(t,s) f(s,u(s),v(s)) \mathrm{d}s = \frac{1}{16} \|T_1(u,v)\|. \end{aligned}$$

In the similar manner, we deduce $\min_{\frac{1}{4}\le t\le \frac{3}{4}} T_2(u,v)(t) \ge \frac{1}{16} \|T_2(u,v)\|$.

Thus we have

$$\min_{\frac{1}{4}\le t\le \frac{3}{4}} (T_1(u,v)(t) + T_2(u,v)(t)) \ge \frac{1}{16} (\|T_1(u,v)\| + \|T_2(u,v)\|) \ge \frac{1}{16} T \|(u,v)\|_Y.$$

Hence $T(u,v) \in P$, that is $T(P) \subset P$.

According to the Arzela-Ascoli theorem, we can easily get that $T : P \to P$ is a completely continuous operator. The proof is completed.

Next, for $\alpha_1, \alpha_2, \overline{\alpha}_1, \overline{\alpha}_2 > 0$ such that $\alpha_1 + \alpha_2 = 1, \overline{\alpha}_1 + \overline{\alpha}_2 = 1$, we define the numbers $L_1, L_2, L_3, L_4$ by

$$L_1 = \frac{\int_{\frac{1}{4}}^{\frac{3}{4}} G_1(1,s)\mathrm{d}s}{256\overline{\alpha}_1}, \qquad L_2 = \frac{\int_0^1 G_1(1,s)\mathrm{d}s}{\alpha_1}, \qquad L_3 = \frac{\int_{\frac{1}{4}}^{\frac{3}{4}} G_2(1,s)\mathrm{d}s}{256\overline{\alpha}_2}, \qquad L_4 = \frac{\int_0^1 G_2(1,s)\mathrm{d}s}{\alpha_2}.$$

**Theorem 3.1.** If $f^0, g^0, f_\infty, g_\infty \in (0,\infty)$, $\frac{1}{L_1 f_\infty} < \frac{1}{L_2 f^0}$ and $\frac{1}{L_3 g_\infty} < \frac{1}{L_4 g^0}$ hold, then for any $\lambda \in \left(\frac{1}{L_1 f_\infty}, \frac{1}{L_2 f^0}\right)$ and $\mu \in (\frac{1}{L_3 g_\infty}, \frac{1}{L_4 g^0})$, BVP (1.1) has at least one positive solution $(u(t), v(t)), t \in [0,1]$.

**Proof.** When $\lambda \in \left(\frac{1}{L_1 f_\infty}, \frac{1}{L_2 f^0}\right)$ and $\mu \in \left(\frac{1}{L_3 g_\infty}, \frac{1}{L_4 g^0}\right)$, choosing $\varepsilon > 0$, such that

$$\frac{1}{L_1(f_\infty - \varepsilon)} \le \lambda \le \frac{1}{L_2(f^0 + \varepsilon)}, \quad \frac{1}{L_3(g_\infty - \varepsilon)} \le \mu \le \frac{1}{L_4(g^0 + \varepsilon)} \tag{3.1}$$

By the definition of $f^0, g^0$, there exists $R_1 > 0$, such that for all $t \in [0,1], u, v \in \mathbb{R}^+$, with $0 \le u + v \le R_1$, we have

$$f(t,u,v) \le (f^0 + \varepsilon)(u+v), \qquad g(t,u,v) \le (g^0 + \varepsilon)(u+v). \tag{3.2}$$

Now define the set $\Omega_1 = \{(u,v) \in Y, \|(u,v)\|_Y < R_1\}$. Let $(u,v) \in P \cap \partial\Omega_1$, that is $(u,v) \in P$ with $\|(u,v)\|_Y = R_1$, so $u(t) + v(t) \le R_1$ for all $t \in [0,1]$, thus

$$\begin{aligned} T_1(u,v)(t) =& \lambda \int_0^1 G_1(t,s) f(s,u(s),v(s)) \mathrm{d}s \le \lambda \int_0^1 G_1(1,s)(f^0+\varepsilon)(u(s)+v(s)) \mathrm{d}s \\ \le& \lambda (f^0+\varepsilon) \int_0^1 G_1(1,s)\mathrm{d}s \|(u,v)\|_Y \le \frac{1}{L_2} \int_0^1 G_1(1,s)\mathrm{d}s \|(u,v)\|_Y \\ \le& \alpha_1 \|(u,v)\|_Y. \end{aligned}$$

Therefore, $\|T_1(u,v)\| \leq \alpha_1 \|(u,v)\|_Y$.

In the similar manner, we deduce

$$T_2(u,v)(t) \leq \mu(g^0+\varepsilon)\int_0^1 G_2(1,s)\mathrm{d}s\,\|(u,v)\|_Y \leq \alpha_2\|(u,v)\|_Y.$$

So, $\|T_2(u,v)\| \leq \alpha_2\|(u,v)\|_Y$.

Then for $(u,v)\in P\cap\partial\Omega_1$, we deduce

$$\|T(u,v)\|_Y = \|T_1(u,v)\| + \|T_2(u,v)\| \leq (\alpha_1+\alpha_2)\|(u,v)\|_Y = \|(u,v)\|_Y.$$

By the definition of $f_\infty, g_\infty$, there exists $R_2>0$, such that for all $t\in[0,1], u,v\in\mathbb{R}^+$, with $u+v\geq R_2$, we have

$$f(t,u,v)\geq(f_\infty-\varepsilon)(u+v), \qquad g(t,u,v)\geq(g_\infty-\varepsilon)(u+v). \tag{3.3}$$

Now define the set $\Omega_2=\{(u,v)\in Y, \|(u,v)\|_Y<R_2\}$. Let $(u,v)\in P\cap\partial\Omega_2$, that is $(u,v)\in P$ with $\|(u,v)\|_Y=R_2$, so $u(t)+v(t)\geq\frac{1}{16}\|(u,v)\|_Y$ for all $t\in[\frac{1}{4},\frac{3}{4}]$, thus, it follows from Lemma 2.4 that

$$\begin{aligned}
T_1(u,v)(t) =&\lambda\int_0^1 G_1(t,s)f(s,u(s),v(s))\mathrm{d}s \geq \lambda\int_{\frac{1}{4}}^{\frac{3}{4}} G_1(t,s)f(s,u(s),v(s))\mathrm{d}s\\
\geq&\lambda\int_{\frac{1}{4}}^{\frac{3}{4}}\frac{1}{16}G_1(1,s)(f_\infty-\varepsilon)(u(s)+v(s))\mathrm{d}s = \frac{\lambda(f_\infty-\varepsilon)}{16}\int_{\frac{1}{4}}^{\frac{3}{4}} G_1(1,s)(u(s)+v(s))\mathrm{d}s\\
\geq&\frac{\lambda(f_\infty-\varepsilon)}{256}\int_{\frac{1}{4}}^{\frac{3}{4}} G_1(1,s)\mathrm{d}s\,\|(u,v)\|_Y \geq \frac{1}{256L_1}\int_{\frac{1}{4}}^{\frac{3}{4}} G_1(1,s)\mathrm{d}s\,\|(u,v)\|_Y \geq \overline{\alpha}_1\|(u,v)\|_Y.
\end{aligned}$$

Therefore, $\|T_1(u,v)\|\geq\overline{\alpha}_1\|(u,v)\|_Y$.

Similarly, we have

$$T_2(u,v)(t)\geq\frac{\mu(g_\infty-\varepsilon)}{256}\int_{\frac{1}{4}}^{\frac{3}{4}} G_2(1,s)\mathrm{d}s\,\|(u,v)\|_Y \geq \overline{\alpha}_2\|(u,v)\|_Y.$$

So, $\|T_2(u,v)\|\geq\overline{\alpha}_2\|(u,v)\|_Y$.

Then for $(u,v)\in P\cap\partial\Omega_2$, we deduce

$$\|T(u,v)\|_Y = \|T_1(u,v)\| + \|T_2(u,v)\| \geq (\overline{\alpha}_1+\overline{\alpha}_2)\|(u,v)\|_Y = \|(u,v)\|_Y.$$

By using Lemma 2.6, we conclude that $T$ has a fixed point $(u,v)\in P\cap(\overline{\Omega}_2\setminus\Omega_1)$ such that $R_1\leq\|u\|+\|v\|\leq R_2$.

**Theorem 3.2.** If $f_0,g_0,f^\infty,g^\infty\in(0,\infty)$, $\frac{1}{L_1f_0}<\frac{1}{L_2f^\infty}$ and $\frac{1}{L_3g_0}<\frac{1}{L_4g^\infty}$ hold, then for any $\lambda\in(\frac{1}{L_1f_0},\frac{1}{L_2f^\infty})$ and $\mu\in(\frac{1}{L_3g_0},\frac{1}{L_4g^\infty})$, BVP (1.1) has at least one positive solution $(u(t),v(t)), t\in[0,1]$.

**Proof.** When $\lambda\in\left(\frac{1}{L_1f_0},\frac{1}{L_2f^\infty}\right)$ and $\mu\in(\frac{1}{L_3g_0},\frac{1}{L_4g^\infty})$, choosing $\varepsilon>0$, such that

$$\frac{1}{L_1(f_0-\varepsilon)}\leq\lambda\leq\frac{1}{L_2(f^\infty+\varepsilon)}, \quad \frac{1}{L_3(g_0-\varepsilon)}\leq\mu\leq\frac{1}{L_4(g^\infty+\varepsilon)}. \tag{3.4}$$

By the definition of $f_0,g_0$, there exists $R_3>0$, such that for all $t\in[0,1], u,v\in\mathbb{R}^+$, with $0\leq u+v\leq R_3$, we have

$$f(t,u,v)\geq(f_0-\varepsilon)(u+v), \qquad g(t,u,v)\geq(g_0-\varepsilon)(u+v). \tag{3.5}$$

Now define the set $\Omega_3=\{(u,v)\in Y, \|(u,v)\|_Y<R_3\}$. Let $(u,v)\in P\cap\partial\Omega_3$, that is $(u,v)\in P$ with $\|(u,v)\|_Y=R_3$, so $u(t)+v(t)\geq\frac{1}{16}\|(u,v)\|_Y$ for all $t\in[\frac{1}{4},\frac{3}{4}]$, thus, it follows from Lemma 2.4 that

$$\begin{aligned}
T_1(u,v)(t) =&\lambda\int_0^1 G_1(t,s)f(s,u(s),v(s))\mathrm{d}s \geq \lambda\int_{\frac{1}{4}}^{\frac{3}{4}} G_1(t,s)f(s,u(s),v(s))\mathrm{d}s\\
\geq&\lambda\int_{\frac{1}{4}}^{\frac{3}{4}}\frac{1}{16}G_1(1,s)(f_0-\varepsilon)(u(s)+v(s))\mathrm{d}s = \frac{\lambda(f_0-\varepsilon)}{16}\int_{\frac{1}{4}}^{\frac{3}{4}} G_1(1,s)(u(s)+v(s))\mathrm{d}s\\
\geq&\frac{\lambda(f_0-\varepsilon)}{256}\int_{\frac{1}{4}}^{\frac{3}{4}} G_1(1,s)\mathrm{d}s\|(u,v)\|_Y \geq \frac{1}{256L_1}\int_{\frac{1}{4}}^{\frac{3}{4}} G_1(1,s)\mathrm{d}s\|(u,v)\|_Y \geq \overline{\alpha}_1\|(u,v)\|_Y.
\end{aligned}$$

Therefore, $\|T_1(u,v)\| \geq \overline{\alpha}_1 \|(u,v)\|_Y$.

In the similar manner, we deduce

$$T_2(u,v)(t) \geq \frac{\mu(g_0-\varepsilon)}{256}\int_{\frac{1}{4}}^{\frac{3}{4}} G_2(1,s)\mathrm{d}s\,\|(u,v)\|_Y \geq \overline{\alpha}_2\|(u,v)\|_Y.$$

So, $\|T_2(u,v)\| \geq \overline{\alpha}_2\|(u,v)\|_Y$.

Then for $(u,v)\in P\cap\partial\Omega_3$, we deduce

$$\|T(u,v)\|_Y = \|T_1(u,v)\| + \|T_2(u,v)\| \geq (\overline{\alpha}_1+\overline{\alpha}_2)\|(u,v)\|_Y = \|(u,v)\|_Y.$$

Next, we define the functions $f^*, g^* : [0,1]\times\mathbb{R}^+ \to \mathbb{R}^+, f^*(t,x) = \max\limits_{0\leq u+v\leq x} f(t,u,v), g^*(t,x) = \max\limits_{0\leq u+v\leq x} g(t,u,v), t\in[0,1], x\in\mathbb{R}^+$. Then $f(t,u,v)\leq f^*(t,x), g(t,u,v)\leq g^*(t,x)$ for all $t\in[0,1], u\geq 0, v\geq 0$ and $u+v\leq x$. The functions $f^*(t,\cdot), g^*(t,\cdot)$ are nondecreasing for every $t\in[0,1]$, and satisfy the conditions

$$\limsup_{x\to\infty}\max_{t\in[0,1]}\frac{f^*(t,x)}{x}\leq f^\infty, \qquad \limsup_{x\to\infty}\max_{t\in[0,1]}\frac{g^*(t,x)}{x}\leq g^\infty.$$

Therefore, for $\varepsilon>0$, there exists $\overline{R}_4>0$, such that for all $x\geq\overline{R}_4$ and $t\in[0,1]$, we can get

$$\frac{f^*(t,x)}{x}\leq\limsup_{x\to\infty}\max_{t\in[0,1]}\frac{f^*(t,x)}{x}+\varepsilon\leq f^\infty+\varepsilon,$$

$$\frac{g^*(t,x)}{x}\leq\limsup_{x\to\infty}\max_{t\in[0,1]}\frac{g^*(t,x)}{x}+\varepsilon\leq g^\infty+\varepsilon,$$

so $f^*(t,x)\leq(f^\infty+\varepsilon)x, g^*(t,x)\leq(g^\infty+\varepsilon)x$.

We consider $R_4\geq\overline{R}_4+R_3$ and define the set $\Omega_4=\{(u,v)\in Y, \|(u,v)\|_Y<R_4\}$. Let $(u,v)\in P\cap\partial\Omega_4$, that is $(u,v)\in P$ with $\|(u,v)\|_Y=R_4$ or equivalently $\|u\|+\|v\|=R_4$. By the definition of $f^*, g^*$, we can get for all $t\in[0,1]$,

$$f(t,u(t),v(t))\leq f^*(t,\|(u,v)\|_Y), \qquad g(t,u(t),v(t))\leq g^*(t,\|(u,v)\|_Y). \tag{3.6}$$

Thus

$$\begin{aligned}T_1(u,v)(t)=&\lambda\int_0^1 G_1(t,s)f(s,u(s),v(s))\mathrm{d}s\leq\lambda\int_0^1 G_1(1,s)f^*(t,\|(u,v)\|_Y)\mathrm{d}s\\ \leq&\lambda\int_0^1 G_1(1,s)(f^\infty+\varepsilon)R_4\mathrm{d}s=\lambda(f^\infty+\varepsilon)\int_0^1 G_1(1,s)\mathrm{d}s\|(u,v)\|_Y\\ \leq&\frac{1}{L_2}\int_0^1 G_1(1,s)\mathrm{d}s\|(u,v)\|_Y\leq\alpha_1\|(u,v)\|_Y.\end{aligned}$$

Therefore, $\|T_1(u,v)\|\leq\alpha_1\|(u,v)\|_Y$.

Similarly, we have

$$T_2(u,v)(t)\leq\mu(g^\infty+\varepsilon)\int_0^1 G_2(1,s)\mathrm{d}s\|(u,v)\|_Y\leq\alpha_2\|(u,v)\|_Y.$$

So, $\|T_2(u,v)\|\leq\alpha_2\|(u,v)\|_Y$.

Then for$(u,v)\in P\cap\partial\Omega_4$, we deduce

$$\|T(u,v)\|_Y=\|T_1(u,v)\|+\|T_2(u,v)\|\leq(\alpha_1+\alpha_2)\|(u,v)\|_Y=\|(u,v)\|_Y.$$

By using Lemma 2.6, we conclude that $T$ has a fixed point $(u,v)\in P\cap(\overline{\Omega}_4\setminus\Omega_3)$ such that $R_3\leq\|u\|+\|v\|\leq R_4$.

# 4 Example

**Example 4.1.** Consider the following system of fractional differential equations

$$\begin{cases} {}^cD_{0+}^{\frac{7}{2}}u(t) + \lambda f(t, u(t), v(t)) = 0, \quad 0 < t < 1, \\ {}^cD_{0+}^{\frac{10}{3}}v(t) + \mu g(t, u(t), v(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = u''(1) = u'''(0) = 0, \\ v(0) = v'(0) = v''(1) = v'''(0) = 0, \end{cases} \tag{4.1}$$

In the system (4.1), $\alpha = \frac{7}{2}, \beta = \frac{10}{3}$ and

$$f(t, u, v) = \frac{(t+1)[p_1(u+v) + q_1 e^{-(u+v)}](u+v)}{u+v+1},$$
$$g(t, u, v) = \frac{(t+1)^2[p_2(u+v) + q_2 e^{-(u+v)}](u+v)}{u+v+1},$$

for $t \in [0, 1], u, v \geq 0$, where $p_1, p_2, q_1, q_2 > 0$.

We deduce $L_1 \approx \frac{0.0007}{\overline{\alpha}_1}, L_2 \approx \frac{0.2902}{\alpha_1}, L_3 \approx \frac{0.0007}{\overline{\alpha}_2}, L_4 \approx \frac{0.3120}{\alpha_2}$. We have $f^0 = 2q_1, g^0 = 4q_2, f_\infty = p_1, g_\infty = p_2$. For $\alpha_1, \alpha_2 > 0$ with $\alpha_1 + \alpha_2 = 1$, we consider $\overline{\alpha}_1 = \alpha_1, \overline{\alpha}_2 = \alpha_2$.

Then, the conditions $\frac{1}{L_1 f_\infty} < \frac{1}{L_2 f^0}$ and $\frac{1}{L_3 g_\infty} < \frac{1}{L_4 g^0}$ become

$$L_1 p_1 > 2L_2 q_1, \qquad L_3 p_2 > 4L_4 q_2.$$

For example, if $\frac{p_1}{q_1} \geq 830$ and $\frac{p_2}{q_2} \geq 1783$, then the above conditions are satisfied. Therefore, by Theorem 3.1, there exists one positive solution $(u(t), v(t)), t \in [0, 1]$.

# References


[1] C. Bai, J. Fang, The existence of a positive solution for a singular coupled system of nonlinear fractional differential equations, Appl. Math. Comput. 150 (2004) 611-621.

[2] Z. Bai, H. Lü, Positive solutions for boundary value problem of nonlinear fractional differential equation, J. Math. Anal. Appl. 311 (2005) 495-505.

[3] Z. Bai, T. Qiu. Existence of positive solutions for singular fractional differential equation. Appl. Math. Comput. 215 (2009) 2761-2767

[4] S. Das, Functional Fractional Calculus for System Identification andControls, Springer, New York (2008).

[5] W. Feng, S. Sun, Z. Han, Y. Zhao, Existence of solutions for a singular system of nonlinear fractional differential equations, Comput. Math. Appl. 62 (2011) 1370-1378.

[6] D. Guo, J. Sun, Z. Liu, Functional Methods in Nonlinear Ordinary Differential Equations, Shandong Science and Technology Press, Jinan, 1995 (in Chinese).

[7] J. Henderson, R. Luca, Positive solutions for a system of nonlocal fractional boundary value problems, Fract. Calc. Appl. Anal. 16 (2013) 985-1008.

[8] I. Podlubny, Fractional Differential Equations, Academic Press, SanDiego (1999).

[9] S. Zhang, Positive solutions for boundary value problems of nonlinear fractional differential equations, Elect. J. Diff. Equ. 36 (2006) 1-12.

[10] Y. Zhao, S. Sun, Z. Han, W. Feng, Positive solutions for a coupled system of nonlinear differential equations of mixed fractional orders, Adv. Diff. Equ. 2011 (2011) 1-13.

# A COMMON FIXED POINT THEOREM FOR A PAIR OF GENERALIZED CONTRACTION MAPPINGS WITH APPLICATIONS

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ABSTRACT. In this article, we introduce abscissa dominating function $\mathbb{F} : [0,\infty)^2 \to \mathbb{R}$ and define a generalized $(\alpha, \mathbb{F}, \psi, \varphi)$-contraction mapping which retrieves Banach's contraction, Geraghty type contraction and weak contraction as particular cases. We establish a common fixed points theorem for a pair of generalized $(\alpha, \mathbb{F}, \psi, \varphi)$-contraction mappings in complete partial metric spaces and apply this theorem to show the existence of solution of system of integral equations. This result and its consequences generalize many existing results both in partial metric spaces and metric spaces. We give examples to illustrate our results and to express the usefulness of these results in the literature.

## 1. INTRODUCTION

A partial metric on a nonempty set $X$ is a function $p : X \times X \to [0,\infty)$ such that

$(p_1)$ $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,
$(p_2)$ $p(x,x) \leq p(x,y)$,
$(p_3)$ $p(x,y) = p(y,x)$,
$(p_4)$ $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

Partial metrics were introduced in [12] as a generalization of the notion of metric to allow non-zero self distance for the purpose of modeling partial objects in reasoning about data flow networks. The self distance $p(x,x)$ is to be understood as a quantification of the extent to which $x$ is unknown. A partial metric space is a pair $(X,p)$ such that $X$ is a nonempty set and $p$ is a partial metric on $X$. Matthews [12] proved an analogue of Banach's fixed point theorem in partial metric spaces. After this remarkable fixed point theorem, many authors took interest in partial metric spaces and its topological properties and established many well known fixed point results successfully (see [1, 2, 3, 4, 5, 6, 7, 8, 11]).

In this paper, continuing the study of fixed point theorems in partial metric spaces, we shall establish a common fixed points theorem for a pair of generalized $(\alpha, \mathbb{F}, \psi, \varphi)$-contraction mappings and shall discuss its consequences. The result proved in this paper generalizes many existing results in the literature (see [5, 7, 8, 14]). We explain hypotheses of our result through an example. In the last section of this paper, we apply this theorem to show the existence of solution of system of integral equations.

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## 2. Preliminaries

Throughout this paper, we denote $(0,\infty)$ by $\mathbb{R}^+$, $[0,\infty)$ by $\mathbb{R}_0^+$, $(-\infty,+\infty)$ by $\mathbb{R}$ and the set of natural numbers by $\mathbb{N}$. Following concepts and results will be required for the proofs of main results. Matthews [12] proved that every partial metric $p$ on $X$ induces a metric $d_p : X \times X \to \mathbb{R}_0^+$ by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y), \tag{2.1}$$

for all $x, y \in X$.

Notice that a metric on a set $X$ is a partial metric $p$ such that $p(x,x) = 0$ for all $x \in X$. Following [12], each partial metric $p$ on $X$ generates a $T_0$ topology $\tau(p)$ on $X$. The base of the topology $\tau(p)$ is the family of open $p$-balls $\{B_p(x,\epsilon) \ : x \in X,\ \epsilon > 0\}$, where $B_p(x,\epsilon) = \{y \in X : p(x,y) < p(x,x) + \epsilon\}$ for all $x \in X$ and $\epsilon > 0$.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ in $(X,p)$ converges to a point $x \in X$ if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$.

**Definition 1.** [12] Let $(X,p)$ be a partial metric space.

(1) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in $(X,p)$ is called a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists and is finite.
(2) A partial metric space $(X,p)$ is said to be complete if every Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in $X$ converges, with respect to $\tau(p)$, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.

**Definition 2.** [15] Let $S : X \to X$ and $\alpha : X \times X \to \mathbb{R}_0^+$ be two functions. Then $S$ is said to be $\alpha$-admissible if

$$\alpha(x,y) \geq 1 \text{ implies } \alpha(S(x),S(y)) \geq 1 \ \forall\ x,y \in X.$$

**Definition 3.** [10] Let $S : X \to X$ and $\alpha : X \times X \to \mathbb{R}_0^+$ be two functions. Then $S$ is said to be a triangular $\alpha$-admissible mapping if

(1) $\alpha(x,y) \geq 1$ implies $\alpha(S(x),S(y)) \geq 1$,
(2) $\alpha(x,z) \geq 1$ and $\alpha(z,y) \geq 1$ imply $\alpha(x,y) \geq 1$

for all $x, y, z \in X$.

**Definition 4.** [1] Let $S, T : X \to X$ and $\alpha : X \times X \to \mathbb{R}_0^+$ be two functions. The pair $(S,T)$ is said to be triangular $\alpha$-admissible if

(1) $\alpha(x,y) \geq 1$ implies $\alpha(S(x),T(y)) \geq 1$ and $\alpha(T(x),S(y)) \geq 1$,
(2) $\alpha(x,z) \geq 1$ and $\alpha(z,y) \geq 1$ imply $\alpha(x,y) \geq 1$

for all $x, y, z \in X$.

The following lemma will be helpful in the sequel.

**Lemma 1.** [12]

(1) A partial metric space $(X,p)$ is complete if and only if the metric space $(X,d_p)$ is complete.
(2) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in $X$ converges to a point $x \in X$, with respect to $\tau(d_p)$ if and only if $\lim_{n\to\infty} p(x,x_n) = p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.
(3) If $\lim_{n\to\infty} x_n = \upsilon$ such that $p(\upsilon,\upsilon) = 0$ then $\lim_{n\to\infty} p(x_n,y) = p(\upsilon,y)$ for every $y \in X$.

**Lemma 2.** [5] Let $S : X \to X$ be a triangular $\alpha$-admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, S(x_0)) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = S(x_n)$. Then we have $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N} \cup \{0\}$ with $n < m$.

**Lemma 3.** [1] Let $S, T : X \to X$ be triangular $\alpha$-admissible mappings. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, S(x_0)) \geq 1$. Define sequence $x_{2i+1} = S(x_{2i})$, and $x_{2i+2} = T(x_{2i+1})$, where $i = 0, 1, 2, \ldots$.. Then we have $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N} \cup \{0\}$ with $n < m$.

**Definition 5.** A continuous function $\mathbb{F} : [0, \infty)^2 \to \mathbb{R}$ is called an abscissa dominating function if for any $u, v \in \mathbb{R}_0^+$, the following conditions hold:

(1) $\mathbb{F}(u, v) < u$,
(2) If $\mathbb{F}(u, v) = u$, then either $u = 0$ or $v = 0$.

An extra condition $\mathbb{F}(0, 0) = 0$ could be imposed in some cases if required. Let $\Delta_c$ denote the class of all abscissa dominating functions.

**Example 1.**
(1) $\mathbb{F}(u, v) = u - v$.
(2) $\mathbb{F}(u, v) = ru$, for some $r \in (0, 1)$.
(3) $\mathbb{F}(u, v) = \dfrac{u}{(1+v)^r}$ for some $r \in (0, \infty)$.
(4) $\mathbb{F}(u, v) = \dfrac{\log(t + a^u)}{(1+v)}$, for some $a > 1$.
(5) $\mathbb{F}(u, v) = (u + l)^{\frac{1}{(1+v)^r}} - l$, $l > 1$, for $r \in (0, \infty)$.
(6) $\mathbb{F}(u, v) = u\beta(u)$, where $\beta : \mathbb{R}_0^+ \to [0, 1)$. and continuous.
(7) $\mathbb{F}(u, v) = u\pi^{-1/2} \int_0^\infty \dfrac{e^{-j}}{\sqrt{j} + v}\, dj$.

Let $\Phi$ denote the class of the functions $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ which satisfy the following conditions:

(a) $\varphi$ is continuous;
(b) $\varphi(t) > 0, t > 0$ and $\varphi(0) \geq 0$,

and $\Psi$ denote the class of all the functions $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ which satisfy the following conditions:

(1) $\psi$ is increasing;
(2) $\psi(t) > 0, t > 0$ and $\psi(t) = 0$ imply $t = 0$.

## 3. Main results

This section contains definitions, a common fixed point result for a pair of generalized $(\alpha, \mathbb{F}, \psi, \varphi)$-contraction mappings in the setting of partial metric spaces and examples to support this result. We begin with following definitions.

**Definition 6.** Let (X,p) be a partial metric space and $\alpha : X \times X \to [0, \infty)$ be a function. Mappings $S, T : X \to X$ are called a pair of generalized $(\alpha, \mathbb{F}, \psi, \varphi)$-contraction mapping if for all $x, y \in X$, the contractive condition

$$\alpha(x, y)\psi\left(p(S(x), T(y))\right) \leq \mathbb{F}(\psi(M(x, y)), \varphi(M(x, y))) \tag{3.1}$$

holds, where $\mathbb{F} \in \Delta_c, \psi \in \Psi, \varphi \in \Phi$ and

$$M(x,y) = \max\left\{p(x,y), \frac{p(x,S(x))p(y,T(y))}{1+p(x,y)}, \frac{p(x,S(x))p(y,T(y))}{1+p(S(x),T(y))}\right\}.$$

If we set $S = T$ in (3.1), then we obtain the following contractive condition

$$\alpha(x,y)\psi\left(p(T(x),T(y))\right) \leq \mathbb{F}(\psi(N(x,y)), \varphi(N(x,y))),$$

where

$$N(x,y) = \max\left\{p(x,y), \frac{p(x,T(x))p(y,T(y))}{1+p(x,y)}, \frac{p(x,T(x))p(y,T(y))}{1+p(T(x),T(y))}\right\}.$$

The following theorem is one of the main results.

**Theorem 1.** *Let $(X,p)$ be a complete partial metric space, $\alpha : X \times X \to \mathbb{R}_0^+$ be a function. Suppose that $S, T : X \to X$ are continuous mappings satisfying the following conditions:*

(1) *$(S,T)$ is a pair of $(\alpha, \mathbb{F}, \psi, \varphi)$-contraction mappings,*
(2) *$(S,T)$ is triangular $\alpha$-admissible,*
(3) *there exists $x_0 \in X$ such that $\alpha(x_0, S(x_0)) \geq 1$,*
(4) *$\alpha(x,y) \geq 1$ for all $x, y \in Fix(T,S)$.*

*Then $(S,T)$ have a unique common fixed point.*

*Proof.* We begin with the following observation. $M(x,y) = 0$ if and only if $x = y$ is a common fixed point of $(S,T)$. Indeed, if $x = y$ is a common fixed point of $(S,T)$, then $T(y) = T(x) = x = y = S(y) = S(x)$ and

$$M(x,y) = \max\left\{p(x,x), \frac{p(x,x)p(x,x)}{1+p(x,x)}, \frac{p(x,x)p(x,x)}{1+p(x,x)}\right\} = p(x,x).$$

From the contractive condition (3.1), we get

$$\psi\left(p(x,x)\right) = \psi\left(p(S(x),T(y))\right) \leq \alpha(x,y)\psi\left(p(S(x),T(y))\right) \leq \mathbb{F}\left(\psi\left(M(x,y)\right), \varphi\left(M(x,y)\right)\right),$$

which is only possible if $p(x,x) = 0$. So $M(x,y) = 0$.

Conversely, if $M(x,y) = 0$, then by $(P_1)$ and $(P_2)$ it is easy to check that $x = y$ is a fixed point of $S$ and $T$.

On the other hand, if $M(x,y) > 0$, we construct an iterative sequence $x_n$ of points in $X$ such a way that $x_{2i+1} = S(x_{2i})$ and $x_{2i+2} = T(x_{2i+1})$ where $i = 0, 1, 2, \ldots$. We observe that if $x_n = x_{n+1}$, then $x_n$ is a common fixed point of $S$ and $T$. Suppose that $x_n \neq x_{n+1}$ for all $n \geq 0$. Since $\alpha(x_0, x_1) \geq 1$ and the pair $(S,T)$ is $\alpha$-admissible, by Lemma 3, we have

$$\alpha(x_n, x_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{3.2}$$

Thus, for $\mathbb{F} \in \Delta_c$, we have

$$\begin{aligned} \psi\left(p(x_{2i+1}, x_{2i+2})\right) &= \psi\left(p(S(x_{2i}), T(x_{2i+1}))\right) \leq \alpha(x_{2i}, x_{2i+1})\psi\left(p(S(x_{2i}), T(x_{2i+1}))\right) \\ &\leq \mathbb{F}\left(\psi\left(M(x_{2i}, x_{2i+1})\right), \varphi\left(M(x_{2i}, x_{2i+1})\right)\right) \end{aligned}$$

for all $i \in \mathbb{N} \cup \{0\}$.

Now

$$\begin{aligned} M(x_{2i}, x_{2i+1}) &= \max\left\{ \begin{array}{l} p(x_{2i}, x_{2i+1}), \dfrac{p(x_{2i}, S(x_{2i}))p(x_{2i+1}, T(x_{2i+1}))}{1+p(x_{2i}, x_{2i+1})}, \\ \dfrac{p(x_{2i}, S(x_{2i}))p(x_{2i+1}, T(x_{2i+1}))}{1+p(S(x_{2i}), T(x_{2i+1}))} \end{array} \right\} \\ &= \max\left\{ p(x_{2i}, x_{2i+1}), \frac{p(x_{2i}, x_{2i+1})p(x_{2i+1}, x_{2i+2})}{1+p(x_{2i}, x_{2i+1})}, \frac{p(x_{2i}, x_{2i+1})p(x_{2i+1}, x_{2i+2})}{1+p(x_{2i+1}, x_{2i+2})} \right\} \\ &\leq \max\left\{ p(x_{2i}, x_{2i+1}), p(x_{2i+1}, x_{2i+2}) \right\}. \end{aligned}$$

From the definition of $\mathbb{F}$, the case $M(x_{2i}, x_{2i+1}) = p(x_{2i+1}, x_{2i+2})$ is impossible. Indeed, if $x_{2i+1} \neq x_{2i+2}$, then

$$\begin{aligned} \psi\left(p(x_{2i+1}, x_{2i+2})\right) &\leq \mathbb{F}\left(\psi\left(M(x_{2i}, x_{2i+1})\right), \varphi\left(M(x_{2i}, x_{2i+1})\right)\right) \\ &< \psi\left(M(x_{2i}, x_{2i+1})\right) = \psi\left(p(x_{2i+1}, x_{2i+2})\right), \end{aligned}$$

which is a contradiction. Therefore, $M(x_{2i}, x_{2i+1}) = p(x_{2i}, x_{2i+1})$. Thus

$$\begin{aligned} \psi\left(p(x_{2i+1}, x_{2i+2})\right) &\leq \mathbb{F}\left(\psi\left(M(x_{2i}, x_{2i+1})\right), \varphi\left(M(x_{2i}, x_{2i+1})\right)\right) \\ &\leq \mathbb{F}\left(\psi\left(p(x_{2i}, x_{2i+1})\right), \varphi\left(p((x_{2i}, x_{2i+1})\right) < \psi\left(p(x_{2i}, x_{2i+1})\right)\right) \end{aligned}$$

and so

$$\psi\left(p(x_{2i+1}, x_{2i+2})\right) < \psi\left(p(x_{2i}, x_{2i+1})\right).$$

The definition of $\psi$ implies that

$$p(x_{2i+1}, x_{2i+2}) < p(x_{2i}, x_{2i+1}).$$

Thus

$$p(x_{n+1}, x_{n+2}) < p(x_n, x_{n+1}), \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{3.3}$$

Hence we deduce that the sequence $\{p(x_n, x_{n+1})\}_{n\in\mathbb{N}}$ is nonnegative and nonincreasing. Consequently, there exists $r \geq 0$ such that $\lim_{n\to\infty} p(x_n, x_{n+1}) = r$. We assert that $r = 0$. Suppose, on contrary, that $r > 0$. If $r > 0$, then letting $n \to +\infty$ in the following inequality

$$\psi\left(p(x_{n+1}, x_{n+2})\right) \leq \mathbb{F}\left(\psi\left(p(x_n, x_{n+1})\right), \varphi\left(p((x_n, x_{n+1}))\right)\right) \leq \psi\left(p(x_n, x_{n+1})\right), \tag{3.4}$$

we get

$$\psi(r) \leq \mathbb{F}(\psi(r), \varphi(r)) < \psi(r),$$

which is a contradiction. Thus $r = 0$. Hence

$$\lim_{n\to+\infty} p(x_{n-1}, x_n) = 0. \tag{3.5}$$

Now, we claim that the sequence $\{x_n\}$ is a Cauchy sequence in $(X, p)$. Suppose, on contrary, that $\{x_n\}$ is not a Cauchy sequence. Then $\lim_{n,m\to\infty} p(x_n, x_m) \neq 0$ and there exists $\epsilon > 0$ for which we can find two subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ such that $m_k$ is the smallest index for $m_k > n_k > k$ satisfying

$$p(x_{m_k}, x_{n_k}) \geq \epsilon. \tag{3.6}$$

This means that

$$p(x_{m_k}, x_{n_{k-1}}) < \epsilon. \tag{3.7}$$

By the triangle inequality, we have

$$\begin{aligned}\epsilon &\leq p(x_{m_k}, x_{n_k}) \\ &\leq p(x_{m_k}, x_{n_{k-1}}) + p(x_{n_{k-1}}, x_{n_k}) - p(x_{n_{k-1}}, x_{n_{k-1}}) \\ &\leq p(x_{m_k}, x_{n_{k-1}}) + p(x_{n_{k-1}}, x_{n_k}) \\ &< \epsilon + p(x_{n_{k-1}}, x_{n_k}).\end{aligned}$$

That is,

$$\epsilon < \epsilon + p(x_{n_{k-1}}, x_{n_k}) \tag{3.8}$$

for all $k \in \mathbb{N}$. In the view of (3.8), (3.5), we have

$$\lim_{k\to\infty} p(x_{m_k}, x_{n_k}) = \epsilon. \tag{3.9}$$

Again using the triangle inequality, we have

$$\begin{aligned}p(x_{m_k}, x_{n_k}) &\leq p(x_{m_k}, x_{m_{k+1}}) + p(x_{m_{k+1}}, x_{n_k}) - p(x_{m_{k+1}}, x_{m_{k+1}}) \\ &\leq p(x_{m_k}, x_{m_{k+1}}) + p(x_{m_{k+1}}, x_{n_k}) \\ &\leq p(x_{m_k}, x_{m_{k+1}}) + p(x_{m_{k+1}}, x_{n_{k+1}}) + p(x_{n_{k+1}}, x_{n_k}) - p(x_{n_{k+1}}, x_{n_{k+1}}) \\ &\leq p(x_{m_k}, x_{m_{k+1}}) + p(x_{m_{k+1}}, x_{n_{k+1}}) + p(x_{n_{k+1}}, x_{n_k})\end{aligned}$$

and

$$\begin{aligned}p(x_{m_{k+1}}, x_{n_{k+1}}) &\leq p(x_{m_{k+1}}, x_{m_k}) + p(x_{m_k}, x_{n_{k+1}}) - p(x_{m_k}, x_{m_k}) \\ &\leq p(x_{m_{k+1}}, x_{m_k}) + p(x_{m_k}, x_{n_{k+1}}) \\ &\leq p(x_{m_{k+1}}, x_{m_k}) + p(x_{m_k}, x_{n_k}) + p(x_{n_k}, x_{n_{k+1}}) - p(x_{n_k}, x_{n_k}) \\ &\leq p(x_{m_{k+1}}, x_{m_k}) + p(x_{m_k}, x_{n_k}) + p(x_{n_k}, x_{n_{k+1}}).\end{aligned}$$

Taking the limit as $k \to +\infty$ and using (3.5) and (3.9), we obtain

$$\lim_{k\to+\infty} p(x_{m_{k+1}}, x_{n_{k+1}}) = \epsilon. \tag{3.10}$$

By Lemma 3 and $\alpha(x_{n_k}, x_{m_{k+1}}) \geq 1$, we have

$$\begin{aligned}\psi\left(p(x_{n_{k+1}}, x_{m_{k+2}})\right) &= \psi\left(p(S(x_{n_k}), T(x_{m_{k+1}}))\right) \leq \alpha(x_{n_k}, x_{m_{k+1}})\psi\left(p(S(x_{n_k}), T(x_{m_{k+1}}))\right) \\ &\leq \mathbb{F}\left(\psi\left(M(x_{n_k}, x_{m_{k+1}})\right), \varphi\left(M(x_{n_k}, x_{m_{k+1}})\right)\right).\end{aligned}$$

This implies that $\lim_{k\to\infty} p(x_{n_k}, x_{m_{k+1}}) = 0 < \epsilon$, which is a contradiction. So $\lim_{n,m\to\infty} p(x_n, x_m) = 0$, which implies that $\{x_n\}$ is a Cauchy sequence in $(X, p)$. From (2.1), we obtain that $d_p(x_n, x_m) \leq 2p(x_n, x_m)$. Therefore, $\lim_{n,m\to\infty} d_p(x_n, x_m) = 0$ and thus by Lemma 1, $\{x_n\}$ is a Cauchy sequence in both $(X, p)$ and $(X, d_p)$. Since $(X, p)$ is a complete partial metric space, by Lemma 1, $(X, d_p)$ is also a complete metric space. Thus there exists $\upsilon \in X$ such that $x_n \to \upsilon$, that is, $\lim_{n\to\infty} d_p(x_n, \upsilon) = 0$. Then again from Lemma 1, we get

$$\lim_{n\to\infty} p(\upsilon, x_n) = p(\upsilon, \upsilon) = \lim_{n,m\to\infty} p(x_n, x_m). \tag{3.11}$$

Due to $\lim_{n,m\to\infty} p(x_n, x_m) = 0$, it follows from (3.11) that $p(\upsilon, \upsilon) = 0$ and $\{x_n\}$ converges to $\upsilon$ with respect to $\tau(p)$. Moreover, $x_{2n+1} \to \upsilon$ and $x_{2n+2} \to \upsilon$. Now the continuity of $T$ implies

$$\upsilon = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} x_{2n+2} = \lim_{n\to\infty} T(x_{2n+1}) = T(\lim_{n\to\infty} x_{2n+1}) = T(\upsilon).$$

Analogously, $\upsilon = S(\upsilon)$. Thus we have $S(\upsilon) = T(\upsilon) = \upsilon$. Hence $(S,T)$ have a common fixed point. Now we show that $\upsilon$ is the unique common fixed point of $S$ and $T$. Assume the contrary, that is, there exists $\omega \in X$ such that $\upsilon \neq \omega$ and $\omega = T(\omega)$. From the contractive condition (3.1), we have

$$\psi\left(p(\upsilon,\omega)\right) \leq \mathbb{F}\left(\psi\left(M(\upsilon,\omega)\right), \varphi\left(M(\upsilon,\omega)\right)\right) < \psi\left(M(\upsilon,\omega)\right),$$

but

$$M(\upsilon,\omega) = \max\left\{p(\upsilon,\omega), \frac{p(\upsilon,S(\upsilon))p(\omega,T(\omega))}{1+p(\upsilon,\omega)}, \frac{p(\upsilon,S(\upsilon))p(\omega,T(\omega))}{1+p(S(\upsilon),T(\omega))}\right\}.$$

This implies that

$$M(\upsilon,\omega) = p(\upsilon,\omega).$$

This means that $p(\upsilon,\omega) < p(\upsilon,\omega)$, which is a contradiction and so $p(\upsilon,\omega) = 0$. Consequently, $\upsilon$ is a unique common fixed point of the pair $(S,T)$. □

It is also possible to remove the continuity of the mappings $S$ and $T$ by replacing a weaker condition:
(C) If $\{x_n\}$ is a sequence in $X$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to \upsilon \in X$ as $n \to +\infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, \upsilon) \geq 1$ for all $k$.

**Theorem 2.** *Let $(X,p)$ be a complete partial metric space and $\alpha : X \times X \to \mathbb{R}_0^+$ be a function. Suppose that $S,T : X \to X$ are mappings such that*

(1) *$(S,T)$ is a pair of $(\alpha,\mathbb{F},\psi,\varphi)$-contraction mappings,*
(2) *$(S,T)$ is triangular $\alpha$-admissible,*
(3) *there exists $x_0 \in X$ such that $\alpha(x_0, S(x_0)) \geq 1$,*
(4) *$\alpha(x,y) \geq 1$ for all $x,y \in Fix(T,S)$,*
(5) *(C) holds.*

*Then $(S,T)$ have a unique common fixed point.*

*Proof.* Following the proof of Theorem 1, we know that $x_{2n+1} \to \upsilon$ and $x_{2n+2} \to \upsilon$ as $n \to +\infty$. We only have to show that $\upsilon$ is a common fixed point of the pair $(S,T)$. Due to the hypothesis (4), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{2n_k}, \upsilon) \geq 1$ for all $k$. Now by using (3.1) for all $k$, we have

$$\begin{aligned}\psi\left(p(x_{2n_k+1}, T(\upsilon))\right) &= \psi\left(p(S(x_{2n_k}), T(\upsilon))\right) \leq \alpha(x_{2n_k},\upsilon)\psi\left(p(S(x_{2n_k}), T(\upsilon))\right) \\ &\leq \mathbb{F}\left(\psi\left(M(x_{2n_k},\upsilon)\right), \varphi\left(M(x_{2n_k},\upsilon)\right)\right)\end{aligned}$$

and so

$$\psi\left(p(x_{2n_k+1}, T(\upsilon))\right) \leq \mathbb{F}\left(\psi\left(M(x_{2n_k},\upsilon)\right), \varphi\left(M(x_{2n_k},\upsilon)\right)\right),$$

which implies that

$$p(x_{2n_k+1}, T(\upsilon)) \leq M(x_{2n_k},\upsilon). \tag{3.12}$$

On the other hand, we obtain

$$M(x_{2n_k},\upsilon) = \max\left\{p(x_{2n_k},\upsilon), \frac{p(x_{2n_k},S(x_{2n_k}))p(\upsilon,T(\upsilon))}{1+p(x_{2n_k},\upsilon)}, \frac{p(x_{2n_k},S(x_{2n_k}))p(\upsilon,T(\upsilon))}{1+p(S(x_{2n_k}),T(\upsilon))}\right\}.$$

Letting $k \to \infty$, we have

$$\lim_{k\to\infty} M(x_{2n_k},\upsilon) \leq \max\left\{p(\upsilon,S(\upsilon)), p(\upsilon,T(\upsilon))\right\}. \tag{3.13}$$

**Case I**. Assume that $\lim_{k\to\infty} M(x_{2n_k}, \upsilon) = p(\upsilon, T(\upsilon))$. Suppose that $p(\upsilon, T(\upsilon)) > 0$. Otherwise, the result is obvious. Letting $k \to \infty$ in (3.12), we obtain that $p(\upsilon, T(\upsilon)) < p(\upsilon, T(\upsilon))$, which is a contradiction. Thus we obtain that $p(\upsilon, T(\upsilon)) = 0$. Due to $(PM1)$ and $(PM2)$, we have $\upsilon = T(\upsilon)$.
**Case II**. Assume that $\lim_{k\to\infty} M(x_{2n_k}, \upsilon) = p(\upsilon, S(\upsilon))$. Then arguing as above, we get $\upsilon = S(\upsilon)$. Thus $\upsilon = T(\upsilon) = S(\upsilon)$. □

If we set $T = S$ and $M(x, y) = \max\left\{p(x, y), \frac{p(x, S(x))p(y, S(y))}{1 + p(x, y)}, \frac{p(x, S(x))p(y, S(y))}{1 + p(S(x), S(y))}\right\}$ in Theorems 1 and 2, then we obtain the following results.

**Corollary 1.** *Let $(X, p)$ be a complete partial metric space and $\alpha : X \times X \to \mathbb{R}_0^+$ be a function. Suppose that $S : X \to X$ is a continuous mapping such that*

(1) *$S$ is a $(\alpha, \mathbb{F}, \psi, \varphi)$-contraction mapping,*
(2) *$S$ is triangular $\alpha$-admissible,*
(3) *there exists $x_0 \in X$ such that $\alpha(x_0, S(x_0) \geq 1$,*
(4) *$\alpha(x, y) \geq 1$ for all $x, y \in Fix(S)$.*

*Then $S$ has a unique fixed point $\upsilon \in X$ and $\{S^n(x)\}$ converges to $\upsilon$ for every $x \in X$.*

**Corollary 2.** *Let $(X, p)$ be a complete partial metric space and $\alpha : X \times X \to \mathbb{R}_0^+$ be a function. Suppose that $S$ satisfies the following conditions:*

(1) *$S$ is a $(\alpha, \mathbb{F}, \psi, \varphi)$-contraction mapping,*
(2) *$S$ is triangular $\alpha$-admissible,*
(3) *there exists $x_0 \in X$ such that $\alpha(x_0, S(x_0)) \geq 1$,*
(4) *$\alpha(x, y) \geq 1$ for all $x, y \in Fix(S)$,*
(5) *(C) holds.*

*Then $S$ has a unique fixed point $\upsilon \in X$ and $\{S^n(x)\}$ converges to $\upsilon$ for every $x \in X$.*

**Remark 1.** *For a partial metric space $(X, p)$, we have the following observations:*

(1) *If we set $p(x, x) = 0$ and $\mathbb{F}(x, y) = \beta(x)x$ for all $x, y \in X$ in Corollaries 1 and 2, then we obtain the results presented by Chandok* [4].

(2) *If we set $M(x, y) = \max\{p(x, y), p(x, S(x)), p(y, S(y))\}$, $p(x, x) = 0$ and $\mathbb{F}(x, y) = \beta(x)x$ for all $x, y \in X$ in Theorems 1 and 2, then the results presented by Cho et al.* [5] *can be viewed as particular cases of Theorems 1 and 2.*

## 4. Consequences

The following corollaries shall support our claim that Theorem 1 is a generalized version of many corresponding results and shorten the proofs of many results presented in the literature.

The results established in [14] can be viewed as particular cases of Corollary 3.

**Corollary 3.** ([14]) *Let $(X, p)$ be a complete partial metric space and $\alpha : X \times X \to \mathbb{R}_0^+$ be a function. Let $S, T : X \to X$ be a pair of self-mappings such that*

(1) *$(S, T)$ is a pair of Geraghty type contraction mappings,*
(2) *$(S, T)$ is triangular $\alpha$-admissible,*

(3) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, S(x_0)) \geq 1$,
(4) $\alpha(x, y) \geq 1$ *for all* $x, y \in Fix(T, S)$,
(5) *either* $S, T$ *are continuous or the condition (C) holds.*

*Then* $(S, T)$ *have a unique common fixed point* $\upsilon$ *in* $X$ *with* $p(\upsilon, \upsilon) = 0$.

*Proof.* Setting $\mathbb{F}(x, y) = x\beta(x)$, $\psi(t) = t$, $\varphi(t) = t$ in Theorem 1, we obtain the required result. □

**Corollary 4.** ([14]) *Let* $(X, p)$ *be a complete partial metric space and* $\alpha : X \times X \to \mathbb{R}_0^+$ *be a function. Let* $S, T : X \to X$ *be a pair of self-mappings such that*

(1) *the pair* $(S, T)$ *satisfies*

$$\alpha(x, y)p(S(x), T(y)) \leq \kappa M(x, y) \text{ where } \kappa \in (0, 1),$$

(2) $(S, T)$ *is triangular* $\alpha$*-admissible,*
(3) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, S(x_0)) \geq 1$,
(4) $\alpha(x, y) \geq 1$ *for all* $x, y \in Fix(T, S)$,
(5) *either* $S, T$ *are continuous or the condition (C) holds.*

*Then* $(S, T)$ *have a unique common fixed point* $\upsilon$ *in* $X$ *with* $p(\upsilon, \upsilon) = 0$.

*Proof.* Setting $\mathbb{F}(x, y) = \kappa x$, $\psi(t) = t$, $\varphi(t) = t$ in Theorem 1, we obtain the required result. □

Corollary 5 generalizes the results proved in [13].

**Corollary 5.** ([13]) *Let* $(X, p)$ *be a complete partial metric space and* $\alpha : X \times X \to \mathbb{R}_0^+$ *be a function. Let* $S, T : X \to X$ *be a pair of self-mappings such that*

(1) *the pair* $(S, T)$ *satisfies*

$$\alpha(x, y)\psi(p(S(x), T(y))) \leq \psi(M(x, y)) - \varphi(M(x, y)),$$

(2) $(S, T)$ *is triangular* $\alpha$*-admissible,*
(3) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, S(x_0)) \geq 1$,
(4) $\alpha(x, y) \geq 1$ *for all* $x, y \in Fix(T, S)$,
(5) *either* $S, T$ *are continuous or thecondition (C) holds.*

*Then* $(S, T)$ *have a unique common fixed point* $\upsilon$ *in* $X$ *with* $p(\upsilon, \upsilon) = 0$.

*Proof.* Setting $\mathbb{F}(x, y) = x - y$ in Theorem 1, we obtain the required result. □

To illustrate the results proved in this paper and to show the superiority of a pair of $(\alpha, \mathbb{F}, \psi, \varphi)$-contraction mappings than the contractions used in [4, 5], we present the following example.

**Example 2.** Let $X = \{1, 2, 3\}$. Define $p : X \times X \to \mathbb{R}_0^+$ by

$$\begin{aligned} p(1,3) &= p(3,1) = \frac{5}{7}, p(1,1) = \frac{1}{10}, p(2,2) = \frac{2}{10}, p(3,3) = \frac{3}{10}, \\ p(1,2) &= p(2,1) = \frac{3}{7}, p(2,3) = p(3,2) = \frac{4}{7}. \end{aligned}$$

It is easy to check that $p$ is a partial metric and define $\alpha : X \times X \to \mathbb{R}_0^+$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in X; \\ 0 & \text{if otherwise.} \end{cases}$$

Define the mappings $S, T : X \to X$ as follows:

$$\begin{aligned} S(x) &= 1 \text{ for each } x \in X, \\ T(1) &= T(3) = 1,\ T(2) = 3. \end{aligned}$$

In addition, define $\mathbb{F}(x, y) = \beta(x)x$ for all $x \in X$, where $\beta : \mathbb{R}_0^+ \to [0, 1)$ defined by $\beta(M(x, y)) = \frac{9}{10}$ for all $x, y \in X$. Note that $S(x)$ and $T(x)$ belong to $X$ and are continuous. The pair $(S, T)$ is $\alpha$-admissible. Indeed, $\alpha(x, y) = 1$ implies $\alpha(S(x), T(y)) = 1$. We shall show that the condition (3.1) in Theorem 2 is satisfied. If $x = 2$, $y = 3$, then $\alpha(2, 3) = 1$ and

$$\begin{aligned} M(2,3) &= \max\left\{p(2,3), \frac{p(2,S(2))p(3,T(3))}{1+p(2,3)}, \frac{p(2,S(2))p(3,T(3))}{1+p(S(2),T(3))}\right\} \\ &= \max\left\{\frac{4}{7}, \frac{9}{20}, \frac{9}{14}\right\} = \frac{9}{14}, \end{aligned}$$

$p(S(2), T(3)) = p(1, 1) = \frac{1}{10}$. Now

$$\frac{1}{10} = \alpha(2,3)p(S(2),T(3)) \leq \beta(M(2,3))M(2,3) = \frac{81}{140}$$

holds.

Similarly, for other cases ($x = 1$, $y = 3$ and $x = 2$, $y = 1$), it is easy to check that the contractive condition (3.1) in Theorem 1 is satisfied. Consequently, all the conditions (1-4) of Theorem 1 are satisfied. Hence $(S, T)$ have a unique common fixed point ($x = 1$). Nevertheless, the contractive condition (3) in [5] does not hold for this particular case. Indeed, for $x = 2$, $y = 3$,

$$\begin{aligned} M(2,3) &= \max\{d(2,3), d(2,T(2)), d(3,T(3))\} \\ &= \max\left\{\frac{4}{7}, \frac{4}{7}, \frac{5}{7}\right\} = \frac{5}{7}, \end{aligned}$$

$$\alpha(2,3)d(T(2),T(3)) = \frac{5}{7} \not\leq \frac{9}{14} = \beta(M(2,3))M(2,3).$$

Similarly, the contractive condition (2.1) in [4] does not hold for this particular case. Indeed, for $x = 2$, $y = 3$ and $\psi(t) = t$,

$$\begin{aligned} M(2,3) &= \max\left\{d(2,3), d(2,T(2)), d(3,T(3)), \frac{d(2,T(2))d(3,T(3))}{1+d(2,3)}, \frac{d(2,T(2))d(3,T(3))}{1+d(T2,T3)}\right\} \\ &= \max\left\{\frac{4}{7}, \frac{4}{7}, \frac{5}{7}, \frac{20}{77}, \frac{5}{21}\right\} = \frac{5}{7}, \end{aligned}$$

$$\alpha(2,3)\psi(d(T(2),T(3))) = \frac{5}{7} \not\leq \frac{9}{14} = \beta(\psi(M(2,3)))\psi(M(2,3)).$$

Here we have assumed that $p(x, y) = d(x, y)$ for all $x, y \in X$ such that $x \neq y$.

## 5. Application to system of integral equations

In this section, we shall apply Theorem 1 to show the existence of solution of a pair of simultaneous Volterra-Hammerstein integral equations

$$x(t) = f(t) + \lambda \int_0^1 K(t,s)F_n(s,x(s))\,ds, \tag{5.1}$$

$$y(t) = f(t) + \lambda \int_0^1 K(t,s)G_n(s,y(s))\,ds \tag{5.2}$$

for all $t \in [0,1]$, where $f(t)$ is known, $K(t,s)$, $F_n(s,x(s))$ and $G_n(s,y(s))$ are real-valued functions that are measurable both in $t$ and $s$ on $[0,1]$, and $\lambda$ is a real number.

Let $X = L^1([0,1],\mathbb{R})$ and $p(x,y) = d(x,y) + c_n$ for all $x, y \in X$, where

$$d(x,y) = \|x(s) - y(s)\|_X = \int_0^1 |x(s) - y(s)|\ ds$$

and $\{c_n\}$ is a sequence of positive real numbers satisfying $c_n \to 0$ as $n \to \infty$. It is easy to verify that $(X,p)$ is a complete partial metric space. We define $\mathbb{F} : [0,\infty)^2 \to \mathbb{R}$ by $\mathbb{F}(x,y) = \beta(x)x$ for all $x \in X$ and $\psi(t) = t$.

Let $\Theta$ represent the class of functions $\phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with the following properties

(1) $\phi$ is increasing,
(2) For each $t > 0$, $\phi(t) < t$,
(3) $\int_0^1 \phi(t)\,dt \le \phi\left(\int_0^1 t\,dt\right)$,
(4) $\beta(t) = \frac{\phi(t)}{t} \in S$.

For examples, $\phi(t) = \frac{1}{5}t$, $\phi(t) = \frac{t}{1+t}$ are elements of $\Theta$.

Now we present the main result regarding application of Theorem 1.

**Theorem 3.** *Assume that the following hypotheses are satisfied:*

$(C_1)$

$$\int_0^1 \sup_{0\le s\le 1} |K(t,s)|\ dt = R_1 < +\infty.$$

$(C_2)$ $F, G \in L^1[0,1]$ *are such that, for all* $s \in [0,1]$ *and* $x, y \in L^1[0,1]$,

$$|F_n(s,x(s)) - G_n(s,y(s))| \le \phi(x(s) - y(s)),\ \text{as } n \to \infty.$$

*Then the system of integral equations* (5.1) *and* (5.2) *has a solution for each* $\lambda$ *with* $\lambda R_1 < 1$.

*Proof.* We define the operators, for all $x, y \in X$,

$$Sx(t) = f(t) + \lambda \int_0^1 K(t,s)F_n(s,x(s))\,ds,$$

$$Ty(t) = f(t) + \lambda \int_0^1 K(t,s)G_n(s,y(s))\,ds.$$

Then $S$ and $T$ are operators from $X$ into itself. Indeed, we have

$$\begin{aligned} |Sx| &\leq |f(t)| + |\lambda| \int_0^1 |K(t,s)F_n(s,x(s))|\, ds \\ &\leq |f(t)| + |\lambda| \sup_{0\leq s\leq 1} |K(t,s)| \int_0^1 |F_n(s,x(s))|\, ds. \end{aligned}$$

By the assumptions $(C_1)$ and $(C_2)$, we obtain

$$\int_0^1 |Sx|\, dt \leq |\lambda| \int_0^1 \sup_{0\leq s\leq 1} |K(t,s)|\, dt \int_0^1 |F_n(s,x(s))|\, ds + \int_0^1 |f(t)|\, dt < +\infty.$$

This implies that $Sx \in X$.

Similarly $Ty \in X$.

Now consider

$$\begin{aligned} p(Sx,Ty) &= d(Sx,Ty) + c_n \\ &= \|Sx - Ty\| + c_n \\ &= \int_0^1 |Sx(t) - Ty(t)|\, dt + c_n \\ &= \int_0^1 \left| \lambda \int_0^1 K(t,s)F_n(s,x(s))\, ds - \lambda \int_0^1 K(t,s)G_n(s,y(s))\, ds \right| dt + c_n \\ &= \int_0^1 \left| \lambda \int_0^1 K(t,s) \left[F_n(s,x(s)) - G_n(s,y(s))\right] ds \right| dt + c_n \\ &\leq |\lambda| \int_0^1 \sup_{0\leq s\leq 1} |K(t,s)|\, dt \int_0^1 |F_n(s,x(s)) - G_n(s,y(s))|\, ds + c_n \end{aligned}$$

for all $x, y \in X$.

Letting $n \to \infty$, we get

$$\begin{aligned} p(Sx,Ty) &\leq |\lambda| R_1 \int_0^1 \phi\left(|x(s) - y(s)|\right) ds \\ &\leq |\lambda| R_1 \phi\left(d(x,y)\right) < \phi\left(d(x,y)\right) \leq \phi\left(p(x,y)\right). \end{aligned}$$

Thus

$$\begin{aligned} p(Sx,Ty) &\leq \phi\left(p(x,y)\right) \leq \phi\left(M(x,y)\right) = \frac{\phi\left(M(x,y)\right)}{M(x,y)} M(x,y), \\ p(Sx,Ty) &\leq \beta\left(M(x,y)\right) M(x,y) \end{aligned}$$

for all $x, y \in X$.

Finally, we define $\alpha : X \times X \to \mathbb{R}_0^+$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x,y \in X; \\ 0 & \text{otherwise.} \end{cases}$$

Hence, for all $x, y \in X$, we have

$$\alpha(x,y)\psi\left(p(S(x),T(y))\right) \leq \mathbb{F}\left(\psi\left(M(x,y)\right), \varphi\left(M(x,y)\right)\right).$$

Apparently, $\alpha(x,y) = 1$ and $\alpha(y,z) = 1$ imply $\alpha(x,z) = 1$ for all $x, y, z \in X$. Moreover, $\alpha(x,y) = 1$ implies $\alpha(S(x),T(y)) = 1$ and $\alpha(T(x),S(y)) = 1$ and so $(S,T)$ is a triangular $\alpha$-admissible pair of

mappings. Hence all the hypotheses of Theorem 1 are satisfied. Consequently, the mappings $S$ and $T$ have a common fixed point which is the solution of system of integral equations (5.1) and (5.2). □

## 6. Conclusion

This paper presents a common fixed point theorem for a pair of generalized $(\alpha, \mathbb{F}, \psi, \varphi)$-contraction mappings. The presented theorem not only generalizes and improvea many new and classical results in fixed point theory but also proves a short method to show the existence of fixed points. The authors believe that the use of abscissa dominating function to find fixed points of various contraction mappings makes significant and useful contribution in the existing literature.

## References

[1] T. Abdeljawad, *Meir-Keeler $\alpha$-contractive fixed and common fixed point theorems*, Fixed Point Theory Appl. **2013**, 2013:100.
[2] T. Abdeljawad, E. Karapnar, K. Tas, *Existence and uniqueness of a common fixed point on partial metric spaces,* Appl. Math. Lett. **24** (2011), 1900–1904.
[3] I. Altun, F. Sola, H. Simsek, *Generalized contractions on partial metric spaces,* Topology Appl. **157** (2010), 2778–2785.
[4] S. Chandok, *Some fixed point theorems for $(\alpha, \beta)$-admissible Geraghty type contractive mappings and related results*, Math. Sci. **9** (2015), 127–135.
[5] S. H. Cho, S. Bae, E. Karapinar *Fixed point theorems for $\alpha$-Geraghty contraction type maps in metric spaces*, Fixed Point Theory Appl. **2013**, 2013:329.
[6] D. M. Erhan, E. Karapinar, D. T́urkoğlu, *Different types Meir-Keeler contractions on partial metric spaces,* J. Comput. Anal. Appl. **14** (2012), 1000-1005.
[7] M. Geraghty, *On contractive mappings*, Proc. Am. Math. Soc. **40** (1973), 604–608.
[8] N. Hussain, E. Karapinar, P. Salimi, F. Akbar, *$\alpha$-admissible mappings and related fixed point theorems*, J. Inequal. Appl. **2013**, 2013:114.
[9] N. Hussain, P. Salimi, A. Latif, *Fixed point results for single and set-valued $\alpha$-$\eta$-$\psi$-contractive mappings*, Fixed Point Theory Appl. **2013**, 2013:212.
[10] E, Karapinar, P. Kumam, P. Salimi, *On $\alpha$-$\psi$-Meir-Keeler contractive mappings*, Fixed Point Theory Appl. **2013**, 2013:94.
[11] E. Karapinar, B. Samet, *Generalized $\alpha$-$\psi$-contractive type mappings and related fixed point theorems with applications*, Abstr. Appl. Anal. **2012** (2012), Article ID 793486.
[12] S. G. Matthews, *Partial metric topology*, in Proceedings of the $11^th$ Summer Conference on General Topology and Applications **728**, pp. 183-197, The New York Academy of Sciences, 1995.
[13] S. Moradi, A.Farajzadeh *On the fixed point of $(\psi - \varphi)$-weak and generalized $(\psi - \varphi)$-weak contraction mappings* Appl. Math. Lett. **25** (2012), 1257–1262
[14] V. L. Rosa, P. Vetro. *Fixed points for Geraghty-contractions in partial metric spaces* J. Nonlinear Sci. Appl. **7** (2014), 1–10.
[15] B. Samet, C. Vetro, P. Vetro, *Fixed point theorems for $\alpha$-$\psi$-contractive type mappings*, Nonlinear Anal. **75** (2012), 2154–2165.

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# Inner-outer factorization on the Nevanlinna space in a strip

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**Abstract** In this paper, we prove a famous harmonic majorant lemma, and by applying this lemma to $\log^+|f|$, we claim that $\log^+|f|$ has indeed a harmonic majorant for every function $f$ in the Nevanlinna space in a strip instead of the usual assumption on $\log^+|f|$ having a harmonic majorant in the same setting. By using the conformal mapping from a strip onto the unit disk and the inner-outer factorization theorem on the Nevanlinna space in the unit disk, we obtain an inner-outer factorization theorem on the Nevanlinna space in such a strip.

**Key words** Nevanlinna space; strip; inner-outer factorization; harmonic majorant

## 1 Introduction

Let $\mathbb{C}$ be the complex plane. We denote the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ by $\mathbb{U}$ and its boundary by $\partial\mathbb{U}$. Let $H(\mathbb{U})$ be the space of all holomorphic functions in $\mathbb{U}$. For $0 < p < \infty$, the Hardy space $H^p(\mathbb{U})$ (see [1–3]) is the set of $f \in H(\mathbb{U})$ for which

$$\|f\|_{H^p}^p = \sup_{0<r<1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

The Nevanlinna space $H^0(\mathbb{U})$ (see [1–3]) is the set of $f \in H(\mathbb{U})$ for which

$$\|f\|_{H^0} = \sup_{0<r<1} \exp\left\{\frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})| d\theta\right\} < \infty.$$

Let $e^{i\theta_0}$ be a point of $\partial\mathbb{U}$. We write $\lim_{z\to e^{i\theta_0}} f(z) = A$ nontangentially if for every open triangular sector $D$ in $\mathbb{U}$ with vertex at $e^{i\theta_0}$, $f(z) \to A$ as $z \to e^{i\theta_0}$ within $D$.

For a sequence $\{z_n\}$ in $\mathbb{U}$ satisfying $\sum_n (1 - |z_n|) < \infty$, the following function

$$B(z) = z^k \prod_n \frac{|z_n|(z_n - z)}{z_n(1 - z\overline{z_n})} \tag{1.1}$$

is called a Blaschke product, where $k$ is a nonnegative integer. Note that $\{z_n\}$ may be finite, or even empty. If $\{z_n\}$ is empty, then denote $B(z) = z^k$.

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A function $g \in H(\mathbb{U})$ is said to be an inner function if it is bounded and has nontangential limit whose modulus is 1 almost everywhere on $\partial\mathbb{U}$. The following function

$$S(z) = \exp\left\{-\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}d\mu(t)\right\} \tag{1.2}$$

is called a singular inner function, where $\mu(t)$ is a bounded nondecreasing singular function. One can show that every inner function has a factorization $e^{i\gamma}B(z)S(z)$, where $B(z)$ is from (1.1), $S(z)$ is from (1.2). For more details on inner function, we refer to [4–8].

A function $h \in H(\mathbb{U})$ is called an outer function if there exists a positive function $\varphi$ with $\log\varphi \in L^1(\partial\mathbb{U})$ and a complex number $c$ with $|c| = 1$ such that

$$h(z) = c\exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}\log\varphi(e^{it})dt\right\}.$$

Let $S_a = \{x+iy : x \in \mathbb{R}, 0 < y < a\}(a > 0)$ be a strip in $\mathbb{C}$. We denote its boundary by $\partial S_a = L_0 \bigcup L_a$, where $L_b = \{t+ib : t \in \mathbb{R}\}$ $(b \in \mathbb{R})$. Let $H(S_a)$ be the space of all analytic functions in $S_a$. The Nevanlinna space $H^0(S_a)$ is the set of $f \in H(S_a)$ with

$$\|f\|_{H^0} = \sup_{0<y<a}\exp\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\log^+|f(x+iy)|dx\right\} < \infty.$$

If $\alpha > 0$ and $z_0 = x_0 + iy_0 \in \partial S_a$, then the angular domain in $S_a$ with vertex $z_0$ and aperture $\alpha > 0$ is the region

$$\Gamma_\alpha(z_0) = \{x+iy \in S_a : |x-x_0| < \alpha|y-y_0|\}.$$

Let $\mathscr{M}(\mathbb{R})$ be the set of finite complex valued Borel measures, then $\mathscr{M}(\mathbb{R})$ is a Banach space with the norm $\|\mu\|_{\mathscr{M}} = \int_{\mathbb{R}} d|\mu|(x)$, where $|\mu|$ is the total variation of $\mu \in \mathscr{M}(\mathbb{R})$. Moreover, by Riesz representation theorem, $\mathscr{M}(\mathbb{R})$ is the dual space of $C_0(\mathbb{R})$ in the sense of isomorphism (see [9]).

It is well known that every $f \in H^p(\mathbb{U})$ $(p > 0)$ has a unique canonical factorization $f(z) = B(z)S(z)F(z)$, where $B(z)$ is a Blaschke product, $S(z)$ is a singular inner function, and $F(z)$ is an outer function. Motivated by this result, inner-outer factorization of analytic functions in some other spaces were studied (see $\mathcal{Q}_p$ spaces [10], Besov-type spaces [11, 12]). However, there is a sharp structural difference between functions in the Hardy space and that in the Nevanlinna space. In factoring functions in the Nevanlinna space $H^0(\mathbb{U})$, the singular factor is replaced by a quotient of two singular inner functions. That is the following theorem, which can be found in [1, 3].

**Theorem A** *If $f \in H^0(\mathbb{U})$, $f \not\equiv 0$, then $f^*(e^{i\theta}) = \lim\limits_{z\to e^{i\theta}} f(z)$ exists nontangentially at almost every $\theta \in [-\pi, \pi)$ and $\log|f^*(e^{i\theta})| \in L^1([-\pi, \pi))$. Moreover, $f(z)$ can be written by*

$$f(z) = cB(z)G(z)S(z),$$

*where $c$ is a constant with $|c| = 1$. $B(z)$ is the Blaschke product of the form (1.1), where $\{z_n\}$ are the zeros of $f(z)$; and*

$$G(z) = \exp\Big\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}\log|f^*(e^{it})|dt\Big\}$$

*is an outer function in $\mathbb{U}$; and*

$$S(z) = \exp\Big\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}d\mu_s(t)\Big\}$$

*is a quotient of two singular inner functions in $\mathbb{U}$, where $\mu_s$ is a singular signed measure on $[-\pi,\pi)$ with finite total variation.*

Inner-outer factorization on the Nevanlinna space in other domains were studied (see [3, 13]). However, most of the results on the unit disk or other domains were obtained on the premise that $\log^+|f|$ had a harmonic majorant (see [1, 13, 14]). In this paper, we prove that $\log^+|f|$ has a harmonic majorant indeed for every function $f$ in the Nevanlinna space in a strip. Based on this fact, we obtain the existence of nontangential limits of $f \in H^0(S_a)$ as follows:

**Theorem 1.1** *If $f \in H^0(S_a)$ and $f(z) \not\equiv 0$, then*

$$f^*(t) = \lim_{\substack{z\to t\\ z\in\Gamma_\alpha(t)}} f(z),\ f^*(t+ia) = \lim_{\substack{z\to t+ia\\ z\in\Gamma_\alpha(t+ia)}} f(z) \tag{1.3}$$

*exist nontangentially at almost every $t \in \mathbb{R}$, and*

$$\int_{-\infty}^{\infty}\frac{\big|\log|f^*(t)|\big| + \big|\log|f^*(t+ia)|\big|}{\cosh\frac{\pi}{a}t}dt < \infty. \tag{1.4}$$

*Further, $f^*(t),\ f^*(t+ia) \neq 0$ at almost every $t \in \mathbb{R}$.*

Next, we give a similar factorization on $H^0(S_a)$ as Theorem A, which is also called inner-outer factorization.

**Theorem 1.2** *Let $f \in H^0(S_a)$, $f \not\equiv 0$, then the zeros $\{z_n\}$ of $f$ satisfy*

$$\sum_n \frac{e^{\frac{\pi}{a}x_n}\sin\frac{\pi}{a}y_n}{1+e^{\frac{2\pi}{a}x_n}} < \infty,\ z_n = x_n + iy_n, \tag{1.5}$$

*and there exist two singular signed measures $\mu_{1,s}$ on $L_0$ and $\mu_{2,s}$ on $L_a$ such that*

$$\int_{\mathbb{R}}\frac{\big|\log|f^*(t)|\big|}{\cosh\frac{\pi}{a}t}dt + \int_{\mathbb{R}}\frac{\big|\log|f^*(t+ia)|\big|}{\cosh\frac{\pi}{a}t}dt + \int_{\mathbb{R}}\frac{d|\mu_{1,s}|(t)}{\cosh\frac{\pi}{a}t} + \int_{\mathbb{R}}\frac{d|\mu_{2,s}|(t)}{\cosh\frac{\pi}{a}t} < \infty. \tag{1.6}$$

*Moreover, $f$ can be written by*

$$f(z) = cG(z)S(z)B(z),$$

*where $c$ is a complex constant with $|c| = 1$,*

$$\begin{aligned} G(z) =& \exp\left\{ \frac{1}{ai}\int_{-\infty}^{\infty}\left(\frac{e^{\frac{\pi}{a}t}}{e^{\frac{\pi}{a}t}-e^{\frac{\pi}{a}z}}-\frac{e^{\frac{2\pi}{a}t}}{1+e^{\frac{\pi}{a}t}}\right)\log|f^*(t)|dt \right. \\ & \left. -\frac{1}{ai}\int_{-\infty}^{\infty}\left(\frac{e^{\frac{\pi}{a}t}}{e^{\frac{\pi}{a}t}+e^{\frac{\pi}{a}z}}-\frac{e^{\frac{2\pi}{a}t}}{1+e^{\frac{2\pi}{a}t}}\right)\log|f^*(t+ia)|dt\right\}; \end{aligned}$$

*and*

$$\begin{aligned} S(z) =& \exp\left\{ i\tau_1 e^{-\frac{\pi}{a}z} - i\tau_2 e^{\frac{\pi}{a}z} + \frac{1}{ai}\int_{-\infty}^{\infty}\left(\frac{e^{\frac{\pi}{a}t}}{e^{\frac{\pi}{a}t}-e^{\frac{\pi}{a}z}}-\frac{e^{\frac{2\pi}{a}t}}{1+e^{\frac{2\pi}{a}t}}\right)d\mu_{1,s}(t) \right. \\ & \left. -\frac{1}{ai}\int_{-\infty}^{\infty}\left(\frac{e^{\frac{\pi}{a}t}}{e^{\frac{\pi}{a}t}+e^{\frac{\pi}{a}z}}-\frac{e^{\frac{2\pi}{a}t}}{1+e^{\frac{2\pi}{a}t}}\right)d\mu_{2,s}(t)\right\}, \end{aligned}$$

*where $\tau_1, \tau_2$ are real numbers; and*

$$B(z) = \left(\frac{e^{\frac{\pi}{a}z}-i}{e^{\frac{\pi}{a}z}+i}\right)^k \prod_n \frac{e^{\frac{\pi}{a}z}-e^{\frac{\pi}{a}z_n}}{e^{\frac{\pi}{a}z}-e^{\frac{\pi}{a}\bar{z}_n}} e^{i\theta(z_n)},$$

*where $k$ is a nonnegative integer and*

$$e^{i\theta(z_n)} = \frac{(e^{\frac{\pi}{a}\bar{z}_n}+i)(e^{\frac{\pi}{a}\bar{z}_n}-i)}{|e^{\frac{\pi}{a}z_n}+i||e^{\frac{\pi}{a}z_n}-i|}.$$

# 2 Proofs of main results

In this section, we will give the proofs of our main results in Section 1. To this end, we need the following lemmas.

**Lemma 2.1** *If $v(z)$ is subharmonic in $S_a$ and it satisfies*

$$C = \sup_{0<y<a}\int_{-\infty}^{\infty}|v(x+iy)|dx < \infty, \tag{2.1}$$

*then*

$$v(x+iy) \le \frac{2C}{\pi\min\{y, a-y\}}.$$

**Proof** Let $z = x + iy \in S_a$, $\rho = \min\{y, a-y\}$, $D(z,\rho) = \{w : |w-z| < \rho\}$. By the mean property, it establishes that

$$\begin{aligned} v(z) &\le \int_{D(z,\rho)} v(\zeta)\frac{d\lambda(\zeta)}{\pi\rho^2} \le \int_{D(z,\rho)} |v(\zeta)|\frac{d\lambda(\zeta)}{\pi\rho^2} \\ &\le \frac{1}{\pi\rho^2}\int_{y-\rho}^{y+\rho}\int_{\mathbb{R}} |v(\xi+i\eta)|d\xi d\eta \\ &\le \frac{2C}{\pi\rho} = \frac{2C}{\pi\min\{y, a-y\}}. \end{aligned}$$

□

**Lemma 2.2** *Let*

$$P_y^{\pm}(x) = \frac{\sin\frac{\pi}{a}y}{2a(\cosh\frac{\pi}{a}x \mp \cos\frac{\pi}{a}y)}$$

*and* $P_y(x) = P_y^+(x) + P_y^-(x)$. *Then*

(1) $P(x,y) = P_y(x)$ *is harmonic in* $S_a$;

(2) $P_y(x) > 0, x + iy \in S_a$;

(3) $\int_{\mathbb{R}} P_y^{\pm}(x)dx = \frac{1}{2} \pm (\frac{1}{2} - \frac{y}{a}), \int_{\mathbb{R}} P_y(x)dx = 1$;

(4) $\int_{|x|>\delta} P_y(x)dx \to 0$ $(y \to 0)$ *and* $\int_{|x|>\delta} P_y(x)dx \to 0$ $(y \to a)$, *where* $\delta > 0$ *is a constant.*

**Proof** The proofs of these facts follow from the following relations (see [15]):

*A.* $P_y^{\pm}(x) = \frac{1}{a}\mathrm{Im}(\pm 1 - e^{\frac{\pi}{a}z})^{-1}$;

*B.* $-\cosh\frac{\pi}{a}x < \cos\frac{\pi}{a}y < \cosh\frac{\pi}{a}x$;

*C.* $\frac{d}{dx}(\frac{1}{\pi}\arctan(\tan\frac{\pi}{2a}y \cdot \tanh\frac{\pi}{2a}x)) = P_y^-(x),\ P_y^+(x) = P_{a-y}^-(x)$;

*D.* $\cos\frac{\pi}{a}y \in (-1,1)$, $e^{-\frac{\pi}{a}|x|}(\cosh\frac{\pi}{a}x - 1)$ is even and increasing for $\delta \le x < +\infty$, which implies that

$$|P_y^{\pm}(x)| \le \frac{e^{-\frac{\pi}{a}|x|+\frac{\pi}{a}\delta}\sin\frac{\pi}{a}y}{2a(\cosh\frac{\pi}{a}\delta - 1)} \quad (|x| \ge \delta).$$

□

**Lemma 2.3** *Let* $\tilde{P}_y^{\pm}(x) = \frac{\sin\frac{\pi}{a-2y_0}y}{2(a-2y_0)(\cosh\frac{\pi}{a-2y_0}x \mp \cos\frac{\pi}{a-2y_0}y)}$, $0 < y_0 < \frac{a}{2}$, *then there exist* $A_1, A_2 > 0$ *depending on* $y_0$ *and* $y$, *such that*

$$\begin{aligned} |P_y^+(x) - \tilde{P}_{y-y_0}^+(x)| \le & \frac{1}{1-\cos\frac{\pi}{a}y}\left|\frac{1}{2a} - \frac{1}{2(a-2y_0)}\right| + A_1\left|\frac{\pi}{a-2y_0} - \frac{\pi}{a}\right| \\ & + \frac{|\frac{\pi}{a}y - \frac{\pi}{a-2y_0}(y-y_0)|}{2(a-2y_0)(1-\cos\frac{\pi}{a}y)} \\ & + \frac{|\frac{\pi}{a}y - \frac{\pi}{a-2y_0}(y-y_0)|}{2(a-2y_0)(1-\cos\frac{\pi}{a}y)(1-\cos\frac{\pi}{a-2y_0}(y-y_0))} \end{aligned} \tag{2.2}$$

*and*

$$\begin{aligned}|P_y^-(x)-\tilde{P}_{y-y_0}^-(x)|\leq&\frac{1}{1+\cos\frac{\pi}{a}y}\left|\frac{1}{2a}-\frac{1}{2(a-2y_0)}\right|+A_2\left|\frac{\pi}{a-2y_0}-\frac{\pi}{a}\right|\\&+\frac{|\frac{\pi}{a}y-\frac{\pi}{a-2y_0}(y-y_0)|}{2(a-2y_0)(1+\cos\frac{\pi}{a}y)}\\&+\frac{|\frac{\pi}{a}y-\frac{\pi}{a-2y_0}(y-y_0)|}{2(a-2y_0)(1+\cos\frac{\pi}{a}y)(1+\cos\frac{\pi}{a-2y_0}(y-y_0))}\end{aligned}\tag{2.3}$$

*hold for every* $y_0<y<a-y_0$.

**Proof** Firstly, we have

$$\begin{aligned}P_y^+(x)-\tilde{P}_{y-y_0}^+(x)=&\frac{\sin\frac{\pi}{a}y}{2a(\cosh\frac{\pi}{a}x-\cos\frac{\pi}{a}y)}-\frac{\sin\frac{\pi}{a-2y_0}(y-y_0)}{2(a-2y_0)(\cosh\frac{\pi}{a-2y_0}x-\cos\frac{\pi}{a-2y_0}(y-y_0))}\\=&I_1+I_2+I_3+I_4,\end{aligned}$$

where

$$\begin{aligned}I_1=&\left[\frac{1}{2a}-\frac{1}{2(a-2y_0)}\right]\frac{\sin\frac{\pi}{a}y}{\cosh\frac{\pi}{a}x-\cos\frac{\pi}{a}y},\\I_2=&\frac{\sin\frac{\pi}{a}y-\sin\frac{\pi}{a-2y_0}(y-y_0)}{2(a-2y_0)(\cosh\frac{\pi}{a}x-\cos\frac{\pi}{a}y)},\\I_3=&\frac{\sin\frac{\pi}{a-2y_0}(y-y_0)}{2(a-2y_0)}\frac{\cos\frac{\pi}{a}y-\cos\frac{\pi}{a-2y_0}(y-y_0)}{(\cosh\frac{\pi}{a}x-\cos\frac{\pi}{a}y)(\cosh\frac{\pi}{a-2y_0}x-\cos\frac{\pi}{a-2y_0}(y-y_0))},\\I_4=&\frac{\sin\frac{\pi}{a-2y_0}(y-y_0)}{2(a-2y_0)}\frac{\cosh\frac{\pi}{a-2y_0}x-\cosh\frac{\pi}{a}x}{(\cosh\frac{\pi}{a}x-\cos\frac{\pi}{a}y)(\cosh\frac{\pi}{a-2y_0}x-\cos\frac{\pi}{a-2y_0}(y-y_0))}.\end{aligned}$$

Obviously,

$$|I_1|\leq\frac{1}{1-\cos\frac{\pi}{a}y}\left|\frac{1}{2a}-\frac{1}{2(a-2y_0)}\right|.$$

By mean value theorem of differentials, it is easy to see that

$$\begin{aligned}|I_2|\leq&\frac{|\frac{\pi}{a}y-\frac{\pi}{a-2y_0}(y-y_0)|}{2(a-2y_0)(1-\cos\frac{\pi}{a}y)},\\|I_3|\leq&\frac{|\frac{\pi}{a}y-\frac{\pi}{a-2y_0}(y-y_0)|}{2(a-2y_0)(1-\cos\frac{\pi}{a}y)(1-\cos\frac{\pi}{a-2y_0}(y-y_0))},\\I_4=&\frac{\sin\frac{\pi}{a-2y_0}(y-y_0)}{2(a-2y_0)}\frac{(\frac{\pi}{a-2y_0}-\frac{\pi}{a})x\cdot\sinh\xi}{(\cosh\frac{\pi}{a}x-\cos\frac{\pi}{a}y)(\cosh\frac{\pi}{a-2y_0}x-\cos\frac{\pi}{a-2y_0}(y-y_0))},\end{aligned}$$

where $\xi$ is between $\frac{\pi}{a}x$ and $\frac{\pi}{a-2y_0}x$. Note that there exists $M > 0$ such that $|x|e^{-\frac{\pi}{a}|x|} \leq \frac{1}{4}$, $\cosh\frac{\pi}{a}x \geq 2$ and $\cosh\frac{\pi}{a-2y_0}x \geq 2$ for all $|x| > M$. Therefore, we have

$$\frac{|x|}{\cosh\frac{\pi}{a}x - \cos\frac{\pi}{a}y} \leq \frac{|x|}{\frac{1}{2}\cosh\frac{\pi}{a}x} \leq \frac{4|x|}{e^{\frac{\pi}{a}|x|}} \leq 1$$

and

$$\frac{|\sinh\xi|}{\cosh\frac{\pi}{a-2y_0}x - \cos\frac{\pi}{a-2y_0}(y-y_0)} \leq \frac{|\sinh\xi|}{\frac{1}{2}\cosh\frac{\pi}{a-2y_0}x} \leq \frac{2e^{|\xi|}}{\cosh\frac{\pi}{a-2y_0}x} \leq 4.$$

Hence

$$|I_4| \leq \frac{2}{a-2y_0}\left|\frac{\pi}{a-2y_0} - \frac{\pi}{a}\right|.$$

If $|x| \leq M$, then

$$|\sinh\xi| \leq \sinh\frac{\pi}{a}|x| + \sinh\frac{\pi}{a-2y_0}|x| \leq \sinh\frac{\pi}{a}M + \sinh\frac{\pi}{a-2y_0}M,$$

which follows that

$$|I_4| \leq \frac{1}{2(a-2y_0)}\frac{M(\sinh\frac{\pi}{a}M + \sinh\frac{\pi}{a-2y_0}M)}{(1-\cos\frac{\pi}{a}y)(1-\cos\frac{\pi}{a-2y_0}(y-y_0))}\left|\frac{\pi}{a-2y_0} - \frac{\pi}{a}\right|.$$

Let

$$A_1 = \max\left\{\frac{2}{a-2y_0}, \frac{M(\sinh\frac{\pi}{a}M + \sinh\frac{\pi}{a-2y_0}M)}{2(a-2y_0)(1-\cos\frac{\pi}{a}y)(1-\cos\frac{\pi}{a-2y_0}(y-y_0))}\right\},$$

then

$$|I_4| \leq A_1\left|\frac{\pi}{a-2y_0} - \frac{\pi}{a}\right|, \quad x \in \mathbb{R},$$

(2.2) is thus proved. Similarly, we can prove (2.3). □

**Lemma 2.4 (Harmonic Majorant)** *Let $v(z)$ be a nonnegative subharmonic function in $S_a$ satisfying (2.1), then*

$$M_v(y) = \int_{\mathbb{R}} v(x+iy)dx$$

*is convex in $(0,a)$ and there exist two positive measures $\mu_1,\ \mu_2 \in \mathscr{M}(\mathbb{R})$ with $\|\mu_1\|,\ \|\mu_2\| \leq C$ such that*

$$u(x+iy) = \int_{-\infty}^{\infty} P_y^+(x-t)d\mu_1(t) + \int_{-\infty}^{\infty} P_y^-(x-t)d\mu_2(t).$$

*Moreover, $v(z) \leq u(z)$ for all $z \in S_a$.*

**Proof** There exists a sequence $\{y_k\}$ such that $\lim_{k\to\infty} y_k = 0$. By (2.1), $\{v_{y_k}\}, \{v_{a-y_k}\}$ are bounded linear functionals on $C_0(\mathbb{R})$ and they are uniformly bounded, where $v_y(x) = v(x+iy)$. Based on Banach-Alaoglu theorem, there exist $\mu_1, \mu_2 \in \mathscr{M}(\mathbb{R})$ and a subsequence

$\{y_{k_j}\}$ such that $\{v_{y_{k_j}}\}$ converges weakly to $\mu_1$ as $j \to \infty$ and $\{v_{a-y_{k_j}}\}$ converges weakly to $\mu_2$ as $j \to \infty$. That is, for each $\varphi \in C_0(\mathbb{R})$,

$$\lim_{j\to\infty} \int_{-\infty}^{\infty} v(t+iy_{k_j})\varphi(t)dt = \int_{-\infty}^{\infty} \varphi(t)d\mu_1(t),$$

$$\lim_{j\to\infty} \int_{-\infty}^{\infty} v(t+i(a-y_{k_j}))\varphi(t)dt = \int_{-\infty}^{\infty} \varphi(t)d\mu_2(t).$$

Accordingly, we obtain that

$$\begin{aligned}\|\mu_1\| &= \sup\left\{\Big|\int_{\mathbb{R}} \varphi(t)d\mu_1(t)\Big| : \varphi \in C_0(\mathbb{R}), \|\varphi\|_\infty = 1\right\}\\ &\le \liminf_{j\to\infty} \int_{\mathbb{R}} v(t+iy_{k_j})dt \le C,\end{aligned}$$

$$\begin{aligned}\|\mu_2\| &= \sup\left\{\Big|\int_{\mathbb{R}} \varphi(t)d\mu_2(t)\Big| : \varphi \in C_0(\mathbb{R}), \|\varphi\|_\infty = 1\right\}\\ &\le \liminf_{j\to\infty} \int_{\mathbb{R}} v(t+i(a-y_{k_j})dt \le C.\end{aligned}$$

Because of $\varphi(t) = P_y^+(x-t)$(or $P_y^-(x-t)$) $\in C_0(\mathbb{R})$, in particular, we have

$$\begin{aligned}&\lim_{j\to\infty} \int_{-\infty}^{\infty} v(t+iy_{k_j})P_y^+(x-t)dt + \lim_{j\to\infty} \int_{-\infty}^{\infty} v(t+i(a-y_{k_j}))P_y^-(x-t)dt\\ &= \int_{-\infty}^{\infty} P_y^+(x-t)d\mu_1(t) + \int_{-\infty}^{\infty} P_y^-(x-t)d\mu_2(t) \triangleq u(x+iy).\end{aligned}$$

For any fixed $0 < y_0 < y_1 < a$, let $r = \frac{y_1-y_0}{a}$, then the function

$$\tilde{u}(z) = \int_{\mathbb{R}} v(t+iy_0)P_y^+(x-t)dt + \int_{\mathbb{R}} v(t+iy_1)P_y^-(x-t)dt$$

is harmonic in $S_a$ (see [15, Theorem 1]). Assume that $\varepsilon > 0$ and $A > \exp\{\frac{C}{\pi\varepsilon \min\{y_0, a-y_1\}}\} + 1$. Since $v(rz+iy_0)$ is subharmonic in the set $\{z = x+iy : x \in \mathbb{R}, -\frac{y_0}{r} < y < \frac{a-y_0}{r}\}$, then there exist two sequences of continuous functions $\{u_n^{(1)}(t)\}$ and $\{u_n^{(2)}(t)\}$ decreasing to $v(rt+iy_0)$ and $v(rt+iy_1)$ on $[-A, A]$, respectively. Let

$$U_n(z) = \int_{-A}^{A} P_y^+(x-t)u_n^{(1)}(t)dt + \int_{-A}^{A} P_y^-(x-t)u_n^{(2)}(t)dt,$$

then by Lemma 2.2, $U_n(z)$ is harmonic in $S_a$ and

$$|U_n(z)| \le \max\{\max_{|t|\le A} u_n^{(1)}(t), \max_{|t|\le A} u_n^{(2)}(t)\} = A_n.$$

Therefore, the function

$$V_n(z) = v(rz + iy_0) - 2\varepsilon \log|z + i| - U_n(z)$$

is subharmonic in $S_a$, and by Lemma 2.1, we speculate that

$$V_n(z) \leq \frac{2C}{\pi \min\{y_0, a - y_1\}} - \varepsilon \log|x^2 + (y+1)^2| + A_n \to -\infty \ (z \to \infty, 0 < y < a).$$

It follows that

$$\limsup_{z \to t, 0<y<a} V_n(z) \leq \frac{2C}{\pi \min\{y_0, a - y_1\}} - 2\varepsilon \log A \leq 0$$

and

$$\limsup_{z \to t+ia, 0<y<a} V_n(z) \leq \frac{2C}{\pi \min\{y_0, a - y_1\}} - 2\varepsilon \log A \leq 0$$

for $t \in \mathbb{R},\ |t| > A$. If $t \in \mathbb{R},\ |t| \leq A$, then

$$\limsup_{z \to t, 0<y<a} V_n(z) \leq v(rt + iy_0) - u_n^{(1)}(t) \leq 0,$$

and

$$\limsup_{z \to t+ia, 0<y<a} V_n(z) \leq v(rt + iy_1) - u_n^{(2)}(t) \leq 0.$$

By [3, Theorem 4.3.11], we derive that $V_n(z) \leq 0$ on $S_a$. Take $n \to \infty$, then $A \to \infty$, and then let $\varepsilon \to 0$, we obtain

$$v(rz + iy_0) \leq \int_{-\infty}^{\infty} P_y^+(x - t) v(rt + iy_0) dt + \int_{-\infty}^{\infty} P_y^-(x - t) v(rt + iy_1) dt.$$

Hence, for every $0 < y_0 < y < y_1 < a$, we have

$$\begin{aligned} v(z) &\leq \int_{-\infty}^{\infty} P^+_{\frac{y-y_0}{r}}\left(\frac{x}{r} - t\right) v(rt + iy_0) dt + \int_{-\infty}^{\infty} P^-_{\frac{y-y_0}{r}}\left(\frac{x}{r} - t\right) v(rt + iy_1) dt \\ &= \int_{-\infty}^{\infty} \frac{1}{r} P^+_{\frac{y-y_0}{r}}\left(\frac{x}{r} - \frac{t}{r}\right) v(t + iy_0) dt + \int_{-\infty}^{\infty} \frac{1}{r} P^-_{\frac{y-y_0}{r}}\left(\frac{x}{r} - \frac{t}{r}\right) v(t + iy_1) dt. \end{aligned} \tag{2.4}$$

Moreover, by Lemma 2.2,

$$\begin{aligned} \int_{-\infty}^{\infty} v(x + iy) dx \leq & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{r} P^+_{\frac{y-y_0}{r}}\left(\frac{x}{r} - \frac{t}{r}\right) v(t + iy_0) dt dx \\ & + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{r} P^-_{\frac{y-y_0}{r}}\left(\frac{x}{r} - \frac{t}{r}\right) v(t + iy_1) dt dx \\ = & \left(1 - \frac{y - y_0}{ar}\right) \int_{-\infty}^{\infty} v(t + iy_0) dt + \frac{y - y_0}{ar} \int_{-\infty}^{\infty} v(t + iy_1) dt \\ = & \frac{y_1 - y}{y_1 - y_0} \int_{-\infty}^{\infty} v(t + iy_0) dt + \frac{y - y_0}{y_1 - y_0} \int_{-\infty}^{\infty} v(t + iy_1) dt. \end{aligned}$$

Thus, we conclude that $M_v(y)$ is convex in $(0,a)$. Let $y_1 = a - y_0$, then $\frac{1}{r}P^{\pm}_{\frac{y-y_0}{r}}(\frac{x}{r} - \frac{t}{r}) = \tilde{P}^{\pm}_{y-y_0}(x-t)$ and (2.4) becomes

$$v(z) \le \int_{-\infty}^{\infty} \tilde{P}^{+}_{y-y_0}(x-t)v(t+iy_0)dt + \int_{-\infty}^{\infty} \tilde{P}^{-}_{y-y_0}(x-t)v(t+i(a-y_0))dt,$$

where $0 < y_0 < y < a - y_0 < a$. By Lemma 2.3, one has

$$\lim_{j\to\infty} \int_{-\infty}^{\infty} |\tilde{P}^{+}_{y-y_{k_j}}(x-t) - P^{+}_{y}(x-t)|v(t+iy_{k_j})dt = 0,$$

$$\lim_{j\to\infty} \int_{-\infty}^{\infty} |\tilde{P}^{-}_{y-y_{k_j}}(x-t) - P^{-}_{y}(x-t)|v(t+i(a-y_{k_j}))dt = 0.$$

Therefore, it is not hard to verify that

$$\begin{aligned} v(z) \le & \lim_{j\to\infty} \int_{-\infty}^{\infty} [v(t+iy_{k_j})P^{+}_{y}(x-t) + v(t+i(a-y_{k_j}))P^{-}_{y}(x-t)]dt \\ = & \int_{-\infty}^{\infty} P^{+}_{y}(x-t)d\mu_1(t) + \int_{-\infty}^{\infty} P^{-}_{y}(x-t)d\mu_2(t) \\ = & u(z) \end{aligned}$$

for all $z = x + iy,\ 0 < y < a$. It completes the proof. □

Next, we will apply this lemma to the function $\log^+|f|$ to prove Theorem 1.1 and Theorem 1.2, where $f \in H^0(S_a)$.

**Lemma 2.5 ([2, 14])** *If $v$ is subharmonic in $\mathbb{U}$, then the following statements are equivalent:*
*(i) $v$ has a harmonic majorant in $\mathbb{U}$;*
*(ii) $\sup\limits_{0<r<1} \{\frac{1}{2\pi}\int_{-\pi}^{\pi} v(re^{i\varphi})d\varphi\} < \infty$.*

**Proof of Theorem 1.1** Since $\log^+|f|$ is subharmonic and it satisfies (2.1), it follows from Lemma 2.4 that there exist two positive measures $\mu_1, \mu_2 \in \mathscr{M}(\mathbb{R})$ such that

$$u(x+iy) = \int_{-\infty}^{\infty} P^{+}_{y}(x-t)d\mu_1(t) + \int_{-\infty}^{\infty} P^{-}_{y}(x-t)d\mu_2(t),$$

and $\log^+|f|(z) \le u(z)$ for all $z \in S_a$. The conformal mapping

$$\beta(z) = \frac{e^{\frac{\pi}{a}z} - i}{e^{\frac{\pi}{a}z} + i} \tag{2.5}$$

from $S_a$ onto $\mathbb{U}$ maps $\partial S_a$ onto $\partial\mathbb{U}\backslash\{1,-1\}$ conformally (to be precise, there exists a continuous and strictly increasing function $\theta_1(t)$ from $\mathbb{R}$ onto $(-\pi, 0)$ such that $e^{i\theta_1(t)} = \beta(t)$; and there exists a continuous and strictly decreasing function $\theta_2(t)$ from $\mathbb{R}$ onto $(0,\pi)$ such that $e^{i\theta_2(t)} = \beta(t+ia)$). Its inverse mapping is

$$\alpha(w) = \frac{a}{\pi}\log\frac{i(1+w)}{1-w}$$

(take the analytic branch $\log 1 = 0$). Then the function $\log^+|f|(\alpha(w))$ has a harmonic majorant $u(\alpha(w))$ in $\mathbb{U}$. According to Lemma 2.5, one has $F(w) = f(\alpha(w)) \in H^0(\mathbb{U})$. Therefore, by Theorem A, $F(w) = f(\alpha(w))$ has a nontangential limit $F^*(e^{i\theta})$ at almost every $\theta \in [-\pi, \pi)$ and $\log|F^*(e^{i\theta})| \in L^1([-\pi, \pi))$. Since $\alpha$ and $\beta$ are conformal, the limits in (1.3) exist nontangentially for almost every $t \in \mathbb{R}$. By virtue of

$$e^{i\theta_1(t)} = \beta(t) = \frac{e^{\frac{\pi}{a}t} - i}{e^{\frac{\pi}{a}t} + i}, \quad e^{i\theta_2(t)} = \beta(t + ia) = \frac{e^{\frac{\pi}{a}(t+ia)} - i}{e^{\frac{\pi}{a}(t+ia)} + i},$$

which follows that $d\theta_1 = \frac{\frac{2\pi}{a}e^{\frac{\pi}{a}t}}{1+e^{\frac{2\pi}{a}t}}dt$ and $d\theta_2 = \frac{-\frac{2\pi}{a}e^{\frac{\pi}{a}t}}{1+e^{\frac{2\pi}{a}t}}dt$. Thus, (1.4) follows immediately from the following identities:

$$\begin{aligned}\int_{-\pi}^{\pi} \big|\log|F^*(e^{i\theta})|\big| d\theta =& \int_{-\infty}^{\infty} \frac{\frac{2\pi}{a}e^{\frac{\pi}{a}t}\big|\log|f^*(t)|\big|}{1 + e^{\frac{2\pi}{a}t}} dt \\ &+ \int_{-\infty}^{\infty} \frac{\frac{2\pi}{a}e^{\frac{\pi}{a}t}\big|\log|f^*(t+ia)|\big|}{1 + e^{\frac{2\pi}{a}t}} dt \\ =& \frac{\pi}{a}\int_{-\infty}^{\infty} \frac{|\log|f^*(t)|| + \big|\log|f^*(t+ia)|\big|}{\cosh\frac{\pi}{a}t} dt.\end{aligned}$$

□

**Proof of Theorem 1.2** According to Theorem 1.1, $f(z)$ has nontangential limits $f^*(t)$ and $f^*(t+ia)$ and they satisfy (1.4). Since $\log^+|f|$ is subharmonic and it satisfies (2.1), then by Lemma 2.4, $\log^+|f|$ has a harmonic majorant in $S_a$. It follows that $F(w) = f(\alpha(w)) \in H(\mathbb{U})$ and $\log^+|F|$ has a harmonic majorant in $\mathbb{U}$. By Lemma 2.5, we have $F \in H^0(\mathbb{U})$. Then, by Theorem A, $F(w)$ has a nontangential limit $F^*(e^{i\theta})$ at almost every $\theta \in [-\pi, \pi)$ and $\log|F^*(e^{i\theta})| \in L^1([-\pi, \pi))$. Furthermore, $F$ can be written by

$$F(w) = c_1 G_1(w) B_1(w) S_1(w),$$

where $c_1$ is a constant with $|c_1| = 1$, and

$$G_1(w) = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta} + w}{e^{i\theta} - w} \log|F^*(e^{i\theta})| d\theta\right\}$$

is an outer function in $\mathbb{U}$; and

$$S_1(w) = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta} + w}{e^{i\theta} - w} d\nu_s(\theta)\right\}$$

is a quotient of two singular inner functions in $\mathbb{U}$, where $\nu_s$ is a singular signed measure on $[-\pi, \pi)$ with finite total variation;

$$B_1(w) = w^k \prod_n \Big(\frac{\beta(z_n) - w}{1 - \overline{\beta(z_n)}w}\Big)\Big(\frac{\overline{\beta(z_n)}}{|\beta(z_n)|}\Big)$$

is a Blaschke product in $\mathbb{U}$, where $k$ is a nonnegative integer and

$$\sum_n (1-|\beta(z_n)|^2) \le 2\sum_n (1-|\beta(z_n)|) < \infty. \tag{2.6}$$

Therefore,

$$\begin{aligned}
\log|G_1(\beta(z))| =& \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(\mathrm{Re}\frac{\beta(t)+\beta(z)}{\beta(t)-\beta(z)}\Big)|\beta'(t)|\log|f^*(t)|dt \\
&+\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(\mathrm{Re}\frac{\beta(t+ia)+\beta(z)}{\beta(t+ia)-\beta(z)}\Big)|\beta'(t+ia)|\log|f^*(t+ia)|dt, \\
\log|S_1(\beta(z))| =& \frac{1}{2\pi}\mathrm{Re}\frac{-1+\beta(z)}{-1-\beta(z)}\nu_s(\{-\pi\})+\frac{1}{2\pi}\mathrm{Re}\frac{1+\beta(z)}{1-\beta(z)}\nu_s(\{0\}) \\
&+\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(\mathrm{Re}\frac{\beta(t)+\beta(z)}{\beta(t)-\beta(z)}\Big)d\nu_s(\theta_1(t)) \\
&+\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(\mathrm{Re}\frac{\beta(t+ia)+\beta(z)}{\beta(t+ia)-\beta(z)}\Big)d\nu_s(\theta_2(t)).
\end{aligned}$$

Making use of (2.5), we have

$$1-|\beta(z_n)|^2 = \frac{4e^{\frac{\pi}{a}x_n}\sin\frac{\pi}{a}y_n}{1+e^{\frac{2\pi}{a}x_n}+2e^{\frac{\pi}{a}x_n}\sin\frac{\pi}{a}y_n} \ge \frac{e^{\frac{\pi}{a}x_n}\sin\frac{\pi}{a}y_n}{1+e^{\frac{2\pi}{a}x_n}}, \tag{2.7}$$

and

$$\Big(\frac{\beta(z_n)-w}{1-\overline{\beta(z_n)}w}\Big)\Big(\frac{\overline{\beta(z_n)}}{|\beta(z_n)|}\Big) = \frac{e^{\frac{\pi}{a}z_n}-e^{\frac{\pi}{a}z}}{e^{\frac{\pi}{a}\bar{z}_n}-e^{\frac{\pi}{a}z}}e^{i\theta(z_n)},$$

where

$$e^{i\theta(z_n)} = \frac{(e^{\frac{\pi}{a}\bar{z}_n}+i)(e^{\frac{\pi}{a}\bar{z}_n}-i)}{|e^{\frac{\pi}{a}z_n}+i||e^{\frac{\pi}{a}z_n}-i|}.$$

Moreover,

$$\begin{aligned}
\frac{1}{2\pi}\mathrm{Re}\frac{-1+\beta(z)}{-1-\beta(z)}\nu_s(\{-\pi\}) &= \frac{\nu_s(\{-\pi\})}{2\pi}\mathrm{Re}(ie^{-\frac{\pi}{a}z}), \\
\frac{1}{2\pi}\mathrm{Re}\frac{1+\beta(z)}{1-\beta(z)}\nu_s(\{0\}) &= \frac{\nu_s(\{0\})}{2\pi}\mathrm{Re}(-ie^{\frac{\pi}{a}z}), \\
\frac{1}{2\pi}\mathrm{Re}\frac{\beta(t)+\beta(z)}{\beta(t)-\beta(z)}|\beta'(t)| &= \mathrm{Re}\left\{\frac{1}{ai}\left(\frac{e^{\frac{\pi}{a}t}}{e^{\frac{\pi}{a}t}-e^{\frac{\pi}{a}z}}-\frac{e^{\frac{2\pi}{a}t}}{1+e^{\frac{2\pi}{a}t}}\right)\right\}, \\
\frac{1}{2\pi}\mathrm{Re}\frac{\beta(t+ia)+\beta(z)}{\beta(t+ia)-\beta(z)}|\beta'(t+ia)| &= \mathrm{Re}\left\{-\frac{1}{ai}\left(\frac{e^{\frac{\pi}{a}t}}{e^{\frac{\pi}{a}t}+e^{\frac{\pi}{a}z}}-\frac{e^{\frac{2\pi}{a}t}}{1+e^{\frac{2\pi}{a}t}}\right)\right\}.
\end{aligned}$$

Let $\tau_1 = \frac{\nu_s(\{-\pi\})}{2\pi}, \tau_2 = \frac{\nu_s(\{0\})}{2\pi}$. Define two singular signed measures $\mu_{1,s}$ on $L_0$ and $\mu_{2,s}$ on $L_a$ by

$$d\mu_{1,s}(t) = |\beta'(t)|^{-1} d\nu_s(\theta_1(t)) = \frac{a}{\pi} \cosh \frac{\pi}{a} t \, d\nu_s(\theta_1(t)),$$
$$d\mu_{2,s}(t) = |\beta'(t+ia)|^{-1} d\nu_s(\theta_2(t)) = \frac{a}{\pi} \cosh \frac{\pi}{a} t \, d\nu_s(\theta_2(t)),$$

then $\log|G_1(\beta(z))| = \log|G(z)|$ and $\log|S_1(\beta(z))| = \log|S(z)|$. Therefore, there exist two constants $c_2, c_3$ with $|c_2| = |c_3| = 1$ such that $G_1(\beta(z)) = c_2 G(z), S_1(\beta(z)) = c_3 S(z)$. Let $c = c_1 c_2 c_3, B(z) = B_1(\beta(z))$, then $f(z) = cG(z)S(z)B(z)$. Accordingly, (1.5) follows instantly from (2.6) and (2.7). Since $\nu_s$ is finite, then (1.6) follows from (1.4). It completes the proof. □

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# References

[1] P.L. Duren, Theory of $H^p$ spaces, Academic Press, New York, 1970.

[2] M. Rosenblum, J. Rovnyak, Topics in Hardy classes and univalent functions, Birkhäuser Verlag, Berlin, 1994.

[3] G.T. Deng, Complex Analysis, Beijing Normal University Press, Beijing, 2010 (in Chinese).

[4] P. Ahern, The mean modulus and the derivative of an inner function, Indiana Univ. Math. J. 1979, **28**(2): 311-347.

[5] H. Kim, Derivatives of Blaschke products, Pacific. J. Math. 1984, **114**(1): 175-190.

[6] A. Gluchoff, On inner functions with derivative in Bergman spaces, Illinois J. Math. 1987, **31**(3): 518-527.

[7] K.M. Dyakonov, Self-improving behaviour of inner functions as multipliers, J. Funct. Anal. 2006, **240**(2): 429-444.

[8] J.Á. Peláez, Inner functions as improving multipliers, J. Funct. Anal. 2008, **255**(6): 1403-1418.

[9] W. Rudin, Real and complex analysis, 3rd ed., McGraw-Hill Book Co., New York, 1987.

[10] R.S. Qian, S.X. Li, Inner-outer factorization on $\mathcal{Q}_p$ spaces, Ann. Funct. Anal. 2015, **6**(3): 1-7.

[11] K.M. Dyakonov, Division and multiplication by inner functions and embedding theorems for star-invariant subspaces, Amer. J. Math. 1993, **115**(4): 881-902.

[12] R.S. Qian, S.X. Li, Inner-outer factorization on Besov-type spaces, J. Comput. Anal. Appl. 2017, **22**(6): 1150-1157.

[13] S.Y. Ke, Completeness of exponential and random exponential, Ph.D. Thesis, Beijing Normal University, Beijing, 2010 (in Chinese).

[14] J. Garnett, Bounded analytic functions, Academic Press, New York, 1981.

[15] D.V. Widder, Functions harmonic in a strip, P. Am. Math. Soc. 1961, **12**(1): 67-72.

# Controllability of Stochastic Evolution Differential Equations Driven by Fractional Brownian Motion and Poisson Jumping Processes *

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**Abstract**

In this article, we investigate stochastic evolution differential equations driven by fractional Brownian motions and Poisson random measure processes. At first, we discuss the existence and unique of the mild solution by using Banach fixed point principle. Secondly, sufficient conditions for the complete controllability of the stochastic evolution systems are formulated and proved by using the $C_0$-semigroup theory and stochastic analysis techniques. In the end, an example is presented to illustrate our main results.

**Key words**: *Stochastic evolution equation; Fractional Brownian motion; Poisson noise process; Mild solution; Complete controllability.*

## 1 Introduction

In this paper, we will study the problem having the following form:

$$\begin{cases} dx(t) = [Ax(t) + Bu(t) + f(t, x(t))]dt + \sigma(t)dB_Q^H(t) + \int_Z h(t, x(t), y)\widetilde{N}(dt, dy), \\ \qquad\qquad t \in J = [0, T] \\ x(0) = x_0. \end{cases} \tag{1.1}$$

where $A$ is the infinitesimal generator of a $C_0$-semigroup $S(t)$ on a separable Hilbert space $X$, $B_Q^H(t)$ is a fractional Brownian motion (fBm for short) with Hurst index $H \in (\frac{1}{2}, 1)$ on a real and separable Hilbert space $K$. $f : J \times X \to X, \sigma : J \to L_2^0(K, X), h : J \times X \times Z \to X$ are Borel measurable functions. Here $L_2^0(K, X)$ denotes the space of all $Q$-Hilbert-Schmidt operators from $K$ into $X$. The control function $u(t)$ takes value in $V = L_2(J, U)$, and $U$ is a Hilbert space, $B$ is a linear operator from $V$ into $L_2(J, X)$. $\widetilde{N}(dt, dy)$ is the compensated Poisson measure which will be given in the below.

Recently, stochastic differential systems have attracted a great attention since it arises naturally in mathematical modeling of various phenomena in the social and natural sciences, such as pricing an option, forecasting the growth of population and determining optimal portfolio of investments, for example one can see [16, 29, 32] and the references therein. Prato and Giuseppe [33] researched stochastic equations in infinite dimensions. Luo and Taniguchi [26] considered the existence and uniqueness of non-Lipschitz stochastic neutral delay evolution equations driven by Poisson jumps. For the literatures on controllability of stochastic system with impulsive effect, one can see the papers [18, 24, 27, 36] and references therein.

It's well known that the noise or perturbations of a stochastic differential system are typically modeled by a Brownian motion as such a process is Gauss-Markov and has independent increments. However, many researchers have found that empirical data from many physical phenomena with the standard Brownian motion is often shown not to be an effective process to use in a model. A family of processes that seems to have wide physical applicability is fractional Brownian motion (fBm). Since it was first introduced by Kolmogorov in 1940, Mandelbrot and Ness discussed the applications of the fBm process in later. Since then, based on different settings, various forms of equations have been studied. For example, the case of finite-dimensional equations has been studied by Besalú and Rovira [5], Jérémie Unterberger [39], Dung [9], León and Tindel [23], for the case of infinite-dimensional systems in a Hilbert space have been considered by Boufoussi and Hajji [7], Caraballo, Garrido-Atienza and Taniguchi [8], and Ahmed [11]. Furthermore, the stochastic differential equations driven by a Poisson process can be widely found in applications from various fields such as storage systems, queueing systems, economic systems and neurophysiology systems, for example, one can see [1, 20, 35]. SPDEs with Poisson jump process is an important step for the study of SPDEs with Lévy process. In recent years, there is quite a substantial amount of work that has been done in this field. Hausenblas[12] dealt with SPDEs driven by Poisson random measures with non-Lipschitz coefficients in Banach spaces. Laukajtys and Słomiński [22] considered the penalization method for a reflected SDE driven by

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Poisson jumps. Later, Ren and Wu [34] established existence and uniqueness for solutions of multivalued, finite dimensional SDEs driven by Poisson point processes, where the maximal monotone operator has the whole space as domain.

On the other hand, one of the well-known qualitative behaviors of a dynamical system is controllability, which was first introduced by Kalman [17] in 1963. It means that it is possible to steer a dynamical control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. Recently, many researchers take attention to the study of the controllability for a variety of differential dynamical systems. For example, Leiva [13] considered the exact controllability of the suspension bridge model proposed by Lazer and McKenna. Liu and Li [25] studied the controllability of impulsive fractional evolution inclusions in Banach spaces. The approximate controllability for a class of semilinear abstract equations was discussed by Zhou [40]. For more detailed, one can see [4, 14, 37]. For the controllability problem there are different methods for various types of nonlinear stochastic systems. Subalakshmi and Balachandran [37] studied the approximate controllability of nonlinear stochastic impulsive systems in Hilbert spaces by using Nussbaum's fixed point theorem. In [19], using a stochastic Lyapunov-like approach, sufficient conditions for stochastic $\epsilon$-controllability are formulated. Balachandran etal.[3] researched the controllability of semilinear stochastic integrodifferential systems by using the Picard type iteration. By using the contraction mapping principle, Mahmudov and Zorlu studied the controllability [28] for non-linear stochastic systems.

By contrast, there has not been very much research of stochastic differential equations driven both by fractional Brownian motion and by Poisson noise processes. By using the extended form of Krylov-type estimate for the combined noise of fBM and compound Poisson, Bai and Ma [2] studied the existence of the strong solutions for the stochastic differential equation driven by fractional Brownian motion and Poisson point processes. Hajji and Lakhel [10] discussed the existence of the neutral stochastic functional differential equation driven by fractional Brownian motion and Poisson point processes. To the best of our knowledge, there is no paper researched the complete controllability of stochastic differential equations driven by fractional Brownian motions and poisson noise processes. Thus, we shall make the first attempt to discuss such problem in this paper.

The rest of this paper is organized as follows. In the next section, we will introduce some useful preliminaries on the data. In Section 3, some sufficient conditions are established to guarantee the existence and uniqueness of mild solutions of the system (1.1). In Section 4, we will study the completely controllability for stochastic evolution systems. Finally, we present an example to illustrate our main results.

# 2 Preliminaries

Now, we introduce some basic definitions and preliminaries which are used throughout this paper. Throughout this article, we use the following notations:

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \in [0,T]\}, P)$ be a complete probability space satisfying the standard conditions, which means that the filtration $\mathcal{F}_t, t \in [0,T]$ is right continuous increasing family and $\mathcal{F}_0$ contains all $P$-null sets. Let $L_2(\Omega, X) = L_2(\Omega, \mathcal{F}_t, X)$ be the Hilbert space of all $\mathcal{F}_t$-measurable square integrable random variables with values in $X$. Moreover, let $L_2^{\mathcal{F}}(J, X)$ be the Hilbert space of all square integrable and $\mathcal{F}_t$-adapted measurable processes with values in $X$. Further, let $C(J, L_2(\Omega, X)) := C(J, L_2(\Omega, \mathcal{F}_t, X))$ be the Banach space of continuous maps from $J$ into $L_2(\Omega, X)$ satisfying $\sup_{t\in J} E\|x(t)\|^2 < \infty$ with the norm $\|x\| = (\sup_{t\in J} E\|x(t)\|_X^2)^{\frac{1}{2}}$.

Now, we present some basic definitions on fractional Brownian motion (fBm).

**Definition 2.1.** The fractional Brownian motion (fBm) with Hurst index $H \in (0,1)$ is a Gaussian process $B_t^H = \{B_t^H, \mathcal{F}_t, t \in [0,T]\}$, having the properties $B_0^H = 0, EB_t^H = 0$, and $EB_t^H B_s^H = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H})$.

Let $T > 0$, for a linear space $\Lambda$, there exists a $R-$valued step function $\phi \in \Lambda$ on $[0,T]$, such that

$$\phi(t) = \sum_{i=1}^{n-1} z_i \chi_{(t_i, t_{i+1}]}(t),$$

where $t \in [0,T], z_i \in R$ and $0 = t_1 < t_2 < \cdots < t_n = T$. For any $\phi \in \Lambda$, the Wiener integral with respect to $B^H$ can be defined as

$$\int_0^T \phi(s) dB^H(s) = \sum_{i=1}^{n-1} z_i (B^H(t_{i+1}) - B^H(t_i)).$$

Let $\mathcal{H}$ be a Hilbert space, which is defined as the closure of $\Lambda$ with respect to the scalar product $\langle \chi_{[0,t]}, \chi_{(0,s]} \rangle_{\mathcal{H}} = R_H(t,s)$. Then the mapping

$$\phi = \sum_{i=1}^{n-1} z_i \chi_{(t_i, t_{i+1}]} \mapsto \int_0^T \phi(s) dB^H(s)$$

is an isometry between $\Lambda$ and the linear space span $\{B^H(t) : t \in [0,T]\}$, which can be extended to an isometry between $\mathcal{H}$ and the first Wiener chaos of the fBm $\overline{span}^{L^2(\Omega)}\{B^H(t) : t \in [0,T]\}$ (see [38] ). The image of an

element $\phi \in \mathcal{H}$ by this isometry is called the Wiener integral of $\phi$ with respect to $B^H$. Our next aim is to give an explicit expression of this integral.

Now, let us consider the Kernel

$$K_H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (z-s)^{H-\frac{3}{2}} z^{H-\frac{1}{2}} dz,$$

where $c_H = \left( \frac{H(2H-1)}{B(2-2H, H-\frac{1}{2})} \right)^{\frac{1}{2}}$ ($B(\cdot,\cdot)$ denote the Beta function), and $t > s$. It is easily shown that

$$\frac{\partial K_H(t,s)}{\partial t} = c_H \left( \frac{t}{s} \right)^{H-\frac{1}{2}} (t-s)^{H-\frac{3}{2}}.$$

Let $\mathcal{K}_H : \Lambda \to L^2([0,T])$ be the linear operator, which is defined as

$$\mathcal{K}_H \phi(s) = \int_s^t \phi(t) \frac{\partial K_H}{\partial t}(t,s) dt.$$

Then $(\mathcal{K}_H \chi_{[0,T]})(s) = K_H(t,s)\chi_{[0,T]}(s)$, and $\mathcal{K}_H$ is an isometry between $\Lambda$ and $L_2([0,T])$ which can be extended to $\mathcal{H}$.

We denote $L_2^{\mathcal{H}}([0,T]) = \{\phi \in \mathcal{H} : \mathcal{K}_H \phi \in L_2([0,T])\}$, then for $H > \frac{1}{2}$, we get

$$L^{\frac{1}{H}}([0,T]) \subset L_2^{\mathcal{H}}([0,T]).$$

Moreover, the following lemma hold:

**Lemma 2.2** ([30]). *For $\phi \in L^{\frac{1}{H}}([0,T])$,*

$$H(2H-1) \int_0^T \int_0^T |\phi(r)||\phi(z)||r-u|^{2H-2} dr dz \le c_H \|\phi\|^2_{L^{\frac{1}{H}}([0,T])}.$$

Let $(X, |\cdot|_X, \langle \cdot,\cdot \rangle_X)$ and $(K, |\cdot|_K, \langle \cdot,\cdot \rangle_K)$ be separable Hilbert spaces. $L(K,X)$ denotes the space of all bounded linear operator from $K$ to $X$ and $Q \in L(K,X)$ is a non-negative self adjoint operator. Denote by $L_2^0(K,X)$ the space of all $\xi \in L(K,X)$ such that $\xi Q^{\frac{1}{2}}$ is a Hilbert-Schmidt operator, the norm is given by

$$|\xi|^2_{L_2^0(K,X)} = |\xi Q^{\frac{1}{2}}|^2_{HS} = tr(\xi Q \xi^*).$$

Then $\xi$ is a $Q$-Hilbert-Schmidt operator from $K$ to $X$.

Let $\{B_n^H(t)\}_{n\in N}$ be a sequence of two-side one-dimensional fBm mutually independent on the complete probability space $(\Omega, \mathcal{F}, P)$, $\{e_n\}_{n\in N}$ be a complete orthonormal basis in $K$. We define the $K$-valued stochastic process $B_Q^H(t)$ as

$$B_Q^H(t) = \sum_{n=1}^{\infty} B_n^H(t) Q^{\frac{1}{2}} e_n, t \ge 0.$$

If $Q$ is a non-negative self-adjoint trace class operator, then the series $\sum_{n=1}^{\infty} B_n^H(t) Q^{\frac{1}{2}} e_n, t \ge 0$ converges in the space $K$, i.e., it holds that $B_Q^H(t) \in L_2(\Omega, K)$. Then, we can say that $B_Q^H(t)$ is a $K$-valued $Q$-cylindrical fBm with covariance operator $Q$.

**Definition 2.3.** Let $\psi : [0,T] \to L_2^0(K,X)$ such that

$$\sum_{n=1}^{\infty} \|\mathcal{K}_H(\psi Q^{\frac{1}{2}}) e_n\|_{L_2([0,T],X)} < \infty. \tag{2.1}$$

Then for $t \ge 0$, its stochastic integral with respect to the fBm $B_Q^H$ is defined as

$$\int_0^t \psi(s) dB_Q^H(s) = \sum_{n=1}^{\infty} \int_0^t \psi(s) Q^{\frac{1}{2}} e_n dB_n^H(s) = \sum_{n=1}^{\infty} \int_0^t \left( \mathcal{K}_H(\psi Q^{\frac{1}{2}} e_n) \right)(s) dw(s),$$

where $w$ is a Wiener process.

Notice that if

$$\sum_{n=1}^{\infty} \|\psi Q^{\frac{1}{2}} e_n\|_{L^{\frac{1}{H}}([0,T],X)} < \infty, \tag{2.2}$$

then in particular (2.2) holds, which follows immediately from (2.1).

The following lemma is obtained as a simple application of Lemma 2.2.

**Lemma 2.4** ([30]). *For any $\psi : [0,T] \to L_2^0(K,X)$ such that $\sum_{n=1}^{\infty} \|\psi Q^{\frac{1}{2}} e_n\|_U$ is uniformly convergent for $t \in [0,T]$, and for any $p,q \in [0,T]$ with $p > q$,*

$$E\left\| \int_q^p \psi(s) dB_Q^H(s) \right\|_X^2 \leq cH(2H-1)(p-q)^{2H-1} \sum_{n=1}^{\infty} \int_q^p \|\psi Q^{\frac{1}{2}} e_n\|_X^2 ds.$$

*Then*

$$E\left\| \int_q^p \psi(s) dB_Q^H(s) \right\|_X^2 \leq cH(2H-1)(p-q)^{2H-1} \int_q^p \|\psi(s)\|_{L_2^0}^2 ds, \tag{2.3}$$

*where $c = c(H)$.*

In the follow, we give the definition of the Poission random measure.

Let $\{p(t) : t \in J\}$ be a Poisson point process, and take its value in a measure in a measurable space $(Z, \mathcal{B}(Z))$ with a $\sigma$-finite intensity measure $\mu(dy)$. We denote the Poisson counting measure as $N(dt,dy)$, which is induced by $p(\cdot)$, and the compensating martingale measure by

$$\widetilde{N}(dt,dy) = N(dt,dy) - \mu(dy)dt.$$

For investigated our main results, we shall give the following lemma.

**Lemma 2.5.** *Let the space $M_\mu^\theta(J \times \Omega \times (K - \{0\}), H), (\theta \geq 2)$ be the set of all random process $L(t,y)$ with values in $H$, predictable with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ such that*

$$E\left( \int_0^T \int_Z \|L(t,y)\|_X^\theta \mu(dy)dt \right) < \infty.$$

*Assume $L \in M_\mu^2(J \times \Omega \times (K - \{0\}), H) \cap M_\mu^4(J \times \Omega \times (K - \{0\}), H)$, then for any $t \in J$, we have*

$$\begin{aligned} E\left[ \sup_{0\leq r\leq t} \left\| \int_0^r \int_Z S(r-s)L(s,y)\widetilde{N}(dy,ds) \right\|_H^2 \right] &\leq l\Big\{ E\left( \int_0^t \int_Z \|L(s,y)\|_H^2 \mu(dy)ds \right) \\ &\quad + E\left( \int_0^t \int_Z \|L(s,y)\|_H^4 \mu(dy)ds \right)^{\frac{1}{2}} \Big\} \end{aligned}$$

*for some number $l > 0$ dependent on $T > 0$.*

Now, we define the mild solution of the system(1.1) as follows.

**Definition 2.6.** A $X$-valued process $x(t)$ is called a mild solution of (1.1), if $x(0) = x_0, x(t) \in C(J, L_2(\Omega, X))$, for each $0 \leq t \leq T$, the following integral equation satisfies:

$$\begin{aligned} x(t) &= S(t)x_0 + \int_0^t S(t-s)Bu(s)ds + \int_0^t S(t-s)f(s,x(s))dt + \int_0^t S(t-s)\sigma(s)dB_Q^H(s) \\ &\quad + \int_0^t \int_Z S(t-s)h(t,x(t),y)N(ds,dy). \end{aligned} \tag{2.4}$$

# 3 Existence result

The purpose of this section is to study the existence of mild solutions for problem (1.1). Our main method is the Banach contraction fixed point theorem.

At first, we assume that the following hypotheses be held:

$(H_1)$ The $C_0$-semigroup $\{S(t)\}_{t\geq 0}$ is linear and bounded in $X$ [31], i.e., there exists a constant $M > 0$, such that

$$\|S(t)\| \leq M.$$

$(H_2)$ There exist constants $L_1, L_2 > 0$ such that

$$\begin{aligned} &\|f(t,x_1) - f(t,x_2)\|^2 \leq L_1 \|x_1 - x_2\|^2 \\ &\|f(t,x)\|^2 \leq L_2(1 + \|x\|^2) \end{aligned}$$

for all $x_1, x_2, x \in X$ and a.e. $t \in J$.

$(H_3)$ There are some constants $L_3, L_4 > 0$ such that

$$\int_Z \|h(t,x_1(t),y) - h(t,x_2(t),y)\|^2 \mu(dy) \leq L_3 \|x_1(t) - x_2(t)\|^2,$$

$$\int_Z \|h(t,x(t),y)-h(t,x(t),y)\|^4\mu(dy) \le L_4\|x_1(t)-x_2(t)\|^4,$$

for all $x_1, x_2, x \in X$ and a.e. $t \in J$.

$(H_4)$ There are some constants $L_5, L_6 > 0$ such that

$$\int_Z \|h(t,x(t),y)\|^2\mu(dy) \le L_5(1+\|x(t)\|^2),$$
$$\int_Z \|h(t,x(t),y)\|^4\mu(dy) \le L_6(1+\|x(t)\|^4)$$

for all $x_1, x_2, x \in X$ and a.e. $t \in J$.

$(H_5)$ The function $\sigma : [0,\infty) \to L_2^0(K,X)$ satisfies $\int_0^T \|\sigma(s)\|_{L_2^0}^2 ds < \infty$.

Now, we consider the existence result for system (1.1).

**Theorem 3.1.** *Assume that hypotheses $(H_1)-(H_5)$ hold. Then for any $u \in L_2(J,U)$ the stochastic system (1.1) has a unique mild solution on $J$, if*

$$2T^2[M^2L_1 + l(L_3+\sqrt{L_4})] < 1.$$

*Proof.* We define an operator $F : C(J, L_2(\Omega,X)) \to C(J, L_2(\Omega,X))$ by

$$\begin{aligned}(Fx)(t) &= S(t)x_0 + \int_0^t S(t-s)Bu(s)ds + \int_0^t S(t-s)f(s,x(s))ds \\ &\quad + \int_0^t S(t-s)\sigma(s)dB_Q^H(s) + \int_0^t\int_Z S(t-s)h(s,x(s),y)N(ds,dy).\end{aligned}$$

Using the contraction mapping principle, we will show that the operator $F$ has a fixed point. To prove this, we subdivide the proof into four steps.

Step 1. For any $x \in C(J, L_2(\Omega,X))$, we show that $F$ maps $C(J, L_2(\Omega,X))$ into itself.

For all $x \in C(J, L_2(\Omega,X))$, we have

$$\begin{aligned}&E\|(Fx)(t)\|^2 \\ \le\ & 5E\|S(t)x_0\|^2 + 5E\left\|\int_0^t S(t-s)Bu(s)ds\right\|^2 + 5E\left\|\int_0^t S(t-s)f(s,x(s))ds\right\|^2 \\ &+5E\left\|\int_0^t S(t-s)\sigma(s)dB_Q^H(s)\right\|^2 + 5E\left\|\int_0^t\int_Z S(t-s)h(t,x(t),y)N(ds,dy)\right\|^2 \\ \le\ & 5M^2\left[E\|x_0\|^2 + E\|Bu\|^2T^2 + TL_2(1+E\|x\|_C^2) + cH(2H-1)T^{2H-1}\int_0^T \|\sigma(s)\|_{L_Q^0(V,U)}^2 ds\right] \\ &+l\left\{E\left(\int_0^t\int_Z \|h(t,x(t),y)\|_H^2\mu(dy)ds\right) + E\left(\int_0^t\int_Z \|h(t,x(t),y)\|_H^4\mu(dy)ds\right)^{\frac{1}{2}}\right\} \\ \le\ & 5M^2\left[E\|x_0\|^2 + E\|Bu\|^2T^2 + TL_2(1+E\|x\|_C^2) + cH(2H-1)T^{2H-1}\int_0^T \|\sigma(s)\|_{L_Q^0(V,U)}^2 ds\right] \\ &+l\left[L_5\int_0^t E(1+\|x(s)\|^2)ds + \sqrt{L_6}\left(\int_0^t E(1+\|x(s)\|^4)ds\right)^{\frac{1}{2}}\right] \\ \le\ & 5M^2\Bigg[E\|x_0\|^2 + E\|Bu\|^2T^2 + TL_2(1+E\|x\|_C^2) \\ &+cH(2H-1)T^{2H-1}\int_0^T \|\sigma(s)\|_{L_Q^0(V,U)}^2 ds\Bigg] + l(L_5T + \sqrt{L_6T})E(1+\|x(s)\|^2) \qquad (3.1)\end{aligned}$$

for all $t \in J$.

From the inequality (3.1) and the assumptions, one can see that there exists $M_1 > 0$ such that

$$E\|(Fx)(t)\|^2 \le M_1(1+T\sup_{s\in J}E\|x(s)\|^2)$$

for all $t \in J$. Thus, $F$ maps $C(J, L_2(\Omega,X))$ into itself.

Step 2. We prove that $F$ is a contraction mapping.

Let $x_1, x_2 \in C(J, L_2(\Omega,X))$, for $t \in J$ we have

$$\begin{aligned}&E\|(Fx_1)(t)-(Fx_2)(t)\|^2 \\ \le\ & E\left\|\int_0^t S(t-s)[f(s,x_1(s))-f(s,x_2(s))]ds\right.\end{aligned}$$

$$+\int_0^t\int_Z S(t-s)[h(t,x_1(t),y)-h(t,x_2(t),y)]N(ds,dy)\Big\|$$

$$\leq 2M^2T^2L_1\sup_{t\in[0,T]}E\|x_1(t)-x_2(t)\|_{X^2}+2l\Big\{E\Big(\int_0^t\int_Z\|h(t,x_1(t),y)-h(t,x_2(t),y)\|_C^2\mu(dy)ds\Big)$$

$$+E\Big(\int_0^t\int_Z\|h(t,x_1(t),y)-h(t,x_2(t),y)\|_C^4\mu(dy)ds\Big)^{\frac{1}{2}}\Big\}$$

$$\leq 2M^2T^2L_1\sup_{s\in J}E\|x_1(s)-x_2(s)\|_C^2+2l(L_3+\sqrt{L_4})\int_0^t E\|x_1(s)-x_2(s)\|^2ds$$

$$\leq 2T^2[M^2L_1+l(L_3+\sqrt{L_4})]\sup_{s\in J}E\|x_1(s)-x_2(s)\|_C^2.$$

Since $2T^2[M^2L_1+l(L_3+\sqrt{L_4})]<1$, then $F$ is a contraction mapping and hence there exists a unique fixed point $x(\cdot)$ in $C(J,L_2(\Omega,X))$ which is the mild solution of problem (1.1). □

# 4 Controllability results

In this section, we discuss the controllability results for System (1.1). Before starting, we consider the following assumption:

$(H_5)$ The linear operator $L_0^T\in L_2(U,X)$ is defined by

$$L_0^Tu=\int_0^T S(T-s)Bu(s)ds.$$

has an inverse operator $(L_0^T)^{-1}$ which takes values in $L_2(J,U)\setminus\ker L_0^T$, where $\ker L_0^T=\{x\in L_2(J,U),L_0^Tx=0\}$, and there are positive constants $M_b,M_L$ such that $\|B\|^2\leq M_b,\|(L_0^T)^{-1}\|^2\leq M_L$.

To the readers' convenience, we give the definitions of controllability as follows.

**Definition 4.1.** System (1.1) is said to be completely controllable on the interval $J$ if

$$\mathcal{R}_t(x_0)=C(J,L_2(\Omega,X)),$$

that is, all the points in $C(J,L_2(\Omega,X))$ can be exactly reached from arbitrary initial condition $x(0)=x_0$ and $x_T$ at time $T$.

**Theorem 4.2.** *Assume that hypotheses $(H_1)-(H_5)$ hold. Then the stochastic system (1.1) is completely controllable on $J$, if*

$$3\Big\{TM^2\Big(L_1T+2M^2M_b^2M_L[M^2L_1T+(L_3+\sqrt{L_4})]T^2\Big)+l(L_3+\sqrt{L_4})\Big\}<1.$$

*Proof.* Fix $T>0$ and let $\mathcal{Z}_T=C(J,L_2(\Omega,X))$ be the Banach space of all functions from $J$ into $L_2(\Omega,X)$, endowed with the supremum norm

$$\|\mu\|_{\mathcal{Z}_T}=\Big(\sup_{t\in[0,T]}E\|\mu(t)\|^2\Big)^{\frac{1}{2}}.$$

Let's consider the set

$$G_T=\{x\in\mathcal{Z}_T:x(0)=x_0\}.$$

We easily know that $G_T$ is a closed subset of $\mathcal{Z}_T$ equipped with norm $\|\cdot\|_{\mathcal{Z}_T}$.

By assumption $(H_5)$, one can choose the feedback control function $u_x(t)$ as

$$u_x(t) = B^*S^*(T-t)E\Big\{(L_0^T)^{-1}(x_T-S(T)x_0-\int_0^T S(T-s)f(s,x(s))ds$$

$$-\Big[\int_0^T S(T-s)\sigma(s)dB_Q^H(s)+\int_0^T\int_Z S(T-s)h(s,x(s),y)N(ds,dy)\Big]|\mathcal{F}_t\Big\}.$$

We will prove that if we use this control $u_x(t)$, the operator $\Phi$ define on $\|\cdot\|_{\mathcal{Z}_T}$ by

$$\Phi(x)(t) = S(t)x_0+\int_0^t S(t-s)BB^*S^*(T-s)E\Big[(L_0^T)^{-1}\Big(x_b-S(T)x_0$$

$$-\int_0^T S(T-\eta)f(s,x(\eta))d\eta-\Big[\int_0^T S(T-\eta)\sigma(\eta)dB_Q^H(\eta)$$

$$+\int_0^T\int_Z S(T-\eta)h(\eta,x(\eta),y)N(d\eta,dy)\Big]|\mathcal{F}_t\Big]ds+\int_0^t S(t-s)f(s,x(s))ds$$

$$+\int_0^t S(t-s)\sigma(s)dB_Q^H(s)+\int_0^t\int_Z S(t-s)h(s,x(s),y)N(ds,dy).$$

has a fixed point on $J$.

To prove that the operator $\Phi$ has a fixed point on $J$, we divide the subsequent proof into the following two steps.

Step 1. For any $x\in G_T$, let's prove that $t\to\Phi(x)(t)$ is continuous on $J$ in the $L_2(\Omega,X)$-sense.

Let $0<t<t+\delta<T$, here $t,t+\delta$ are belong to $J$, and $\delta>0$ is sufficiently small. Then we have

$$\begin{aligned}
&E\|\Phi(x)(t+\delta)-\Phi(x)(t)\|^2\\
\le\ & 9E\|S(t+\delta)x_0-S(t)x_0]\|^2+9E\Big\|\int_0^t[S(t+\delta-s)-S(t-s)]f(s,x(s))ds\Big\|^2\\
&+9E\Big\|\int_t^{t+\delta}S(t+\delta-s)f(s,x(s))ds\Big\|^2+9E\Big\|\int_0^t[S(t+\delta-s)-S(t-s)]B\\
&\times B^*S^*(T-s)(L_0^T)^{-1}\Big(x_T-S(T)x_0-\int_0^T S(T-\eta)f(\eta,x(\eta))d\eta\\
&-\int_0^T S(T-\eta)\sigma(\eta)dB_Q^H(\eta)-\int_0^T\int_Z S(T-\eta)h(\eta,x(\eta),y)N(d\eta,dy)\Big)ds\Big\|^2\\
&+9E\Big\|\int_t^{t+\delta}S(t+\delta-s)BB^*S^*(T-s)(L_0^T)^{-1}\Big(x_T-S(T)x_0-\int_0^T S(T-\eta)f(\eta,x(\eta))d\eta\\
&-\int_0^T S(T-\eta)\sigma(\eta)dB_Q^H(\eta)-\int_0^T\int_Z S(T-\eta)h(\eta,x(\eta),y)N(d\eta,dy)\Big)ds\Big\|^2\\
&+9E\Big\|\int_0^t[S(t+\delta-s)-S(t-s)]\sigma(s)dB_Q^H(s)\Big\|^2+9E\Big\|\int_t^{t+\delta}S(t-s)\sigma(s)dB_Q^H(s)\Big\|^2.\\
&+9E\Big\|\int_0^t\int_Z[S(t+\delta-s)-S(t-s)]h(t,x(t),y)N(ds,dy)\Big\|^2\\
&+9E\Big\|\int_t^{t+\delta}\int_Z S(t+\delta-s)h(t,x(t),y)N(ds,dy)\Big\|^2\\
\le\ & 9\sum_{i=1}^9 I_9.
\end{aligned}$$

We can easily know that

$$I_1\le\|S(t+\delta)-S(t)\|^2E\|x_0\|^2\to 0\qquad\text{as }\delta\to 0.$$

By using the well-known Hölder's inequality, we get

$$I_2\le t\int_0^t\|S(t+\delta-s)-S(t-s)\|^2\sup_{s\in J}E(1+\|x(s)\|^2)ds\to 0\qquad\text{as }\delta\to 0.$$

$$I_3\le M^2t\int_t^{t+\delta}\sup_{s\in J}E(1+\|x(s)\|^2)ds\to 0\qquad\text{as }\delta\to 0.$$

By Hölder's inequality again, Lemma 2.4 and the condition $(H_3)$, we get

$$\begin{aligned}
I_4\ \le\ & 5t\int_0^t\|S(t+\delta-s)-S(t-s)\|^2\|B\|^4\|L_0^T\|^2\Big(E\|x_T\|^2+M^2E\|x_0\|^2\\
&+E\Big\|\int_0^T S(T-\eta)f(\eta,x(\eta))d\eta\Big\|^2+E\Big\|\int_0^T S(T-\eta)\sigma(\eta)dB_Q^H(\eta)\Big\|^2\\
&+E\Big\|\int_0^T\int_Z S(T-\eta)h(\eta,x(\eta),y)N(d\eta,dy)\Big\|^2\Big)ds\\
\le\ & 5M_b^2M_L\int_0^t\|S(t+\delta-s)-S(t-s)\|^2\Big(E\|x_T\|^2+M^2E\|x_0\|^2\\
&+M^2T\int_0^T\sup_{\eta\in J}E(1+\|x(\eta)\|^2)d\eta+M^2cH(2H-1)T^{2H-1}\int_0^T\|\sigma(\eta)\|_{L_2^0}^2d\eta\\
&+l\Big\{E\big(\int_0^T\int_Z E\|h(\eta,x(\eta),y)\|^2\mu(dy)d\eta\big)+E\big(\int_0^T\int_Z E\|h(\eta,x(\eta),y)\|^4\mu(dy)d\eta\big)^{\frac12}\Big\}\Big)ds\\
\le\ & 5M_b^2M_L\int_0^t\|S(t+\delta-s)-S(t-s)\|^2\Big(E\|x_T\|^2+M^2E\|x_0\|^2
\end{aligned}$$

$$+M^2T\int_0^T \sup_{\eta\in J}E(1+\|x(\eta)\|^2)d\eta + M^2cH(2H-1)T^{2H-1}\int_0^T \|\sigma(\eta)\|^2_{L_2^0}d\eta$$
$$+l(L_5T+\sqrt{L_6T})E(1+\|x(s)\|^2)\Big)ds.$$

Hence, by Lebesgue's dominated convergence, one can know that $I_4 \to 0$ as $\delta \to 0$.

For a similar way, we obtain

$$\begin{aligned} I_5 &\leq 5M^2M_b^2M_L\int_t^{t+\delta}\Big(E\|x_T\|^2 + M^2E\|x_0\|^2 + M^2T\int_0^T \sup_{\eta\in J}E(1+\|x(\eta)\|^2)d\eta \\ &\quad +M^2cH(2H-1)T^{2H-1}\int_0^T \|\sigma(\eta)\|^2_{L_2^0}d\eta \\ &\quad +L_4M^2\int_0^T \sup_{\eta\in J}E(1+\|x(\eta)\|^2)d\eta\Big)ds \to 0 \text{ as } \delta\to 0. \end{aligned}$$

$$I_6 \leq cH(2H-1)t^{2H-1}\int_0^t \|S(t+\delta-s)-S(t-s)\|^2\|\sigma(s)\|^2_{L_2^0}ds \text{ as } \delta\to 0.$$

$$I_7 \leq cH(2H-1)M^2\delta^{2H-1}\int_t^{t+\delta}\|\sigma(s)\|^2_{L_2^0}ds \text{ as } \delta\to 0.$$

$$I_8 \leq (L_5T+\sqrt{L_6T})\int_0^t \|S(t+\delta-s)-S(t-s)\|^2 \sup_{s\in J}E(1+\|x(s)\|^2)ds \text{ as } \delta\to 0.$$

$$I_9 \leq l(L_5T+\sqrt{L_6T})\int_t^{t+\delta}\sup_{s\in J}E(1+\|x(s)\|^2)ds \text{ as } \delta\to 0.$$

Then, by the strong continuous of $S(t)$ and the Lebesgue's dominated convergence theorem, we know that the right hand of $I_i(i=1,\cdots,9)$ tends to 0 as $\delta\to 0$. Hence, $\Phi(x)(t)$ is continuous on $J$ in the $L_2(\Omega,X)$-sense.

Next, we prove that $\Phi$ is a contraction mapping. Let $x,z\in C(J,L_2(\Omega,X))$ are two mild solution of (1.1), then

$$\begin{aligned} &E\|\Phi(x)(t)-\Phi(z)(t)\|_H^2 \\ \leq\ & 3E\Big\|\int_0^t S(t-s)[f(s,x(s))-f(s,z(s))]ds\Big\|^2 \\ &+3E\Big\|\int_0^t S(t-s)B(s)[u_x(s)-u_z(s)]ds\Big\|^2 \\ &+3E\Big\|\int_0^t\int_Z S(t-s)[h(s,x(s),y)-h(s,z(s),y)]N(ds,dy)\Big\|^2 \\ \leq\ & 3J_1+3J_2+3J_3. \end{aligned}$$

We can easily show that

$$J_1 \leq TM^2L_1T\sup_{t\in J}E\|x(t)-z(t)\|_H^2, \tag{4.1}$$

$$J_3 \leq l(L_3+\sqrt{L_4})\sup_{t\in J}E\|x(t)-z(t)\|^2. \tag{4.2}$$

Since

$$\begin{aligned} &E\|u_x(t)-u_z(t)\|^2 \\ \leq\ & E\Big\|B^*S^*(T-t)(L_0^T)^{-1}\Big(\int_0^T S(T-s)(f(s,x(s))-f(s,z(s)))ds \\ &-\int_0^T\int_Z S(T-s)[h(s,x(s),y)-h(s,z(s),y)]N(ds,dy)\Big)\Big\|^2 \\ \leq\ & 2M^2M_bM_L\Big(M^2T\int_0^T E\|f(s,x(s))-f(s,z(s))\|^2ds \\ &+l\Big\{E\Big(\int_0^T\int_Z E\|h(\eta,x(\eta),y)\|^2\mu(dy)d\eta\Big)+E\Big(\int_0^T\int_Z E\|h(\eta,x(\eta),y)\|^4\mu(dy)d\eta\Big)^{\frac{1}{2}}\Big\}\Big) \\ \leq\ & 2M^2M_bM_L[M^2L_1T+(L_3+\sqrt{L_4})]\int_0^T\sup_{s\in[0,t]}E\|x(s)-z(s)\|^2ds, \end{aligned}$$

we have

$$\sup_{t\in J}E\|u_x(t)-u_z(t)\|^2 \leq 2M^2M_bM_L[M^2L_1T+(L_3+\sqrt{L_4})]T\sup_{t\in J}E\|x(t)-z(t)\|^2.$$

Hence

$$J_2 \leq 2M^4M_b^2M_L[M^2L_1T+(L_3+\sqrt{L_4})]T^2\sup_{t\in J}E\|x(t)-z(t)\|^2, \tag{4.3}$$

By inequalities (4.1)-(4.3), we get

$$\begin{aligned} &E\|\Psi(x)(t)-\Psi(z)(t)\|^2 \\ \leq\ & 3\left\{TM^2\left(L_1T+2M^2M_b^2M_L[M^2L_1T+(L_3+\sqrt{L_4})]T^2\right)+l(L_3+\sqrt{L_4})\right\}\|x-z\|_C^2. \end{aligned}$$

Since $3\left\{TM^2\left(L_1T+2M^2M_b^2M_L[M^2L_1T+(L_3+\sqrt{L_4})]T^2\right)+l(L_3+\sqrt{L_4})\right\}<1$. Therefore, $\Phi$ is a contraction mapping and hence there exists a unique fixed point $x(\cdot)$ in $C(J,L_2(\Omega,X))$ which is the mild solution of system (1.1). Thus, system (1.1) is complete controllable on $J$.

□

# 5 An example

Let's consider the following stochastic partial differential equations driven by fractional Brownian motion and Poisson noise process:

$$\begin{cases} dx(\theta,t)=\left[\dfrac{\partial^2x(\theta,t)}{\partial\theta^2}+F(\theta,t,x(\theta,t))+g(\theta,t)\right]dt+\sigma(t)dB^H(t) \\ \qquad\qquad +\displaystyle\int_Z\cos tx(\theta,t)\mu\widetilde{N}(dt,d\mu),\ \text{in } \Omega\times[0,\tau], \\ x(\theta,t)=0, \qquad \text{on } \partial\Omega\times[0,\tau], \\ x(\theta,0)=x_0(\theta), \qquad \theta\in\Omega, \end{cases} \tag{5.1}$$

where $B^H$ is a fractional Brownian motion, $\Omega$ is a bounded open set in $R, F:\Omega\times J\times R\to R$ is nonlinear function, measurable with respect to $\theta$ and almost everywhere continuous with respect to $t$. Let $\{q(t),t\in[0,\tau]\}$ be the Poisson jump process taking values in the space $H=[0,\infty)$ with a $\sigma$-finite intensity measure $\lambda(d\mu)$ on the completely probability space $(\Sigma,\mathcal{F},P)$. We denote the Poisson counting measure as $N(dt,d\mu)$, which is induced by $q(\cdot)$, and compensating martingale measure given by

$$\widetilde{N}(dt,d\mu)=N(dt,d\mu)-\lambda(d\mu)dt.$$

Take $X=Y=U=L^2([0,\tau])$ and the operator $A:D(A)\subset X\to X$ is defined by

$$Ax=x'',$$

$$D(A)=\{x\in X: x,\ x' \text{ are absolutely continuous},\ x''\in X,\ x(0)=x(\pi)=0\}.$$

Then, $A$ can be written as

$$Ax=-\sum_{n=1}^{\infty}n^2(x,x_n)x_n, \qquad x\in D(A),$$

where $x_n(x)=\sqrt{2/\pi}\sin ny(n=1,2,\cdots)$ is an orthonormal basis of $X$. It is well known that $A$ is the infinitesimal generator of a differentiable semigroup $T(t)(t>0)$ in $X$ given by

$$T(t)x=\sum_{n=1}^{\infty}\exp^{-n^2t}(x,x_n)x_n, \quad x\in X, \qquad \text{and} \quad \|T(t)\|\leq e^{-1}<1=M.$$

In order to define the operator $Q:Y\to R$, we choose a sequence $\{l_n\}_{n\in N}\subset R^+$, let $Qe_n=l_ne_n$, and assume that

$$\mathrm{tr}(Q)=\sum_{n=1}^{\infty}\sqrt{l_n}<\infty.$$

Thus, we define the fractional Brownian motion in $Y$ as

$$B^H(t)=\sum_{n=1}^{\infty}\sqrt{l_n}\gamma_n^H(t)e_n,$$

where $H\in(\frac{1}{2},1)$ and $\{\gamma_n^H\}_{n\in n}$ is a sequence of one-dimensional fractional Brownian motion mutually independent.

Let $x(t)(\cdot)=x(\cdot,t), f(t,x)(\cdot)=F(\cdot,t,x(\cdot))$. Define the bounded operator $B:U\to X$ by $Bu(t)(\theta)=g(\theta,t),\theta\in\Omega,u\in U$. Hence, by the above choice, it's easily known that the system (5.1) can be written into (1.1) and all the conditions of Theorem 4.2 are satisfied. Then by the Theorem 4.2, the stochastic partial differential equations driven by fractional Brownian motion and Poisson noise process is completely controllable on $[0,\tau]$.

# References


[1] D. Applebaum, F. Tang, Stochastic flows of diffeomorphisms on manifolds driven by infinite-dimensional semimartingales with jumps, Stoch. Proc. Appl., 92 (2001) 219-236

[2] L.H. Bai, J. Ma, Stochastic differential equations driven by fractional Brownian motion and Poisson point process, Bernoulli, 21(1) (2015) 303-334.

[3] K. Balachandran, S. Karthikeyan, J.H. Kim, Controllability of semilinear stochastic integrodifferential systems, Kybernetika, 43(1) (2007) 31-44.

[4] D. Barraez, H. Leiva, N. Merentes, M. Narváez, Exact Controllability of Semilinear Stochastic Evolution Equations, African Diaspora J. Math, 11(1) (2011) 124-139.

[5] M. Besalú, C. Rovira, Stochastic delay equations with non-negativity constraints driven by fractional Brownian motion, Bernoulli, 18(1) (2012) 24-45.

[6] W. Bian, Controllability of nonlinear evolution systems with preassigned responses, J. Optim. Theory Appl., 100 (1999) 265-285.

[7] B. Boufoussi, S. Hajji, Neutral stochastic functional differential equations driven by a fractional Brownian motion in a Hilbert space, Statistics and Probability Letters, 82(8) (2012) 1549-1558.

[8] T. Caraballo, M.J. Garrido-Atienza, The existence and exponential behavior of solutions to stochastic delay evolution equations with a fractional Brownian motion, Nonlinear Analysis, 74 (2011) 3671-3684.

[9] N.T. Dung, Mackey-Glass equation driven by fractional Brownian motion, Physica A, 391(2012) 5465-5472.

[10] S. Hajji, E. Lakhel, Neutral stochastic functional differential equation driven by fractional Brownian motion and Poisson point processes, (2013) 18 pages. arXiv admin note: text overlap with arXiv:1312.6147

[11] Hamdy M. Ahmed, Controllability of impulsive neutral stochastic differential equations with fractional Brownian motion, IMA Journal of Mathematical Control and Information, (2014) 1-14, doi:10.1093/imamci/dnu019

[12] E. Hausenblas, SPDEs driven by Poisson random measure with non-Lipschitz coefficients: Existence results, Probab. Th. Relat. Fields 137 (2007) 161-200.

[13] H. Leiva, Exact controllability of the suspension bridge model proposed by Lazer and McKenna, J. Math. Anal. Appl. 309 (2005) 404-419.

[14] H. Leiva, Exact controllability of semilinear evolution equation and applications, Int. J. Systems Control Communications, 1 (1) (2008) 1-12.

[15] N. Ikeda, S. Watanabe, Stochastic Differential Equations and Diffusion Processes, North-Holland, 1981.

[16] K. Itô, Stochastic Analysis, North-Holland Publishing Company Amsterdam, New York, Oxford, 1982.

[17] R.E. Kalman, Controllablity of linear dynamical systems, Contributions Differ. Equat., 1 (1963) 190-213.

[18] S. Karthikeyan, K. Balachandran, Controllability of nonlinear stochastic neutral impulsive systems, Nonlinear Analysis: HS, 3(3)(2009) 266-276.

[19] J. Klamka, L. Socha, Some remarks about stochastic controllability, IEEE Trans. Automat. Control, 22 (1977) 880-881.

[20] C. Knoche, SPDE's in infinite dimensions with Poisson noise, Comptes Rendus Mathématique, Académie des Sciences. Paris, Serie I, 339 (2004) 647-652.

[21] C. Knoche, Mild solutions of SPDEs driven by Poisson noise in infinite dimensions and their dependence on initial conditions, Preprint no. 05-003 of the Collaborative Research Centre 701, Faculty of Mathematics, University of Bielefeld, Bielefeld, Germany, 2005.

[22] W. Laukajtys, L. Slomiński, Penalization methods for reflecting stochastic differential equations with jumps, Stoch. Stoch. Rep. 75 (5) (2003) 275-293.

[23] J. León, S. Tindel, Malliavin calculus for fractional delay equations, J. Theoretical Probability, 25(3) (2012) 854-889.

[24] B. Liu, Stability of solutions for stochastic impulsive systems via comparison approach, IEEE Trans. Auto. Control, 53 (9) (2008) 2128-2133.

[25] Z.H. Liu, X.W. Li, On the controllability of impulsive fractional evolution inclusions in Banach spaces, J. Optim. Theory Appl., 156 (2013) 167-182.

[26] J. Luo, T. Taniguchi, The existence and uniqueness for non-Lipschitz stochastic neutral delay evolution equations driven by Poisson jumps, Stochastics Dyna., 9 (1) (2009) 135-152.

[27] N.I. Mahmudov, Controllability of linear stochastic systems in Hilbert spaces, J. Math. Anal. Appl. 259 (2001) 64-82.

[28] N.I. Mahmudov, S. Zorlu, Controllability of non-linear stochastic systems. Inter. J. Control, 76 (2) (2003) 95-104.

[29] X.R. Mao, Stochastic Differential Equations and Their Applications, Horwood, Chichester, UK, 1997.

[30] Y. Mishura, Stochastic calculus for fractional Brownian motion and related processes, Springer, Berlin, 2008.

[31] A. Pazy, Semigroups of linear operators and applications to partial differential equations, Springer-Verlag, New York, 1983.

[32] P.E. Protter, Stochastic Integration and Differential Equations, Springer-Verlag, Berlin, Heidelberg, New York, 2004.

[33] G.D. Prato, J. Zabczyk, Stochastic Equations In Infinite Dimensions. Encyclopedia Math. Appl. vol 44, Cambridge University Press, Cambridge, UK, 1992.

[34] J. Ren, J. Wu, Multivalued stochastic differential equations driven by Poisson point processes, in: Proc. of Workshop on Stochastic Analysis and Finance, July 2009.

[35] M. Royer, Backward stochastic differential equations with jumps and related non-linear expectations, Stochastic Process. Appl. 116 (2006) 1358-1376.

[36] L.J. Shen, J.P. Shi, J.T. Sun, Complete controllability of impulsive stochastic integro-differential systems, Automatica 46 (2010) 1068-1073.

[37] R. Subalakshmi, K. Balachandran, Approximate controllability of nonlinear stochastic impulsive integrodifferential systems in hilbert spaces, Chaos Solitons Fractals, 42(4) (2009) 2035-2046.

[38] S. Tindel, C. Tudor, F. Viens, Stochastic evolution equations with fractional brownian motion, Probab. Theory Related Fields, 127 (2) (2003) 186-204.

[39] J. Unterberger, Stochastic calculus for fractinal Brownian montion with Hurst Exponent $H > \frac{1}{4}$ : A rough path method by analytic extension, The Annals of Probability, 27 (2) (2009) 565-614.

[40] H.X. Zhou, Approximate controllability for a class of semilinear abstract equations, SIMA J. Control Optim. 22 (1983) 405-422.

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# A note on non-instantaneous impulsive fractional neutral integro-differential systems with state-dependent delay in Banach spaces

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**Abstract**

In this research, we establish the existence results for non-instantaneous impulsive fractional neutral integro-differential systems with state-dependent delay in Banach space. By utilizing the Banach contraction principle and condensing fixed point theorem coupled with semigroup theory, we build up the desired results. To acquire the main results, our working concepts are that the functions deciding the equation fulfill certain Lipschitz conditions of local type which is similar to the hypotheses [5]. In the end, an example is given to show the abstract theory.

**Keywords**: Fractional order differential systems, Caputo fractional integral operator, non-instantaneous impulses , state-dependent delay, fixed point theorem, semigroup theory.
**MSC 2010**: Primary 34K30, 26A33; Secondary 35R10, 47D06.

## 1 Introduction

Fractional calculus may be considered an old and yet novel topic. In fact, the concepts are almost as old as their more familiar integer-order counterparts. In 1965 Leibniz and L'Hopital had correspondence where they discussed the meaning of the derivative of order one half. Since then, many famous mathematicians have worked on this and related questions, creating the field which is known today as fractional calculus.

The fractional calculus is also considered a novel topic, since it is only during the last three decades that it has been the subject of specialised conferences and treatises. This was stimulated by the fact that many important applications of fractional calculus have been found in numerous diverse and widespread

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fields in science, engineering and finance. Many authors have pointed out that fractional derivatives and integrals are very suitable for modelling the memory and hereditary properties of various materials and processes that are governed by anomalous diffusion. This represents the main advantage of using the fractional derivatives in comparison with classical integer-order models, in which such effects are not taken into account. For more details, we suggest the reader to refer the monographs [6, 24, 47], and the papers [1, 3, 10, 23, 28, 34, 44], and the references cited therein.

Due to the diverse applications in science and technology, functional differential equations turn out to be the most essential branch of research in mathematical sciences. The work on non-integer order functional differential equations with state-dependent delay (abbreviate, SDD), is going on last few years. Furthermore, the study for such kind of the differential equations with SDD, we refer the papers [2, 4, 9, 11–13, 15, 16, 21, 41, 42, 45, 46].

An important feature of real-world dynamic processes that has attracted considerable interest by scientists is the effect of abrupt changes. Hereby, "abrupt" is meant in the sense of a multi-scale problem, i.e. the state of a system changes only slowly for a long time interval, and then undergoes a drastic change within a very short time interval. For example, a football may be flying through the air for several seconds before it changes its flight direction within milliseconds during a collision with a goal post. For the mathematical description of this system, the specification of two sets of equations is appropriate: one for the flight phase, and one for the collision phase.

Several mathematical models can be developed for the football example. In a simplified setting, the motion of the football could be described by the position and velocity of its center of mass, and the encounter with the goal post could be treated as an inelastic collision (i.e. by an immediate change of the football's velocity).

For the description of the collision of the ball with the goal post leads to differential equations in which the velocity experiences, at the time of the collision, a so-called impulse. For additional information on this concept and pertinent advancements of impulsive differential equations (abbreviated, IDEs), for instance [7, 8, 26, 39].

However, in [22, 32], the authors suggested a new class of abstract IDEs for which the impulses are not instantaneous. In particular, in [22], the authors investigate the new type of IDEs with NII of the form

$$u'(t) = Au(t) + f(t, u(t)), \quad t \in (s_i, t_{i+1}],\ i = 0, 1, 2, \ldots, N, \tag{1.1}$$

$$u(t) = g_i(t, u(t)), \quad t \in (t_i, s_i],\ i = 1, 2, \cdots, N, \tag{1.2}$$

$$u(0) = x_0, \tag{1.3}$$

and set up the existence and uniqueness solutions of mild and classical solutions by applying well-known fixed point theorems. From the above system (1.1)-(1.3), we observe that the impulses start instantly at the points $t_i$ and their action continue on a finite time interval $[t_i, s_i]$. As indicated in [22], there are actually several distinct aspirations for the research of this kind of model. For more details on this theory and on its applications, we suggest the reader to refer [14, 22, 30, 32].

Moreover, Pierri et al. [32] have generalized the results of [22], by employing the theory of analytic

semigroup and fractional power of closed operators and proven the existence results of solutions for a class of semilinear IDEs with NII in Banach space. Furthermore, in [17, 19, 20, 25], the authors analyzed the different types of IFDEs with NII in Banach spaces under appropriate fixed point theorem. Recently, Suganya et al.[40] researched the existence results for fractional neutral integro-differential system with SDD and NII in Banach space through the utilization of the Hausdorff's measures of non-compactness and Darbo-Sadovskii fixed point theorem.

On the other hand, the existence results for impulsive fractional neutral integro-differential systems(abbreviated, IFNIDS) with SDD and NII in $\mathscr{B}_h$ phase space axioms have not yet been completely examined. This persuade us to explore the existence results of these types of structures with NII in Banach spaces.

Motivated by the effort of the aforementioned papers [5, 17, 21, 22], the principle motivation behind this manuscript is to research the existence of mild solutions for an IFNIDS with SDD of the model

$$\begin{aligned} {}^{C}D_t^{\alpha}\left[z(t)-\mathscr{Q}_1\left(t,z_{\zeta(t,z_t)},\mathcal{C}z_{\zeta(t,z_t)}\right)\right] &= \mathscr{A}z(t)+\mathscr{Q}_2\left(t,z_{\zeta(t,z_t)},\mathcal{C}z_{\zeta(t,z_t)}\right) \\ &\quad +\mathscr{Q}_3\left(t,z_{\zeta(t,z_t)},\mathcal{C}z_{\zeta(t,z_t)}\right),\ t\in(s_i,t_{i+1}],\ i=0,1,2,\ldots,N, \qquad (1.4)\\ z(t) &= g_i(t,z_{\zeta(t,z_t)}),\quad t\in(t_i,s_i],\ i=1,2,\ldots,N, \qquad (1.5)\\ z_0 &= \varphi(t)\in\mathscr{B}_h,\quad t\in(-\infty,0], \qquad (1.6)\end{aligned}$$

where $\mathscr{A}$ denotes the infinitesimal generator of an analytic semigroup $\{\mathbb{T}(t)\}_{t\geq 0}$ in a Banach space $\mathbb{X}$; ${}^{C}D_t^{\alpha}$ is the Caputo fractional derivative operator of order $\alpha$ with $0<\alpha\leq 1$; $\mathscr{I}=[0,T]$ is an operational interval; $\mathscr{Q}_1,\mathscr{Q}_2,\mathscr{Q}_3:\mathscr{I}\times\mathscr{B}_h\times\mathscr{B}_h\to\mathbb{X}$, $\zeta:\mathscr{I}\times\mathscr{B}_h\to\mathbb{R}$ are appropriate functions, and $\mathscr{B}_h$ is a phase space outlined in next section. The term $\mathcal{C}z_{\zeta(t,z_t)}$ is given by $\mathcal{C}z_{\zeta(t,z_t)}=\int_0^t K(t,s)(z_{\zeta(s,z_s)})ds$, where $K\in\mathcal{C}(\mathscr{D},\mathbb{R}^+)$ is the set of all positive functions which are continuous on $\mathscr{D}=\{(t,s)\in\mathscr{I}\times\mathscr{I}:0\leq s\leq t\leq T\}$ and $\mathcal{C}^*=\sup_{t\in[0,T]}\int_0^t K(t,s)ds$. Here $0=t_0=s_0<t_1\leq s_1<t_2\leq s_2<\cdots<t_N\leq s_N<t_{N+1}=T$, are prefixed numbers, and $g_i\in C((t_i,s_i]\times\mathscr{B}_h,\mathbb{X})$ for all $i=1,2,\ldots,N$, is stand for impulsive conditions.

For almost any continuous function $z$ characterized on $(-\infty,T]$ and for almost any $t\geq 0$, we designate by $z_t$ the part of $\mathscr{B}_h$ characterized by $z_t(\theta)=z(t+\theta)$ for $\theta\leq 0$. Now $z_t(\cdot)$ speaks to the historical backdrop of the state from every $\theta\in(-\infty,0]$ likely the current time $t$.

Contrary to the recent results, this paper has some useful features including the integral term in the involved functions $\mathscr{Q}_1,\mathscr{Q}_2,\mathscr{Q}_3$ and define a suitable mild solution of the model (1.4)-(1.6) with the help of probability density function. Then, based on local Lipschitz conditions of the involved functions, we establish the existence results for IFNIDS with SDD and NII of the problem (1.4)-(1.6) under appropriate fixed point theorem, and the outcomes in [17, 21] might be viewed as the particular situations.

We organize the paper as follows. We provide some basis definitions, lemmas and theorems in Section 2 as these are useful for establish our results. Section 3 focuses on the existence of mild solutions for the model (1.4)-(1.6) with the help of the fixed point theorem. Section 4 provides an example to illustrate the acquired abstract concept.

## 2 Preliminaries

From now on, $\mathbb{X}$ represents Banach space with norm $\|\cdot\|$, $C(\mathscr{I},\mathbb{X})$ denotes the space of all $\mathbb{X}$-valued continuous functions on $\mathscr{I}$ and $\mathscr{L}(\mathbb{X})$ is the Banach space of all linear and bounded operators on $\mathbb{X}$. Furthermore, the notation $B_r(z,\mathbb{X})$ stands for the closed ball with center at $z$ and the radius $r>0$ in $\mathbb{X}$.

Let $\mathscr{A}: D(\mathscr{A})\subset\mathbb{X}\to\mathbb{X}$ is the infinitesimal generator of an analytic semigroup of uniformly bounded linear operator on $\mathbb{X}$. Let $0\in\varrho(\mathscr{A})$, then it is possible to describe the fractional power $\mathscr{A}^{\mu}, 0<\mu\leq 1$, as a closed linear operator on its domain $D(\mathscr{A}^{\mu})$. Moreover, the subspace $D(\mathscr{A}^{\mu})$ is dense in $\mathbb{X}$ and the expression $\|z\|_{\mu}=\|\mathscr{A}^{\mu}z\|$, $z\in D(\mathscr{A}^{\mu})$, defines a norm on $D(\mathscr{A}^{\mu})$. For $0<\nu\leq\mu\leq 1$, $\mathbb{X}_{\mu}\to\mathbb{X}_{\nu}$ and the imbedding is compact whenever the resolvent operator of $\mathscr{A}$ is compact. Also for every $0<\mu\leq 1$, there exists a positive constant $\mathcal{M}_{\mu}$ such that

$$\|\mathscr{A}^{\mu}\mathbb{T}(t)\|\leq\frac{\mathcal{M}_{\mu}}{t^{\mu}},\quad 0<t\leq T.$$

For additional information about the above preliminaries, we refer to [31, 35].

To portray properly our system, we claim that a function $z:[\sigma,\tau]\to\mathbb{X}$ is a normalized piecewise continuous function on $[\sigma,\tau]$ if $z$ is piecewise continuous and left continuous on $(\sigma,\tau]$. By the symbol $\mathcal{PC}([\sigma,\tau];\mathbb{X})$, we mean the space of normalized piecewise continuous functions from $[\sigma,\tau]$ into $\mathbb{X}$. Specifically, we signify the space $\mathcal{PC}$ established by all functions $z:[0,T]\to\mathbb{X}$ in ways that $z$ is continuous at $t\neq t_i$, $z(t_i^-)=z(t_i)$ and $z(t_i^+)$ exists, for all $i=1,2,\cdots,N$. It is not difficult to find out that $\mathcal{PC}$ is a Banach space having the norm $\|z\|_{\mathcal{PC}}=\sup\limits_{s\in[0,T]}\|z(s)\|$.

Once the delay is infinite, then we should talk about the theoretical phase space $\mathscr{B}_h$ in a beneficial way. Thus, in this manuscript, we deliberate phase spaces $\mathscr{B}_h$ which are same as it was described in [21]. As a result, we bypass the details.

We assume that the phase space $(\mathscr{B}_h,\|\cdot\|_{\mathscr{B}_h})$ is a semi-normed linear space of functions mapping $(-\infty,0]$ into $\mathbb{X}$, and fulfilling the subsequent elementary adages as a result of Fu et al. [18] and Ganga Ram Gautam et al. [21].

If $z:(-\infty,T]\to\mathbb{X}, T>0$, is continuous on $\mathscr{I}$ and $z_0\in\mathscr{B}_h$, then for every $t\in\mathscr{I}$ the accompanying conditions hold:

$(P_1)$ $z_t$ is in $\mathscr{B}_h$;

$(P_2)$ $\|z(t)\|_{\mathbb{X}}\leq H\|z_t\|_{\mathscr{B}_h}$;

$(P_3)$ $\|z_t\|_{\mathscr{B}_h}\leq\mathscr{E}_1(t)\sup\{\|z(s)\|_{\mathbb{X}}:0\leq s\leq t\}+\mathscr{E}_2(t)\|z_0\|_{\mathscr{B}_h}$, where $H>0$ is a constant and $\mathscr{E}_1(\cdot):[0,+\infty)\to[0,+\infty)$ is continuous, $\mathscr{E}_2(\cdot):[0,+\infty)\to[0,+\infty)$ is locally bounded, and $\mathscr{E}_1,\mathscr{E}_2$ are independent of $z(\cdot)$.
For our convenience, denote $\mathscr{E}_1^*=\sup\limits_{s\in\mathscr{I}}\mathscr{E}_1(s)$, $\mathscr{E}_2^*=\sup\limits_{s\in\mathscr{I}}\mathscr{E}_2(s)$.

Define the space

$$\mathscr{B}_T=\{z:(-\infty,T]\to\mathbb{X}\quad\text{such that}\quad z_0\in\mathscr{B}_h\quad\text{and the constraint}\quad z|_{\mathscr{I}}\in\mathcal{PC}\},$$

where $z|_{\mathscr{I}}$ is the constraint of $z$ to the real compact interval on $\mathscr{I}$. The function $\|\cdot\|_{\mathscr{B}_T}$ to be a seminorm in $\mathscr{B}_T$, it is described by

$$\|z\|_{\mathscr{B}_T} = \|\varphi\|_{\mathscr{B}_h} + \sup\{\|z(s)\|_{\mathbb{X}} : s \in [0,T]\},\ z \in \mathscr{B}_T.$$

To stay away from the reiterations of a few definitions utilized as a part of this paper we refer the readers: such as for the definition of the fractional integral, Riemann-Liouville fractional integral operator, the generalized Mittag-Leffler special function, Wright-type function and the Caputo's derivative one can see the papers [17, 35, 40] and the monographs [24, 33, 47].

Currently, we are have the ability to define the mild solution for the problem (1.4)-(1.6). For this, initially we treat the following model

$$^{C}D_t^{\alpha} z(t) = \mathscr{A} z(t) + \mathscr{Q}_2(t), \tag{2.1}$$

$$z(0) = z_0, \tag{2.2}$$

where $^{C}D_t^{\alpha}$ and $\mathscr{A}$ are just like described in (1.4)-(1.6).

By thinking the proofs as in [35, Lemma 6 and Lemma 9], we directly define the mild solution for the model (2.1)-(2.2).

**Definition 2.1.** *A function $z : \mathscr{I} \to \mathbb{X}$ is considered to be a mild solution of problem (2.1)-(2.2) if $z \in C(\mathscr{I}, \mathbb{X})$ fulfills the accompanying integral equation:*

$$z(t) \quad = \mathbb{T}_{\alpha}(t) z_0 + \int_0^t \mathbb{S}_{\alpha}(t-s) \mathscr{Q}_2(s) ds, \quad t \in \mathscr{I}.$$

For additional reference about this concept, we suggest the reader to refer[35, 38, 40].

Before we characterize the mild solution for the structure (1.4)-(1.6), finally, we treat the following system

$$^{C}D_t^{\alpha}\big[z(t) - \mathscr{Q}_1(t, z(t))\big] = \mathscr{A} z(t) + \mathscr{Q}_2(t, z(t)) + \mathscr{Q}_3(t, z(t)),\ t \in (s_i, t_{i+1}],\ i = 0, 1, \ldots, N, \tag{2.7}$$

$$z(t) = g_i(t, z(t)),\ t \in (t_i, s_i],\ i = 1, 2, \cdots, N, \tag{2.8}$$

$$z(0) = z_0, \tag{2.9}$$

where $^{C}D_t^{\alpha}, g_i(t, z(t))$ and $\mathscr{A}$ are same as defined in (1.4)-(1.6) and $z_0 \in \mathbb{X}, \mathscr{Q}_1, \mathscr{Q}_2, \mathscr{Q}_3$ are appropriate functions.

We remark that, the impulses in problem (2.7)-(2.9) start abruptly at the points $t_i$ and their action continues on the interval $[t_i, s_i]$. In addition, the function $z$ takes an abrupt impulse at $t_i$ and follows different rules in the two subintervals $(t_i, s_i]$ and $(s_i, t_{i+1}]$ of the interval $(t_i, t_{i+1}]$. At the point $s_i$, the function $z$ is continuous.

On the results received in the papers [35–37, 43, 48], first we define the mild solution for the system

(2.7)-(2.9) is given by

$$x(t)=\begin{cases}\mathbb{T}_\alpha(t)[z_0-\mathscr{Q}_1(0,z_0)]+\mathscr{Q}_1(t,z(t))+\displaystyle\int_0^t \mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{Q}_1(s,z(s))ds\\ \quad+\displaystyle\int_0^t \mathbb{S}_\alpha(t-s)\Big[\mathscr{Q}_2(s,z(s))+\mathscr{Q}_3(s,z(s))\Big]ds, \quad t\in[0,t_1],\\ g_1(t,z(t)), \quad t\in(t_1,s_1],\\ \mathbb{T}_\alpha(t-s_1)d_1+\mathscr{Q}_1(t,z(t))+\displaystyle\int_0^t \mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{Q}_1(s,z(s))ds\\ \quad+\displaystyle\int_0^t \mathbb{S}_\alpha(t-s)\Big[\mathscr{Q}_2(s,z(s))+\mathscr{Q}_3(s,z(s))\Big]ds, \quad t\in(s_1,t_2],\\ \cdots,\\ g_i(t,z(t)), \quad t\in(t_i,s_i],\ i=1,2,\cdots,N,\\ \mathbb{T}_\alpha(t-s_i)d_i+\mathscr{Q}_1(t,z(t))+\displaystyle\int_0^t \mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{Q}_1(s,z(s))ds\\ \quad+\displaystyle\int_0^t \mathbb{S}_\alpha(t-s)\Big[\mathscr{Q}_2(s,z(s))+\mathscr{Q}_3(s,z(s))\Big]ds, \quad t\in(s_i,t_{i+1}],\end{cases}$$

where

$$\begin{aligned}d_i=g_i(s,z(s_i))-\mathscr{Q}_1(s_i,z(s_i))-\int_0^{s_i}\mathscr{A}\mathbb{S}_\alpha(s_i-s)\mathscr{Q}_1(s,z(s))ds\\ -\int_0^{s_i}\mathbb{S}_\alpha(s_i-s)\Big[\mathscr{Q}_2(s,z(s))+\mathscr{Q}_3(s,z(s))\Big]ds,\ i=1,2,\cdots,N.\end{aligned}$$

**Remark 2.1.** *From the discussion in [40], we clearly see that our definition of mild solution fulfills the given model (2.7)-(2.9).*

In accordance with the above discussion, we determine the mild solution of the model (1.4)-(1.6).

**Definition 2.2.** *[40, Definition 2.8] A function $z:(-\infty,T]\to\mathbb{X}$ is called a mild solution of the model (1.1)-(1.3) if $z_0=\varphi\in\mathscr{B}_h$, $z(\cdot)|_{\mathscr{I}}\in\mathcal{PC}$ and for each $s\in[0,t)$ the function $\mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{Q}_1(s,z_{\zeta(s,z_s)},\mathcal{C}z_{\zeta(s,z_s)})$ is integrable and*

$$\begin{aligned}z(t)=\mathbb{T}_\alpha(t)[\varphi(0)-\mathscr{Q}_1(0,\varphi(0),0)]+\mathscr{Q}_1(t,z_{\zeta(t,z_t)},\mathcal{C}z_{\zeta(t,z_t)})\\ +\int_0^t\mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{Q}_1(s,z_{\zeta(s,z_s)},\mathcal{C}z_{\zeta(s,z_s)})ds\\ +\int_0^t\mathbb{S}_\alpha(t-s)\mathscr{Q}_2\left(s,z_{\zeta(s,z_s)},\mathcal{C}z_{\zeta(s,z_s)}\right)ds\\ +\int_0^t\mathbb{S}_\alpha(t-s)\mathscr{Q}_3\left(s,z_{\zeta(s,z_s)},\mathcal{C}z_{\zeta(s,z_s)}\right)ds,\ t\in[0,t_1],\\ g_i(t,z_{\zeta(t,z_t)}),\quad t\in(t_i,s_i],\ i=1,2,\cdots,N,\\ \mathbb{T}_\alpha(t-s_i)d_i+\mathscr{Q}_1(t,z_{\zeta(t,z_t)},\mathcal{C}z_{\zeta(t,z_t)})\\ +\int_0^t\mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{Q}_1(s,z_{\zeta(s,z_s)},\mathcal{C}z_{\zeta(s,z_s)})ds\end{aligned}\tag{2.16}$$

$$+\int_0^t \mathbb{S}_\alpha(t-s)\mathscr{Q}_2\left(s, z_{\zeta(s,z_s)}, \mathcal{C}z_{\zeta(s,z_s)}\right) ds$$
$$+\int_0^t \mathbb{S}_\alpha(t-s)\mathscr{Q}_3\left(s, z_{\zeta(s,z_s)}, \mathcal{C}z_{\zeta(s,z_s)}\right) ds,\ t \in (s_i, t_{i+1}],$$

*where*

$$d_i = g_i(s_i, z_{\zeta(s_i,z_{s_i})}) - \mathscr{Q}_1(s_i, z_{\zeta(s_i,z_{s_i})}, \mathcal{C}z_{\zeta(s_i,z_{s_i})}) - \int_0^{s_i} \mathscr{A}\mathbb{S}_\alpha(s_i-s)\mathscr{Q}_1(s, z_{\zeta(s,z_s)}, \mathcal{C}z_{\zeta(s,z_s)})ds$$
$$-\int_0^{s_i} \mathbb{S}_\alpha(s_i-s)\mathscr{Q}_2\left(s, z_{\zeta(s,z_s)}, \mathcal{C}z_{\zeta(s,z_s)}\right) ds - \int_0^{s_i} \mathbb{S}_\alpha(s_i-s)\mathscr{Q}_3\left(s, z_{\zeta(s,z_s)}, \mathcal{C}z_{\zeta(s,z_s)}\right) ds,$$
$$i = 1, 2, \cdots, N. \tag{2.17}$$

Now, we turn to the statement of Condensing fixed point theorem [21, Theorem 2.9].

**Theorem 2.1.** *Let $B$ be a convex, bounded and closed subset of Banach space $\mathbb{X}$ and let $P : B \to B$ be a condensing map. Then $P$ has a fixed point.*

## 3 Existence results

In this section, we show and demonstrate the existence of solutions for the model (1.4)-(1.6) under different fixed point theorems and we consider $\varphi \in \mathscr{B}_h$ a fixed function, $\mathscr{I} = [0, T]$. To simplify writing of the text, in what follows, we assume that $0 \le \zeta(t, \psi) \le t$ for all $\psi \in \mathscr{B}_h$.

Presently, we itemizing the subsequent suppositions:

(H1) The function $\mathscr{Q}_1 : \mathscr{I} \times \mathscr{B}_h \times \mathscr{B}_h \to \mathbb{X}$ is continuous and we can find constants $\beta \in (0, 1), L_{\mathscr{Q}_1} > 0$, $\widetilde{L}_{\mathscr{Q}_1} > 0$ and $L^*_{\mathscr{Q}_1} > 0$ in ways that $\mathscr{Q}_1$ is $\mathbb{X}_\beta$ -valued and fulfills the subsequent assumptions:

$$\| (\mathscr{A})^\beta \mathscr{Q}_1(t, \varphi_1, \psi_1) - (\mathscr{A})^\beta \mathscr{Q}_1(t, \varphi_2, \psi_2) \|_{\mathbb{X}} \le L_{\mathscr{Q}_1} \|\varphi_1 - \varphi_2\|_{\mathscr{B}_h} + \widetilde{L}_{\mathscr{Q}_1} \| \psi_1 - \psi_2 \|_{\mathscr{B}_h},$$

$$\| (\mathscr{A})^\beta \mathscr{Q}_1(t, \varphi, \psi) \|_{\mathbb{X}} \le L_{\mathscr{Q}_1} \| \psi \|_{\mathscr{B}_h} + L^*_g,$$

where

$$L^*_{\mathscr{Q}_1} = \max_{t \in \mathscr{I}} \|\mathscr{Q}_1(t, 0, 0)\|_{\mathbb{X}}, \quad \text{for all} \quad t \in \mathscr{I} \quad \text{and} \quad \psi, \varphi_1, \varphi_2, \psi_1, \psi_2 \in \mathscr{B}_h.$$

(H2) The function $\mathscr{Q}_2 : \mathscr{I} \times \mathscr{B}_h \times \mathscr{B}_h \to \mathbb{X}$ is continuous and we can find positive constants $L_{\mathscr{Q}_2}, \widetilde{L}_{\mathscr{Q}_2} > 0$ and $L^*_{\mathscr{Q}_2} > 0$ in ways that

$$\|\mathscr{Q}_2(t, \varphi_1, \psi_1) - \mathscr{Q}_2(t, \varphi_2, \psi_2)\|_{\mathbb{X}} \le L_{\mathscr{Q}_2} \|\varphi_1 - \varphi_2\|_{\mathscr{B}_h} + \widetilde{L}_{\mathscr{Q}_2} \|\psi_1 - \psi_2\|_{\mathscr{B}_h},$$

and

$$L^*_{\mathscr{Q}_2} = \max_{t \in \mathscr{I}} \|\mathscr{Q}_2(t, 0, 0)\|_{\mathbb{X}}, \quad \text{for all} \quad t \in \mathscr{I} \quad \text{and} \quad \varphi_1, \varphi_2, \psi_1, \psi_2 \in \mathscr{B}_h.$$

(H3) The function $\mathscr{Q}_3 : \mathscr{I}\times\mathscr{B}_h\times\mathscr{B}_h \to \mathbb{X}$ is continuous and we can find positive constants $L_{\mathscr{Q}_3}, \widetilde{L}_{\mathscr{Q}_3} > 0$ and $L^*_{\mathscr{Q}_3} > 0$ in ways that

$$\|\mathscr{Q}_3(t,\varphi_1,\psi_1) - \mathscr{Q}_3(t,\varphi_2,\psi_2)\|_{\mathbb{X}} \le L_{\mathscr{Q}_3}\|\varphi_1-\varphi_2\|_{\mathscr{B}_h} + \widetilde{L}_{\mathscr{Q}_3}\|\psi_1-\psi_2\|_{\mathscr{B}_h},$$

and

$$L^*_{\mathscr{Q}_3} = \max_{t\in\mathscr{I}}\|\mathscr{Q}_3(t,0,0)\|_{\mathbb{X}}, \quad \text{for all} \quad t\in\mathscr{I} \quad \text{and} \quad \varphi_1,\varphi_2,\psi_1,\psi_2\in\mathscr{B}_h.$$

(H4) The function $g_i : (t_i,s_i]\times\mathscr{B}_h \to \mathbb{X}, i = 1,2,\cdots,N$ are continuous and there exist positive constants $L_{g_i} > 0, L^*_{g_i} > 0$ such that

$$\|g_i(t,\varphi_1) - g_i(t,\varphi_2)\|_{\mathbb{X}} \le L_{g_i}\|\varphi_1-\varphi_2\|_{\mathscr{B}_h},$$
$$\|g_i(t,\varphi)\|_{\mathbb{X}} \le L_{g_i}\|\varphi\|_{\mathscr{B}_h} + L^*_{g_i},$$

where

$$L^*_{g_i} = \max_{t\in(t_i,s_i]}\|g_i(t,0)\|_{\mathbb{X}}, \quad \text{for all} \quad t\in(t_i,s_i] \quad \text{and} \quad \varphi,\varphi_1,\varphi_2\in\mathscr{B}_h.$$

(H5) For every $r > 0$, there exist constants $L_{\mathscr{Q}_1}(r) > 0, L_{\mathscr{Q}_2}(r) > 0, L_{\mathscr{Q}_3}(r) > 0$ and $L_{g_i}(r) > 0$ such that

$$\|\ (\mathscr{A})^\beta\mathscr{Q}_1(t,\varphi_{t_2},\mathcal{C}\psi_{t_2}) - (\mathscr{A})^\beta\mathscr{Q}_1(t,\varphi_{t_1},\mathcal{C}\psi_{t_1})\ \| \le L_{\mathscr{Q}_1}(r)(1+\mathcal{C}^*)|t_2-t_1|,$$
$$\|\ \mathscr{Q}_2(t,\varphi_{t_2},\mathcal{C}\psi_{t_2}) - \mathscr{Q}_2(t,\varphi_{t_1},\mathcal{C}\psi_{t_1})\ \| \le L_{\mathscr{Q}_2}(r)(1+\mathcal{C}^*)|t_2-t_1|,$$
$$\|\ \mathscr{Q}_3(t,\varphi_{t_2},\mathcal{C}\psi_{t_2}) - \mathscr{Q}_3(t,\varphi_{t_1},\mathcal{C}\psi_{t_1})\ \| \le L_{\mathscr{Q}_3}(r)(1+\mathcal{C}^*)|t_2-t_1|,$$
$$\text{and } \|g_i(t,\varphi_{t_2}) - g_i(t,\varphi_{t_1})\| \le L_{g_i}(r)|t_2-t_1|, \quad t,t_1,t_2\in\mathscr{I},$$

for all function $z : (-\infty,T] \to \mathbb{X}$ such that $z_0 = \psi \in \mathscr{B}_h, z : \mathscr{I} \to \mathbb{X}$ is continuous and $\max_{0\le s\le T}\|z(s)\| \le r$.

(H6) The function $\zeta : \mathscr{I}\times\mathscr{B}_h \to [0,\infty)$ satisfies:

(i) For every $\psi\in\mathscr{B}_h$, the function $t\mapsto\zeta(t,\psi)$ is continuous.

(ii) There exists a constant $L_\zeta > 0$ such that
$|\zeta(t,\varphi_2) - \zeta(t,\varphi_1)| \le L_\zeta\|\varphi_2-\varphi_1\|_{\mathscr{B}_h}$, $\varphi_1,\varphi_2\in\mathscr{B}_h$ for all $t\in\mathscr{I}$.

(H7) The following inequalities holds:

(i) Let

$$\max_{1\le i\le N}\Bigg\{\mathcal{M}\mathcal{M}_0L_{\mathscr{Q}_1}\|\varphi\|_{\mathscr{B}_h} + \mathcal{M}L^*_{g_i} + (\mathcal{M}+1)\left(\mathcal{M}_0 + \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)}\cdot\frac{T^{\alpha\beta}}{\beta}\right)L^*_{\mathscr{Q}_1}$$
$$+ \frac{\mathcal{M}(\mathcal{M}+1)T^\alpha}{\Gamma(\alpha+1)}\{L^*_{\mathscr{Q}_2} + L^*_{\mathscr{Q}_3}\} + (\mathscr{E}^*_1 r + c_n)\Bigg[\mathcal{M}L_{g_i}$$
$$+ (\mathcal{M}+1)\left(\mathcal{M}_0 + \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)}\cdot\frac{T^{\alpha\beta}}{\beta}\right)(L_{\mathscr{Q}_1} + \mathcal{C}^*\widetilde{L}_{\mathscr{Q}_1})$$
$$+ \frac{\mathcal{M}(\mathcal{M}+1)T^\alpha}{\Gamma(\alpha+1)}\{(L_{\mathscr{Q}_2} + L_{\mathscr{Q}_3}) + \mathcal{C}^*(\widetilde{L}_{\mathscr{Q}_2} + \widetilde{L}_{\mathscr{Q}_3})\}\Bigg]\Bigg\} \le r, \quad \text{for some}\, r > 0.$$

(ii) Let

$$\Lambda = \mathscr{E}_1^* \max_{1\le i\le N}\Bigg[\mathcal{M}(L_{g_i}+2L_{g_i}(r)L_\zeta)+(\mathcal{M}+1)\left(\mathcal{M}_0+\frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)T^{\alpha\beta}}{\beta\Gamma(\alpha\beta+1)}\right)(L_{\mathscr{Q}_1}+\mathcal{C}^*\widetilde{L}_{\mathscr{Q}_1}$$
$$+L_{\mathscr{Q}_1}(r)L^*)+\frac{\mathcal{M}(\mathcal{M}+1)T^\alpha}{\Gamma(\alpha+1)}\{(L_{\mathscr{Q}_2}+L_{\mathscr{Q}_3})+\mathcal{C}^*(\widetilde{L}_{\mathscr{Q}_2}+\widetilde{L}_{\mathscr{Q}_3})+(L_{\mathscr{Q}_2}(r)+L_{\mathscr{Q}_3}(r))L^*\}\Bigg]<1$$

be such that $0\le\Lambda<1$, where $2L_\zeta(1+\mathcal{C}^*)=L^*$.

(H8) The functions $\mathscr{Q}_1,\mathscr{Q}_2,\mathscr{Q}_3$ and $g_i, i=1,2,\cdots,N$ are continuous and there exist $\mu_{\mathscr{Q}_1}(t),\widetilde{\mu}_{\mathscr{Q}_1}(t),\mu_{\mathscr{Q}_2}(t),\widetilde{\mu}_{\mathscr{Q}_2}(t),\mu_{\mathscr{Q}_3}(t),\widetilde{\mu}_{\mathscr{Q}_3}(t),\mu_{g_i}\in C(\mathscr{I},[0,\infty))$ in a way that

$$\begin{aligned}\|(\mathscr{A})^\beta\mathscr{Q}_1(t,\varphi_1,\varphi_2)\|_{\mathbb{X}}&\le\mu_{\mathscr{Q}_1}(t)\|\varphi_1\|_{\mathscr{B}_h}+\widetilde{\mu}_{\mathscr{Q}_1}(t)\|\varphi_2\|_{\mathscr{B}_h},\\ \|\mathscr{Q}_2(t,\varphi_1,\varphi_2)\|_{\mathbb{X}}&\le\mu_{\mathscr{Q}_2}(t)\|\varphi_1\|_{\mathscr{B}_h}+\widetilde{\mu}_{\mathscr{Q}_2}(t)\|\varphi_2\|_{\mathscr{B}_h},\\ \|\mathscr{Q}_3(t,\varphi_1,\varphi_2)\|_{\mathbb{X}}&\le\mu_{\mathscr{Q}_3}(t)\|\varphi_1\|_{\mathscr{B}_h}+\widetilde{\mu}_{\mathscr{Q}_3}(t)\|\varphi_2\|_{\mathscr{B}_h},\\ \|g_i(t,\varphi)\|_{\mathbb{X}}&\le\mu_{g_i}(t)\|\varphi\|_{\mathscr{B}_h},\ i=1,2,\cdots,N,\end{aligned}$$

for all $t\in\mathscr{I}$ and $\varphi,\varphi_j\in\mathscr{B}_h$, $j=1,2$.

**Theorem 3.1.** *Assume that the conditions (H1)-(H7) hold. Then the structure (1.4)-(1.6) has a unique mild solution on* $(-\infty,T]$.

*Proof.* We will transform the model (1.4)-(1.6) into a fixed-point problem. Recognize the operator $\Upsilon:\mathscr{B}_T\to\mathscr{B}_T$ specified by

$$(\Upsilon x)(t)=\begin{cases}\mathbb{T}_\alpha(t)[\varphi(0)-\mathscr{Q}_1(0,\varphi(0),0)]+\mathscr{Q}_1(t,z_{\zeta(t,z_t)},\mathcal{C}z_{\zeta(t,z_t)})\\ +\displaystyle\int_0^t\mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{Q}_1(s,z_{\zeta(s,z_s)},\mathcal{C}z_{\zeta(s,z_s)})ds\\ +\displaystyle\int_0^t\mathbb{S}_\alpha(t-s)\mathscr{Q}_2\left(s,z_{\zeta(s,z_s)},\mathcal{C}z_{\zeta(s,z_s)}\right)ds\\ +\displaystyle\int_0^t\mathbb{S}_\alpha(t-s)\mathscr{Q}_3\left(s,z_{\zeta(s,z_s)},\mathcal{C}z_{\zeta(s,z_s)}\right)ds,\ t\in[0,t_1],\\ g_i(t,z_{\zeta(t,z_t)}),\quad t\in(t_i,s_i],\ i=1,2,\cdots,N,\\ \mathbb{T}_\alpha(t-s_i)d_i+\mathscr{Q}_1(t,z_{\zeta(t,z_t)},\mathcal{C}z_{\zeta(t,z_t)})\\ +\displaystyle\int_0^t\mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{Q}_1(s,z_{\zeta(s,z_s)},\mathcal{C}z_{\zeta(s,z_s)})ds\\ +\displaystyle\int_0^t\mathbb{S}_\alpha(t-s)\mathscr{Q}_2\left(s,z_{\zeta(s,z_s)},\mathcal{C}z_{\zeta(s,z_s)}\right)ds\\ +\displaystyle\int_0^t\mathbb{S}_\alpha(t-s)\mathscr{Q}_3\left(s,z_{\zeta(s,z_s)},\mathcal{C}z_{\zeta(s,z_s)}\right)ds,\ t\in(s_i,t_{i+1}],\end{cases}\tag{3.1}$$

with $d_i$, $i=1,2,3,\cdots,N$, defined by (2.17).

In perspective of [40, Theorem 2.1] and for any $z\in\mathbb{X}$ and $\beta\in(0,1)$, we obtain

$$\begin{aligned}&\|\mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{Q}_1(s,z_{\zeta(s,z_s)},\mathcal{C}z_{\zeta(s,z_s)})\|_{\mathbb{X}}\\ &=\|\mathscr{A}^{1-\beta}\mathbb{S}_\alpha(t-s)\mathscr{A}^\beta\mathscr{Q}_1(s,z_{\zeta(s,z_s)},\mathcal{C}z_{\zeta(s,z_s)})\|_{\mathbb{X}}\end{aligned}$$

$$\leq \left\| \left[ \alpha \int_0^\infty r\phi_\alpha(r)(t-s)^{\alpha-1} \mathscr{A}^{1-\beta} \mathbb{T}((t-s)^\alpha r)dr \right] \mathscr{A}^\beta \mathscr{Q}_1(s, z_{\zeta(s,z_s)}, \mathcal{C}z_{\zeta(s,z_s)}) \right\|_{\mathbb{X}}$$
$$\leq \alpha \mathcal{M}_{1-\beta}(t-s)^{\alpha\beta-1} \left[ \int_0^\infty r^\beta \phi_\alpha(r)dr \right] \|\mathscr{A}^\beta \mathscr{Q}_1(s, z_{\zeta(s,z_s)}, \mathcal{C}z_{\zeta(s,z_s)})\|_{\mathbb{X}}. \tag{3.2}$$

On the other hand, from $\int_0^\infty r^{-q}\psi_\alpha(r)dr = \frac{\Gamma(1+\frac{q}{\alpha})}{\Gamma(1+q)}$, for all $q \in [0,1]$ (see [48, Lemma 3.2]), we have

$$\int_0^\infty r^\beta \phi_\alpha(r)dr = \int_0^\infty \frac{1}{r^{\beta\alpha}}\psi_\alpha(r)dr = \frac{\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}. \tag{3.3}$$

From (3.2) and (3.3), we conclude that

$$\|\mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{Q}_1(s, z_{\zeta(s,z_s)}, \mathcal{C}z_{\zeta(s,z_s)})\|_{\mathbb{X}}$$
$$\leq \frac{\alpha\mathcal{M}_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)(t-s)^{1-\alpha\beta}} \|\mathscr{A}^\beta \mathscr{Q}_1(s, z_{\zeta(s,z_s)}, \mathcal{C}z_{\zeta(s,z_s)})\|_{\mathbb{X}}. \tag{3.4}$$

It is obvious that the function $s \to \mathscr{A}\mathbb{T}_\alpha(t-s)\mathscr{Q}_1(s, z_{\zeta(s,z_s)}, \mathcal{C}z_{\zeta(s,z_s)})$ is integrable on $[0,t)$ for every $t > 0$.

It is clear that the fixed points of the operator $\Upsilon$ are mild solutions of the model (1.4)-(1.6). We express the function $\widetilde{y}(\cdot) : (-\infty, T] \to \mathbb{X}$ as

$$\widetilde{y}(t) = \begin{cases} \varphi(t), & t \leq 0; \\ \mathbb{T}_\alpha(t)\varphi(0), & t \in \mathscr{I}, \end{cases}$$

then $\widetilde{y}_0 = \varphi$. For every function $x \in C(\mathscr{I}, \mathbb{R})$ with $x(0) = 0$, we allocate as $\widetilde{x}$ is characterized by

$$\widetilde{x}(t) = \begin{cases} 0, & t \leq 0; \\ x(t), & t \in \mathscr{I}. \end{cases}$$

If $z(\cdot)$ obeys (2.16), we are able to split it as $z(t) = \widetilde{y}(t) + x(t)$, $t \in \mathscr{I}$, which suggests $z_t = \widetilde{y}_t + x_t$, for each $t \in \mathscr{I}$ and also the function $x(\cdot)$ obeys

$$x(t) = \begin{cases} -\mathbb{T}_\alpha(t)\mathscr{Q}_1(0, \varphi(0), 0) + \mathscr{Q}_1\left(t, x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}, \mathcal{C}x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}\right) \\ + \int_0^t \mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{Q}_1(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)})ds \\ + \int_0^t \mathbb{S}_\alpha(t-s)\mathscr{Q}_2\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right) ds \\ + \int_0^t \mathbb{S}_\alpha(t-s)\mathscr{Q}_3\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right) ds, \ t \in [0, t_1], \\ g_i(t, x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}), \quad t \in (t_i, s_i], \ i = 1, 2, \cdots, N, \\ \mathbb{T}_\alpha(t-s_i)\widetilde{d}_i + \mathscr{Q}_1(t, x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}, \mathcal{C}x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}) \\ + \int_0^t \mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{Q}_1(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)})ds \\ + \int_0^t \mathbb{S}_\alpha(t-s)\mathscr{Q}_2\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right) ds \\ + \int_0^t \mathbb{S}_\alpha(t-s)\mathscr{Q}_3\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right) ds, \ t \in (s_i, t_{i+1}], \end{cases}$$

where

$$\begin{aligned}\widetilde{d}_i &= g_i(s_i, x_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})}) \\ &- \mathscr{Q}_1(s_i, x_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})}, \mathcal{C}x_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})}) \\ &- \int_0^{s_i} \mathscr{A}\mathbb{S}_\alpha(s_i-s)\mathscr{Q}_1(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)})ds \\ &- \int_0^{s_i} \mathbb{S}_\alpha(s_i-s)\mathscr{Q}_2\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right) ds \\ &- \int_0^{s_i} \mathbb{S}_\alpha(s_i-s)\mathscr{Q}_3\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right) ds,\ i = 1,2,\cdots,N. \quad (3.5)\end{aligned}$$

Let $\mathscr{B}_T^0 = \{x \in \mathscr{B}_T\colon\ x_0 = 0 \in \mathscr{B}_h\}$. Let $\|\cdot\|_{\mathscr{B}_T^0}$ be the seminorm in $\mathscr{B}_T^0$ described by

$$\|x\|_{\mathscr{B}_T^0} = \sup_{t\in\mathscr{I}}\|x(t)\|_{\mathbb{X}} + \|x_0\|_{\mathscr{B}_h} = \sup_{t\in\mathscr{I}}\|x(t)\|_{\mathbb{X}}, \quad x \in \mathscr{B}_T^0,$$

as a result $(\mathscr{B}_T^0, \|\cdot\|_{\mathscr{B}_T^0})$ is a Banach space. Set $B_r = \{x \in \mathscr{B}_T^0 : \|x\|_{\mathbb{X}} \le r\}$ for some $r \ge 0$; then for each $r, B_r \subset \mathscr{B}_T^0$ is clearly a bounded closed convex set. For $x \in B_r(0, \mathscr{B}_T^0)$, from phase space axioms $(P_1) - (P_3)$ and along with the above discussion, we receive

$$\begin{aligned}&\|x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\|_{\mathscr{B}_h} \\ &\le \|x_{\zeta(s,x_s+\widetilde{y}_s)}\|_{\mathscr{B}_h} + \|\widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\|_{\mathscr{B}_h} \\ &\le \mathscr{E}_1^* \sup_{0\le\tau\le\zeta(s,x_s+\widetilde{y}_s)}\|x(\tau)\|_{\mathbb{X}} + \mathscr{E}_2^*\|x_0\|_{\mathscr{B}_h} + \mathscr{E}_1^* \sup_{0\le\tau\le\zeta(s,x_s+\widetilde{y}_s)}\|\widetilde{y}(\tau)\| + \mathscr{E}_2^*\|\widetilde{y}_0\|_{\mathscr{B}_h} \\ &\le \mathscr{E}_1^* \sup_{0\le\tau\le s}\|x(\tau)\|_{\mathbb{X}} + \mathscr{E}_1^*\|\mathbb{T}_\alpha(t)\|_{\mathscr{L}(\mathbb{X})}\|\varphi(0)\| + \mathscr{E}_2^*\|\varphi\|_{\mathscr{B}_h} \\ &\le \mathscr{E}_1^* \sup_{0\le\tau\le s}\|x(\tau)\|_{\mathbb{X}} + (\mathscr{E}_1^*\mathcal{M}H + \mathscr{E}_2^*)\|\varphi\|_{\mathscr{B}_h}.\end{aligned}$$

In the event that $\|x\|_{\mathbb{X}} < r$, $r > 0$, then

$$\|x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\|_{\mathscr{B}_h} \le \mathscr{E}_1^* r + c_n, \quad s \in \mathscr{I}, \quad (3.6)$$

where $c_n = (\mathscr{E}_1^*\mathcal{M}H + \mathscr{E}_2^*)\|\varphi\|_{\mathscr{B}_h}$. We delimit the operator $\overline{\Upsilon} : \mathscr{B}_T^0 \to \mathscr{B}_T^0$ by

$$(\overline{\Upsilon}z)(t) = \begin{cases} -\mathbb{T}_\alpha(t)\mathscr{Q}_1(0,\varphi(0),0) + \mathscr{Q}_1\left(t, x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}, \mathcal{C}x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}\right) \\ + \int_0^t \mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{Q}_1(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)})ds \\ + \int_0^t \mathbb{S}_\alpha(t-s)\mathscr{Q}_2\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right) ds \\ + \int_0^t \mathbb{S}_\alpha(t-s)\mathscr{Q}_3\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right) ds,\ t \in [0,t_1], \\ g_i(t, x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}), \quad t \in (t_i, s_i],\ i = 1,2,\cdots,N, \\ \mathbb{T}_\alpha(t-s_i)\widetilde{d}_i + \mathscr{Q}_1(t, x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}, \mathcal{C}x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}) \\ + \int_0^t \mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{Q}_1(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)})ds \\ + \int_0^t \mathbb{S}_\alpha(t-s)\mathscr{Q}_2\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right) ds \\ + \int_0^t \mathbb{S}_\alpha(t-s)\mathscr{Q}_3\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right) ds,\ t \in (s_i, t_{i+1}], \end{cases}$$

with $\widetilde{d}_i, i = 1, 2, \cdots, N$, defined by (3.5).

It is vindicated that the operator $\Upsilon$ has a fixed point if and only if $\overline{\Upsilon}$ admits a fixed point.

**Remark 3.1.** *As a result, we have the following estimations:*

$$
\begin{aligned}
I_1 &= \|\mathbb{T}_\alpha(t)\mathscr{Q}_1(0,\varphi(0),0)\|_{\mathbb{X}} \\
&\le \left\| \int_0^\infty \phi_\alpha(r)\mathbb{T}(t^\alpha r)dr\,\mathscr{Q}_1(0,\varphi,0) \right\|_{\mathbb{X}} \\
&= \mathcal{M}\|(\mathscr{A})^{-\beta}\|\|(\mathscr{A})^{\beta}\mathscr{Q}_1(0,\varphi,0)\|_{\mathbb{X}} \\
&\le \mathcal{M}\mathcal{M}_0\Big[L_{\mathscr{Q}_1}\|\varphi\|_{\mathscr{B}_h} + L^*_{\mathscr{Q}_1}\Big], \quad \textit{where} \quad \mathcal{M}_0 = \|(\mathscr{A})^{-\beta}\|. \\
I_2 &= \left\| \mathscr{Q}_1\left(t, x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}, \mathcal{C}x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}\right)\right\|_{\mathbb{X}} \\
&\le \|(\mathscr{A})^{-\beta}\|\Big[\|(\mathscr{A})^{\beta}\mathscr{Q}_1\left(t, x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}, \mathcal{C}x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}\right) - (\mathscr{A})^{\beta}\mathscr{Q}_1(t,0,0)\|_{\mathbb{X}} \\
&\quad + \|(\mathscr{A})^{\beta}\mathscr{Q}_1(t,0,0)\|_{\mathbb{X}}\Big] \\
&\le \mathcal{M}_0\Big[(L_{\mathscr{Q}_1} + \widetilde{L}_{\mathscr{Q}_1}\mathcal{C}^*)(\mathscr{E}_1^* r + c_n) + L^*_{\mathscr{Q}_1}\Big]. \\
I_3 &= \left\| \int_0^t \mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{Q}_1(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)})ds \right\|_{\mathbb{X}} \\
&\le \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)} \cdot \frac{t_1^{\alpha\beta}}{\beta}\Big[(L_{\mathscr{Q}_1} + \widetilde{L}_{\mathscr{Q}_1}\mathcal{C}^*)(\mathscr{E}_1^* r + c_n) + L^*_{\mathscr{Q}_1}\Big],\ t \in [0, t_1]. \\
I_4 &= \left\| \int_0^t \mathbb{S}_\alpha(t-s)\mathscr{Q}_2\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right) ds \right\|_{\mathbb{X}} \\
&\le \Big\| \int_0^t \alpha \int_0^\infty r\phi_\alpha(r)(t-s)^{\alpha-1}\mathbb{T}((t-s)^\alpha r)dr \\
&\quad \times \mathscr{Q}_2\Big(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\Big)ds \Big\|_{\mathbb{X}} \\
&\le \alpha\left[\int_0^\infty r\phi_\alpha(r)dr\right]\int_0^t (t-s)^{\alpha-1}\|\mathbb{T}((t-s)^\alpha r)\|_{\mathscr{L}(\mathbb{X})} \\
&\quad \times \left\|\mathscr{Q}_2\Big(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\Big)ds\right\|_{\mathbb{X}} \\
&\le \frac{\mathcal{M}(t_1)^\alpha}{\Gamma(\alpha+1)}\Big[(L_{\mathscr{Q}_2} + \widetilde{L}_{\mathscr{Q}_2}\mathcal{C}^*)(\mathscr{E}_1^* r + c_n) + L^*_{\mathscr{Q}_2}\Big],\ t \in [0, t_1]. \\
I_5 &= \left\| \int_0^t \mathbb{S}_\alpha(t-s)\mathscr{Q}_3\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right) ds \right\|_{\mathbb{X}} \\
&\le \frac{\mathcal{M}(t_1)^\alpha}{\Gamma(\alpha+1)}\Big[(L_{\mathscr{Q}_3} + \widetilde{L}_{\mathscr{Q}_3}\mathcal{C}^*)(\mathscr{E}_1^* r + c_n) + L^*_{\mathscr{Q}_3}\Big],\ t \in [0, t_1]. \\
I_6 &= \|g_i(t, x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)})\|_{\mathbb{X}} \\
&\le \|g_i(t, x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}) - g_i(t,0)\|_{\mathbb{X}} + \|g_i(t,0)\|_{\mathbb{X}} \\
&\le L_{g_i}(\mathscr{E}_1^* r + c_n) + L^*_{g_i},\ t \in (t_i, s_i]. \\
I_7 &= \|g_i(s_i, x_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})})\|_{\mathbb{X}}
\end{aligned}
$$

$$\leq L_{g_i}(\mathscr{E}_1^* r + c_n) + L_{g_i}^*,\ t \in (s_i, t_{i+1}].$$

$$I_8 = \|\mathscr{Q}_1(s_i, x_{\zeta(s_i, x_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i, x_{s_i}+\widetilde{y}_{s_i})}, \mathcal{C}x_{\zeta(s_i, x_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i, x_{s_i}+\widetilde{y}_{s_i})})\|_{\mathbb{X}}$$
$$\leq \mathcal{M}_0\left[(L_{\mathscr{Q}_1} + \widetilde{L}_{\mathscr{Q}_1}\mathcal{C}^*)(\mathscr{E}_1^* r + c_n) + L_{\mathscr{Q}_1}^*\right],\ t \in (s_i, t_{i+1}].$$

$$I_9 = \left\| \int_0^{s_i} \mathscr{A}\mathbb{S}_\alpha(s_i - s)\mathscr{Q}_1(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)})ds \right\|_{\mathbb{X}}$$
$$\leq \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)} \cdot \frac{(s_i)^{\alpha\beta}}{\beta}\left[(L_{\mathscr{Q}_1} + \widetilde{L}_{\mathscr{Q}_1}\mathcal{C}^*)(\mathscr{E}_1^* r + c_n) + L_{\mathscr{Q}_1}^*\right],\ t \in (s_i, t_{i+1}].$$

$$I_{10} = \left\| \int_0^{s_i} \mathbb{S}_\alpha(s_i - s)\mathscr{Q}_2\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right) ds \right\|_{\mathbb{X}}$$
$$\leq \frac{\mathcal{M}(s_i)^\alpha}{\Gamma(\alpha+1)}\left[(L_{\mathscr{Q}_2} + \widetilde{L}_{\mathscr{Q}_2}\mathcal{C}^*)(\mathscr{E}_1^* r + c_n) + L_{\mathscr{Q}_2}^*\right],\ t \in (s_i, t_{i+1}].$$

$$I_{11} = \left\| \int_0^{s_i} \mathbb{S}_\alpha(s_i - s)\mathscr{Q}_3\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right) ds \right\|_{\mathbb{X}}$$
$$\leq \frac{\mathcal{M}(s_i)^\alpha}{\Gamma(\alpha+1)}\left[(L_{\mathscr{Q}_3} + \widetilde{L}_{\mathscr{Q}_3}\mathcal{C}^*)(\mathscr{E}_1^* r + c_n) + L_{\mathscr{Q}_3}^*\right],\ t \in (s_i, t_{i+1}].$$

$$I_{12} = \left\| \int_0^{t} \mathscr{A}\mathbb{S}_\alpha(t - s)\mathscr{Q}_1(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)})ds \right\|_{\mathbb{X}}$$
$$\leq \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)} \cdot \frac{(t_{i+1})^{\alpha\beta}}{\beta}\left[(L_{\mathscr{Q}_1} + \widetilde{L}_{\mathscr{Q}_1}\mathcal{C}^*)(\mathscr{E}_1^* r + c_n) + L_{\mathscr{Q}_1}^*\right],\ t \in (s_i, t_{i+1}].$$

$$I_{13} = \left\| \int_0^{t} \mathbb{S}_\alpha(t - s)\mathscr{Q}_2\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right) ds \right\|_{\mathbb{X}}$$
$$\leq \frac{\mathcal{M}(t_{i+1})^\alpha}{\Gamma(\alpha+1)}\left[(L_{\mathscr{Q}_2} + \widetilde{L}_{\mathscr{Q}_2}\mathcal{C}^*)(\mathscr{E}_1^* r + c_n) + L_{\mathscr{Q}_2}^*\right],\ t \in (s_i, t_{i+1}].$$

$$I_{14} = \left\| \int_0^{t} \mathbb{S}_\alpha(t - s)\mathscr{Q}_3\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right) ds \right\|_{\mathbb{X}}$$
$$\leq \frac{\mathcal{M}(t_{i+1})^\alpha}{\Gamma(\alpha+1)}\left[(L_{\mathscr{Q}_3} + \widetilde{L}_{\mathscr{Q}_3}\mathcal{C}^*)(\mathscr{E}_1^* r + c_n) + L_{\mathscr{Q}_3}^*\right],\ t \in (s_i, t_{i+1}].$$

$$I_{15} = \Big\| \mathscr{Q}_1\left(t, x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}, \mathcal{C}x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}\right)$$
$$- \mathscr{Q}_1\left(t, \overline{x}_{\zeta(t,\overline{x}_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,\overline{x}_t+\widetilde{y}_t)}, \mathcal{C}\overline{x}_{\zeta(t,\overline{x}_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,\overline{x}_t+\widetilde{y}_t)}\right) \Big\|_{\mathbb{X}}$$
$$\leq \Big\| \mathscr{Q}_1\left(t, x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}, \mathcal{C}x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}\right)$$
$$- \mathscr{Q}_1\left(t, \overline{x}_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}, \mathcal{C}\overline{x}_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}\right)$$
$$+ \mathscr{Q}_1\left(t, \overline{x}_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}, \mathcal{C}\overline{x}_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}\right)$$
$$- \mathscr{Q}_1\left(t, \overline{x}_{\zeta(t,\overline{x}_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,\overline{x}_t+\widetilde{y}_t)}, \mathcal{C}\overline{x}_{\zeta(t,\overline{x}_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,\overline{x}_t+\widetilde{y}_t)}\right) \Big\|_{\mathbb{X}}$$
$$\leq \mathcal{M}_0\mathscr{E}_1^*(L_{\mathscr{Q}_1} + \widetilde{L}_{\mathscr{Q}_1}\mathcal{C}^* + L_{\mathscr{Q}_1}(r)L^*)\|x - \overline{x}\|_{\mathscr{B}_T^0},$$

$$I_{16} = \Big\| \int_0^{t} \mathscr{A}\mathbb{S}_\alpha(t - s)\Big[\mathscr{Q}_1(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)})$$
$$- \mathscr{Q}_1(s, \overline{x}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)}, \mathcal{C}\overline{x}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)})\Big] ds \Big\|_{\mathbb{X}}$$

$$\leq \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)} \cdot \frac{t_1^{\alpha\beta}}{\beta}\mathscr{E}_1^*(L_{\mathscr{Q}_1} + \widetilde{L}_{\mathscr{Q}_1}\mathcal{C}^* + L_{\mathscr{Q}_1}(r)L^*)\|x-\overline{x}\|_{\mathscr{B}_T^0},\ t\in[0,t_1].$$

$$I_{17} = \Big\| \int_0^t \mathbb{S}_\alpha(t-s)\Big[\mathscr{Q}_2\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right)$$
$$- \mathscr{Q}_2\left(s, \overline{x}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)}, \mathcal{C}\overline{x}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)}\right)\Big]ds\Big\|_{\mathbb{X}}$$
$$\leq \frac{\mathcal{M}(t_1)^\alpha}{\Gamma(\alpha+1)}\mathscr{E}_1^*(L_{\mathscr{Q}_2} + \widetilde{L}_{\mathscr{Q}_2}\mathcal{C}^* + L_{\mathscr{Q}_2}(r)L^*)\|x-\overline{x}\|_{\mathscr{B}_T^0},\ t\in[0,t_1].$$

$$I_{18} = \Big\| \int_0^t \mathbb{S}_\alpha(t-s)\Big[\mathscr{Q}_3\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right)$$
$$- \mathscr{Q}_3\left(s, \overline{x}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)}, \mathcal{C}\overline{x}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)}\right)\Big]ds\Big\|_{\mathbb{X}}$$
$$\leq \frac{\mathcal{M}(t_1)^\alpha}{\Gamma(\alpha+1)}\mathscr{E}_1^*(L_{\mathscr{Q}_3} + \widetilde{L}_{\mathscr{Q}_3}\mathcal{C}^* + L_{\mathscr{Q}_3}(r)L^*)\|x-\overline{x}\|_{\mathscr{B}_T^0},\ t\in[0,t_1].$$

$$I_{19} = \|g_i(t, x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}) - g_i(t, \overline{x}_{\zeta(t,\overline{x}_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,\overline{x}_t+\widetilde{y}_t)})\|_{\mathbb{X}}$$
$$\leq \mathscr{E}_1^*[L_{g_i} + 2L_{g_i}(r)L_\zeta]\|x-\overline{x}\|_{\mathscr{B}_T^0},\ t\in(t_i,s_i].$$

$$I_{20} = \|g_i(s_i, x_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})}) - g_i(s_i, \overline{x}_{\zeta(s_i,\overline{x}_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i,\overline{x}_{s_i}+\widetilde{y}_{s_i})})\|_{\mathbb{X}}$$
$$\leq \mathscr{E}_1^*[L_{g_i} + 2L_{g_i}(r)L_\zeta]\|x-\overline{x}\|_{\mathscr{B}_T^0},\ t\in(s_i,t_{i+1}].$$

$$I_{21} = \|\mathscr{Q}_1(s_i, x_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})}, \mathcal{C}x_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})})$$
$$- \mathscr{Q}_1(s_i, \overline{x}_{\zeta(s_i,\overline{x}_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i,\overline{x}_{s_i}+\widetilde{y}_{s_i})}, \mathcal{C}\overline{x}_{\zeta(s_i,\overline{x}_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i,\overline{x}_{s_i}+\widetilde{y}_{s_i})})\|_{\mathbb{X}}$$
$$\leq \mathcal{M}_0\mathscr{E}_1^*(L_{\mathscr{Q}_1} + \widetilde{L}_{\mathscr{Q}_1}\mathcal{C}^* + L_{\mathscr{Q}_1}(r)L^*)\|x-\overline{x}\|_{\mathscr{B}_T^0},\ t\in(s_i,t_{i+1}].$$

$$I_{22} = \Big\| \int_0^{s_i} \mathscr{A}\mathbb{S}_\alpha(s_i-s)\Big[\mathscr{Q}_1(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)})$$
$$- \mathscr{Q}_1(s, \overline{x}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)}, \mathcal{C}\overline{x}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)})\Big]ds\Big\|_{\mathbb{X}}$$
$$\leq \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)} \cdot \frac{(s_i)^{\alpha\beta}}{\beta}\mathscr{E}_1^*(L_{\mathscr{Q}_1} + \widetilde{L}_{\mathscr{Q}_1}\mathcal{C}^* + L_{\mathscr{Q}_1}(r)L^*)\|x-\overline{x}\|_{\mathscr{B}_T^0},\ t\in(s_i,t_{i+1}].$$

$$I_{23} = \Big\| \int_0^{s_i} \mathbb{S}_\alpha(s_i-s)\Big[\mathscr{Q}_2\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right)$$
$$- \mathscr{Q}_2\left(s, \overline{x}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)}, \mathcal{C}\overline{x}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)}\right)\Big]ds\Big\|_{\mathbb{X}}$$
$$\leq \frac{\mathcal{M}(s_i)^\alpha}{\Gamma(\alpha+1)}\mathscr{E}_1^*(L_{\mathscr{Q}_2} + \widetilde{L}_{\mathscr{Q}_2}\mathcal{C}^* + L_{\mathscr{Q}_2}(r)L^*)\|x-\overline{x}\|_{\mathscr{B}_T^0},\ t\in(s_i,t_{i+1}].$$

$$I_{24} = \Big\| \int_0^{s_i} \mathbb{S}_\alpha(s_i-s)\Big[\mathscr{Q}_3\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right)$$
$$- \mathscr{Q}_3\left(s, \overline{x}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)}, \mathcal{C}\overline{x}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)}\right)\Big]ds\Big\|_{\mathbb{X}}$$
$$\leq \frac{\mathcal{M}(s_i)^\alpha}{\Gamma(\alpha+1)}\mathscr{E}_1^*(L_{\mathscr{Q}_3} + \widetilde{L}_{\mathscr{Q}_3}\mathcal{C}^* + L_{\mathscr{Q}_3}(r)L^*)\|x-\overline{x}\|_{\mathscr{B}_T^0},\ t\in(s_i,t_{i+1}].$$

$$I_{25} = \Big\| \int_0^t \mathscr{A}\mathbb{S}_\alpha(t-s)\Big[\mathscr{Q}_1(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)})$$

$$- \mathscr{Q}_1(s, \overline{x}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)}, \mathcal{C}\overline{x}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)})\Big] ds\Big\|_{\mathbb{X}}$$

$$\leq \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)} \cdot \frac{(t_{i+1})^{\alpha\beta}}{\beta} \mathscr{E}_1^*(L_{\mathscr{Q}_1} + \widetilde{L}_{\mathscr{Q}_1}\mathcal{C}^* + L_{\mathscr{Q}_1}(r)L^*)\|x-\overline{x}\|_{\mathscr{B}_T^0},\ t \in (s_i, t_{i+1}].$$

$$I_{26} = \Big\| \int_0^t \mathbb{S}_\alpha(t-s)\Big[\mathscr{Q}_2\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right)$$

$$- \mathscr{Q}_2\left(s, \overline{x}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)}, \mathcal{C}\overline{x}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)}\right)\Big] ds\Big\|_{\mathbb{X}}$$

$$\leq \frac{\mathcal{M}(t_{i+1})^\alpha}{\Gamma(\alpha+1)} \mathscr{E}_1^*(L_{\mathscr{Q}_2} + \widetilde{L}_{\mathscr{Q}_2}\mathcal{C}^* + L_{\mathscr{Q}_2}(r)L^*)\|x-\overline{x}\|_{\mathscr{B}_T^0},\ t \in (s_i, t_{i+1}].$$

$$I_{27} = \Big\| \int_0^t \mathbb{S}_\alpha(t-s)\Big[\mathscr{Q}_3\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right)$$

$$- \mathscr{Q}_3\left(s, \overline{x}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)}, \mathcal{C}\overline{x}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,\overline{x}_s+\widetilde{y}_s)}\right)\Big] ds\Big\|_{\mathbb{X}}$$

$$\leq \frac{\mathcal{M}(t_{i+1})^\alpha}{\Gamma(\alpha+1)} \mathscr{E}_1^*(L_{\mathscr{Q}_3} + \widetilde{L}_{\mathscr{Q}_3}\mathcal{C}^* + L_{\mathscr{Q}_3}(r)L^*)\|x-\overline{x}\|_{\mathscr{B}_T^0},\ t \in (s_i, t_{i+1}].$$

Now, we start proving the main proof of this Theorem. We demonstrate that $\overline{\Upsilon}$ maps $B_r(0, \mathscr{B}_T^0)$ into $B_r(0, \mathscr{B}_T^0)$. For any $x(\cdot) \in \mathscr{B}_T^0$, by employing Remark 3.1, we sustain

$$\|(\overline{\Upsilon}x)(t)\|_{\mathbb{X}} = \sum_{i=1}^{5} I_i$$

$$\leq \mathcal{M}\mathcal{M}_0 L_{\mathscr{Q}_1}\|\varphi\|_{\mathscr{B}_h} + \left(\mathcal{M}_0(\mathcal{M}+1) + \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)} \cdot \frac{t_1^{\alpha\beta}}{\beta}\right) L_{\mathscr{Q}_1}^*$$

$$+ \frac{\mathcal{M}(t_1)^\alpha}{\Gamma(\alpha+1)}\{L_{\mathscr{Q}_2}^* + L_{\mathscr{Q}_3}^*\} + (\mathscr{E}_1^* r + c_n)\Bigg[\left(\mathcal{M}_0 + \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)} \cdot \frac{t_1^{\alpha\beta}}{\beta}\right)(L_{\mathscr{Q}_1} + \mathcal{C}^*\widetilde{L}_{\mathscr{Q}_1})$$

$$+ \frac{\mathcal{M}(t_1)^\alpha}{\Gamma(\alpha+1)}\{(L_{\mathscr{Q}_2} + L_{\mathscr{Q}_3}) + \mathcal{C}^*(\widetilde{L}_{\mathscr{Q}_2} + \widetilde{L}_{\mathscr{Q}_3})\}\Bigg]$$

$$\leq r, \quad t \in [0, t_1].$$

$$\|(\overline{\Upsilon}x)(t)\|_{\mathbb{X}} = I_6 \leq L_{g_i}(\mathscr{E}_1^* r + c_n) + L_{g_i}^*,\ t \in (t_i, s_i], \quad i = 1, 2, \cdots, N.$$

$$\|(\overline{\Upsilon}x)(t)\|_{\mathbb{X}} = \sum_{i=7}^{14} I_i$$

$$\leq \max_{1\leq i\leq N}\Bigg\{\mathcal{M}L_{g_i}^* + \left(\mathcal{M}_0(\mathcal{M}+1) + \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\beta\Gamma(\alpha\beta+1)} \cdot \left[\mathcal{M}(s_i)^{\alpha\beta} + (t_{i+1})^{\alpha\beta}\right]\right) L_{\mathscr{Q}_1}^*$$

$$+ \frac{\mathcal{M}}{\Gamma(\alpha+1)}\{L_{\mathscr{Q}_2}^* + L_{\mathscr{Q}_3}^*\}\left[\mathcal{M}(s_i)^\alpha + (t_{i+1})^\alpha\right] + (\mathscr{E}_1^* r + c_n)\Bigg[\mathcal{M}L_{g_i}$$

$$+ \left(\mathcal{M}_0(\mathcal{M}+1) + \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\beta\Gamma(\alpha\beta+1)} \cdot \{\mathcal{M}(s_i)^{\alpha\beta} + (t_{i+1})^{\alpha\beta}\}\right)(L_{\mathscr{Q}_1} + \mathcal{C}^*\widetilde{L}_{\mathscr{Q}_1})$$

$$+ \frac{\mathcal{M}}{\Gamma(\alpha+1)}\{(L_{\mathscr{Q}_2} + L_{\mathscr{Q}_3}) + \mathcal{C}^*(\widetilde{L}_{\mathscr{Q}_2} + \widetilde{L}_{\mathscr{Q}_3})\}\{\mathcal{M}(s_i)^\alpha + (t_{i+1})^\alpha\}\Bigg]\Bigg\} \leq r, \quad t \in (s_i, t_{i+1}],$$

Then, for all $t \in \mathscr{I}$, we conclude that

$$\begin{aligned}
&\|(\overline{\Upsilon}x)(t)\|_{\mathbb{X}} \\
&\leq \max_{1\leq i\leq N}\Bigg\{\mathcal{M}\mathcal{M}_0 L_{\mathscr{Q}_1}\|\varphi\|_{\mathscr{B}_h} + \mathcal{M}L^*_{g_i} + (\mathcal{M}+1)\left(\mathcal{M}_0 + \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)}\cdot\frac{T^{\alpha\beta}}{\beta}\right)L^*_{\mathscr{Q}_1} \\
&\quad + \frac{\mathcal{M}(\mathcal{M}+1)T^{\alpha}}{\Gamma(\alpha+1)}\{L^*_{\mathscr{Q}_2} + L^*_{\mathscr{Q}_3}\} + (\mathscr{E}^*_1 r + c_n)\Bigg[\mathcal{M}L_{g_i} \\
&\quad + (\mathcal{M}+1)\left(\mathcal{M}_0 + \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)}\cdot\frac{T^{\alpha\beta}}{\beta}\right)(L_{\mathscr{Q}_1} + \mathcal{C}^*\widetilde{L}_{\mathscr{Q}_1}) \\
&\quad + \frac{\mathcal{M}(\mathcal{M}+1)T^{\alpha}}{\Gamma(\alpha+1)}\{(L_{\mathscr{Q}_2} + L_{\mathscr{Q}_3}) + \mathcal{C}^*(\widetilde{L}_{\mathscr{Q}_2} + \widetilde{L}_{\mathscr{Q}_3})\}\Bigg]\Bigg\} \\
&\leq r.
\end{aligned}$$

Thus, $\overline{\Upsilon}$ maps the ball $B_r(0,\mathscr{B}^0_T)$ into itself. Now, we prove that $\overline{\Upsilon}$ is a contraction on $B_r(0,\mathscr{B}^0_T)$. Let us consider $x,\overline{x} \in B_r(0,\mathscr{B}^0_T)$, then from estimations $I_j, j = 15,\cdots,27$, we sustain

$$\begin{aligned}
\|(\overline{\Upsilon}x)(t) - (\overline{\Upsilon}\overline{x})(t)\|_{\mathbb{X}} &\leq \mathscr{E}^*_1\Bigg[\left(\mathcal{M}_0 + \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)}\cdot\frac{t_1^{\alpha\beta}}{\beta}\right)(L_{\mathscr{Q}_1} + \widetilde{L}_{\mathscr{Q}_1}\mathcal{C}^* + L_{\mathscr{Q}_1}(r)L^*) + \frac{\mathcal{M}(t_1)^{\alpha}}{\Gamma(\alpha+1)} \\
&\qquad \{(L_{\mathscr{Q}_2} + L_{\mathscr{Q}_3}) + \mathcal{C}^*(\widetilde{L}_{\mathscr{Q}_2} + \widetilde{L}_{\mathscr{Q}_3}) + L^*(L_{\mathscr{Q}_2}(r) + L_{\mathscr{Q}_3}(r))\}\Bigg]\|x-\overline{x}\|_{\mathscr{B}^0_T},\ t\in[0,t_1].
\end{aligned}$$

$$\|(\overline{\Upsilon}x)(t) - (\overline{\Upsilon}\overline{x})(t)\|_{\mathbb{X}} \leq \mathscr{E}^*_1[L_{g_i} + 2L_{g_i}(r)L_{\zeta}]\|x-\overline{x}\|_{\mathscr{B}^0_T},\ t\in(t_i,s_i].$$

$$\begin{aligned}
\|(\overline{\Upsilon}x)(t) - (\overline{\Upsilon}\overline{x})(t)\|_{\mathbb{X}} &\leq \mathscr{E}^*_1 \max_{1\leq i\leq N}\Bigg[\mathcal{M}L_{g_i} + \left(\mathcal{M}_0(\mathcal{M}+1) + \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\beta\Gamma(\alpha\beta+1)}\cdot\{\mathcal{M}(s_i)^{\alpha\beta} + (t_{i+1})^{\alpha\beta}\}\right) \\
&\qquad (L_{\mathscr{Q}_1} + \mathcal{C}^*\widetilde{L}_{\mathscr{Q}_1} + L_{\mathscr{Q}_1}(r)L^*) + \frac{\mathcal{M}}{\Gamma(\alpha+1)}\{(L_{\mathscr{Q}_2} + L_{\mathscr{Q}_3}) + \mathcal{C}^*(\widetilde{L}_{\mathscr{Q}_2} + \widetilde{L}_{\mathscr{Q}_3}) \\
&\qquad + L^*(L_{\mathscr{Q}_2}(r) + L_{\mathscr{Q}_3}(r))\}\{\mathcal{M}(s_i)^{\alpha} + (t_{i+1})^{\alpha}\}\Bigg]\|x-\overline{x}\|_{\mathscr{B}^0_T},\ t\in(s_i,t_{i+1}].
\end{aligned}$$

As a result, for all $t \in \mathscr{I}$, we conclude that

$$\begin{aligned}
&\|(\overline{\Upsilon}x)(t) - (\overline{\Upsilon}\overline{x})(t)\|_{\mathbb{X}} \\
&\leq \mathscr{E}^*_1 \max_{1\leq i\leq N}\Bigg[\mathcal{M}L_{g_i} + (\mathcal{M}+1)\left(\mathcal{M}_0 + \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)T^{\alpha\beta}}{\beta\Gamma(\alpha\beta+1)}\right)(L_{\mathscr{Q}_1} + \mathcal{C}^*\widetilde{L}_{\mathscr{Q}_1} + L_{\mathscr{Q}_1}(r)L^*) \\
&\quad + \frac{\mathcal{M}(\mathcal{M}+1)T^{\alpha}}{\Gamma(\alpha+1)}\{(L_{\mathscr{Q}_2} + L_{\mathscr{Q}_3}) + \mathcal{C}^*(\widetilde{L}_{\mathscr{Q}_2} + \widetilde{L}_{\mathscr{Q}_3}) + L^*(L_{\mathscr{Q}_2}(r) + L_{\mathscr{Q}_3}(r))\}\Bigg]\|x-\overline{x}\|_{\mathscr{B}^0_T} \\
&\leq \Lambda\|x-\overline{x}\|_{\mathscr{B}^0_T}.
\end{aligned}$$

From the assumption (H7) and in the perspective of the contraction mapping principle, we understand that $\overline{\Upsilon}$ includes a unique fixed point $x \in \mathscr{B}^0_T$ which is a mild solution of the model (1.4)-(1.6) on $(-\infty, T]$. The proof is now completed. □

**Theorem 3.2.** *Let the assumption (H8) hold and*

$$\Lambda_1 = \mathscr{E}_1^* \max_{1\le i\le N}\left\{\mathcal{M}\|\mu_{g_i}\|_\infty + (\mathcal{M}+1)\left(\mathcal{M}_0 + \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)T^{\alpha\beta}}{\beta\Gamma(\alpha\beta+1)}\right)(\|\mu_{\mathscr{Q}_1}\|_\infty + \mathcal{C}^*\|\widetilde{\mu}_{\mathscr{Q}_1}\|_\infty)\right\} < 1. \tag{3.7}$$

*Then the system (1.4)-(1.6) has a mild solution on $\mathscr{I}$.*

*Proof.* Let $\overline{\Upsilon} : \mathscr{B}_T^0 \to \mathscr{B}_T^0$ be the operator same as defined in Theorem 3.1. Now, we demonstrate that $\overline{\Upsilon}$ has a fixed point.

**Remark 3.2.** *From the hypothesis (H8) along with the above discussion, we sustain*

$$\begin{aligned}
I_{28} &= \|\mathbb{T}_\alpha(t)\mathscr{Q}_1(0,\varphi(0),0)\|_{\mathbb{X}}\\
&\le \mathcal{M}\mathcal{M}_0\|\mu_{\mathscr{Q}_1}\|_\infty\|\varphi\|_{\mathscr{B}_h}.\\
I_{29} &= \left\|\mathscr{Q}_1\left(t, x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}, \mathcal{C}x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}\right)\right\|_{\mathbb{X}}\\
&\le \mathcal{M}_0(\mathscr{E}_1^* r + c_n)\left\{\|\mu_{\mathscr{Q}_1}\|_\infty + \|\widetilde{\mu}_{\mathscr{Q}_1}\|_\infty\mathcal{C}^*\right\}.\\
I_{30} &= \left\|\int_0^t \mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{Q}_1(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)})ds\right\|_{\mathbb{X}}\\
&\le \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)}\cdot\frac{t_1^{\alpha\beta}}{\beta}(\mathscr{E}_1^* r + c_n)\left\{\|\mu_{\mathscr{Q}_1}\|_\infty + \|\widetilde{\mu}_{\mathscr{Q}_1}\|_\infty\mathcal{C}^*\right\},\ t\in[0,t_1].\\
I_{31} &= \left\|\int_0^t \mathbb{S}_\alpha(t-s)\mathscr{Q}_2\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right)ds\right\|_{\mathbb{X}}\\
&\le \frac{\mathcal{M}(t_1)^\alpha}{\Gamma(\alpha+1)}(\mathscr{E}_1^* r + c_n)\left\{\|\mu_{\mathscr{Q}_2}\|_\infty + \|\widetilde{\mu}_{\mathscr{Q}_2}\|_\infty\mathcal{C}^*\right\},\ t\in[0,t_1].\\
I_{32} &= \left\|\int_0^t \mathbb{S}_\alpha(t-s)\mathscr{Q}_3\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right)ds\right\|_{\mathbb{X}}\\
&\le \frac{\mathcal{M}(t_1)^\alpha}{\Gamma(\alpha+1)}(\mathscr{E}_1^* r + c_n)\left\{\|\mu_{\mathscr{Q}_3}\|_\infty + \|\widetilde{\mu}_{\mathscr{Q}_3}\|_\infty\mathcal{C}^*\right\},\ t\in[0,t_1].\\
I_{33} &= \|g_i(t, x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)})\|_{\mathbb{X}}\\
&\le \|\mu_{g_i}\|_\infty(\mathscr{E}_1^* r + c_n),\ t\in(t_i,s_i].\\
I_{34} &= \|g_i(s_i, x_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})})\|_{\mathbb{X}}\\
&\le \|\mu_{g_i}\|_\infty(\mathscr{E}_1^* r + c_n),\ t\in(s_i,t_{i+1}].\\
I_{35} &= \|\mathscr{Q}_1(s_i, x_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})}, \mathcal{C}x_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})})\|_{\mathbb{X}}\\
&\le \mathcal{M}_0(\mathscr{E}_1^* r + c_n)\left\{\|\mu_{\mathscr{Q}_1}\|_\infty + \|\widetilde{\mu}_{\mathscr{Q}_1}\|_\infty\mathcal{C}^*\right\},\ t\in(s_i,t_{i+1}].\\
I_{36} &= \left\|\int_0^{s_i} \mathscr{A}\mathbb{S}_\alpha(s_i-s)\mathscr{Q}_1(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)})ds\right\|_{\mathbb{X}}\\
&\le \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)}\cdot\frac{(s_i)^{\alpha\beta}}{\beta}(\mathscr{E}_1^* r + c_n)\left\{\|\mu_{\mathscr{Q}_1}\|_\infty + \|\widetilde{\mu}_{\mathscr{Q}_1}\|_\infty\mathcal{C}^*\right\},\ t\in(s_i,t_{i+1}].\\
I_{37} &= \left\|\int_0^{s_i} \mathbb{S}_\alpha(s_i-s)\mathscr{Q}_2\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right)ds\right\|_{\mathbb{X}}
\end{aligned}$$

$$\leq \frac{\mathcal{M}(s_i)^{\alpha}}{\Gamma(\alpha+1)}(\mathscr{E}_1^* r + c_n)\Big\{\|\mu_{\mathscr{Q}_2}\|_{\infty} + \|\widetilde{\mu}_{\mathscr{Q}_2}\|_{\infty}\mathcal{C}^*\Big\},\ t \in (s_i, t_{i+1}].$$

$$I_{38} = \left\| \int_0^{s_i} \mathbb{S}_{\alpha}(s_i - s)\mathscr{Q}_3\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right) ds \right\|_{\mathbb{X}}$$

$$\leq \frac{\mathcal{M}(s_i)^{\alpha}}{\Gamma(\alpha+1)}(\mathscr{E}_1^* r + c_n)\Big\{\|\mu_{\mathscr{Q}_3}\|_{\infty} + \|\widetilde{\mu}_{\mathscr{Q}_3}\|_{\infty}\mathcal{C}^*\Big\},\ t \in (s_i, t_{i+1}].$$

$$I_{39} = \left\| \int_0^{t} \mathscr{A}\mathbb{S}_{\alpha}(t - s)\mathscr{Q}_1(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)})ds \right\|_{\mathbb{X}}$$

$$\leq \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)} \cdot \frac{(t_{i+1})^{\alpha\beta}}{\beta}(\mathscr{E}_1^* r + c_n)\Big\{\|\mu_{\mathscr{Q}_1}\|_{\infty} + \|\widetilde{\mu}_{\mathscr{Q}_1}\|_{\infty}\mathcal{C}^*\Big\},\ t \in (s_i, t_{i+1}].$$

$$I_{40} = \left\| \int_0^{t} \mathbb{S}_{\alpha}(t - s)\mathscr{Q}_2\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right) ds \right\|_{\mathbb{X}}$$

$$\leq \frac{\mathcal{M}(t_{i+1})^{\alpha}}{\Gamma(\alpha+1)}(\mathscr{E}_1^* r + c_n)\Big\{\|\mu_{\mathscr{Q}_2}\|_{\infty} + \|\widetilde{\mu}_{\mathscr{Q}_2}\|_{\infty}\mathcal{C}^*\Big\},\ t \in (s_i, t_{i+1}].$$

$$I_{41} = \left\| \int_0^{t} \mathbb{S}_{\alpha}(t - s)\mathscr{Q}_3\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right) ds \right\|_{\mathbb{X}}$$

$$\leq \frac{\mathcal{M}(t_{i+1})^{\alpha}}{\Gamma(\alpha+1)}(\mathscr{E}_1^* r + c_n)\Big\{\|\mu_{\mathscr{Q}_3}\|_{\infty} + \|\widetilde{\mu}_{\mathscr{Q}_3}\|_{\infty}\mathcal{C}^*\Big\},\ t \in (s_i, t_{i+1}].$$

Now, we will give the main proof of this theorem. We remark that $(\overline{\Upsilon}x)(t) \subset \mathscr{B}_T^0$. Let $B_r$ be the set same as defined in Theorem 3.1, where

$$r \geq \max_{1\leq i\leq N}\Bigg\{\mathcal{M}\mathcal{M}_0\|\mu_{\mathscr{Q}_1}\|_{\infty}\|\varphi\|_{\mathscr{B}_h} + (\mathscr{E}_1^* r + c_n)\Bigg[\mathcal{M}\|\mu_{g_i}\|_{\infty}$$
$$+ (\mathcal{M}+1)\left(\mathcal{M}_0 + \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)} \cdot \frac{T^{\alpha\beta}}{\beta}\right)(\|\mu_{\mathscr{Q}_1}\|_{\infty} + \mathcal{C}^*\|\widetilde{\mu}_{\mathscr{Q}_1}\|_{\infty})$$
$$+ \frac{\mathcal{M}(\mathcal{M}+1)T^{\alpha}}{\Gamma(\alpha+1)}\{(\|\mu_{\mathscr{Q}_2}\|_{\infty} + \|\mu_{\mathscr{Q}_3}\|_{\infty}) + \mathcal{C}^*(\|\widetilde{\mu}_{\mathscr{Q}_2}\|_{\infty} + \|\widetilde{\mu}_{\mathscr{Q}_3}\|_{\infty})\}\Bigg]\Bigg\}.$$

It is obvious that $B_r$ is closed bounded and convex subset of $\mathscr{B}_T^0$. Let $x \in B_r(0, \mathscr{B}_T^0)$ then for $t \in [0, t_1]$, we receive

$$\|(\overline{\Upsilon}x)\|_{\mathscr{B}_T^0} \leq \mathcal{M}\mathcal{M}_0\|\mu_{\mathscr{Q}_1}\|_{\infty}\|\varphi\|_{\mathscr{B}_h} + (\mathscr{E}_1^* r + c_n)\Bigg[\left(\mathcal{M}_0 + \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)} \cdot \frac{(t_1)^{\alpha\beta}}{\beta}\right)(\|\mu_{\mathscr{Q}_1}\|_{\infty}$$
$$+ \mathcal{C}^*\|\widetilde{\mu}_{\mathscr{Q}_1}\|_{\infty}) + \frac{\mathcal{M}(t_1)^{\alpha}}{\Gamma(\alpha+1)}\{(\|\mu_{\mathscr{Q}_2}\|_{\infty} + \|\mu_{\mathscr{Q}_3}\|_{\infty}) + \mathcal{C}^*(\|\widetilde{\mu}_{\mathscr{Q}_2}\|_{\infty} + \|\widetilde{\mu}_{\mathscr{Q}_3}\|_{\infty})\}\Bigg].$$

In the similar manner for $t \in (t_i, s_i]$, we sustain

$$\|(\overline{\Upsilon}x)\|_{\mathscr{B}_T^0} \leq \|\mu_{\mathscr{Q}_1}\|_{\infty}\|\varphi\|_{\mathscr{B}_h}(\mathscr{E}_1^* r + c_n),\ i = 1, 2, \cdots, N.$$

Similarly, for $t \in (s_i, t_{i+1}]$, we find that

$$\begin{aligned}\|(\overline{\Upsilon}x)\|_{\mathscr{B}_T^0} \leq \max_{1\leq i\leq N}(\mathscr{E}_1^* r + c_n)\Bigg\{ & \mathcal{M}\|\mu_{g_i}\|_\infty \\ & + \left(\mathcal{M}_0(\mathcal{M}+1) + \frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\beta\Gamma(\alpha\beta+1)}\cdot\{\mathcal{M}(s_i)^{\alpha\beta} + (t_{i+1})^{\alpha\beta}\}\right)(\|\mu_{\mathscr{Q}_1}\|_\infty + \mathcal{C}^*\|\widetilde{\mu}_{\mathscr{Q}_1}\|_\infty) \\ & + \frac{\mathcal{M}}{\Gamma(\alpha+1)}\{(\|\mu_{\mathscr{Q}_2}\|_\infty + \|\mu_{\mathscr{Q}_3}\|_\infty) + \mathcal{C}^*(\|\widetilde{\mu}_{\mathscr{Q}_2}\|_\infty + \|\widetilde{\mu}_{\mathscr{Q}_3}\|_\infty)\}\{\mathcal{M}(s_i)^\alpha + (t_{i+1})^\alpha\}\Bigg\}.\end{aligned}$$

From this, we notice that $\|\overline{\Upsilon}x\|_{\mathscr{B}_T^0} \leq r$ for every $t \in \mathscr{J}$. Therefore, $\overline{\Upsilon}(B_r) \subseteq B_r$. In order to utilizing the Theorem 2.2, we have to prove that the operator $\overline{\Upsilon}$ is a condensing operator. For this, we split $\overline{\Upsilon}$ by $\overline{\Upsilon} = \overline{\Upsilon}_1 + \overline{\Upsilon}_2$, where

$$(\overline{\Upsilon}_1 x)(t) = \begin{cases} -\mathbb{T}_\alpha(t)\mathscr{Q}_1(0,\varphi(0),0) + \mathscr{Q}_1\left(t, x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}, \mathcal{C}x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}\right) \\ + \displaystyle\int_0^t \mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{Q}_1(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)})ds,\ t\in[0,t_1], \\ g_i(t, x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}), \quad t\in(t_i,s_i],\ i=1,2,\cdots,N, \\ \mathbb{T}_\alpha(t-s_i)\Big[g_i(s_i, x_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})}) \\ -\mathscr{Q}_1(s_i, x_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})}, \mathcal{C}x_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})}) \\ -\displaystyle\int_0^{s_i}\mathscr{A}\mathbb{S}_\alpha(s_i-s)\mathscr{Q}_1(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)})ds\Big] \\ +\mathscr{Q}_1(t, x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}, \mathcal{C}x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}) \\ + \displaystyle\int_0^t \mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{Q}_1(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)})ds,\ t\in(s_i,t_{i+1}], \end{cases}$$

and

$$(\overline{\Upsilon}_2 x)(t) = \begin{cases} \displaystyle\int_0^t \mathbb{S}_\alpha(t-s)\mathscr{Q}_2\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right)ds \\ + \displaystyle\int_0^t \mathbb{S}_\alpha(t-s)\mathscr{Q}_3\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right)ds,\ t\in[0,t_1], \\ 0, \quad t\in(t_i,s_i],\ i=1,2,\cdots,N, \\ -\mathbb{T}_\alpha(t-s_i)\Big[\displaystyle\int_0^{s_i}\mathbb{S}_\alpha(s_i-s)\mathscr{Q}_2\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right)ds \\ \quad -\displaystyle\int_0^{s_i}\mathbb{S}_\alpha(s_i-s)\mathscr{Q}_3\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right)ds\Big] \\ + \displaystyle\int_0^t \mathbb{S}_\alpha(t-s)\mathscr{Q}_2\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right)ds \\ + \displaystyle\int_0^t \mathbb{S}_\alpha(t-s)\mathscr{Q}_3\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right)ds,\ t\in(s_i,t_{i+1}]. \end{cases}$$

Firstly, we show that $\overline{\Upsilon}_1$ is continuous, so we consider a sequence $x^n \to x \in B_r$. In perspective of (3.1), we notice that

$$\|x^n_{\zeta(t,x_t^n+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t^n+\widetilde{y}_t)}\|_{\mathscr{B}_h} \leq \mathscr{E}_1^* r + c_n.$$

**Remark 3.3.** *By utilizing the hypothesis (H8) and Definition 2.2, we receive:*

*(i) For every $t \in [0, t_1]$, we obtain*

$$\mathscr{Q}_1\Big(t, x^n_{\zeta(t,x^n_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x^n_t+\widetilde{y}_t)}, \mathcal{C}x^n_{\zeta(t,x^n_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x^n_t+\widetilde{y}_t)}\Big)$$
$$\to \mathscr{Q}_1\Big(t, x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}, \mathcal{C}x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}\Big)$$

*and since*

$$\Big\|\mathscr{Q}_1\Big(t, x^n_{\zeta(t,x^n_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x^n_t+\widetilde{y}_t)}, \mathcal{C}x^n_{\zeta(t,x^n_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x^n_t+\widetilde{y}_t)}\Big)$$
$$-\mathscr{Q}_1\Big(t, x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}, \mathcal{C}x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}\Big)\Big\|_{\mathbb{X}} \leq 2I_{29}.$$

*(ii) For every $t \in [0, t_1]$, we obtain*

$$\mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{Q}_1\Big(s, x^n_{\zeta(s,x^n_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x^n_s+\widetilde{y}_s)}, \mathcal{C}x^n_{\zeta(s,x^n_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x^n_s+\widetilde{y}_s)}\Big)$$
$$\to \mathscr{A}\mathbb{S}_\alpha(t-s)\mathscr{Q}_1\Big(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\Big)$$

*and since*

$$\Big\|\int_0^t \mathscr{A}\mathbb{S}_\alpha(t-s)\Big[\mathscr{Q}_1\Big(s, x^n_{\zeta(s,x^n_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x^n_s+\widetilde{y}_s)}, \mathcal{C}x^n_{\zeta(s,x^n_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x^n_s+\widetilde{y}_s)}\Big)$$
$$-\mathscr{Q}_1\Big(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\Big)\Big]ds\Big\|_{\mathbb{X}} \leq 2I_{30}.$$

*(iii) For each $t \in (t_i, s_i]$, we sustain*

$$g_i\Big(t, x^n_{\zeta(t,x^n_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x^n_t+\widetilde{y}_t)}, \mathcal{C}x^n_{\zeta(t,x^n_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x^n_t+\widetilde{y}_t)}\Big)$$
$$\to g_i\Big(t, x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}, \mathcal{C}x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}\Big)$$

*and since*

$$\Big\|g_i\Big(t, x^n_{\zeta(t,x^n_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x^n_t+\widetilde{y}_t)}, \mathcal{C}x^n_{\zeta(t,x^n_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x^n_t+\widetilde{y}_t)}\Big)$$
$$-g_i\Big(t, x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}, \mathcal{C}x_{\zeta(t,x_t+\widetilde{y}_t)} + \widetilde{y}_{\zeta(t,x_t+\widetilde{y}_t)}\Big)\Big\|_{\mathbb{X}} \leq 2I_{33}.$$

*(iv) For every $t \in (s_i, t_{i+1}]$, we receive*

$$g_i(s_i, x^n_{\zeta(s_i,x^n_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i,x^n_{s_i}+\widetilde{y}_{s_i})}) \to g_i(s_i, x_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})})$$

*and since*

$$\Big\|g_i(s_i, x^n_{\zeta(s_i,x^n_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i,x^n_{s_i}+\widetilde{y}_{s_i})}) - g_i(s_i, x_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})})\Big\|_{\mathbb{X}} \leq 2I_{34}.$$

*(v) For all $t \in (s_i, t_{i+1}]$, we get*

$$\begin{aligned}&\mathscr{Q}_1(s_i, x^n_{\zeta(s_i,x^n_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i,x^n_{s_i}+\widetilde{y}_{s_i})}, \mathcal{C}x^n_{\zeta(s_i,x^n_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i,x^n_{s_i}+\widetilde{y}_{s_i})})\\ &\to \mathscr{Q}_1(s_i, x_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})}, \mathcal{C}x_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})})\end{aligned}$$

*and since*

$$\begin{aligned}&\Big\|\mathscr{Q}_1(s_i, x^n_{\zeta(s_i,x^n_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i,x^n_{s_i}+\widetilde{y}_{s_i})}, \mathcal{C}x^n_{\zeta(s_i,x^n_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i,x^n_{s_i}+\widetilde{y}_{s_i})})\\ &-\mathscr{Q}_1(s_i, x_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})}, \mathcal{C}x_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})} + \widetilde{y}_{\zeta(s_i,x_{s_i}+\widetilde{y}_{s_i})})\Big\|_{\mathbb{X}} \leq 2I_{35}.\end{aligned}$$

*(vi) For every $t \in (s_i, t_{i+1}]$, we obtain*

$$\begin{aligned}&\mathscr{A}\mathbb{S}_\alpha(s_i - s)\mathscr{Q}_1\Big(s, x^n_{\zeta(s,x^n_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x^n_s+\widetilde{y}_s)}, \mathcal{C}x^n_{\zeta(s,x^n_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x^n_s+\widetilde{y}_s)}\Big)\\ &\to \mathscr{A}\mathbb{S}_\alpha(s_i - s)\mathscr{Q}_1\Big(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\Big)\end{aligned}$$

*and since*

$$\begin{aligned}&\Big\|\int_0^{s_i} \mathscr{A}\mathbb{S}_\alpha(s_i - s)\Big[\mathscr{Q}_1\Big(s, x^n_{\zeta(s,x^n_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x^n_s+\widetilde{y}_s)}, \mathcal{C}x^n_{\zeta(s,x^n_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x^n_s+\widetilde{y}_s)}\Big)\\ &-\mathscr{Q}_1\Big(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\Big)\Big]ds\Big\|_{\mathbb{X}} \leq 2I_{36}.\end{aligned}$$

*(vii) For every $t \in (s_i, t_{i+1}]$, we find that*

$$\begin{aligned}&\mathscr{A}\mathbb{S}_\alpha(t - s)\mathscr{Q}_1\Big(s, x^n_{\zeta(s,x^n_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x^n_s+\widetilde{y}_s)}, \mathcal{C}x^n_{\zeta(s,x^n_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x^n_s+\widetilde{y}_s)}\Big)\\ &\to \mathscr{A}\mathbb{S}_\alpha(t - s)\mathscr{Q}_1\Big(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\Big)\end{aligned}$$

*and since*

$$\begin{aligned}&\Big\|\int_0^{t} \mathscr{A}\mathbb{S}_\alpha(t - s)\Big[\mathscr{Q}_1\Big(s, x^n_{\zeta(s,x^n_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x^n_s+\widetilde{y}_s)}, \mathcal{C}x^n_{\zeta(s,x^n_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x^n_s+\widetilde{y}_s)}\Big)\\ &-\mathscr{Q}_1\Big(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\Big)\Big]ds\Big\|_{\mathbb{X}} \leq 2I_{39}.\end{aligned}$$

From the Remark 3.3, for all $[0, t_1]$, we have

$$\|(\overline{\Upsilon}_1 x^n) - (\overline{\Upsilon}_1 x)\|_{\mathscr{B}^0_T} \leq 2(I_{29} + I_{30}),$$

and for every $t \in (s_i, t_{i+1}]$, we receive

$$\|(\overline{\Upsilon}_1 x^n) - (\overline{\Upsilon}_1 x)\|_{\mathscr{B}^0_T} \leq 2\Big[\mathcal{M}(I_{34} + I_{35} + I_{36}) + I_{29} + I_{39}\Big].$$

Since the functions $\mathscr{Q}_1$ and $g_i, i = 1, 2, \cdots, N$ are continuous, so we conclude that

$$\|(\overline{\Upsilon}_1 x^n) - (\overline{\Upsilon}_1 x)\|_{\mathscr{B}^0_T} \to 0 \quad \text{as} \quad n \to \infty.$$

Hence $\overline{\Upsilon}_1$ is continuous.

Next, we prove that the operator $\overline{\Upsilon}_1$ is contraction on $B_r(0,\mathscr{B}_T^0)$. Indeed, let $x,\overline{x}\in B_r(0,\mathscr{B}_T^0)$, for $[0,t_1]$, we sustain

$$\|(\overline{\Upsilon}_1x^n)-(\overline{\Upsilon}_1x)\|_{\mathscr{B}_T^0}\leq \mathscr{E}_1^*(\|\mu_{\mathscr{Q}_1}\|_\infty+\mathcal{C}^*\|\widetilde{\mu}_{\mathscr{Q}_1}\|_\infty)\left(\mathcal{M}_0+\frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)}\cdot\frac{(t_1)^{\alpha\beta}}{\beta}\right)\|x-\overline{x}\|_{\mathscr{B}_T^0},$$

and for $t\in(t_i,s_i]$, we get

$$\|(\overline{\Upsilon}_1x^n)-(\overline{\Upsilon}_1x)\|_{\mathscr{B}_T^0}\leq \mathscr{E}_1^*\|\mu_{g_i}\|_\infty\|x-\overline{x}\|_{\mathscr{B}_T^0},$$

and for every $t\in(s_i,t_{i+1}]$, we sustain

$$\begin{aligned}
&\|(\overline{\Upsilon}_1x^n)-(\overline{\Upsilon}_1x)\|_{\mathscr{B}_T^0}\\
&\leq \mathscr{E}_1^*\max_{1\leq i\leq N}\Bigg\{\mathcal{M}\|\mu_{g_i}\|_\infty\\
&\quad+\left(\mathcal{M}_0(\mathcal{M}+1)+\frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)}{\beta\Gamma(\alpha\beta+1)}\cdot\{\mathcal{M}(s_i)^{\alpha\beta}+(t_{i+1})^{\alpha\beta}\}\right)(\|\mu_{\mathscr{Q}_1}\|_\infty+\mathcal{C}^*\|\widetilde{\mu}_{\mathscr{Q}_1}\|_\infty)\Bigg\}\|x-\overline{x}\|_{\mathscr{B}_T^0}.
\end{aligned}$$

Then for all $t\in\mathscr{I}$, we find that

$$\begin{aligned}
&\|(\overline{\Upsilon}_1x^n)-(\overline{\Upsilon}_1x)\|_{\mathscr{B}_T^0}\\
&\leq \mathscr{E}_1^*\max_{1\leq i\leq N}\left\{\mathcal{M}\|\mu_{g_i}\|_\infty+(\mathcal{M}+1)\left(\mathcal{M}_0+\frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)T^{\alpha\beta}}{\beta\Gamma(\alpha\beta+1)}\right)(\|\mu_{\mathscr{Q}_1}\|_\infty+\mathcal{C}^*\|\widetilde{\mu}_{\mathscr{Q}_1}\|_\infty)\right\}\|x-\overline{x}\|_{\mathscr{B}_T^0}\\
&\leq \Lambda_1\|x-\overline{x}\|_{\mathscr{B}_T^0}.
\end{aligned}$$

Since $\Lambda_1<1$, which implies that $\overline{\Upsilon}_1$ is a contraction.

Next, we prove that the operator $\overline{\Upsilon}_2$ is completely continuous on $B_r(0,\mathscr{B}_T^0)$. First, we prove $\overline{\Upsilon}_2$ is continuous, so consider a sequence $x^n\to x\in B_r$. By applying the condition (H8), $I_{31},I_{32},I_{37},I_{38},I_{40},I_{41}$ and in perspective of Remark 3.3, for all $t\in[0,t_1]$, we get

$$\|(\overline{\Upsilon}_2x^n)-(\overline{\Upsilon}_2x)\|_{\mathscr{B}_T^0}\leq 2(I_{31}+I_{32}),$$

and for all $t\in(s_i,t_{i+1}]$, we receive

$$\|(\overline{\Upsilon}_2x^n)-(\overline{\Upsilon}_2x)\|_{\mathscr{B}_T^0}\leq 2\Big[\mathcal{M}(I_{37}+I_{38})+I_{40}+I_{41}\Big].$$

Since the functions $\mathscr{Q}_2$ and $\mathscr{Q}_3$ are continuous, so we conclude that

$$\|(\overline{\Upsilon}_2x^n)-(\overline{\Upsilon}_2x)\|_{\mathscr{B}_T^0}\to 0\quad\text{as}\quad n\to\infty.$$

Hence $\overline{\Upsilon}_2$ is continuous.

Next, we show that the operator $\overline{\Upsilon}_2$ maps bounded sets into bounded set in $B_r$. It is enough to show that there exists a positive constant $\Lambda_2$ such that for each $x\in B_r$ one has $\|\overline{\Upsilon}_2x\|_{\mathscr{B}_T^0}\leq\Lambda_2$. For all $t\in\mathscr{I}$, we obtain

$$\|\overline{\Upsilon}_2x\|_{\mathscr{B}_T^0}\leq\frac{\mathcal{M}(\mathcal{M}+1)T^\alpha}{\Gamma(\alpha+1)}(\mathscr{E}_1^*r+c_n)\Big\{(\|\mu_{\mathscr{Q}_2}\|_\infty+\|\mu_{\mathscr{Q}_3}\|_\infty)+\mathcal{C}^*(\|\widetilde{\mu}_{\mathscr{Q}_2}\|_\infty+\|\widetilde{\mu}_{\mathscr{Q}_3}\|_\infty)\Big\}\leq\Lambda_2.$$

Finally, we show that $\overline{\Upsilon}_2$ is a family of equi-continuous functions. Let $\tau_1, \tau_2 \in [0, t_1]$ be such that $0 \leq \tau_1 < \tau_2 \leq t_1$. Then

$$
\begin{aligned}
&\|(\overline{\Upsilon}_2 x)(\tau_2) - (\overline{\Upsilon}_2 x)(\tau_1)\|_{\mathbb{X}} \\
&\leq \int_0^{\tau_1} \|\mathbb{S}_\alpha(\tau_2 - s) - \mathbb{S}_\alpha(\tau_1 - s)\|_{\mathscr{L}(\mathbb{X})} \|\mathscr{Q}_2\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right)\|_{\mathbb{X}} ds \\
&\quad + \int_{\tau_1}^{\tau_2} \|\mathbb{S}_\alpha(\tau_2 - s)\|_{\mathscr{L}(\mathbb{X})} \|\mathscr{Q}_2\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right)\|_{\mathbb{X}} ds \\
&\quad + \int_0^{\tau_1} \|\mathbb{S}_\alpha(\tau_2 - s) - \mathbb{S}_\alpha(\tau_1 - s)\|_{\mathscr{L}(\mathbb{X})} \|\mathscr{Q}_3\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right)\|_{\mathbb{X}} ds \\
&\quad + \int_{\tau_1}^{\tau_2} \|\mathbb{S}_\alpha(\tau_2 - s)\|_{\mathscr{L}(\mathbb{X})} \|\mathscr{Q}_3\left(s, x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}, \mathcal{C}x_{\zeta(s,x_s+\widetilde{y}_s)} + \widetilde{y}_{\zeta(s,x_s+\widetilde{y}_s)}\right)\|_{\mathbb{X}} ds \\
&\leq (\mathscr{E}_1^* r + c_n)\Big\{(\|\mu_{\mathscr{Q}_2}\|_\infty + \|\mu_{\mathscr{Q}_3}\|_\infty) + \mathcal{C}^*(\|\widetilde{\mu}_{\mathscr{Q}_2}\|_\infty + \|\widetilde{\mu}_{\mathscr{Q}_3}\|_\infty)\Big\}\Bigg(\int_0^{\tau_1} \|\mathbb{S}_\alpha(\tau_2 - s) - \mathbb{S}_\alpha(\tau_1 - s)\|_{\mathscr{L}(\mathbb{X})} ds \\
&\quad + \frac{\mathcal{M}(\tau_2 - \tau_1)^\alpha}{\Gamma(\alpha + 1)}\Bigg),
\end{aligned}
$$

and for all $\tau_1, \tau_2 \in (s_i, t_{i+1}]$, we have

$$
\begin{aligned}
&\|(\overline{\Upsilon}_2 x)(\tau_2) - (\overline{\Upsilon}_2 x)(\tau_1)\|_{\mathbb{X}} \\
&\leq (\mathscr{E}_1^* r + c_n)\Big\{(\|\mu_{\mathscr{Q}_2}\|_\infty + \|\mu_{\mathscr{Q}_3}\|_\infty) + \mathcal{C}^*(\|\widetilde{\mu}_{\mathscr{Q}_2}\|_\infty + \|\widetilde{\mu}_{\mathscr{Q}_3}\|_\infty)\Big\}\Bigg(\int_0^{\tau_1} \|\mathbb{S}_\alpha(\tau_2 - s) - \mathbb{S}_\alpha(\tau_1 - s)\|_{\mathscr{L}(\mathbb{X})} ds \\
&\quad + \frac{\mathcal{M}(\tau_2 - \tau_1)^\alpha}{\Gamma(\alpha + 1)} + \frac{\mathcal{M}(s_i)^\alpha}{\Gamma(\alpha + 1)} \|\mathbb{T}_\alpha(\tau_2 - s_i) - \mathbb{T}_\alpha(\tau_1 - s_i)\|_{\mathscr{L}(\mathbb{X})}\Bigg).
\end{aligned}
$$

Since $T_\alpha$ and $\mathbb{S}_\alpha$ are strongly continuous, so $\lim_{\tau_2 \to \tau_1} \|\mathbb{S}_\alpha(\tau_2 - s) - \mathbb{S}_\alpha(\tau_1 - s)\|_{\mathscr{L}(\mathbb{X})} = 0$, $\lim_{\tau_2 \to \tau_1} \|\mathbb{T}_\alpha(\tau_2 - s_i) - \mathbb{T}_\alpha(\tau_1 - s_i)\|_{\mathscr{L}(\mathbb{X})} = 0$. From this, we conclude that $\|(\overline{\Upsilon}_2 x)(\tau_2) - (\overline{\Upsilon}_2 x)(\tau_1)\|_{\mathbb{X}} \to 0$ as $\tau_2 \to \tau_1$. This proves that $\overline{\Upsilon}_2$ is a family of equi-continuous functions. Hence, the operator $\overline{\Upsilon}_2$ is completely continuous. Therefore the operator $\overline{\Upsilon} = \overline{\Upsilon}_1 + \overline{\Upsilon}_2$ is a condensing operator from $\mathscr{B}_T^0$ into $\mathscr{B}_T^0$, where $\overline{\Upsilon}_1$ is contraction and $\overline{\Upsilon}_2$ is completely continuous. Finally, from Theorem 2.2, we infer that there exists a mild solution of the structure (1.4)-(1.6). This completes the proof. □

## 4 Example

To prove our theoretical results, we treat the IFNIDS with SDD of the model

$$
\begin{aligned}
D_t^\alpha \Bigg[ u(t,z) + \int_{-\infty}^t e^{2(s-t)} \frac{u(s - \zeta_1(s)\zeta_2(\|u(s)\|), z)}{49} ds \\
+ \int_0^t \sin(t-s) \int_{-\infty}^s e^{2(\tau - s)} \frac{u(\tau - \zeta_1(\tau)\zeta_2(\|u(\tau)\|), z)}{36} d\tau ds \Bigg] = \frac{\partial^2}{\partial z^2} u(t,z) \\
+ \int_{-\infty}^t e^{2(s-t)} \frac{u(s - \zeta_1(s)\zeta_2(\|u(s)\|), z)}{9} ds
\end{aligned}
$$

$$+\int_0^t \sin(t-s)\int_{-\infty}^{s} e^{2(\tau-s)}\frac{u(\tau-\zeta_1(\tau)\zeta_2(\|u(\tau)\|),z)}{25}d\tau ds$$
$$+\int_{-\infty}^{t} e^{2(s-t)}\frac{u(s-\zeta_1(s)\zeta_2(\|u(s)\|),z)}{64}ds$$
$$+\int_0^t \sin(t-s)\int_{-\infty}^{s} e^{2(\tau-s)}\frac{u(\tau-\zeta_1(\tau)\zeta_2(\|u(\tau)\|),z)}{16}d\tau ds,\ (t,z)\in\bigcup_{i=1}^{N}(s_i,t_{i+1}]\times[0,\pi], \quad (4.1)$$

$$u(t,0)=0=u(t,\pi),\quad t\in[0,T], \quad (4.2)$$

$$u(t,z)=\varphi(t,z),\quad t\le 0,\quad z\in[0,\pi], \quad (4.3)$$

$$u(t,z)=\int_{-\infty}^{t} e^{2(s-t)}\frac{u(s-\zeta_1(s)\zeta_2(\|u(s)\|),z)}{81}ds,\ (t,z)\in(t_i,s_i]\times[0,\pi],\ i=1,2,\ldots,N, \quad (4.4)$$

where ${}^{C}D_t^q$ is Caputo's fractional derivative of order $0<q<1$, $0=t_0=s_0<t_1<t_2<\cdots<t_{N-1}\le S_N\le t_N\le t_{N+1}=T$ are pre-fixed real numbers and $\varphi\in\mathscr{B}_h$. We consider $\mathbb{X}=L^2[0,\pi]$ having the norm $\|\cdot\|_{L^2}$ and determine the operator $\mathscr{A}:D(\mathscr{A})\subset\mathbb{X}\to\mathbb{X}$ by $\mathscr{A}w=w''$ having the domain

$$D(\mathscr{A})=\{w\in\mathbb{X}: w, w' \quad \text{are absolutely continuous},\quad w''\in\mathbb{X},\ w(0)=w(\pi)=0\}.$$

Then

$$\mathscr{A}w=\sum_{n=1}^{\infty}n^2\langle w,w_n\rangle w_n,\quad w\in D(\mathscr{A}),$$

where $w_n(s)=\sqrt{\frac{2}{\pi}}\sin(ns)$, $n=1,2,\ldots,.$ denotes the orthogonal set of eigenvectors of $\mathscr{A}$. It is long familiar that $\mathscr{A}$ is the infinitesimal generator of an analytic semigroup $\{\mathbb{T}(t)\}_{t\ge0}$ in $\mathbb{X}$ and is provided by

$$\mathbb{T}(t)w=\sum_{n=1}^{\infty}e^{-n^2t}\langle w,w_n\rangle w_n,\quad \text{for all}\quad w\in\mathbb{X},\quad \text{and every}\quad t>0.$$

We can find a constant $\mathcal{M}>0$ in a way that $\|\mathbb{T}(t)\|\le\mathcal{M}$. If we fix $\beta=\frac{1}{2}$, then the operator $(\mathscr{A})^{\frac{1}{2}}$ is given by

$$(\mathscr{A})^{\frac{1}{2}}w=\sum_{n=1}^{\infty}-n^2\langle w,w_n\rangle w_n,\quad w\in(D(\mathscr{A})^{\frac{1}{2}}),$$

in which $(D(\mathscr{A})^{\frac{1}{2}})=\left\{\omega(\cdot)\in\mathbb{X}:\sum_{n=1}^{\infty}n^2\langle\omega,w_n\rangle w_n\in\mathbb{X}\right\}$. Then

$$\mathbb{S}_\alpha(t)w=\alpha\int_0^{\infty}r\phi_\alpha(r)t^{\alpha-1}\mathbb{T}(t^\alpha r)dr,$$
$$=\sum_{n=1}^{\infty}E_{\alpha,\alpha}(-n^2t^\alpha)\langle w,w_n\rangle w_n,\ w\in\mathbb{X}.$$

For the phase space, we choose $h=e^{2s}$, $s<0$, then $l=\int_{-\infty}^{0}h(s)ds=\frac{1}{2}<\infty$, for $t\le0$ and determine

$$\|\varphi\|_{\mathscr{B}_h}=\int_{-\infty}^{0}h(s)\sup_{\theta\in[s,0]}\|\varphi(\theta)\|_{L^2}ds.$$

Hence, for $(t,\varphi)\in[0,T]\times\mathscr{B}_h$, where $\varphi(\theta)(z)=\varphi(\theta,z)$, $(\theta,z)\in(-\infty,0]\times[0,\pi]$. Set

$$u(t)(z)=u(t,z),\quad \zeta(t,\varphi)=\zeta_1(t)\zeta_2(\|\varphi(0)\|),$$

we have

$$\begin{aligned}
\mathscr{Q}_1(t,\varphi,\overline{\mathscr{H}}\varphi)(z)&=\int_{-\infty}^{0}e^{2(s)}\frac{\varphi}{49}ds+(\overline{\mathscr{H}}\varphi)(z),\\
\mathscr{Q}_2(t,\varphi,\widetilde{\mathscr{H}}\varphi)(z)&=\int_{-\infty}^{0}e^{2(s)}\frac{\varphi}{9}ds+(\widetilde{\mathscr{H}}\varphi)(z),\\
\mathscr{Q}_3(t,\varphi,\widehat{\mathscr{H}}\varphi)(z)&=\int_{-\infty}^{0}e^{2(s)}\frac{\varphi}{64}ds+(\widehat{\mathscr{H}}\varphi)(z),\\
g_i(t,\varphi)(z)&=\int_{-\infty}^{0}e^{2(s)}\frac{\varphi}{81}ds,\ i=1,2,\cdots,N,
\end{aligned}$$

where

$$\begin{aligned}
(\overline{\mathscr{H}}\varphi)(z)&=\int_0^t\sin(t-s)\int_{-\infty}^{0}e^{2(\tau)}\frac{\varphi}{36}d\tau ds,\\
(\widetilde{\mathscr{H}}\varphi)(z)&=\int_0^t\sin(t-s)\int_{-\infty}^{0}e^{2(\tau)}\frac{\varphi}{25}d\tau ds,\\
(\widehat{\mathscr{H}}\varphi)(z)&=\int_0^t\sin(t-s)\int_{-\infty}^{0}e^{2(\tau)}\frac{\varphi}{16}d\tau ds,
\end{aligned}$$

then using these configurations, the system (4.1)-(4.4) is usually written in the theoretical form of design (1.4)-(1.6).

To treat this system we assume that $\zeta_i:[0,\infty)\to[0,\infty)$, $i=1,2$ are continuous. Now, we can see that for $t\in[0,1],\varphi,\overline{\varphi}\in\mathscr{B}_h$, we have

$$\begin{aligned}
\|(\mathscr{A})^{\frac{1}{2}}\mathscr{Q}_1(t,\varphi,\overline{\mathscr{H}}\varphi)\|_{\mathbb{X}}&\leq\left(\int_0^{\pi}\left(\int_{-\infty}^{0}e^{2(s)}\left\|\frac{\varphi}{49}\right\|ds+\int_0^t\|\sin(t-s)\|\int_{-\infty}^{0}e^{2(\tau)}\left\|\frac{\varphi}{36}\right\|d\tau ds\right)^2dz\right)^{\frac{1}{2}}\\
&\leq\left(\int_0^{\pi}\left(\frac{1}{49}\int_{-\infty}^{0}e^{2(s)}\sup\|\varphi\|ds+\frac{1}{36}\int_{-\infty}^{0}e^{2(s)}\sup\|\varphi\|ds\right)^2dz\right)^{\frac{1}{2}}\\
&\leq\frac{\sqrt{\pi}}{49}\|\varphi\|_{\mathscr{B}_h}+\frac{\sqrt{\pi}}{36}\|\varphi\|_{\mathscr{B}_h}\\
&\leq L_{\mathscr{Q}_1}\|\varphi\|_{\mathscr{B}_h}+\widetilde{L}_{\mathscr{Q}_1}\|\varphi\|_{\mathscr{B}_h},
\end{aligned}$$

where $L_{\mathscr{Q}_1}+\widetilde{L}_{\mathscr{Q}_1}=\frac{85\sqrt{\pi}}{1764}$, and

$$\begin{aligned}
&\|(\mathscr{A})^{\frac{1}{2}}\mathscr{Q}_1(t,\varphi,\overline{\mathscr{H}}\varphi)-(\mathscr{A})^{\frac{1}{2}}\mathscr{Q}_1(t,\overline{\varphi},\overline{\mathscr{H}}\overline{\varphi})\|_{\mathbb{X}}\\
&\leq\left(\int_0^{\pi}\left(\int_{-\infty}^{0}e^{2(s)}\left\|\frac{\varphi}{49}-\frac{\overline{\varphi}}{49}\right\|ds+\int_0^t\|\sin(t-s)\|\int_{-\infty}^{0}e^{2(\tau)}\left\|\frac{\varphi}{36}-\frac{\overline{\varphi}}{36}\right\|d\tau ds\right)^2dz\right)^{\frac{1}{2}}\\
&\leq\left(\int_0^{\pi}\left(\frac{1}{49}\int_{-\infty}^{0}e^{2(s)}\sup\|\varphi-\overline{\varphi}\|ds+\frac{1}{36}\int_{-\infty}^{0}e^{2(s)}\sup\|\varphi-\overline{\varphi}\|ds\right)^2dz\right)^{\frac{1}{2}}
\end{aligned}$$

$$\leq \frac{\sqrt{\pi}}{49}\|\varphi-\overline{\varphi}\|_{\mathscr{B}_h}+\frac{\sqrt{\pi}}{36}\|\varphi-\overline{\varphi}\|_{\mathscr{B}_h}$$
$$\leq L_{\mathscr{Q}_1}\|\varphi-\overline{\varphi}\|_{\mathscr{B}_h}+\widetilde{L}_{\mathscr{Q}_1}\|\varphi-\overline{\varphi}\|_{\mathscr{B}_h}.$$

Similarly, we conclude

$$\|\mathscr{Q}_2(t,\varphi,\widetilde{\mathscr{H}}\varphi)\|_{L^2} \leq \left(\int_0^\pi\left(\int_{-\infty}^0 e^{2(s)}\left\|\frac{\varphi}{9}\right\|ds+\int_0^t\|\sin(t-s)\|\int_{-\infty}^0 e^{2(\tau)}\left\|\frac{\varphi}{25}\right\|d\tau ds\right)^2 dz\right)^{\frac{1}{2}}$$
$$\leq \left(\int_0^\pi\left(\frac{1}{9}\int_{-\infty}^0 e^{2(s)}\sup\|\varphi\|ds+\frac{1}{25}\int_{-\infty}^0 e^{2(s)}\sup\|\varphi\|ds\right)^2 dz\right)^{\frac{1}{2}}$$
$$\leq \frac{\sqrt{\pi}}{9}\|\varphi\|_{\mathscr{B}_h}+\frac{\sqrt{\pi}}{25}\|\varphi\|_{\mathscr{B}_h}$$
$$\leq L_{\mathscr{Q}_2}\|\varphi\|_{\mathscr{B}_h}+\widetilde{L}_{\mathscr{Q}_2}\|\varphi\|_{\mathscr{B}_h},$$

where $L_{\mathscr{Q}_2}+\widetilde{L}_{\mathscr{Q}_2}=\frac{34\sqrt{\pi}}{225}$, and

$$\|\mathscr{Q}_2(t,\varphi,\widetilde{\mathscr{H}}\varphi)-\mathscr{Q}_2(t,\overline{\varphi},\widetilde{\mathscr{H}}\overline{\varphi})\|_{L^2}$$
$$\leq \left(\int_0^\pi\left(\int_{-\infty}^0 e^{2(s)}\left\|\frac{\varphi}{9}-\frac{\overline{\varphi}}{9}\right\|ds+\int_0^t\|\sin(t-s)\|\int_{-\infty}^0 e^{2(\tau)}\left\|\frac{\varphi}{25}-\frac{\overline{\varphi}}{25}\right\|d\tau ds\right)^2 dz\right)^{\frac{1}{2}}$$
$$\leq \left(\int_0^\pi\left(\frac{1}{9}\int_{-\infty}^0 e^{2(s)}\sup\|\varphi-\overline{\varphi}\|ds+\frac{1}{25}\int_{-\infty}^0 e^{2(s)}\sup\|\varphi-\overline{\varphi}\|ds\right)^2 dz\right)^{\frac{1}{2}}$$
$$\leq \frac{\sqrt{\pi}}{9}\|\varphi-\overline{\varphi}\|_{\mathscr{B}_h}+\frac{\sqrt{\pi}}{25}\|\varphi-\overline{\varphi}\|_{\mathscr{B}_h}$$
$$\leq L_{\mathscr{Q}_2}\|\varphi-\overline{\varphi}\|_{\mathscr{B}_h}+\widetilde{L}_{\mathscr{Q}_2}\|\varphi-\overline{\varphi}\|_{\mathscr{B}_h}.$$

Correspondingly, we have

$$\|\mathscr{Q}_3(t,\varphi,\widehat{\mathscr{H}}\varphi)\|_{L^2} \leq \left(\int_0^\pi\left(\int_{-\infty}^0 e^{2(s)}\left\|\frac{\varphi}{64}\right\|ds+\int_0^t\|\sin(t-s)\|\int_{-\infty}^0 e^{2(\tau)}\left\|\frac{\varphi}{16}\right\|d\tau ds\right)^2 dz\right)^{\frac{1}{2}}$$
$$\leq \left(\int_0^\pi\left(\frac{1}{64}\int_{-\infty}^0 e^{2(s)}\sup\|\varphi\|ds+\frac{1}{16}\int_{-\infty}^0 e^{2(s)}\sup\|\varphi\|ds\right)^2 dz\right)^{\frac{1}{2}}$$
$$\leq \frac{\sqrt{\pi}}{64}\|\varphi\|_{\mathscr{B}_h}+\frac{\sqrt{\pi}}{16}\|\varphi\|_{\mathscr{B}_h}$$
$$\leq L_{\mathscr{Q}_3}\|\varphi\|_{\mathscr{B}_h}+\widetilde{L}_{\mathscr{Q}_3}\|\varphi\|_{\mathscr{B}_h},$$

where $L_{\mathscr{Q}_3}+\widetilde{L}_{\mathscr{Q}_3}=\frac{80\sqrt{\pi}}{1024}$, and

$$\|\mathscr{Q}_3(t,\varphi,\widehat{\mathscr{H}}\varphi)-\mathscr{Q}_3(t,\overline{\varphi},\widehat{\mathscr{H}}\overline{\varphi})\|_{L^2}$$
$$\leq \left(\int_0^\pi\left(\int_{-\infty}^0 e^{2(s)}\left\|\frac{\varphi}{64}-\frac{\overline{\varphi}}{64}\right\|ds+\int_0^t\|\sin(t-s)\|\int_{-\infty}^0 e^{2(\tau)}\left\|\frac{\varphi}{16}-\frac{\overline{\varphi}}{16}\right\|d\tau ds\right)^2 dz\right)^{\frac{1}{2}}$$
$$\leq \left(\int_0^\pi\left(\frac{1}{64}\int_{-\infty}^0 e^{2(s)}\sup\|\varphi-\overline{\varphi}\|ds+\frac{1}{16}\int_{-\infty}^0 e^{2(s)}\sup\|\varphi-\overline{\varphi}\|ds\right)^2 dz\right)^{\frac{1}{2}}$$

$$\leq \frac{\sqrt{\pi}}{64}\|\varphi-\overline{\varphi}\|_{\mathscr{B}_h}+\frac{\sqrt{\pi}}{16}\|\varphi-\overline{\varphi}\|_{\mathscr{B}_h}$$
$$\leq L_{\mathscr{Q}_3}\|\varphi-\overline{\varphi}\|_{\mathscr{B}_h}+\widetilde{L}_{\mathscr{Q}_3}\|\varphi-\overline{\varphi}\|_{\mathscr{B}_h}.$$

Finally,

$$\|g_i(t,\varphi)\|_{\mathbb{X}}=\left(\int_0^{\pi}\left(\int_{-\infty}^{0}e^{2(s)}\left\|\frac{\varphi}{81}\right\|ds\right)^2dz\right)^{\frac{1}{2}}$$
$$\leq\left(\int_0^{\pi}\left(\frac{1}{81}\int_{-\infty}^{0}e^{2(s)}\sup\|\varphi\|ds\right)^2dz\right)^{\frac{1}{2}}$$
$$\leq L_{g_i}\|\varphi\|_{\mathscr{B}_h},\ i=1,2,\cdots,N,$$

where $L_{g_i}=\frac{\sqrt{\pi}}{81}$, and

$$\|g_i(t,\varphi)-g_i(t,\overline{\varphi})\|_{\mathbb{X}}$$
$$=\left(\int_0^{\pi}\left(\int_{-\infty}^{0}e^{2(s)}\left\|\frac{\varphi}{81}-\frac{\overline{\varphi}}{81}\right\|ds\right)^2dz\right)^{\frac{1}{2}}$$
$$\leq\left(\int_0^{\pi}\left(\frac{1}{81}\int_{-\infty}^{0}e^{2(s)}\sup\|\varphi-\overline{\varphi}\|ds\right)^2dz\right)^{\frac{1}{2}}$$
$$\leq L_{g_i}\|\varphi-\overline{\varphi}\|_{\mathscr{B}_h}.$$

Therefore the conditions (H1)-(H6) are all fulfilled. Furthermore, we assume that $\mathscr{E}_1*=1,\mathcal{M}=1,\mathcal{M}_0=1,\mathcal{M}_{\frac{1}{2}}=1,\alpha=\frac{1}{2},T=1,\mathcal{C}^*=1$ and $L_\zeta=1$. Moreover, the appropriate values of the constants $L_{g_i}(r),L_{\mathscr{Q}_1}(r)$, $L_{\mathscr{Q}_2}(r)$ and $L_{\mathscr{Q}_3}(r)$, obtain

$$\Lambda=\mathscr{E}_1^*\max_{1\leq i\leq N}\Bigg[\mathcal{M}(L_{g_i}+2L_{g_i}(r)L_\zeta)+(\mathcal{M}+1)\left(\mathcal{M}_0+\frac{\mathcal{M}_{1-\beta}\Gamma(\beta+1)T^{\alpha\beta}}{\beta\Gamma(\alpha\beta+1)}\right)(L_{\mathscr{Q}_1}+\mathcal{C}^*\widetilde{L}_{\mathscr{Q}_1}$$
$$+L_{\mathscr{Q}_1}(r)L^*)+\frac{\mathcal{M}(\mathcal{M}+1)T^{\alpha}}{\Gamma(\alpha+1)}\{(L_{\mathscr{Q}_2}+L_{\mathscr{Q}_3})+\mathcal{C}^*(\widetilde{L}_{\mathscr{Q}_2}+\widetilde{L}_{\mathscr{Q}_3})+(L_{\mathscr{Q}_2}(r)+L_{\mathscr{Q}_3}(r))L^*\}\Bigg]<1$$

be such that $0\leq\Lambda<1$, where $2L_\zeta(1+\mathcal{C}^*)=L^*$. Thus the condition (H7) holds. Hence by Theorem 3.1, we realize that the system (4.1)-(4.4) has a unique mild solution on [0,1].

## References

[1] R. P. Agarwal, V. Lupulescu, D. O'Regan and G. Rahman, Fractional calculus and fractional differential equations in nonreflexive Banach spaces, *Communications in Nonlinear Science and Numerical Simulation*, 20(1)(2015), 59–73.

[2] R. P. Agarwal and B. D. Andrade, On fractional integro-differential equations with state-dependent delay, *Comp. Math. App.,* 62(2011), 1143–1149.

[3] B. Ahmad, S. K. Ntouyas and A. Alsaed, Existence of solutions for fractional q-integro-difference inclusions with fractional q-integral boundary conditions, *Advances in Difference Equations*, 257(2014), 1–18.

[4] K. Aissani and M. Benchohra, Fractional integro-differential equations with state-dependent delay, *Advances in Dynamical Systems and Applications*, 9(1)(2014), 17–30.

[5] F. Andrade, C.Cuevas and H.R. Henriquez, Periodic solutions of abstract functional differential equations with state-dependent delay, *Mathematical Methods in the Applied Sciences,* DOI: 10.1002/mma. 3837, 2016.

[6] D. Baleanu, J. A. T. Machado and A. C. J. Luo, *Fractional Dynamics and Control*, Springer, New York, USA, 2012.

[7] D. Bainov and V. Covachev, *Impulsive Differential Equations With a Small Parameter*, World Scientific Publishing Corporation, Singapore, 1995.

[8] M. Benchohra, J. Henderson and S. K. Ntouyas, *Impulsive Differential Equations and Inclusions, in: Contemporary Mathematics and its Applications*, Vol. 2, Hindawi Publishing Corporation, New York, 2006.

[9] M. Benchohra and F. Berhoun, Impulsive fractional differential equations with state-dependent delay, *Commun. Appl. Anal.,* 14(2)(2010), 213–224.

[10] G. Bonanno, R. Rodriguez-Lopez and S. Tersian, Existence of solutions to boundary value problem for impulsive fractional differential equations, *Fractional Calculus and Applied Analysis*, 17(3)(2014), 717–744.

[11] J. Dabas and G.R. Gautam, Impulsive neutral fractional integro-differential equation with state-dependent delay and integral boundary condition, *Electronic Journal of Differential Equations,* Vol. 2013 (2013), No. 273, 1–13.

[12] M.A. Darwish and S.K. Ntouyas, Functional differential equations of fractional order with state-dependent delay, *Dyn. Syst. Appl.,* (18)(2009), 539–550.

[13] M.A. Darwish and S.K. Ntouyas, Semilinear functional differential equations of fractional order with state-dependent delay, *Electron. J. Diff. Equ.,* 2009(2009), No. 38, 1–10.

[14] S. Das, D. N. Pandey and N. Sukavanam, Existence of solution and approximate controllability for neutral differential equation with state dependent delay, *International Journal of Partial Differential Equations*, Volume 2014 (2014), Article ID 787092, 12 pages.

[15] J. P. C. Dos Santos, C. Cuevas and B. de Andrade, Existence results for a fractional equations with state-dependent delay, *Advances in Difference Equations*, Volume 2011, Article ID 642013, 15 pages.

[16] J. P. C. Dos Santos, M. Mallika Arjunan and C. Cuevas, Existence results for fractional neutral integrodifferential equations with state-dependent delay, *Comput. Math. Appl.*, 62(2011), 1275–1283.

[17] X. Fu, X. Liu and B. Lu, On a new class of impulsive fractional evolution equations, *Advances in Difference Equations*, (2015), 2015:227.

[18] X. Fu and R. Huang, Existence of solutions for neutral integro-differential equations with state-dependent delay, *Appl. Math. Comp.,* 224(2013), 743-759.

[19] G. R. Gautam and Jaydev Dabas, Mild solution for fractional functional integro-differential equation with not instantaneous impulse, *Malaya J. Mat.,* 2(3)(2014), 428–437.

[20] G. R. Gautam and J. Dabas, Existence result of fractional functional integrodifferential equation with not instantaneous impulse, *Int. J. Adv. Appl. Math. and Mech.,* 1(3)(2014), 11–21.

[21] G. R. Gautam and J. Dabas, Mild solutions for class of neutral fractional functional differential equations with not instantaneous impulses, *Applied Mathematics and Computation*, 259(2015), 480–489.

[22] E. Hernandez and D. O'Regan, On a new class of abstract impulsive differential equations, *Proc. Amer. Math. Soc.*, 141(5)(2013), 1641–1649.

[23] E. Keshavarz, Y. Ordokhani and M. Razzaghi, Bernoulli wavelet operational matrix of fractional order integration and its applications in solving the fractional order differential equations, *Applied Mathematical Modelling*, 38(2014), 6038–6051.

[24] A.A. Kilbas, H.M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amesterdam, 2006.

[25] P. Kumara, D. N. Pandey and D. Bahuguna, On a new class of abstract impulsive functional differential equations of fractional order, *J. Nonlinear Sci. Appl.*, 7(2014), 102–114.

[26] V. Lakshmikantham, D. D Bainov, and P. S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, Teaneck, London, 1989.

[27] A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, Birkhauser, Basel, Switzerland, 1995.

[28] Z. Lv and B. Chen, Existence and uniqueness of positive solutions for a fractional switched system, *Abstract and Applied Analysis*, Volume 2014 (2014), Article ID 828721, 7 pages.

[29] F. Mainardi, P. Paradisi and R. Gorenflo, *Probability distributions generated by fractional diffusion equations*, in Econo-physics: An Emerging Science, J. Kertesz and I. Kondor, Eds., Kluwer Academic, Dordrecht, The Netherlands, 2000.

[30] D. N. Pandey, S. Das and N. Sukavanam, Existence of solution for a second-order neutral differential equation with state dependent delay and non-instantaneous impulses, *International Journal of Nonlinear Science*, 18(2)(2014), 145–155.

[31] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[32] M. Pierri, D. O'Regan and V. Rolnik, Existence of solutions for semi-linear abstract differential equations with not instantaneous impulses, *Appl. Math. Comp.*, 219(12)(2013), 6743–6749.

[33] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, 1999.

[34] R. Rodriguez-Lopez and S. Tersian, Multiple solutions to boundary value problem for impulsive fractional differential equations, *Fractional Calculus and Applied Analysis*, 17(4)(2014), 1016–1038.

[35] X.B. Shu and F. Xu, The existence of solutions for impulsive fractional partial neutral differential equations, *Journal of Mathematics*, Volume 2013, Article ID 147193, 9 pages.

[36] X.B. Shu, Y. Lai and Y. Chen, The existence of mild solutions for impulsive fractional partial differential equations, *Nonlinear Analysis: Theory, Methods & Applications*, 74(5)(2011), 2003–2011.

[37] X.B. Shu and Q. Wang, The existence and uniqueness of mild solutions for fractional differential equations with nonlocal conditions of order $1 < \alpha < 2$, *Computers and Mathematics with Applications*, 64(2012), 2100-2110.

[38] X.B. Shu, F. Xu and Y. Shi, $S$-asymptotically $\omega$-positive periodic solutions for a class of neutral fractional differential equations, *Applied Mathematics and Computation*, 270(1)(2015), 768-776.

[39] I. M. Stamova, *Stability Analysis of Impulsive Functional Differential Equations*, De Gruyter, 2009.

[40] S. Suganya, D. Baleanu, P. Kalamani and M. Mallika Arjunan, On fractional neutral integro-differential systems with state-dependent delay and non-instantaneous impulses, *Advances in Difference Equations*, 2015, 2015:372.

[41] S. Suganya, M. Mallika Arjunan and J. J. Trujillo, Existence results for an impulsive fractional integro-differential equation with state-dependent delay, *Applied Mathematics and Computation*, 266(2015), 54-69.

[42] V. Vijayakumar, C. Ravichandran and R. Murugesu, Approximate controllability for a class of fractional neutral integro-differential inclusions with state-dependent delay, *Nonlinear Studies*, 20(4)(2013), 513-532.

[43] X. Wang and X.B. Shu, The existence of positive mild solutions for fractional differential evolution equations with nonlocal conditions of order $1 < \alpha < 2$, *Advances in Difference Equations*, 2015, 2015:159, 15 pp.

[44] Y. Wang, L. Liu and Y. Wu, Positive solutions for a class of higher-order singular semipositone fractional differential systems with coupled integral boundary conditions and parameters, *Advances in Difference Equations*, 268(2014), 1–24.

[45] Z. Yan, Approximate controllability of fractional neutral integro-differential inclusions with state-dependent delay in Hilbert spaces, *IMA Journal of Mathematical Control and Information*, 30(2013), 443–462.

[46] Z. Yan and X. Jia, Approximate controllability of partial fractional neutral stochastic functional integro-differential inclusions with state-dependent delay, *Collect. Math.,* 66(2015), 93–124.

[47] Y. Zhou, *Basic Theory of Fractional Differential Equations*, World Scientific, Singapore, 2014.

[48] Y. Zhou and F. Jiao, Existence of mild solutions for fractional neutral evolution equations, *Comput. Math. Appl.*, 59(3)(2010), 1063–1077.

# Characterizations of positive implicative superior ideals induced by superior mappings

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**Abstract.** The notion of positive implicative superior ideals of BCK-algebras is introduced, and their properties are investigated. Relations between a superior ideal and a positive implicative superior ideal in BCK-algebras are studied, and conditions for a superior ideal to be a positive implicative superior ideal are provided. Characterizations of positive implicative superior ideals induced by superior mappings are discussed.

## 1. Introduction

Algebras have played an important role in pure and applied mathematics and have its comprehensive applications in many aspects including dynamical systems and genetic code of biology (see [1], [2], [7], and [12]). Starting from the four DNA bases order in the Boolean lattice, Sáanchez et al. [11] proposed a novel Lie Algebra of the genetic code which shows strong connections among algebraic relationship, codon assignments and physicochemical properties of amino acids. A BCK/BCI-algebra (see [3, 4, 10]) is an important class of logical algebras introduced by Iséki and was extensively investigated by several researchers. Jun and Song [5] introduced the notion of BCK-valued functions and investigated several properties. They established block-codes by using the notion of BCK-valued functions, and shown that every finite BCK-algebra determines a block-code. In [6], Jun and Song introduced the notion of superior mapping by using partially ordered sets. Using the superior mapping, they introduced the concept of superior subalgebras and (commutative) superior ideals in BCK/BCI-algebras, and investigated related properties. They also discussed relations among a superior subalgebra, a superior ideal and a commutative superior ideal.

In this paper, we introduce the notion of positive implicative superior ideals of BCK-algebras, and investigate properties. We investigate relations between a superior ideal and a positive implicative superior ideal in BCK-algebras. We provide conditions for a superior ideal to be a positive implicative superior ideal, and discuss characterizations of positive implicative superior ideals.

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## 2. **Preliminaries**

We display basic definitions and properties of BCK/BCI-algebras that will be used in this paper. For more details of BCK/BCI-algebras, we refer the reader to [3], [8], [9] and [10].

An algebra $\mathcal{L} := (L; *, 0)$ is called a BCI-algebra if it satisfies the following conditions:

(I) $(\forall x, y, z \in L)\ (((x * y) * (x * z)) * (z * y) = 0)$,
(II) $(\forall x, y \in L)\ ((x * (x * y)) * y = 0)$,
(III) $(\forall x \in L)\ (x * x = 0)$,
(IV) $(\forall x, y \in L)\ (x * y = 0, y * x = 0 \ \Rightarrow\ x = y)$.

If a BCI-algebra $\mathcal{L}$ satisfies the following identity:

(V) $(\forall x \in L)\ (0 * x = 0)$,

then $\mathcal{L}$ is called a BCK-algebra.

A BCK-algebra $\mathcal{L}$ is said to be positive implicative if it satisfies:

$$(\forall x, y, z \in L)\,((x * y) * z = (x * z) * (y * z))\,. \tag{2.1}$$

Any BCK/BCI-algebra $\mathcal{L}$ satisfies the following conditions:

$$(\forall x \in L)\,(x * 0 = x)\,, \tag{2.2}$$

$$(\forall x, y, z \in L)\,(x \le y \ \Rightarrow\ x * z \le y * z,\ z * y \le z * x)\,, \tag{2.3}$$

$$(\forall x, y, z \in L)\,((x * y) * z = (x * z) * y)\,, \tag{2.4}$$

$$(\forall x, y, z \in L)\,((x * z) * (y * z) \le x * y) \tag{2.5}$$

where $x \le y$ if and only if $x * y = 0$.

A subset $A$ of a BCK/BCI-algebra $\mathcal{L}$ is called an ideal of $\mathcal{L}$ if it satisfies:

$$0 \in A, \tag{2.6}$$

$$(\forall x, y \in L)\,(x * y \in A,\ y \in A \ \Rightarrow\ x \in A)\,. \tag{2.7}$$

A subset $A$ of a BCK-algebra $\mathcal{L}$ is called a positive implicative ideal of $\mathcal{L}$ if it satisfies (2.6) and

$$(\forall x, y, z \in L)\,((x * y) * z \in A,\ y * z \in A \ \Rightarrow\ x * z \in A)\,. \tag{2.8}$$

Let $L$ be a set of parameters and let $U$ be a partially ordered set with the partial ordering $\preceq$ and the first element $e$. For a mapping $\tilde{f} : L \to \mathcal{P}(U)$, we consider the mapping

$$||\tilde{f}|| : L \to U,\ x \mapsto \begin{cases} \sup \tilde{f}(x) & \text{if } \exists \sup \tilde{f}(x), \\ e & \text{otherwise,} \end{cases} \tag{2.9}$$

which is called the superior mapping of $L$ with respect to $\left(\tilde{f}, L\right)$. In this case, we say that $\left(\tilde{f}, L\right)$ is a pair on $(U, \preceq)$ (see [6]).

**Definition 2.1** ([6]). Let $\mathcal{L} := (L, *, 0)$ be a BCK/BCI-algebra. By a superior ideal on $(\mathcal{L}, \tilde{f})$, we mean the superior mapping $||\tilde{f}||$ of $\mathcal{L}$ with respect to $\left(\tilde{f}, L\right)$ which satisfies the following conditions:

$$(\forall x \in L)\left(||\tilde{f}||(0) \preceq ||\tilde{f}||(x)\right), \tag{2.10}$$

$$(\forall x, y \in L)\left(||\tilde{f}||(x) \preceq \sup\{||\tilde{f}||(x * y), ||\tilde{f}||(y)\}\right). \tag{2.11}$$

**Proposition 2.2** ([6]). *If $||\tilde{f}||$ is a superior ideal on $(\mathcal{L}, \tilde{f})$, then $||\tilde{f}||(x) \preceq ||\tilde{f}||(y)$ for all $x, y \in L$ with $x \leq y$.*

## 3. **Positive implicative superior ideals**

In what follows, let $\mathcal{L} := (L, *, 0)$ be a BCK-algebra unless otherwise specified, where $L$ is a set of parameters.

**Definition 3.1.** By a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$, we mean the superior mapping $||\tilde{f}||$ of $\mathcal{L}$ with respect to $\left(\tilde{f}, L\right)$ which satisfies the condition (2.10) and

$$(\forall x, y, z \in L)\left(||\tilde{f}||(x * z) \preceq \sup\{||\tilde{f}||((x * y) * z), ||\tilde{f}||(y * z)\}\right). \tag{3.1}$$

**Example 3.2.** Let $L = \{0, 1, 2, 3\}$ be a set with a binary operation '$*$' shown in Table 1.

TABLE 1. Cayley table for the binary operation '$*$'

| $*$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 1 |
| 2 | 2 | 2 | 0 | 2 |
| 3 | 3 | 3 | 3 | 0 |

Then $\mathcal{L} := (L, *, 0)$ is a BCK-algebra (see [10]). Let $U = \{a, b, c, d, e, f\}$ be ordered as pictured in Figure A3.

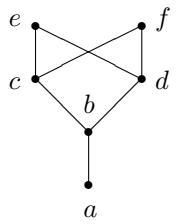



Figure A3

Seok Zun Song[1,†], Young Bae Jun[2] and Hee Sik Kim[3,*]

(1) Let $\left(\tilde{f}, L\right)$ be a pair on $(U, \preceq)$ where $\tilde{f}$ is given as follows:

$$\tilde{f} : L \to \mathcal{P}(U),\ x \mapsto \begin{cases} \{a, b\} & \text{if } x = 0, \\ \{a, d, f\} & \text{if } x = 1, \\ \{b, c, d, f\} & \text{if } x = 2, \\ \{a, b, c\} & \text{if } x = 3. \end{cases}$$

Then the superior mapping $||\tilde{f}||$ of $\mathcal{L}$ with respect to $\left(\tilde{f}, L\right)$ is described as follows: $||\tilde{f}||(0) = b$, $||\tilde{f}||(1) = ||\tilde{f}||(2) = f$ and $||\tilde{f}||(3) = c$. By routine calculations, we know that $||\tilde{f}||$ is a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$.

(2) Let $(\tilde{g}, L)$ be a pair on $(U, \preceq)$ where $\tilde{g}$ is given as follows:

$$\tilde{g} : L \to \mathcal{P}(U),\ x \mapsto \begin{cases} \{a, b\} & \text{if } x = 0, \\ \{b, c, d, e\} & \text{if } x = 3, \\ \{b, c, d, f\} & \text{if } x \in \{1, 2\}. \end{cases}$$

Then the superior mapping $||\tilde{g}||$ of $\mathcal{L}$ with respect to $(\tilde{g}, L)$ is described as follows: $||\tilde{g}||(0) = b$, $||\tilde{f}||(1) = ||\tilde{f}||(2) = f$ and $||\tilde{f}||(3) = e$. It is not a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$ since there does not exist $\sup\{||\tilde{f}||((3 * 2) * 1), ||\tilde{f}||(2 * 1)\}$ because $||\tilde{f}||((3 * 2) * 1) = e$ and $||\tilde{f}||(2 * 1) = f$ are noncomparable.

**Example 3.3.** Let $U = \{a, b, c, d, e, f\}$ be ordered as pictured in Figure B3.

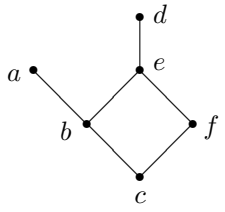



Figure B3

Let $L = \{0, 1, 2, 3, 4\}$ be a set with a binary operation '$*$' shown in Table 2.

TABLE 2. Cayley table for the binary operation '$*$'

| $*$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 1 | 0 | 1 |
| 2 | 2 | 2 | 0 | 2 | 0 |
| 3 | 3 | 1 | 3 | 0 | 3 |
| 4 | 4 | 4 | 4 | 4 | 0 |

Then $\mathcal{L} := (L, *, 0)$ is a BCK-algebra (see [10]). Let $\left(\tilde{f}, L\right)$ be a pair on $(U, \preceq)$ where $\tilde{f}$ is defined by

$$\tilde{f} : L \to \mathcal{P}(U),\ x \mapsto \begin{cases} \{b, c\} & \text{if } x = 0, \\ \{a, b, e\} & \text{if } x = 1, \\ \{b, e, f\} & \text{if } x = 2, \\ \{a, c, e, f\} & \text{if } x = 3, \\ \{d, e\} & \text{if } x = 4. \end{cases}$$

Then the superior mapping of $\mathcal{L}$ with respect to $\left(\tilde{f}, L\right)$ is described as follows: $||\tilde{f}||(0) = b$, $||\tilde{f}||(1) = c$, $||\tilde{f}||(2) = e$, $||\tilde{f}||(3) = c$ and $||\tilde{f}||(4) = d$, and it is neither a superior ideal nor a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$.

**Theorem 3.4.** *Let $\left(\tilde{f}, L\right)$ be a pair on $(U, \preceq)$. If $||\tilde{f}||$ is a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$, then the nonempty set*

$$||\tilde{f}||_\alpha := \{x \in L \mid ||\tilde{f}||(x) \preceq \alpha\}$$

*is a positive implicative ideal of $\mathcal{L}$ for all $\alpha \in U$.*

*Proof.* Let $\alpha \in U$ be such that $||\tilde{f}||_\alpha \neq \emptyset$. Clearly $0 \in ||\tilde{f}||_\alpha$. Let $x, y, z \in L$ be such that $(x*y)*z \in ||\tilde{f}||_\alpha$ and $y * z \in ||\tilde{f}||_\alpha$. Then $||\tilde{f}||((x * y) * z) \preceq \alpha$ and $||\tilde{f}||(y * z) \preceq \alpha$. It follows from (3.1) that

$$||\tilde{f}||(x * z) \preceq \sup\{||\tilde{f}||((x * y) * z), ||\tilde{f}||(y * z)\} \preceq \alpha.$$

Thus $x * z \in ||\tilde{f}||_\alpha$, and therefore $||\tilde{f}||_\alpha$ is a positive implicative ideal of $\mathcal{L}$. □

**Corollary 3.5.** *Let $\left(\tilde{f}, L\right)$ be a pair on $(U, \preceq)$. If $||\tilde{f}||$ is a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$, then the set*

$$A := \{x \in L \mid ||\tilde{f}||(x) = ||\tilde{f}||(0)\}$$

*is a positive implicative ideal of $\mathcal{L}$.*

**Theorem 3.6.** *Every positive implicative superior ideal is a superior ideal.*

*Proof.* Let $||\tilde{f}||$ be a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$. If we take $z = 0$ in (3.1) and use (2.2), then

$$||\tilde{f}||(x) = ||\tilde{f}||(x * 0) \preceq \sup\{||\tilde{f}||((x * y) * 0), ||\tilde{f}||(y * 0)\} = \sup\{||\tilde{f}||(x * y), ||\tilde{f}||(y)\}$$

for all $x, y \in L$. Hence $||\tilde{f}||$ is a superior ideal on $(\mathcal{L}, \tilde{f})$. □

The converse of Theorem 3.6 is not true in general as seen in the following example.

**Example 3.7.** Let $L = \{0, a, b, c\}$ be a set with a binary operation '$*$' shown in Table 3.
Then $\mathcal{L} := (L, *, 0)$ is a BCK-algebra (see [10]). Let $U = \{1, 2, 3, \cdots, 8\}$ be ordered as pictured in Figure 1.

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TABLE 3. Cayley table for the binary operation '$*$'

| $*$ | 0 | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| $a$ | $a$ | 0 | 0 | $a$ |
| $b$ | $b$ | $a$ | 0 | $b$ |
| $c$ | $c$ | $c$ | $c$ | 0 |

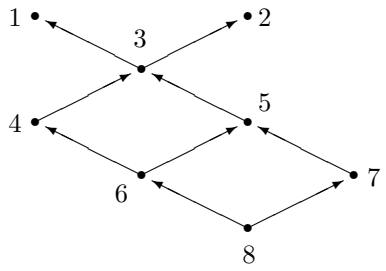



Figure 1

Consider a pair $\left(\tilde{f}, L\right)$ in which $\tilde{f}$ is given as follows:

$$\tilde{f} : L \to \mathcal{P}(U),\ x \mapsto \begin{cases} \{6,8\} & \text{if } x \in \{0\}, \\ \{4,6,7\} & \text{if } x \in \{a,b\}, \\ \{2,3,5,6,7\} & \text{if } x = c. \end{cases}$$

Then the superior mapping $||\tilde{f}||$ on $(\mathcal{L}, \tilde{f})$ is described as follows: $||\tilde{f}||(0) = 6$, $||\tilde{f}||(a) = ||\tilde{f}||(b) = 3$ and $||\tilde{f}||(c) = 2$. Routine calculations show that $||\tilde{f}||$ is a superior ideal on $(\mathcal{L}, \tilde{f})$. But it is not a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$ since

$$||\tilde{f}||(b * a) = 3 \not\succeq 6 = \sup\{||\tilde{f}||((b * a) * a), ||\tilde{f}||(a * a)\}.$$

We provide conditions for a superior ideal to be a positive implicative superior ideal.

**Theorem 3.8.** *For a superior ideal $||\tilde{f}||$ on $(\mathcal{L}, \tilde{f})$, the following are equivalent.*

(i) *$||\tilde{f}||$ is a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$.*

(ii) $(\forall x, y \in L)\left(||\tilde{f}||(x * y) \preceq ||\tilde{f}||((x * y) * y)\right)$.

*Proof.* Assume that $||\tilde{f}||$ is a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$. If we put $z = y$ in (3.1), then

$$\begin{aligned} ||\tilde{f}||(x * y) &\preceq \sup\{||\tilde{f}||((x * y) * y), ||\tilde{f}||(y * y)\} \\ &= \sup\{||\tilde{f}||((x * y) * y), ||\tilde{f}||(0)\} \\ &= ||\tilde{f}||((x * y) * y) \end{aligned}$$

for all $x, y \in L$.

Conversely, let $||\tilde{f}||$ be a superior ideal on $(\mathcal{L}, \tilde{f})$ which satisfies the condition (ii). Note that

$$((x * z) * z) * (y * z) \leq (x * z) * y = (x * y) * z$$

for all $x, y, z \in L$. It follows from (ii), (2.11) and Proposition 2.2 that

$$\begin{aligned} ||\tilde{f}||(x * z) &\preceq ||\tilde{f}||((x * z) * z) \\ &\preceq \sup\{||\tilde{f}||(((x * z) * z) * (y * z)), ||\tilde{f}||(y * z)\} \\ &\preceq \sup\{||\tilde{f}||((x * y) * z), ||\tilde{f}||(y * z)\} \end{aligned}$$

for all $x, y, z \in L$. Therefore $||\tilde{f}||$ is a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$. □

**Theorem 3.9.** *For a superior ideal $||\tilde{f}||$ on $(\mathcal{L}, \tilde{f})$, the following are equivalent.*

(i) *$||\tilde{f}||$ is a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$.*

(ii) $(\forall x, y, z \in L)\left(||\tilde{f}||((x * z) * (y * z)) \preceq ||\tilde{f}||((x * y) * z)\right)$.

*Proof.* Suppose that $||\tilde{f}||$ is a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$. Then $||\tilde{f}||$ is a superior ideal on $(\mathcal{L}, \tilde{f})$ by Theorem 3.6. Note that

$$((x * (y * z)) * z) * z = ((x * z) * (y * z)) * z \leq (x * y) * z$$

for all $x, y, z \in L$. It follows from (2.4), Theorem 3.8 and Proposition 2.2 that

$$\begin{aligned} ||\tilde{f}||((x * z) * (y * z)) &= ||\tilde{f}||((x * (y * z)) * z) \\ &\preceq ||\tilde{f}||(((x * (y * z)) * z) * z) \\ &\preceq ||\tilde{f}||((x * y) * z) \end{aligned}$$

for all $x, y, z \in L$.

Conversely, let $||\tilde{f}||$ be a superior ideal on $(\mathcal{L}, \tilde{f})$ which satisfies the second condition. Using (2.11) and the second condition, we have

$$\begin{aligned} ||\tilde{f}||(x * z) &= \sup\{||\tilde{f}||((x * z) * (y * z)), ||\tilde{f}||(y * z)\} \\ &\preceq \sup\{||\tilde{f}||((x * y) * z), ||\tilde{f}||(y * z)\} \end{aligned}$$

for all $x, y, z \in L$. Therefore $||\tilde{f}||$ is a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$. □

**Theorem 3.10.** *Let $||\tilde{f}||$ be the superior mapping of $\mathcal{L}$ with respect to $\left(\tilde{f}, L\right)$. Then $||\tilde{f}||$ is a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$ if and only if it satisfies the condition* (2.10) *and*

$$(\forall x, y, z \in L)\left(||\tilde{f}||(x * y) \preceq \sup\{||\tilde{f}||(((x * y) * y) * z), ||\tilde{f}||(z)\}\right). \tag{3.2}$$

*Proof.* Assume that $||\tilde{f}||$ is a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$. Then $||\tilde{f}||$ is a superior ideal on $(\mathcal{L}, \tilde{f})$ by Theorem 3.6, and so $||\tilde{f}||$ satisfies the condition (2.10). Using (2.11), (III), (2.2), (2.4) and Theorem 3.9, we have

$$\begin{aligned} ||\tilde{f}||(x * y) &\preceq \sup\{||\tilde{f}||((x * y) * z), ||\tilde{f}||(z)\} \\ &= \sup\{||\tilde{f}||(((x * z) * y) * (y * y)), ||\tilde{f}||(z)\} \\ &\preceq \sup\{||\tilde{f}||(((x * z) * y) * y), ||\tilde{f}||(z)\} \\ &= \sup\{||\tilde{f}||(((x * y) * y) * z), ||\tilde{f}||(z)\} \end{aligned}$$

for all $x, y, z \in L$.

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Conversely, suppose that $||\tilde{f}||$ satisfies two conditions (2.10) and (3.2). Then

$$\begin{aligned} ||\tilde{f}||(x) &= ||\tilde{f}||(x*0) \\ &\preceq \sup\{||\tilde{f}||(((x*0)*0)*z), ||\tilde{f}||(z)\} \\ &= \sup\{||\tilde{f}||(x*z), ||\tilde{f}||(z)\} \end{aligned}$$

for all $x, z \in L$, and so $||\tilde{f}||$ is a superior ideal on $(\mathcal{L}, \tilde{f})$. If we take $z = 0$ in (3.2) and use (2.2) and (2.10), then

$$\begin{aligned} ||\tilde{f}||(x*y) &\preceq \sup\{||\tilde{f}||(((x*y)*y)*0), ||\tilde{f}||(0)\} \\ &= \sup\{||\tilde{f}||((x*y)*y), ||\tilde{f}||(0)\} \\ &= ||\tilde{f}||((x*y)*y) \end{aligned}$$

for all $x, y \in L$. Therefore $||\tilde{f}||$ is a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$ by Theorem 3.8. □

**Lemma 3.11.** *Let $||\tilde{f}||$ be the superior mapping of $\mathcal{L}$ with respect to a pair $\left(\tilde{f}, L\right)$ on $(U, \preceq)$. Then $||\tilde{f}||$ is a superior ideal on $(\mathcal{L}, \tilde{f})$ if and only if it satisfies the following assertion:*

$$(\forall x, y, z \in L)\left((x*y)*z = 0 \ \Rightarrow \ ||\tilde{f}||(x) \preceq \sup\{||\tilde{f}||(y), ||\tilde{f}||(z)\}\right). \tag{3.3}$$

*Proof.* Assume that $||\tilde{f}||$ is a superior ideal on $(\mathcal{L}, \tilde{f})$. Let $x, y, z \in L$ be such that $x * y \leq z$. Then $(x*y)*z = 0$, and so

$$||\tilde{f}||(x*y) \preceq \sup\{||\tilde{f}||((x*y)*z), ||\tilde{f}||(z)\} = \sup\{||\tilde{f}||(0), ||\tilde{f}||(z)\} = ||\tilde{f}||(z)$$

by (2.11) and (2.10). It follows that

$$||\tilde{f}||(x) \preceq \sup\{||\tilde{f}||(x*y), ||\tilde{f}||(y)\} \preceq \sup\{||\tilde{f}||(z), ||\tilde{f}||(y)\}.$$

Conversely, suppose that the assertion (3.3) is valid. Since

$$(0*x)*x = 0 \text{ and } (x*(x*y))*y = 0$$

for all $x, y \in L$, it follows from (3.3) that

$$||\tilde{f}||(0) \preceq \sup\{||\tilde{f}||(x), ||\tilde{f}||(x)\} = ||\tilde{f}||(x)$$

and

$$||\tilde{f}||(x) \preceq \sup\{||\tilde{f}||(x*y), ||\tilde{f}||(y)\}$$

for all $x, y \in L$. Therefore $||\tilde{f}||$ is a superior ideal on $(\mathcal{L}, \tilde{f})$. □

**Corollary 3.12.** *Let $||\tilde{f}||$ be the superior mapping of $\mathcal{L}$ with respect to a pair $\left(\tilde{f}, L\right)$ on $(U, \preceq)$. Then $||\tilde{f}||$ is a superior ideal on $(\mathcal{L}, \tilde{f})$ if and only if it satisfies the following assertion:*

$$||\tilde{f}||(x) \preceq \sup\{||\tilde{f}||(a_1), ||\tilde{f}||(a_2), \cdots, ||\tilde{f}||(a_n)\} \tag{3.4}$$

*for all $x, a_1, a_2, \cdots, a_n \in L$ with $(\cdots((x*a_1)*a_2)*\cdots)*a_n = 0$.*

**Theorem 3.13.** *Let $||\tilde{f}||$ be the superior mapping of $\mathcal{L}$ with respect to a pair $\left(\tilde{f}, L\right)$ on $(U, \preceq)$. Then $||\tilde{f}||$ is a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$ if and only if it satisfies the following assertion:*

$$||\tilde{f}||(x * y) \preceq \sup\{||\tilde{f}||(a), ||\tilde{f}||(b)\} \tag{3.5}$$

*for all $x, y, a, b \in L$ with $(((x * y) * y) * a) * b = 0$.*

*Proof.* Assume that $||\tilde{f}||$ is a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$. Then $||\tilde{f}||$ is a superior ideal on $(\mathcal{L}, \tilde{f})$ by Theorem 3.6. Let $x, y, a, b \in L$ be such that $(((x * y) * y) * a) * b = 0$. Using Theorem 3.8(ii) and (3.3), we have

$$||\tilde{f}||(x * y) \preceq ||\tilde{f}||((x * y) * y) \preceq \sup\{||\tilde{f}||(a), ||\tilde{f}||(b)\}.$$

Conversely, suppose that $||\tilde{f}||$ satisfies the condition (3.5) for all $x, y, a, b \in L$ with $(((x*y)*y)*a)*b = 0$. Assume that $(x*u)*v = 0$ for all $x, u, v \in L$. Then $(((x*0)*0)*u)*v = 0$, and so $||\tilde{f}||(x) = ||\tilde{f}||(x*0) \preceq \sup\{||\tilde{f}||(u), ||\tilde{f}||(v)\}$ by (3.5). It follows from Lemma 3.11 that $||\tilde{f}||$ is a superior ideal on $(\mathcal{L}, \tilde{f})$. Note that

$$(((x * y) * y) * ((x * y) * y)) * 0 = 0$$

for all $x, y \in L$. Using (3.5) and (2.10), we have

$$||\tilde{f}||(x * y) \preceq \sup\{||\tilde{f}||((x * y) * y), ||\tilde{f}||(0)\} = ||\tilde{f}||((x * y) * y)$$

for all $x, y \in L$. Therefore $||\tilde{f}||$ is a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$ by Theorem 3.8. □

**Corollary 3.14.** *Let $||\tilde{f}||$ be the superior mapping of $\mathcal{L}$ with respect to a pair $\left(\tilde{f}, L\right)$ on $(U, \preceq)$. Then $||\tilde{f}||$ is a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$ if and only if it satisfies the following assertion:*

$$||\tilde{f}||(x * y) \preceq \sup\{||\tilde{f}||(a_1), ||\tilde{f}||(a_2), \cdots, ||\tilde{f}||(a_n)\} \tag{3.6}$$

*for all $x, y, a_1, a_2, \cdots, a_n \in L$ with $(\cdots((((x * y) * y) * a_1) * a_2) * \cdots) * a_n = 0$.*

**Theorem 3.15.** *Let $||\tilde{f}||$ be the superior mapping of $\mathcal{L}$ with respect to a pair $\left(\tilde{f}, L\right)$ on $(U, \preceq)$. Then $||\tilde{f}||$ is a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$ if and only if it satisfies the following assertion:*

$$||\tilde{f}||((x * z) * (y * z)) \preceq \sup\{||\tilde{f}||(a), ||\tilde{f}||(b)\} \tag{3.7}$$

*for all $x, y, z, a, b \in L$ with $(((x * y) * z) * a) * b = 0$.*

*Proof.* Assume that $||\tilde{f}||$ is a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$. Then $||\tilde{f}||$ is a superior ideal on $(\mathcal{L}, \tilde{f})$ by Theorem 3.6. Suppose that $(((x * y) * z) * a) * b = 0$ for all $x, y, z, a, b \in L$. Then

$$||\tilde{f}||((x * z) * (y * z)) \preceq ||\tilde{f}||((x * y) * z) \preceq \sup\{||\tilde{f}||(a), ||\tilde{f}||(b)\}$$

by Theorem 3.9 and Lemma 3.11.

Conversely, suppose that $||\tilde{f}||$ satisfies the condition (3.7) for all $x, y, z, a, b \in L$ with $(((x*y)*z)*a)*b = 0$. Let $x, y, a, b \in L$ be such that $(((x * y) * y) * a) * b = 0$. Then

$$||\tilde{f}||(x * y) = ||\tilde{f}||((x * y) * (y * y)) \preceq \sup\{||\tilde{f}||(a), ||\tilde{f}||(b)\}$$

by (3.7). It follows from Theorem 3.13 that $||\tilde{f}||$ is a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$. □

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**Corollary 3.16.** *Let $||\tilde{f}||$ be the superior mapping of $\mathcal{L}$ with respect to a pair $\left(\tilde{f}, L\right)$ on $(U, \preceq)$. Then $||\tilde{f}||$ is a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$ if and only if it satisfies the following assertion:*

$$||\tilde{f}||((x * z) * (y * z)) \preceq \sup\{||\tilde{f}||(a_1), ||\tilde{f}||(a_2), \cdots, ||\tilde{f}||(a_n)\} \tag{3.8}$$

*for all $x, y, z, a_1, a_2, \cdots, a_n \in L$ with $(\cdots((((x * y) * z) * a_1) * a_2) * \cdots) * a_n = 0$.*

**Theorem 3.17.** *Let $||\tilde{f}||$ and $||\tilde{g}||$ be superior ideals on $(\mathcal{L}, \tilde{f})$ and $(\mathcal{L}, \tilde{g})$, respectively, such that $||\tilde{f}||(0) = ||\tilde{g}||(0)$ and $||\tilde{g}||(x) \preceq ||\tilde{f}||(x)$ for all $x(\neq 0) \in L$. If $||\tilde{f}||$ is a positive implicative superior ideal on $(\mathcal{L}, \tilde{f})$, then $||\tilde{g}||$ is a positive implicative superior ideal on $(\mathcal{L}, \tilde{g})$.*

*Proof.* For any $x, y, z \in L$, let $u := (x * y) * z$. Then

$$\begin{aligned}
&||\tilde{g}||(((x * z) * (y * z)) * ((x * y) * z)) = ||\tilde{g}||(((x * z) * (y * z)) * u) \\
&= ||\tilde{g}||(((x * u) * z) * (y * z)) \preceq ||\tilde{f}||(((x * u) * z) * (y * z)) \\
&\preceq ||\tilde{f}||(((x * u) * y) * z) = ||\tilde{f}||(((x * y) * z) * u) \\
&= ||\tilde{f}||(0) = ||\tilde{g}||(0),
\end{aligned}$$

and so $||\tilde{g}||(((x * z) * (y * z)) * ((x * y) * z)) = ||\tilde{g}||(0)$. It follows from (2.11) that

$$\begin{aligned}
&||\tilde{g}||((x * z) * (y * z)) \preceq \sup\{||\tilde{g}||(((x * z) * (y * z)) * ((x * y) * z)), ||\tilde{g}||((x * y) * z)\} \\
&= \sup\{||\tilde{g}||(0), ||\tilde{g}||((x * y) * z)\} = ||\tilde{g}||((x * y) * z).
\end{aligned}$$

Therefore $||\tilde{g}||$ is a positive implicative superior ideal on $(\mathcal{L}, \tilde{g})$ by Theorem 3.9. □

## References


[1] J. D. Bashford and P. D. Jarvis, *The genetic code as a peridic table: algebraic aspects*, BioSystems **57** (2000), 147–161.

[2] L. Frappat, A. Sciarrino and P. Sorba, *Crystalizing the genetic code*, J. Biological Physics **27** (2001), 1–34.

[3] Y. Huang, *BCI*-algebra, Science Press, Beijing, 2006.

[4] K. Iséki and S. Tanaka, *An introduction to the theory of BCK-algebras*, Math. Japonica **23** (1978), 1–26.

[5] Y. B. Jun and S. Z. Song, *Codes based on BCK-algebras*, Inform. Sci. **181** (2011), 5102–5109.

[6] Y. B. Jun and S. Z. Song, *Superior subalgebras and ideals of BCK/BCI-algerbas*, Discuss. Math. Gen. Algebra Appl. **36** (2016), 85–99.

[7] M. K. Kinyon and A. A. Sagle, *Quadratic dynamical systems and algebras*, J. Differential Equations **117** (1995), 67–126.

[8] J. Meng, *Commutative ideals in BCK-algebras*, Pure Appl. Math. (in China) **9** (1991), 49–53.

[9] J. Meng, *On ideals in BCK-algebras*, Math. Japonica **40** (1994), 143–154.

[10] J. Meng and Y. B. Jun, BCK-algebras, Kyungmoon Sa, Seoul, 1994.

[11] R. Sáanchez, R. Grau and E. Morgado, *A novel Lie algebra of the genetic code over the Galois field of four DNA bases*, Mathematical Biosciences **202** (2006), 156–174.

[12] J. J. Tian and B. L. Li, *Coalgebraic structure of genetics inheritance*, Mathematical Biosciences and Engineering **1** (2004), 243–266.

# The uniqueness of meromorphic functions sharing sets in an angular domain *

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**Abstract**

By using the Tsuji's characteristic, we deal with the uniqueness problem of meromorphic functions sharing sets in an angular domain and obtain some theorems which improve and extend the results given by Zheng, Xuan.

**Key words:** Meromorphic function; Angular domain; Uniqueness; Tsuji's characteristic.

**Mathematical Subject Classification (2010):** 30D30 30D35.

## 1 Introduction and main results

The purpose of this paper is to investigate the uniqueness of meromorphic functions sharing sets in an angular domain by using the Tsuji's characteristic functions of angular domain. It is assumed that the readers are familiar with the notations of the Nevanlinna theory such as $T(r,f), m(r,f), N(r,f)$ and so on, that can be found, for instance, in [5, 17].

We use $\mathbb{C}$ to denote the open complex plane, $\widehat{\mathbb{C}}(=\mathbb{C}\bigcup\{\infty\})$ to denote the extended complex plane, and $\Omega(\subset\mathbb{C})$ to denote an angular domain. Let $S$ be a set of distinct elements in $\widehat{\mathbb{C}}$ and $\Omega\subseteq\mathbb{C}$. Define

$$E(S,\Omega,f)=\bigcup_{a\in S}\{z\in\Omega|f_a(z)=0,\quad counting\quad multiplicities\},$$

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$$\overline{E}(S,\Omega,f)=\bigcup_{a\in S}\{z\in\Omega|f_a(z)=0,\quad ignoring\quad multiplicities\},$$

where $f_a(z)=f(z)-a$ if $a\in\mathbb{C}$ and $f_\infty(z)=1/f(z)$.

Let $f$ and $g$ be two non-constant meromorphic functions in $\mathbb{C}$. If $E(S,\Omega,f)=E(S,\Omega,g)$, we say that $f$ and $g$ share the set $S$ $CM$ (counting multiplicities) in $\Omega$. If $\overline{E}(S,\Omega,f)=\overline{E}(S,\Omega,g)$, we say $f$ and $g$ share the set $S$ $IM$(ignoring multiplicities) in $\Omega$. In particular, when $S=\{a\}$, where $a\in\widehat{\mathbb{C}}$, we say $f$ and $g$ share the value $a$ $CM$ in $\Omega$ if $E(S,\Omega,f))=E(S,\Omega,g)$, and we say $f$ and $g$ share the value $a$ $IM$ in $\Omega$ if $\overline{E}(S,\Omega,f)=\overline{E}(S,\Omega,g)$. When $\Omega=\mathbb{C}$, we give the simple notation as before, $E(S,f),\overline{E}(S,f)$ and so on(see [13]).

Let $l$ be a nonnegative integer or infinity. For $a\in\mathbb{C}\cup\{\infty\}$, we denote by $\overline{E}_l(a,\Omega,f)$ the set of all $a$-points of $f$ in $\Omega$, where an $a$-point of multiplicity $k$ is counted one times if $k\leq l$ and zero times if $k>l$.

R.Nevanlinna(see [9]) proved the following well-known theorem.

**Theorem 1.1** (see [9].) *If $f$ and $g$ are two non-constant meromorphic functions that share five distinct values $a_1,a_2,a_3,a_4,a_5$ $IM$ in $\Omega=\mathbb{C}$, then $f(z)\equiv g(z)$.*

After his theorems, the uniqueness problems of meromorphic functions sharing values in the whole complex plane attracted many investigations (see [15]). In 2004, Zheng [19] studied the uniqueness problem under the condition that five values are shared in some angular domain in $\mathbb{C}$. It is an interesting topic to investigate the uniqueness with shared values in the remaining part of the complex plane removing an unbounded closed set, see [3, 4, 7, 8, 10, 13, 18, 19, 20]. Zheng [20], Cao and Yi [2], Xu and Yi [13] continued to investigate the uniqueness of meromorphic functions sharing five values and four values, Lin, Mori and Tohge [7] and Lin, Mori and Yi [8] investigated the uniqueness of meromorphic and entire functions sharing sets in an angular domain. To state theirs results, we need the following basic notations and definitions of meromorphic functions in an angular domain(see [5, 19, 20]).

In 2009, the present author [14] investigated the uniqueness of meromorphic functions with finite order sharing some values in an angular domain and obtained the following theorem

**Theorem 1.2** *(see [14]). Let $f(z)$ and $g(z)$ be both transcendental meromorphic functions, and let $f(z)$ be of finite order $\lambda$ (lower order $\mu$) and such that for some $a\in\widehat{\mathbb{C}}$ and an integer $p\geq 0,\delta=\delta(a,f^{(p)})>0$. For $m$ pair of real numbers $\{\alpha_j,\beta_j\}$ satisfying*

$$-\pi\leq\alpha_1\leq\beta_1\leq\alpha_2\leq\beta_2\leq\cdots\leq\alpha_m\leq\beta_m\leq 2\pi$$

*and*

$$\sum_{j=1}^{m}(\alpha_{j+1}-\beta_j)<\frac{4}{\sigma}\arcsin\sqrt{\frac{\delta}{2}},$$

*where $\sigma=\max\{\omega,\mu\}$, $\omega=\max\{\frac{\pi}{\beta_1-\alpha_1},\ldots,\frac{\pi}{\beta_m-\alpha_m}\}$, assume that $a_j(j=1,2,\ldots,q)$ be $q$ distinct complex numbers, and let $k_j(j=1,2,\ldots,q)$ be positive integers or $\infty$ satisfying*

$$k_1\geq k_2\geq\cdots\geq k_p,\tag{1}$$

$$\overline{E}_{k_j}(a_j,\Omega,f)=\overline{E}_{k_j}(a_j,\Omega,g),$$
$$\sum_{j=3}^{q}\frac{k_j}{k_j+1}>2, \tag{2}$$

*where* $\Omega=\bigcup_{j=1}^{m}\{z:\ \alpha_j\le\arg z\le\beta_j\}$. *If* $\omega<\lambda(f)$, *then* $f(z)\equiv g(z)$.

In 2009, Cao and Yi [2] investigated the uniqueness problem of two transcendental meromorphic functions $f,g$ sharing five values $IM$ in an angular domain and obtained the following result which extended Theorem 1.1 to an angular domain.

**Theorem 1.3** (see [2, Theorem 1.3].) *Let* $f$ *and* $g$ *be two transcendental meromorphic functions. Given one angular domain* $\Omega=\{z:\alpha<\arg z<\beta\}$ *with* $0<\beta-\alpha\le 2\pi$, *we assume that* $f$ *and* $g$ *share five distinct values* $a_j(j=1,2,3,4,5)$ $IM$ *in* $\Omega$. *Then* $f(z)\equiv g(z)$, *provided that*

$$\lim_{r\to\infty}\frac{S_{\alpha,\beta}(r,f)}{\log(rT(r,f))}=\infty,\quad (r\notin E),$$

*where* $S_{\alpha,\beta}(r,f)$ *is called the Nevanlinna's angular characteristic.*

Moreover, Cao and Yi [2] also investigated the two uniqueness problems of two transcendental meromorphic functions $f,g$ sharing four distinct values $CM$ in an angular domain $X$ and $f,g$ sharng two distinct values $CM$ in an angular domain $X$ and the other two distinct values $IM$ in an angular domain $X$, and they obtained two interesting results which extended the analogous results as in the whole complex plane to an angular domain. In 2011, Xu and Cao [11, 12] improve the results given by Cao and Yi[1, 2] to some extent.

Most recently, Zheng [21] prove the following theorem by using the Tsuji's characteristic to extend the five $IM$ theorem of Nevanlinna's to an angular domain. *The Tsuji's characteristic will be introduced in Section 2.*

**Theorem 1.4** *(see [21]). Let* $f(z)$ *and* $g(z)$ *be both meromorphic functions in an angular domain* $\Omega=\{z:\alpha<\arg z<\beta\}$ *with* $0\le\alpha<\beta\le 2\pi$ *and* $f(z)$ *be transcendental in the Tsuji's sense. Assume that* $a_j(j=1,2,\ldots,5)$ *be* 5 *distinct complex numbers. If* $\overline{E}(a_j,\Omega,f)=\overline{E}(a_j,\Omega,g)$, *then* $f(z)\equiv g(z)$.

In this paper, we will deal with the uniqueness of meromorphic functions sharing sets in an angular domain by using the Tsuji's characteristic and obtain the following results which are improvement of Theorem 1.4.

**Theorem 1.5** *Let* $f(z)$ *and* $g(z)$ *be both meromorphic functions in an angular domain* $\Omega=\{z:\alpha<\arg z<\beta\}$ *with* $0\le\alpha<\beta\le 2\pi$ *and* $f(z)$ *be transcendental in the Tsuji's sense. Suppose that*

$$S_j=\{a_j,a_j+b,\ldots,a_j+(l-1)b\},\quad j=1,2,\ldots,q,$$

*with* $b\neq 0$, $S_i\cap S_j=\emptyset$, $(i\neq j)$. *Let* $k_j$ $(j=1,2,\ldots,q)$ *be positive integers or* $\infty$ *satisfying (1) and*

$$\overline{E}_{k_j)}(S_j,\Omega,f)=\overline{E}_{k_j)}(S_j,\Omega,g),\quad (j=1,2,\ldots,q). \tag{3}$$

*Furthermore, let*

$$\Theta_T(f)=\sum_a \Theta_T(0,f-a)-\sum_{j=1}^{q}\sum_{s=0}^{l-1}\Theta_T(0,f-(a_j+sb)),$$

$$\begin{aligned}A_1 &= \frac{\sum_{j=1}^{m-1}\sum_{s=0}^{l-1}\delta_T(0,f-(a_j+sb))}{k_m+1}+\sum_{j=m}^{q}\sum_{s=0}^{l-1}\frac{k_j+\delta_T(0,f-(a_j+sb))}{k_j+1}\\ &\quad+\frac{(lm-3l+1)k_m}{k_m+1}-\frac{(2l-1)k_n}{k_n+1}+\Theta_0(f)-2\end{aligned}$$

*and*

$$\begin{aligned}A_2 &= \frac{\sum_{j=1}^{n-1}\sum_{s=0}^{l-1}\delta_T(0,g-(a_j+sb))}{k_n+1}+\sum_{j=n}^{q}\sum_{s=0}^{l-1}\frac{k_j+\delta_T(0,g-(a_j+sb))}{k_j+1}\\ &\quad+\frac{(ln-3l+1)k_n}{k_n+1}-\frac{(2l-1)k_m}{k_m+1}+\Theta_0(g)-2,\end{aligned}$$

*where $m$ and $n$ are positive integers in $\{1,2,\ldots,q\}$ and $a$ is an arbitrary complex number or $\infty$. If*

$$\min\{A_1,A_2\}\geq 0,\quad \textit{and}\quad \max\{A_1,A_2\}>0. \tag{4}$$

*Then $f_1(z)\equiv f_2(z)$.*

From Theorem 1.5, we can get the following corollaries.

**Corollary 1.1** *Let $f(z)$ and $g(z)$ be both meromorphic functions in an angular domain $\Omega=\{z:\alpha<\arg z<\beta\}$ with $0\leq\alpha<\beta\leq 2\pi$ and $f(z)$ be transcendental in the Tsuji's sense. Suppose that*

$$S_j=\{a_j,a_j+b,\ldots,a_j+(l-1)b\},\quad j=1,2,\ldots,q,$$

*with $b\neq 0$, $S_i\cap S_j=\emptyset$, $(i\neq j)$. Let $k_j$ $(j=1,2,\ldots,q)$ be positive integers or $\infty$ satisfying (1) and*

$$\overline{E}_{k_j)}(S_j,\Omega,f)=\overline{E}_{k_j)}(S_j,\Omega,g),\quad (j=1,2,\ldots,q).$$

*If*

$$\sum_{j=3}^{q}\sum_{s=0}^{l-1}\frac{k_j}{k_j+1}+\frac{(2-2l)k_3}{k_3+1}>2.$$

*Then $f(z)\equiv g(z)$.*

*Proof:* Let $m=n=3$. Since $\Theta_T(f)\geq 0,\Theta_T(g)\geq 0$, $\delta_T(0,f-(a_j+sb))\geq 0$ and $\delta_T(0,g-(a_j+sb))\geq 0$ for $j=1,2,\ldots,q$, one can deduce from Theorem 1.5 that Corollary 1.1 follows. □

The following corollary is an analog of a result due to Yi (Theorem 10.7 in [15], see also [16]) on $\mathbb{C}$.

**Corollary 1.2** *Let $f(z)$ and $g(z)$ be both meromorphic functions in an angular domain $\Omega = \{z : \alpha < \arg z < \beta\}$ with $0 \leq \alpha < \beta \leq 2\pi$ and $f(z)$ be transcendental in the Tsuji's sense. Suppose that*

$$S_j = \{a_j, a_j + b, \ldots, a_j + (l-1)b\}, \quad j = 1, 2, \ldots, q,$$

*with $b \neq 0$, $q > 4$, $S_i \cap S_j = \emptyset$, $(i \neq j)$. If $\overline{E}(S_j, \Omega, f) = \overline{E}(S_j, \Omega, g), (j = 1, 2, \ldots, q)$. Then $f(z) \equiv g(z)$.*

*Proof:* Let $k_1 = k_2 = \ldots = k_q = \infty$. One can deduce from Corollary 1.1 that Corollary 1.2 follows immediately. □

Let $l = 1$. Then it is easily derived the following corollary from Corollary 1.1, which is an analog of the Corollary of Theorem 3.15 in [15].

**Corollary 1.3** *Let $f(z)$ and $g(z)$ be both meromorphic functions in an angular domain $\Omega = \{z : \alpha < \arg z < \beta\}$ with $0 \leq \alpha < \beta \leq 2\pi$ and $f(z)$ be transcendental in the Tsuji's sense. Let $a_j$ $(j = 1, 2, \ldots, q)$ be $q$ distinct complex numbers in $\widehat{\mathbb{C}}$, and $k_j (j = 1, 2, \ldots, q)$ be positive integers or $\infty$ satisfying (1) and $\overline{E}_{k_j)}(a_j, \Omega, f) = \overline{E}_{k_j)}(a_j, \Omega, g), (j = 1, 2, \ldots, q)$. Then*

*(i) if $q = 7$, then $f(z) \equiv g(z)$.*
*(ii) if $q = 6$ and $k_3 \geq 2$, then $f(z) \equiv g(z)$.*
*(iii) if $q = 5$, $k_3 \geq 3$ and $k_5 \geq 2$, then $f(z) \equiv g(z)$.*
*(iv) if $q = 5$ and $k_4 \geq 4$, then $f(z) \equiv g(z)$.*
*(v) if $q = 5$, $k_3 \geq 5$ and $k_4 \geq 3$, then $f(z) \equiv g(z)$.*
*(vi) if $q = 5$, $k_3 \geq 6$ and $k_4 \geq 2$, then $f(z) \equiv g(z)$.*

Another main theorem of this paper is listed as follows.

**Theorem 1.6** *Let $f(z)$ and $g(z)$ be both meromorphic functions in an angular domain $\Omega = \{z : \alpha < \arg z < \beta\}$ with $0 \leq \alpha < \beta \leq 2\pi$ and $f(z)$ be transcendental in the Tsuji's sense. Suppose that*

$$S_j = \{c + a_j, c + a_j w, \ldots, c + a_j w^{l-1}\}, \quad j = 1, 2, \ldots, q,$$

*with $a_j \neq 0$, $(j = 1, 2, \ldots, q)$, $w = \exp(\frac{2\pi i}{l})$, $S_i \cap S_j = \emptyset$, $(i \neq j)$. Let $k_j$ $(j = 1, 2, \ldots, q)$ be positive integers or $\infty$ satisfying (1) and*

$$\overline{E}_{k_j)}(S_j, \Omega, f) = \overline{E}_{k_j)}(S_j, \Omega, g), \quad (j = 1, 2, \ldots, q). \tag{5}$$

*Furthermore, let*

$$\Theta_T(f) = \sum_a \Theta_T(0, f - a) - \sum_{j=1}^{q} \sum_{s=0}^{l-1} \Theta_T(0, f - (c + a_j w^s)),$$

$$\begin{aligned} A_1 \quad = \quad & \frac{\sum_{j=1}^{m-1} \sum_{s=0}^{l-1} \delta_T(0, f - (c + a_j w^s))}{k_m + 1} + \sum_{j=m}^{q} \sum_{s=0}^{l-1} \frac{k_j + \delta_T(0, f - (c + a_j w^s))}{k_j + 1} \\ & + \frac{l(m-2)k_m}{k_m + 1} - \frac{lk_n}{k_n + 1} + \Theta_T(f) - 2 \end{aligned}$$

*and*

$$A_2 = \frac{\sum_{j=1}^{n-1}\sum_{s=0}^{l-1}\delta_T(0,g-(c+a_jw^s))}{k_n+1}+\sum_{j=n}^{q}\sum_{s=0}^{l-1}\frac{k_j+\delta_T(0,g-(c+a_jw^s))}{k_j+1}$$
$$+\frac{l(n-2)k_n}{k_n+1}-\frac{lk_m}{k_m+1}+\Theta_T(g)-2,$$

*where $m$ and $n$ are positive integers in $\{1,2,\ldots,q\}$ and $a$ is an arbitrary complex number or $\infty$. If*

$$\min\{A_1,A_2\}\geq 0, \quad \textit{and} \quad \max\{A_1,A_2\}>0. \tag{6}$$

*Then $(f(z)-c)^l\equiv(g(z)-c)^l$.*

From Theorem 1.6, we can get the following corollary immediately.

**Corollary 1.4** *Let $f(z)$ and $g(z)$ be both meromorphic functions in an angular domain $\Omega=\{z:\alpha<\arg z<\beta\}$ with $0\leq\alpha<\beta\leq 2\pi$ and $f(z)$ be transcendental in the Tsuji's sense. Suppose that*

$$S_j=\{c+a_j,c+a_jw,\ldots,c+a_jw^{l-1}\},\quad j=1,2,\ldots,q,$$

*with $a_j\neq 0$, $(j=1,2,\ldots,q)$, $q>2+\frac{2}{l}$, $w=\exp(\frac{2\pi i}{l})$, $S_i\cap S_j=\emptyset$, $(i\neq j)$. If $\overline{E}(S_j,\Omega,f)=\overline{E}(S_j,\Omega,g)$ for $j=1,2,\ldots,q$, then $(f(z)-c)^l\equiv(g(z)-c)^l$.*

*Proof:* Set $m=n=1$ and $k_1=k_2=\ldots=\infty$. Since $\Theta_T(f)\geq 0,\Theta_T(g)\geq 0$, $\delta_T(0,f-(a_j+sb))\geq 0$ and $\delta_T(0,g-(a_j+sb))\geq 0$ for $j=1,2,\ldots,q$. Then Corollary 1.4 follows immediately from Theorem 1.6. □

## 2 Preliminaries

In this section, we will introduce some notations of Tsuji's characteristic in an angular domain (see [6, 21]). For meromorphic function $f$ in an angular domain $\Omega$ and $\omega=\frac{\pi}{\beta-\alpha}$, we define

$$\mathfrak{M}_{\alpha,\beta}(r,f)=\frac{1}{2\pi}\int_{\arcsin(r^{-\omega})}^{\pi-\arcsin(r^{-\omega})}\log^+\left|f(re^{i(\alpha+\omega^{-1}\theta)}\sin^{\omega^{-1}}\theta)\right|\frac{1}{r^\omega\sin^2\theta}d\theta,$$

$$\mathfrak{N}_{\alpha,\beta}(r,f)=\sum_{1<|b_n|<r(\sin(\omega(\beta_n-\alpha)))^{\omega^{-1}}}\left(\frac{\sin\omega(\beta_n-\alpha)}{|b_n|^\omega}-\frac{1}{r^\omega}\right),$$

where $b_n$ are the poles of $f(z)$ in $\Xi(\alpha,\beta;r)=\{z=re^{i\theta}\theta:\alpha<\theta<\beta,1<t\leq r(\sin(\omega(\beta_n-\alpha)))^{\omega^{-1}}$ appearing often according to their multiplicities and then Tsuji characteristic of $f$ is

$$\mathfrak{T}_{\alpha,\beta}(r,f)=\mathfrak{M}_{\alpha,\beta}(r,f)+\mathfrak{N}_{\alpha,\beta}(r,f).$$

We denote by $\mathfrak{n}_{\alpha,\beta}(r,f)$ the number of poles of $f(z)$ in $\Xi(\alpha,\beta;r)$, and then

$$\mathfrak{N}_{\alpha,\beta}(r,f)=\int_1^r\left(\frac{1}{t^\omega}-\frac{1}{r^\omega}\right)d\mathfrak{n}_{\alpha,\beta}(r,f)=\omega\int_1^r\frac{\mathfrak{n}_{\alpha,\beta}(t,f)}{t^{\omega+1}}dt,$$

when pole $b_n$ occurs in the sum $\sum_{1<|b_n|<r(\sin(\omega(\beta_n-\alpha)))^{\omega^{-1}}}$ only once, we denote it by $\overline{\mathfrak{N}}_{\alpha,\beta}(r,f)$. For meromorphic function $f$ in $\Omega$ and for all complex numbers $a$, if

$$\limsup_{r\to\infty}\frac{\mathfrak{T}_{\alpha,\beta}(r,f)}{\log r}=\infty,$$

then $f$ is called transcendental with respect to the Tsuji characteristic[21], and we have the Tsuji deficiency of $f(z)$ as follows

$$\delta_T(a,f;\alpha,\beta)=\liminf_{r\to\infty}\frac{\mathfrak{M}_{\alpha,\beta}\left(r,\frac{1}{f-a}\right)}{\mathfrak{T}_{\alpha,\beta}(r,f)}=1-\limsup_{r\to\infty}\frac{\mathfrak{N}_{\alpha,\beta}\left(r,\frac{1}{f-a}\right)}{\mathfrak{T}_{\alpha,\beta}(r,f)},$$

and

$$\Theta_T(a,f;\alpha,\beta)=1-\limsup_{r\to\infty}\frac{\overline{\mathfrak{N}}_{\alpha,\beta}\left(r,\frac{1}{f-a}\right)}{\mathfrak{T}_{\alpha,\beta}(r,f)},$$

for $a\not\equiv\infty$ and $\delta_T(\infty,f;\alpha,\beta)$ is defined by the above formula with $\mathfrak{M}_{\alpha,\beta}(r,f)$ and $\mathfrak{N}_{\alpha,\beta}(r,f)$ in place of $\mathfrak{M}_{\alpha,\beta}(r,\frac{1}{f-a})$ and $\mathfrak{N}_{\alpha,\beta}(r,\frac{1}{f-a})$, $\Theta_T(\infty,f;\alpha,\beta)$ is defined by the above formula with $\overline{\mathfrak{N}}_{\alpha,\beta}(r,f)$ in place of $\overline{\mathfrak{N}}_{\alpha,\beta}(r,\frac{1}{f-a})$. If no confusion occur in the context, then we simply write $\delta_T(a,f)$ for $\delta_T(a,f;\alpha,\beta)$ and $\Theta_T(a,f)$ for $\Theta_T(a,f;\alpha,\beta)$. $\delta_T(a,f)$ is called the Tsuji deficiency of $f$ at $a$ and if $\delta_T(a,f)>0$, then $a$ is said to be a Tsuji deficient value of $f$. In addition, from ref.[21], we have the following properties of this Tsuji's characteristic

$$\mathfrak{T}_{\alpha,\beta}\left(r,\frac{1}{f-a}\right)=\mathfrak{T}_{\alpha,\beta}(r,f)+O(1),\tag{7}$$

and the fundamental inequalities

$$(q-2)\mathfrak{T}_{\alpha,\beta}(r,f)\le\sum_{j=1}^{q}\overline{\mathfrak{N}}_{\alpha,\beta}\left(r,\frac{1}{f-a_j}\right)+Q_{\alpha,\beta}(r,f),\tag{8}$$

hold for $q$ distinct points $a_j\in\widehat{\mathbb{C}}$,

$$Q_{\alpha,\beta}(r,f)=O(\log^{+}\mathfrak{T}_{\alpha,\beta}(r,f)+\log r),\quad r\notin E$$

where $E$ denotes a set of r with finite linear measure. It is not necessarily the same for every occurrence in the context. For sake of simplicity, we omit the subscript in all notations and use $\mathfrak{M}(r,f),\mathfrak{N}(r,f),Q(r,f)$ and $\mathfrak{T}(r,f)$ instead of $\mathfrak{M}_{\alpha,\beta}(r,f),\mathfrak{N}_{\alpha,\beta}(r,f),Q_{\alpha,\beta}(r,f)$ and $\mathfrak{T}_{\alpha,\beta}(r,f)$, respectively.

By using Lo Yang's method in dealing with the multiple values problem, we can get the following lemma

**Lemma 2.1** *For meromorphic function $f$ in an angular domain $\Omega$ and $\omega=\frac{\pi}{\beta-\alpha}$, Let $a$ be an arbitrary complex number, and $k$ be a positive integer. Then*

$$(i)\quad\overline{\mathfrak{N}}_{\alpha,\beta}(r,\frac{1}{f-a})\le\frac{k}{k+1}\overline{\mathfrak{N}}^{k)}_{\alpha,\beta}(r,\frac{1}{f-a})+\frac{1}{k+1}\mathfrak{N}_{\alpha,\beta}(r,\frac{1}{f-a}),$$

$$(ii)\quad\overline{\mathfrak{N}}_{\alpha,\beta}(r,\frac{1}{f-a})\le\frac{k}{k+1}\overline{\mathfrak{N}}^{k)}_{\alpha,\beta}(r,\frac{1}{f-a})+\frac{1}{k+1}\mathfrak{T}_{\alpha,\beta}(r,f)+O(1),$$

*where* $\overline{\mathfrak{N}}^{k)}_{\alpha,\beta}(r,\frac{1}{f-a})$ *to denote the zeros of* $f(z)-a$ *in* $\Omega$*, whose multiplicities are no greater than* $k$ *and are counted only once. Likewise, we use* $\overline{\mathfrak{N}}^{(k}_{\alpha,\beta}(r,\frac{1}{f-a})$ *to denote the zeros of* $f(z)-a$ *in* $\Omega$*, whose multiplicities are greater than* $k$ *and are counted only once.*

By using Lemma 2.1 and (8), we can obtain the following lemma

**Lemma 2.2** *For meromorphic function* $f$ *in an angular domain* $\Omega$ *and* $\omega=\frac{\pi}{\beta-\alpha}$*. Let* $a_1$, $a_2$, …, $a_q$ *be* $q$ *distinct complex numbers in the extended complex plane* $\widehat{\mathbb{C}}$*, let* $k_1$, $k_2$, …, $k_q$ *be* $q$ *positive integers. Then*

$$(i)\quad (q-2)\mathfrak{T}(r,f)<\sum_{j=1}^{q}\frac{k_j}{k_j+1}\overline{\mathfrak{N}}^{k_j)}(r,\frac{1}{f-a_j})+\sum_{j=1}^{q}\frac{1}{k_j+1}\mathfrak{N}(r,\frac{1}{f-a_j})+Q(r,f),$$

$$(ii)\quad (q-2-\sum_{j=1}^{q}\frac{1}{k_j+1})\mathfrak{T}(r,f)<\sum_{j=1}^{q}\frac{k_j}{k_j+1}\overline{\mathfrak{N}}^{k_j)}(r,\frac{1}{f-a_j})+Q(r,f),$$

*where*

$$\mathfrak{N}^{(1)}(r,f)=\mathfrak{N}(r,\frac{1}{f'})+2\mathfrak{N}(r,f)-\mathfrak{N}(r,f'),$$

*and* $Q(r,f)$ *is stated as in (8).*

From (8) and the definition of transcendental in Tsuji sense, we can get the following lemma.

**Lemma 2.3** *(Picard theorem for angular domain) Let* $f$ *be an transcendental meromorphic function in* $\Omega$ *in the Tsuji sense. Then* $f$ *has at most two Picard exceptional values in* $\Omega$*.*

# 3 Proof of Theorem 1.5

Suppose that $f(z)\not\equiv g(z)$. Without loss of generality, we assume that there exist infinitely many $d$ such that $\Theta_T(0,f-d)>0$ and $d\notin\{a_j+sb: j=1,2,\ldots,q \text{ and } s=0,1,\ldots,l-1\}$. We denote them by $d_k$ $(k=1,2,\ldots,\infty)$. Obviously, $\Theta_T(f)=\sum_{k=1}^{\infty}\Theta_T(0,f-d_k)$. Thus there exits a $p$ such that $\sum_{k=1}^{p}\Theta_T(0,f-d_k)>\Theta_T(f)-\varepsilon$ holds for any given $\varepsilon(>0)$. From (8) we have

$$(ql+p-2)\mathfrak{T}(r,f)<\sum_{j=1}^{q}\sum_{s=0}^{n-1}\overline{\mathfrak{N}}(r,\frac{1}{f-(a_j+sb)})+\sum_{k=1}^{p}\overline{\mathfrak{N}}(r,\frac{1}{f-d_k})+Q(r,f).$$

From the definition of deficiency in Tsuji sense, we have

$$\overline{\mathfrak{N}}(r,\frac{1}{f-d_k})<(1-\Theta_T(0,f-d_k))\,\mathfrak{T}(r,f)+Q(r,f).$$

From Lemma 2.1 and the definition of deficiency in Tsuji sense, it follows that for $s \in \{0,1,\ldots,l-1\}$

$$\begin{aligned}
& \overline{\mathfrak{N}}(r,\frac{1}{f-(a_j+sb)}) \\
\leq\ & \frac{k_j}{k_j+1}\overline{\mathfrak{N}}^{k_j)}(r,\frac{1}{f-(a_j+sb)})+\frac{1}{k_j+1}\mathfrak{N}(r,\frac{1}{f-(a_j+sb)}) \\
<\ & \frac{k_j}{k_j+1}\overline{\mathfrak{N}}^{k_j)}(r,\frac{1}{f-(a_j+sb)})+\frac{1}{k_j+1}\left(1-\delta_T(0,f-(a_j+sb))\right)\mathfrak{T}(r,f) \\
& +Q(r,f).
\end{aligned}$$

Thus, from Lemma 2.2, we have

$$\begin{aligned}
& (ql+p-2)\mathfrak{T}(r,f) \\
<\ & \left\{\sum_{k=1}^{p}\left(1-\Theta_T(0,f-d_k)\right)\right\}\mathfrak{T}(r,f)+\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{k_j}{k_j+1}\overline{\mathfrak{N}}^{k_j)}(r,\frac{1}{f-(a_j+sb)}) \\
& +\left\{\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{1}{k_j+1}\left(1-\delta_T(0,f-(a_j+sb))\right)\right\}\mathfrak{T}(r,f)+Q(r,f).
\end{aligned}$$

Since

$$1\geq\frac{k_1}{k_1+1}\geq\frac{k_2}{k_2+1}\geq\cdots\geq\frac{k_q}{k_q+1}\geq\frac{1}{2},$$

we can deduce that

$$\begin{aligned}
& (ql+p-2)\mathfrak{T}(r,f) \\
<\ & \left(p-\Theta_T(f)+\varepsilon\right)\mathfrak{T}(r,f)+\frac{k_m}{k_m+1}\sum_{j=1}^{q}\sum_{s=0}^{l-1}\overline{\mathfrak{N}}^{k_j)}(r,\frac{1}{f-(a_j+sb)}) \\
& +\left\{\sum_{j=1}^{m-1}\sum_{s=0}^{l-1}\left(\frac{k_j}{k_j+1}-\frac{k_m}{k_m+1}\right)\left(1-\delta_T(0,f-(a_j+sb))\right)\right\}\mathfrak{T}(r,f) \\
& +\left\{\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{1-\delta_T(0,f-(a_j+sb))}{k_j+1}\right\}\mathfrak{T}(r,f)+Q(r,f),
\end{aligned}$$

that is,

$$\left(\frac{l(m-1)k_m}{k_m+1}+B_1-\varepsilon\right)\mathfrak{T}(r,f)<\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{k_m}{k_m+1}\overline{\mathfrak{N}}^{k_j)}(r,\frac{1}{f-(a_j+sb)})+Q(r,f),$$

where

$$B_1=\frac{\sum_{j=1}^{m-1}\sum_{s=0}^{l-1}\delta_T(0,f-(a_j+sb))}{k_m+1}+\sum_{j=m}^{q}\sum_{s=0}^{l-1}\frac{k_j+\delta_T(0,f-(a_j+sb))}{k_j+1}+\Theta_T(f)-2.$$

Similar to the above discussion, we also have

$$\left(\frac{l(n-1)k_n}{k_n+1}+B_2-\varepsilon\right)\mathfrak{T}(r,g)<\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{k_n}{k_n+1}\overline{N}_0^{k_j)}(r,\frac{1}{g-(a_j+sb)})+Q(r,g),$$

where

$$B_2=\frac{\sum_{j=1}^{n-1}\sum_{s=0}^{l-1}\delta_T(0,g-(a_j+sb))}{k_n+1}+\sum_{j=n}^{q}\sum_{s=0}^{l-1}\frac{k_j+\delta_T(0,g-(a_j+sb))}{k_j+1}+\Theta_T(g)-2.$$

Thus,

$$\begin{aligned}&\left(\frac{l(m-1)k_m}{k_m+1}+B_1-\varepsilon\right)\mathfrak{T}(r,f)+\left(\frac{l(n-1)k_n}{k_n+1}+B_2-\varepsilon\right)\mathfrak{T}(r,g)\\ <\ &\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{k_m}{k_m+1}\overline{\mathfrak{N}}^{k_j)}(r,\frac{1}{f-(a_j+sb)})+\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{k_n}{k_n+1}\overline{\mathfrak{N}}^{k_j)}(r,\frac{1}{g-(a_j+sb)})\\ &+Q(r,f)+Q(r,g).\end{aligned}$$

We will prove that $f(z)-g(z)\not\equiv sb$, $s=1,2,\ldots,l-1$. Suppose that $f(z)-g(z)\equiv sb$, $s=1,2,\ldots,l-1.$, we get that $a_j$ $(j=1,2,\ldots,q)$ are the Picard exceptional values of $f$, and that $a_j+(l-1)b$ $(j=1,2,\ldots,q)$ are the Picard exceptional values of $g$ in $\Omega$. By Lemma 2.3, we can get a contradiction. Similarly, we have $g(z)-f(z)\not\equiv sb$, $s=1,2,\ldots,l-1$.

By using (7) and condition (4), we have

$$\begin{aligned}&\sum_{j=1}^{q}\sum_{s=0}^{l-1}\overline{\mathfrak{N}}^{k_j)}(r,\frac{1}{f-(a_j+sb)})\\ \le\ &\overline{\mathfrak{N}}(r,\frac{1}{f-g})+\sum_{s=1}^{l-1}\overline{\mathfrak{N}}(r,\frac{1}{f-g-sb})+\sum_{s=1}^{l-1}\overline{\mathfrak{N}}(r,\frac{1}{g-f-sb})\\ \le\ &(2l-1)(\mathfrak{T}(r,f)+\mathfrak{T}(r,g))+O(1).\end{aligned}$$

and

$$\begin{aligned}&\sum_{j=1}^{q}\sum_{s=0}^{l-1}\overline{\mathfrak{N}}^{k_j)}(r,\frac{1}{g-(a_j+sb)})\\ \le\ &\overline{\mathfrak{N}}(r,\frac{1}{f-g})+\sum_{s=1}^{l-1}\overline{\mathfrak{N}}(r,\frac{1}{f-g-sb})+\sum_{s=1}^{l-1}\overline{\mathfrak{N}}(r,\frac{1}{g-f-sb})\\ \le\ &(2l-1)(\mathfrak{T}(r,f)+\mathfrak{T}(r,g))+O(1).\end{aligned}$$

Therefore, from the above discussion we obtain

$$\begin{aligned}&\left(\frac{l(m-1)k_m}{k_m+1}+B_1-\varepsilon\right)\mathfrak{T}(r,f)+\left(\frac{l(n-1)k_n}{k_n+1}+B_2-\varepsilon\right)\mathfrak{T}(r,g)\\ <\ &(2l-1)\left(\frac{k_m}{k_m+1}+\frac{k_n}{k_n+1}\right)(\mathfrak{T}(r,f)+\mathfrak{T}(r,g))+Q(r,f)+Q(r,g),\end{aligned}$$

that is,

$$(A_1-\varepsilon)\,\mathfrak{T}(r,f)+(A_2-\varepsilon)\,\mathfrak{T}(r,g)<Q(r,f)+Q(r,g).$$

Since $f$ and $g$ are transcendental in Tsuji sense and $\varepsilon$ is arbitrary, the above inequality contradicts the conditions (4).

Therefore, the proof of Theorem 1.5 is completed

# 4 The proof of Theorem 1.6

Suppose that $(f(z)-c)^l \not\equiv (g(z)-c)^l$. Without loss of generality, we assume that there exist infinitely many $d$ such that $\Theta_T(0,f-d)>0$ and $d\notin\{c+a_jw^s : j=1,2,\ldots,q \text{ and } s=0,1,\ldots,l-1\}$. We denote them by $d_k$ $(k=1,2,\ldots,\infty)$. Obviously, $\Theta_T(f)=\sum_{k=1}^{\infty}\Theta_T(0,f-d_k)$. Thus there exits a $p$ such that $\sum_{k=1}^{p}\Theta_T(0,f-d_k)>\Theta_T(f)-\varepsilon$ holds for any given $\varepsilon(>0)$.

Using a similar discussion as in the proof of Theorem 1.5, we obtain

$$\begin{aligned}&\left(\frac{l(m-1)k_m}{k_m+1}+B_1-\varepsilon\right)\mathfrak{T}(r,f)+\left(\frac{l(n-1)k_n}{k_n+1}+B_2-\varepsilon\right)\mathfrak{T}(r,g)\\ <\;&\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{k_m}{k_m+1}\overline{\mathfrak{N}}^{k_j)}(r,\frac{1}{f-(c+a_jw^s)})+\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{k_n}{k_n+1}\overline{\mathfrak{N}}^{k_j)}(r,\frac{1}{g-(c+a_jw^s)})\\ &+Q(r,f)+Q(r,g),\end{aligned}$$

where

$$B_1=\frac{\sum_{j=1}^{m-1}\sum_{s=0}^{l-1}\delta_T(0,f-(c+a_jw^s))}{k_m+1}+\sum_{j=m}^{q}\sum_{s=0}^{l-1}\frac{k_j+\delta_T(0,f-(c+a_jw^s))}{k_j+1}+\Theta_T(f)-2,$$

$$B_2=\frac{\sum_{j=1}^{n-1}\sum_{s=0}^{l-1}\delta_T(0,g-(c+a_jw^s))}{k_n+1}+\sum_{j=n}^{q}\sum_{s=0}^{l-1}\frac{k_j+\delta_T(0,g-(c+a_jw^s))}{k_j+1}+\Theta_T(g)-2.$$

Furthermore, from the condition (5), (7) and Lemma 2.1, we have

$$\sum_{j=1}^{q}\sum_{s=0}^{l-1}\overline{\mathfrak{N}}^{k_j)}(r,\frac{1}{f-(c+a_jw^s)})<\overline{\mathfrak{N}}(r,\frac{1}{(f-c)^l-(g-c)^l})\le l(\mathfrak{T}(r,f)+\mathfrak{T}(r,g))+O(1),$$

and

$$\sum_{j=1}^{q}\sum_{s=0}^{l-1}\overline{\mathfrak{N}}^{k_j)}(r,\frac{1}{g-(c+a_jw^s)})<\overline{\mathfrak{N}}(r,\frac{1}{(f-c)^l-(g-c)^l})\le l(\mathfrak{T}(r,f)+\mathfrak{T}(r,g))+O(1).$$

Therefore, from the above discussion we obtain

$$\begin{aligned}&\left(\frac{l(m-1)k_m}{k_m+1}+B_1-\varepsilon\right)\mathfrak{T}(r,f)+\left(\frac{l(n-1)k_n}{k_n+1}+B_2-\varepsilon\right)\mathfrak{T}(r,g)\\ <\;&l\left(\frac{k_m}{k_m+1}+\frac{k_n}{k_n+1}\right)(\mathfrak{T}(r,f)+\mathfrak{T}(r,g))+Q(r,f)+Q(r,g),\end{aligned}$$

that is,

$$(A_1-\varepsilon)\,\mathfrak{T}(r,f)+(A_2-\varepsilon)\,\mathfrak{T}(r,g)<Q(r,f)+Q(r,g).$$

Since $f$ and $g$ are transcendental and $\varepsilon$ is arbitrary, the above inequality contradicts (6). Therefore, the proof of Theorem 1.6 is completed.

# 5 Remarks

Zheng [21] had proved the results related to Tsuji's characteristic and Nevanlinna's characteristic as follows

**Lemma 5.1** *(see [21, lemma 2.3.3]) Let $f(z)$ be a meromorphic function in $\Omega(\alpha,\beta)$, for any real number $\varepsilon>0$, $\Omega_\varepsilon=\Omega(\alpha+\varepsilon,\beta-\varepsilon)$. Then for $\varepsilon>0$, we have*

$$\mathfrak{N}(r,f)\le\omega\frac{N(r,\Omega,f)}{r^\omega}+\omega^2\int_1^r\frac{N(t,\Omega,f)}{t^{\omega+1}}dt,$$

*and*

$$\mathfrak{N}(r,f)\ge\omega c^\omega\frac{N(cr,\Omega_\varepsilon,f)}{r^\omega}+\omega^2c^\omega\int_1^{cr}\frac{N(t,\Omega_\varepsilon,f)}{t^{\omega+1}}dt$$

*where $0<c<1$ is a constant depending on $\varepsilon$, $\omega=\frac{\pi}{\beta-\alpha}$ and $N(t,\Omega,f)=\int_1^r\frac{n(t,\Omega,f)}{t}dt$, $n(t,\Omega,f)$ is the number of poles of $f(z)$ in $\Omega\cap\{z:1<|z|\le t\}$.*

From Lemma 5.1, we can get that $f$ is transcendental in Tsuji sense if $f$ satisfies condition (9). Thus, we can get the following results

**Theorem 5.2** *Let the assumptions of Theorems 1.5-1.6 and Corollaries 1.1-1.4 be given with the exception of that $f(z)$ is transcendental in Tsuji sense. Assume that for some $a\in\widehat{\mathbb{C}}$ and $\varepsilon>0$,*

$$\limsup_{r\to\infty}\frac{N(r,\Omega_\varepsilon,f=a)}{r^\omega\log r}=\infty,\tag{9}$$

*where $\omega, N(t,\Omega,f)$ are stated as in Lemma 5.1. Then $f(z)\equiv g(z)$.*

# References


[1] T. B. Cao, H. X. Yi, Analytic functions sharing three values $DM$ in one angular domain, J.Korean Math. Soc. 45(6)(2008), 1523-1534.

[2] T. B. Cao, H. X. Yi, On the uniqueness of meromorphic functions that share four values in one angular domain, J. Math. Anal. Appl. 358(2009), 81-97.

[3] A. A. Gol'dberg, Nevanlinna's lemma on the logarithmic derivative of a meromorphic function, Mathematical Notes 17(4) (1975), 310-312.

[4] A. A. Gol'dberg, I. V. Ostrovskii, The Distribution of Values of Meromorphic Function, Nauka, Moscow, 1970 (in Russian).

[5] W. K. Hayman, Meromorphic Functions, Oxford Univ. Press, London, 1964.

[6] J. K. Langley, An application of the Tsuji characteristic, J. Fac. Sci. Univ. Tokyo Sect. IA, Math. 38 (1991), 299-318.

[7] W. C. Lin, S. Mori, K. Tohge, Uniqueness theorems in an angular domain, Tohoku Math. J. 58 (2006), 509-527.

[8] W. C. Lin, S. Mori, H. X. Yi, Uniqueness theorems of entire functions with shared-set in an angular domain, Acta Mathematica Sinica 24(2008), 1925-1934.

[9] R. Nevanlinna, Le théorème de Picard-Borel et la théorie des fonctions méromorphes, Repringting of the 1929 original, Chelsea Publishing Co., New York, 1974(in Frech).

[10] Z. J. Wu, A remark on uniqueness theorems in an angular domain, Proc, Japan Acad. Ser. A 64(6)(2008), 73-77.

[11] H. Y. Xu, T. B. Cao, Uniqueness of two analyticfunctions sharing four values in an angular domain, Ann. Polon.Math. 99 (2010), 55-65.

[12] H. Y. Xu, T. B. Cao, Uniqueness of meromorphicfunctions sharing four values IM and one set in an angular domain,Bulletin of the Belgian Mathematical Society -Simon Stevin, 17(5) (2010), 937-948.

[13] J. F. Xu, H. X. Yi, On uniqueness of meromorphic functions with shared four values in some angular domains, Bull. Malays. Math. Sci. Soc. 31(1) (2008), 57-65.

[14] Z. X. Xuan, On uniqueness of meromorphic functions with multiple values in some angular domains, Journal of Inequalities and Applications, Volume 2009 (2009), Article ID 208516, 10 pages.

[15] H. X. Yi, C. C. Yang, Uniqueness Theory of Meromorphic Functions, Science Press/Kluwer., Beijing, 2003.

[16] H. X. Yi , On the uniqueness of meromorphic functions, Acta Math. Sinica (Chin. Ser.) 31(4) (1988), 570-576.

[17] L. Yang, Value Distribution Theory, Springer/Science Press, Berlin/Beijing, 1993/1982.

[18] Q. C. Zhang, Meromorphic functions sharing values in an angular domain, J. Math. Anal. Appl. 349(2009), 100-112.

[19] J. H. Zheng, On uniqueness of meromorphic functions with shared values in some angular domains, Canad J. Math. 47(2004), 152-160.

[20] J. H. Zheng, On uniqueness of meromorphic functions with shared values in one angular domains, Complex Var. Elliptic Equ. 48(2003), 777-785.

[21] J. H. Zheng, Value distrbution of meromorphic functions, Springer, Beijing: Tsinghua Univ. Press. 2010.

# A new generalization of Fibonacci and Lucas $p-$numbers

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**Abstract**

In this paper, we define a new generalization of the Fibonacci and Lucas $p-$numbers. Further, we build up the tree diagrams for generalized Fibonacci and Lucas $p-$sequence and derive the recurrence relations of these sequences by using these diagrams. Also, we show that the generalized Fibonacci and Lucas $p-$sequences can be reduced into the various number sequences. Finally, we develop Binet formulas for the generalized Fibonacci and Lucas $p-$numbers and present the numerical and graphical results, which obtained by means of the Binet formulas, for specific values of $a$, $b$ and $p$.

*Keywords:* The generalized Fibonacci $p-$numbers, The generalized Lucas $p-$numbers, Binet formula.
*2010 MSC:* 11B39

## 1. Introduction

Fibonacci and Lucas sequences are one of the most popular and fascinating sequences that arise in various situations, especially in mathematics, physics and related fields. The classical Fibonacci and Lucas sequences are defined by $F_{n+2} = F_{n+1} + F_n$ and $L_{n+2} = L_{n+1} + L_n$, for $n \in \mathbb{N}$, with initial conditions $F_0 = 0, \quad F_1 = 1$ and $L_0 = 2, \quad L_1 = 1$, respectively. One of the most important sources of this area is [1], which was written by Thomas Koshy, and contains numerous applications, generalizations and recurrence relations of Fibonacci and Lucas numbers. In recent years, many authors have studied generalizations of the Fibonacci and Lucas sequences [2–12]. For instance, in [8, 10] the authors defined the generalized Fibonacci $\{q_n\}_{n\in N_0}$ sequence as

$$q_0 = 0, \;\; q_1 = 1, \;\; q_{n+2} = \begin{cases} aq_{n+1} + q_n, \text{if } n \equiv 0 \pmod 2 \\ bq_{n+1} + q_n, \text{if } n \equiv 1 \pmod 2, \end{cases} \tag{1}$$

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and the generalized Lucas $\{l_n\}_{n\in N_0}$ sequence as in the form

$$l_0 = 2,\ \ l_1 = a,\ \ l_{n+2} = \begin{cases} bl_{n+1} + l_n, \text{if } n \equiv 0 \pmod 2 \\ al_{n+1} + l_n, \text{if } n \equiv 1 \pmod 2. \end{cases} \tag{2}$$

Stakhov and Rozin introduced Fibonacci and Lucas $p-$numbers, one of the most significant mathematical discoveries of the modern Fibonacci numbers theory, and they presented some properties of this sequence,

$$F_p(n) = F_p(n-1) + F_p(n-p-1) \tag{3}$$

and

$$L_p(n) = L_p(n-1) + L_p(n-p-1), \tag{4}$$

in [13], with the initial conditions $F_p(0) = 0, F_p(1) = 1, F_p(2) = 1, \ldots, F_p(p) = 1$ and $L_p(0) = p+1, L_p(1) = 1, L_p(2) = 1, \ldots, L_p(p) = 1$, respectively. After that, Kocer et al. defined the $m-$extension of the Fibonacci and Lucas $p-$numbers,

$$F_{p,m}(n+p+1) = mF_{p,m}(n+p) + F_{p,m}(n) \tag{5}$$

and

$$L_{p,m}(n+p+1) = mL_{p,m}(n+p) + L_{p,m}(n), \tag{6}$$

with initial conditions $F_{p,m}(0) = 0, F_{p,m}(1) = 1, F_{p,m}(2) = m, F_{p,m}(3) = m^2, \ldots, F_{p,m}(p+1) = m^p$ and $L_{p,m}(0) = p+1, L_{p,m}(1) = m, L_{p,m}(2) = m^2, L_{p,m}(3) = m^3, \ldots, L_{p,m}(p+1) = m^{p+1}$, where $p$ and $n$ are nonnegative integers and $m$ is a positive real number [14]. The main purpose of the present article is to give a wider generalization of the generalized Fibonacci and Lucas sequence given by (1) and (2), the Fibonacci and Lucas $p$-sequences given by (3) and (4) and the $m-$extension of the Fibonacci and Lucas $p$-sequences given by (5) and (6) to introduce a new class of the recurrence numerical sequences called the *generalization of Fibonacci and Lucas $p-$numbers.*

## 2. Generalized Fibonacci and Lucas $p-$numbers

**Definition 2.1.** For any positive real numbers $a$, $b$ and positive integer $p$, the generalized Fibonacci $p-$sequence $\{f_n\}_{n=0}^{\infty}$ and Lucas $p-$sequence $\{\ell_n\}_{n=0}^{\infty}$ are defined recursively by

$$f_n = \begin{cases} af_{n-1} + f_{n-p-1}, \text{if } n \equiv 0 \pmod 2 \\ bf_{n-1} + f_{n-p-1}, \text{if } n \equiv 1 \pmod 2, \end{cases} \quad \text{and} \quad \ell_n = \begin{cases} b\ell_{n-1} + \ell_{n-p-1}, \text{if } n \equiv 0 \pmod 2 \\ a\ell_{n-1} + \ell_{n-p-1}, \text{if } n \equiv 1 \pmod 2, \end{cases}$$

where $n \geq p+1$ and the initial conditions of $f_n$ and $\ell_n$ are

$$f_0 = 0, f_1 = 1, f_2 = a, \ldots, f_p = a^{\lfloor \frac{p}{2} \rfloor} b^{\lfloor \frac{p-1}{2} \rfloor} \tag{7}$$

and

$$\ell_0 = p+1, \ell_1 = a, \ell_2 = ab, \ldots, \ell_p = a^{\lfloor \frac{p+1}{2} \rfloor} b^{\lfloor \frac{p}{2} \rfloor}, \tag{8}$$

respectively.

Note that, these sequences can be reduced to different sequences for specific values of $p$, $a$ and $b$. It is not difficult to see from the following table that Fibonacci, Lucas, Pell, Pell−Lucas, $k$−Fibonacci, $k$−Lucas, Fibonacci $p$, Lucas $p$, Pell $p$, Pell−Lucas $p$, $m$−extension of Fibonacci $p$ and $m$−extension of Lucas $p$-sequences are special cases of generalized Fibonacci and Lucas $p$−sequence.

| $p$ | $a$ | $b$ | $f_n$ | $\ell_n$ |
|---|---|---|---|---|
| 1 | 1 | 1 | Classical Fibonacci sequence $F_n$ | Classical Lucas sequence $L_n$ |
| 1 | 2 | 2 | Classical Pell sequence $P_n$ | Classical Pell-Lucas sequence $Q_n$ |
| 1 | $k$ | $k$ | $k$−Fibonacci numbers $\{F_{k,n}\}_{n=0}^{\infty}$ | $k$−Lucas numbers $\{L_{k,n}\}_{n=0}^{\infty}$ |
| $p$ | 1 | 1 | Fibonacci $p$−sequence $F_{p,n}$ | Lucas $p$−sequence $L_{p,n}$ |
| $p$ | 2 | 2 | Pell $p$−sequence $F_{p,n}$ | Pell-Lucas $p$−sequence $L_{p,n}$ |
| $p$ | $m$ | $m$ | $m$−extension of Fibonacci $p$−numbers $F_{p,m,n}$ | $m$−extension of Lucas $p$−numbers $L_{p,m,n}$ |

Let $a$ and $b$ be positive real numbers, $p$ be a positive integer and $\xi(n) = n - 2\lfloor \frac{n}{2} \rfloor$. We can construct the tree diagrams for the generalized Fibonacci and Lucas $p$−numbers as:

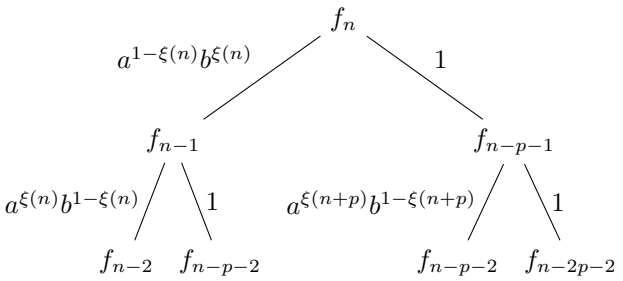



Figure 1: Tree diagram for generalized Fibonacci $p$−numbers

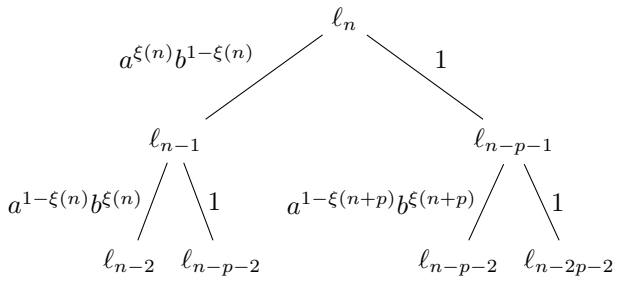



Figure 2: Tree diagram for generalized Lucas $p$−numbers.

By considering Figures 1 and 2, we will derive the recurrence relations for $f_n$ and $\ell_n$. First we suppose that $p$ is even. Then, $\{f_n\}$ satisfies the recurrence relation

$$\begin{aligned}
f_n &= a^{1-\xi(n)}b^{\xi(n)}f_{n-1} + f_{n-p-1} \\
&= a^{1-\xi(n)}b^{\xi(n)}\left(a^{\xi(n)}b^{1-\xi(n)}f_{n-2} + f_{n-p-2}\right) + a^{\xi(n+p)}b^{1-\xi(n+p)}f_{n-p-2} + f_{n-2p-2} \\
&= abf_{n-2} + a^{1-\xi(n)}b^{\xi(n)}f_{n-p-2} + a^{\xi(n)}b^{1-\xi(n)}f_{n-p-2} + f_{n-2p-2} \\
&= abf_{n-2} + \left(a^{1-\xi(n)}b^{\xi(n)} + a^{\xi(n)}b^{1-\xi(n)}\right)f_{n-p-2} + f_{n-2p-2} \\
&= abf_{n-2} + (a+b)\,f_{n-p-2} + f_{n-2p-2}.
\end{aligned}$$

Next, we suppose that $p$ is odd. Then, $f_n$ also satisfies the recurrence relation

$$\begin{aligned} f_n &= a^{1-\xi(n)}b^{\xi(n)}f_{n-1} + f_{n-p-1} \\ &= a^{1-\xi(n)}b^{\xi(n)}\left(a^{\xi(n)}b^{1-\xi(n)}f_{n-2} + f_{n-p-2}\right) + a^{\xi(n+p)}b^{1-\xi(n+p)}f_{n-p-2} + f_{n-2p-2} \\ &= abf_{n-2} + a^{1-\xi(n)}b^{\xi(n)}f_{n-p-2} + a^{1-\xi(n)}b^{\xi(n)}f_{n-p-2} + f_{n-2p-2} \\ &= abf_{n-2} + 2a^{1-\xi(n)}b^{\xi(n)}f_{n-p-2} + f_{n-2p-2} \\ &= abf_{n-2} + 2\left(f_{n-p-1} - f_{n-2p-2}\right) + f_{n-2p-2} \\ &= abf_{n-2} + 2f_{n-p-1} - f_{n-2p-2}. \end{aligned}$$

In a similar way, we can easily obtain the same recurrence relation for $\ell_n$. Let

$$\alpha_n = \begin{cases} f_n, & \text{if } \alpha_0 = 0, \alpha_1 = 1, \alpha_2 = a, \ldots, \alpha_p = a^{\lfloor \frac{p}{2} \rfloor} b^{\lfloor \frac{p-1}{2} \rfloor} \\ \ell_n, & \text{if } \alpha_0 = p+1, \alpha_1 = a, \alpha_2 = ab, \ldots, \alpha_p = a^{\lfloor \frac{p+1}{2} \rfloor} b^{\lfloor \frac{p}{2} \rfloor} \end{cases}$$

be a sequence that satisfies both $f_n$ and $\ell_n$. Thereby, $\alpha_n$ satisfies the recurrence relation

$$\alpha_n = \begin{cases} ab\alpha_{n-2} + (a+b)\,\alpha_{n-p-2} + \alpha_{n-2p-2}, & \text{if } p \text{ is even}, \\ ab\alpha_{n-2} + 2\alpha_{n-p-1} - \alpha_{n-2p-2}, & \text{if } p \text{ is odd}. \end{cases} \tag{9}$$

By considering eq. (9), the characteristic polynomial of $\alpha_n$ is

$$\alpha_p(x) = \begin{cases} x^{2p+2} - abx^{2p} - (a+b)\,x^p - 1, & \text{if } p \text{ is even}, \\ x^{2p+2} - abx^{2p} - 2x^{p+1} + 1, & \text{if } p \text{ is odd}. \end{cases} \tag{10}$$

By taking $r = x^2$, we can express the characteristic equation (10) as

$$\beta_p(r) = \begin{cases} r^{p+1} - abr^p - (a+b)\,r^{\frac{p}{2}} - 1, & \text{if } p \text{ is even}, \\ r^{p+1} - abr^p - 2r^{\frac{p+1}{2}} + 1, & \text{if } p \text{ is odd}. \end{cases} \tag{11}$$

**Lemma 2.1.** Assume that $p$ is odd. Then the characteristic equation of the generalized Fibonacci and Lucas $p-$numbers $\alpha_p(x)$ does not have multiple roots.

For the other case, it can easily seen that there are no multiple real roots. However, whether there exists complex multiple roots or not is an open problem, and we suggest that interested readers study it with us.

*Proof of Lemma.* The characteristic equation of the generalized Fibonacci and Lucas $p-$numbers for odd $p$ can be written in the form

$$\alpha_p(x) = (x^{p+1} - 1)^2 - abx^{2p}$$

and its derivative is

$$\alpha_p'(x) = 2(p+1)x^p(x^{p+1} - 1) - 2pabx^{2p-1}.$$

Then, $\alpha_p(x) = 0$ if and only if

$$ab = \frac{(x^{p+1}-1)^2}{x^{2p}}$$

and $\alpha_p'(x) = 0$ if and only if

$$ab = \frac{(p+1)x^p(x^{p+1}-1)}{px^{2p-1}} = \frac{(p+1)x}{p}\frac{x^{p+1}-1}{x^p}.$$

Upon simplifying, we obtain $ab = \left(\frac{p+1}{p}\right)x\sqrt{ab}$. Therefore, $\alpha_p(x)$ and $\alpha_p'(x)$ vanish for the same $x$ if and only if for some root $x$ of $\alpha_p(x)$,

$$ab = \left(\frac{p+1}{p}\right)x\sqrt{ab} \quad \text{or equivalently, } x = \frac{p\sqrt{ab}}{p+1}.$$

So, for every $p$ and $ab$, if such an $x$ is a root, it is a multiple root. Let $t$ be a multiple root. Then, $ab = \left(\frac{p+1}{p}\right)^2 t^2$. Since $\alpha_p(t) = 0$, we have

$$\begin{aligned} t^{2p+2} - \left(\frac{p+1}{p}\right)^2 t^{2p+2} - 2t^{p+1} + 1 &= 0 \\ -\frac{(2p+1)}{p^2}t^{2p+2} - 2t^{p+1} + 1 &= 0 \\ (2p+1)t^{2(p+1)} + 2p^2t^{p+1} - p^2 &= 0. \end{aligned}$$

When treated as a quadratic equation in $t^{p+1}$, the discriminant is

$$4p^4 + 4p^2(2p+1) = 4p^2(p+1)^2,$$

and therefore, the solutions are

$$t^{p+1} = -p, \frac{p}{2p+1}.$$

But substituting the same $ab$ in $\alpha_p'(t) = 0$, we get

$$\begin{aligned} &2(p+1)t^p(t^{p+1}-1) - 2p\left(\frac{p+1}{p}\right)^2 t^{2p+1} = 0 \\ &p(t^{p+1}-1) - (p+1)t^{p+1} = 0 \\ &t^{p+1} = -p \end{aligned}$$

Then, by $ab = \left(\frac{p+1}{p}\right)^2 t^2$, we have

$$ab = \frac{(p+1)^2}{(-p)^{\frac{2p}{p+1}}}.$$

The equation has multiple roots exactly when $ab$ and $p$ are related as above. Note that for odd values of $p$, $ab$ will be a real number, and then, $ab < 0$. This is a contradiction. Therefore the characteristic equation $\alpha_p(x)$ has distinct roots. The proof is complete. □

We will describe the terms of the sequence $\{\alpha_n\}$ clearly by using the Binet formula. So, we can give the generalized Binet formula for the generalized Fibonacci and Lucas $p-$numbers with the following theorem.

**Theorem 2.1.** *Suppose that the characteristic equation* (11) *has* $(p+1)$ *distinct roots,* $r_1, r_2, \ldots, r_{p+1}$. *Then* $\alpha_n$ *satisfies the relation*

$$\alpha_n = \sum_{i=1}^{p+1}\left(\sum_{j=1}^{p}(-1)^j \frac{\sum\limits_{\substack{1\le k_1<k_2<\ldots<k_{p+1-j}\le p+1\\ k_1,k_2,\ldots,k_{p+1-j}\neq i}} r_{k_1}r_{k_2}\ldots r_{k_{p+1-j}}}{\prod\limits_{\substack{1\le k\le p+1\\ k\neq i}}(r_i-r_k)}\alpha_{2j-2+\xi(n)} + \frac{1}{\prod\limits_{\substack{1\le k\le p+1\\ k\neq i}}(r_i-r_k)}\alpha_{2p+\xi(n)}\right) r_i^{\lfloor \frac{n}{2}\rfloor}.$$

*Proof.* Let $r_1, r_2, \ldots, r_{p+1}$ be $(p+1)$ distinct roots of the characteristic equation (11). There are $(2p+2)$ coefficients $k_1, k_2, \ldots, k_{2p+1}, k_{2p+2}$ such that

$$\alpha_n = k_1(\sqrt{r_1})^n + k_2(-\sqrt{r_1})^n + k_3(\sqrt{r_2})^n + k_4(-\sqrt{r_2})^n + \cdots + k_{2p+1}(\sqrt{r_{p+1}})^n + k_{2p+2}(-\sqrt{r_{p+1}})^n.$$

First, we suppose that $n$ is even. Then we obtain

$$\begin{aligned}\alpha_n &= (k_1+k_2)(\sqrt{r_1})^n + (k_3+k_4)(\sqrt{r_2})^n + \cdots + (k_{2p+1}+k_{2p+2})(\sqrt{r_{p+1}})^n \\ &= \sum_{i=1}^{p+1}(k_{2i-1}+k_{2i})(\sqrt{r_i})^n. \end{aligned} \tag{12}$$

In order to determine the coefficients $k_1, k_2, \ldots, k_{2p+1}, k_{2p+2}$, we must solve the linear equation system $\mathbf{V}\boldsymbol{\gamma} = \boldsymbol{\alpha}$, where $\mathbf{V}_{i,j} = r_j^{i-1}$ is a Vandermonde matrix, $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_{p+1})^T$ and $\boldsymbol{\alpha} = (\alpha_0, \alpha_2, \ldots, \alpha_{2p})^T$ are the column vectors. Considering the cases $n = 0, 2, 4, \ldots, 2p$ in (12), we have the linear equation system

$$\begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ r_1 & r_2 & r_3 & \ldots & r_{p+1} \\ r_1^2 & r_2^2 & r_3^2 & \ldots & r_{p+1}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_1^p & r_2^p & r_3^p & \ldots & r_{p+1}^p \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \vdots \\ \gamma_{p+1} \end{pmatrix} = \begin{pmatrix} \alpha_0 \\ \alpha_2 \\ \alpha_4 \\ \vdots \\ \alpha_{2p} \end{pmatrix}, \tag{13}$$

where $\gamma_p = k_{2p-1} + k_{2p}$. The $(p+1)\times(p+1)$ Vandermonde matrix can be factorized as $\mathbf{V} = \mathbf{LU}$, (see [15]), where $\mathbf{L}$ is a lower triangular matrix with units on its main diagonal, $(i,j)$-th element of $\mathbf{L}$ is

$$\mathbf{L}_{i,j} = \begin{cases} 1 & , \quad \text{if } i = j\ , \\ \sum\limits_{1\le k_1\le k_2\le\ldots\le k_{i-j}\le j} r_{k_1}r_{k_2}\ldots r_{k_j} & , \quad \text{if } i > j \ge 1, \\ 0 & , \quad \text{if } i < j\ , \end{cases}$$

and $\mathbf{U}$ is an upper triangular matrix, $(i,j)$-th element of $\mathbf{U}$ is

$$\mathbf{U}_{i,j} = \begin{cases} 1 & , \quad \text{if } i = 1 \ , \\ \prod\limits_{\substack{k=1 \\ k \neq j}}^{i-1} (r_j - r_k) & , \quad \text{if } i \leq j, \\ 0 & , \quad \text{if } i > j \ . \end{cases}$$

Now we suppose that $\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \cdots + \lambda_p \mathbf{v}_p + \lambda_{p+1} \mathbf{v}_{p+1} = (0, 0, \ldots, 0)$, where $\mathbf{v}_k = (1, r_k^1, r_k^2, \ldots, r_k^p)$ is the $k$-th column vector and $\lambda_1, \lambda_2, \ldots, \lambda_{p+1}$ are real numbers. Then the $k$-th coordinate

$$\lambda_1 + \lambda_2 r_k + \lambda_3 r_k^2 + \ldots + \lambda_{p+1} r_k^p = 0,$$

which means that $r_k$ is a zero of the polynomial $\xi(r) = \lambda_1 + \lambda_2 r + \lambda_3 r^2 + \ldots + \lambda_{p+1} r^p$. If the polynomial $\xi(r)$ of degree at most $p$ has $(p+1)$ distinct zeros $r_1, r_2, \ldots, r_{p+1}$ then it must be zero polynomial and we get $\lambda_1 = \lambda_2 = \ldots = \lambda_{p+1} = 0$. So it is easily seen that the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{p+1}$ are linearly independent. This proves that $\mathbf{V}$ is invertible. So, we can factorize the inverse of the Vandermonde matrix as $\mathbf{V}^{-1} = \mathbf{U}^{-1}\mathbf{L}^{-1}$, (see [16]), where $\mathbf{L}^{-1}$ is a lower triangular matrix with units on its main diagonal, $(i,j)$-th element of $\mathbf{L}^{-1}$ is

$$\mathbf{L}_{i,j}^{-1} = \begin{cases} 1 & , \quad \text{if } i = j \ , \\ \mathbf{L}_{i-1,j-1}^{-1} - \mathbf{L}_{i-1,j}^{-1} r_{i-1} & , \quad i = 2, 3, \ldots, p+1; j = 2, 3, \ldots, i-1, \\ 0 & , \quad \text{if } i < j \ , \end{cases}$$

and $\mathbf{U}^{-1}$ is an upper triangular matrix, $(i,j)$-th element of $\mathbf{U}^{-1}$ is

$$\mathbf{U}_{i,j}^{-1} = \begin{cases} \prod\limits_{\substack{k=1 \\ k \neq i}}^{j} \dfrac{1}{r_i - r_k} & , \quad \text{if } i \leq j, \\ 0 & , \quad \text{if } i > j \ . \end{cases}$$

Therefore $(i,j)$-th element of $\mathbf{V}^{-1}$ can be written as

$$\mathbf{V}_{i,j}^{-1} = \begin{cases} (-1)^j \dfrac{\sum\limits_{\substack{1 \leq k_1 < k_2 < \ldots < k_{p+1-j} \leq p+1 \\ k_1, k_2, \ldots, k_{p+1-j} \neq i}} r_{k_1} r_{k_2} \ldots r_{k_{p+1-j}}}{\prod\limits_{\substack{1 \leq k \leq p+1 \\ k \neq i}} (r_i - r_k)} & , \quad \text{if } 1 \leq j < p+1, \\ \dfrac{1}{\prod\limits_{\substack{1 \leq k \leq p+1 \\ k \neq i}} (r_i - r_k)} & , \quad \text{if } j = p+1 \ . \end{cases}$$

Since $\boldsymbol{\gamma} = \mathbf{V}^{-1}\boldsymbol{\alpha}$, the $i$-th element of $\boldsymbol{\gamma}$ is $\boldsymbol{\gamma_i} = \sum_{j=1}^{p+1} \mathbf{V}_{i,j}^{-1}\alpha_{2j-2}$. So, we have

$$\alpha_n = \sum_{i=1}^{p+1}\left(\sum_{j=1}^{p}(-1)^j \frac{\sum\limits_{\substack{1\le k_1<k_2<\ldots<k_{p+1-j}\le p+1\\ k_1,k_2,\ldots,k_{p+1-j}\neq i}} r_{k_1}r_{k_2}\ldots r_{k_{p+1-j}}}{\prod\limits_{\substack{1\le k\le p+1\\ k\neq i}}(r_i-r_k)}\alpha_{2j-2} + \frac{\alpha_{2p}}{\prod\limits_{\substack{1\le k\le p+1\\ k\neq i}}(r_i-r_k)}\right) r_i^{\frac{n}{2}}. \quad (14)$$

Next, we suppose that $n$ is odd. Then we obtain

$$\begin{aligned}\alpha_n &= (k_1-k_2)(\sqrt{r_1})^n + (k_3-k_4)(\sqrt{r_2})^n + \cdots + (k_{2p+1}-k_{2p+2})(\sqrt{r_{p+1}})^n \\ &= \sum_{i=1}^{p+1}(k_{2i-1}-k_{2i})(\sqrt{r_i})^n. \end{aligned} \quad (15)$$

Considering the cases $n = 1, 3, 5, \ldots, 2p+1$ in (15), we have

$$\begin{pmatrix} \sqrt{r_1} & \sqrt{r_2} & \sqrt{r_3} & \cdots & \sqrt{r_{p+1}} \\ \sqrt{r_1}^3 & \sqrt{r_2}^3 & \sqrt{r_3}^3 & \cdots & \sqrt{r_{p+1}}^3 \\ \sqrt{r_1}^5 & \sqrt{r_2}^5 & \sqrt{r_3}^5 & \cdots & \sqrt{r_{p+1}}^5 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sqrt{r_1}^{2p+1} & \sqrt{r_2}^{2p+1} & \sqrt{r_3}^{2p+1} & \cdots & \sqrt{r_{p+1}}^{2p+1} \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \vdots \\ \gamma_{p+1} \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_3 \\ \alpha_5 \\ \vdots \\ \alpha_{2p+1} \end{pmatrix}, \quad (16)$$

where $\gamma_p = k_{2p-1} - k_{2p}$. It can be easily seen that $\mathbf{L}_{i,j}$ and $\mathbf{L}_{i,j}^{-1}$ are the same as in the previous case. In a similar way, we can find the $(i,j)-$th element of the matrices $\mathbf{U}_{i,j}$ and $\mathbf{U}_{i,j}^{-1}$, respectively, as

$$\mathbf{U}(i,j) = \begin{cases} \sqrt{r_j} & , \quad \text{if } i = 1\ , \\ \sqrt{r_j}\prod\limits_{\substack{k=1\\ k\neq j}}^{i-1}(r_j - r_k) & , \quad \text{if } i \le j, \\ 0 & , \quad \text{if } i > j \end{cases} \quad \text{and} \quad \mathbf{U}_{i,j}^{-1} = \begin{cases} \prod\limits_{\substack{k=1\\ k\neq i}}^{j}\frac{1}{\sqrt{r_i}(r_i-r_k)} & , \quad \text{if } i \le j, \\ 0 & , \quad \text{if } i > j\ . \end{cases}$$

Using these identities, we find the $(i,j)-$th element of $\mathbf{V}^{-1}$ as

$$\mathbf{V}_{i,j}^{-1} = \begin{cases} (-1)^j \dfrac{\sum\limits_{\substack{1\le k_1<k_2<\ldots<k_{p+1-j}\le p+1\\ k_1,k_2,\ldots,k_{p+1-j}\neq i}} r_{k_1}r_{k_2}\ldots r_{k_{p+1-j}}}{\sqrt{r_i}\prod\limits_{\substack{1\le k\le p+1\\ k\neq i}}(r_i-r_k)} & , \quad \text{if } 1 \le j < p+1, \\ \dfrac{1}{\sqrt{r_i}\prod\limits_{\substack{1\le k\le p+1\\ k\neq i}}(r_i-r_k)} & , \quad \text{if } j = p+1\ . \end{cases}$$

Thus, $\alpha_n$ satisfies the recurrence relation

$$\alpha_n = \sum_{i=1}^{p+1}\left(\sum_{j=1}^{p}(-1)^j \frac{\sum\limits_{\substack{1\le k_1<k_2<\ldots<k_{p+1-j}\le p+1\\ k_1,k_2,\ldots,k_{p+1-j}\neq i}} r_{k_1}r_{k_2}\ldots r_{k_{p+1-j}}}{\prod\limits_{\substack{1\le k\le p+1\\ k\neq i}}(r_i-r_k)}\alpha_{2j-1} + \frac{\alpha_{2p+1}}{\prod\limits_{\substack{1\le k\le p+1\\ k\neq i}}(r_i-r_k)}\right) r_i^{\frac{n-1}{2}}. \quad (17)$$

Finally, by combining (14) and (17), we have the generalized Binet formula

$$\alpha_n = \sum_{i=1}^{p+1}\left(\sum_{j=1}^{p}(-1)^j \frac{\sum\limits_{\substack{1\le k_1<k_2<\ldots<k_{p+1-j}\le p+1\\ k_1,k_2,\ldots,k_{p+1-j}\neq i}} r_{k_1}r_{k_2}\ldots r_{k_{p+1-j}}}{\prod\limits_{\substack{1\le k\le p+1\\ k\neq i}}(r_i-r_k)}\alpha_{2j-2+\xi(n)} + \frac{\alpha_{2p+\xi(n)}}{\prod\limits_{\substack{1\le k\le p+1\\ k\neq i}}(r_i-r_k)}\right) r_i^{\lfloor\frac{n}{2}\rfloor}, \quad (18)$$

which proves the theorem. □

**Corollary 2.1.** *If we take the initial conditions* $\left\{\alpha_0=0,\alpha_1=1,\alpha_2=a,\ldots,\alpha_p=a^{\lfloor\frac{p}{2}\rfloor}b^{\lfloor\frac{p-1}{2}\rfloor}\right\}$ *in* (18)*, we obtain the Binet formula of the generalized Fibonacci* $p-$*numbers as*

$$f_n = \sum_{i=1}^{p+1}\left(\sum_{j=1}^{p}(-1)^j \frac{\sum\limits_{\substack{1\le k_1<k_2<\ldots<k_{p+1-j}\le p+1\\ k_1,k_2,\ldots,k_{p+1-j}\neq i}} r_{k_1}r_{k_2}\ldots r_{k_{p+1-j}}}{\prod\limits_{\substack{1\le k\le p+1\\ k\neq i}}(r_i-r_k)}f_{2j-2+\xi(n)} + \frac{f_{2p+\xi(n)}}{\prod\limits_{\substack{1\le k\le p+1\\ k\neq i}}(r_i-r_k)}\right) r_i^{\lfloor\frac{n}{2}\rfloor}. \quad (19)$$

*If we take the initial conditions* $\left\{\alpha_0=p+1,\alpha_1=a,\alpha_2=ab,\ldots,\alpha_p=a^{\lfloor\frac{p+1}{2}\rfloor}b^{\lfloor\frac{p}{2}\rfloor}\right\}$ *in* (18)*, we obtain the Binet formula of the generalized Lucas* $p-$*numbers as*

$$\ell_n = \sum_{i=1}^{p+1}\left(\sum_{j=1}^{p}(-1)^j \frac{\sum\limits_{\substack{1\le k_1<k_2<\ldots<k_{p+1-j}\le p+1\\ k_1,k_2,\ldots,k_{p+1-j}\neq i}} r_{k_1}r_{k_2}\ldots r_{k_{p+1-j}}}{\prod\limits_{\substack{1\le k\le p+1\\ k\neq i}}(r_i-r_k)}\ell_{2j-2+\xi(n)} + \frac{\ell_{2p+\xi(n)}}{\prod\limits_{\substack{1\le k\le p+1\\ k\neq i}}(r_i-r_k)}\right) r_i^{\lfloor\frac{n}{2}\rfloor}. \quad (20)$$

## 3. Examples

In this section, we present the numerical results of generalized Fibonacci and Lucas $p-$sequence by specifying $p$, $a$ and $b$.

### *3.1. Case* $p=1$

By considering (18), we obtain the Binet formulas of the generalized Fibonacci and Lucas $1-$numbers as

$$f_n = \left(\frac{f_{2+\xi(n)}-f_{\xi(n)}r_2}{r_1-r_2}\right)r_1^{\lfloor\frac{n}{2}\rfloor} - \left(\frac{f_{2+\xi(n)}-f_{\xi(n)}r_1}{r_1-r_2}\right)r_2^{\lfloor\frac{n}{2}\rfloor} \quad (21)$$

and

$$\ell_n = \left(\frac{\ell_{2+\xi(n)} - \ell_{\xi(n)} r_2}{r_1 - r_2}\right) r_1^{\lfloor \frac{n}{2} \rfloor} - \left(\frac{\ell_{2+\xi(n)} - \ell_{\xi(n)} r_1}{r_1 - r_2}\right) r_2^{\lfloor \frac{n}{2} \rfloor}, \tag{22}$$

where $r_1 = \left(\frac{ab+2-\sqrt{a^2b^2+4ab}}{2}\right)$ and $r_2 = \left(\frac{ab+2+\sqrt{a^2b^2+4ab}}{2}\right)$. We give the first few terms of the generalized Fibonacci and Lucas 1−numbers with the following table as

Table 1: Generalized Fibonacci and Lucas 1−numbers for different $a$ and $b$

| $(a,b)$ | $(1,1)$ | $(1,2)$ | $(2,1)$ | $(2,2)$ |
|---|---|---|---|---|
| $f_n$ | $\{0,1,1,2,3,5,8\ldots\}$ | $\{0,1,1,3,4,11,15,\ldots\}$ | $\{0,1,2,3,8,11,30,\ldots\}$ | $\{0,1,2,5,12,29,70,\ldots\}$ |
| $\ell_n$ | $\{2,1,3,4,7,11,18,\ldots\}$ | $\{2,1,4,5,14,19,52,\ldots\}$ | $\{2,2,4,10,14,38,52,\ldots\}$ | $\{2,2,6,14,34,82,198,\ldots\}$ |

Moreover, we give the graphical illustration of the first 30 terms of the generalized Fibonacci and Lucas 1−numbers for different values of $a$ and $b$, ( see Figure 3 and Figure 4 ).

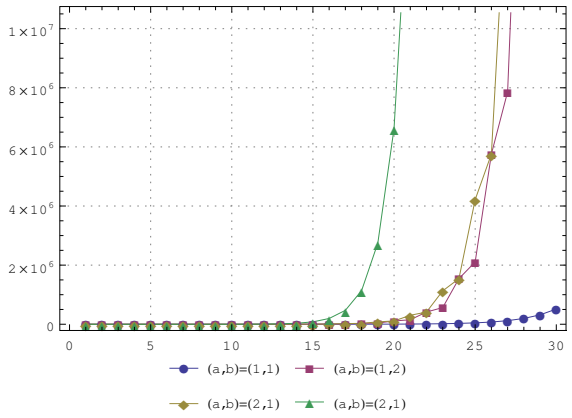



Figure 3: generalized Fibonacci 1−numbers

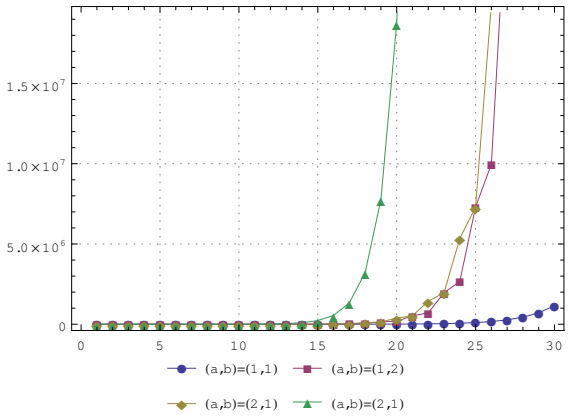



Figure 4: generalized Lucas 1−numbers

### *3.2. Case $p = 2$*

Recall the equation (18). The Binet formulas for the generalized Fibonacci and Lucas 2−numbers are

$$\begin{aligned} f_n &= \left(\frac{f_{4+\xi(n)} - (r_2+r_3) f_{2+\xi(n)} + r_2 r_3 f_{\xi(n)}}{(r_1-r_3)(r_1-r_2)}\right) r_1^{\lfloor \frac{n}{2} \rfloor} + \left(\frac{f_{4+\xi(n)} - (r_1+r_3) f_{2+\xi(n)} + r_1 r_3 f_{\xi(n)}}{(r_2-r_3)(r_2-r_1)}\right) r_2^{\lfloor \frac{n}{2} \rfloor} \\ &+ \left(\frac{f_{4+\xi(n)} - (r_1+r_2) f_{2+\xi(n)} + r_1 r_2 f_{\xi(n)}}{(r_3-r_2)(r_3-r_1)}\right) r_3^{\lfloor \frac{n}{2} \rfloor} \end{aligned} \tag{23}$$

and

$$\begin{aligned} \ell_n &= \left(\frac{\ell_{4+\xi(n)} - (r_2+r_3) \ell_{2+\xi(n)} + r_2 r_3 \ell_{\xi(n)}}{(r_1-r_3)(r_1-r_2)}\right) r_1^{\lfloor \frac{n}{2} \rfloor} + \left(\frac{\ell_{4+\xi(n)} - (r_1+r_3) \ell_{2+\xi(n)} + r_1 r_3 \ell_{\xi(n)}}{(r_2-r_3)(r_2-r_1)}\right) r_2^{\lfloor \frac{n}{2} \rfloor} \\ &+ \left(\frac{\ell_{4+\xi(n)} - (r_1+r_2) \ell_{2+\xi(n)} + r_1 r_2 \ell_{\xi(n)}}{(r_3-r_2)(r_3-r_1)}\right) r_3^{\lfloor \frac{n}{2} \rfloor}, \end{aligned} \tag{24}$$

where

$$r_1 = \frac{\sqrt[3]{2a^3b^3+9a^2b+\sqrt{\left(2a^3b^3+9a^2b+9ab^2+27\right)^2-4\left(a^2b^2+3a+3b\right)^3}+9ab^2+27}}{3\sqrt[3]{2}} + \frac{\sqrt[3]{2}\left(a^2b^2+3a+3b\right)}{3\sqrt[3]{2a^3b^3+9a^2b+\sqrt{\left(2a^3b^3+9a^2b+9ab^2+27\right)^2-4\left(a^2b^2+3a+3b\right)^3}+9ab^2+27}} + \frac{ab}{3},$$

$$r_2 = \frac{\left(-1+i\sqrt{3}\right)\sqrt[3]{2a^3b^3+9a^2b+\sqrt{\left(2a^3b^3+9a^2b+9ab^2+27\right)^2-4\left(a^2b^2+3a+3b\right)^3}+9ab^2+27}}{6\sqrt[3]{2}} + \frac{\left(1+i\sqrt{3}\right)\left(-a^2b^2-3a-3b\right)}{3\ 2^{2/3}\sqrt[3]{2a^3b^3+9a^2b+\sqrt{\left(2a^3b^3+9a^2b+9ab^2+27\right)^2-4\left(a^2b^2+3a+3b\right)^3}+9ab^2+27}} + \frac{ab}{3}$$

and

$$r_3 = -\frac{\left(1+i\sqrt{3}\right)\sqrt[3]{2a^3b^3+9a^2b+\sqrt{\left(2a^3b^3+9a^2b+9ab^2+27\right)^2-4\left(a^2b^2+3a+3b\right)^3}+9ab^2+27}}{6\sqrt[3]{2}} + \frac{\left(1-i\sqrt{3}\right)\left(-a^2b^2-3a-3b\right)}{3\ 2^{2/3}\sqrt[3]{2a^3b^3+9a^2b+\sqrt{\left(2a^3b^3+9a^2b+9ab^2+27\right)^2-4\left(a^2b^2+3a+3b\right)^3}+9ab^2+27}} + \frac{ab}{3}.$$

We give the the first few terms of the generalized Fibonacci and Lucas 2−numbers with the following table as

Table 2: Generalized Fibonacci and Lucas 2−numbers for different $a$ and $b$

| $(a,b)$ | $(1,1)$ | $(1,2)$ | $(2,1)$ | $(2,2)$ |
|---|---|---|---|---|
| $f_n$ | $\{0,1,1,1,2,3,4\ldots\}$ | $\{0,1,1,2,3,7,9,\ldots\}$ | $\{0,1,2,2,5,7,16,\ldots\}$ | $\{0,1,2,4,9,20,44,\ldots\}$ |
| $\ell_n$ | $\{3,1,1,4,5,6,10,\ldots\}$ | $\{3,1,2,5,11,13,31,\ldots\}$ | $\{3,2,2,7,9,20,27,\ldots\}$ | $\{3,2,4,11,24,52,115,\ldots\}$ |

Moreover, we give the graphical illustration of the first 30 terms of the generalized Fibonacci and Lucas 2−numbers for different values of $a$ and $b$, ( see Figure 5 and Figure 6 ).

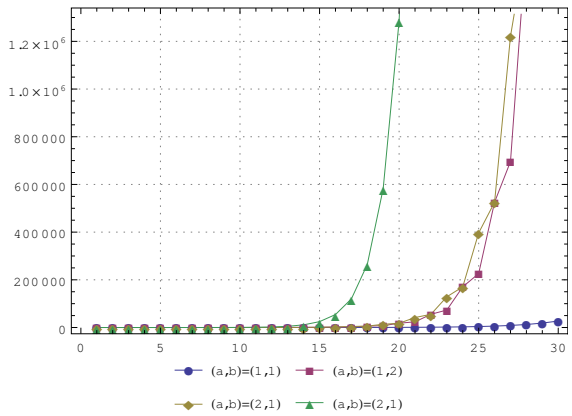



Figure 5: generalized Fibonacci 2−numbers

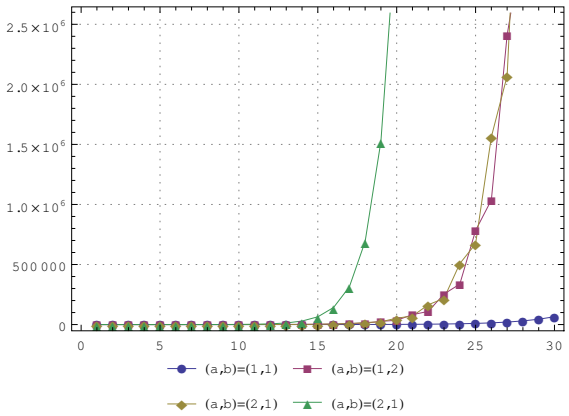



Figure 6: generalized Lucas 2−numbers

## 4. Conclusion

In this study, for the first time in literature we define a new generalization of Fibonacci and Lucas $p-$numbers. Also, we produce a Binet formula for these sequences by taking different initial conditions. The generalized Binet formula of the Fibonacci and Lucas $p-$numbers can be reduced to different Binet formulas in the literature. For example, if we take $p = 1$, we reduce the Binet formula of $\{f_n\}$ to the bi-periodic Fibonacci numbers $q_n = \left(\frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}}\right) \frac{\alpha^n - \beta^n}{\alpha - \beta}$, in [8]. If we take $p = 2$, we reduce the Binet formulas of $\{f_n\}$ and $\{\ell_n\}$ to the $\gamma_n$, in [12], Theorem 2.1. As a result, this study contributes to the literature by providing essential information for the generalization of Fibonacci and Lucas $p-$numbers.

## 5. Acknowledgments

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## References

[1] T. Koshy, Fibonacci and Lucas numbers with applications, Vol. 51, John Wiley & Sons, 2011.

[2] S. Falcon, A. Plaza, On the Fibonacci $k-$numbers, Chaos, Solitons & Fractals 32 (5) (2007) 1615–1624.

[3] S. Falcon, On the $k-$Lucas numbers, International Journal of Contemporary Mathematical Sciences 6 (21) (2011) 1039–1050.

[4] K. Uslu, N. Taskara, H. Kose, The generalized $k-$Fibonacci and $k-$Lucas numbers, Ars Combinatoria 99 (2011) 25–32.

[5] A. Horadam, Basic properties of a certain generalized sequence of numbers, The Fibonacci Quarterly 3 (3) (1965) 161–176.

[6] Y. Yazlik, N. Taskara, A note on generalized $k$-Horadam sequence, Computers & Mathematics with Applications 63 (1) (2012) 36–41.

[7] N. Yilmaz, Y. Yazlik, N. Taskara, On the $k-$generalized Fibonacci numbers, Selcuk Journal of Applied Mathematics 13 (1) (2012) 83–88.

[8] M. Edson, O. Yayenie, A new generalization of Fibonacci sequence & extended binet's formula, Integers 9 (6) (2009) 639–654.

[9] O. Yayenie, A note on generalized Fibonacci sequences, Applied Mathematics and Computation 217 (12) (2011) 5603–5611.

[10] G. Bilgici, Two generalizations of Lucas sequence, Applied Mathematics and Computation 245 (2014) 526–538.

[11] C. Cooper, An identity for period $k$ second order linear recurrence systems, Congr. Numer 200 (2010) 95–106.

[12] M. Alp, N. Irmak, L. Szalay, Two-periodic ternary recurrences and their binet-formula, Acta Math. Univ. Comenianae 81 (2) (2012) 227–232.

[13] A. Stakhov, B. Rozin, Theory of binet formulas for Fibonacci and Lucas $p-$numbers, Chaos, Solitons & Fractals 27 (5) (2006) 1162–1177.

[14] E. G. Kocer, N. Tuglu, A. Stakhov, On the $m-$extension of the Fibonacci and Lucas $p-$numbers, Chaos, Solitons & Fractals 40 (4) (2009) 1890–1906.

[15] S.-l. Yang, On the lu factorization of the vandermonde matrix, Discrete applied mathematics 146 (1) (2005) 102–105.

[16] F. Soto-Eguibar, H. Moya-Cessa, Inverse of the vandermonde and vandermonde confluent matrices, Appl. Math. Inf. Sci 5 (2011) 361–366.

# SOME FAMILIES OF GENERATING FUNCTIONS FOR THE GENERALIZED CESÀRO POLYNOMIALS

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Abstract. The present study deals with some new properties for the generalized Cesàro polynomials. The results obtained here include various families of multilinear and multilateral generating functions, miscellaneous properties and also some special cases for these polynomials. In addition, we derive a theorem giving certain families of bilateral generating functions for the generalized Cesàro polynomials and the generalized Lauricella functions. Finally, we get several interesting results of this theorem.

## 1. Introduction

The Cesàro polynomials $g_n^{(s)}(x)$ are defined by the generating relation (see, for example,[2], p. 449, Problem 20)

$$\sum_{n=0}^{\infty} g_n^{(s)}(x)t^n = (1-t)^{-s-1}(1-xt)^{-1}. \tag{1.1}$$

It is from (1.1) that

$$g_n^{(s)}(x) = \binom{s+n}{n} {}_2F_1\left[-n, 1; -s-n; x\right], \tag{1.2}$$

where ${}_2F_1$ denotes Gauss's hypergeometric series whose natural generalization of an arbitrary number of $p$ numerator and $q$ denominator parameters ($p, q \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$) is called and denoted by the generalized hypergeometric series ${}_pF_q$ defined by

$$\begin{aligned} {}_pF_q\left[\begin{array}{l} \alpha_1, ..., \alpha_p; \\ \beta_1, ..., \beta q; \end{array} \quad z\right] &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ ... \ (\alpha_p)_n}{(\beta_1)_n \ ... \ (\beta_q)_n} \frac{z^n}{n!} \\ &= {}_pF_q\left(\alpha_1, ..., \alpha_p; \beta_1, ..., \beta q; z\right). \end{aligned}$$

Here $(\lambda)_\nu$ denotes the Pochhammer symbol defined (in terms of gamma function) by

$$\begin{aligned} (\lambda)_\nu &= \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} \quad (\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-) \\ &= \begin{cases} 1, & \text{if } \nu = 0; \ \lambda \in \mathbb{C}\setminus\{0\} \\ \lambda(\lambda+1)...(\lambda+n-1), & \text{if } \nu = n \in \mathbb{N}; \ \lambda \in \mathbb{C} \end{cases}. \end{aligned}$$

and $\mathbb{Z}_0^-$ denotes the set of nonpositive integers and $\Gamma(\lambda)$ is the familiar Gamma function.

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In addition, we have the following relationship between the Cesàro polynomials $g_n^{(s)}(x)$ and the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ [2] :

$$g_n^{(s)}(x) = P_n^{(s+1,-s-n-1)}(2x-1).$$

In 2011, Lin et. al. [3] introduced the generalized Cesàro polynomials as follows:

$$g_n^{(s)}(\lambda, x) = \binom{s+n}{n} {}_2F_1\left[-n, \lambda; -s-n; x\right]. \tag{1.3}$$

It is noted that the special case $\lambda = 1$ of (1.3) reduces immediately to the Cesàro polynomials defined by (1.2).

The four Appell functions of two variables, denoted by $F_1, F_2, F_3$ and $F_4$, were generalized by Lauricella functions of $n$ variables which are denoted by $F_A^{(n)}, F_B^{(n)}, F_C^{(n)}$ and $F_D^{(n)}$ [2] and

$$F_A^{(2)} = F_2,\ F_B^{(2)} = F_3,\ F_C^{(2)} = F_4,\ F_D^{(2)} = F_1.$$

A further generalization of the familiar Kampé de Fériet hypergeometric function in two variables is due to Srivastava and Daoust [4] who defined the generalized Lauricella (or the Srivastava-Daoust ) function as follows:

$$F_{C:D^{(1)};\ldots;D^{(n)}}^{A:B^{(1)};\ldots;B^{(n)}}\left(\begin{array}{lll} \left[(a):\theta^{(1)},\ldots,\theta^{(n)}\right]: & \left[(b^{(1)}):\phi^{(1)}\right]; & \ldots; \quad \left[(b^{(n)}):\phi^{(n)}\right]; \\ \left[(c):\psi^{(1)},\ldots,\psi^{(n)}\right]: & \left[(d^{(1)}):\delta^{(1)}\right]; & \ldots; \quad \left[(d^{(n)}):\delta^{(n)}\right]; \end{array} z_1,\ldots,z_n\right)$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \Omega(m_1,\ldots,m_n)\frac{z_1^{m_1}}{m_1!}\ldots\frac{z_n^{m_n}}{m_n!},$$

where, for convenience,

$$\Omega(m_1,\ldots,m_n) := \frac{\prod_{j=1}^{A}(a_j)_{m_1\theta_j^{(1)}+\ldots+m_n\theta_j^{(n)}}}{\prod_{j=1}^{C}(c_j)_{m_1\psi_j^{(1)}+\ldots+m_n\psi_j^{(n)}}} \frac{\prod_{j=1}^{B^{(1)}}(b_j^{(1)})_{m_1\phi_j^{(1)}}}{\prod_{j=1}^{D^{(1)}}(d_j^{(1)})_{m_1\delta_j^{(1)}}} \ldots \frac{\prod_{j=1}^{B^{(n)}}(b_j^{(n)})_{m_n\phi_j^{(n)}}}{\prod_{j=1}^{D^{(n)}}(d_j^{(n)})_{m_n\delta_j^{(n)}}}$$

the coefficients

$$\theta_j^{(k)}\ (j=1,\ldots,A;\ k=1,\ldots,n),\ \text{and}\ \phi_j^{(k)}\ (j=1,\ldots,B^{(k)};\ k=1,\ldots,n),$$

$$\psi_j^{(k)}\ (j=1,\ldots,C;\ k=1,\ldots,n),\ \ \text{and}\ \delta_j^{(k)}\ (j=1,\ldots,D^{(k)};\ k=1,\ldots,n)$$

are real constants and $\left(b_{B^{(k)}}^{(k)}\right)$ abbreviates the array of $B^{(k)}$ parameters

$$b_j^{(k)}\ (j=1,\ldots,B^{(k)};\ k=1,\ldots,n)$$

with similar interpretations for other sets of parameters [1].

For a suitably bounded non-vanishing multiple sequence $\{\Omega(m_1;m_2,\ldots,m_s)\}_{m_1,m_2,\ldots,m_s\in\mathbb{N}_0}$ of real or complex parameters, let $\phi_n(u_1;u_2,\ldots,u_s)$ of $s$ (real or complex) variables

$u_1; u_2, ..., u_s$ defined by

$$\phi_n(u_1; u_2, ..., u_s) \quad : \quad = \sum_{m_1=0}^{n} \sum_{m_2,...,m_s=0}^{\infty} \frac{(-n)_{m_1}((b))_{m_1\phi}}{((d))_{m_1\delta}} \times \Omega\left(f(m_1, m_2, ..., m_s); m_2, ..., m_s\right) \frac{u_1^{m_1}}{m_1!} ... \frac{u_s^{m_s}}{m_s!} \tag{1.4}$$

where, for convenience,

$$((b))_{m_1\phi} = \prod_{j=1}^{B} (b_j)_{m_1\phi_j} \quad \text{and} \quad ((d))_{m_1\delta} = \prod_{j=1}^{D} (d_j)_{m_1\delta_j}.$$

The main object of this paper to study different properties of the generalized Cesàro polynomials. We give various families of multilinear and multilateral generating functions, miscellaneous properties and also some special cases for these polynomials. In addition, we derive a theorem giving certain families of bilateral generating functions for the generalized Cesàro polynomials and the generalized Lauricella functions.

## 2. Generating Functions

**Theorem 2.1.** *The generalized Cesàro polynomials $g_n^{(s)}(\lambda, x)$ have the following generating function:*

$$\sum_{n=0}^{\infty} g_n^{(s)}(\lambda, x) t^n = (1-t)^{-s-1}(1-xt)^{-\lambda}. \tag{2.1}$$

*Proof.* If we denote the right-hand side of (2.1) by $G$, then we obtain

$$\begin{aligned} G &= \sum_{n=0}^{\infty} (s+1)_n \frac{t^n}{n!} \sum_{m=0}^{\infty} (\lambda)_m \frac{(xt)^m}{m!} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(s+1)_n (\lambda)_m}{n!m!} x^m t^{n+m} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{(s+1)_{n-m} (\lambda)_m}{(n-m)!m!} x^m t^n. \end{aligned}$$

By using the formula

$$(s+1)_{n-m} = \frac{(s+n-m)!}{s!},$$

we get

$$\begin{aligned} G &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{(s+1)_{n-m} (\lambda)_m}{(n-m)!m!} x^m t^n \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{s+n}{n} (-n)_m \frac{1}{(-s-n)_m} \frac{(\lambda)_m}{m!} x^m t^n \\ &= \sum_{n=0}^{\infty} \binom{s+n}{n} {}_2F_1[-n, \lambda; -s-n; x] t^n \\ &= \sum_{n=0}^{\infty} g_n^{(s)}(\lambda, x) t^n. \end{aligned}$$

□

**Lemma 2.2.** *The following generating function holds true:*

$$\sum_{n=0}^{\infty}\binom{n+m}{n}g_{n+m}^{(s)}(\lambda,x)t^n=(1-t)^{-s-m-1}(1-xt)^{-\lambda}g_m^{(s)}\left(\lambda,\frac{x(1-t)}{1-xt}\right). \quad (2.2)$$

*Proof.* If we write $t+u$ instead of $t$ in (2.1) , we get

$$\begin{aligned}\sum_{n=0}^{\infty}g_n^{(s)}(\lambda,x)(t+u)^n &= (1-t-u)^{-s-1}(1-xt-xu)^{-\lambda}\\ \sum_{n=0}^{\infty}g_n^{(s)}(\lambda,x)\sum_{m=0}^{n}\binom{n}{m}t^{n-m}u^m &= (1-t)^{-s-1}\left(1-\frac{u}{1-t}\right)^{-s-1}(1-xt)^{-\lambda}\left(1-\frac{xu}{1-xt}\right)^{-\lambda}.\end{aligned}$$

Replacing $n$ by $n+m$ in last relation, we may write that

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\binom{n+m}{m}g_{n+m}^{(s)}(\lambda,x)t^nu^m=(1-t)^{-s-1}(1-xt)^{-\lambda}\sum_{m=0}^{\infty}(1-t)^{-m}g_m^{(s)}\left(\lambda,\frac{x(1-t)}{1-xt}\right)u^m$$

From the coefficients of $u^m$ on the both sides of the equality, one can get the desired result. □

**Lemma 2.3.** *The following addition formula holds for the generalized Cesàro polynomials $g_n^{(s)}(\lambda,x)$:*

$$g_n^{(s_1+s_2+1)}(\lambda_1+\lambda_2,x)=\sum_{k=0}^{n}g_{n-k}^{(s_1)}(\lambda_1,x)g_k^{(s_2)}(\lambda_2,x). \quad (2.3)$$

*Proof.* Replacing $s$ by $s_1+s_2+1$ and $\lambda$ by $\lambda_1+\lambda_2$ in (2.1), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}g_n^{(s_1+s_2+1)}(\lambda_1+\lambda_2,x)t^n &= (1-t)^{-s_1-s_2-2}(1-xt)^{-(\lambda_1+\lambda_2)}\\ &= (1-t)^{-s_1-1}(1-xt)^{-\lambda_1}(1-t)^{s_2-1}(1-xt)^{-\lambda_2}\\ &= \sum_{n=0}^{\infty}g_n^{(s_1)}(\lambda_1,x)t^n\sum_{k=0}^{\infty}g_k^{(s_2)}(\lambda_2,x)t^k\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}g_n^{(s_1)}(\lambda_1,x)g_k^{(s_2)}(\lambda_2,x)t^{n+k}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}g_{n-k}^{(s_1)}(\lambda_1,x)g_k^{(s_2)}(\lambda_2,x)t^n.\end{aligned}$$

From the coefficients of $t^n$ on the both sides of the last equality, one can get the desired result. □

## 3. Bilinear and Bilateral Generating Functions

In this section, we derive several families of bilinear and bilateral generating fuctions for the generalized Cesàro polynomials $g_n^{(s)}(\lambda,x)$ given by (1.3) .

We begin by stating the following theorem.

**Theorem 3.1.** *Corresponding to an identically non-vanishing function* $\Omega_{\mu}(y_1,...,y_r)$ *of* $r$ *complex variables* $y_1,...,y_r$ $(r\in\mathbb{N})$ *and of complex order* $\mu,\psi$, *let*

$$\Lambda_{\mu,\psi}(y_1,...,y_r;\zeta):=\sum_{k=0}^{\infty}a_k\Omega_{\mu+\psi k}(y_1,...,y_r)\zeta^k \qquad (a_k\neq 0) \tag{3.1}$$

*and*

$$\Theta_{n,p}^{\mu,\psi}(\lambda,x;y_1,...,y_r;\xi):=\sum_{k=0}^{[n/p]}a_k g_{n-pk}^{(s)}(\lambda,x)\,\Omega_{\mu+\psi k}(y_1,...,y_r)\xi^k.$$

*Then, for* $p\in\mathbb{N}$, *we have*

$$\sum_{n=0}^{\infty}\Theta_{n,p}^{\mu,\psi}\left(\lambda,x;y_1,...,y_r;\frac{\eta}{t^p}\right)t^n=(1-t)^{-s-1}(1-xt)^{-\lambda}\Lambda_{\mu,\psi}(y_1,...,y_r;\eta) \tag{3.2}$$

*provided that each member of (3.2) exists.*

*Proof.* For convenience, let $S$ denote the first member of the assertion (3.2) of Theorem 3.1. Then,

$$S=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}a_k g_{n-pk}^{(s)}(\lambda,x)\,\Omega_{\mu+\psi k}(y_1,...,y_r)\eta^k t^{n-pk}.$$

Replacing $n$ by $n+pk$, we may write that

$$\begin{aligned}S &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}a_k\,g_n^{(s)}(\lambda,x)\,\Omega_{\mu+\psi k}(y_1,...,y_r)\eta^k t^n\\ &= \sum_{n=0}^{\infty}g_n^{(s)}(\lambda,x)\,t^n\sum_{k=0}^{\infty}a_k\Omega_{\mu+\psi k}(y_1,...,y_r)\eta^k\\ &= (1-t)^{-s-1}(1-xt)^{-\lambda}\Lambda_{\mu,\psi}(y_1,...,y_r;\eta)\end{aligned}$$

which completes the proof. □

By using a similar idea, we also get the next result immediately.

**Theorem 3.2.** *Corresponding to an identically non-vanishing function* $\Omega_{\mu}(y_1,...,y_r)$ *of* $r$ *complex variables* $y_1,...,y_r$ $(r\in\mathbb{N})$ *and of complex order* $\mu,\psi$, *let*

$$\Lambda_{\mu,\psi}^{n,p}(\lambda_1+\lambda_2,x;y_1,...,y_r;z):=\sum_{k=0}^{[n/p]}a_k g_{n-pk}^{(s_1+s_2+1)}(\lambda_1+\lambda_2,x)\Omega_{\mu+\psi k}(y_1,...,y_r)z^k,$$

*where* $a_k\neq 0\,,n,p\in\mathbb{N}$ *and the notation* $[n/p]$ *means the greatest integer less than or equal* $n/p$.

*Then, for* $p\in\mathbb{N}$ *, we have*

$$\sum_{k=0}^{n}\sum_{l=0}^{[k/p]}a_l g_{n-k}^{(s_1)}(\lambda_1,x)g_{k-pl}^{(s_2)}(\lambda_2,x)\Omega_{\mu+\psi l}(y_1,...,y_r)z^l=\Lambda_{\mu,\psi}^{n,p}(\lambda_1+\lambda_2,x;y_1,...,y_r;z) \tag{3.3}$$

*provided that each member of (3.3) exists.*

*Proof.* For convenince, let $T$ denote the first member of the assertion (3.3). Then, upon substituting for the polynomials $g_n^{(s_1+s_2+1)}(\lambda_1+\lambda_2,x)$ from the (2.3) into the left-hand side of (3.3), we obtain

$$\begin{aligned} T &= \sum_{l=0}^{[n/p]}\sum_{k=0}^{n-pl} a_l g_{n-k-pl}^{(s_1)}(\lambda_1,x) g_k^{(s_2)}(\lambda_2,x)\Omega_{\mu+\psi l}(y_1,...,y_r)z^l \\ &= \sum_{l=0}^{[n/p]} a_l \left(\sum_{k=0}^{n-pl} g_{n-k-pl}^{(s_1)}(\lambda_1,x) g_k^{(s_2)}(\lambda_2,x)\right)\Omega_{\mu+\psi l}(y_1,...,y_r)z^l \\ &= \sum_{l=0}^{[n/p]} a_l g_{n-pl}^{(s_1+s_2+1)}(\lambda_1+\lambda_2,x)\Omega_{\mu+\psi l}(y_1,...,y_r)z^l \\ &= \Lambda_{\mu,\psi}^{n,p}(\lambda_1+\lambda_2,x;y_1,...,y_r;z). \end{aligned}$$

□

**Theorem 3.3.** *Corresponding to an identically non-vanishing function* $\Omega_\mu(y_1,...,y_r)$ *of* $r$ *complex variables* $y_1,...,y_r$ $(r\in\mathbb{N})$ *and of complex order* $\mu$*, let*

$$\Lambda_{\mu,p,q}[\lambda,x;y_1,...,y_r;t] := \sum_{n=0}^{\infty} a_n g_{m+qn}^{(s)}(\lambda,x)\Omega_{\mu+pn}(y_1,...,y_r)t^n$$

*where* $a_n\neq 0$ *and*

$$\theta_{n,p,q}(y_1,...,y_r;z) := \sum_{k=0}^{[n/q]}\binom{m+n}{n-qk} a_k \Omega_{\mu+pk}(y_1,...,y_r)z^k.$$

*Then, for* $p\in\mathbb{N}$*; we have*

$$\begin{aligned} &\sum_{n=0}^{\infty} g_{m+n}^{(s)}(\lambda,x)\theta_{n,p,q}(y_1,...,y_r;z)t^n \\ &= (1-t)^{-s-m-1}(1-xt)^{-\lambda}\Lambda_{\mu,p,q}\left(\lambda,\frac{x(1-t)}{1-xt};y_1,...,y_r;z(\frac{t}{1-t})^q\right) \quad (3.4) \end{aligned}$$

*provided that each member of (3.4) exists.*

*Proof.* For convenience, let $T$ denote the first member of the assertion (3.4) of Theorem 3.3. Then,

$$T=\sum_{n=0}^{\infty} g_{m+n}^{(s)}(\lambda,x)\sum_{k=0}^{[n/q]}\binom{m+n}{n-qk} a_k\Omega_{\mu+pk}(y_1,...,y_r)z^k t^n.$$

Replacing $n$ by $n+qk$ and then using (2.2), we may write that

$$\begin{aligned}
T &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\binom{m+n+qk}{n} g_{m+n+qk}^{(s)}(\lambda,x)a_k\,\Omega_{\mu+pk}(y_1,...,y_r)z^k t^{n+qk} \\
&= \sum_{k=0}^{\infty}\left(\sum_{n=0}^{\infty}\binom{m+n+qk}{n} g_{m+n+qk}^{(s)}(\lambda,x)t^n\right) a_k\Omega_{\mu+pk}(y_1,...,y_r)(zt^q)^k \\
&= \sum_{k=0}^{\infty}(1-t)^{-s-m-qk-1}(1-xt)^{-\lambda} g_{m+qk}^{(s)}\left(\lambda,\frac{x(1-t)}{1-xt}\right) a_k\Omega_{\mu+pk}(y_1,...,y_r)(zt^q)^k \\
&= (1-t)^{-s-m-1}(1-xt)^{-\lambda}\sum_{k=0}^{\infty}(1-t)^{-qk} g_{m+qk}^{(s)}\left(\lambda,\frac{x(1-t)}{1-xt}\right) a_k\Omega_{\mu+pk}(y_1,...,y_r)(zt^q)^k \\
&= (1-t)^{-s-m-1}(1-xt)^{-\lambda}\Lambda_{\mu,p,q}\left(\lambda,\frac{x(1-t)}{1-xt};y_1,...,y_r;z(\frac{t}{1-t})^q\right)
\end{aligned}$$

which completes the proof. □

## 4. Special Cases

When the multivariable function $\Omega_{\mu+\psi k}(y_1,...,y_r)$, $k\in\mathbb{N}_0$, $r\in\mathbb{N}$, is expressed in terms of simpler functions of one and more variables, then we can give further applications of the above theorems. We first set

$$\Omega_{\mu+\psi k}(y_1,...,y_r) = \Phi_{\mu+\psi k}^{(\alpha)}(y_1,...,y_r)$$

in Theorem 3.1 , where the multivariable polynomials $\Phi_{\mu+\psi k}^{(\alpha)}(x_1,...,x_r)$ [5], generated by

$$\begin{gathered}
(1-x_1t)^{-\alpha}e^{(x_2+...+x_r)\,t} = \sum_{n=0}^{\infty}\Phi_n^{(\alpha)}(x_1,...,x_r)\,t^n \\
\left(\alpha\in\mathbb{C}\;;\;|t|<\left\{|x_1|^{-1}\right\}\right).
\end{gathered} \tag{4.1}$$

Thus, we have the following result which provides a class of bilateral generating functions for the multivariable polynomials $\Phi_{\mu+\psi k}^{(\alpha)}(x_1,...,x_r)$ and the generalized Cesàro polynomials.

**Corollary 4.1.** *If*

$$\begin{aligned}
\Lambda_{\mu,\psi}(y_1,...,y_r;w) &: = \sum_{k=0}^{\infty}a_k\Phi_{\mu+\psi k}^{(\alpha)}(y_1,...,y_r)w^k \\
&\quad (a_k\neq 0\,,\ \mu,\psi\in\mathbb{C})\,,
\end{aligned}$$

*then, we have*

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}a_k g_{n-pk}^{(s)}(\lambda,x)\,\Phi_{\mu+\psi k}^{(\alpha)}(y_1,...,y_r)\frac{w^k}{t^{pk}}t^n \\
&= (1-t)^{-s-1}(1-xt)^{-\lambda}\Lambda_{\mu,\psi}(y_1,...,y_r;w)
\end{aligned} \tag{4.2}$$

*provided that each member of (4.2) exists.*

**Remark 4.1.** *Using the generating relation (4.1) for the multivariable polynomials* $\Phi_n^{(\alpha)}(x_1,...,x_r)$ *and getting* $a_k=1$, $\mu=0$, $\psi=1$ *in Corollary 4.1, we find that*

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} g_{n-pk}^{(s)}(\lambda,x)\,\Phi_k^{(\alpha)}(y_1,...,y_r)w^k t^{n-pk}\\ = (1-t)^{-s-1}(1-xt)^{-\lambda}(1-y_1w)^{-\alpha}e^{(y_2+...+y_r)\,w},\\ \left(|w|<\left\{|y_1|^{-1}\right\}\right).\end{aligned}$$

If we set $r=1$ and

$$\Omega_{\mu+\psi k}(y_1)=g_{\mu+\psi k}^{(s_3)}(\lambda_3,y)$$

in Theorem 3.2, we have the following bilinear generating functions for the generalized Cesàro polynomials.

**Corollary 4.2.** *If*

$$\begin{aligned}\Lambda_{\mu,\psi}^{n,p}(\lambda_1+\lambda_2,x;\lambda_3,y;z) &:= \sum_{k=0}^{[n/p]} a_k g_{n-pk}^{(s_1+s_2+1)}(\lambda_1+\lambda_2,x)g_{\mu+\psi k}^{(s_3)}(\lambda_3,y)z^k\\ (a_k &\neq 0\,,\,\mu,\psi\in\mathbb{C})\end{aligned}$$

*then, we have*

$$\sum_{k=0}^{n}\sum_{l=0}^{[k/p]} a_l g_{n-k}^{(s_1)}(\lambda_1,x)g_{k-pl}^{(s_2)}(\lambda_2,x)g_{\mu+\psi l}^{(s_3)}(\lambda_3,y)z^l=\Lambda_{\mu,\psi}^{n,p}(\lambda_1+\lambda_2,x;\lambda_3,y;z) \quad (4.3)$$

*provided that each member of (4.3) exists.*

**Remark 4.2.** *Using (4.3) and taking* $a_l=1$, $\mu=0$, $\psi=1$, $x=y$ , $p=1$ *in Corollary 4.2, we have*

$$\sum_{n=0}^{\infty}\sum_{l=0}^{n} g_{n-l}^{(s_1+s_2+1)}(\lambda_1+\lambda_2,x)g_l^{(s_3)}(\lambda_3,x)z^n=(1-z)^{-(s_1+s_2+s_3)-3}(1-xz)^{-(\lambda_1+\lambda_2+\lambda_3)}.$$

Finally, choosing

$$s=r \text{ and } \Omega_{\mu+\psi k}(y_1,...,y_r)=u_{\mu+\psi k}^{(\alpha_1,...,\alpha_r)}(y_1,...,y_r)$$

in Theorem 3.3, where the Erkus-Srivastava polynomials $u_n^{(\alpha_1,...,\alpha_r)}(y_1,...,y_r)$ is generated by [7]

$$\begin{aligned}\prod_{j=1}^{r}\left\{(1-x_jt^{m_j})^{-\alpha_j}\right\} &= \sum_{n=0}^{\infty} u_n^{(\alpha_1,...,\alpha_r)}(x_1,...,x_r)\,t^n,\\ (\alpha_j &\in \mathbb{C}\ \ (j=1,...,r)\ ;\ |t|<\min\left\{|x_1|^{-1/m_1},...,|x_r|^{-1/m_r}\right\}\end{aligned}$$

we get a family of the bilateral generating functions for the Erkus-Srivastava polynomials and the generalized Cesàro polynomials as follows:

**Corollary 4.3.** *If*

$$\begin{aligned}\Lambda_{m,q}[\lambda,x;y_1,...,y_r;z] &:= \sum_{k=0}^{\infty} a_k g_{m+qk}^{(s)}(\lambda,x)u_{\mu+\psi k}^{(\alpha_1,...,\alpha_r)}(y_1,...,y_r)z^k\\ &(a_k\neq 0,\ m\in\mathbb{N}_0,k\neq 0)\end{aligned}$$

*and*

$$\theta_{n,m,q}^{p,\mu}(y_1,...,y_r;z) := \sum_{k=0}^{[n/q]} \binom{m+n}{n-qk} a_k u_{\mu+\psi k}^{(\alpha_1,...,\alpha_r)}(y_1,...,y_r) z^k$$

*where* $n, p \in \mathbb{N}$*, then we have*

$$\sum_{n=0}^{\infty} g_{m+n}^{(s)}(\lambda,x)\theta_{n,m,q}^{p,\mu}(y_1,...,y_r;z)t^n \tag{4.4}$$

$$= (1-t)^{-s-m-1}(1-xt)^{-\lambda}\Lambda_{m,q}\left[\lambda, \frac{x(1-t)}{1-xt}; y_1,...,y_r; z\left(\frac{t}{1-t}\right)^q\right]$$

*provided that each member of (4.4) exists.*

Furthermore, for every suitable choice of the coefficients $a_k$ $(k \in \mathbb{N}_0)$, if the multivariable functions $\Omega_{\mu+\psi k}(y_1,...,y_r)$, $r \in \mathbb{N}$, are expressed as an appropriate product of several simpler functions, the assertions of Theorem 3.1, Theorem 3.2, Theorem 3.3 can be applied in order to derive various families of multilinear and multilateral generating functions for the family of the generalized Cesáro polynomials given explicitly by (1.3).

## 5. Miscellaneous Properties

In this section we give some properties for the generalized Cesàro polynomials $g_n^{(s)}(\lambda,x)$ given by (1.3).

**Theorem 5.1.** *The generalized Cesàro polynomials* $g_n^{(s)}(\lambda,x)$ *have the following integral representation:*

$$g_n^{(s)}(\lambda,x) = \frac{1}{\Gamma(s+1)\Gamma(\lambda)}\int_0^\infty\int_0^\infty e^{-(u_1+u_2)}\frac{(u_1+u_2x)^n}{n!}u_1^s u_2^{\lambda-1}du_1du_2. \tag{5.1}$$

*Proof.* If we use the identity

$$a^{-v} = \frac{1}{\Gamma(v)}\int_0^\infty e^{-at}t^{v-1}dt, \qquad (Re(v)>0)$$

on the right-hand side of the generating function (2.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} g_n^{(s)}(\lambda,x)t^n &= \frac{1}{\Gamma(s+1)}\int_0^\infty e^{-(1-t)u_1}u_1^s du_1 \frac{1}{\Gamma(\lambda)}\int_0^\infty e^{-(1-xt)u_2}u_2^{\lambda-1}du_2\\ &= \frac{1}{\Gamma(s+1)\Gamma(\lambda)}\int_0^\infty\int_0^\infty e^{-(u_1+u_2)}e^{(u_1+u_2x)t}u_1^s u_2^{\lambda-1}du_1du_2\\ &= \frac{1}{\Gamma(s+1)\Gamma(\lambda)}\int_0^\infty\int_0^\infty e^{-(u_1+u_2)}\sum_{n=0}^{\infty}\frac{(u_1+u_2x)^n}{n!}t^n u_1^s u_2^{\lambda-1}du_1du_2.\end{aligned}$$

From the coefficents of $t^n$ on the both sides of the last equality, one can get the desired result. □

We now discuss some miscellaneous recurrence relations of the generalized Cesàro polynomials. By differentiating each member of the generating function relation (2.1) with respect to $x$ and using

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n-k), \tag{5.2}$$

we arrive at the following (differential) recurrence relation for the generalized Cesàro polynomials:

$$\frac{\partial}{\partial x}g_n^{(s)}(\lambda,x)-x\frac{\partial}{\partial x}g_{n-1}^{(s)}(\lambda,x)=\lambda g_{n-1}^{(s)}(\lambda,x),\qquad n\geq 1. \tag{5.3}$$

On the other hand, by differentiating each member of the generating function relation (2.1) with respect to $x$, we have

$$\frac{\partial}{\partial x}g_n^{(s)}(\lambda,x)=\lambda\sum_{k=0}^{n-1}x^{n-k-1}g_k^{(s)}(\lambda,x). \tag{5.4}$$

If we consider (5.3) and (5.4), we can easily get the following recurrence relation for the generalized Cesàro polynomials :

$$\sum_{k=0}^{n-1}x^{n-k-1}g_k^{(s)}(\lambda,x)-\sum_{k=0}^{n-2}x^{n-k-1}g_k^{(s)}(\lambda,x)=g_{n-1}^{(s)}(\lambda,x).$$

Besides, by differentiating each member of the generating function relation (2.1) with respect to $t$, we have the following another recurrence relation for these polynomials:

$$(n+1)g_{n+1}^{(s)}(\lambda,x)=\sum_{m=0}^{n}\left[(s+1)\,g_{n-m}^{(s)}(\lambda,x)+\lambda x^{m+1}g_{n-m}^{(s)}(\lambda,x)\right].$$

## 6. The Generalized Lauricella Functions

In the present section, we derive various families of bilateral generating functions for the generalized Cesàro polynomials and the generalized Lauricella (or the Srivastava-Daoust) functions.

**Theorem 6.1.** *The following bilateral generating function holds true:*

$$\begin{aligned}&\sum_{n=0}^{\infty}g_n^{(s)}(\lambda,x)\phi_n(u_1;u_2,...,u_k)t^n\\ =\;&(1-t)^{-s-1}(1-xt)^{-\lambda}\sum_{m_1,p,m_2,...,m_k=0}^{\infty}\frac{((b))_{(m_1+p)\phi}(s+1)_{m_1}(\lambda)_p}{((d))_{(m_1+p)\delta}}\\ &\times\Omega(f((m_1+p),m_2,...,m_k);m_2,...,m_k)\frac{(\frac{u_1t}{t-1})^{m_1}}{m_1!}\frac{\left(\frac{u_1xt}{xt-1}\right)^p}{p!}\frac{u_2^{m_2}}{m_2!}...\frac{u_k^{m_k}}{m_k!},\end{aligned}$$

*where $\phi_n(u_1;u_2,...,u_k)$ is given by (1.4).*

*Proof.* By using the relationship (2.2), it is easily observed that

$$\begin{aligned}
&\sum_{n=0}^{\infty} g_n^{(s)}(\lambda,x)\phi_n(u_1;u_2,...,u_k)t^n \\
&= \sum_{n=0}^{\infty} g_n^{(s)}(\lambda,x) \sum_{m_1=0}^{n} \sum_{m_2,...,m_k=0}^{\infty} \frac{(-n)_{m_1}((b))_{m_1\phi}}{((d))_{m_1\delta}} \\
&\qquad \times \Omega\left(f(m_1,m_2,...,m_k);m_2,...,m_k\right) \frac{u_1^{m_1}}{m_1!}...\frac{u_k^{m_k}}{m_k!}t^n \\
&= \sum_{m_1,m_2,...,m_k=0}^{\infty} \frac{((b))_{m_1\phi}}{((d))_{m_1\delta}} \Omega\left(f(m_1,m_2,...,m_k);m_2,...,m_k\right) \\
&\qquad \times (-u_1t)^{m_1}\frac{u_2^{m_2}}{m_2!}...\frac{u_s^{m_k}}{m_k!}\ (1-t)^{-s-m_1-1}(1-xt)^{-\lambda} g_{m_1}^{(s)}\left(\lambda,\frac{x(1-t)}{1-xt}\right) \\
&= (1-t)^{-s-1}(1-xt)^{-\lambda} \sum_{m_1,m_2,...,m_k=0}^{\infty} \frac{((b))_{m_1\phi}}{((d))_{m_1\delta}} \Omega\left(f(m_1,m_2,...,m_k);m_2,...,m_k\right) \\
&\qquad \times (-u_1t)^{m_1}\frac{u_2^{m_2}}{m_2!}...\frac{u_k^{m_k}}{m_k!}(1-t)^{-m_1}\binom{s+m_1}{m_1}\sum_{p=0}^{m_1}\frac{(-m_1)_p(\lambda)_p}{(-s-m_1)_p p!}\left(\frac{x(1-t)}{1-xt}\right)^p \\
&= (1-t)^{-s-1}(1-xt)^{-\lambda} \\
&\quad \times \sum_{m_1,p,m_2,...,m_k=0}^{\infty} \frac{((b))_{(m_1+p)\phi}}{((d))_{(m_1+p)\delta}} \Omega\left(f((m_1+p),m_2,...,m_k);m_2,...,m_k\right)(s+1)_{m_1}(\lambda)_p \\
&\qquad \frac{(\frac{u_1t}{t-1})^{m_1}}{m_1!}\frac{\left(\frac{u_1xt}{xt-1}\right)^p}{p!}\frac{u_2^{m_2}}{m_2!}...\frac{u_k^{m_k}}{m_k!}.
\end{aligned}$$

□

By appropriately choosing the multiple sequence $\Omega(m_1,m_2,...,m_s)$ in Theorem 6.1, we obtain several interesting results as follows which give bilateral generating functions for the generalized Cesàro polynomials and the generalized Lauricella (or the Srivastava-Daoust) functions.

**I**.By letting

$$\begin{aligned}
&\Omega\left(f(m_1,m_2,...,m_k);m_2,...,m_k\right) \\
&= \frac{\prod_{j=1}^{A}(a_j)_{m_1\theta_j^{(1)}+...+m_k\theta_j^{(k)}}}{\prod_{j=1}^{E}(c_j)_{m_1\psi_j^{(1)}+...+m_k\psi_j^{(k)}}} \frac{\prod_{j=1}^{B^{(2)}}(b_j^{(2)})_{m_2\phi_j^{(2)}}}{\prod_{j=1}^{D^{(2)}}(d_j^{(2)})_{m_2\delta_j^{(2)}}}...\frac{\prod_{j=1}^{B^{(k)}}(b_j^{(k)})_{m_k\phi_j^{(k)}}}{\prod_{j=1}^{D^{(k)}}(d_j^{(k)})_{m_k\delta_j^{(k)}}}
\end{aligned}$$

in Theorem 6.1,we get the following result.

**Corollary 6.2.** *The following bilateral generating function holds true:*

$$\sum_{n=0}^{\infty} g_n^{(s)}(\lambda,x) F_{E:D;D^{(2)};...;D^{(k)}}^{A:B+1;B^{(2)};...;B^{(k)}}$$
$$\left( \begin{array}{lllll} \left[(a):\theta^{(1)},...,\theta^{(k)}\right]: & [-n:1], & [(b):\phi]; & [(b^{(2)}):\phi^{(2)}]; & ...; \quad [(b^{(k)}):\phi^{(k)}]; \\ \left[(c):\psi^{(1)},...,\psi^{(k)}\right]: & & [(d):\delta]; & [(d^{(2)}):\delta^{(2)}]; & ...; \quad [(d^{(k)}):\delta^{(k)}]; \\ & & & & u_1,u_2,...,u_k \end{array} \right) t^n$$

$$= (1-t)^{-s-1}(1-xt)^{-\lambda} F_{E+D:0;0;D^{(2)};...;D^{(k)}}^{A+B:1;1;B^{(2)};...;B^{(k)}}$$
$$\left( \begin{array}{lllll} \left[(e):\varphi^{(1)},...,\varphi^{(k+1)}\right]: & [s+1:1], & [\lambda:1]; & [(b^{(2)}):\phi^{(2)}]; & ...; \quad [(b^{(k)}):\phi^{(k)}]; \\ \left[(f):\xi^{(1)},...,\xi^{(k+1)}\right]: & - & -; & [(d^{(2)}):\delta^{(2)}]; & ...; \quad [(d^{(k)}):\delta^{(k)}]; \\ & & & & \left(\frac{u_1 t}{t-1}\right),\left(\frac{u_1 xt}{xt-1}\right),u_2,...,u_k \end{array} \right)$$

*where the coefficients* $e_j, f_j, \varphi_j^{(k)}$ *and* $\xi_j^{(k)}$ *are given by*

$$e_j = \begin{cases} a_j & (1 \le j \le A) \\ b_{j-A} & (A < j \le A+B), \end{cases}$$

$$f_j = \begin{cases} c_j & (1 \le j \le E) \\ d_{j-E} & (E < j \le E+D), \end{cases}$$

$$\varphi_j^{(r)} = \begin{cases} \theta_j^{(1)} & (1 \le j \le A;\ 1 \le r \le 2) \\ \theta_j^{(r-1)} & (1 \le j \le A;\ 2 < r \le k+1) \\ \phi_{j-A} & (A < j \le A+B;\ 1 \le r \le 2) \\ 0 & (A < j \le A+B;\ 2 < r \le k+1) \end{cases}$$

*and*

$$\xi_j^{(r)} = \begin{cases} \psi_j^{(1)} & (1 \le j \le E;\ 1 \le r \le 2) \\ \psi_j^{(r-1)} & (1 \le j \le E;\ 2 < r \le k+1) \\ \delta_{j-E} & (E < j \le E+D;\ 1 \le r \le 2) \\ 0 & (E < j \le E+D;\ 2 < r \le k+1) \end{cases}$$

*respectively.*

***II****.Upon setting*

$$\Omega\left(f(m_1,m_2,...,m_k);m_2,...,m_k\right) = \frac{(a)_{m_1+...+m_k}(b_2)_{m_2}...(b_k)_{m_k}}{(c_1)_{m_1}...(c_k)_{m_k}}$$

*and*

$$\phi = \delta = 0 \quad (\textit{that is, } \phi_1 = ... = \phi_B = \delta_1 = ... = \delta_D = 0)$$

*in Theorem 6.1,we obtain the following result.*

**Corollary 6.3.** *The following bilateral generating function holds true:*

$$\begin{aligned}
&\sum_{n=0}^{\infty} g_n^{(s)}(\lambda,x) F_A^{(k)}\left[a,-n,b_2,...,b_k;c_1,...,c_k;u_1,u_2,...,u_k\right]t^n \\
&= (1-t)^{-s-1}(1-xt)^{-\lambda} F_{1:0;0;1;...;1}^{1:1;1;1;...;1} \\
&\left(\begin{array}{cccccc}
[(a):1,...,1]: & [s+1:1]; & [\lambda:1]; & [b_2:1]; & ...; & [b_k:1]; \\
\left[(c_1):\psi^{(1)},...,\psi^{(k+1)}\right]: & -; & -; & [c_2:1]; & ...; & [c_k:1]; \\
& & & & & (\frac{u_1 t}{t-1}),(\frac{u_1 xt}{xt-1}),u_2,...,u_k
\end{array}\right),
\end{aligned}$$

*where the coefficients $\psi^{(\eta)}$ are given by*

$$\psi^{(\eta)} = \begin{cases} 1, & (1 \leq \eta \leq 2) \\ 0, & (2 < \eta \leq k+1) \end{cases}.$$

***III.*** *If we put*

$$\Omega\left(f(m_1,m_2,...,m_k);m_2,...,m_k\right) = \frac{(a_1^{(1)})_{m_2}...(a_1^{(k-1)})_{m_k}(a_2^{(1)})_{m_2}...(a_2^{(k-1)})_{m_k}}{(c)_{m_1+\cdots+m_k}}$$

*and*

$$B = 1, b_1 = b, \phi_1 = 1 \text{ and } \delta = 0$$

*in Theorem 6.1,we obtain Corollary 6.4 below.*

**Corollary 6.4.** *The following bilateral generating function holds true:*

$$\begin{aligned}
&\sum_{n=0}^{\infty} g_n^{(s)}(\lambda,x) F_B^{(k)}\left[-n,a_1^{(1)},...,a_1^{(k-1)},b,a_2^{(1)},...,a_2^{(k-1)};c;u_1,u_2,...,u_k\right]t^n \\
&= (1-t)^{-s-1}(1-xt)^{-\lambda} F_{1:0;0;0;...;0}^{1:1;1;2;...;2} \\
&\left(\begin{array}{cccccc}
\left[(b):\theta^{(1)},...,\theta^{(k+1)}\right]: & [s+1:1]; & [\lambda:1]; & \left[a^{(1)}:1\right]; & ...; & \left[a^{(k-1)}:1\right]; \\
[(c):1,...,1]: & -; & -; & -; & ...; & -; \\
& & & & & (\frac{u_1 t}{t-1}),(\frac{u_1 xt}{xt-1}),u_2,...,u_k
\end{array}\right),
\end{aligned}$$

*where the coefficients $\theta^{(\eta)}$ are given by*

$$\theta^{(\eta)} = \begin{cases} 1, & (1 \leq \eta \leq 2) \\ 0, & (2 < \eta \leq k+1) \end{cases}.$$

***IV****.By letting*

$$\Omega\left(f(m_1,m_2,...,m_k);m_2,...,m_k\right) = \frac{(a)_{m_1+...+m_k}(b_2)_{m_2}...(b_s)_{m_k}}{(c)_{m_1+...+m_k}}$$

*and*

$$\phi = \delta = 0,$$

*in Theorem 6.1,we obtain the following result.*

**Corollary 6.5.** *The following bilateral generating function holds true:*

$$\sum_{n=0}^{\infty} g_n^{(s)}(\lambda, x) F_D^{(k)} \left[a, -n, b_2, ..., b_k; c; u_1, u_2, ..., u_k\right] t^n$$

$$= (1-t)^{-s-1}(1-xt)^{-\lambda} F_D^{(k+1)} \left[a, s+1, \lambda, b_2, ..., b_k; c; \left(\frac{u_1 t}{t-1}\right), \left(\frac{u_1 xt}{xt-1}\right), u_2, ..., u_k\right].$$

## References


[1] Liu, S.-J. , Lin, S.-D., Srivastava, H.M. and Wong, M.-M. , Bilateral generating functions for the Erkus-Srivastava polynomials and the generalized Lauricella functions, App. Mathematcis and Comp., 218 (2012) 7685-7693.

[2] Srivastava, H. M. and Manocha, H. L. A Treatise on Generating Functions, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, 1984.

[3] Lin, Shy-Der,Liu, S.-J., Lu, H.-C. and Srivastava, H. M. Integral representations for the generalized Bedient polynomials and the generalized Cesáro polynomials,Appl. Mathematics and Computation 218 (2011), 1330-1341.

[4] Srivastava, H. M. and Daoust, M.C. Certain generalized Neumann expansions associated with the Kampé de Fériet function, Nederl. akad. Westensch. Indag. Math. 31 (1969), pp. 449-457.

[5] Ozmen, N., Erkus-Duman, E., Some results for a family of multivariable polynomials, AIP Conference Proceedings, 1558, 1124 (2013).

[6] Agarwal, A.K. , and Manocha, H.L., On Some new generating functions, Indian J. Pure Appl. Math., 13(11):1369-1375, November 1982

[7] Erkus, E. and Srivastava, H. M, A unified presentation of some families of multivariable polynomials, Integral Transform Spec. Funct. 17 (2006), 267-273



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# On uniqueness of meromorphic functions sharing one small function

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Abstract: In this paper, we prove the uniqueness theorems of meromorphic functions concerning shared small functions. Let $f$ and $g$ be meromorphic functions with finite order. We prove $f \equiv tg$ under some conditions, when $f^n f^{(k)}$ and $g^n g^{(k)}$ share a small function $a(z)$. We improve the results of Wang and Gao in [7].

*Keywords*: meromorphic function; shared value; uniqueness.

## 1. Introduction and results

In this article, a meromorphic function means meromorphic in the whole complex plane. We shall use the standard notations in the Nevanlinna value distribution theorem of meromorphic functions such as $T(r,f)$, $N(r,f)$, $m(r,f)$, $\overline{N}(r,f)$, etc. (see [1],[2]) For any nonconstant meromorphic function $f$, we denote by $S(r,f)$ any quantity satisfying

$$\lim_{r\to\infty}\frac{S(r,f)}{T(r,f)}=0,$$

possibly outside of a finite linear measure in $R^+$.

Let $f$ and $g$ be two nonconstant meromorphic functions. A meromorphic function $a(z)$ is called a small function with respect to $f$ provided that $T(r,a)=S(r,f)$. Note that the set of all small functions of $f$ is a field. Let $a(z)$ be a small function with respect to $f$ and $g$. We say that $f$ and $g$ share $a(z)$ CM(IM) provided that $f-a(z)$ and $g-a(z)$ have same zeros counting multiplicities (ignoring multiplicities).

Throughout this paper, we need the following definitions.

$$\Theta(b,f)=1-\limsup_{r\to\infty}\frac{\overline{N}(r,b;f)}{T(r,f)},$$

where $b$ is a value in the extended complex plane.

The order of growth $\rho$ of $f$:

$$\rho(f)=\limsup_{r\to\infty}\frac{\log^+ T(r,f)}{\log r}.$$

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The exponent of convergence of $a-$value of $f$:

$$\begin{aligned}\lambda(a,f) &= \inf\left\{\tau>0:\sum_{n=1}^{\infty}\frac{1}{|a_n|^{\tau}}<\infty\right\}\\ &= \limsup_{r\to\infty}\frac{\log^{+}n(r,\frac{1}{f-a})}{\log r}\\ &= \limsup_{r\to\infty}\frac{\log^{+}N(r,\frac{1}{f-a})}{\log r}.\end{aligned}$$

where $\{a_n\}$ is the sequence of $a$-value of $f(z)$. If $a=0$, we denote $\lambda(0,f)$ by $\lambda(f)$. If $a=\infty$, we denote $\lambda(\infty,f)$ by $\lambda(\frac{1}{f})$.

In 1920's, Nevanlinna [3] proved the famous four-valued theorem, which is an important result about uniqueness of meromorphic functions. Then many results about meromorphic functions that share more than or equal to two values have been obtained (see [4]). In 1997, Yang and Hua [5] studied meromorphic functions sharing one value.

**Theorem A.** (see [5]). Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions and let $n\geq 11$ be a positive integer. If $f^nf'$ and $g^ng'$ share 1 CM, then either $f(z)=c_1e^{cz}$, $g(z)=c_2e^{-cz}$, where $c_1$, $c_2$, and $c$ are three constants satisfying $(c_1c_2)^{n+1}c^2=-1$, or $f(z)\equiv tg(z)$ for a constant such that $t^{n+1}=1$.

In 2002, Fang and Qiu [6] investigated meromorphic functions sharing fixed point.

**Theorem B.** (see [6]). Let $f(z)$ and $g(z)$ be two nonconstant meromorphic(entire) functions and let $n\geq 11(n\geq 6)$ be a positive integer. If $f^nf'$ and $g^ng'$ share $z$ CM, then either $f(z)=c_1e^{cz^2}$, $g(z)=c_2e^{-cz^2}$, where $c_1$, $c_2$, and $c$ are three constants satisfying $4(c_1c_2)^{n+1}c^2=-1$, or $f(z)\equiv tg(z)$ for a constant such that $t^{n+1}=1$.

In 2007, Wang and Gao [7] extended Theorem B, in which they studied meromorphic functions sharing a small function.

**Theorem C.** (see [7]). Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions, and let $a(z)(\not\equiv 0)$ be a common small function with respect to them, and let $n\geq 11$ be a positive integer. If $f^nf'$ and $g^ng'$ share $a(z)$ CM, then either $f^nf'g^ng'\equiv a(z)^2$, or $f(z)\equiv tg(z)$ for a constant such that $t^{n+1}=1$.

In this paper we replace $f'$ by $f^{(k)}$ in Theorem C and obtain the following results which improve Theorem C.

**Theorem 1.** Let $f$ and $g$ be two transcendental meromorphic functions with finite order and $\max\{\lambda(f),\lambda(\frac{1}{f})\}<1$. Let $k$, $n$ be two positive integers and $a(z)(\not\equiv 0)$ be a small function of $f$ and $g$. If $f^nf^{(k)}$ and $g^ng^{(k)}$ share $a(z)$ CM, $f$ and $g$ share 0 CM, and $n>k+4+\sqrt{k^2+4k+25}$, then $f\equiv tg$ for a constant such that $t^{n+1}=1$ or $f^nf^{(k)}g^ng^{(k)}\equiv a(z)^2$.

Remark: When $k=1$, $n\geq 11>5+\sqrt{30}$. Let $a(z)=z$, $f(z),g(z)$ in the Theorem B satisfy the conditions of Theorem 1 and $f^nf'g^ng'\equiv z^2$.

**Theorem 2.** Let $f$ and $g$ be two transcendental meromorphic functions with finite order and $\max\left\{\lambda(f),\lambda(\frac{1}{f})\right\}<1$. Let $k$, $n$ be two positive integers and $a(z)(\not\equiv 0)$ be a small function of $f$ and $g$. If $f^nf^{(k)}$ and $g^ng^{(k)}$ share $a(z)$ IM, $f$ and $g$ share 0 CM and $\Delta=\min\{\Delta_1,\Delta_2\}>6k+20-n$, where

$$\begin{aligned}\Delta_1 &= (3k+4)\Theta(\infty,f)+(2k+3)\Theta(\infty,g)+4\Theta(0,f)+3\Theta(0,g)\\ &+ \delta_{k+2}(0,f)+\delta_{k+2}(0,g)+2\delta_{k+1}(0,f)+\delta_{k+1}(0,g),\end{aligned}$$

$$\begin{aligned}\Delta_2 &= (3k+4)\Theta(\infty,g)+(2k+3)\Theta(\infty,f)+4\Theta(0,g)+3\Theta(0,f)\\ &+ \delta_{k+2}(0,g)+\delta_{k+2}(0,f)+2\delta_{k+1}(0,g)+\delta_{k+1}(0,f),\end{aligned}$$

then $f\equiv tg$ for a constant such that $t^{n+1}=1$ or $f^nf^{(k)}g^ng^{(k)}\equiv a(z)^2$.

**2.Lemmas**

In this section, we present some lemmas which will be needed in the sequel. We now explain some definitions and notions which are used in this paper.

**Definition 1**. Let $k$ be a positive integer. We denote by $N_{k)}\left(r,\frac{1}{f-a}\right)$ the counting function for the zeros of $f-a$ with multiplicity $\leq k$ and by $\overline{N}_{k)}\left(r,\frac{1}{f-a}\right)$ the corresponding one for which the multiplicity is not counted. Let $N_{(k}\left(r,\frac{1}{f-a}\right)$ be the counting function for the zeros of $f-a$ with multiplicity $\geq k$, and $\overline{N}_{(k}\left(r,\frac{1}{f-a}\right)$ be the corresponding one for which the multiplicity is not counted. Set

$$N_k\left(r,\tfrac{1}{f-a}\right)=\overline{N}\left(r,\tfrac{1}{f-a}\right)+\overline{N}_{(2}\left(r,\tfrac{1}{f-a}\right)+\cdots+\overline{N}_{(k}\left(r,\tfrac{1}{f-a}\right).$$

**Definition 2**. For $b\in\mathbb{C}\cup\{\infty\}$ we put

$$\delta_k(b,f)=1-\limsup_{r\to\infty}\frac{N_k(r,b;f)}{T(r,f)}.$$

**Definition 3**. Let $f$ and $g$ be two non-constant meromorphic functions such that $f$ and $g$ share the value 1IM. Let $z_0$ be a zero of $f-1$ with multiplicity $p$, and a zero of $g-1$ with multiplicity $q$. We denote by $\overline{N}_L(r,1;f)$ the reduced counting function of those common zeros of $f-1$ and $g-1$ satisfying $p>q$. Similarly we define $\overline{N}_L(r,1;g)$. In addition, we denote by $N_E^{1)}\left(r,\frac{1}{f-1}\right)$ the counting function of those common simple 1-points of $f(z)$ and $g(z)$.

**Lemma 2.1**([9], Lemma 3) Let $f$ be a nonconstant meromorphic function and $k$ be a positive integer. Then

$$N_p\left(r,\frac{1}{f^{(k)}}\right)\leq N_{p+k}\left(r,\frac{1}{f}\right)+k\overline{N}(r,f)+S(r,f) \tag{2.1}$$

**Lemma 2.2**([10], Lemma 2) Let $f$ be a nonconstant meromorphic function. If $f$ and $g$ share 1 IM, then

$$\overline{N}_L\left(r,\frac{1}{f^{(k)}-1}\right)<\overline{N}\left(r,\frac{1}{f^{(k)}}\right)+\overline{N}(r,f)+S(r,f). \tag{2.2}$$

**Lemma 2.3**([7,11], Lemma 3) Let $f$ and $g$ be two transcendental meromorphic functions defined in the complex plane $\mathbb{C}$. If $f$ and $g$ share 1 CM, then one of the following cases will occur:

(i)$T(r,f)\leq N_2(r,\frac{1}{f})+N_2(r,\frac{1}{g})+N_2(r,f)+N_2(r,g)+S(r,f)$, (2.3)

(ii)$f\equiv g$ or $fg\equiv 1$.

**Lemma 2.4** Let $f$ and $g$ be two meromorphic functions with finite order defined in the complex plane $\mathbb{C}$ and let $k$, $n$ be two positive integers. If $f$ and $g$ share 0 CM, $\max\left\{\lambda(f),\lambda(\frac{1}{f})\right\}<1$ and $f^nf^{(k)}\equiv g^ng^{(k)}$, then $f\equiv tg$ for a constant such that $t^{n+1}=1$.

Proof: Let $z_0$ be a pole of $f$ with multiplicity $p$, then by $f^nf^{(k)}\equiv g^ng^{(k)}$, $z_0$ is also a pole of $g$ with multiplicity $p$, thus $f$ and $g$ share $\infty$ CM. Since $f$ and $g$ share 0 CM, $f$ and $g$ have same zeros and poles. Suppose $f$ and $g$ have the forms $f=\frac{h_1(z)}{h_2(z)}e^{\alpha(z)}$, $g=\frac{h_1(z)}{h_2(z)}e^{\beta(z)}$, where $h_1(z)$, $h_2(z)$ are entire functions, and $\alpha(z)$, $\beta(z)$ are polynomials.

Let $h(z)=\frac{h_1(z)}{h_2(z)}$. Thus $f$ and $g$ have the forms $f=h(z)e^{\alpha(z)}$, $g=h(z)e^{\beta(z)}$. Since $\max\left\{\lambda(f),\lambda(\frac{1}{f})\right\}<1$, $\rho(h)<1$.

First we assume $k=1$, then $f^nf'\equiv g^ng'$, by integration we have $f^{n+1}=g^{n+1}+c$, where $c$ is a constant. From the condition that $f$ and $g$ share 0 CM we have $c=0$, thus $f\equiv tg$ where $t^{n+1}=1$.

Next we assume $k=2$, then

$$f^nf''=e^{(n+1)\alpha}h^n[h(\alpha')^2+2h'\alpha'+h\alpha''+h''],$$

$$g^ng''=e^{(n+1)\beta}h^n[h(\beta')^2+2h'\beta'+h\beta''+h''],$$

thus $f^nf''\equiv g^ng''$ gives

$$e^{(n+1)\alpha}[h(\alpha')^2+2h'\alpha'+h\alpha''+h'']\equiv e^{(n+1)\beta}[h(\beta')^2+2h'\beta'+h\beta''+h'']. \tag{2.4}$$

If $\alpha\not\equiv\beta$, we set $\alpha-\beta=C$ where $C$ is constant or $\alpha-\beta=P(z)$ where $P(z)$ is a polynomial.

Let $\alpha-\beta=C$. We have $f=C_1g$ and replace (2.4) by the following equality

$$C_1^{n+1}g^ng''=g^ng'',$$

then $f\equiv tg$ where $t^{n+1}=1$.

Let $\alpha-\beta=P(z)$. We replace (2.4) by the following equality

$$e^{(n+1)P(z)}[h(\alpha')^2+2h'\alpha'+h\alpha''+h'']\equiv[h(\beta')^2+2h'\beta'+h\beta''+h''].$$

$P(z)$ is a nonconstant polynomial, then $\rho(e^{(n+1)P(z)})\geq 1$. Since $\rho(h)<1$ and $\alpha,\beta$ are polynomials,

$$\rho\left(\frac{h(\alpha')^2+2h'\alpha'+h\alpha''+h''}{h(\beta')^2+2h'\beta'+h\beta''+h''}\right)<1.$$

It is a contradiction.

At last we consider $f^nf^{(k)}\equiv g^ng^{(k)}$. By calculation we get

$$f^nf^{(k)}=e^{(n+1)\alpha}h^n[h(\alpha')^k+D(\alpha',\alpha'',\cdots,\alpha^{(k)},h,h',h'',\cdots,h^{(k)})],$$

where $D$ is a differential polynomial concerning $\alpha$ and $h$.
Similarly,

$$g^ng^{(k)}=e^{(n+1)\beta}h^n[h(\beta')^k+D(\beta',\beta'',\cdots,\beta^{(k)},h,h',h'',\cdots,h^{(k)})].$$

Thus $f^nf^{(k)}\equiv g^ng^{(k)}$ yields

$$\begin{aligned} & e^{(n+1)(\alpha-\beta)}[h(\alpha')^k+D(\alpha',\alpha'',\cdots,\alpha^{(k)},h,h',h'',\cdots,h^{(k)})] \\ \equiv\ & [h(\beta')^k+D(\beta',\beta'',\cdots,\beta^{(k)},h,h',h'',\cdots,h^{(k)})] \end{aligned} \tag{2.5}$$

If $\alpha-\beta=C$, as the same proof of the case where $k=2$, we obtain $f\equiv tg$ where $t^{n+1}=1$.
If $\alpha-\beta=P(z)$ where $P(z)$ is a polynomial, from (2.5) we get a contraction. The proof of Lemma 2.4 is completed.

## 3. The Proof of Theorem 1

Let

$$F = \frac{f^n f^{(k)}}{a(z)}, \quad G = \frac{g^n g^{(k)}}{a(z)}. \tag{3.1}$$

Then $F$ and $G$ share 1 CM. This and Lemma 2.1 imply that

$$\begin{aligned}
N_2\left(r,\frac{1}{F}\right) &= N_2\left(r,\frac{a(z)}{f^n f^{(k)}}\right) \le N_2\left(r,\frac{1}{f^n f^{(k)}}\right) + N_2(r,a(z)) \\
&\le 2\overline{N}\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f^{(k)}}\right) + S(r,f) \\
&= \frac{2}{n}\left[n\overline{N}\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f^{(k)}}\right)\right] \\
&+ \left(1-\frac{2}{n}\right)N\left(r,\frac{1}{f^{(k)}}\right) + S(r,f) \\
&\le \frac{2}{n}N(r,\frac{1}{f^n f^{(k)}}) + S(r,f) \\
&+ (1-\frac{2}{n})\frac{1}{n+1}\left[nN_k\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f^{(k)}}\right) + nk\overline{N}(r,f)\right] \\
&\le \frac{2}{n}N\left(r,\frac{1}{f^n f^{(k)}}\right) + \left(1-\frac{2}{n}\right)\frac{1}{n+1}N\left(r,\frac{1}{f^n f^{(k)}}\right) \\
&+ \left(1-\frac{2}{n}\right)\frac{nk}{n+1}\overline{N}(r,f) + S(r,f) \\
&\le \frac{3}{n+1}N\left(r,\frac{1}{f^n f^{(k)}}\right) + \frac{(n-2)k}{n+1}\overline{N}(r,f) + S(r,f).
\end{aligned} \tag{3.2}$$

Thus we have

$$\begin{aligned}
N_2\left(r,\frac{1}{F}\right) + N_2(r,F) &\le \frac{3}{n+1}N\left(r,\frac{1}{f^n f^{(k)}}\right) + \frac{(n-2)k}{n+1}\overline{N}(r,f) \\
&+ 2\overline{N}(r,f) + S(r,f) \\
&= \frac{3}{n+1}N\left(r,\frac{1}{f^n f^{(k)}}\right) + \frac{(n-2)k+2n+2}{n+1}\overline{N}(r,f) + S(r,f) \\
&\le \frac{3}{n+1}N\left(r,\frac{1}{F}\right) + \frac{(n-2)k+2n+2}{(n+1)^2}N(r,f^n f^{(k)}) + S(r,f) \\
&\le \frac{3}{n+1}N(r,\frac{1}{F}) + \frac{(n-2)k+2n+2}{(n+1)^2}N(r,F) + S(r,f) \\
&\le \frac{(n-2)k+5n+5}{(n+1)^2}T(r,F) + S(r,f).
\end{aligned} \tag{3.3}$$

Since

$$\begin{aligned}
nT(r,f) &= T\left(r,\frac{f^n f^{(k)}}{a(z)}\cdot\frac{a(z)}{f^{(k)}}\right) + S(r,f) \\
&\le T(r,F) + T\left(r,\frac{a(z)}{f^{(k)}}\right) + S(r,f) \\
&\le T(r,F) + T(r,f^{(k)}) + S(r,f) \\
&\le T(r,F) + T(r,\frac{f^{(k)}}{f}) + T(r,f) + S(r,f) \\
&\le T(r,F) + N\left(r,\frac{f^{(k)}}{f}\right) + m\left(r,\frac{f^{(k)}}{f}\right) + T(r,f) + S(r,f) \\
&\le T(r,F) + N(r,f) + k\overline{N}(r,f) + T(r,f) + S(r,f) \\
&\le T(r,F) + (k+2)T(r,f) + S(r,f),
\end{aligned} \tag{3.4}$$

we have $S(r,f)=S(r,F)$. Combining with (3.3), we obtain

$$N_2\left(r,\frac{1}{F}\right)+N_2(r,F)\leq\frac{(n-2)k+5n+5}{(n+1)^2}T(r,F)+S(r,F). \tag{3.5}$$

Similarly, we get

$$N_2\left(r,\frac{1}{G}\right)+N_2(r,G)\leq\frac{(n-2)k+5n+5}{(n+1)^2}T(r,G)+S(r,G) \tag{3.6}$$

By Lemma 2.3, if (i) holds, then we have

$$T(r,F)+T(r,G)\leq\frac{2(n-2)k+10n+10}{(n+1)^2}[T(r,F)+T(r,G)]+S(r,F)+S(r,G),$$

this contradicts with the assumption $n>k+4+\sqrt{k^2+4k+25}$, therefore, $f^nf^{(k)}\equiv g^ng^{(k)}$ or $f^nf^{(k)}g^ng^{(k)}\equiv a(z)^2$. If $f^nf^{(k)}\equiv g^ng^{(k)}$, by Lemma 2.4, we obtain $f\equiv tg$. This completes the proof of Theorem 1.

## 4. The Proof of Theorem 2

Let

$$F=\frac{f^nf^{(k)}}{a(z)},\quad G=\frac{g^ng^{(k)}}{a(z)}, \tag{4.1}$$

$$H=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right). \tag{4.2}$$

Suppose that $H\not\equiv 0$, from the condition that $f^nf^{(k)}$ and $g^ng^{(k)}$ share $a(z)$IM, we have $F$ and $G$ share 1IM, so the common simple 1-point of $F$ and $G$ is the zero of $H$, and

$$\begin{aligned}\overline{N}_E^{1)}\left(r,\frac{1}{F-1}\right)&=\overline{N}_E^{1)}\left(r,\frac{1}{G-1}\right)\\&\leq N\left(r,\frac{1}{H}\right)\leq T(r,H)+O(1)\\&\leq N(r,H)+S(r,f)+S(r,g),\end{aligned} \tag{4.3}$$

From (4.2) we have

$$\begin{aligned}N(r,H)&\leq\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+\overline{N}(r,F)+\overline{N}(r,G)+\overline{N}_L\left(r,\frac{1}{F-1}\right)\\&+\overline{N}_L\left(r,\frac{1}{G-1}\right)+\overline{N}_0\left(r,\frac{1}{F'}\right)+\overline{N}_0\left(r,\frac{1}{G'}\right)+S(r,f)+S(r,g).\end{aligned} \tag{4.4}$$

where $\overline{N}_0\left(r,\frac{1}{F'}\right)$ is the counting function of the zeros of $F'$, which are not the zeros of $F$ and $F-1$ and $\overline{N}_0\left(r,\frac{1}{G'}\right)$ is the counting function of the zeros of $G'$, which are not the zeros of $G$ and $G-1$. Noting

that $F$ and $G$ share 1 IM, we get

$$\begin{aligned}\overline{N}\left(r,\frac{1}{F-1}\right) &+ \overline{N}\left(r,\frac{1}{G-1}\right)\\ &= 2\overline{N}_E^{1)}\left(r,\frac{1}{F-1}\right)+2\overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right)+2\overline{N}_L\left(r,\frac{1}{F-1}\right)\\ &+ 2\overline{N}_L\left(r,\frac{1}{G-1}\right)+S(r,f)+S(r,g)\\ &\leq \overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+\overline{N}(r,F)+\overline{N}(r,G)\\ &+ 3\overline{N}_L\left(r,\frac{1}{F-1}\right)+3\overline{N}_L\left(r,\frac{1}{G-1}\right)\\ &+ 2\overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right)+\overline{N}_E^{1)}\left(r,\frac{1}{F-1}\right)\\ &+ \overline{N}_0\left(r,\frac{1}{F'}\right)+\overline{N}_0\left(r,\frac{1}{G'}\right)+S(r,f)+S(r,g). \qquad (4.5)\end{aligned}$$

It is easy to see that

$$\begin{aligned}\overline{N}_L\left(r,\frac{1}{F-1}\right) &+ 2\overline{N}_L\left(r,\frac{1}{G-1}\right)+2\overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right)+\overline{N}_E^{1)}\left(r,\frac{1}{F-1}\right)\\ &\leq N\left(r,\frac{1}{G-1}\right)\leq T(r,G)+O(1). \qquad (4.6)\end{aligned}$$

This and (4.5) give

$$\begin{aligned}\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right) &\leq \overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)\\ &+ \overline{N}(r,F)+\overline{N}(r,G)+2\overline{N}_L(r,\frac{1}{F-1})\\ &+ \overline{N}_L(r,\frac{1}{G-1})+T(r,G)+\overline{N}_0(r,\frac{1}{F'})\\ &+ \overline{N}_0(r,\frac{1}{G'})+S(r,f)+S(r,g). \qquad (4.7)\end{aligned}$$

By the second fundamental theorem and (4.7), we have

$$\begin{aligned}T(r,F)+T(r,G) &\leq \overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}(r,F)+\overline{N}(r,G)+\overline{N}\left(r,\frac{1}{F-1}\right)\\ &+ \overline{N}\left(r,\frac{1}{G-1}\right)-N_0\left(r,\frac{1}{F'}\right)-N_0\left(r,\frac{1}{G'}\right)+S(r,f)+S(r,g)\\ &\leq T(r,G)+2\overline{N}(r,F)+2\overline{N}(r,G)+N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)\\ &+ 2\overline{N}_L\left(r,\frac{1}{F-1}\right)+\overline{N}_L\left(r,\frac{1}{G-1}\right)+S(r,f)+S(r,g).\end{aligned}$$

Thus

$$\begin{aligned}T(r,F) &\leq 2\overline{N}(r,F)+2\overline{N}(r,G)+N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)\\ &+ 2\overline{N}_L\left(r,\frac{1}{F-1}\right)+\overline{N}_L\left(r,\frac{1}{G-1}\right)+S(r,f)+S(r,g). \qquad (4.8)\end{aligned}$$

From Lemma 2.2, we obtain

$$\overline{N}_L\left(r,\frac{1}{F-1}\right)\leq\overline{N}\left(r,F\right)+\overline{N}\left(r,\frac{1}{F}\right), \tag{4.9}$$

$$\overline{N}_L\left(r,\frac{1}{G-1}\right)\leq\overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G}\right). \tag{4.10}$$

Taking (4.9) and (4.10) into (4.8) and combining with Lemma 2.1 , we get

$$\begin{aligned}
T(r,F) &\leq 4\overline{N}(r,F)+3\overline{N}(r,G)+N_2\left(r,\frac{1}{F}\right)\\
&+ N_2\left(r,\frac{1}{G}\right)+2\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+S(r,f)+S(r,g)\\
&\leq 4\overline{N}(r,f)+3\overline{N}(r,g)+N_2\left(r,\frac{1}{f^n}\right)\\
&+ N_2\left(r,\frac{1}{f^{(k)}}\right)+N_2\left(r,\frac{1}{g^n}\right)+N_2\left(r,\frac{1}{g^{(k)}}\right)\\
&+ 2\overline{N}\left(r,\frac{1}{f^n}\right)+2\overline{N}\left(r,\frac{1}{f^{(k)}}\right)+\overline{N}\left(r,\frac{1}{g^n}\right)\\
&+ \overline{N}\left(r,\frac{1}{g^{(k)}}\right)+S(r,f)+S(r,g)\\
&\leq 4\overline{N}(r,f)+3\overline{N}(r,g)+4\overline{N}(r,\frac{1}{f})\\
&+ 3\overline{N}\left(r,\frac{1}{g}\right)+N_2\left(r,\frac{1}{f^{(k)}}\right)+N_2\left(r,\frac{1}{g^{(k)}}\right)\\
&+ 2\overline{N}(r,\frac{1}{f^{(k)}})+\overline{N}\left(r,\frac{1}{g^{(k)}}\right)+S(r,f)+S(r,g)\\
&\leq (3k+4)\overline{N}(r,f)+(2k+3)\overline{N}(r,g)+4\overline{N}\left(r,\frac{1}{f}\right)\\
&+ 3\overline{N}\left(r,\frac{1}{g}\right)+N_{2+k}\left(r,\frac{1}{f}\right)+N_{2+k}\left(r,\frac{1}{g}\right)\\
&+ 2N_{1+k}\left(r,\frac{1}{f}\right)+N_{1+k}\left(r,\frac{1}{g}\right)+S(r,f)+S(r,g). \qquad (4.11)
\end{aligned}$$

On the other hand, from (3.4) we have

$$T(r,F)\geq(n-k-1)T(r,f)+S(r,f). \tag{4.12}$$

Thus from (4.11) and (4.12), it follows that

$$\begin{aligned}
(n-k-1)T(r,f) &\leq (3k+4)\overline{N}(r,f)+(2k+3)\overline{N}(r,g)\\
&+ 4\overline{N}\left(r,\frac{1}{f}\right)+3\overline{N}\left(r,\frac{1}{g}\right)+N_{2+k}\left(r,\frac{1}{f}\right)\\
&+ N_{2+k}\left(r,\frac{1}{g}\right)+2N_{1+k}\left(r,\frac{1}{f}\right)+N_{1+k}\left(r,\frac{1}{g}\right)\\
&+ S(r,f)+S(r,g). \qquad (4.13)
\end{aligned}$$

In a similar way, we can get

$$\begin{aligned}(n-k-1)T(r,g) &\leq (3k+4)\overline{N}(r,g)+(2k+3)\overline{N}(r,f)\\ &+ 4\overline{N}\left(r,\frac{1}{g}\right)+3\overline{N}\left(r,\frac{1}{f}\right)+N_{2+k}\left(r,\frac{1}{g}\right)\\ &+ N_{2+k}\left(r,\frac{1}{f}\right)+2N_{1+k}\left(r,\frac{1}{g}\right)+N_{1+k}\left(r,\frac{1}{f}\right)\\ &+ S(r,f)+S(r,g). \qquad (4.14)\end{aligned}$$

We suppose that there exists a set $I$ of infinite measure such that $T(r,g)\leq T(r,f)$ for $r\in I$. Hence by (4.13) we have

$$\begin{aligned}nT(r,f) &\leq [(6k+20)-(3k+3)\Theta(\infty,f)-(2k+3)\Theta(\infty,g)-4\Theta(0,f)\\ &- 3\Theta(0,g)-\delta_{k+2}(0,f)-\delta_{k+2}(0,g)-2\delta_{k+1}(0,f)-\delta_{k+1}(0,g)+\epsilon]T(r,f)+S(r,f)\end{aligned}$$

for $r\in I$ and $0<\epsilon<\Delta_1-(6k+20-n)$. And we obtain $T(r,f)\leq S(r,f)$ for $r\in I$, which is a contradiction.

Similarly, if $T(r,f)\leq T(r,g)$ for $r\in I$, by (4.14) we have

$$\begin{aligned}nT(r,g) &\leq [(6k+20)-(3k+3)\Theta(\infty,g)-(2k+3)\Theta(\infty,f)-4\Theta(0,g)\\ &- 3\Theta(0,f)-\delta_{k+2}(0,g)-\delta_{k+2}(0,f)-2\delta_{k+1}(0,g)-\delta_{k+1}(0,f)+\epsilon]T(r,g)+S(r,g)\end{aligned}$$

for $r\in I$ and $0<\epsilon<\Delta_2-(6k+20-n)$, which also is a contradiction.

Hence $H(z)\equiv 0$. That is

$$\frac{F''}{F'}-\frac{2F'}{F-1}\equiv\frac{G''}{G'}-\frac{2G'}{G-1}.$$

Without loss of generality, next we suppose that there exists a set $I$ of infinite measure such that $T(r,g)\leq T(r,f)$ for $r\in I$. By integrating two sides of the above equality we get

$$\frac{1}{G-1}=\frac{A}{F-1}+B, \qquad (4.15)$$

where $A(\neq 0)$ and $B$ are constants.

We consider the following three cases:

Case 1. Let $B\neq 0$ and $A=B$. If $B=-1$, from (4.15) we obtain $FG\equiv 1$, that is $f^nf^{(k)}g^ng^{(k)}\equiv a(z)^2$.

If $B\neq -1$, from (4.15) we get $\frac{1}{G}=\frac{BF}{(1+B)F-1}$, thus $\overline{N}(r,\frac{1}{F-\frac{1}{B+1}})=\overline{N}(r,\frac{1}{G})$. By the second funda-

mental theorem, (4.12) and Lemma 2.1, we have

$$\begin{aligned}
(n-k-1)T(r,f) &\leq T(r,F)+S(r,F)\\
&\leq \overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{F-\frac{1}{B+1}}\right)+S(r,F)\\
&= \overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+S(r,F)\\
&\leq \overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}}\right)\\
&+ \overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{g^{(k)}}\right)+S(r,f)\\
&\leq (k+1)\overline{N}(r,f)+k\overline{N}(r,g)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{g}\right)\\
&+ N_{1+k}\left(r,\frac{1}{f}\right)+N_{1+k}\left(r,\frac{1}{g}\right)+S(r,f)\\
&\leq (3k+4)\overline{N}(r,f)+(2k+3)\overline{N}(r,g)\\
&+ 4\overline{N}\left(r,\frac{1}{f}\right)+3\overline{N}\left(r,\frac{1}{g}\right)+N_{2+k}\left(r,\frac{1}{f}\right)\\
&+ N_{2+k}\left(r,\frac{1}{g}\right)+2N_{1+k}\left(r,\frac{1}{f}\right)\\
&+ N_{1+k}\left(r,\frac{1}{g}\right)+S(r,f)+S(r,g).
\end{aligned}$$

That is,

$$\begin{aligned}
(n-k-1)T(r,f) &\leq (3k+4)\overline{N}(r,f)+(2k+3)\overline{N}(r,g)\\
&+ 4\overline{N}\left(r,\frac{1}{f}\right)+3\overline{N}\left(r,\frac{1}{g}\right)+N_{2+k}\left(r,\frac{1}{f}\right)\\
&+ N_{2+k}\left(r,\frac{1}{g}\right)+2N_{1+k}\left(r,\frac{1}{f}\right)\\
&+ N_{1+k}\left(r,\frac{1}{g}\right)+S(r,f)+S(r,g).
\end{aligned}$$

Thus we obtain

$$[\Delta_1-(5k+18-n)]T(r,f)\leq S(r,f),$$

which is a contradiction.

Case 2. Let $B\neq 0$ and $A=B$. If $B=-1$, we have from (4.15) that $G=\frac{A}{-F+A+1}$, therefore $\overline{N}(r,\frac{1}{-F+A+1})=\overline{N}(r,G)$. By the second fundamental theorem and the same argument as in case 1, we get a contradiction.

If $B\neq -1$, we have from (4.15) that $G-(1+\frac{1}{B})=\frac{-A}{B^2(F+\frac{A-B}{B})}$, therefore $\overline{N}\left(r,\frac{1}{F+\frac{A-B}{B}}\right)=\overline{N}(r,G)$. Next by the second fundamental theorem and the same argument as in case 1, we get a contradiction.

Case 3. Let $B=0$, then we obtain from (4.15) that $G=\frac{F}{A}+1-\frac{1}{A}$. If $A\neq 1$, then $\overline{N}\left(r,\frac{1}{G-\frac{A-1}{A}}\right)=\overline{N}\left(r,\frac{1}{F}\right)$. By the second fundamental theorem and the same argument as in case 1, we get a contradiction.

Thus $A=1$, so we have $F\equiv G$, that is $f^nf^{(k)}\equiv g^ng^{(k)}$. By Lemma 2.4 we have $f\equiv tg$ for a constant such that $t^{n+1}=1$. From the above proof we obtain $f\equiv tg$ or $f^nf^{(k)}g^ng^{(k)}\equiv a(z)^2$.

The proof of Theorem 2 is completed.

# References

[1] G. G. Gundersen and L. Z. Yang, Entire functions that share one value with one or two of their derivatives, J. Math. Anal. Appl., 223(1998),88-95.

[2] C. C. Yang and H. X. Yi, Uniqueness theory of meromorphic functions, Kluwer Academic Publishers, The Netherlands, 2003.

[3] R. Nevanlinna, Les Théoréms de Picard-Borel et la théorie des fonction méromorphes, Gauthiers-Villars, Paris, France, 1929.

[4] C. C. Yang and H. X. Yi, Uniqueness Theorem of Meromorphic Functions, vol. 557 of Mathematics and Its Appilications, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2003.

[5] C. C. Yang and X. Hua, "Uniqueness and value-sharing of meromorphic function," Annales Academiæ Scientiarum Fennicæ, vol. 22, pp. 395-406, 1997.

[6] F. L. Fang and H. L. Qiu, "Meromorphic functions sharing one value," Journal of Matheatical Analysis and Applications, vol. 268, no. 2, pp. 426-439, 2002.

[7] S. M. Wang and Z. Sh. gao, "Meromorphic Functions Sharing a Small Function," Hindawi Publishing Corporation, Abstract and Applied Analysis, Volume 2007, Artical ID 60718, 6 pages, doi: 10. 1155/2007/60718.

[8] C. C. Yang and X. H. Hua, "Uniqueness and value-sharing of meromorphic functions," Annales Academiæ Scientiarum Fennicæ, Mathematica, Volumen 22, 1997, 395-406.

[9] J. L. Zhang and L. Zh. Yang, "Some Results Related To a Conjecture of R. BRÜCK Concerning Meromorphic Functions Sharing One Small Function with Their Derivatives," Annales Acamediæ Sciential Fennicæ, Mathematica, Volume 32, 2007, 141-149.

[10] H. X. Yi, "Meromorphic function that share one or two value II", Kodai Math. J., 22(1999), 264-272.

[11] E.Mues and M.Reindevs, "Meromrophic functions share one value and unique range sets", Kodai Math. J., 18(1995), 515-522.

# Stochastic elastic equation in $d$-dimensional space driven by multiplicative multi-parameter fractional white noise

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**Abstract**

Stochastic elastic equation in $d$-dimensional space driven by multiplicative multi-parameter fractional white noise are considered. By using the Wiener chaos expansion and undetermined coefficient methods, we obtain the existence and uniqueness of the solution in a distribution space. The Lyapunov exponents and the Hölder continuity in the distribution space of the solution are also estimated.

**Key Words**: Fractional Brownian motion; Multi-parameter fractional white noise; Winner Chaos expansion; Asymptotic behavior.

## 1 Introduction

The subject of stochastic calculus with respect to fractional Brownian motion has gained considerable popularity and importance due to its frequent appearance in a wide variety of physical phenomena, such as hydrology, economic, telecommunications and medicine. Many contributions for stochastic calculus with respect to fractional Brownian motion have emerged in the last decades, see [3, 7, 23, 24]. Since some physical phenomena are naturally modeled by stochastic partial differential equations and the randomness can be described by multi-parameter fractional white noise, it is important to study the problems of solutions of stochastic partial differential equations with multi-parameter fractional white noise. Many studies of the solutions of stochastic equations with fractional white noise have emerged recently, see [8, 15, 19, 25, 28] and reference therein.

In [14], Y. Hu studied the existence and uniqueness of the solution for a stochastic heat equation by chaos expansion, and estimated the Lyapunov exponent in a distribution space of the solution. In [11], the authors considered a class of hyperbolic stochastic

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partial differential equation driven by a space time fractional noise. In [1, 27], some authors considered stochastic wave equation with fractional Gaussian noise. A stochastic generalized Burgers equations driven by fractional noises has been studied in [20]. In [3], the authors investigated stochastic Poisson equation and quasi-linear heat equation driven by multi-parameter fractional white noise. In a recent paper [30], the authors considered a 1-dimensional stochastic Burgers' equation driven by a genuine cylindrical fBm with Hurst parameter $H > \frac{1}{4}$. They proved the regularities of the solution to the linear stochastic problem corresponding to the stochastic Burgers' equation and then obtained the global existence and uniqueness results of the stochastic Burgers' equation. For more contributions on stochastic calculus with fractional noise, we refer the reader to [4, 16–18, 20, 22, 26, 29, 30] and reference therein.

It should be noted that most of the papers and books on stochastic partial differential equations with fractional noise are devoted to the case of additive noise. However, there are few papers that consider the case of multiplicative fractional noise [2, 9, 14, 18]. Enlightened by the above contributions, in this paper we will consider a stochastic elastic equation driven by multiplicative multi-parameter white noise.

Let $H = (h_0, h_1, \ldots, h_d)$ with $\frac{1}{2} < h_i < 1, i = 0, 1, \ldots, d$. Consider the following stochastic elastic equation

$$\frac{\mathrm{d}u_t(t,x)}{\mathrm{d}t} + \Delta^2 u(t,x) = u(t,x) \diamond W^H(t,x), \tag{1.1}$$

where $t > 0, x \in D = (0,1)^d$ with initial and boundary conditions

$$u(0,x) = u_0(x), \frac{\partial u}{\partial t}(0,x) = \upsilon_0(x), \quad x \in D,$$

$$u(t,x) = \Delta u(t,x) = 0, \quad \text{on } \partial D.$$

We assume that $u_0$ and $\upsilon_0$ are deterministic functions defined on $[0,1]^d$, $u_0$ and $\Delta u_0$ both vanish at $\partial D$, $W^H$ is a $(d+1)$-dimensional fractional white noise and $u \diamond W^H$ is Wick product which will be defined in section 2.

Stochastic elastic equation driven by Brownian motion has been studied by many authors, (see [5, 6, 12, 21, 31]). However, there are few papers consider stochastic elastic equation driven by fractional Brownian motion. Such equation is a fourth order partial differential equation and has very wide applications in structural engineering. As an engineering problem, it has its applications in beams, bridges and other structures, see [5, 26].

Our aim in this paper is to obtain the existence, uniqueness, the asymptotic behavior and the Hölder continuity of the mild solution of problem (1.1) in a distribution space. The keys to the proof are the Wiener chaos expansion of the solution and the undetermined coefficient method.

Throughout the paper, we use the letter $C$ denotes a constant that may not be the same form one occurrence to anther, even in the same line. We express the dependence on some parameters by writing the parameters as arguments, e.g. $C = C(H)$.

The remaining of this paper is organized as follows. In section 2, we give preliminaries of the stochastic integral with respect to multi-parameter fractional white noise. In section 3, we prove the the existence, uniqueness, asymptotic behavior and Hölder continuity in a distribution space of the solutions of problem (1.1).

## 2 Stochastic integral with respect to multi-parameter fractional white noise

In this section, we introduce the definition of stochastic integral with respect to the $d-$parameter fractional Brownian fields for Hurst index $H = (h_1, \cdots, h_d), (\frac{1}{2} < h_i < 1, i = 1, \cdots, d)$ by using the fractional white noise analysis method. For more contributions about white noise analysis, we refer the reader to [3, 13].

Let $h \in (0, 1)$. A fractional Brownian motion $B_t^h, t \geq 0$, of Hurst index $h$ is a continuous Gaussian stochastic process, such that for all $s, t \in \mathbb{R}_+$,

$$B_0^h = 0, \quad \mathbb{E}(B_t^h) = 0, \quad \mathbb{E}(B_t^h B_s^h) = \frac{1}{2}(|t|^{2h} + |s|^{2h} - |t-s|^{2h}). \tag{2.1}$$

Since we are concerned with the fractional Brownian motions of multi-parameter, some notations must be introduced. Let $u = (u_1, \cdots, u_d) \in \mathbb{R}^d$, denote $\mathrm{d}u = \mathrm{d}u_1 \cdots \mathrm{d}u_d$. Fix $h$ with $\frac{1}{2} < h < 1$. We put

$$\phi_h(s, t) = h(2h-1)|s-t|^{2h-2}, \quad s, t \in \mathbb{R}, \tag{2.2}$$

and

$$\phi_H(u, v) = \prod_{i=1}^{d} \phi_{h_i}(u_i, v_i), \quad u = (u_1, \cdots, u_d), v = (v_1, \cdots, v_d) \in \mathbb{R}^d \tag{2.3}$$

for $H = (h_1, \cdots, h_d), (\frac{1}{2} < h_i < 1, i = 1, \cdots, d)$.

Let $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}^d$ and $\mathcal{S}'(\mathbb{R}^d)$ be the dual of $\mathcal{S}(\mathbb{R}^d)$, i.e., $\mathcal{S}'(\mathbb{R}^d)$ is the space of tempered distributions on $\mathbb{R}^d$.

The action of $\omega \in \mathcal{S}'(\mathbb{R}^d)$ on $f \in \mathcal{S}(\mathbb{R}^d)$ is given by $\langle \omega, f\rangle = \int_{\mathbb{R}^d} \omega(x) f(x) \mathrm{d}x$. Denote

$$\langle f, g\rangle_H = \int_{\mathbb{R}^{2d}} f(u) g(v) \phi_H(u, v) \mathrm{d}u \mathrm{d}v, \quad f, g \in \mathcal{S}(\mathbb{R}^d). \tag{2.4}$$

If we equip $\mathcal{S}(\mathbb{R}^d)$ with the inner product $\langle \cdot, \cdot \rangle_H$ and the norm $\|f\|_H = \langle f, f\rangle_H$, then the completion of $\mathcal{S}(\mathbb{R}^d)$, denote by $L^2_{\phi_H}(\mathbb{R}^d)$, becomes a separable Hilbert space. Now let $\Omega = \mathcal{S}'(\mathbb{R}^d)$. The map $f \to e^{-\frac{1}{2}\|f\|_H^2}$ is positive definite on $\mathcal{S}(\mathbb{R}^d)$, by the Bochner-Minlos theorem [13], there exists a probability measure $\mathbb{P}^H$ on the Borel subset $\mathcal{B}(\Omega)$ of $\Omega$ such that

$$\int_\Omega e^{i\langle \omega, f\rangle} \mathrm{d}\mathbb{P}^H(\omega) = e^{-\frac{1}{2}\|f\|_H^2}, \quad \forall f \in \mathcal{S}(\mathbb{R}^d). \tag{2.5}$$

Let $\mathbb{E}$ denotes the expectation under the probability measure $\mathbb{P}^H$, then

$$\mathbb{E}[\langle \cdot, f\rangle] = 0, \quad \mathbb{E}[\langle \cdot, f\rangle^2] = \|f\|_H^2. \tag{2.6}$$

Now define a square integrable stochastic field $B^H(x), x \in \mathbb{R}^d$ as

$$B^H(x) = B^H(x, \omega) = \langle \omega, I_{[0,x]}(\cdot)\rangle, \tag{2.7}$$

where $x = (x_1, \cdots, x_d) \in \mathbb{R}^d, I_{[0,x]} = I_{[0,x_1]} \cdots I_{[0,x_d]}$ and for every $i \in \{1, 2, \cdots, d\}$,

$$I_{[0,x_i]}(y_i) = \begin{cases} 1 & 0 \le y_i \le x_i, \\ -1 & x_i \le y_i \le 0, \text{ excetp } x_i = y_i = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.8}$$

From (2.7), we see that $B^H(x)$ is a Gaussian field and for every $x, y \in \mathbb{R}^d$,

$$\mathbb{E}[B^H(x)] = 0, \quad \mathbb{E}[B^H(x) B^H(y)] = \frac{1}{2^d} \prod_{i=1}^d (|x_i|^{2h_i} + |y_i|^{2h_i} - |x_i - y_i|^{2h_i}), \tag{2.9}$$

where we have used the well known identity

$$\int_0^{x_i} \int_0^{y_i} \phi_h(s, t) \mathrm{d}s \mathrm{d}t = \frac{1}{2}(|x_i|^{2h_i} + |y_i|^{2h_i} - |x_i - y_i|^{2h_i}). \tag{2.10}$$

By Kolmogorov's continuity theorem, $B^H(x)$ has a continuous version. The fractional Brownian field is defined as the continuous version of $B^H(x)$, which we sill denote it by $B^H(x)$.

Similarly to the case of stochastic integral with respect to fractional Brownian motion for deterministic function [3, 7, 10], we have the following lemma.

**Lemma 2.1** *If $f \in L^2_H(\mathbb{R}^d)$, then $\int_{\mathbb{R}^d} f(x) \mathrm{d}B^H(x)$ is well-defined Gaussian random variable and*

$$\mathbb{E}\left[\int_{\mathbb{R}^d} f(x) \mathrm{d}B^H(x)\right] = 0, \quad \mathbb{E}\left[\int_{\mathbb{R}^d} f(x) \mathrm{d}B^H(x)\right]^2 = \int_{\mathbb{R}^{2d}} f(u) f(v) \phi_H(u, v) \mathrm{d}u \mathrm{d}v. \tag{2.11}$$

The Hermite polynomials $h_n(x)$ are defined by

$$h_n(x) = (-1)^n e^{\frac{1}{2}x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n}(e^{-\frac{1}{2}x^2}), \quad n = 0, 1, 2, \ldots. \tag{2.12}$$

For example, the first Hermite polynomials are

$$h_0(x) = 1, \quad h_1(x) = x, \quad h_2(x) = x^2 - 1, \quad h_3(x) = x^3 - 3x, \cdots. \tag{2.13}$$

Let $\xi_n(x)$ be the Hermite functions defined by

$$\xi_n(x) = \pi^{-\frac{1}{4}}((n-1)!)^{-\frac{1}{2}} h_{n-1}(\sqrt{2}x)e^{-\frac{x^2}{2}}, \quad n = 1, 2, \ldots. \tag{2.14}$$

It is proved in [13] that $\xi_n \in \mathcal{S}(\mathbb{R})$ and the collection $\{\xi_n\}_{n=1}^{\infty}$ constitutes an orthonormal basis of $L^2(\mathbb{R})$. The most important property of $\xi_n$ used in this paper is

$$|\xi_n(x)| \leq \begin{cases} Cn^{-\frac{1}{12}}, & |x| \leq 2\sqrt{n}, \\ Ce^{-\gamma x^2}, & |x| > 2\sqrt{n}, \end{cases} \quad n = 1, 2, \ldots, \tag{2.15}$$

where constants $C$ and $\gamma$ are independent of $n$.

**Lemma 2.2** *[3] Let $\frac{1}{2} < h < 1$. The fractional integral $I_-^{h-\frac{1}{2}}$ is defined by*

$$I_-^{h-\frac{1}{2}} f(u) = c_h \int_u^{\infty} (t-u)^{h-\frac{3}{2}} f(t)\mathrm{d}t, \tag{2.16}$$

*where $c_h = \sqrt{h(2h-1)\Gamma(3/2-h)/(\Gamma(h-1/2)\Gamma(2-2h))}$ and $\Gamma$ denotes the gamma function. Then $I_-^{h-\frac{1}{2}}$ is an isometry from $L^2_{\phi_h}(\mathbb{R})$ to $L^2(\mathbb{R})$.*

Now we define $\eta_n^h(u) = (I_-^{h-\frac{1}{2}})^{-1}(\xi_n)(u)$. Then by Lemma 2.2 and the properties of $\xi_n$, $\{\eta_n^h\}_{n=1}^{\infty}$ is an orthonormal basis of $L^2_{\phi_h}(\mathbb{R})$.

Let $\delta = (\delta_1, \ldots, \delta_d)$ denote $d$-dimensional multi-indices with $\delta_i \in \mathbb{N}, i = 1, \ldots, d$, where $\mathbb{N}$ is the set of natural numbers. Then the family of tensor products

$$\eta_\delta(x_1, x_2, \ldots, x_d) := \eta_{\delta_1}^{h_1}(x_1)\eta_{\delta_2}^{h_2}(x_2)\cdots\eta_{\delta_d}^{h_d}(x_d), \quad \delta = (\delta_1, \ldots, \delta_d) \in \mathbb{N}^d \tag{2.17}$$

forms an orthonormal basis of $L^2_{\phi_H}(\mathbb{R}^d)$. Let $\{\delta^i = (\delta_1^i, \ldots, \delta_d^i)\}_{i=1}^{\infty}$ be a fixed ordering of $\mathbb{N}^d$. From a detailed proof in [13], we can assume that the ordering has the properties that

$$i < j \Rightarrow \delta_1^i + \delta_2^i + \cdots + \delta_d^i \leq \delta_1^j + \delta_2^j + \cdots + \delta_d^j. \tag{2.18}$$

and

$$j^{\frac{1}{d}} \leq \delta_1^j \cdot \delta_2^j \cdots \delta_d^j \leq j. \tag{2.19}$$

Let $\mathcal{J} = \left(\mathbb{N}_0^{\mathbb{N}}\right)_c$ denote the set of all multi-indices $\alpha = (\alpha_1, \alpha_2, \ldots)$ with elements $\alpha_i \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and with compact support, i.e., with only finitely many $\alpha_i \neq 0$. Denote $|\alpha| = \sum_{i=1}^{\infty} \alpha_i$. Define

$$e_j(x) = \eta_{\delta^j}(x) = \eta_{\delta_1^j}^{h_1}(x_1)\eta_{\delta_2^j}^{h_2}(x_2)\cdots\eta_{\delta_d^j}^{h_d}(x_d), \quad x = (x_1, \ldots, x_d), j = 1, 2, \ldots. \tag{2.20}$$

Then $\{e_j\}$ forms an orthonormal basis of $L^2_{\phi_H}(\mathbb{R}^d)$. If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathcal{J}$, define

$$\mathcal{H}_\alpha(\omega) = \prod_{i=1}^{m} h_{\alpha_i}(\langle \omega, e_i \rangle). \tag{2.21}$$

Then we have the following fractional Wiener Ito chaos expansion theorem.

**Theorem 2.1** *[3] The family $\{\mathcal{H}_\alpha\}_{\alpha\in\mathcal{J}}$ constitutes an orthogonal basis for $L^2(\mathbb{P}^H)$ and for $\alpha = (\alpha_1, \alpha_2, \ldots) \in \mathcal{J}$,*

$$\|\mathcal{H}_\alpha\|^2_{L^2(\mathbb{P}^H)} = \mathbb{E}[\mathcal{H}_\alpha^2] = \alpha! = \alpha_1!\alpha_2!\cdots. \tag{2.22}$$

*Moreover, if $F \in L^2(\mathbb{P}^H)$, then there exist constants $c_\alpha \in \mathbb{R}, \alpha \in \mathcal{J}$, such that*

$$F(\omega) = \sum_{\alpha\in\mathcal{J}} c_\alpha \mathcal{H}_\alpha(\omega), \tag{2.23}$$

*where the convergence holds in $L^2(\mathbb{P}^H)$ and*

$$\|F\|^2_{L^2(\mathbb{P}^H)} = \sum_{\alpha\in\mathcal{J}} \alpha! c_\alpha^2. \tag{2.24}$$

Now we compute the Wiener Ito chaos expansion of the fractional Brownian field $B^H(x)$. By (2.7),

$$\begin{aligned} B^H(x) &= \langle \omega, I_{[0,x]} \rangle = \langle \omega, \sum_{i=1}^{\infty} \langle I_{[0,x]}, e_i \rangle_H e_i \rangle = \sum_{i=1}^{\infty} \langle I_{[0,x]}, e_i \rangle_H \langle \omega, e_i \rangle \\ &= \sum_{i=1}^{\infty} \left[ \int_0^x \int_{\mathbb{R}^d} e_i(v)\phi_H(u, v)\mathrm{d}v du \right] \mathcal{H}_{\varepsilon^{(i)}}(\omega), \end{aligned} \tag{2.25}$$

where $\varepsilon^{(i)} = (0, \ldots, 0, 1, 0, \ldots)$ denote the $i$th unit vector.

The fractional Hida test function and distribution spaces are defined as follows.

**Definition 2.1** *The fractional Hida test function space $(\mathcal{S})_H = \bigcap_{k=0}^{\infty} (\mathcal{S})_{H,k}$, where $(\mathcal{S})_{H,k}$ is the set of all $\psi(\omega) = \sum_{\alpha\in\mathcal{J}} a_\alpha \mathcal{H}_\alpha(\omega) \in L^2(\mathbb{P}^H)$ such that*

$$\|\psi\|^2_{H,k} = \sum_{\alpha\in\mathcal{J}} \alpha! a_\alpha^2 (2\mathbb{N})^{k\alpha} < \infty, \ k \in \mathbb{N}, \tag{2.26}$$

*where* $(2\mathbb{N})^{\gamma} = \prod_{j=1}^{\infty}(2j)^{\gamma_j}$ *if* $\gamma = (\gamma_1, \gamma_2, \ldots) \in \mathcal{J}$. *The fractional Hida distribution space* $(\mathcal{S})_H^* = \bigcup_{q=0}^{\infty}(\mathcal{S})_{H,-q}^*$, *where* $(\mathcal{S})_{H,-q}^*$ *is the set of all formal expansions* $G(\omega) = \sum_{\alpha\in\mathcal{J}} b_\alpha \mathcal{H}_\alpha(\omega)$ *such that*

$$\|G\|_{H,-q}^2 = \sum_{\alpha\in\mathcal{J}} \alpha! a_\alpha^2 (2\mathbb{N})^{-q\alpha} < \infty,\ q \in \mathbb{N}. \tag{2.27}$$

The family of seminorms $\|\cdot\|_{H,k}, k \in \mathbb{N}$ gives rise to a topology on $(\mathcal{S})_H$ and $(\mathcal{S})_H^*$ can be identified with the dual of $(\mathcal{S})_H$ by the action

$$\ll G, \psi \gg = \sum_{\alpha\in\mathcal{J}} \alpha! a_\alpha b_\alpha. \tag{2.28}$$

**Definition 2.2** $G : \mathbb{R}^d \to (\mathcal{S})_H^*$ *is* $\mathrm{d}x$*-integrable in* $(\mathcal{S})_H^*$ *if*

$$\ll G(x), \psi \gg \in L^1(\mathbb{R}^d), \quad \textit{for all } \psi \in (\mathcal{S})_H. \tag{2.29}$$

*If* $G : \mathbb{R}^d \to (\mathcal{S})_H^*$ *is* $\mathrm{d}x$*-integrable in* $(\mathcal{S})_H^*$, *We define* $\int_{\mathbb{R}^d} G(x)\mathrm{d}x$ *to be the unique element of* $(\mathcal{S})_H^*$ *such that*

$$\ll \int_{\mathbb{R}^d} G(x)\mathrm{d}x, \psi \gg = \int_{\mathbb{R}^d} \ll G(x), \psi \gg \mathrm{d}x, \quad \textit{for all } \psi \in (\mathcal{S})_H. \tag{2.30}$$

The fractional noise $W^H(x)$ is defined by the formal derivative of $B^H(x)$ in $(\mathcal{S})_H^*$,

$$W^H(x) = \sum_{i=1}^{\infty} \left[\int_{\mathbb{R}^d} e_i(\upsilon)\phi_H(x,\upsilon)\mathrm{d}\upsilon\right] \mathcal{H}_{\varepsilon^{(i)}}(\omega). \tag{2.31}$$

Then, $W^H(x) \in (\mathcal{S})_H^*$ and $\frac{\mathrm{d}}{\mathrm{d}x}B^H(x) = W^H(x)$ in $(\mathcal{S})_H^*$.

**Definition 2.3** *The Wick product* $F \diamond G$ *of* $F(\omega) = \sum_{\alpha\in\mathcal{J}} a_\alpha \mathcal{H}_\alpha(\omega) \in (\mathcal{S})_H^*$ *and* $G(\omega) = \sum_{\alpha\in\mathcal{J}} b_\alpha \mathcal{H}_\alpha(\omega) \in (\mathcal{S})_H^*$ *is defined by*

$$F \diamond G(\omega) = \sum_{\alpha,\beta\in\mathcal{J}} a_\alpha b_\beta \mathcal{H}_{\alpha+\beta}(\omega) = \sum_{\gamma\in\mathcal{J}} \left(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta\right) \mathcal{H}_\gamma(\omega). \tag{2.32}$$

Based on the preparations above, now we define the fractional Wick Ito Skorohod integral as follows.

**Definition 2.4** *Suppose* $G : \mathbb{R}^d \to (\mathcal{S})_H^*$ *is a given function and* $G(x) \diamond W^H(x)$ *is* $\mathrm{d}x$*-integrable in* $(\mathcal{S})_H^*$. *Then the fractional Wick Ito Skorohod integral* $\int_{\mathbb{R}^d} G(x)\mathrm{d}B^H(x)$ *is defined by*

$$\int_{\mathbb{R}^d} G(x)\mathrm{d}B^H(x) = \int_{\mathbb{R}^d} G(x) \diamond W^H(x)\mathrm{d}x. \tag{2.33}$$

*For a interval in* $\mathbb{R}^d$, *the integral can be defined as*

$$\int_0^x G(y)\mathrm{d}B^H(y) = \int_{\mathbb{R}^d} G(y) I_{[0,x]}(y)\mathrm{d}B^H(y). \tag{2.34}$$

## 3 Main results

Let $H = (h_0, h_1, \ldots, h_d)$ with $\frac{1}{2} < h_i < 1, i = 0, 1, \ldots, d$. In this section, we consider the following stochastic elastic equation

$$\frac{\mathrm{d}u_t(t,x)}{\mathrm{d}t} + \Delta^2 u(t,x) = u(t,x) \diamond W^H(t,x), \tag{3.1}$$

where $t > 0, x \in D = (0,1)^d$, with initial and boundary conditions

$$u(0,x) = u_0(x), \frac{\partial u}{\partial t}(0,x) = \upsilon_0(x), \quad x \in D,$$

$$u(t,x) = \Delta u(t,x) = 0, \quad \text{on } \partial D.$$

We assume that $u_0$ and $\upsilon_0$ are deterministic functions defined on $[0,1]^d$, $u_0$ and $\Delta u_0$ both vanish at $x = 0$ and $x = 1$, and that $W^H$ is a $(d+1)$-dimensional fractional white noise has been defined in section 2.

Let $r = (r_1, \ldots, r_d) \in \mathbb{N}^d, x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. Define $\varphi_r(x) = \sqrt{2^d} \prod_{i=1}^d \sin(r_i \pi x_i)$. Then $\{\varphi_r\}_{r \in \mathbb{N}^d}$ satisfy the boundary conditions of (3.1) and compose of a complete orthonormal system on $L^2(D)$ which diagonalize $\Delta$ with

$$\lambda_r = \pi^2 |r|^2 = \pi^2 \sum_{i=1}^d r_i^2, \tag{3.2}$$

the corresponding eigenvalues.

For a given function $g : D \to \mathbb{R}$ and $\rho \in \mathbb{R}$, define

$$\|g\|_{\rho,2} := \left( \sum_{r \in \mathbb{N}^d} (1 + |r|^2)^\rho |\langle g, \varphi_r \rangle|^2 \right)^{\frac{1}{2}},$$

where $\langle \cdot, \cdot \rangle$ stands for the usual scalar product in $L^2(D)$, and denote by $H^{\rho,2}(D)$ the set of functions $g : D \to \mathbb{R}$ such that $\|g\|_{\rho,2} < \infty$. Notice that $H^{\rho,2}(D)$ is a subspace of the fractional Sobolev space of fractional differential order $\alpha$ and integrability order $p = 2$ (see [27]). For a special case $\rho = 0$, it is clear that $H^{\rho,2}(D) = L^2(D)$ and we will denote $\|\cdot\|_{0,2}$ by $\|\cdot\|$. By Parseval's identity, we have

$$\|g\|^2 = \sum_{r \in \mathbb{N}^d} |\langle g, \varphi_r \rangle|^2, \quad \forall g \in L^2(D). \tag{3.3}$$

Since the fundamental solution of

$$\upsilon_{tt} + \Delta^2 \upsilon = 0,$$
$$\upsilon = \Delta u = 0 \text{ on } \partial D, \quad \upsilon|_{t=0} = \phi(x), \ \upsilon_t|_{t=0} = \psi(x) \text{ on } D$$

is given by

$$\upsilon(t,x)=\sum_{r\in\mathbb{N}^d}\frac{\sin(\lambda_r t)}{\lambda_r}\varphi_r(x)\int_D\psi(y)\varphi_r(y)\mathrm{d}y+\sum_{r\in\mathbb{N}^d}\cos(\lambda_r t)\varphi_r(x)\int_D\phi(y)\varphi_r(y)\mathrm{d}y, \tag{3.4}$$

we can define the solution of (3.1) as follows.

**Definition 3.1** *A random field $u=u(t,x):\mathbb{R}^+\times D\times\Omega\to\mathbb{R}$ is said to be a solution of (3.1), if*

*(i)* $u=u(t,x):\mathbb{R}^+\times D\times\Omega\to\mathbb{R}$ *is jointly measurable.*

*(ii) There exists constant $q\in\mathbb{N}$, such that for almost all $x\in D$ and $t\geq 0$,*

$$\sum_{r\in\mathbb{N}^d}\int_0^t\int_D\frac{\sin(\lambda_r(t-s))}{\lambda_r}\varphi_r(x)\varphi_r(y)u(s,y)\mathrm{d}B^H(s,y)$$

*is well defined as an element of $(\mathcal{S})^*_{H,-q}$, and that*

$$\int_D\left\|\sum_{r\in\mathbb{N}^d}\int_0^t\int_D\frac{\sin(\lambda_r(t-s))}{\lambda_r}\varphi_r(x)\varphi_r(y)u(s,y)\mathrm{d}B^H(s,y)\right\|^2_{H,-q}\mathrm{d}x<\infty,\forall t\geq 0.$$

*(iii) It holds in $(\mathcal{S})^*_{H,-q}$ that*

$$\begin{aligned}u(t,x) &= \sum_{r\in\mathbb{N}^d}\frac{\sin(\lambda_r t)}{\lambda_r}\varphi_r(x)\int_D\upsilon_0(y)\varphi_r(y)\mathrm{d}y\\ &\quad+\sum_{r\in\mathbb{N}^d}\cos(\lambda_r t)\varphi_r(x)\int_D u_0(y)\varphi_r(y)\mathrm{d}y\\ &\quad+\sum_{r\in\mathbb{N}^d}\int_0^t\int_D\frac{\sin(\lambda_r(t-s))}{\lambda_r}\varphi_r(x)\varphi_r(y)u(s,y)\mathrm{d}B^H(s,y).\end{aligned} \tag{3.5}$$

Let

$$\begin{aligned}U_0(t,x) &= \sum_{r\in\mathbb{N}^d}\frac{\sin(\lambda_r t)}{\lambda_r}\varphi_r(x)\int_D\upsilon_0(y)\varphi_r(y)\mathrm{d}y\\ &\quad+\sum_{r\in\mathbb{N}^d}\cos(\lambda_r t)\varphi_r(x)\int_D u_0(y)\varphi_r(y)\mathrm{d}y.\end{aligned}$$

Then by the definition of the stochastic integral, the solution of (3.1), if it exists, can be written as

$$u(t,x)=U_0(t,x)+\sum_{r\in\mathbb{N}^d}\int_0^t\int_D\frac{\sin(\lambda_r(t-s))}{\lambda_r}\varphi_r(x)\varphi_r(y)u(s,y)\diamond W^H(s,y)\mathrm{d}s\mathrm{d}y. \tag{3.6}$$

If we assume that the solution of (3.1) exists in $(\mathcal{S})_H^*$ and it has a formal expansions

$$u(t,x)=\sum_{\alpha\in\mathcal{J}}c_\alpha(t,x)\mathcal{H}_\alpha, \tag{3.7}$$

where $c_\alpha(t,x),\alpha\in\mathcal{J}$ are coefficients of Wiener chaos expansion of $u$, which are undetermined. Let

$$c_{\alpha-\varepsilon^{(i)}}=0 \ \text{ if } \alpha_i=0. \tag{3.8}$$

Then by the formal expansion of $W^H$, we obtain that

$$u(t,x)\diamond W^H(t,x)=\sum_{\alpha\in\mathcal{J}}\left[\sum_{i=1}^{\infty}c_{\alpha-\varepsilon^{(i)}}(t,x)\left(\int_{\mathbb{R}^{d+1}}e_i(\upsilon)\phi_H(t,x;\upsilon)\mathrm{d}\upsilon\right)\right]\mathcal{H}_\alpha(\omega). \tag{3.9}$$

Brought (3.9) into (3.6), we derive that

$$\begin{aligned}u(t,x)-U_0(t,x) &= \sum_{r\in\mathbb{N}^d}\int_0^t\int_D\frac{\sin(\lambda_r(t-s))}{\lambda_r}\varphi_r(x)\varphi_r(y)u(s,y)\diamond W^H(s,y)\mathrm{d}s\mathrm{d}y.\\ &= \sum_{|\alpha|\geq1}\left[\sum_{r\in\mathbb{N}^d}\int_0^t\int_D\frac{\sin(\lambda_r(t-s))}{\lambda_r}\varphi_r(x)\varphi_r(y)\sum_{i=1}^{\infty}c_{\alpha-\varepsilon^{(i)}}(s,y)\right.\\ &\quad\left.\times\left(\int_{\mathbb{R}^{d+1}}e_i(\upsilon)\phi_H(s,y;\upsilon)\mathrm{d}\upsilon\right)\mathrm{d}s\mathrm{d}y\right]\mathcal{H}_\alpha.\end{aligned} \tag{3.10}$$

Therefore, by (3.10) and (3.7), we get $c_\alpha(t,x)=U_0(t,x)$ if $\alpha=0$ and for $|\alpha|\geq1$,

$$\begin{aligned}c_\alpha(t,x) &= \sum_{r\in\mathbb{N}^d}\int_0^t\int_D\frac{\sin(\lambda_r(t-s))}{\lambda_r}\varphi_r(x)\varphi_r(y)\sum_{i=1}^{\infty}c_{\alpha-\varepsilon^{(i)}}(s,y)\\ &\quad\times\left(\int_{\mathbb{R}^{d+1}}e_i(\upsilon)\phi_H(s,y;\upsilon)\mathrm{d}\upsilon\right)\mathrm{d}s\mathrm{d}y.\end{aligned} \tag{3.11}$$

We will need the following preliminaries and lemmas to estimate $c_\alpha$.

The boundedness and Hölder continuity of $U_0$ are given by the following lemma.

**Lemma 3.1** *Assume that $\upsilon_0\in H^{\varrho,2}(D)$ for some $\varrho\geq-2$ and $u_0\in H^{\rho,2}(D)$ for some $\rho\geq0$, then $U_0(t,\cdot)\in L^2(D)$, and*

$$\sup_{t\in\mathbb{R}^+}\|U_0(t,\cdot)\|<+\infty. \tag{3.12}$$

*Moreover, if $\upsilon_0\in H^{\varrho,2}(D)$ for some $\varrho\geq0$ and $u_0\in H^{\rho,2}(D)$ for some $\rho\geq2$, then, for any $t,s\in\mathbb{R}^+$ with $|t-s|<1$,*

$$\|U_0(t,\cdot)-U_0(s,\cdot)\|^2\leq C|t-s|^2. \tag{3.13}$$

*Proof.* It is clear that $\|U_0(t,\cdot)\| \leq \|I(t,\cdot)\| + \|J(t,\cdot)\|$, with

$$\begin{aligned} I(t,x) &= \sum_{r\in\mathbb{N}^d} \frac{\sin(\lambda_r t)}{\lambda_r}\varphi_r(x)\int_D \upsilon_0(y)\varphi_r(y)\mathrm{d}y, \\ J(t,x) &= \sum_{r\in\mathbb{N}^d} \cos(\lambda_r t)\varphi_r(x)\int_D u_0(y)\varphi_r(y)\mathrm{d}y. \end{aligned}$$

By (3.2), (3.3) and the assumptions of $\upsilon_0$, we have

$$\begin{aligned} &\|I(t,\cdot)\|^2 = \sum_{r\in\mathbb{N}^d} |< I(t,\cdot),\varphi_r >|^2 \\ = &\sum_{r\in\mathbb{N}^d} \frac{\sin^2(\lambda_r t)}{\lambda_r^2}|<\varphi_r,\upsilon_0>|^2 \leq C\sum_{r\in\mathbb{N}^d}\frac{|<\varphi_r,\upsilon_0>|^2}{|r|^4} < +\infty. \end{aligned}$$

Similarly, we have

$$\|J(t,\cdot)\|^2 \leq C\sum_{r\in\mathbb{N}^d}|<\varphi_r,u_0>|^2 < +\infty.$$

Thus, the first part of the lemma if proved. For the second part, we have

$$\|U_0(t,\cdot)-U_0(s,\cdot)\|^2 \leq 2(\|I_1(t,s;\cdot)\|^2 + \|I_2(t,s;\cdot)\|^2),$$

where

$$\begin{aligned} I_1(t,s;x) &= \sum_{r\in\mathbb{N}^d}\frac{\sin(\lambda_r t)-\sin(\lambda_r s)}{\lambda_r}\varphi_r(x)\int_D \upsilon_0(y)\varphi_r(y)\mathrm{d}y, \\ I_2(t,s;x) &= \sum_{r\in\mathbb{N}^d}(\cos(\lambda_r t)-\cos(\lambda_r s))\varphi_r(x)\int_D u_0(y)\varphi_r(y)\mathrm{d}y. \end{aligned}$$

By the assumptions on $u_0,\upsilon_0$, we get

$$\begin{aligned} \|I_1(t,s;\cdot)\|^2 &= \sum_{r\in\mathbb{N}^d}|<I_1(t,s;\cdot),\varphi_r>|^2 \\ &= \sum_{r\in\mathbb{N}^d}\frac{|\sin(\lambda_r t)-\sin(\lambda_r s)|^2}{\lambda_r^2}|<\varphi_r,\upsilon_0>|^2 \\ &\leq C(t-s)^2\sum_{r\in\mathbb{N}^d}|<\varphi_r,\upsilon_0>|^2 \leq C(t-s)^2, \end{aligned}$$

and

$$\begin{aligned} \|I_2(t,s;\cdot)\|^2 &= \sum_{r\in\mathbb{N}^d}|<I_2(t,s;\cdot),\varphi_r>|^2 \\ &= \sum_{r\in\mathbb{N}^d}|\cos(\lambda_r t)-\cos(\lambda_r s)|^2|<\varphi_r,\upsilon_0>|^2 \\ &\leq C(t-s)^2\sum_{r\in\mathbb{N}^d}|\lambda_r|^2|<\varphi_r,\upsilon_0>|^2 \\ &\leq C(t-s)^2\sum_{r\in\mathbb{N}^d}(1+|r|^2)^2|<\varphi_r,\upsilon_0>|^2 \leq C(t-s)^2. \end{aligned}$$

Bringing together the above estimates, we obtain

$$\|U_0(t,\cdot)-U_0(s,\cdot)\|^2 \le C(t-s)^2,$$

with a constant $C$ independent of $t$. Thus the lemma is proved. ¶

Let $\Phi_i(t,x)=\int_{\mathbb{R}^{d+1}} e_i(s,y)\phi_H(t,x;s,y)\mathrm{d}s\mathrm{d}y$, then we have the following lemma.

**Lemma 3.2** *There exists constant $C=C(h)$, such that*

$$\sup_{x\in D}|\Phi_i(t,x)| \le C(h)i^{\sigma(h)}(1\vee t^{h-\frac{1}{2}}), \quad i\in\mathbb{N}, \tag{3.14}$$

*where $h=\max\{h_0,h_1,\ldots,h_d\}, 1\vee t^{h-\frac{1}{2}}=\max\{1,t^{h-\frac{1}{2}}\}$ and*

$$\sigma(h)=\begin{cases} \frac{1}{4}(h-\frac{2}{3}), & \text{if } h<\frac{2}{3},\\ \frac{1}{2}(h-\frac{2}{3}), & \text{if } h\ge\frac{2}{3}.\end{cases} \tag{3.15}$$

*Proof.* By the definitions of $e_i$ and $\phi_H$, we have

$$\begin{aligned}&\int_{\mathbb{R}^{d+1}} e_i(s,y)\phi_H(t,x;s,y)\mathrm{d}s\mathrm{d}y\\ =\ &\int_{\mathbb{R}}\eta^{h_0}_{\delta^i_0}(s)\phi_{h_0}(t,s)\mathrm{d}s\prod_{k=1}^{d}\int_{\mathbb{R}}\eta^{h_k}_{\delta^i_k}(y)\phi_{h_k}(x,y)\mathrm{d}y.\end{aligned} \tag{3.16}$$

Let $\frac{1}{2}<h<1$ and $n\in\mathbb{N}$. It is proved in [**?** ] that

$$\phi_h(t,s)=c_h^2\int_{-\infty}^{t\wedge s}(s-u)^{h-\frac{3}{2}}(t-u)^{h-\frac{3}{2}}\mathrm{d}u. \tag{3.17}$$

where $c_h=\sqrt{h(2h-1)\Gamma(3/2-h)/(\Gamma(h-1/2)\Gamma(2-2h))}$. Therefore, by (2.15) and Lemma 3.2, we have

$$\begin{aligned}&\left|\int_{\mathbb{R}}\eta^h_n(s)\phi_h(t,s)\mathrm{d}s\right| = c_h^2\left|\int_{\mathbb{R}}\eta^h_n(s)\int_{-\infty}^{t\wedge s}(s-u)^{h-\frac{3}{2}}(t-u)^{h-\frac{3}{2}}\mathrm{d}u\mathrm{d}s\right|\\ =\ & c_h^2\left|\int_{-\infty}^{t}(t-u)^{h-\frac{3}{2}}\mathrm{d}u\int_{u}^{+\infty}(s-u)^{h-\frac{3}{2}}\eta^h_n(s)\mathrm{d}s\right|\\ =\ & c_h^2\left|\int_{-\infty}^{t}(t-u)^{h-\frac{3}{2}}\frac{1}{c_h}I_-^{h-\frac{1}{2}}\eta^h_n(u)\mathrm{d}u\right| = c_h\left|\int_{-\infty}^{t}(t-u)^{h-\frac{3}{2}}\xi_n(u)\mathrm{d}u\right|\\ \le\ & c_h\left[\int_{|u|\le 2\sqrt{n}}(t-u)^{h-\frac{3}{2}}n^{-\frac{1}{12}}\mathrm{d}u+\int_{|u|>2\sqrt{n}}(t-u)^{h-\frac{3}{2}}e^{-\gamma u^2}\mathrm{d}u\right].\end{aligned} \tag{3.18}$$

If $0\le t\le 2\sqrt{n}$, it follows from (3.18) that

$$\begin{aligned}&\int_{\mathbb{R}}\eta^h_n(s)\phi_h(t,s)\mathrm{d}s\\ \le\ & c_h\left[\int_{-\infty}^{-2\sqrt{n}}\left[(t-u)^{h-\frac{3}{2}}\frac{1}{u}\right]ue^{-\gamma u^2}\mathrm{d}u+\int_{-2\sqrt{n}}^{t}(t-u)^{h-\frac{3}{2}}n^{-\frac{1}{12}}\mathrm{d}u\right]\\ \le\ & c_h\left[\frac{e^{-4\gamma n}}{4\gamma}n^{-\frac{1}{2}}(t+2\sqrt{n})^{h-\frac{3}{2}}+\frac{1}{h-\frac{1}{2}}n^{-\frac{1}{12}}(t+2\sqrt{n})^{h-\frac{1}{2}}\right]\\ \le\ & C(h)n^{-\frac{1}{12}}(t+2\sqrt{n})^{h-\frac{1}{2}}\le C(h)n^{\frac{1}{2}(h-\frac{2}{3})}(1\vee t^{h-\frac{1}{2}}).\end{aligned} \tag{3.19}$$

For the case of $t > 2\sqrt{n}$, we have

$$\begin{aligned}
&\int_{\mathbb{R}} \eta_n^h(s)\phi_h(t,s)\mathrm{d}s \le c_h \left[\int_{-2\sqrt{n}}^{2\sqrt{n}} (t-u)^{h-\frac{3}{2}} n^{-\frac{1}{12}}\mathrm{d}u \right.\\
+ &\left. \int_{-\infty}^{-2\sqrt{n}} \left[(t-u)^{h-\frac{3}{2}}\frac{1}{u}\right] u e^{-\gamma u^2}\mathrm{d}u + \int_{2\sqrt{n}}^{t} (t-u)^{h-\frac{3}{2}} e^{-\gamma u^2}\mathrm{d}u\right]\\
\le\ & c_h \left[\frac{e^{-4\gamma n}(t+2\sqrt{n})^{h-\frac{3}{2}}}{4\gamma n^{\frac{1}{2}}} + \frac{n^{-\frac{1}{12}}(t+2\sqrt{n})^{h-\frac{1}{2}} - (n^{-\frac{1}{12}} - e^{-4\gamma n})(t-2\sqrt{n})^{h-\frac{1}{2}}}{h-\frac{1}{2}}\right]\\
\le\ & C(h) n^{-\frac{1}{12}}(t+2\sqrt{n})^{h-\frac{1}{2}} \le C(h) n^{\frac{1}{2}(h-\frac{2}{3})}(1 \vee t^{h-\frac{1}{2}}). \qquad (3.20)
\end{aligned}$$

With the estimates of (3.19), (3.20), we obtain from (3.16) that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^{d+1}} e_i(s,y)\phi_H(t,x;s,y)\right| \le C(h_0,h_1,\cdots,h_d)\prod_{k=0}^{d}(\delta_k^i)^{\frac{1}{2}(h_k-\frac{2}{3})}(1\vee t^{h_1-\frac{1}{2}})\\
\le\ & C(h_0,h_1,\cdots,h_d)(\prod_{k=0}^{d}\delta_k^i)^{\frac{1}{2}(h-\frac{2}{3})}(1\vee t^{h-\frac{1}{2}}) \le C(h) i^{\sigma(h)}(1\vee t^{h-\frac{1}{2}}), \qquad (3.21)
\end{aligned}$$

where $h = \max\{h_0, h_1, \cdots, h_d\}$ and $\sigma(h)$ is given by (3.15). This proves the lemma. ¶

Using the above lemmas, now we can estimate $c_\alpha(t,x)$.

**Lemma 3.3** *Assume that all the assumptions of Lemma 3.1 are satisfied, then for every $\alpha \in \mathcal{J}$ and $t \ge 0$, $c_\alpha(t,\cdot) \in L^2(D)$. Moreover, there exists constant $C = C(h)$, such that*

$$\|c_\alpha(t,\cdot)\|^2 \le C(h)^{2|\alpha|}\frac{(\mathbb{N})^{(2\sigma(h)+1)\alpha}}{\alpha!}(t^2 \vee t^{2h+1})^{|\alpha|}. \qquad (3.22)$$

*Proof.* Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ and $C_\alpha(t) = \|c_\alpha(t,\cdot)\|^2$. By Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}
C_\alpha(t) &= \sum_{r\in\mathbb{N}^d} |< c_\alpha(t,\cdot), \varphi_r >|^2\\
&= \sum_{r\in\mathbb{N}^d}\left|\int_0^t\int_D \frac{\sin(\lambda_r(t-s))}{\lambda_r}\varphi_r(y)\sum_{i=1}^{m} c_{\alpha-\varepsilon^{(i)}}(s,y)\Phi_i(s,y)\mathrm{d}s\mathrm{d}y\right|^2\\
&\le m\sum_{r\in\mathbb{N}^d}\sum_{i=1}^{m}\left|\int_0^t \frac{\sin(\lambda_r(t-s))}{\lambda_r}\int_D \varphi_r(y)c_{\alpha-\varepsilon^{(i)}}(s,y)\Phi_i(s,y)\mathrm{d}y\mathrm{d}s\right|^2\\
&\le m\sum_{r\in\mathbb{N}^d}\sum_{i=1}^{m}\int_0^t\left|\frac{\sin(\lambda_r(t-s))}{\lambda_r}\right|^2\mathrm{d}s\int_0^t\left|\int_D \varphi_r(y)c_{\alpha-\varepsilon^{(i)}}(s,y)\Phi_i(s,y)\mathrm{d}y\right|^2\mathrm{d}s\\
&\le Ctm\sum_{i=1}^{m}\int_0^t\sum_{r\in\mathbb{N}^d}|<\varphi_r, c_{\alpha-\varepsilon^{(i)}}(s,\cdot)\Phi_i(s,\cdot)>|^2\mathrm{d}s\\
&\le Ctm\sum_{i=1}^{m}\int_0^t \|c_{\alpha-\varepsilon^{(i)}}(s,\cdot)\Phi_i(s,\cdot)\|^2\mathrm{d}s. \qquad (3.23)
\end{aligned}$$

Owing to Lemma 3.2, $\sup_{x\in D}|\Phi_i(t,x)| \le C(h)i^{\sigma(h)}(1\vee t^{h-\frac{1}{2}})$. Hence, by (3.23),

$$\begin{aligned} C_\alpha(t) &\le Ctm\sum_{i=1}^m \int_0^t \|c_{\alpha-\varepsilon^{(i)}}(s,\cdot)\Phi_i(s,\cdot)\|^2 \mathrm{d}s \\ &\le C(h)^2 tm\sum_{i=1}^m i^{2\sigma(h)}\int_0^t C_{\alpha-\varepsilon^{(i)}}(s)(1\vee(s^{h-\frac{1}{2}}))^2\mathrm{d}s \\ &\le C(h)^2(t\vee t^{2h})m\sum_{i=1}^m i^{2\sigma(h)}\int_0^t C_{\alpha-\varepsilon^{(i)}}(s)\mathrm{d}s. \end{aligned} \tag{3.24}$$

Now let $n=|\alpha|=\alpha_1+\cdots+\alpha_m$. By iterating the above equation, we get

$$\begin{aligned} & C_\alpha(t) \le C(h)^2(t\vee t^{2h})m\sum_{i=1}^m i^{2\sigma(h)}\int_0^t C_{\alpha-\varepsilon^{(i)}}(s)\mathrm{d}s \\ \le\ & C(h)^{2n}(t\vee t^{2h})^n[1^{\alpha_1}2^{\alpha-2}\cdots m^{\alpha_m}]\sum_{i_1,i_2,\ldots,i_n=1}^m (i_1i_2\cdots i_n)^{2\sigma(h)}\times \\ & \int_0^t\int_0^s\int_0^{s_1}\cdots\int_0^{s_{n-2}} C_{\alpha-\varepsilon^{(i_1)}-\varepsilon^{(i_2)}\ldots-\varepsilon^{(i_n)}}(s_{n-1})\mathrm{d}s_{n-1}\cdots\mathrm{d}s_1\mathrm{d}s. \end{aligned} \tag{3.25}$$

Since for some $\beta\in\mathcal{J}$ with $\beta_j=0$ we have $C_{\beta-\varepsilon^{(j)}}=0$, and by Lemma 3.1, $C_0(t)=\|U_0(t,\cdot)\|$ is bounded on $\mathbb{R}^+$, thus we derive from (3.25) that

$$\begin{aligned} C_\alpha(t) &\le C(h)^{2n}(t\vee t^{2h})^n[1^{\alpha_1}2^{\alpha-2}\cdots m^{\alpha_m}]\frac{n!}{\alpha_1!\cdots\alpha_m!}[1^{\alpha_1}2^{\alpha_2}\cdots m^{\alpha_m}]^{2\sigma(h)}\times \\ &\quad \int_0^t\int_0^s\int_0^{s_1}\cdots\int_0^{s_{n-2}} C_0(s_{n-1})\mathrm{d}s_{n-1}\cdots\mathrm{d}s_1\mathrm{d}s \\ &\le C(h)^{2n}(t\vee t^{2h})^n\frac{n!}{\alpha_1!\cdots\alpha_m!}[1^{\alpha_1}2^{\alpha_2}\cdots m^{\alpha_m}]^{2\sigma(h)+1}\frac{t^n}{n!} \\ &= C(h)^{2|\alpha|}\frac{(\mathbb{N})^{(2\sigma(h)+1)\alpha}}{\alpha!}(t^2\vee t^{2h+1})^{|\alpha|}. \end{aligned} \tag{3.26}$$

This proves the lemma. ¶

The following lemma will be used in the proof of our main results.

**Lemma 3.4** *Let constants $a>0, b\in\mathbb{R}, q\in\mathbb{R}$. Then the sufficient and necessary condition for*

$$\sum_{\alpha\in\mathcal{J}}(a\mathbb{N})^{b\alpha}(2\mathbb{N})^{-q\alpha}<\infty \tag{3.27}$$

*is $q>\max\{b+1, b\log_2 a\}$.*

*Proof.* For $\alpha=(\alpha_1,\alpha_2,\ldots,)\in\mathcal{J}$, let $\alpha_0=\max\{j;\alpha_j\neq 0\}$. Then

$$\sum_{\alpha\in\mathcal{J}}(a\mathbb{N})^{b\alpha}(2\mathbb{N})^{-q\alpha}=\sum_{n=0}^\infty\sum_{\alpha\in\mathcal{J},\alpha_0=n}(a\mathbb{N})^{b\alpha}(2\mathbb{N})^{-q\alpha}:=\sum_{n=0}^\infty a_n.$$

We assume $q > b\log_2 a$ and $q > b+1$, then $a_0 = 1$ and for $n \geq 1$,

$$\begin{aligned} a_n &= \sum_{\alpha\in\mathcal{J},\alpha_0=n} (a\mathbb{N})^{b\alpha}(2\mathbb{N})^{-q\alpha} = \sum_{\alpha_1,\ldots,\alpha_{n-1}\geq 0,\alpha_n\geq 1} (a\mathbb{N})^{b\alpha}(2\mathbb{N})^{-q\alpha} \\ &= \left(\sum_{\alpha_n=1}^{\infty}(an)^{b\alpha_n}(2n)^{-q\alpha_n}\right)\left[\prod_{j=1}^{n-1}\left(\sum_{\alpha_j=0}^{\infty}(aj)^{b\alpha_j}(2j)^{-q\alpha_j}\right)\right] \\ &= \left(\sum_{\alpha_n=1}^{\infty}\left[\frac{(an)^b}{(2n)^q}\right]^{\alpha_n}\right)\left[\prod_{j=1}^{n-1}\left(\sum_{\alpha_j=0}^{\infty}\left[\frac{(aj)^b}{(2j)^q}\right]^{\alpha_j}\right)\right] \\ &= \frac{(an)^b}{(2n)^q-(an)^b}\prod_{j=1}^{n-1}\frac{(2j)^q}{(2j)^2-(aj)^b} = \frac{(an)^b}{(2n)^q}\prod_{j=1}^{n}\frac{(2j)^q}{(2j)^2-(aj)^b}. \end{aligned} \tag{3.28}$$

By (3.28), we have

$$\begin{aligned} \frac{a_n}{a_{n+1}} &= \frac{(an)^b(2n+2)^q}{(an+a)^b(2n)^q}\times\frac{(2n+2)^q-(an+a)^b}{(2n+2)^q} \\ &= \left(\frac{n}{n+1}\right)^b\left[\left(\frac{n+1}{n}\right)^q-\frac{(an+a)^b}{(2n)^q}\right] = \left(1+\frac{1}{n}\right)^{q-b}-\frac{(an)^b}{(2n)^q}. \end{aligned} \tag{3.29}$$

This gives

$$\frac{a_n}{a_{n+1}}-1 \geq \frac{q-b}{n}-\frac{(an)^b}{(2n)^q}. \tag{3.30}$$

Hence

$$\liminf_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right) \geq q-b > 1. \tag{3.31}$$

Therefore, by Abel's criterion for convergence, $\sum_{n=0}^{\infty} a_n < \infty$. Conversely, if $q \leq b\log_2 a$, then, by (3.28), $a_n = \infty$. If $q = b+1$, then, by (3.29), $\liminf_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right) = 1-\frac{1}{2}(\frac{a}{2})^b < 1$, by Abel's criterion, $\sum_{n=0}^{\infty} a_n = \infty$. Hence, condition $q > \max\{b+1, b\log_2 a\}$ is sufficient and necessary for the convergence of (3.27). ¶

We now present our main results.

**Theorem 3.1** *Assume that all the conditions in Lemma 3.1 are satisfied, then there exists a unique solution of (3.1) in the sense of Definition 3.1.*

*Proof.* From the above analyses and lemmas, we know that the stochastic field $u(t,x)$ with a formal expansions

$$u(t,x) = \sum_{\alpha\in\mathcal{J}} c_\alpha(t,x)\mathcal{H}_\alpha, \tag{3.32}$$

where $c_\alpha(t,x) = U_0(t,x)$ if $\alpha = 0$ and for $|\alpha| \geq 1$,

$$\begin{aligned} c_\alpha(t,x) &= \sum_{r\in\mathbb{N}^d}\int_0^t\int_D \frac{\sin(\lambda_r(t-s))}{\lambda_r}\varphi_r(x)\varphi_r(y)\sum_{i=1}^{\infty} c_{\alpha-\varepsilon^{(i)}}(s,y) \\ &\quad\times\left(\int_{\mathbb{R}^{d+1}} e_i(\upsilon)\phi_H(s,y;\upsilon)\mathrm{d}\upsilon\right)\mathrm{d}s\mathrm{d}y, \end{aligned} \tag{3.33}$$

is a solution of (3.1) in the sense of Definition 3.1, if for almost every $x \in D$ and $t \geq 0$, it belongs to $(\mathcal{S})^*_{H,-q}$ for some $q \in \mathbb{N}$ and $\int_D \|u(t,x)\|^2_{H,-q}\mathrm{d}x < +\infty, (\forall t \geq 0)$. Now we compute $\int_D \|u(t,x)\|^2_{H,-q}\mathrm{d}x$. By Lemma 3.3, we have

$$\begin{aligned}\int_D \|u(t,x)\|^2_{H,-q}\mathrm{d}x &= \sum_{\alpha\in\mathcal{J}} \alpha!\|c_\alpha(t,\cdot)\|^2(2\mathbb{N})^{-q\alpha} \\ \leq \sum_{\alpha\in\mathcal{J}} &\frac{C(h)^{2|\alpha|}(\mathbb{N})^{(2\sigma(h)+1)\alpha}(t^2\vee t^{2h+1})^{|\alpha|}(2\mathbb{N})^{-q\alpha}}{\alpha!}. \end{aligned} \tag{3.34}$$

It is prove in [13] that for $\alpha \in \mathcal{J}$,

$$|\alpha|! \leq \alpha!(2\mathbb{N})^{2\alpha}. \tag{3.35}$$

Therefore,

$$\begin{aligned}\int_D \|u(t,x)\|^2_{H,-q}\mathrm{d}x &\leq \sum_{\alpha\in\mathcal{J}} \frac{C(h)^{2|\alpha|}(\mathbb{N})^{(2\sigma(h)+1)\alpha}(t^2\vee t^{2h+1})^{|\alpha|}(2\mathbb{N})^{-q\alpha}(2\mathbb{N})^{2\alpha}}{|\alpha|!} \\ = \sum_{n=0}^{\infty} \frac{(t^2\vee t^{2h+1})^n}{n!} &\sum_{|\alpha|=n} \left(C(h)^{\frac{2}{2\sigma(h)+1}}\mathbb{N}\right)^{(2\sigma(h)+1)\alpha}(2\mathbb{N})^{-(q-2)\alpha}. \end{aligned} \tag{3.36}$$

If we choose a natural number $q > \max\{4+2\sigma(h), 2+2\log_2 C(h)\}$, then by Lemma 3.4, $\sum_{|\alpha|=n}\left(C(h)^{\frac{1}{2\sigma(h)+1}}\mathbb{N}\right)^{(2\sigma(h)+1)\alpha}(2\mathbb{N})^{-(q-2)\alpha} < \infty$. Thus

$$\int_D \|u(t,x)\|^2_{H,-q}\mathrm{d}x \leq C(h)\sum_{n=0}^{\infty}\frac{(t^2\vee t^{2h+1})^n}{n!} = C(h)e^{t^2\vee t^{2h+1}}. \tag{3.37}$$

Now we prove the uniqueness. Assume that stochastic fields $u(t,x), \upsilon(t,x)$ with formal expansions

$$u(t,x) = \sum_{\alpha\in\mathcal{J}} a_\alpha(t,x)\mathcal{H}_\alpha,\ \upsilon(t,x) = \sum_{\alpha\in\mathcal{J}} b_\alpha(t,x)\mathcal{H}_\alpha \tag{3.38}$$

are two solutions of (3.1) in the sense of Definition 3.1. Then $u(t,x) - \upsilon(t,x)$ is a solution of of (3.1) in the sense of Definition 3.1 with zero initial conditions, that is, $a_0(t,x) - b_0(t,x) = 0$. Therefore, similarly to the proof of Lemma 3.3, we can derive that

$$\begin{aligned}&\|a_\alpha(t,\cdot) - b_\alpha(t,\cdot)\|^2 \\ \leq\ & C(h)^{2n}(t\vee t^{2h})^n[1^{\alpha_1}2^{\alpha-2}\cdots m^{\alpha_m}]\frac{n!}{\alpha_1!\cdots\alpha_m!}[1^{\alpha_1}2^{\alpha_2}\cdots m^{\alpha_m}]^{2\sigma(h)}\times \\ &\int_0^t\int_0^s\int_0^{s_1}\cdots\int_0^{s_{n-2}} \|a_0(s_{n-1},\cdot) - b_0(s_{n-1},\cdot)\|^2\mathrm{d}s_{n-1}\cdots\mathrm{d}s_1\mathrm{d}s = 0. \end{aligned} \tag{3.39}$$

Thus the theorem follows. ¶

The following corollary deals with the asymptotic properties of $u(t,x)$ in $(\mathcal{S})^*_{H,-q}$.

**Corollary 3.1** *Let $u(t,x)$ be the solution of (3.1) and $q > \max\{4+2\sigma(h), 2+2\log_2 C(h)\}$, where $C(h)$ is given by (3.37), then $u(t,x) \in (\mathcal{S})^*_{H,-q}$ for almost all $(t,x) \in \mathbb{R}^+ \times [0,1]$, and*

$$\limsup_{t\to+\infty} \frac{\ln(\int_D \|u(t,x)\|^2_{H,-q}\mathrm{d}x)}{t^{2h+1}} < \infty. \tag{3.40}$$

*Proof.* By (3.37), we know that for sufficiently large $t$,

$$\int_D \|u(t,x)\|^2_{H,-q}\mathrm{d}x \quad \leq \quad C(h)e^{t^{2h+1}}. \tag{3.41}$$

Therefore

$$\limsup_{t\to+\infty} \frac{\ln(\int_D \|u(t,x)\|^2_{H,-q}\mathrm{d}x)}{t^{2h+1}} \leq \limsup_{t\to+\infty} \frac{\ln(C(h)) + t^{2h+1}}{t^{2h+1}} = 1. \tag{3.42}$$

This shows the corollary. ¶

We now study the Hölder property of the trajectories of the solution of equation (3.1) in the distribution space $(\mathcal{S})^*_{H,-q}$.

**Theorem 3.2** *Fix $T > 0$ and let $u(t,x)$ be the solution of (3.1) in time interval $[0,T]$ with initial conditions $\upsilon_0 \in H^{\beta_1,2}(D)$ for some $\beta_1 \geq 0$ and $u_0 \in H^{\beta_2,2}$ for some $\beta_2 \geq 2$, then there exists $q \in \mathbb{N}$, such that for any $t,\tau \in [0,T]$ with $|t-\tau| < 1$,*

$$\int_D \|u(t,x) - u(\tau,x)\|^2_{H,-q}\mathrm{d}x \leq C|t-\tau|. \tag{3.43}$$

*Proof.* Let $0 \leq \tau < t \leq T$. First we estimate the term $\|c_\alpha(t,\cdot) - c_\alpha(\tau,\cdot)\|$. Since

$$\begin{aligned}
& c_\alpha(t,x) - c_\alpha(\tau,x) \\
= & \sum_{r\in\mathbb{N}^d} \int_0^t \int_D \frac{\sin(\lambda_r(t-s))}{\lambda_r} \varphi_r(x)\varphi_r(y) \sum_{i=1}^\infty c_{\alpha-\varepsilon^{(i)}}(s,y)\Phi_i(s,y)\mathrm{d}s\mathrm{d}y \\
& - \sum_{r\in\mathbb{N}^d} \int_0^\tau \int_D \frac{\sin(\lambda_r(\tau-s))}{\lambda_r} \varphi_r(x)\varphi_r(y) \sum_{i=1}^\infty c_{\alpha-\varepsilon^{(i)}}(s,y)\Phi_i(s,y)\mathrm{d}s\mathrm{d}y \\
= & \sum_{r\in\mathbb{N}^d} \int_\tau^t \int_D \frac{\sin(\lambda_r(t-s))}{\lambda_r} \varphi_r(x)\varphi_r(y) \sum_{i=1}^\infty c_{\alpha-\varepsilon^{(i)}}(s,y)\Phi_i(s,y)\mathrm{d}s\mathrm{d}y \\
& + \sum_{r\in\mathbb{N}^d} \int_0^\tau \int_D \frac{\sin(\lambda_r(t-s)) - \sin(\lambda_r(\tau-s))}{\lambda_r} \varphi_r(x)\varphi_r(y) \\
& \times \sum_{i=1}^\infty c_{\alpha-\varepsilon^{(i)}}(s,y)\Phi_i(s,y)\mathrm{d}s\mathrm{d}y \\
\doteq & I(t,\tau;x) + J(t,\tau;x).
\end{aligned}$$

By using the methods used in the proof of Lemma 3.3, we can derive that

$$\begin{aligned}
\|I(t,\tau;\cdot)\|^2 \quad &\leq \quad (t-\tau)^2 C(h)^{2|\alpha|} \frac{(\mathbb{N})^{(2\sigma(h)+1)\alpha}}{\alpha!} (T^2 \vee T^{2h+1})^{|\alpha|}, \\
\|J(t,\tau;\cdot)\|^2 \quad &\leq \quad (t-\tau) C(h)^{2|\alpha|} \frac{(\mathbb{N})^{(2\sigma(h)+1)\alpha}}{\alpha!} (T^2 \vee T^{2h+1})^{|\alpha|}.
\end{aligned}$$

Thus we have $\|c_\alpha(t,\cdot)-c_\alpha(\tau,\cdot)\|^2 \le C(t-\tau)C(h)^{2|\alpha|}\frac{(\mathbb{N})^{(2\sigma(h)+1)\alpha}}{\alpha!}(T^2\vee T^{2h+1})^{|\alpha|}$. By the definition of the norm $\|\cdot\|^2_{H,-q}$, we can divide the term $\int_D \|u(t,x)-u(\tau,x)\|^2_{H,-q}\mathrm{d}x$ into two terms, that is

$$\begin{aligned}
&\int_D \|u(t,x)-u(\tau,x)\|^2_{H,-q}\mathrm{d}x\\
\le\ & 2\left(\|U_0(t,\cdot)-U_0(\tau,\cdot)\|^2+\sum_{\alpha\in\mathcal{J},|\alpha|\ge 1}\alpha!\|c_\alpha(t,\cdot)-c_\alpha(\tau,\cdot)\|^2(2\mathbb{N})^{-q\alpha}\right)\\
\le\ & C(t-\tau)\sum_{\alpha\in\mathcal{J}}\left[(C(h)(T\vee T^{h+\frac{1}{2}}))^{\frac{2}{2\sigma(h)+1}}\mathbb{N}\right]^{(2\sigma(h)+1)\alpha}(2\mathbb{N})^{-q\alpha}.
\end{aligned}$$

Bringing together the above estimates and by Lemma 3.1, Lemma 3.3, we obtain that

$$\int_D \|u(t,x)-u(\tau,x)\|^2_{H,-q}\mathrm{d}x \ \le\ C(h,T)(t-\tau),$$

if $q>\max\{2\sigma(h)+2, 2\log_2 C(h)(T\vee T^{h+\frac{1}{2}})\}$, where $C(h)$ is given by (3.37). The proof is finished. ¶

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## References

[1] R.M. Balan, C.A. Tudor, The stochastic wave equation with fractional noise: A random field approach, Stochastic Process. Appl. 120 (2010) 2468–2494.

[2] R.M. Balan, C.A. Tudor, Stochastic heat equation with multiplicative fractional-colored noise, J. Theor. Probab. 23 (2010) 834–870.

[3] F. Biagini, Y. Hu, B. Øksendal, T. Zhang, Stochastic Calculus for Fractional Brownian Motion and Applications, Springer-Verlag, London, 2008.

[4] L. Bo, Y. Jiang, Y. Wang, Stochastic Cahn-Hilliard equation with fractional noise, Stoch. Dyn. 4 (2008) 643–665.

[5] Z. Brzeźniak, B. Maslowski, J. Seidler, Stochastic nonlinear beam equations, Probab. Theory Relat. Field. 132 (2005) 119–149.

[6] P.L. Chow, J.L. Menaldi, Stochastic PDE for nonlinear Vibration of elastic panels, Diff. Int. Eqns. 12(3) (1999) 419–434.

[7] T.E. Duncan, J. Jakubowski, B. Pasik-Duncan, Stochastic integration for fractional Brownian motion in a Hilbert space, Stoch. Dyn. 6 (2006) 53–75.

[8] T.E. Duncan, B. Maslowski, B. Pasik-Duncan, Semilinear stochastic equations in a Hilbert space with a fractional Brownian motion, SIAM J. Math. Anal. 40 (2009) 2286–2315.

[9] T.E. Duncan, B. Maslowski, B. Pasik-Duncan, Stochastic equation in Hilbert space with a multiplicative fractional Gaussian noise, Stoch. Process. Appl. 115 (2005) 1357–1383.

[10] T.E. Duncan, Y. Hu, B. Pasik-Duncan, Stochastic calculus for fractional Brownian motion. I. Theory, SIAM J. Control Optim. 38 (2000) 582–612.

[11] M. Erraoui, D. Nualart, Y. Ouknine, Hyperbolic stochastic partial differential equations with additive fractional Brownian sheet, Stoch. Dyn. 3 (2003) 121–139.

[12] O.H. Galal, M.A. El-Tawil, A.A. Mahmoud, Stochastic beam equations under random dynamic loads, Inter. J. Solid. Struct. 39 (2002) 1031–1040.

[13] H. Holden, B. Øksendal, J. Uboe, T. Zhang, Stochastic Partial Differential Equations, a Modeling, White Noise Functional Approach, Second Edition published by Springer Science+Business Media, LLC, 2010.

[14] Y. Hu, Heat equation with fractional white noise potential, Appl. Math. Optim. 43 (2001) 221–243.

[15] Y. Hu, D. Nualart, Stochastic heat equation driven by fractional noise and local time, Probab. Theory Related Fields 143 (2009) 285–328.

[16] Y. Hu, J. Huang, D. Nualart, S. Tindel, Stochastic heat equations with general multiplicative Gaussian noise: Hölder continuity and intermittency, Electron. J. Probab. 20 (2015), 1–50.

[17] J.W. Jeon, Y.T. Kim, Stochastic Green's theorem for fractional Brownian sheet and its application, J. Korean Statistical Society, 41 (2012) 225–234.

[18] Y. Jiang, K. Shi, Y. Wang, Stochastic fractional Anderson models with fractional noises, Chin. Ann. Math. 31B(1) (2010) 101–118.

[19] Y. Jiang, X. Wang, Y. Wang, On a stochastic heat equation with first order fractional noises and applications to finance, J. Math. Anal. Appl. 396 (2012) 656–669.

[20] Y. Jiang, T. Wei, X. Zhou, Stochastic generalized Burgers equation driven by fractional noise, J. Diff. Eqn. 252 (2012) 1934–1961.

[21] J.U. Kim, On a stochastic plate equation, Appl. Math. Optim. 44 (2001) 33–48.

[22] J. Liu, L. Yan, Solving a nonlinear fractional stochastic partial differential equation with fractional noise, J. Theor. Probab. 29 (2016), 307–347.

[23] Y.S. Mishura, Stochastic Calculus for Fractional Brownian Motion and Related Processes, Springer-Verlag, Berlin Heidelberg, 2008.

[24] D. Nualart, Malliavin Calculus and Related Topics, Springer-Verlag, 1995.

[25] B. Pasik-Duncan, T.E. Duncan, B. Maslowski, Linear stochastic equations in a Hilbert space with fractional Brownian motion, in: Control Theory Applications in Financial Engineering and Manufacturing, Springer-Verlag, 2006, pp. 201–222 (Chapter 11).

[26] G.D. Prato, J. Zabczyk, Stochastic Equations in Infinite Dimensions, Cambridge University press, Cambridge, UK, 1992.

[27] L. Quer-Sardanyons, S. Tindel, The 1-d stochastic wave equation driven by a fractional Brownian sheet, Stochastic Process. Appl. 117 (2007) 1448–1472.

[28] S. Tindel, C.A. Tudor, F. Viens, Stochastic evolution equations with fractional Brownian motion, Probab. Theory Related Fields 127 (2003) 186–204.

[29] J.B. Walsh, An introduction to stochastic partial differential equations, Lecture Notes in Mathematics, 1180, Springer-Verlag, Berlin, (1986) 265–439.

[30] G. Wang, M. Zeng, B. Guo, Stochastic Burger's equation driven by fractional Brownian motion, J. Math. Anal. Appl. 371 (2010) 210–222.

[31] T. Zhang, Large deviations for stochastic nonlinear beam equations, J. Funct. Anal. 248 (2007) 175–201.

# FOURIER SERIES OF SUMS OF PRODUCTS OF EULER FUNCTIONS AND THEIR APPLICATIONS

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Abstract. We consider three types of sums of products of Euler functions and derive their Fourier series expansions. In addition, we express each of them in terms of Bernoulli functions.

## 1. Introduction

Let $E_m(x)$ be the Euler polynomials given by the generating function

$$\frac{2}{e^t+1}e^{xt}=\sum_{m=0}^{\infty}E_m(x)\frac{t^m}{m!},\quad (\text{see } [1,2,14]). \tag{1.1}$$

For any real number $x$, we let

$$< x >= x-[x]\in[0,1) \tag{1.2}$$

denote the fractional part of $x$.

Here we will consider the following three types of sums of products of Euler functions and derive their Fourier series expansions. Further, we will express each of them in terms of Bernoulli functions $B_m(< x >)$.

(1) $\alpha_m(< x >)=\sum_{k=0}^{m}E_k(< x >)E_{m-k}(< x >),(m\geq 1)$;
(2) $\beta_m(< x >)=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}E_k(< x >)E_{m-k}(< x >),(m\geq 1)$;
(3) $\gamma_m(< x >)=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k(< x >)E_{m-k}(< x >),(m\geq 2)$.

For elementary facts about Fourier analysis, the reader may refer to any book (for example, see [8,13,15,18]).

As to $\gamma_m(< x >)$, we note that the polynomial identity (1.3 ) follows immediately from (4.16), which is in turn derived from the Fourier series expansion of $\gamma_m(< x >)$.

$$\begin{aligned}&\sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k(x)E_{m-k}(x)\\&=-\frac{4}{m}\sum_{s=0,s\neq 1}^{m}\binom{m}{s}\frac{E_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s})B_s(x),\end{aligned} \tag{1.3}$$

where $H_m=\sum_{j=1}^{m}\frac{1}{j}$ are the harmonic numbers.

The obvious polynomial identities can be derived also for $\alpha_m(< x >)$ and $\beta_m(< x >)$ from (2.21) and (2.24), and (3.15) and (3.18), respectively. It is remarkable that from the Fourier series expansion of the

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function $\sum_{k=1}^{m-1}\frac{1}{k(m-k)}B_k(<x>)B_{m-k}(<x>)$ we can derive the following corresponding polynomial identity:

$$\begin{aligned}&\sum_{k=1}^{m-1}\frac{1}{k\left(m-k\right)}B_k\left(x\right)B_{m-k}\left(x\right) \\ =&\frac{2}{m^2}\left(B_m+\frac{1}{2}\right)+\frac{2}{m}\sum_{k=1}^{m-2}\frac{1}{m-k}\binom{m}{k}B_{m-k}B_k\left(x\right)+\frac{2}{m}H_{m-1}B_m\left(x\right),\quad (m\geq 2).\end{aligned} \tag{1.4}$$

Simple modification of (1.3) yields

$$\begin{aligned}&\sum_{k=1}^{m-1}\frac{1}{2k\left(2m-2k\right)}B_{2k}\left(x\right)B_{2m-2k}\left(x\right)+\frac{2}{2m-1}B_1\left(x\right)B_{2m-1}\left(x\right) \\ =&\frac{1}{m}\sum_{k=1}^{m}\frac{1}{2k}\binom{2m}{2k}B_{2k}B_{2m-2k}\left(x\right)+\frac{1}{m}H_{2m-1}B_{2m}\left(x\right) \\ &+\frac{2}{2m-1}B_1\left(x\right)B_{2m-1},\quad (m\geq 2)\,.\end{aligned} \tag{1.5}$$

Letting $x=0$ in (1.4) gives a slightly different version of the well-known Miki's identity (see [3,6,16,17]):

$$\begin{aligned}&\sum_{k=1}^{m-1}\frac{1}{2k\left(2m-2k\right)}B_{2k}B_{2m-2k} \\ =&\frac{1}{m}\sum_{k=1}^{m}\frac{1}{2k}\binom{2m}{2k}B_{2k}B_{2m-2k}+\frac{1}{m}H_{2m-1}B_{2m},\quad (m\geq 2)\,.\end{aligned} \tag{1.6}$$

Setting $x=\frac{1}{2}$ in (1.5) with $\overline{B}_m=\left(\frac{1-2^{m-1}}{2^{m-1}}\right)B_m=\left(2^{1-m}-1\right)B_m=B_m\left(\frac{1}{2}\right)$, we have

$$\begin{aligned}&\sum_{k=1}^{m-1}\frac{1}{2k\left(2m-2k\right)}\overline{B}_{2k}\overline{B}_{2m-2k} \\ =&\frac{1}{m}\sum_{k=1}^{m}\frac{1}{2k}\binom{2m}{2k}B_{2k}\overline{B}_{2m-2k}+\frac{1}{m}H_{2m-1}\overline{B}_{2m},\quad (m\geq 2)\,,\end{aligned} \tag{1.7}$$

which is the Faber-Pandharipande-Zagier identity (see [4]). Some related works can be found in [9-12].

## 2. Fourier series of functions of the first type

In this section, we consider the function

$$\alpha_m(<x>)=\sum_{k=0}^{m}E_k(<x>)E_{m-k}(<x>),\ \ (m\geq 1) \tag{2.1}$$

defined on $(-\infty,-\infty)$ which is periodic of period 1. The Fourier series of $\alpha_m(<x>)$ is

$$\sum_{n=-\infty}^{\infty}A_n^{(m)}e^{2\pi inx}, \tag{2.2}$$

where

$$\begin{aligned} A_n^{(m)} &= \int_0^1 \alpha_m(< x >)e^{-2\pi inx}dx \\ &= \int_0^1 \alpha_m(x)e^{-2\pi inx}dx. \end{aligned} \tag{2.3}$$

Before proceeding further, we observe the following.

$$\begin{aligned} \alpha_m'(x) &= \sum_{k=0}^{m}(kE_{k-1}(x)E_{m-k}(x) + (m-k)E_k(x)E_{m-k-1}(x)) \\ &= \sum_{k=1}^{m}(kE_{k-1}(x)E_{m-k}(x) + \sum_{k=0}^{m-1}(m-k)E_k(x)E_{m-k-1}(x)) \\ &= \sum_{k=0}^{m-1}(k+1)E_k(x)E_{m-k-1}(x) + \sum_{k=0}^{m-1}(m-k)E_k(x)E_{m-k-1}(x)) \\ &= (m+1)\sum_{k=0}^{m-1}E_k(x)E_{m-1-k}(x) \\ &= (m+1)\alpha_{m-1}(x). \end{aligned} \tag{2.4}$$

From this, we have $\left(\frac{\alpha_{m+1}(x)}{m+2}\right)' = \alpha_m(x)$.Then we have

$$\int_0^1 \alpha_m(x)dx = \frac{1}{m+2}(\alpha_{m+1}(1) - \alpha_{m+1}(0)). \tag{2.5}$$

Noting that $E_m(x+1) + E_m(x) = 2x^m$, we see that $E_m(1) + E_m(0) = 2\delta_{m,0}$. So, we have

$$\begin{aligned} &\alpha_m(1) - \alpha_m(0) \\ &= \sum_{k=0}^{m}(E_k(1)E_{m-k}(1) - E_k(0)E_{m-k}(0)) \\ &= \sum_{k=0}^{m}((-E_k(0) + 2\delta_{k,0})(-E_{m-k}(0) + 2\delta_{m-k,0}) - E_k(0)E_{m-k}(0)) \\ &= \sum_{k=0}^{m}(-2\delta_{m-k,0}E_k(0) - 2E_{m-k}(0)\delta_{k,0} + 4\delta_{k,0}\delta_{m-k,0}) \\ &= 4\delta_{m,0} - 4E_m(0), \quad (m \geq 0). \end{aligned} \tag{2.6}$$

Thus, for $m \geq 1$,

$$\alpha_m(1) - \alpha_m(0) = -4E_m. \tag{2.7}$$

Also,

$$\begin{aligned} \int_0^1 \alpha_m(x)dx &= \frac{1}{m+2}(\alpha_{m+1}(1) - \alpha_{m+1}(0)) \\ &= -\frac{4}{m+2}E_{m+1}. \end{aligned} \tag{2.8}$$

Now, we would like to determine the Fourier coefficients $A_n^{(m)}$.

Case 1 : $n \neq 0$.

$$\begin{aligned}
A_n^{(m)} &= \int_0^1 \alpha_m(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n} \left[\alpha_m(x) e^{-2\pi i n x}\right]_0^1 + \frac{1}{2\pi i n} \int_0^1 \alpha_m'(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n} (\alpha_m(1) - \alpha_m(0)) + \frac{m+1}{2\pi i n} \int_0^1 \alpha_{m-1}(x) e^{-2\pi i n x} dx \\
&= \frac{m+1}{2\pi i n} A_n^{(m-1)} + \frac{2}{\pi i n} E_m \\
&= \frac{m+1}{2\pi i n} \left( \frac{m}{2\pi i n} A_n^{(m-2)} + \frac{2}{\pi i n} E_{m-1} \right) + \frac{2}{\pi i n} E_n \\
&= \frac{(m+1)m}{(2\pi i n)^2} A_n^{(m-2)} + \frac{m+1}{2\pi i n} \frac{2}{\pi i n} E_{m-1} + \frac{2}{\pi i n} E_n \\
&= \frac{(m+1)m}{(2\pi i n)^2} \left( \frac{m-1}{2\pi i n} A_n^{(m-3)} + \frac{2}{\pi i n} E_{m-2} \right) + \frac{m+1}{2\pi i n} \frac{2}{\pi i n} E_{m-1} + \frac{2}{\pi i n} E_m \\
&= \frac{(m+1)m(m-1)}{(2\pi i n)^3} A_n^{(m-3)} + \frac{(m+1)m}{(2\pi i n)^2} \frac{2}{\pi i n} E_{m-2} + \frac{m+1}{2\pi i n} \frac{2}{\pi i n} E_{m-1} + \frac{2}{\pi i n} E_m \\
&= \cdots \\
&= \frac{(m+1)_{m-1}}{(2\pi i n)^{m-1}} A_n^{(1)} + \sum_{k=1}^{m-1} \frac{(m+1)_{k-1}}{(2\pi i n)^{k-1}} \frac{2}{\pi i n} E_{m-k+1},
\end{aligned} \tag{2.9}$$

where

$$A_n^{(1)} = \int_0^1 \alpha_1(x) e^{-2\pi i n x} dx = \int_0^1 (2x-1)\, e^{-2\pi i n x} dx = -\frac{1}{\pi i n}. \tag{2.10}$$

Hence

$$\begin{aligned}
A_n^{(m)} &= -\frac{2(m+1)_{m-1}}{(2\pi i n)^m} + 4 \sum_{k=1}^{m-1} \frac{(m+1)_{k-1}}{(2\pi i n)^k} E_{m-k+1} \\
&= \frac{4}{m+2} \sum_{k=1}^{m} \frac{(m+2)_k}{(2\pi i n)^k} E_{m-k+1}.
\end{aligned} \tag{2.11}$$

Case 2: $n = 0$.

$$A_0^{(m)} = \int_0^1 \alpha_m(x) dx = -\frac{4}{m+2} E_{m+1}. \tag{2.12}$$

We recall here that

$$B_1 = -\frac{1}{2}, B_{2n+1} = 0, \text{ for } n \geq 1, (-1)^{n+1} B_{2n} > 0, \tag{2.13}$$

(see [5], Proposition 15.1.1), and

$$E_n = -\frac{1}{n+1}(2^{n+2} - 2) B_{n+1} \ (n \geq 0), \quad \text{(see [3])}. \tag{2.14}$$

From these, we see that

$$E_{2n} = 0 \ (n \geq 1), \ \ E_{2n-1} \neq 0 \ (n \geq 1), \ \text{ and } E_0 = 1. \tag{2.15}$$

From these and (2.7), we observe that

$$\begin{aligned}\alpha_m(1) = \alpha_m(0)(\alpha_m(1) \neq \alpha_m(0)) &\Longleftrightarrow E_m = 0(E_m \neq 0)\\ &\Longleftrightarrow m \text{ is an even positive integer } (m \text{ is an odd positive integer}).\end{aligned} \tag{2.16}$$

Here $\alpha_m(< x >)$ is piecewise $C^\infty$. In addition, $\alpha_m(< x >)$ is continuous for all even positive integers $m$ and discontinuous with jump discontinuities at integers for all odd positive integers $m$.

We now recall the following facts about Bernoulli functions $B_m(< x >)$:

(a) for $m \geq 2$,

$$B_m(< x >) = -m! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^m}. \tag{2.17}$$

(b) for $m = 1$,

$$-\sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{2\pi i n} = \begin{cases} B_1(< x >), & \text{for} \quad x \in \mathbb{Z}^c, \\ 0, & \text{for} \quad x \in \mathbb{Z}, \end{cases} \tag{2.18}$$

where $\mathbb{Z}^c = \mathbb{R} - \mathbb{Z}$. Assume first that $m$ is an even positive integer. Then $\alpha_m(1) = \alpha_m(0)$. Thus $\alpha_m(< x >)$ is piecewise $C^\infty$, and continuous. Hence the Fourier series of $\alpha_m(< x >)$ converges uniformly to $\alpha_m(< x >)$, and

$$\begin{aligned}&\alpha_m(< x >)\\ &= -\frac{4}{m+2}E_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \left( \frac{4}{m+2} \sum_{k=1}^{m} \frac{(m+2)_k}{(2\pi i n)^k} E_{m-k+1} \right) e^{2\pi i n x}\\ &= -\frac{4}{m+2}E_{m+1} - \frac{4}{m+2} \sum_{k=1}^{m} \binom{m+2}{k} E_{m-k+1} \left( -k! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^k} \right)\\ &= -\frac{4}{m+2}E_{m+1} - \frac{4}{m+2} \sum_{k=2}^{m} \binom{m+2}{k} E_{m-k+1} B_k(< x >)\\ &\quad - \frac{4}{m+2}\binom{m+2}{1} E_m \times \begin{cases} B_1(< x >), & \text{for} \quad x \in \mathbb{Z}^c, \\ 0, & \text{for} \quad x \in \mathbb{Z} \end{cases}\\ &= -\frac{4}{m+2} \sum_{k=0, k\neq 1}^{m} \binom{m+2}{k} E_{m-k+1} B_k(< x >),\end{aligned} \tag{2.19}$$

for all $x \in (-\infty, \infty)$. Hence we get the following theorem.

**Theorem 2.1.** *Let $m$ be an even positive integer. Then we have the following.*

*(a) $\sum_{k=0}^{m} E_k(< x >)E_{m-k}(< x >)$ has the Fourier series expansion*

$$\begin{aligned}&\sum_{k=0}^{m} E_k(< x >)E_{m-k}(< x >)\\ &= -\frac{4}{m+2}E_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \left( \frac{4}{m+2} \sum_{k=1}^{m} \frac{(m+2)_k}{(2\pi i n)^k} E_{m-k+1} \right) e^{2\pi i n x},\end{aligned} \tag{2.20}$$

*for all $x \in (-\infty, \infty)$, where the convergence is uniform.*

*(b)*

$$\begin{aligned}&\sum_{k=0}^{m} E_k(< x >)E_{m-k}(< x >)\\ &= -\frac{4}{m+2}\sum_{k=0,k\neq 1}^{m}\binom{m+2}{k}E_{m-k+1}B_k(< x >)\end{aligned} \tag{2.21}$$

*for all $x \in (-\infty, \infty)$, where $B_k(< x >)$ is the Bernoulli function.*

Assume next that $m$ is an odd positive integer. Then $\alpha_m(1) \neq \alpha_m(0)$, and hence $\alpha_m(< x >)$ is piecewise $C^\infty$ and discontinuous with jump discontinuities at integers. Thus the Fourier series of $\alpha_m(< x >)$ converges pointwise to $\alpha_m(< x >)$, for $x \in \mathbb{Z}^c$, and converges to

$$\frac{1}{2}(\alpha_m(0) + \alpha_m(1)) = \alpha_m(0) - 2E_m, \tag{2.22}$$

for $x \in \mathbb{Z}$., Thus we get the following theorem.

**Theorem 2.2.** *Let $m$ be an odd positive integer. Then we have the following.*
*(a)*

$$\begin{aligned}&\sum_{n=-\infty,n\neq 0}^{\infty}\left(\frac{4}{m+2}\sum_{k=1}^{m}\frac{(m+2)_k}{(2\pi in)^k}E_{m-k+1}\right)e^{2\pi inx}\\ &= \begin{cases}\sum_{k=0}^{m} E_k(< x >)E_{m-k}(< x >), & \text{for} \quad x \in \mathbb{Z}^c,\\ \sum_{k=0}^{m} E_kE_{m-k} - 2E_m, & \text{for} \quad x \in \mathbb{Z}.\end{cases}\end{aligned} \tag{2.23}$$

*(b)*

$$\begin{aligned}&-\frac{4}{m+2}\sum_{k=1}^{m}\binom{m+2}{k}E_{m-k+1}B_k(< x >)\\ &= \sum_{k=0}^{m} E_k(< x >)E_{m-k}(< x >),\end{aligned} \tag{2.24}$$

*for $x \in \mathbb{Z}^c$;*

$$\begin{aligned}&-\frac{4}{m+2}\sum_{k=2}^{m}\binom{m+2}{k}E_{m-k+1}E_k(< x >)\\ &= \sum_{k=0}^{m} E_kE_{m-k} - 2E_m,\end{aligned} \tag{2.25}$$

*for $x \in \mathbb{Z}$.*

## 3. Fourier series of functions of the second type

In this section, we consider the function

$$\beta_m(< x >) = \sum_{k=0}^{m}\frac{1}{k!(m-k)!}E_k(< x >)E_{m-k}(< x >), \quad (m \geq 1) \tag{3.1}$$

defined on $(-\infty,-\infty)$ which is periodic of period 1. The Fourier series of $\beta_m(< x >)$ is

$$\sum_{n=-\infty}^{\infty} B_n^{(m)} e^{2\pi i n x}, \tag{3.2}$$

where

$$\begin{aligned} B_n^{(m)} &= \int_0^1 \beta_m(< x >) e^{-2\pi i n x} dx \\ &= \int_0^1 \beta_m(x) e^{-2\pi i n x} dx. \end{aligned} \tag{3.3}$$

Before proceeding further, we observe the following.

$$\begin{aligned} \beta_m'(x) &= \sum_{k=0}^{m} \left\{ \frac{k}{k!(m-k)!} E_{k-1}(x) E_{m-k}(x) + \frac{m-k}{k!(m-k)!} E_k(x) E_{m-k-1}(x) \right\} \\ &= \sum_{k=1}^{m} \frac{1}{(k-1)!(m-k)!} E_{k-1}(x) E_{m-k}(x) + \sum_{k=0}^{m-1} \frac{1}{k!(m-k-1)!} E_k(x) E_{m-k-1}(x) \\ &= \sum_{k=0}^{m-1} \frac{1}{k!(m-1-k)!} E_k(x) E_{m-1-k}(x) + \sum_{k=0}^{m-1} \frac{1}{k!(m-1-k)!} E_k(x) E_{m-1-k}(x) \\ &= 2\beta_{m-1}(x). \end{aligned} \tag{3.4}$$

So, $\beta_m'(x) = 2\beta_{m-1}(x)$. From this, we have

$$\left( \frac{\beta_{m+1}(x)}{2} \right)' = \beta_m(x) \tag{3.5}$$

and

$$\int_0^1 \beta_m(x) dx = \frac{1}{2}(\beta_{m+1}(1) - \beta_{m+1}(0)). \tag{3.6}$$

Using $E_m(1) + E_m(0) = 2\delta_{m,0}$, we observe that

$$\begin{aligned} \beta_m(1) - \beta_m(0) &= \sum_{k=0}^{m} \frac{1}{k!(m-k)!} \left( E_k(1) E_{m-k}(1) - E_k(0) E_{m-k}(0) \right) \\ &= \sum_{k=0}^{m} \frac{1}{k!(m-k)!} \left\{ (-E_k(0) + 2\delta_{k,0})(-E_{m-k}(0) + 2\delta_{m-k,0}) - E_k(0) E_{m-k}(0) \right\} \\ &= \sum_{k=0}^{m} \frac{1}{k!(m-k)!} \left\{ -2\delta_{k,0} E_{m-k}(0) - 2E_k(0)\delta_{m-k,0} + 4\delta_{k,0}\delta_{m-k,0} \right\} \\ &= -\frac{4}{m!}(E_m(0) - \delta_{m,0}). \end{aligned} \tag{3.7}$$

So, for $m \geq 1$,

$$\beta_m(1) - \beta_m(0) = -\frac{4}{m!} E_m. \tag{3.8}$$

Also, we have

$$\begin{aligned}\int_0^1 \beta_m(x)dx &= \frac{1}{2}(\beta_{m+1}(1)-\beta_{m+1}(0)) \\ &= \frac{1}{2}\left(-\frac{4}{(m+1)!}\right)(E_{m+1}(0)-\delta_{m+1,0}) \\ &= -\frac{2}{(m+1)!}E_{m+1}.\end{aligned} \tag{3.9}$$

Now, we are ready to determine the Fourier coefficients $B_n^{(m)}$.

Case 1: $n \neq 0$.

$$\begin{aligned}B_n^{(m)} &= \int_0^1 \beta_m(x)e^{-2\pi inx}dx \\ &= -\frac{1}{2\pi in}[\beta_m(x)e^{-2\pi inx}]_0^1 + \frac{1}{2\pi in}\int_0^1 \beta_m'(x)e^{-2\pi inx}dx \\ &= -\frac{1}{2\pi in}(\beta_m(1)-\beta_m(0)) + \frac{2}{2\pi in}\int_0^1 \beta_{m-1}(x)e^{-2\pi inx}dx \\ &= \frac{1}{\pi in}B_n^{(m-1)} + \frac{2}{m!}\frac{1}{\pi in}E_m \\ &= \frac{1}{\pi in}\left(\frac{1}{\pi in}B_n^{(m-2)} + \frac{2}{(m-1)!}\frac{1}{\pi in}E_{m-1}\right) + \frac{2}{m!}\frac{1}{\pi in}E_m \\ &= \frac{1}{(\pi in)^2}B_n^{(m-2)} + \frac{2}{(m-1)!}\frac{1}{(\pi in)^2}E_{m-1} + \frac{2}{m!}\frac{1}{\pi in}E_m \\ &= \frac{1}{(\pi in)^2}\left(\frac{1}{\pi in}B_n^{(m-3)} + \frac{2}{(m-2)!}\frac{1}{\pi in}E_{m-2}\right) + \frac{2}{(m-1)!}\frac{1}{(\pi in)^2}E_{m-1} + \frac{2}{m!}\frac{1}{\pi in}E_m \\ &= \frac{1}{(\pi in)^3}B_n^{(m-3)} + \frac{2}{(m-2)!}\frac{1}{(\pi in)^3}E_{m-2} + \frac{2}{(m-1)!}\frac{1}{(\pi in)^2}E_{m-1} + \frac{2}{m!}\frac{1}{\pi in}E_m \\ &= \cdots \\ &= \frac{1}{(\pi in)^{m-1}}B_n^{(1)} + \sum_{k=1}^{m-1}\frac{2}{(m-k+1)!}\frac{1}{(\pi in)^k}E_{m-k+1} \\ &= -\frac{1}{(\pi in)^m} + \sum_{k=1}^{m-1}\frac{2}{(m-k+1)!}\frac{1}{(\pi in)^k}E_{m-k+1} \\ &= 2\sum_{k=1}^{m}\frac{E_{m-k+1}}{(m-k+1)!}\frac{1}{(\pi in)^k}.\end{aligned} \tag{3.10}$$

Case 2: $n = 0$. By (3.9), we see that

$$B_0^{(m)} = \int_0^1 \beta_m(x)dx = -\frac{2}{(m+1)!}E_{m+1}. \tag{3.11}$$

From (3.8), we observe that

$$\begin{aligned}\beta_m(1) = \beta_m(0)(\beta_m(1) \neq \beta_m(0)) &\iff E_m = 0 (E_m \neq 0) \\ &\iff m \text{ is an even positive integer } (m \text{ is an odd positive integer}).\end{aligned} \tag{3.12}$$

Here $\beta_m(< x >)$ is piecewise $C^\infty$. In addition, $\beta_m(< x >)$ is continuous for all even positive integers $m$ and discontinuous with jump discontinuities at integers for all odd positive integers $m$.

Assume first that $m$ is an even positive integer. Then $\beta_m(1) = \beta_m(0)$. So $\beta_m(< x >)$ is piecewise $C^\infty$, and continuous. Thus the Fourier series of $\beta_m(< x >)$ converges uniformly to $\beta_m(< x >)$, and

$$\begin{aligned}
&\beta_m(< x >) \\
&= -\frac{2}{(m+1)!}E_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty}\left(2\sum_{k=1}^{m}\frac{E_{m-k+1}}{(m-k+1)!}\frac{1}{(\pi i n)^k}\right)e^{2\pi i n x} \\
&= -\frac{2}{(m+1)!}E_{m+1} - \frac{1}{(m+1)!}\sum_{k=1}^{m}2^{k+1}\binom{m+1}{k}E_{m-k+1}\left(-k!\sum_{n=-\infty, n\neq 0}^{\infty}\frac{e^{2\pi i n x}}{(2\pi i n)^k}\right) \\
&= -\frac{2}{(m+1)!}E_{m+1} - \frac{1}{(m+1)!}\sum_{k=2}^{m}2^{k+1}\binom{m+1}{k}E_{m-k+1}B_k(< x >) \\
&\quad - \frac{4}{(m+1)!}\binom{m+1}{1}E_m \times \begin{cases} B_1(< x >), & \text{for } x \in \mathbb{Z}^c, \\ 0, & \text{for } x \in \mathbb{Z} \end{cases} \\
&= -\frac{1}{(m+1)!}\sum_{k=0, k\neq 1}^{m}2^{k+1}\binom{m+1}{k}E_{m-k+1}B_k(< x >),
\end{aligned} \tag{3.13}$$

for all $x \in (-\infty, \infty)$.

Hence we get the following theorem.

**Theorem 3.1.** *Let $m$ be an even positive integer. Then we have the following.*
*(a) $\sum_{k=0}^{m}\frac{1}{k!(m-k)!}E_k(< x >)E_{m-k}(< x >)$ has the Fourier series expansion*

$$\begin{aligned}
&\sum_{k=0}^{m}\frac{1}{k!(m-k)!}E_k(< x >)E_{m-k}(< x >) \\
&= -\frac{2}{(m+1)!}E_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty}\left(2\sum_{k=1}^{m}\frac{E_{m-k+1}}{(m-k+1)!}\frac{1}{(\pi i n)^k}\right)e^{2\pi i n x},
\end{aligned} \tag{3.14}$$

*for all $x \in (-\infty, \infty)$, where the convergence is uniform.*
*(b)*

$$\begin{aligned}
&\sum_{k=0}^{m}\frac{1}{k!(m-k)!}E_k(< x >)E_{m-k}(< x >) \\
&= -\frac{1}{(m+1)!}\sum_{k=0, k\neq 1}^{m}2^{k+1}\binom{m+1}{k}E_{m-k+1}B_k(< x >),
\end{aligned} \tag{3.15}$$

*for all $x \in (-\infty, \infty)$, where $B_k(< x >)$ is the Bernoulli function.*

Assume next that $m$ is an odd positive integer. Then $\beta_m(1) \neq \beta_m(0)$, and hence $\beta_m(< x >)$ is piecewise $C^\infty$ and discontinuous with jump discontinuities at integers. Thus the Fourier series of $\beta_m(< x >)$ converges pointwise to $\beta_m(< x >)$, for $x \in \mathbb{Z}^c$, and converges to

$$\frac{1}{2}(\beta_m(0) + \beta_m(1)) = \beta_m(0) - \frac{2}{m!}E_m, \tag{3.16}$$

for $x \in \mathbb{Z}$., Thus we get the following theorem.

**Theorem 3.2.** *Let $m$ be an odd positive integer. Then we have the following.*

*(a)*

$$\begin{aligned}&\sum_{n=-\infty,n\neq 0}^{\infty}\left(2\sum_{k=1}^{m}\frac{E_{m-k+1}}{(m-k+1)!}\frac{1}{(\pi in)^k}\right)e^{2\pi inx}\\&=\begin{cases}\sum_{k=0}^{m}\frac{1}{k!(m-k)!}E_k(<x>)E_{m-k}(<x>), & \text{for}\quad x\in\mathbb{Z}^c,\\ \sum_{k=0}^{m}\frac{1}{k!(m-k)!}E_kE_{m-k}-\frac{2}{m!}E_m, & \text{for}\quad x\in\mathbb{Z}.\end{cases}\end{aligned}\tag{3.17}$$

*Here the convergence is pointwise.*

*(b)*

$$\begin{aligned}&-\frac{1}{(m+1)!}\sum_{k=1}^{m}2^{k+1}\binom{m+2}{k}E_{m-k+1}B_k(<x>)\\&=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}E_k(<x>)E_{m-k}(<x>),\end{aligned}\tag{3.18}$$

*for $x\in\mathbb{Z}^c$;*

$$\begin{aligned}&-\frac{1}{(m+1)!}\sum_{k=2}^{m}2^{k+1}\binom{m+1}{k}E_{m-k+1}B_k(<x>)\\&=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}E_kE_{m-k}-\frac{2}{m!}E_m,\end{aligned}\tag{3.19}$$

*for $x\in\mathbb{Z}$.*

## 4. Fourier series of functions of the third type

In this section, we consider the function

$$\gamma_m(<x>)=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k(<x>)E_{m-k}(<x>)\tag{4.1}$$

defined on $(-\infty,-\infty)$ which is periodic of period 1. The Fourier series of $\gamma_m(<x>)$ is

$$\sum_{n=-\infty}^{\infty}C_n^{(m)}e^{2\pi inx},\tag{4.2}$$

where

$$C_n^{(m)}=\int_0^1\gamma_m(<x>)e^{-2\pi inx}dx=\int_0^1\gamma_m(x)e^{-2\pi inx}dx.\tag{4.3}$$

To proceed further, we note the following.

$$\begin{aligned}\gamma_m'(x) &= \sum_{k=1}^{m-1}\frac{1}{m-k}E_{k-1}(x)E_{m-k}(x)+\sum_{k=1}^{m-1}\frac{1}{k}E_k(x)E_{m-k-1}(x)\\ &= \sum_{k=0}^{m-2}\frac{1}{m-1-k}E_k(x)E_{m-1-k}(x)+\sum_{k=1}^{m-1}\frac{1}{k}E_k(x)E_{m-1-k}(x)\\ &= \frac{2}{m-1}E_{m-1}(x)+(m-1)\sum_{k=1}^{m-2}\frac{1}{k(m-1-k)}E_k(x)E_{m-1-k}(x)\\ &= (m-1)\gamma_{m-1}(x)+\frac{2}{m-1}E_{m-1}(x).\end{aligned} \tag{4.4}$$

So,

$$\gamma_m'(x) = (m-1)\gamma_m(x)+\frac{2}{m-1}E_{m-1}(x). \tag{4.5}$$

From this, we note that

$$\frac{1}{m}\left(\gamma_{m+1}(x)-\frac{2}{m(m+1)}E_{m+1}(x)\right)' = \gamma_m(x). \tag{4.6}$$

$$\begin{aligned}&\int_0^1\gamma_m(x)dx\\ &= \left[\frac{1}{m}(\gamma_{m+1}(x)-\frac{2}{m(m+1)}E_{m+1}(x))\right]_0^1\\ &= \frac{1}{m}(\gamma_{m+1}(1)-\gamma_{m+1}(0))-\frac{2}{m^2(m+1)}(E_{m+1}(1)-E_{m+1}(0))\\ &= \frac{1}{m}(\gamma_{m+1}(1)-\gamma_{m+1}(0))-\frac{2}{m^2(m+1)}(-2E_{m+1}(0)+2\delta_{m+1,0})\\ &= \frac{1}{m}(\gamma_{m+1}(1)-\gamma_{m+1}(0))+\frac{4}{m^2(m+1)}E_{m+1}.\end{aligned} \tag{4.7}$$

Observe that

$$\begin{aligned}\gamma_m(1)-\gamma_m(0) &= \sum_{k=1}^{m-1}\frac{1}{k(m-k)}(E_k(1)E_{m-k}(1)-E_k(0)E_{m-k}(0))\\ &= \sum_{k=1}^{m-1}\frac{1}{k(m-k)}\left((-E_k(0)+2\delta_{k,0})(-E_{m-k}(0)+2\delta_{m-k,0})-E_k(0)E_{m-k}(0)\right)\\ &= \sum_{k=1}^{m-1}\frac{1}{k(m-k)}\left(-2\delta_{k,0}E_{m-k}(0)-2E_k(0)\delta_{m-k,0}+4\delta_{k,0}\delta_{m-k,0}\right)\\ &= 0.\end{aligned} \tag{4.8}$$

Thus, for $m\geq 2$, $\gamma_m(1)-\gamma_m(0)=0$. Also,

$$\int_0^1\gamma_m(x)dx\frac{1}{m}(\gamma_{m+1}(x)-\gamma_{m+1}(0))+\frac{4}{m^2(m+1)}E_{m+1}(0) = \frac{4}{m^2(m+1)}E_{m+1}. \tag{4.9}$$

We can show that

$$\int_0^1 E_{m-1}(x)e^{-2\pi inx}dx = \sum_{k=1}^{m-1}\frac{(m-1)_{k-1}}{(2\pi in)^k}E_{m-k}. \tag{4.10}$$

Now, we are ready to determine the Fourier coefficients $C_n^{(m)}$.

Case 1: $n \neq 0$

$$\begin{aligned}
C_n^{(m)} &= \int_0^1 \gamma_m(x)e^{-2\pi inx}dx \\
&= -\frac{1}{2\pi in}[\gamma_m(x)e^{-2\pi inx}]_0^1 + \frac{1}{2\pi in}\int_0^1 \gamma_m'(x)e^{-2\pi inx}dx \\
&= -\frac{1}{2\pi in}(\gamma_m(1)-\gamma_m(0)) + \frac{1}{2\pi in}\int_0^1 (m-1)\gamma_{m-1}(x) + \frac{2}{m-1}E_{m-1}(x))e^{-2\pi inx}dx \\
&= \frac{1}{2\pi in}\left\{(m-1)\int_0^1 \gamma_{m-1}(x)e^{-2\pi inx}dx + \frac{2}{m-1}\int_0^1 E_{m-1}(x)e^{-2\pi inx}dx\right\} \\
&= \frac{m-1}{2\pi in}C_n^{(m-1)} + \frac{1}{2\pi in}\frac{4}{m-1}\sum_{k=1}^{m-1}\frac{(m-1)_{k-1}}{(2\pi in)^k}E_{m-k} \\
&= \frac{m-1}{2\pi in}\left(\frac{m-2}{2\pi in}C_n^{(m-2)} + \frac{4}{m-2}\frac{1}{2\pi in}\sum_{k=1}^{m-2}\frac{(m-2)_{k-1}}{(2\pi in)^k}E_{m-k-1}\right) + \frac{1}{2\pi in}\frac{4}{m-1}\sum_{k=1}^{m-1}\frac{(m-1)_{k-1}}{(2\pi in)^k}E_{m-k} \\
&= \frac{(m-1)(m-2)}{(2\pi in)^2}C_n^{(m-2)} + \frac{m-1}{(2\pi in)^2}\frac{4}{m-2}\sum_{k=1}^{m-2}\frac{(m-2)_{k-1}}{(2\pi in)^k}E_{m-k-1} \\
&\quad + \frac{1}{2\pi in}\frac{4}{m-1}\sum_{k=1}^{m-1}\frac{(m-1)_{k-1}}{(2\pi in)^k}E_{m-k} \\
&= \cdots \\
&= \frac{(m-1)!}{(2\pi in)^{m-2}}C_n^{(2)} + \sum_{l=1}^{m-2}\frac{(m-1)_{l-1}}{(2\pi in)^l}\frac{4}{m-l}\sum_{k=1}^{m-l}\frac{(m-l)_{k-1}}{(2\pi in)^k}E_{m-k-l+1} \\
&= -\frac{2(m-1)!}{(2\pi in)^m} + \sum_{l=1}^{m-2}\frac{(m-1)_{l-1}}{(2\pi in)^l}\frac{4}{m-l}\sum_{k=1}^{m-l}\frac{(m-l)_{k-1}}{(2\pi in)^k}E_{m-k-l+1} \\
&= \sum_{l=1}^{m-1}\frac{(m-1)_{l-1}}{(2\pi in)^l}\frac{4}{m-l}\sum_{k=1}^{m-l}\frac{(m-l)_{k-1}}{(2\pi in)^k}E_{m-k-l+1} \\
&= \frac{4}{m}\sum_{l=1}^{m-1}\frac{1}{m-l}\sum_{k=1}^{m-l}\frac{(m)_{k+l-1}}{(2\pi in)^{k+l}}E_{m-k-l+1} \\
&= \frac{4}{m}\sum_{s=2}^{m}\frac{(m)_{s-1}}{(2\pi in)^s}E_{m-s+1}\sum_{l=1}^{s-1}\frac{1}{m-l} \\
&= \frac{4}{m}\sum_{s=2}^{m}\frac{(m)_s}{(2\pi in)^s}\frac{E_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s}),
\end{aligned} \tag{4.11}$$

where

$$c_n^{(2)} = \int_0^1 \gamma_2(x) e^{-2\pi i n x} dx = \int_0^1 \left(x^2 - x + \frac{1}{4}\right) e^{-2\pi i n x} dx = -\frac{2}{(2\pi i n)^2}. \tag{4.12}$$

Case 2: $n = 0$.

$$C_0^{(m)} = \int_0^1 \gamma_m(x) dx = \frac{4}{m^2(m+1)} E_{m+1}. \tag{4.13}$$

As $\gamma_m(1) = \gamma_m(0)$, for all $m \geq 2$, $\gamma_m(< x >)$ is piecewise $C^\infty$, and continuous. Hence the Fourier series of $\gamma_m(< x >)$ converges uniformly to $\gamma_m(< x >)$, and

$$\begin{aligned}
&\gamma_m(< x >) \\
&= \frac{4}{m^2(m+1)} E_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \left( \frac{4}{m} \sum_{s=2}^{m} \frac{(m)_s}{(2\pi i n)^s} \frac{E_{m-s+1}}{m-s+1} (H_{m-1} - H_{m-s}) \right) e^{2\pi i n x} \\
&= \frac{4}{m^2(m+1)} E_{m+1} - \frac{4}{m} \sum_{s=2}^{m} \binom{m}{s} \frac{E_{m-s+1}}{m-s+1} (H_{m-1} - H_{m-s}) \left( -s! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^s} \right) \\
&= \frac{4}{m^2(m+1)} E_{m+1} - \frac{4}{m} \sum_{s=2}^{m} \binom{m}{s} \frac{E_{m-s+1}}{m-s+1} (H_{m-1} - H_{m-s}) B_s(< x >) \\
&= -\frac{4}{m} \sum_{s=0, s\neq 1}^{m} \binom{m}{s} \frac{E_{m-s+1}}{m-s+1} (H_{m-1} - H_{m-s}) B_s(< x >)
\end{aligned} \tag{4.14}$$

Finally, we obtain the following theorem.

**Theorem 4.1.** *Let $m$ be an integer $\geq 2$. Then we have the following.*
*(a) $\sum_{k=0}^{m} \frac{1}{k(m-k)} E_k(< x >) E_{m-k}(< x >)$ has the Fourier expansion*

$$\begin{aligned}
&\sum_{k=1}^{m-1} \frac{1}{k(m-k)} E_k(< x >) E_{m-k}(< x >) \\
&= \frac{4}{m^2(m+1)} E_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \left( \frac{4}{m} \sum_{s=2}^{m} \frac{(m)_s}{(2\pi i n)^s} \frac{E_{m-s+1}}{m-s+1} (H_{m-1} - H_{m-s}) \right) e^{2\pi i n x},
\end{aligned} \tag{4.15}$$

*for all $x \in (-\infty, \infty)$, where the convergence is uniform.*
*(b)*

$$\begin{aligned}
&\sum_{k=1}^{m-1} \frac{1}{k(m-k)} E_k(< x >) E_{m-k}(< x >) \\
&= -\frac{4}{m} \sum_{s=0, s\neq 1}^{m} \binom{m}{s} \frac{E_{m-s+1}}{m-s+1} (H_{m-1} - H_{m-s}) B_s(< x >),
\end{aligned} \tag{4.16}$$

*for all $x \in (-\infty, \infty)$, where $B_s(< x >)$ is the Bernoulli function.*

## References


1. A. Bayad, T. Kim, *Higher recurrences for Apostol-Bernoulli-Euler numbers*, Russ. J. Math. Phys., **19(1)** (2012), 1-10.
2. D. Ding, J. Yang, *Some identities related to the Apostol-Euler and Apostol-Bernoulli polynomials,* Adv. Stud. Contemp. Math. (Kyungshang), **20(1)**(2010), 7-21.
3. G. V. Dunne, C. Schubert, *Bernoulli number identities from quantum field theory and topological string theory,* Commun. Number Theory Phys., **7(2)**(2013), 225-249.
4. C. Faber, R. Pandharipande, *Hodge integrals and Gromov-Witten theory,* Invent. Math. **139(1)**(2000), 173-199.

5. G. J. Fox, *Congruence relating rational values of Bernoulli and Euler polynomials,* Fibonacci Quart. **39(1)**(2001), 50-57.
6. I. M. Gessel, *On Miki's identities for Bernoulli numbers,* J. Number Theory **110(1)**(2005), 75-82.
7. K. Ireland, M. I. Rosen, *A classical introduction to modern number theory,* 2nd ed., Springer, 1990.
8. L. C. Jang, T. Kim, D. J. Kang, *A note on the Fourier transform of fermionic p -adic integral on* $\mathbb{Z}_p$, J. Comput. Anal. Appl., **11(3)** (2009), 571-575.
9. D.S. Kim, T. Kim, *Identities arising from higher-order Daehee polynomial bases,* Open Math. **13(**2015), 196-208.
10. D.S. Kim, T. Kim, *Euler basis, identities, and their applications,* Int. J. Math. Math. Sci. 2012, Art. ID 343981.
11. D.S. Kim, T. Kim, *Some identities of higher order Euler polynomials arising from Euler basis,* Integral Transforms Spec. Funct., **24(9)** (2013), 734-738.
12. D.S. Kim, T. Kim, S.-H. Lee, D.V. Dolgy, S.-H. Rim, Some new identities on the Bernoulli and Euler numbers, Discrete Dyn. Nat. Soc. 2011, Art. ID 856132.
13. T. Kim, *A note on the Fourier transform of p-adic q-integrals on* $\mathbb{Z}_p$, J. Comput. Anal. Appl., **11(1)** (2009), 81-85.
14. T. Kim, *Some identities for the Bernoulli, the Euler and the Genocchi numbers and polynomials,* Adv. Stud. Contemp. Math. (Kyungshang), **20(1)**(2010), 23-28.
15. J. E. Marsden, *Elementary classical analysis,* W. H. Freeman and Company, 1974.
16. H. Miki, *A relation between Bernoulli numbers,* J. Number Theory **10(3)**(1978), 297-302.
17. K. Shiratani, S. Yokoyama, *An application of p-adic convolutions,* Mem. Fac. Sci. Kyushu Univ. Ser. A **36(1)**(1982), 7383.
18. D. G. Zill, M. R. Cullen, *Advanced Engineering Mathematics,* Jones and Bartlett Publishers 2006.



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# Quotient subtraction algebras by an int-soft ideal

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**Abstract.** The aim of this article is to lay a foundation for providing a soft algebraic tool in considering many problems that contain uncertainties. In order to provide these soft algebraic structures, the notion of an intersectional soft subalgebra and an intersectional soft ideal of a subtraction algebra are introduced, and related properties are investigated. A quotient structure of a subtraction algebra using an intersectional soft ideal is constructed.

## 1. Introduction

The real world is inherently uncertain, imprecise and vague. Various problems in system identification involve characteristics which are essentially non-probabilistic in nature [16]. In response to this situation Zadeh [17] introduced *fuzzy set theory* as an alternative to probability theory. Uncertainty is an attribute of information. In order to suggest a more general framework, the approach to uncertainty is outlined by Zadeh [18]. To solve complicated problem in economics, engineering, and environment, we can't successfully use classical methods because of various uncertainties typical for those problems. There are three theories: theory of probability, theory of fuzzy sets, and the interval mathematics which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. Uncertainties can't be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, theory of (intuitionistic) fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets. However, all of these theories have their own difficulties which are pointed out in [14]. Maji et al. [13] and Molodtsov [14] suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory. To overcome these difficulties, Molodtsov [14] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. Worldwide, there has been a rapid growth in interest in soft set theory and its applications in recent years. Evidence of this can be found in the increasing number

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of high-quality articles on soft sets and related topics that have been published in a variety of international journals, symposia, workshops, and international conferences in recent years. Maji et al. [13] described the application of soft set theory to a decision making problem. Maji et al. [12] also studied several operations on the theory of soft sets. Aktaş and Çağman [4] studied the basic concepts of soft set theory, and compared soft sets to fuzzy and rough sets, providing examples to clarify their differences. They also discussed the notion of soft groups. Jun [9] discussed the union soft sets with applications in $BCK/BCI$-algebras. We refer the reader to the papers [1, 5, 7, 8, 15] for further information regarding algebraic structures/properties of soft set theory.

In this paper, we discuss applications of the an intersectional soft sets in a subalgebra (an ideal) of a subtraction algebra. We introduce the notion of an intersectional soft subalgebra (ideal) of a subtraction algebra, and investigate some related properties. We consider a new construction of a quotient subtraction algebra induced by an int-soft ideal. Also we investigated some related properties.

## 2. **Preliminaries**

We review some definitions and properties that will be useful in our results (see [10]).

By a *subtraction algebra* we mean an algebra $(X, *, 0)$ with a single binary operation " $-$ " that satisfies the following conditions: for any $x, y, z \in S$,

(S1) $x - (y - x) = x$,
(S2) $x - (x - y) = y - (y - x)$,
(S3) $(x - y) - z = (x - z) - y$.

The subtraction determines an order relation on $X$: $a \leq b$ if and only if $a - b = 0$, where $0 = a - a$ ia an element that does not depend on the choice of $a \in X$. The ordered set $(X; \leq)$ is a semi-Boolean algebras in the sense of [2], that is, it is a meet semilattice with zero 0 in which every interval $[0, a]$ is a Boolean algebra with respect to the induced order. Hence $a \wedge b = a - (a - b)$; the complement of an element $b \in [0, a]$ is $a - b$; and if $b, c \in [0, a]$, then

$$\begin{aligned} b \vee c &= (b' \wedge c')' = a - ((a - b) \wedge (a - c)) \\ &= a - ((a - b) - ((a - b) - (a - c))). \end{aligned}$$

In a subtraction algebra, the following are true:

(a1) $(x - y) - y = x - y$,
(a2) $x - 0 = x$ and $0 - x = 0$,
(a3) $(x - y) - x = 0$,

(a4) $x-(x-y)\leq y$,
(a5) $(x-y)-(y-x)=x-y$,
(a6) $x-(x-(x-y))=x-y$,
(a7) $(x-y)-(z-y)\leq x-z$,
(a8) $x\leq y$ if and only if $x=y-w$ for some $w\in X$,
(a9) $x\leq y$ implies $x-z\leq y-z$ and $z-y\leq z-x$ for all $z\in X$,
(a10) $x,y\leq z$ implies $x-y=x\wedge(z-y)$,
(a11) $(x\wedge y)-(x\wedge z)\leq x\wedge(y-z)$.

A non-empty subset $A$ of a subtraction algebra $X$ is called a *subalgebra* ([10]) of $X$ if $x-y\in A$ for any $x,y\in A$. A non-empty subset $I$ of a subtraction algebra $X$ is called an *ideal* ([10]) of $X$ if

(I1) $0\in I$,
(I2) $x-y,y\in I$ imply $x\in I$ for any $x,y,z\in X$.

A mapping $f:X\to Y$ of subtraction algebras is called a *homomorphism* if $f(x-y)=f(x)-f(y)$ for all $x,y\in X$.

Molodtsov [12] defined the soft set in the following way: Let $U$ be an initial universe set and let $E$ be a set of parameters. We say that the pair $(U,E)$ is a *soft universe.* Let $\mathscr{P}(U)$ denotes the power set of $U$ and $A,B,C,\cdots\subseteq E$.

A fair $(\tilde{f},A)$ is called a *soft set* over $U$, where $\tilde{f}$ is a mapping given by $\tilde{f}:X\to\mathscr{P}(U)$.

In other words, a soft set over $U$ is parameterized family of subsets of the universe $U$. For $\varepsilon\in A$, $\tilde{f}(\varepsilon)$ may be considered as the set of $\varepsilon$-approximate elements of the set $(\tilde{f},A)$. A soft set over $U$ can bd represented by the set of ordered pairs:

$$(\tilde{f},A)=\{(x,\tilde{f}(x))|x\in A,\tilde{f}(x)\in\mathscr{P}(U)\},$$

where $\tilde{f}:X\to\mathscr{P}(U)$ such that $\tilde{f}(x)=\emptyset$ if $x\notin A$. Clearly, a soft set is not a set.

For a soft set $(\tilde{f},A)$ of $X$ and a subset $\gamma$ of $U$, the *$\gamma$-inclusive set* of $(\tilde{f},A)$, defined to be the set

$$i_A(\tilde{f};\gamma):=\{x\in A|\gamma\subseteq\tilde{f}(x)\}.$$

## 3. **Intersectional soft subalgebras**

In what follows let $X$ denote a subtraction algebra unless otherwise specified.

**Definition 3.1.** A soft set $(\tilde{f},X)$ over $U$ is called an *intersectional soft subalgebra* (briefly, *int-soft subalgebra* of $X$ if it satisfies:

(3.1) $\tilde{f}(x)\cap\tilde{f}(y)\subseteq\tilde{f}(x-y)$ for all $x,y\in X$.

**Proposition 3.2.** *Every int-soft subalgebra $(\tilde{f}, X)$ of a subtraction algebra $X$ satisfies the following inclusion:*

(3.2) $\tilde{f}(x) \subseteq \tilde{f}(0)$ *for all* $x \in X$.

*Proof.* Using (3.1), we have $\tilde{f}(x) = \tilde{f}(x) \cap \tilde{f}(x) \subseteq \tilde{f}(x - x) = \tilde{f}(0)$ for all $x \in X$. □

**Example 3.3.** Let $(U = \mathbb{Z}, X)$ where $X = \{0, 1, 2, 3\}$ is a subtraction algebra ([11]) with the following Cayley table:

| $*$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 1 | 0 |
| 2 | 2 | 2 | 0 | 0 |
| 3 | 3 | 2 | 1 | 0 |

Let $(\tilde{f}, X)$ be a soft set over $U$ defined as follows:

$$\tilde{f} : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \mathbb{Z} & \text{if } x = 0 \\ 2\mathbb{Z} & \text{if } x \in \{1, 2\} \\ 4\mathbb{Z} & \text{if } x = 3. \end{cases}$$

It is easy to check that $(\tilde{f}, X)$ is an int-soft subalgebra over $U$.

**Theorem 3.4.** *A soft set $(\tilde{f}, X)$ of a subtraction algebra $X$ over $U$ is an int-soft subalgebra of $X$ over $U$ if and only if the $\gamma$-inclusive set $i_X(\tilde{f}; \gamma)$ is a subalgebra of $X$ for all $\gamma \in \mathscr{P}(U)$ with $i_X(\tilde{f}; \gamma) \neq \emptyset$.*

*Proof.* Assume that $(\tilde{f}, X)$ is an int-soft subalgebra over $U$. Let $x, y \in X$ and $\gamma \in \mathscr{P}(U)$ be such that $x, y \in i_X(\tilde{f}; \gamma)$. Then $\gamma \subseteq \tilde{f}(x)$ and $\gamma \subseteq \tilde{f}(y)$. It follows from (3.1) that $\gamma \subseteq \tilde{f}(x) \cap \tilde{f}(y) \subseteq \tilde{f}(x - y)$ Hence $x - y \in i_X(\tilde{f}; \gamma)$. Thus $i_X(\tilde{f}, X)$ is a subalgebra of $X$.

Conversely, suppose that $i_X(\tilde{f}; \gamma)$ is a subalgebra $X$ for all $\gamma \in \mathscr{P}(U)$ with $i_X(\tilde{f}; \gamma) \neq \emptyset$. Let $x, y \in X$, be such that $\tilde{f}(x) = \gamma_x$ and $\tilde{f}(y) = \gamma_y$. Take $\gamma = \gamma_x \cap \gamma_y$. Then $x, y \in i_X(\tilde{f}; \gamma)$ and so $x - y \in i_X(\tilde{f}; \gamma)$ by assumption. Hence $\tilde{f}(x) \cap \tilde{f}(y) = \gamma_x \cap \gamma_y = \gamma \subseteq \tilde{f}(x - y)$. Thus $(\tilde{f}, X)$ is an int-soft subalgebra over $U$. □

The subalgebra $i_X(\tilde{f}; \gamma)$ in Theorem 3.4 is called the *inclusive subalgebra* of $X$.

**Theorem 3.5.** *Every subalgebra of a subtraction algebra can be represented as a $\gamma$-inclusive set of an int-soft subalgebra.*

*Proof.* Let $A$ be a subalgebra of a subtraction algebra $X$. For a subset $\gamma$ of $U$, define a soft set $(\tilde{f}, X)$ over $U$ by

$$\tilde{f} : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma & \text{if } x \in A \\ \emptyset & \text{if } x \notin A \end{cases}$$

Obviously, $A = i_X(\tilde{f}; \gamma)$. We now prove that $(\tilde{f}, X)$ is an int-soft subalgebra over $U$. Let $x, y \in X$. If $x, y \in A$, then $x - y \in A$ because $A$ is a subalgebra of $X$. Hence $\tilde{f}(x) = \tilde{f}(y) = \tilde{f}(x - y) = \gamma$, and so $\tilde{f}(x) \cap \tilde{f}(y) \subseteq \tilde{f}(x - y)$. If $x \in A$ and $y \notin A$, then $\tilde{f}(x) = \gamma$ and $\tilde{f}(y) = \emptyset$ which imply that $\tilde{f}(x) \cap \tilde{f}(y) = \gamma \cap \emptyset = \emptyset \subseteq \tilde{f}(x - y)$. Similarly, if $x \notin A$ and $y \in A$, then $\tilde{f}(x) \cap \tilde{f}(y) \subseteq \tilde{f}(x - y)$. Obviously, if $x \notin A$ and $y \notin A$, then $\tilde{f}(x) \cap \tilde{f}(y) \subseteq \tilde{f}(x - y)$. Therefore $(\tilde{f}, X)$ is an int-soft subalgebra over $U$. □

Any subalgebra of a subtraction algebra $X$ may not be represented as a $\gamma$-inclusive set of an int-soft subalgebra $(\tilde{f}, X)$ over $U$ in general (see the following example).

**Example 3.6.** Consider a subtraction algebra $X = \{0, 1, 2, 3\}$ which is given Example 3.3. Consider a soft set $(\tilde{f}, X)$ which is given by

$$\tilde{f} : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \{0, 1\} & \text{if } x = 0 \\ \{1\} & \text{if } x \in \{1, 2, 3\} \end{cases}$$

Then $(\tilde{f}, X)$ is an int-soft subalgebra over $U$. The $\gamma$-inclusive set of $(\tilde{f}, X)$ are described as follows:

$$i_X(\tilde{f}; \gamma) = \begin{cases} X & \text{if } \gamma \in \{\emptyset, \{1\}\} \\ \{0\} & \text{if } \gamma \in \{\{0\}, \{0, 1\}\} \\ \emptyset & \text{otherwise.} \end{cases}$$

The subalgebra $\{0, 2\}$ cannot be a $\gamma$-inclusive set $i_X(\tilde{f}; \gamma)$ since there is no $\gamma \subseteq U$ such that $i_X(\tilde{f}; \gamma) = \{0, 2\}$.

We make a new int-soft subalgebra from old one.

**Theorem 3.7.** *Let $(\tilde{f}, X)$ be a soft set of a subtraction algebra $X$ over $U$. Define a soft set $(\tilde{f}^*, X)$ of $X$ over $U$ by*

$$\tilde{f}^* : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \tilde{f}(x) & \text{if } x \in i_X(\tilde{f}; \gamma) \\ \emptyset & \text{otherwise} \end{cases}$$

*where $\gamma$ is a non-empty subset subset of $U$. If $(\tilde{f}, X)$ is an int-soft subalgebra of $X$, then so is $(\tilde{f}^*, X)$.*

*Proof.* If $(\tilde{f}, X)$ is an int-soft subalgebra over $U$, then $i_X(\tilde{f};\gamma)$ is a subalgebra of $X$ for all $\gamma \subseteq U$ by Theorem 3.6. Let $x, y \in X$. If $x, y \in i_X(\tilde{f};\gamma)$, then $x - y \in i_X(\tilde{f};\gamma)$. Hence we have

$$\tilde{f}^*(x) \cap \tilde{f}^*(y) = \tilde{f}(x) \cap \tilde{f}(y) \subseteq \tilde{f}(x-y) = \tilde{f}^*(x-y).$$

If $x \notin i_X(\tilde{f};\gamma)$ or $y \notin i_X(\tilde{f};\gamma)$, then $\tilde{f}^*(x) = \emptyset$ or $\tilde{f}^*(y) = \emptyset$. Thus

$$\tilde{f}^*(x) \cap \tilde{f}^*(y) = \emptyset \subseteq \tilde{f}^*(x-y).$$

Therefore $(\tilde{f}^*, X)$ is an int-soft subalgebra over $U$. □

**Definition 3.8.** A soft set $(\tilde{f}, X)$ over $U$ is called an *intersectional ideal* (briefly, *int-soft ideal*) of $X$ if it satisfies (3.2) and

(3.3) $\tilde{f}(x-y) \cap \tilde{f}(y) \subseteq \tilde{f}(x)$ for all $x, y \in X$.

**Example 3.9.** (1) Let $E = X$ be the set of parameters and let $U = X$ be the universe set where $X = \{0, a, b, c\}$ is a subtraction algebra ([3]) with the following Cayley table:

| $*$ | $0$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $a$ | $a$ | $0$ | $a$ | $a$ |
| $b$ | $b$ | $b$ | $0$ | $b$ |
| $c$ | $c$ | $c$ | $c$ | $0$ |

Let $(\tilde{f}, X)$ be a soft set over $U$ defined as follows:

$$\tilde{f} : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma_2 & \text{if } x \in \{0, a\} \\ \gamma_1 & \text{if } x \in \{b, c\} \end{cases}$$

where $\gamma_1$ and $\gamma_2$ are subsets of $U$ with $\gamma_1 \subsetneq \gamma_2$ It is easy to check that $(\tilde{f}, X)$ is an int-soft ideal of $X$.

(2) In Example 3.3, $(\tilde{f}, X)$ is an int-soft subalgebra of $X$. But it is not an int-soft ideal of $X$, since $\tilde{f}(3-1) \cap \tilde{f}(2) = 2\mathbb{Z} \not\subseteq 4\mathbb{Z} = \tilde{f}(3)$.

**Proposition 3.10.** *Every int-soft ideal $(\tilde{f}, X)$ of a subtraction algebra $X$ satisfies the following inclusion:*

(i) $(\forall x, y \in X)(x \leq y \Rightarrow \tilde{f}(y) \subseteq \tilde{f}(x))$,

(ii) $(\forall x, y, z \in X)(\tilde{f}((x-y)-z) \cap \tilde{f}(y) \subseteq \tilde{f}(x-z))$.

*Proof.* (i) Let $x, y \in X$ be such that $x \leq y$. Then $x - y = 0$. Hence $\tilde{f}(y) = \tilde{f}(0) \cap \tilde{f}(y) = \tilde{f}(x-y) \cap \tilde{f}(y) \subseteq \tilde{f}(x)$.

(ii) Let $x, y, z \in X$. Using (S3) and (3.3), we have $\tilde{f}((x-y)-z)) \cap \tilde{f}(y) = \tilde{f}((x-z)-y)) \cap \tilde{f}(y) \subseteq \tilde{f}(x-z)$. □

**Theorem 3.11.** *Let $(\tilde{f}, X)$ be a soft set of $X$ over $U$. Then $(\tilde{f}, X)$ is an int-soft ideal of $X$ over $U$ if and only if*

(3.4) $(\forall x, y, z \in X)(x - y \leq z \Rightarrow \tilde{f}(z) \cap \tilde{f}(y) \subseteq \tilde{f}(x))$.

*Proof.* Assume that $(\tilde{f}, X)$ is an int-soft ideal of $X$ over $U$. Let $x, y$ and $z \in X$ be such that $x - y \leq z$. By Proposition 3.10(i) and (3.3), we have $\tilde{f}(z) \cap \tilde{f}(y) \subseteq \tilde{f}(x - y) \cap \tilde{f}(y) \subseteq \tilde{f}(x)$.

Conversely, suppose that $(\tilde{f}, X)$ satisfies (3.4). By (a2), we get $0 \leq x$ for any $x \in X$. Using Proposition 3.10(i), we have $\tilde{f}(x) \subseteq \tilde{f}(0)$ for any $x \in X$. By (a4), we have $x - (x - y) \leq y$ for any $x, y \in X$. It follows from (3.4) that $\tilde{f}(y) \cap \tilde{f}(x - y) \subseteq \tilde{f}(x)$. Hence (3.3) hold. Therefore $(\tilde{f}, X)$ is an int-soft ideal of $X$ over $U$. □

**Theorem 3.12.** *A soft set $(\tilde{f}, X)$ of $X$ over $U$ is an int-soft ideal of a subtraction algebra $X$ over $U$ if and only if the $\gamma$-inclusive set $i_X(\tilde{f}; \gamma)$ is an ideal of $X$ for all $\gamma \in \mathscr{P}(U)$ with $i_X(\tilde{f}; \gamma) \neq \emptyset$.*

*Proof.* Similar to Theorem 3.4. □

The ideal $i_X(\tilde{f}; \gamma)$ in Theorem 3.12 is called the *inclusive ideal* of $X$.

**Proposition 3.13.** *Let $(\tilde{f}, X)$ be a soft set of a subtraction algebra $X$ over $U$. Then the set $X_{\tilde{f}} := \{x \in X | \tilde{f}(x) = \tilde{f}(0)\}$ is an ideal of $X$.*

*Proof.* Obviously, $0 \in X_{\tilde{f}}$. Let $x, y \in X$ be such that $x - y \in X_{\tilde{f}}$ and $y \in X_{\tilde{f}}$. Then $\tilde{f}(x - y) = \tilde{f}(0)$ and $\tilde{f}(y) = \tilde{f}(0)$. By (3.3), we have $\tilde{f}(0) = \tilde{f}(x - y) \cap \tilde{f}(y) \subseteq \tilde{f}(x)$. It follows from (3.2) that $\tilde{f}(x) = \tilde{f}(0)$. Hence $x \in X_{\tilde{f}}$. Therefore $X_{\tilde{f}}$ is an ideal of $X$. □

## 4. **Quotient subtraction algebras induced by an int-soft ideal**

Let $(\tilde{f}, X)$ be an int-soft ideal of a subtraction algebra $X$. For any $x, y \in X$, we define a binary operation " $\sim^{\tilde{f}}$ " on $X$ as follows:

$$x \sim^{\tilde{f}} y \Leftrightarrow \tilde{f}(x - y) = \tilde{f}(y - x) = \tilde{f}(0).$$

**Lemma 4.1.** *The operation $\sim^{\tilde{f}}$ is an equivalence relation on a subtraction algebra $X$.*

*Proof.* Obviously, it is reflexive and symmetric. Let $x, y$ and $z \in X$ be such that $x \sim^{\tilde{f}} y$ and $y \sim^{\tilde{f}} z$. Then $\tilde{f}(x - y) = \tilde{f}(y - x) = \tilde{f}(0)$ and $\tilde{f}(y - z) = \tilde{f}(z - y) = \tilde{f}(0)$. By (a7), we have $(x - z) - (y - z) \leq x - y$ and $(z - x) - (y - x) \leq z - y$. Using (3.4) and (3.2), we have $\tilde{f}(0) = \tilde{f}(x - y) \cap \tilde{f}(y - z) \subseteq \tilde{f}(x - z) \subseteq \tilde{f}(0)$ and $\tilde{f}(0) = \tilde{f}(z - y) \cap \tilde{f}(y - x) \subseteq \tilde{f}(z - x) \subseteq \tilde{f}(0)$. Hence $\tilde{f}(x - z) = \tilde{f}(z - x) = \tilde{f}(0)$. Thus $x \sim^{\tilde{f}} z$, that is, $\sim^{\tilde{f}}$ is transitive. Therefore $\sim^{\tilde{f}}$ is an equivalence relation. □

**Lemma 4.2.** *For any $x, y, u, v \in X$, if $x \sim^{\tilde{f}} y$ and $u \sim^{\tilde{f}} v$, then $x - u \sim^{\tilde{f}} y - v$.*

*Proof.* Let $x, y, u, v \in X$ be such that $x \sim^{\tilde{f}} y$ and $u \sim^{\tilde{f}} v$. Then $\tilde{f}(x-y) = \tilde{f}(y-x) = \tilde{f}(0)$ and $\tilde{f}(u-v) = \tilde{f}(v-u) = \tilde{f}(0)$. Since $(x-u)-(y-u) \leq x-y$ and $(y-u)-(x-u) \leq y-x$, it follows from Proposition 3.10(i) that $\tilde{f}(0) = \tilde{f}(x-y) \leq \tilde{f}((x-u)-(y-u))$ and $\tilde{f}(0) = \tilde{f}(y-x) \leq \tilde{f}((y-u)-(x-u))$. By (3.2), we have $\tilde{f}((x-u)-(y-u)) = \tilde{f}(0)$ and $\tilde{f}((y-u)-(x-u)) = \tilde{f}(0)$. Hence $x-u \sim^{\tilde{f}} y-u$.

By (a4), (a9) and (S3), we have $(y-(y-v))-u = (y-u)-(y-v) \leq v-u$. Using Proposition 3.10(i), we obtain $\tilde{f}(0) = \tilde{f}(v-u) \leq \tilde{f}((y-u)-(y-v))$. It follows from (3.2) that $\tilde{f}((y-u)-(y-v)) = \tilde{f}(0)$. By a similar way, we get $\tilde{f}((y-v)-(y-u)) = \tilde{f}(0)$. Hence $y-v \sim^{\tilde{f}} y-u$. Since $\sim^{\tilde{f}}$ is symmetric, we have $y-u \sim^{\tilde{f}} y-v$. Since $\sim^{\tilde{f}}$ is transitive, $x-u \sim^{\tilde{f}} y-v$. Therefore $\sim^{\tilde{f}}$ is a congruence relation on $X$. □

Denote $\tilde{f}_x$ and $X/\tilde{f}$ the equivalence class containing $x$ and the set of all equivalence classes of $X$, respectively, i.e.,

$$\tilde{f}_x := \{y \in X | y \sim^{\tilde{f}} x\} \text{ and } X/\tilde{f} := \{\tilde{f}_x | x \in X\}.$$

Define a binary relation $-$ on $X/\tilde{f}$ as follows:

$$\tilde{f}_x - \tilde{f}_y = \tilde{f}_{x-y}$$

for all $\tilde{f}_x, \tilde{f}_y \in X/\tilde{f}$. Then this operation is well-defined by Lemma 4.2.

**Theorem 4.3.** *If $(\tilde{f}, X)$ is an int-soft ideal of a subtraction algebra $X$, then the quotient $X/\tilde{f} := (X/\tilde{f}; -)$ is a subtraction algebra.*

*Proof.* Straightforward. □

**Proposition 4.4.** *Let $\mu : X \to Y$ be an epimorphism of subtraction algebras. If $(\tilde{f}, Y)$ is an int-soft ideal of $Y$, then $(\tilde{f} \circ \mu, X)$ is an int-soft ideal of $X$.*

*Proof.* For any $x \in X$, we have $(\tilde{f} \circ \mu)(x) = \tilde{f}(\mu(x)) \subseteq \tilde{f}(0_Y) = \tilde{f}(\mu(0_X)) = (\tilde{f} \circ \mu)(0_X)$ and $(\tilde{f} \circ \mu)(x) = \tilde{f}(\mu(x)) \supseteq \tilde{f}(\mu(x) -_Y a) \cap \tilde{f}(a)$ for any $a \in Y$. Let $y$ be any preimage of $a$ under $\mu$. Then we have

$$\begin{aligned}(\tilde{f} \circ \mu)(x) &\supseteq \tilde{f}(\mu(x) -_Y a)) \cap \tilde{f}(a) \\ &= \tilde{f}(\mu(x) - \mu(y)) \cap \tilde{f}(\mu(y)) \\ &= \tilde{f}(\mu(x -_X y) \cap \tilde{f}(\mu(y)) \\ &= (\tilde{f} \circ \mu)(x -_X y) \cap (\tilde{f} \circ \mu)(y).\end{aligned}$$

Hence $\tilde{f} \circ \mu$ is an int-soft ideal of $X$. □

**Proposition 4.5.** *Let $(\tilde{f}, X)$ be an int-soft ideal of a subtraction algebra $X$. The mapping $\gamma : X \to X/\tilde{f}$, given by $\gamma(x) := \tilde{f}_x$, is a surjective homomorphism, and $Ker\gamma = \{x \in X | \gamma(x) = \tilde{f}_0\} = X_{\tilde{f}}$.*

*Proof.* Let $\tilde{f}_x \in X/\tilde{f}$. Then there exists an element $x \in X$ such that $\gamma(x) = \tilde{f}_x$. Hence $\gamma$ is surjective. For any $x, y \in X$, we have

$$\gamma(x-y) = \tilde{f}_{x-y} = \tilde{f}_x - \tilde{f}_y = \gamma(x) - \gamma(y).$$

Thus $\gamma$ is a homomorphism. Moreover, $Ker\ \gamma = \{x \in X | \gamma(x) = \tilde{f}_0\} = \{x \in X | x \in \tilde{f}_0\} = \{x \in X | x \sim^{\tilde{f}} 0\} = \{x \in X | \tilde{f}(x) = \tilde{f}(0)\} = X_{\tilde{f}}$. It completes the proof. □

**Proposition 4.6.** *Let a soft set $(\tilde{f}, X)$ over $U$ of a subtraction algebra $X$ be an int-soft ideal of $X$. If $J$ is an ideal of $X$, then $J/\tilde{f}$ is an ideal of $X/\tilde{f}$.*

*Proof.* Let a soft set $(\tilde{f}, X)$ over $U$ of a subtraction algebras $X$ be an int-soft ideal of $X$ and let $J$ be an ideal of $X$. Then $0 \in J$. Hence $\tilde{f}_0 \in J/\tilde{f}$. Let $\tilde{f}_x, \tilde{f}_y \in J/\tilde{f}$ such that $\tilde{f}_x - \tilde{f}_y \in J/\tilde{f}$ and $\tilde{f}_y \in J/\tilde{f}$. Since $\tilde{f}_{x-y} = \tilde{f}_x - \tilde{f}_y$, we have $x - y, y \in J$. Since $J$ is an ideal of $X$, we have $x \in J$. Hence $\tilde{f}_x \in J/\tilde{f}$. □

**Theorem 4.7.** *If $J^*$ is an ideal of a quotient subtraction algebra $X/\tilde{f}$, then there exists an ideal $J = \{x \in X | \tilde{f}_x \in J^*\}$ in $X$ such that $J/\tilde{f} = J^*$.*

*Proof.* Since $J^*$ is an ideal of $X/\tilde{f}$, $\tilde{f}_0 \in J^*$. Hence $0 \in J$. Let $\tilde{f}_x, \tilde{f}_y \in J/\tilde{f}$ be such that $\tilde{f}_x - \tilde{f}_y, \tilde{f}_y \in J^*$. Since $\tilde{f}_{x-y} = \tilde{f}_x - \tilde{f}_y$, we have $x - y, y \in J$. Since $J^*$ is an ideal of $J/\tilde{f}$, $\tilde{f}_x \in J/\tilde{f}$ and so $x \in J$. Therefore $J$ is an ideal of $X$. By Proposition 4.6, we have

$$\begin{aligned} J/\tilde{f} &= \{\tilde{f}_j | j \in J\} \\ &= \{\tilde{f}_j | \exists \tilde{f}_x \in J^* \text{ such that } j \sim^{\tilde{f}} x\} \\ &= \{\tilde{f}_j | \exists \tilde{f}_x \in J^* \text{ such that } \tilde{f}_x = \tilde{f}_j\} \\ &= \{\tilde{f}_j | \tilde{f}_j \in J^*\} = J^*. \end{aligned}$$

□

**Theorem 4.8.** *Let a soft set $(\tilde{f}, X)$ over $U$ be an int-soft ideal of a subtraction algebra $X$. If $J$ is an ideal of $X$, then $\dfrac{X/\tilde{f}}{J/\tilde{f}} \cong X/J$.*

*Proof.* Note that $\dfrac{X/\tilde{f}}{J/\tilde{f}} = \{[\tilde{f}_x]_{J/\tilde{f}} | \tilde{f} \in X/\tilde{f}\}$. If we define $\varphi : \dfrac{X/\tilde{f}}{J/\tilde{f}} \to X/J$ by $\varphi([\tilde{f}_x]_{J/\tilde{f}}) = [x]_J = \{y \in X | x \sim^J y\}$, then it is well defined. In fact, suppose that $[\tilde{f}_x]_{J/\tilde{f}} = [\tilde{f}_y]_{J/\tilde{f}}$. Then $\tilde{f}_x \sim^{J/\tilde{f}} \tilde{f}_y$ and so $\tilde{f}_{x-y} = \tilde{f}_x - \tilde{f}_y, \tilde{f}_{y-x} = \tilde{f}_y - \tilde{f}_x \in J/\tilde{f}$. Hence $y - x, x - y \in J$. Therefore

$x \sim^J y$, i.e., $[x]_J = [y]_J$. Given $[\tilde{f}_x]_{J/\tilde{f}}, [\tilde{f}_y]_{J/\tilde{f}} \in \dfrac{X/\tilde{f}}{J/\tilde{f}}$, we have

$$\begin{aligned}\varphi([\tilde{f}_x]_{J/\tilde{f}} - [\tilde{f}_y]_{J/\tilde{f}}) &= \varphi([\tilde{f}_x - \tilde{f}_y]_{J/\tilde{f}}) = [x-y]_J \\ &= [x]_J - [y]_J = \varphi([\tilde{f}_x]_{J/\tilde{f}}) - \varphi([\tilde{f}_y]_{J/\tilde{f}}).\end{aligned}$$

Hence $\varphi$ is a homomorphism.

Obviously, $\varphi$ is onto. Finally, we show that $\varphi$ is one-to-one. If $\varphi([\tilde{f}_x]_{J/\tilde{f}}) = \varphi([\tilde{f}_y]_{J/\tilde{f}})$, then $[x]_J = [y]_J$, i.e., $x \sim^J y$. If $\tilde{f}_a \in [\tilde{f}_x]_{J/\tilde{f}}$, then $\tilde{f}_a \sim^{J/\tilde{f}} \tilde{f}_x$ and hence $\tilde{f}_{a-x}, \tilde{f}_{x-a} \in J/\tilde{f}$. It follows that $a-x, x-a \in J$, i.e., $a \sim^J x$. Since $\sim^J$ is an equivalence relation, $a \sim^J y$ and so $J_a = J_y$. Hence $a-y, y-a \in J$ and so $\tilde{f}_{a-y}, \tilde{f}_{y-a} \in J/\tilde{f}$. Therefore $\tilde{f}_a \sim^{J/\tilde{f}} \tilde{f}_y$. Hence $\tilde{f}_a \in [\tilde{f}_y]_{J/\tilde{f}}$. Thus $[\tilde{f}_x]_{J/\tilde{f}} \subseteq [\tilde{f}_y]_{J/\tilde{f}}$. Similarly, we obtain $[\tilde{f}_y]_{J/\tilde{f}} \subseteq [\tilde{f}_x]_{J/\tilde{f}}$. Therefore $[\tilde{f}_x]_{J/\tilde{f}} = [\tilde{f}_y]_{J/\tilde{f}}$. It is completes the proof. □

## References


[1] U. Acar, F. Koyuncu and B. Tanay, *Soft sets and soft rings,* Comput. Math. Appl. **59**(2010) 3458-3463.

[2] J. C. Abbott, *Sets, Lattices and Boolean Algebras*, Allyn and Bacon, Boston 1969.

[3] S. S. Ahn, Y. H. Kim and K. J. Lee, *A relation on subtraction algebras,* Sci. Math. Jpn. Online (e-2006), 51-55.

[4] H. Aktaş and N. Çağman, *Soft sets and soft groups,* Inform. Sci. **177**(2007) 2726-2735.

[5] A. O. Atagün and A. Sezgin, *Soft substructures of rings, fields and modules,* Comput. Math. Appl. **61** (2011) 592-601.

[6] Y. Çeven and S. Küçükkoç, *Quotient subtraction algebras,* Inter. Math. Forum **6**(2011) 1241-1247.

[7] F. Feng, Y. B. Jun and X. Zhao, *Soft semirings,* Comput. Math. Appl. **56** (2008) 2621-2628.

[8] Y. B. Jun, *Soft $BCK/BCI$-algebras,* Comput. Math. Appl. **56** (2008) 1408-1413.

[9] Y. B. Jun, *Union soft sets with applications in $BCK/BCI$-algebras,* Bull. Korean Math. Soc. **50** (2013), 1937-1956.

[10] Y. B. Jun, H. S. Kim and E. H. Roh, *Ideal theory of subtraction algebras,* Sci. Math. Jpn. Online (e-2004), 397-402.

[11] Y. B. Jun, Y. H. Kim and K. A. Oh, *Subtraction algebras with additional conditions,* Commun. Korrean Math. Soc. **22** (2007), 1-7.

[12] P. K. Maji, R. Biswas and A. R. Roy, *Soft set theory,* Comput. Math. Appl. **45** (2003) 555-562.

[13] P. K. Maji, A. R. Roy and R. Biswas, *An application of soft sets in a decision making problem,* Comput. Math. Appl. **44** (2002) 1077-1083.

[14] D. Molodtsov, *Soft set theory - First results,* Comput. Math. Appl. **37** (1999) 19-31.

[15] K. S. Yang and S. S. Ahn, *Union soft q-ideals in $BCI$-algebras*, Applied Mathematical Scineces **8**(2014), 2859-2869.

[16] L. A. Zadeh, *From circuit theory to system theory*, Proc. Inst. Radio Eng. **50** (1962) 856-865.

[17] L. A. Zadeh, *Fuzzy sets,* Inform. Control **8** (1965) 338-353.

[18] L. A. Zadeh, *Toward a generalized theory of uncertainty (GTU) - an outline*, Inform. Sci. **172** (2005) 1-40.

# Characterization of weak sharp solutions for generalized variational inequalities in Banach spaces

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**Abstract**

In this paper, we study the solution set of variational like-inequalities (in this sense we are called $\eta$-variational inequalities) and introduce the notion of a weak sharp set of solutions to $\eta$-variational inequality problem in reflexive, strictly convex and smooth Banach space. We also present sufficient conditions for the relevant mapping to be constant on the solutions. Moreover, we characterize the weak sharpness of the solutions of $\eta$-variational inequality by primal gap function.

*Keywords:* $\eta$-variational inequality, Gap function, Weakly sharp solution

## 1. Introduction

Burke and Ferris [2] introduced the concept of a weak sharp minimum to present sufficient conditions for the finite identification, by iterative algorithm, of local minima associated with mathematical programming in space $\mathbb{R}^n$. Patriksson [7] has generalized the concept of the weak sharpness of the solution set of a variational inequality problem (in short, VIP). Their concepts have been extended by Marcotte and Zhu [6] to introduce another the notion of weak sharp solutions for variational inequalities. They also characterized the weak sharp solutions in terms of a dual gap function for variational inequalities. The relevant results have been obtained by Zhang et al. [12]. It is further study by Wu and Wu [9–11]. Hu and Song [4] have extended the results of weak sharpness for the solutions of VIP under some continuity and monotonicity assumptions in Banach space. They also introduce the notion of weak sharp set of solutions to a variational inequality problem in a reflexive, strictly convex and smooth Banach space and present its several equivalent conditions. Liu and Wu [5] studied weak sharp solutions for the variational inequality in terms of its primal gap function. They also characterized the weak sharpness of the solution set of VIP in terms of primal gap function. Recently, AL-Hamidan et al. [1] give some characterization of weak sharp solutions for the VIP without considering the primal or dual gap function.

In this paper, we provide some general two concepts of Liu and Wu [5] and Hu and Song [4] to study the weak sharpness of solution set of $\eta$-variational inequality problem in Banach space. We also give some characterizations of weak sharp solutions for the $\eta$-VIP and also present its several equivalent conditions. Our purpose in this paper is to develop the weak sharpness result in space $\mathbb{R}^n$.

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The paper is organized as follows. Section 2 discuss the new concepts of the Gateaux differentiable and Lipschitz continuity of the primal gap function and we also introduce the main definitions. Several equivalent conditions for $F$ to be constant are discuss and present some relationship among $C^\eta$, $C_\eta$, $\Gamma(x^*)$, and $\Lambda(x^*)$ in Section 3. Finally, section 4 addresses the weak sharpness of $C^\eta$ in terms of the primal gap function is characterized.

## 2. Preliminaries and formulations

Let $E$ be a real Banach space with is topological dual space $E^*$ and $\langle \cdot, \cdot \rangle$ denote the pairing between $E$ and $E^*$ respectively. For a mapping from $\eta : E \times E$ to $E$ . Let $g$ be a mapping from $E$ into Banach space $Y$. The mapping $g$ is called directionally differentiable at a point $x \in E$ in a direction $v \in E$ if the limit

$$g'(x, v) := \lim_{t \downarrow 0} \frac{g(x + tv) - g(x)}{t}$$

exists. We say that $g$ is directionally differentiable at $x$, if $g$ is directionally differentiable at $x$ in every direction $v \in E$.

The mapping $g$ is called Gateaux differentiable at $x$ if $g$ is directionally differentiable at $x$ and the directional derivative $g'(x, v)$ is linear and continuous in $v$ and we denote this operator by $\nabla g(x)$, i.e. $\langle \nabla g(x), v \rangle = g'(x, v)$.

**Definition 2.1.** Let $g$ be a mapping from $E$ into Banach space $Y$. The mapping $g$ is called $\eta$-Gateaux differentiable at $x$ if $g$ is Gateaux differentiable at $x$ and there exists a unique $\xi \in E^*$ such that $\langle \xi, \eta(v, 0) \rangle = \langle \nabla g(x), v \rangle \ \ , \forall v \in E$. We denote this operator by $\nabla_\eta g(x)$ i.e. $\langle \nabla_\eta g(x), \eta(v, 0) \rangle = g'(x, v)$.

We defined $\eta$-subdifferential of a proper convex function $f$ at $x \in E$ is given by

$$\partial_\eta f(x) := \{x^* \in E^* : \langle x^*, \eta(y, x) \rangle \leq f(y) - f(x), \forall y \in E\}.$$

Let $C$ be a closed convex subset of $E$. The mapping $P_C : E \to 2^C$ defined by

$$P_C(x) := \{y \in C : \|x - y\| = d(x, C)\},$$

is called *the metric projection operator*.

We known that if $E$ is a reflexive and strictly convex Banach space, $P_C$ is a single-valued mapping.

The duality mappings $J : E \to 2^{E^*}$ and $J^* : E^* \to 2^E$ are defined by

$$J(x) = \{x^* \in E^* : \langle x^*, x \rangle = \|x^*\|_*^2 = \|x\|^2\}, \forall x \in E$$

and

$$J^*(x^*) = \{x \in E : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|_*^2\}, \forall x^* \in E^*.$$

We know the following (see [8])

(i) if $E$ is smooth, then $J$ is single-valued;
(ii) if $E$ is reflexive, then $J$ is onto;
(iii) if $E$ is strictly convex, then $J$ is one-to-one;
(iv) if $E$ is strictly convex, then $J$ is strictly monotone.

Thought out this paper, we let $\eta : E \times E$ to $E$ be satisfy the following condition;

(i) $\eta$ is continuous on $E \times E$;
(ii) for any $x, y \in E$, $\eta(x, y) = -\eta(y, x)$;

(iii) for any $x, y \in E$ and $\alpha, \beta$ are scalars, $\eta(\alpha x + \beta y, 0) = \alpha\eta(x, 0) + \beta\eta(y, 0)$;
(iv) there exists $k > 0$ such that $\|\eta(x, y)\| = k\|x - y\|$ for all $x, y \in E$;
(v) $\eta(E \times \{0\}) = E$.

For a mapping $g$ from a Banach space $E$ into Banach space $Y$, we say that $g$ is $\eta$-locally Lipschitz on $E$ if for any $\overline{x} \in E$ there exist $\delta > 0$ and $L \geq 0$ such that

$$\|g(x) - g(y)\| \leq L\|\eta(x, y)\|, \quad \text{for all} \quad x, y \in B(\overline{x}, \delta).$$

The following results are importance:

**Lemma 2.2** ([3]). *Let $E$ be a Banach space, $J : E \to 2^{E^*}$ a duality mapping and $\Phi(\|x\|) = \int_0^{\|x\|} ds$, $0 \neq x \in X$. Then $J(x) = \partial\Phi(\|x\|)$.*

**Lemma 2.3.** *Assume that $E$ is a reflexive, strictly convex and smooth Banach space. Let $C$ be a closed convex subset of $E$ and $\hat{x} \in C$. Then the following are equivalent:*

(i) *$\hat{x}$ is a best approximation to $x$ :$\|\eta(x, \hat{x})\| = \inf_{y \in C} \|\eta(x, y)\|$.*
(ii) *the inequality $\langle J(\eta(x, \hat{x})), \eta(y, \hat{x})\rangle \leq 0, \quad \forall y \in C$ holds.*

*Proof.* $(ii) \Rightarrow (i)$ For each $x \in E$. Let $\hat{x} \in C$ such that

$$\langle J(\eta(x, \hat{x})), \eta(y, \hat{x})\rangle \leq 0 \quad \forall y \in C.$$

Then

$$\begin{aligned} \|\eta(x, \hat{x})\|\|J(\eta(x, \hat{x}))\|_* &= \langle J(\eta(x, \hat{x})), \eta(x, \hat{x})\rangle \\ &\leq \langle J(\eta(x, \hat{x})), \eta(x, \hat{x})\rangle + \langle J(\eta(x, \hat{x})), \eta(\hat{x}, y)\rangle, \quad \forall y \in C \\ &= \langle J(\eta(x, \hat{x})), \eta(x, y)\rangle, \quad \forall y \in C \\ &\leq \|J(\eta(x, \hat{x}))\|_*\|\eta(x, y)\|, \quad \forall y \in C. \end{aligned}$$

Hence, $\|\eta(x, \hat{x})\| = \inf_{y \in C} \|\eta(x, y)\|$.

$(i) \Rightarrow (ii)$ For each $x \in E$. Suppose that $\hat{x} \in C$ such that

$$\|\eta(x, \hat{x})\| = \inf_{y \in C} \|\eta(x, y)\|.$$

Since $C$ is convex, we obtain that

$$\|\eta(x, \hat{x})\| \leq \|\eta(x, (1 - t)\hat{x} + ty)\|, \quad \forall y \in C \quad \text{and} \quad t \in [0, 1],$$

which implies that

$$\Phi(\|\eta(x, \hat{x})\|) - \Phi(\|\eta(x, (1 - t)\hat{x} + ty)\|), \quad \forall y \in C \quad \text{and} \quad t \in [0, 1],$$

where $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ give by $\Phi(x) = \int_0^x ds$, for all $x \in \mathbb{R}_+$.

By Lemma 2.2, $J(z) = \partial\Phi(\|z\|)$. It follows that

$$\begin{aligned} \langle J(\eta(x, (1 - t)\hat{x} + ty)), \eta(x, \hat{x}) - \eta(x, (1 - t)\hat{x} + ty)\rangle &\leq \Phi(\|\eta(x, \hat{x})\|) - \Phi(\|\eta(x, (1 - t)\hat{x} + ty)\|) \\ &\leq 0, \quad \forall y \in C \quad \text{and} \quad t \in [0, 1], \end{aligned}$$

that is,

$$t\langle J(\eta(x, (1 - t)\hat{x} + ty)), \eta(y, \hat{x})\rangle \leq 0, \quad \forall y \in C \quad \text{and} \quad t \in [0, 1]$$

Therefore,

$$\langle J(\eta(x,(1-t)\hat{x}+ty)),\eta(y,\hat{x})\rangle \leq 0, \quad \forall y \in C \text{ and } t \in [0,1].$$

Taking $t \to 0$, we have

$$\langle J(\eta(x,\hat{x})),\eta(y,\hat{x})\rangle \leq 0, \quad \forall y \in C.$$

□

**Remark 2.4.** By definition of $\eta$ for each $x \in E$ if $\hat{x} = P_C(x)$ then $\|\eta(x,\hat{x})\| = \inf_{y\in C} \|\eta(x,y)\|$.

If $C$ is a closed convex subset of $E$ and $\overline{x} \in C$, then the $\eta$-tangent cone to $C$ at $\overline{x}$ has the form

$$\begin{aligned} T_C^\eta(\overline{x}) &= \{d \in E : \text{there exists a bounded sequence } \{d_k\} \subseteq X \text{ with } \eta(d_k,0) \to d, t_k \downarrow 0 \\ &\quad \text{such that } \overline{x} + t_k d_k \in C, \quad \forall k \in \mathbb{N}\}. \end{aligned}$$

In the above, denote $x_k = \overline{x} + t_k d_k \in C$. Taking the limit as $k \to +\infty$, $t_k \to 0$, which implies that $t_k d_k \to 0$, thereby leading to $x_k \to \overline{x}$. Also from construction,

$$\frac{\eta(x_k,\overline{x})}{t_k} = \eta(d_k,0) \to d.$$

Thus, the $\eta$-tangent cone can be equivalently expressed as

$$\begin{aligned} T_C^\eta(\overline{x}) &= \{d \in E : \text{there exists sequence } \{x_k\} \subseteq C \text{ with } x_k \to \overline{x}, t_k \downarrow 0 \\ &\quad \text{such that } \frac{\eta(x_k,\overline{x})}{t_k} \to d\}. \end{aligned}$$

**Proposition 2.5.** *Consider a set $C \subseteq E$ and $\overline{x} \in C$. Then the following hold:*

(i) *$T_C^\eta(\overline{x})$ is closed;*
(ii) *If $C$ is convex, $T_C^\eta(\overline{x})$ is the closure of the cone generated by $\eta(C \times \{\overline{x}\})$, that is, $T_C^\eta(\overline{x}) = \overline{cone(\eta(C \times \{\overline{x}\}))}$*

*Proof.* (i) Suppose that $\{d_k\} \subseteq T_C^\eta(\overline{x})$ such that $d_k \to d$. Since $d_k \in T_C^\eta(\overline{x})$, there exist $\{x_k^r\} \subseteq C$ with $x_k^r \to \overline{x}$ and $\{t_k^r\} \subseteq \mathbb{R}_+$ with $t_k^r \to 0$ such that

$$\frac{\eta(x_k^r,\overline{x})}{t_k^r} \to d_k, \quad \forall k \in \mathbb{N}.$$

For a fixed $k$, there exists $\overline{r}$ such that

$$\|\frac{\eta(x_k^r,\overline{x})}{t_k^r} - d_k\| < \frac{1}{k}, \quad \forall r \geq \overline{r}.$$

Taking $k \to +\infty$, one can generate a sequence $\{x_k\} \subseteq C$ with $x_k \to \overline{x}$ and $t_k \downarrow 0$ such that

$$\frac{\eta(x_k,\overline{x})}{t_k} \to d.$$

Thus, $d \in T_C^\eta(\overline{x})$. Hence, $T_C^\eta(\overline{x})$ is closed.

(ii) Suppose that $d \in T_C^\eta(\overline{x})$, which implies that there exist $\{x_k\} \subseteq C$ with $x_k \to \overline{x}$ and $\{t_k\} \subseteq \mathbb{R}_+$ with $t_k \to 0$ such that

$$\frac{\eta(x_k,\overline{x})}{t_k} \to d.$$

Observe that $\eta(x_k,\overline{x}) \in \eta(C \times \{\overline{x}\})$. Since $t_k > 0$, $\frac{1}{t_k} > 0$. Therefore,

$$\frac{\eta(x_k,\overline{x})}{t_k} \in cone\ \eta(C \times \{\overline{x}\}).$$

Thus, $d \in \overline{cone(\eta(C \times \{\overline{x}\}))}$. Hence, $T_C^\eta(\overline{x}) \subseteq \overline{cone(\eta(C \times \{\overline{x}\}))}$.

Conversely, for each $x \in C$. Define a sequence

$$x_k = \overline{x} + \frac{1}{k}(x - \overline{x}), \quad \forall k \in \mathbb{N}.$$

By the convexity of $C$, it is obvious that $\{x_k\} \subseteq C$. Taking $k \to +\infty$, $x_k \to \overline{x}$, by construction, we obtain that

$$k\eta(x_k, \overline{x}) = \eta(x, \overline{x}).$$

Set $t_k = \frac{1}{k} > 0$, $t_k \to 0$ such that $\frac{\eta(x_k,\overline{x})}{t_k} \to \eta(x, \overline{x})$, which implies that $\eta(x, \overline{x}) \in T_C^\eta(\overline{x})$.

Since $x \in C$ is arbitrary, $\eta(C \times \{\overline{x}\}) \subseteq T_C^\eta(\overline{x})$. Because $T_C^\eta(\overline{x})$ is cone, we have $cone(\eta(C \times \{\overline{x}\})) \subseteq T_C^\eta(\overline{x})$. By (i), $T_C^\eta(\overline{x})$ is closed, which implies that $\overline{cone(\eta(C \times \{\overline{x}\}))} \subseteq T_C^\eta(\overline{x})$. □

The $\eta$-normal cone to $C$ at $\overline{x}$ is defined by $N_C^\eta(\overline{x}) := [T_C^\eta(\overline{x})]^\circ$, where

$$A^\circ := \{x^* \in E^* : \langle x^*, x\rangle \leq 0, \forall x \in A\}.$$

If $C$ is convex, then

$$N_C^\eta(\overline{x}) := \begin{cases} \{x^* \in E^* : \langle x^*, \eta(c, \overline{x})\rangle \leq 0 \text{ for all } c \in C\} & \text{if } \overline{x} \in C, \\ \emptyset, & \text{if } \overline{x} \notin C. \end{cases}$$

Let $C$ be a nonempty closed convex subset of reflexive, strictly convex and smooth Banach space $E$. For a mapping $F$ from $E$ into $E^*$, the *$\eta$-variational inequality problem* [$\eta$-VIP] is to find a vector $x^* \in C$ such that

$$\langle F(x^*), \eta(x, x^*)\rangle \geq 0 \text{ for all } x \in C. \tag{2.1}$$

We denote the solution set of the $\eta$-VIP by $C^\eta$

The *$\eta$-dual variational inequality problem* [$\eta$-DVIP] is to find a vector $x_* \in C$ such that

$$\langle F(x), \eta(x, x_*)\rangle \geq 0 \text{ for all } x \in C. \tag{2.2}$$

We denote the solution set of the $\eta$-DVIP by $C_\eta$

**Definition 2.6.** The mapping $F : E \to E^*$ is said to be:

(i) *$\eta$-monotone* on $C$ if $\langle F(x) - F(y), \eta(y, x)\rangle \geq 0$ for all $x, y \in C$;

(ii) *$\eta$-pseudomonotone* at $x \in C$ if for each $y \in C$ there holds

$$\langle F(x), \eta(y, x)\rangle \geq 0 \Rightarrow \langle F(y), \eta(y, x)\rangle \geq 0;$$

(iii) *$\eta$-pseudomonotone$^+$* on $C$ if it is $\eta$-pseudomonotone at each point in $C$ and, for all $x, y \in C$,

$$\left.\begin{array}{l} \langle F(y), \eta(x, y)\rangle \geq 0 \\ \langle F(x), \eta(x, y)\rangle = 0 \end{array}\right\} \Rightarrow F(x) = F(y).$$

Now, we define the *primal gap function* $g(x)$ associated with $\eta$-VIP (2.1) as

$$g(x) := \sup_{y \in C}\{\langle F(x), \eta(x, y)\rangle\}, \text{ for all } x \in E,$$

and we setting

$$\Gamma(x) := \{y \in C : \langle F(x, \eta(x, y))\rangle = g(x)\}.$$

Similarly, we define the *dual gap function* $G(x)$ associated with $\eta$-DVIP (2.2) as

$$G(x) := \sup_{y \in C}\{\langle F(y), \eta(x, y)\rangle\}, \text{ for all } x \in E,$$

and we setting

$$\Lambda(x) := \{y \in C : \langle F(y, \eta(x, y))\rangle = G(x)\}.$$

## 3. Sufficient condition for constancy of $F$ on $C^\eta$ and some properties of the primal gap function

In this section, we discuss about relations among $C^\eta$, $C_\eta$, $\Gamma(x^*)$, and $\Lambda(x^*)$. We study sufficient condition for $F$ to be constant on $C^\eta$ and also study the $\eta$-Lipschitz continuity and $\eta$-subdifferentiability of the primal gap function $g$ in terms of the mapping $F$.

**Proposition 3.1.** *Let $\hat{x} \in C$. Then*

*(i)* $\hat{x} \in C^\eta \Leftrightarrow g(\hat{x}) = 0 \Leftrightarrow \hat{x} \in \Gamma(\hat{x})$;
*(ii)* $\hat{x} \in C_\eta \Leftrightarrow G(\hat{x}) = 0 \Leftrightarrow \hat{x} \in \Lambda(\hat{x})$.

*Proof.* (i) Consider

$$\begin{aligned} \hat{x} \in C^\eta &\Leftrightarrow \langle F(\hat{x}), \eta(y, \hat{x})\rangle \geq 0, \quad \forall y \in C \\ &\Leftrightarrow \langle F(\hat{x}), \eta(\hat{x}, y)\rangle \leq 0, \quad \forall y \in C \\ &\Leftrightarrow g(\hat{x}) = 0. \end{aligned}$$

And we also consider

$$\begin{aligned} \hat{x} \in \Gamma(\hat{x}) &\Leftrightarrow \langle F(\hat{x}), \eta(\hat{x}, \hat{x})\rangle = g(\hat{x}) \\ &\Leftrightarrow 0 = g(\hat{x}). \end{aligned}$$

Similarly, we can obtain (ii). □

**Proposition 3.2.** *If $F$ is $\eta$-pseudomonotone on $C$, $C^\eta \subseteq C_\eta$.*

*Proof.* Immediate from the definitions. □

The following proposition we present a sufficient condition for $F$ to be constant on $C^\eta$.

**Proposition 3.3.** *Let $F$ be $\eta$-pseudomonotone$^+$ on $C^\eta$. Then $F$ is constant on $C^\eta$*

*Proof.* Let $x_1, x_2 \in C^\eta$. Since $F$ is $\eta$-pseudomonotone$^+$ on $C^\eta$, we have

$$\langle F(x_1), \eta(x_2, x_1)\rangle \geq 0 \text{ and } \langle F(x_2), \eta(x_1, x_2)\rangle \geq 0.$$

By pseudomonotonicity of $F$ on $C^\eta$, we have

$$\langle F(x_1), \eta(x_1, x_2)\rangle \geq 0 \text{ it follows that } \langle F(x_1), \eta(x_1, x_2)\rangle = 0.$$

Since $F$ is $\eta$-pseudomonotone$^+$ on $C^\eta$ and $\langle F(x_2), \eta(x_1, x_2)\rangle \geq 0$, implies that $F(x_1) = F(x_2)$. Hence, $F$ is constant on $C^\eta$. □

**Proposition 3.4.** *Let $F$ be $\eta$-pseudomonotone$^+$ on $C$ and $x^* \in C^\eta$. Then $C^\eta = \Lambda(x^*)$ and $F$ is constant on $\Lambda(x^*)$.*

*Proof.* First, we prove that $F$ is constant on $\Lambda(x^*)$. For $x^* \in C^\eta$ and $c \in C$, we have

$\langle F(x^*), \eta(c, x^*)\rangle \geq 0$. Since $F$ is pseudomonotone, we get that

$$\langle F(c), \eta(c, x^*)\rangle \geq 0, \forall c \in C. \text{ It follows that } G(x^*) = 0.$$

For $c \in \Lambda(x^*)$, we have

$$\langle F(c), \eta(c, x^*)\rangle = -G(x^*) = 0 \text{ and hence } F(c) = F(x^*).$$

It sufficient to show that $C^\eta = \Lambda(x^*)$.
($\subseteq$): Let $y^* \in C^\eta$. Then $\langle F(y^*), \eta(x^*, y^*)\rangle \geq 0$. Since $x^* \in C^\eta \subseteq C_\eta$, we have

$$\langle F(z), \eta(z, x^*)\rangle \geq 0, \forall z \in C.$$

It follows that $G(x^*) = 0$, and $\langle F(y^*), \eta(y^*, x^*)\rangle \geq 0$. Therefore,

$$\langle F(y^*), \eta(x^*, y^*)\rangle = 0 = G(x^*), \quad \text{that is,} \quad y^* \in \Lambda(x^*). \quad \text{Thus} \quad C^\eta \subseteq \Lambda(x^*).$$

($\supseteq$): Let $y^* \in \Lambda(x^*)$. Then $\langle F(y^*), \eta(x^*, y^*)\rangle = G(x^*) \geq 0$. Since $x^* \in C^\eta$, we have

$$\langle F(x^*), \eta(y, x^*)\rangle \geq 0, \forall y \in C.$$

Note that $x^* \in \Lambda(x^*)$, we have $F(x^*) = F(y^*)$. Consider, for all $y \in C$,

$$0 \leq \langle F(y^*), \eta(y, x^*)\rangle = \langle F(y^*), \eta(y, y^*)\rangle + \langle F(y^*), \eta(y^*, x^*)\rangle$$

implies $0 \leq \langle F(y^*), \eta(x^*, y^*)\rangle \leq \langle F(y^*), \eta(y, y^*)\rangle, \forall y \in C$. Therefore, $C^\eta = \Lambda(x^*)$. □

**Proposition 3.5.** *Suppose that $F$ be $\eta$-pseudomonotone on $C$ and $x^* \in C^\eta$. If $F$ is constant on $\Gamma(x^*)$ then $F$ is constant on $C^\eta$. And hence*

$$C^\eta = C_\eta = \Gamma(x^*) = \Lambda(x^*).$$

*Proof.* Since $F$ is $\eta$-pseudomonotone on $C$, we have $C^\eta \subseteq C_\eta$. Let $y^* \in C_\eta$. Then

$$\langle F(x^*), \eta(x^*, y^*)\rangle \geq 0.$$

By assumption, we obtain that $g(x^*) = 0$ and hence $\langle F(x^*), \eta(y^*, x^*)\rangle \geq 0$. It follows that

$$\langle F(x^*), \eta(x^*, y^*)\rangle = 0 = g(x^*). \quad \text{Thus} \quad y^* \in \Gamma(x^*).$$

Therefore $C^\eta \subseteq C_\eta \subseteq \Gamma(x^*)$. Let $z^* \in \Gamma(x^*)$. Then $\langle F(x^*), \eta(x^*, z^*)\rangle = g(x^*) = 0$.
From above $x^* \in C^\eta \subseteq \Gamma(x^*)$ and $F$ is constant on $\Gamma(x^*)$, we obtain that $F(x^*) = F(z^*)$. Since $x^* \in C^\eta$, we have $\langle F(x^*), \eta(z, x^*)\rangle \geq 0, \forall z \in C$.

It follows that, for all $z \in C$,

$$\begin{aligned} 0 &\leq \langle F(z^*), \eta(z, x^*)\rangle \\ &= \langle F(z^*), \eta(z, z^*)\rangle + \langle F(z^*), \eta(z^*, x^*)\rangle \\ &= \langle F(z^*), \eta(z, z^*)\rangle + \langle F(x^*), \eta(z^*, x^*)\rangle \\ &= \langle F(z^*), \eta(z, z^*)\rangle. \end{aligned}$$

This implies that $z^* \in C^\eta$. Thus $\Gamma(x^*) \subseteq C^\eta$ and hence

$$C^\eta = C_\eta = \Gamma(x^*).$$

It sufficient to prove that $\Gamma(x^*) = \Lambda(x^*)$. For $c \in \Gamma(x^*)$, we have

$$\langle F(x^*), \eta(x^*, c)\rangle = g(x^*) = 0,$$

so $\langle F(c), \eta(x^*, c)\rangle = 0 = G(x^*)$. Therefore,

$$c \in \Lambda(x^*), \quad \text{which implies that} \quad \Gamma(x^*) \subseteq \Lambda(x^*).$$

Now let $c \in \Lambda(x^*)$. Then

$$\langle F(c), \eta(x^*, c)\rangle = G(x^*) = 0.$$

The pseudomonotonicity of $F$ on $C$ implies that $\langle F(x^*), \eta(x^*, c)\rangle \geq 0$. In this case,

$$\langle F(x^*), \eta(x^*, c)\rangle = 0 = g(x^*) \text{ since } x^* \in C^\eta.$$

Thus $c \in \Gamma(x^*)$ and hence $\Lambda(x^*) \subseteq \Gamma(x^*)$. Therefore

$$C^\eta = C_\eta = \Gamma(x^*) = \Lambda(x^*).$$

□

**Proposition 3.6.** *Let $F$ be $\eta$-pseudomonotone$^+$ on $C$. Then, for $x^* \in C^\eta$, $F$ is constant on $\Gamma(x^*)$ if and only if*

$$C^\eta = C_\eta = \Gamma(x^*) = \Lambda(x^*).$$

*Proof.* ($\Rightarrow$)Suppose that $F$ is constant on $\Gamma(x^*)$. By Proposition 3.5, we obtain that

$$C^\eta = C_\eta = \Gamma(x^*) = \Lambda(x^*).$$

($\Leftarrow$)Assume that $C^\eta = C_\eta = \Gamma(x^*) = \Lambda(x^*)$. Let $x_1, x_2 \in \Gamma(x^*)$. Then

$$\langle F(x_1), \eta(x_2, x_1)\rangle \geq 0 \text{ and } \langle F(x_2), \eta(x_1, x_2)\rangle \geq 0 \text{ , because } x_1, x_2 \in C^\eta.$$

Since $F$ is $\eta$-pseudomonotone and $\langle F(x_2), \eta(x_1, x_2)\rangle \geq 0$, we obtain that

$$\langle F(x_1), \eta(x_1, x_2)\rangle \geq 0 \text{ , that is , } \langle F(x_1), \eta(x_2, x_1)\rangle \leq 0.$$

Thus $\langle F(x_1), \eta(x_2, x_1)\rangle = 0$. Since $F$ is $\eta$-pseudomonotone$^+$ on $C$, we have $F(x_1) = F(x_2)$.

□

**Proposition 3.7.** *Let $F$ be $\eta$-pseudomonotone$^+$ on $C$. Then the following are equivalent:*

(i) *$F$ is constant on $\Gamma(x^*)$ for each $x^* \in C^\eta$.*
(ii) *$C^\eta = C_\eta = \Gamma(x^*) = \Lambda(x^*)$ for each $x^* \in C^\eta$.*
(iii) *$C^\eta = \Gamma(x^*) = \Lambda(x^*)$ for each $x^* \in C^\eta$.*
(iv) *$C^\eta = \Gamma(x^*)$ for each $x^* \in C^\eta$.*

*Proof.* $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$ are immediate. It suffices to show $(iv) \Rightarrow (i)$. Suppose that $C^\eta = \Gamma(x^*)$ for each $x^* \in C^\eta$. Let $x^* \in C^\eta$ and $x_1, x_2 \in \Gamma(x^*)$. Then $x_1, x_2 \in C^\eta$ and $\langle F(x_1), \eta(x_2, x_1)\rangle \geq 0$ and $\langle F(x_2), \eta(x_1, x_2)\rangle \geq 0$.

Since $F$ is $\eta$-pseudomonotone and $\langle F(x_2), \eta(x_1, x_2)\rangle \geq 0$, we obtain that

$$\langle F(x_1), \eta(x_1, x_2)\rangle \geq 0 \text{ , that is , } \langle F(x_1), \eta(x_2, x_1)\rangle \leq 0.$$

Thus $\langle F(x_1), \eta(x_2, x_1)\rangle = 0$. Since $F$ is $\eta$-pseudomonotone$^+$ on $C$, we have $F(x_1) = F(x_2)$. □

Next we prove that if $F$ is $\eta$-locally Lipschitz on $C^\eta$, then so is $g$.

**Lemma 3.8.** *Let $C$ be compact. If $F$ is $\eta$-locally Lipschitz on $C^\eta$, then $g$ is also $\eta$-locally Lipschitz on $C^\eta$.*

*Proof.* Suppose that $F$ is $\eta$-locally Lipschitz on $C^\eta$. Let $x^*$ be any element in $C^\eta$. Then there exist $\delta > 0$ and $L_0 \geq 0$ such that

$$\|F(x) - F(y)\| \leq L_0\|\eta(x, y)\| \text{ and } \|F(x)\| \leq L_0 \text{ for all } x, y \in B(x^*, \delta).$$

Let $c \in \Gamma(x)$ with $x \in B(x^*, \delta)$. Then

$$\begin{aligned} g(x) - g(y) &\leq \langle F(x), \eta(x,c)\rangle - \langle F(y), \eta(y,c)\rangle \\ &= \langle F(x), \eta(x,y)\rangle + \langle F(x), \eta(y,c)\rangle - \langle F(y), \eta(y,c)\rangle \\ &= \langle \eta(x,y)\rangle + \langle F(x) - F(y), \eta(y,c)\rangle \\ &\leq \|F(x)\|\|\eta(x,y)\| + \|F(x) - F(y)\|\|\eta(y,c)\| \\ &\leq L_0\|\eta(x,y)\| + L_0\|\eta(x,y)\|\|\eta(y,c)\|. \end{aligned}$$

By the compactness of $C$ and definition of $\eta$ implies that there exists a constant $M \geq 0$ such that

$$\|\eta(y,c)\| \leq M \quad \text{for all} \quad y \in B(x^*, \delta) \quad \text{and} \quad c \in C.$$

We set $L = L_0 + L_0 M$, we obtain that

$$g(x) - g(y) \leq L\|\eta(x,y)\|.$$

We can conclude that $g$ is $\eta$-locally Lipschitz on $C^\eta$. □

The following Proposition 3.2 we present the $\eta$-subdifferential of $g$ at $x^* \in C^\eta$ is a singleton under sufficient condition.

**Proposition 3.9.** *Let $F$ be $\eta$-monotone on $X$ and $x^* \in C^\eta$. Suppose that $g$ is finite on $X$ and $\eta$-Gateaux differentiable at $x^*$. Then $\partial_\eta g(x^*) = \{F(x^*)\}$.*

*Proof.* Since $x^* \in C^\eta$, we have $g(x^*) = 0$. For each $y \in X$ and $F$ is $\eta$-monotone, we obtain that

$$g(y) - g(x^*) \geq \langle F(y), \eta(y, x^*)\rangle \geq \langle F(x^*), \eta(y, x^*)\rangle.$$

Hence $F(x^*) \in \partial_\eta g(x^*)$.

Let $z \in \partial_\eta g(x^*)$. Then for each $v \in X$ and $t > 0$, we get that

$$g(x^* + tv) - g(x^*) \geq \langle z, \eta(x^* + tv, x^*)\rangle = t\langle z, \eta(v, 0)\rangle,$$

that is,

$$\frac{g(x^* + tv) - g(x^*)}{t} \geq \langle z, \eta(v,0)\rangle.$$

By the $\eta$-Gateaux differentiability of $g$ at $x^*$ implies that

$$\langle \nabla_\eta g(x^*), \eta(v,0)\rangle = \lim_{t\to 0} \frac{g(x^* + tv) - g(x^*)}{t} \geq \langle z, \eta(v,0)\rangle.$$

Therefore, $\langle z - \nabla_\eta g(x^*), \eta(v,0)\rangle \leq 0$, for all $v \in X$. By definition of $\eta$ we can set $\eta(v,0) = z - \nabla_\eta g(x^*)$, we have $\|z - \nabla_\eta g(x^*)\|^2 \leq 0$. This implies that $z = \nabla_\eta g(x^*)$, and hence $\partial_\eta g(x^*) = \{\nabla_\eta g(x^*)\} = \{F(x^*)\}$. □

## 4. Weak sharpness of $C^\eta$

Throughout this paper, we assume that $C^\eta$ and $C_\eta$ are nonempty and that $E$ is a reflexive, strictly convex, and smooth Banach space. We introduce the notion of weak sharpness solution for generalized variational inequality($\eta$-VIP).

**Definition 4.1.** The solution set $C^\eta$ is said to be weakly sharp, if $F$ satisfies

$$-F(x^*) \in int \bigcap_{x \in C^\eta} [T_C(x) \cap J^* N_{C^\eta}(x)]^\circ \quad \text{for all} \quad x^* \in C^\eta.$$

**Theorem 4.2.** *Let $F$ be $\eta$-monotone on $E$ and constant on $\Gamma(x^*)$ for some $x^* \in C^\eta$. Suppose that $g$ is $\eta$-Gateaux differentiable, $\eta$-locally Lipschitz on $C^\eta$, and $g(x) < +\infty$ for all $x \in E$. Then $C^\eta$ is weakly sharp if and only if there exists a positive number $\alpha$ such that*

$$\alpha d^\eta_{C^\eta}(x) \leq g(x) \quad \textit{for all} \quad x \in C, \tag{4.1}$$

*where* $d^\eta_{C^\eta}(x) := \inf_{y \in C^\eta} \|\eta(x,y)\|$.

*Proof.* On the given assumption and by Proposition 3.5, we obtain that

$$C^\eta = C_\eta = \Gamma(x^*) = \Lambda(x^*).$$

If $C^\eta$ is weakly sharp, then for any $x^* \in C^\eta$ there exists $\alpha > 0$ such that

$$\alpha B_{E^*} \subseteq F(x^*) + \bigcap_{x \in C^\eta} [T_C(x) \cap J^* N_{C^\eta}(x)]^\circ, \tag{4.2}$$

where $B_{E^*}$ is the open unit ball in $E^*$.

Since $F$ is constant on $\Gamma(x^*)$, $\alpha$ satisfies (4.2) for all $x^* \in C^\eta$. Therefore, for every $y \in B_{E^*}$, we have

$$\alpha y - F(x^*) \in \bigcap_{x \in C^\eta} [T_C(x) \cap J^* N_{C^\eta}(x)]^\circ \subseteq [T_C(x^*) \cap J^* N_{C^\eta}(x^*)]^\circ.$$

Thus, for every $z \in [T_C(x^*) \cap J^* N_{C^\eta}(x^*)]$. It follows that

$$\langle \alpha y - F(x^*), z \rangle \leq 0. \tag{4.3}$$

Taking $y = \frac{Jz}{\|Jz\|_*}$ in (4.3), we get that, for each $x^* \in C^\eta$,

$$\alpha \|Jz\|_* = \frac{\alpha}{\|Jz\|_*} \langle Jz, z \rangle \leq \langle F(x^*), z \rangle.$$

This implies that for every $z \in [T_C(x^*) \cap J^* N_{C^\eta}(x^*)]$, we have

$$\alpha \|z\| \leq \langle F(x^*), z \rangle.$$

For any $x \in C$, set $\overline{x} = P_{C^\eta}(x)$, we have $\eta(x, \overline{x}) \in T_C(\overline{x}) \cap J^* N_{C^\eta}(\overline{x})$ by Proposition 2.5 and lemma 2.3. Therefore,

$$\langle F(x^*), \eta(x, \overline{x}) \rangle \geq \alpha \|\eta(x, \overline{x})\| = \alpha d^\eta_{C^\eta}(x).$$

Conversely, suppose that there exists $\alpha > 0$ such that

$$\alpha d^\eta_{C^\eta}(x) \leq g(x) \quad \text{for each} \quad x \in C.$$

We claim that

$$\alpha B_{E^*} \subseteq F(x^*) + [T_C(x^*) \cap J^* N_{C^\eta}(x^*)]^\circ \quad \text{for each} \quad x^* \in C^\eta. \tag{4.4}$$

If $T_C(x^*) \cap J^* N_{C^\eta}(x^*) = \{0\}$ for $x^* \in C^\eta$, then $[T_C(x^*) \cap J^* N_{C^\eta}(x^*)]^\eta = E$ and $\alpha B_{E^*} \subseteq F(x^*) + [T_C(x^*) \cap J^* N_{C^\eta}(x^*)]^\eta$, trivially. So it suffices to prove (4.4) to hold if $T_C(x^*) \cap J^* N_{C^\eta}(x^*) \neq \{0\}$ for $x^* \in C^\eta$. Now let $0 \neq z \in T_C(x^*) \cap J^* N_{C^\eta}(x^*)$. By definition of $\eta$ there exists a unique $v \in E$ such that $z = \eta(v, 0)$. Then

$$\langle J(\eta(v,0)), \eta(v,0) \rangle > 0 \quad \text{and} \quad \langle J(\eta(v,0)), \eta(y^*, x^*) \rangle \leq 0 \quad \text{for each} \quad y^* \in C^\eta,$$

which implies that $C^\eta$ is separated from $x^* + v$ by the hyperplane

$$H_v = \{x \in E : \langle J(\eta(v,0)), \eta(x, x^*) \rangle = 0\} = \{x \in E : \langle J(\eta(v,0)), \eta(x,0) \rangle = \langle J(\eta(v,0)), \eta(x^*,0) \rangle\}.$$

Thus we can write

$$H_v = \{x \in E : \langle J(\eta(v,0)), \eta(x,0)\rangle = \beta\}, \quad \text{where} \quad \beta = \langle J(\eta(v,0)), \eta(x^*,0)\rangle.$$

Since $\eta(v,0) \in T_C(x^*)$, for each positive sequence $\{t_i\}$ decreasing to 0, there exists a sequence $\{v_i\}$ such that $\{\eta(v_i,0)\}$ converging to $\eta(v,0)$ and $x^* + t_i v_i \in C$ for sufficiently large $i$ . By definition of $\eta$, we obtain that $v_i$ converging to $v$. Thus $\langle J(\eta(v,0)), \eta(v_i,0)\rangle > 0$ holds for sufficiently large $i$, and hence we suppose that $x^* + t_i v_i$ lies in the open set $\{x \in E : \langle J(\eta(v,0)), \eta(x,x^*)\rangle > 0\}$. Therefore,

$$d^{\eta}_{C^{\eta}}(x^* + t_i v_i) \geq d^{\eta}_{H_v}(x^* + t_i v_i). \tag{4.5}$$

For each $x \in E$. We set

$$y := x - \left[\frac{\langle J(\eta(v,0)), \eta(x,0)\rangle - \beta}{\|J(\eta(v,0))\|^2_*}\right] v.$$

A straightforward computation show that $\langle J(\eta(v,0)), \eta(y,0)\rangle = \beta$, i.e., $y \in H_v$.
Furthermore, for any $z \in H_v$, we have

$$\begin{aligned}\big\langle J(\eta(x,y)), \eta(z,y)\big\rangle &= \left[\frac{\langle J(\eta(v,0)), \eta(x,0)\rangle - \beta}{\|J(\eta(v,0))\|^2_*}\right]\big(\langle J(\eta(v,0)), \eta(z,0)\rangle - \langle J(\eta(v,0)), \eta(y,0)\rangle\big)\\ &= 0.\end{aligned}$$

By Lemma 2.3, we have

$$\begin{aligned} d^{\eta}_{H_v}(x^* + t_i v_i) &= \|\eta(x^* + t_i v_i, y)\| \\ &= \frac{t_i\langle J(\eta(v,0)), \eta(v_i,0)\rangle}{\|J(\eta(v,0))\|^2_*}\|\eta(v,0)\| \\ &= \frac{t_i\langle J(\eta(v,0)), \eta(v_i,0)\rangle}{\|\eta(v,0)\|}, \end{aligned}$$

and hence, by (4.1),

$$g(x^* + t_i v_i) \geq \alpha d_{C^{\eta}}(x^* + t_i v_i) \geq \alpha t_i \frac{\langle J\eta(v,0), \eta(v_i,0)\rangle}{\|\eta(v,0)\|}.$$

By Proposition 3.1, $g(x^*) = 0$ for any $x^* \in C^{\eta}$, so

$$g(x^* + t_i v_i) - g(x^*) = g(x^* + t_i v_i) \geq \alpha t_i \frac{\langle J\eta(v,0), \eta(v_i,0)\rangle}{\|\eta(v,0)\|}.$$

Since $g$ is $\eta$-locally Lipschitz and $\eta$-Gateaux differentiable on $C^{\eta}$, there hold

$$\|g(x^* + t_i v_i) - g(x^* + t_i v)\| \leq L t_i \|\eta(v_i, v)\|$$

for some $L > 0$ and all sufficiently large $i$ and

$$\begin{aligned}\langle \nabla_{\eta} g(x^*), \eta(v,0)\rangle &= \lim_{i\to\infty} \frac{g(x^* + t_i v) - g(x^*)}{t_i} \\ &= \lim_{i\to\infty} \frac{g(x^* + t_i v_i) - g(x^*)}{t_i} \geq \alpha\|\eta(v,0)\|.\end{aligned}$$

By Proposition 3.9, $\nabla_{\eta} g(x^*) = F(x^*)$. Thus

$$\langle F(x^*), \eta(v,0)\rangle \geq \alpha\|\eta(v,0)\|.$$

This implies that for each $w \in B_{E^*}$,

$$\langle \alpha w - F(x^*), \eta(v,0)\rangle = \langle \alpha w, \eta(v,0)\rangle - \langle F(x^*), \eta(v,0)\rangle \leq \alpha\|\eta(v,0)\| - \alpha\|\eta(v,0)\| = 0.$$

Hence $\alpha B_{E^*} - F(x^*) \subseteq [T_C(x^*) \cap J^* N_{C^\eta}(x^*)]^\circ$, that is,

$$\alpha B_{E^*} \subseteq F(x^*) + [T_C(x^*) \cap J^* N_{C^\eta}(x^*)]^\circ.$$

This shows that $C^\eta$ is weakly sharp since $F$ is constant on $C^\eta$. □

**Corollary 4.3** ([5]). *Let $F$ be monotone on $R^n$ and constant on $\Gamma(x^*)$ for some $x^* \in C^*$. Suppose that $g$ is Gateaux differentiable, locally Lipschitz on $C^*$, and $g(x) < +\infty$ for all $x \in R^n$. Then $C^*$ is weakly sharp if and only if there exists a positive number $\alpha$ such that*

$$\alpha d_{C^*}(x) \leq g(x) \quad \textit{for all} \quad x \in C.$$

*Proof.* By applying above Theorem 4.2, if we define $\eta(x,y) = x - y$, for all $x, y \in E$ and space $E = \mathbb{R}^n$, then $C^\eta$ can be reduce to $C^*$, where $C^*$ is the solution set of variational inequalities. Moreover, the mapping $g$ is Gateaux differentiable and locally Lipschitz on $C^*$. □

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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**References**

[1] AI-Homidan, S., Ansari, Q.H., Nguyen, L.V.: Finite convergence analysis and weak sharp solutions for variational inequalities. Optim. Lett. 10, 805-819 (2016)

[2] Burke, J.V., Ferris, M.C.:Weak sharp minima in mathematical programming. SIAM J. Control Optim. 31, 1340-1359 (1993)

[3] Dhara, A., Dutta, J.:Optimality conditions in convex optimization a finite-dimensional view. CRC press Taylor and Francis Group, New York(2012)

[4] Hu, Y.H., Song,W.:Weak sharp solutions for variational inequalities in Banach spaces. J.Math. Anal. Appl. 374, 118-132 (2011)

[5] Liu, Y., Wu, Z.: Characterization of weakly sharp solutions of a variational inequality by its primal gap function. Optim. Lett. 10, 563-576 (2016)

[6] Marcotte, P., Zhu, D.L.: Weak sharp solutions of variational inequalities. SIAM J. Optim. 9, 179-189 (1998)

[7] Patriksson, M.: A Unified Framework of Descent Algorithms for Nonlinear Programs and Variational Inequalities, Ph.D. thesis, Department of Mathematics, Linkping Institute of Technology, Linkping, Sweden (1993)

[8] Takahashi, W. :Nonlinear functional analysis, Fixed point theory and its applications, Yokohama Publ., Yokohama,(2000). 2

[9] Wu, Z.L.,Wu, S.Y.:Weak sharp solutions of variational inequalities in Hilbert spaces. SIAM J. Optim. 14, 1011-1027 (2004)

[10] Wu, Z.L., Wu, S.Y.: Gateaux differentiability of the dual gap function of a variational inequality. Eur. J. Oper. Res. 190, 328-344 (2008)

[11] Wu, Z.L.,Wu, S.Y.: Characterizations of the solution sets of convex programs and variational inequality problems. J. Optim. Theory Appl. 130, 339-358 (2006)

[12] Zhang, J.Z., Wan, C.Y., Xiu, N.H.: The dual gap functions for variational inequalities. Appl. Math. Optim. 48, 129-148 (2003)

# INEQUALITIES OF HERMITE-HADAMARD TYPE FOR $n$-TIMES DIFFERENTIABLE $(\alpha, m)$-LOGARITHMICALLY CONVEX FUNCTIONS

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Abstract. In this paper, some new integral inequalities of Hermite-Hadamard type are presented for functions whose $n$th derivatives in absolute value are $(\alpha, m)$-logarithmically convex. From our results, several inequalities of Hermite-Hadamard type can be derived in terms of functions whose first and second derivatives in absolute value are $(\alpha, m)$-logarithmically convex functions as special cases. Our results may provide refinements of some results for $(\alpha, m)$-logarithmically convex functions already exist in the most recent concerned literature of inequalities.

## 1. Introduction

Let us first refresh our knowledge how the following definition of classical convex functions is generalized.

**Definition 1.** *A function $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, is said to be convex on $I$ if the inequality*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1.1}$$

*holds for all $x, y \in I$ and $t \in [0,1]$. The inequalities in (1.1) are swapped if $f$ is a concave function.*

The definition of convex functions plays an important role in the theory of convex analysis and in many other branches of pure and applied mathematics. A number of remarkable and significant results in the theory of inequality hinge on this definition.

One of the momentous results which uses the notion of convexity is stated as follows:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}, \tag{1.2}$$

where $f : \emptyset \neq I \subseteq \mathbb{R} \to \mathbb{R}$, is a convex function of single variable, $a$, $b \in I$ with $a < b$. The inequalities in (1.2) are celebrated as Hermite-Hadamard inequality and are overturned if $f$ is a concave function.

The inequalities (1.2) have been target of extensive research because of its usefulness and usages in the theory of inequalities and in various other branches of mathematics. A vast literature is reported on the Hermite-Hadamard type inequalities during the past few years which generalize, improve and extend the inequalities (1.2), see for example [6, 12, 13, 14, 15, 17, 19, 23, 28] and closely related references therein.

The classical convexity has been generalized in diverse ways such as $s$-convexity, $m$-convexity, $(\alpha, m)$-convexity, $h$-convexity, logarithmic-convexity, $s$- logarithmic convexity, $(\alpha, m)$- logarithmic convexity and $h$-logarithmic-convexity but we will focus on the following generalizations of the classical convexity to prove our results.

**Definition 2.** [2, 33, 34] *If a function $f : I \subseteq \mathbb{R} \to (0, \infty)$ satisfies*

$$f(\lambda x + (1-\lambda)y) \leq [f(x)]^{\lambda}[f(y)]^{1-\lambda} \tag{1.3}$$

*for all $x$, $y \in I$, $\lambda \in [0,1]$, the function $f$ is called logarithmically convex on $I$. If the inequality (1.3) reverses, the function $f$ is called logarithmically concave on $I$.*

The above stated concept logarithmically convex functions is further generalized as in the definitions below.

**Definition 3.** [9] *A function $f : [0,b] \to (0,\infty)$ is said to be $m$-logarithmically convex if*

$$f(tx + m(1-t)y) \leq [f(x)]^{t}[f(y)]^{m(1-t)}$$

*holds for all $x$, $y \in [0,b]$, $t \in [0,1]$ and $m \in (0,1]$.*

**Definition 4.** [9] *A function $f : [0,b] \to (0,\infty)$ is said to be $(\alpha, m)$-logarithmically convex if*

$$f(tx + m(1-t)y) \leq [f(x)]^{t^{\alpha}}[f(y)]^{m(1-t^{\alpha})}$$

*holds for all $x$, $y \in [0,b]$, $t \in [0,1]$ and $(\alpha, m) \in (0,1] \times (0,1]$.*

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It is also obvious that if $m=1$ in Definition 3 and if $(\alpha,m)=(1,1)$ in Definition 4, the notion of $m$-logarithmic convexity and $(\alpha,m)$-logarithmic convexity recapture the notion of usual logarithmic convexity.

Many papers have been written by a number of mathematicians concerning Hermite-Hadamard type inequalities for different classes of convex functions see for instance the recent papers [2, 3, 4, 7, 8, 9, 16, 18, 24, 25, 27, 29, 31, 32, 33, 35] and the references within these papers.

The main purpose of the present paper is to establish new Hermite-Hadamard type integral inequalities by using the notion of $m$- and $(\alpha,m)$-logarithmically convex functions and a new identity for $n$-times differentiable functions from [19]in Section 2.

## 2. Main Results

We will use the following Lemmas to establish our main results in this section.

**Lemma 1.** [19] *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a function such that $f^{(n)}$ exists on $I^{\circ}$ and $f^{(n)}\in L([a,b])$ for $n\in\mathbb{N}$, where $a,\ b\in I^{\circ}$ with $a<b$, we have the identity*

$$\begin{aligned}(2.1)\quad &\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\\ &=\frac{(b-a)^n}{2^{n+1}n!}\int_0^1(1-t)^{n-1}(n-1+t)\,f^{(n)}\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt\\ &\qquad+\frac{(-1)^n(b-a)^n}{2^{n+1}n!}\int_0^1(1-t)^{n-1}(n-1+t)\,f^{(n)}\left(\frac{1-t}{2}b+\frac{1+t}{2}a\right)dt,\end{aligned}$$

*where an empty sum is understood to be nil.*

**Lemma 2.** [20] *If $\mu>0$ and $n\in\mathbb{N}\cup\{0\}$, then*

$$(2.2)\quad \int_0^1 t^n\mu^t dt=\begin{cases}\frac{(-1)^{n+1}n!}{(\ln\mu)^{n+1}}+n!\mu\sum_{k=0}^n\frac{(-1)^k}{(n-k)!(\ln\mu)^{k+1}}, & \mu\neq 1\\ \frac{1}{n+1}, & \mu=1.\end{cases}$$

**Lemma 3.** *If $\mu>0$ and $\mathbb{N}\cup\{0\}$, then*

$$(2.3)\quad E(n;\mu):=\int_0^1(1-t)^n\mu^t dt=\begin{cases}\frac{n!\mu}{(\ln\mu)^{n+1}}-n!\sum_{k=0}^n\frac{1}{(n-k)!(\ln\mu)^{k+1}}, & \mu\neq 1\\ \frac{1}{n+1}, & \mu=1.\end{cases}$$

*Proof.* By making the substitution $t=1-u$ in Lemma 2, we get (2.3). □

**Lemma 4.** [7] *For $\alpha>0$ and $\mu>0$, we have*

$$(2.4)\quad G(\alpha;\mu):=\int_0^1(1-t)^{\alpha-1}\mu^t dt=\sum_{k=1}^{\infty}\frac{(\ln\mu)^{k-1}}{(\alpha)_k}<\infty,$$

*where*

$$(\alpha)_k=\alpha(\alpha+1)(\alpha+2)...(\alpha+k-1).$$

From Lemma 3 and Lemma 4, by simple computations we get the following results.

**Lemma 5.** *If $\mu>0$ and $n\in\mathbb{N}$, then*

$$(2.5)\quad F(n;\mu):=nE(n-1;\mu)-E(n;\mu)=\begin{cases}\frac{n!\mu(\ln\mu-1)}{(\ln\mu)^{n+1}}+\frac{1}{\ln\mu}-n!\sum_{k=1}^n\frac{\ln\mu-1}{(n-k)!(\ln\mu)^{k+1}}, & \mu\neq 1\\ \frac{n}{n+1}, & \mu=1.\end{cases}$$

**Lemma 6.** *For $\alpha>0$ and $\mu>0$, we have*

$$(2.6)\quad H(\alpha;\mu):=nG(\alpha;\mu)-G(\alpha+1;\mu)=\sum_{k=1}^{\infty}\frac{(n\alpha+nk-\alpha)(\ln\mu)^{k-1}}{(\alpha)_{k+1}}<\infty,$$

*where*

$$(\alpha)_{k+1}=\alpha(\alpha+1)(\alpha+2)...(\alpha+k).$$

**Lemma 7.** [35] *Let $0<\xi\leq 1\leq\eta$, $0\leq\lambda\leq 1$ and $0<s\leq 1$. Then*

$$(2.7)\quad \xi^{\lambda^s}\leq\xi^{s\lambda}\ \text{and}\ \eta^{\lambda^s}\leq\eta^{s\lambda+1-s}.$$

**Theorem 1.** *Let $I \supset [0,\infty)$ be an open interval and let $f: I \to (0,\infty)$ be a function such that $f^{(n)}$ exists on $I$. If $f^{(n)} \in L([a,b])$ for $n \in \mathbb{N}$, where $0 \le a < b < \infty$ and $\left|f^{(n)}\right|^q$ is $(\alpha,m)$-logarithmically convex on $\left[0,\frac{b}{m}\right]$ for $(\alpha,m) \in (0,1] \times (0,1]$, $q \in [1,\infty)$. Then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx - \sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right| \tag{2.8}$$
$$\le \frac{(b-a)^n}{2^{n+1}n!}\left(\frac{n}{n+1}\right)^{1-\frac{1}{q}}\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m \mu^\theta\left\{\left[F\left(n;\mu^{-\frac{\alpha q}{2}}\right)\right]^{1/q} + \left[F\left(n;\mu^{\frac{\alpha q}{2}}\right)\right]^{1/q}\right\},$$

*where $F(n;\xi)$ is defined in Lemma 5, $\mu = \frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}$ and*

$$\theta = \begin{cases} \frac{\alpha}{2}, & 0 < \mu \le 1 \\ 1-\frac{\alpha}{2}, & \mu > 1. \end{cases}$$

*Proof.* From Lemma 1, the Hölder inequality and using the fact that $\left|f^{(n)}\right|^q$ is $(\alpha,m)$-logarithmically convex on $\left[0,\frac{b}{m}\right]$, we have

$$\left|\frac{f(a)+f(b)}{2} - \sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \tag{2.9}$$
$$\le \frac{(b-a)^n}{2^{n+1}n!}\left[f^{(n)}\left(\frac{b}{m}\right)\right]^m\left(\int_0^1 (1-t)^{n-1}(n-1+t)\,dt\right)^{1-\frac{1}{q}}$$
$$\times\left\{\left(\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^{q\left(\frac{1-t}{2}\right)^\alpha}dt\right)^{1/q} + \left(\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^{q\left(\frac{1+t}{2}\right)^\alpha}dt\right)^{1/q}\right\},$$

where $\mu = \frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}$.
It is obvious that

$$\int_0^1 (1-t)^{n-1}(n-1+t)\,dt = \frac{n}{n+1}. \tag{2.10}$$

When $0 < \mu \le 1$, by using Lemma 5 and Lemma 7, we obtain

$$\left(\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^{q\left(\frac{1-t}{2}\right)^\alpha}dt\right)^{\frac{1}{q}} + \left(\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^{q\left(\frac{1+t}{2}\right)^\alpha}dt\right)^{\frac{1}{q}} \tag{2.11}$$
$$\le \left(\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^{\alpha q\left(\frac{1-t}{2}\right)}dt\right)^{\frac{1}{q}} + \left(\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^{\alpha q\left(\frac{1+t}{2}\right)}dt\right)^{\frac{1}{q}}$$
$$= \mu^{\frac{\alpha}{2}}\left(\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^{-\frac{\alpha q t}{2}}dt\right)^{\frac{1}{q}} + \mu^{\frac{\alpha}{2}}\left(\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^{\frac{\alpha q t}{2}}dt\right)^{\frac{1}{q}}$$
$$= \mu^{\frac{\alpha}{2}}\left\{\left[F\left(n;\mu^{-\frac{\alpha q}{2}}\right)\right]^{1/q} + \left[F\left(n;\mu^{\frac{\alpha q}{2}}\right)\right]^{1/q}\right\}.$$

When $\mu > 1$, by using Lemma 5 and Lemma 7, we have

$$\left(\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^{q\left(\frac{1-t}{2}\right)^\alpha}dt\right)^{\frac{1}{q}} + \left(\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^{q\left(\frac{1+t}{2}\right)^\alpha}dt\right)^{\frac{1}{q}} \tag{2.12}$$
$$\le \left(\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^{\alpha q\left(\frac{1-t}{2}\right)+q-\alpha q}dt\right)^{\frac{1}{q}} + \left(\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^{\alpha q\left(\frac{1+t}{2}\right)+q-\alpha q}dt\right)^{\frac{1}{q}}$$
$$= \mu^{1-\frac{\alpha}{2}}\left(\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^{-\frac{\alpha q t}{2}}dt\right)^{\frac{1}{q}} + \mu^{1-\frac{\alpha}{2}}\left(\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^{\frac{\alpha q t}{2}}dt\right)^{\frac{1}{q}}$$
$$= \mu^{1-\frac{\alpha}{2}}\left\{\left[F\left(n;\mu^{-\frac{\alpha q}{2}}\right)\right]^{1/q} + \left[F\left(n;\mu^{\frac{\alpha q}{2}}\right)\right]^{1/q}\right\}.$$

A combination of (2.9)-(2.12) gives the desired result. □

**Corollary 1.** *Suppose the assumptions of Theorem 1 are satisfied and if $q=1$, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right| \tag{2.13}$$
$$\leq \frac{(b-a)^n}{2^{n+1}n!}\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m \mu^\theta\left\{F\left(n;\mu^{-\frac{\alpha}{2}}\right)+F\left(n;\mu^{\frac{\alpha}{2}}\right)\right\},$$

*where $F(n;\xi)$ is defined in Lemma 5, $\mu=\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}$ and $\theta$ is defined in Theorem 1.*

**Corollary 2.** *Under the assumptions of Theorem 1, if $n=1$, we have the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \tag{2.14}$$
$$\leq \frac{(b-a)}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left|f'\left(\frac{b}{m}\right)\right|^m \mu^\theta\left\{\left[F\left(1;\mu^{-\frac{\alpha q}{2}}\right)\right]^{1/q}+\left[F\left(1;\mu^{\frac{\alpha q}{2}}\right)\right]^{1/q}\right\},$$

*where $\mu=\frac{\left|f'(a)\right|}{\left|f'\left(\frac{b}{m}\right)\right|^m}$, $\theta$ is defined in Theorem 1 and*

$$F(1;\xi)=\begin{cases}\frac{1}{\ln\xi}\left[\xi+\frac{1-\xi}{\ln\xi}\right], & \xi\neq 1\\ \frac{1}{2}, & \xi=1.\end{cases}$$

**Corollary 3.** *Corollary 2 with $q=1$ gives the following result*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{(b-a)}{4}\left|f'\left(\frac{b}{m}\right)\right|^m\mu^\theta\left\{\left[F\left(1;\mu^{-\frac{\alpha}{2}}\right)\right]+\left[F\left(1;\mu^{\frac{\alpha}{2}}\right)\right]\right\}, \tag{2.15}$$

*where $F(1;\xi)$ and $\mu$ are defined as in Corollary 2 and $\theta$ is as defined in Theorem 1.*

**Corollary 4.** *Suppose the assumptions of Theorem 1 are fulfilled and if $n=2$, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \tag{2.16}$$
$$\leq \frac{(b-a)^2}{16}\left(\frac{2}{3}\right)^{1-\frac{1}{q}}\left|f''\left(\frac{b}{m}\right)\right|^m \mu^\theta\left\{\left[F\left(2;\mu^{-\frac{\alpha q}{2}}\right)\right]^{1/q}+\left[F\left(2;\mu^{\frac{\alpha q}{2}}\right)\right]^{1/q}\right\},$$

*where $\mu=\frac{\left|f''(a)\right|}{\left|f''\left(\frac{b}{m}\right)\right|^m}$, $\theta$ is as defined in Theorem 1 and*

$$F(2;\xi)=\begin{cases}\frac{2\xi\ln\xi-(\ln\xi)^2-2\xi+2}{(\ln\xi)^3}, & \xi\neq 1,\\ \frac{2}{3}, & \xi=1.\end{cases}$$

**Corollary 5.** *If $q=1$ in Corollary 4, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{(b-a)^2}{16}\left|f''\left(\frac{b}{m}\right)\right|^m\mu^\theta\left\{F\left(2;\mu^{-\frac{\alpha}{2}}\right)+F\left(2;\mu^{\frac{\alpha}{2}}\right)\right\}, \tag{2.17}$$

*where $\theta$ is defined in Theorem 1 and $\mu$, $F(2;\xi)$ are defined in Corollary 4.*

**Theorem 2.** *Let $I\supset[0,\infty)$ be an open interval and let $f:I\to(0,\infty)$ be a function such that $f^{(n)}$ exists on $I$. If $f^{(n)}\in L([a,b])$ for $n\in\mathbb{N}$, where $0\leq a<b<\infty$ and $\left|f^{(n)}\right|^q$ is $(\alpha,m)$-logarithmically convex on $\left[0,\frac{b}{m}\right]$ for $(\alpha,m)\in(0,1]\times(0,1]$, $q\in(1,\infty)$, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right| \tag{2.18}$$
$$\leq\frac{(b-a)^n\left[n^{(2q-1)/(q-1)}-(n-1)^{(2q-1)/(q-1)}\right]^{1-\frac{1}{q}}}{2^{n+1}n!}\left(\frac{q-1}{2q-1}\right)^{1/q}\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m\mu^\theta$$
$$\times\left\{\left[G\left(nq-q+1;\mu^{-\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}}+\left[G\left(nq-q+1;\mu^{\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}}\right\},$$

*where $\mu=\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}$, $G(\alpha;\xi)$ is defined in Lemma 4 and $\theta$ is defined in Theorem 1.*

*Proof.* Using Lemma 1, the Hölder inequality and the $(\alpha,m)$-logarithmic convexity of $\left|f^{(n)}\right|^q$ on $\left[0,\frac{b}{m}\right]$, we have

$$\begin{aligned}(2.19)\quad &\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right|\\ &\leq\frac{(b-a)^n}{2^{n+1}n!}\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m\left(\int_0^1(n-1+t)^{\frac{q}{q-1}}\,dt\right)^{1-\frac{1}{q}}\\ &\qquad\times\left\{\left(\int_0^1(1-t)^{q(n-1)}\mu^{q\left(\frac{1-t}{2}\right)^\alpha}dt\right)^{1/q}+\left(\int_0^1(1-t)^{q(n-1)}\mu^{q\left(\frac{1+t}{2}\right)^\alpha}dt\right)^{1/q}\right\}.\end{aligned}$$

The proof follows by using similar arguments as in proving Theorem 1, using Lemma 4 and Lemma 7. □

**Corollary 6.** *Under the assumptions of Theorem 2, if $n=1$, we have the inequality*

$$\begin{aligned}(2.20)\quad &\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\ &\leq\frac{(b-a)}{4}\left(\frac{q-1}{2q-1}\right)^{1/q}\left|f'\left(\frac{b}{m}\right)\right|^m\mu^\theta\left\{\left[G\left(1;\mu^{-\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}}+\left[G\left(1;\mu^{\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}}\right\},\end{aligned}$$

*where* $\mu=\frac{\left|f'(a)\right|}{\left|f''\left(\frac{b}{m}\right)\right|^m}$,

$$G(1;\xi)=\sum_{k=1}^{\infty}\frac{(\ln\xi)^{k-1}}{k!}<\infty$$

*and $\theta$ is defined in Theorem 1.*

**Corollary 7.** *Under the assumptions of Theorem 2, if $n=2$, we have the inequality*

$$\begin{aligned}(2.21)\quad &\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\ &\leq\frac{(b-a)^2\left[2^{(2q-1)/(q-1)}-1\right]^{1-\frac{1}{q}}}{16}\left(\frac{q-1}{2q-1}\right)^{1/q}\left|f''\left(\frac{b}{m}\right)\right|^m\mu^\theta\\ &\qquad\times\left\{\left[G\left(q+1;\mu^{-\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}}+\left[G\left(q+1;\mu^{\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}}\right\},\end{aligned}$$

*where* $\mu=\frac{\left|f''(a)\right|}{\left|f''\left(\frac{b}{m}\right)\right|^m}$,

$$G(q+1;\xi)=\sum_{k=1}^{\infty}\frac{(\ln\xi)^{k-1}}{(q+1)_k}<\infty$$

*and $\theta$ is defined in Theorem 1.*

**Theorem 3.** *Let $I\supset[0,\infty)$ be an open interval and let $f:I\to\mathbb{R}$ be a function such that $f^{(n)}$ exists on $I$. If $f^{(n)}\in L([a,b])$ for $n\in\mathbb{N}$, where $0\leq a<b<\infty$ and $\left|f^{(n)}\right|^q$ is $(\alpha,m)$-logarithmically convex on $\left[0,\frac{b}{m}\right]$ for $(\alpha,m)\in(0,1]\times(0,1]$, $q\in(1,\infty)$, we have*

$$\begin{aligned}(2.22)\quad &\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right|\\ &\leq\frac{n^{n+1-\frac{1}{q}}(b-a)^n}{2^{n+1}n!}\left[B\left(\frac{1}{n};\frac{nq-1}{q-1},\frac{2q-1}{q-1}\right)\right]^{1-\frac{1}{q}}\mu^\theta\left\{\left[F_3\left(\mu^{-\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}}+\left[F_3\left(\mu^{\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}}\right\},\end{aligned}$$

*where* $\mu=\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}$,

$$F_3(\xi)=\begin{cases}\frac{\xi-1}{\ln\xi}, & \xi\neq1\\ 1, & \xi=1\end{cases},$$

$$B(z;\alpha,\beta)=\int_0^z t^{\alpha-1}(1-t)^{\beta-1}\,dt, 0\leq z\leq1,\alpha>0,\beta>0$$

*is the incomplete Beta function and $\theta$ is defined in Theorem 1.*

*Proof.* Using Lemma 1, the Hölder inequality and the $(\alpha,m)$-logarithmic convexity of $\left|f^{(n)}\right|^q$ on $\left[0,\frac{b}{m}\right]$, we have

$$\begin{aligned}(2.23)\quad &\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right|\\ &\leq\frac{(b-a)^n}{2^{n+1}n!}\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m\left(\int_0^1(1-t)^{q(n-1)/(q-1)}(n-1+t)^{q/(q-1)}\,dt\right)^{1-1/q}\\ &\qquad\times\left\{\left(\int_0^1\mu^{q\left(\frac{1-t}{2}\right)^\alpha}dt\right)^{1/q}+\left(\int_0^1\mu^{q\left(\frac{1+t}{2}\right)^\alpha}dt\right)^{1/q}\right\}.\end{aligned}$$

By using Lemma 7 and the fact that

$$\begin{aligned}&\int_0^1(1-t)^{q(n-1)/(q-1)}(n-1+t)^{q/(q-1)}\,dt\\ &=n^{\frac{nq+q-1}{q-1}}\int_0^{\frac{1}{n}}t^{\frac{(n-1)q}{q-1}}(1-t)^{\frac{q}{q-1}}\,dt=n^{\frac{nq+q-1}{q-1}}B\left(\frac{1}{n};\frac{nq-1}{q-1},\frac{2q-1}{q-1}\right),\end{aligned}$$

we get the required inequality (2.22) from (2.23). □

**Corollary 8.** *Suppose the assumptions of Theorem 3 are satisfied and if $n=1$, we have the inequality*

$$(2.24)\quad \left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{(b-a)}{4}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\mu^\theta\left\{\left[F_3\left(\mu^{-\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}}+\left[F_3\left(\mu^{\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}}\right\},$$

*where* $\mu=\frac{\left|f'(a)\right|}{\left|f'\left(\frac{b}{m}\right)\right|^m}$,

$$F_3(\xi)=\begin{cases}\frac{\xi-1}{\ln\xi}, & \xi\neq1\\ 1, & \xi=1\end{cases}$$

*and $\theta$ are defined as in Theorem 1.*

**Corollary 9.** *Suppose the assumptions of Theorem 3 are satisfied and if $n=2$, we have the inequality*

$$\begin{aligned}(2.25)\quad &\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\ &\leq\frac{(b-a)^2}{2^{1+\frac{1}{q}}}\left[B\left(\frac{1}{2};\frac{2q-1}{q-1},\frac{2q-1}{q-1}\right)\right]^{1-\frac{1}{q}}\mu^\theta\left\{\left[F_3\left(\mu^{-\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}}+\left[F_3\left(\mu^{\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}}\right\},\end{aligned}$$

*where* $\mu=\frac{\left|f''(a)\right|}{\left|f''\left(\frac{b}{m}\right)\right|^m}$,

$$F_3(\xi)=\begin{cases}\frac{\xi-1}{\ln\xi}, & \xi\neq1\\ 1, & \xi=1\end{cases},$$

*$B(z;\alpha,\beta)$ is the incomplete Beta function as defined in Theorem 3 and $\theta$ is defined as in Theorem 1.*

**Theorem 4.** *Let $I\supset[0,\infty)$ be an open interval and let $f:I\subset[0,\infty)\to(0,\infty)$ be a function such that $f^{(n)}$ exists on $I$. If $f^{(n)}\in L([a,b])$ for $n\in\mathbb{N}$, where $0\leq a<b<\infty$ and $\left|f^{(n)}\right|^q$ is $(\alpha,m)$-logarithmically convex on $\left[0,\frac{b}{m}\right]$ for $(\alpha,m)\in(0,1]\times(0,1]$, $q\in(1,\infty)$ for $0\leq r\leq(n-1)q$. Then*

$$\begin{aligned}(2.26)\quad &\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right|\\ &\leq\frac{(b-a)^n}{2^{n+1}n!}\left[\frac{(q-1)\left(n^2q-nr-2n+r+1\right)}{(nq-r-1)(nq+q-r-2)}\right]^{1-\frac{1}{q}}\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m\mu^\theta\\ &\qquad\times\left\{\left[H\left(r+1;\mu^{-\frac{\alpha q}{2}}\right)\right]^{1/q}+\left[H\left(r+1;\mu^{\frac{\alpha q}{2}}\right)\right]^{1/q}\right\}.\end{aligned}$$

*$\mu=\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}$, $\theta$ is defined in Theorem 1 and $H(\alpha;\xi)$ is defined in Lemma 6.*

*Proof.* From Lemma 1, the Hölder inequality and using the fact that $\left|f^{(n)}\right|^q$ is $(\alpha,m)$-logarithmically convex on $\left[0,\frac{b}{m}\right]$, we have

$$\begin{aligned}(2.27)\quad &\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right|\\ &\leq\frac{(b-a)^n}{2^{n+1}n!}\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m\left(\int_0^1(1-t)^{(nq-q-r)/(q-1)}(n-1+t)\,dt\right)^{1-\frac{1}{q}}\\ &\quad\times\left\{\left(\int_0^1(1-t)^r(n-1+t)\,\mu^{q\left(\frac{1-t}{2}\right)^\alpha}dt\right)^{1/q}+\left(\int_0^1(1-t)^r(n-1+t)\,\mu^{q\left(\frac{1+t}{2}\right)^\alpha}dt\right)^{1/q}\right\},\end{aligned}$$

The rest of the proof is similar to that of the proof of Theorem 2 by using Lemma 6 and Lemma 7. □

**Corollary 10.** *Suppose the assumptions of Theorem 4 are fulfilled and if $r=0$, we have*

$$\begin{aligned}(2.28)\quad &\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right|\\ &\leq\frac{(b-a)^n}{2^{n+1}n!}\left[\frac{(q-1)\left(n^2q-2n+1\right)}{(nq-1)(nq+q-2)}\right]^{1-\frac{1}{q}}\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m\mu^\theta\left\{\left[H\left(1;\mu^{-\frac{\alpha q}{2}}\right)\right]^{1/q}+\left[H\left(1;\mu^{\frac{\alpha q}{2}}\right)\right]^{1/q}\right\}.\end{aligned}$$

**Corollary 11.** *Suppose the assumptions of Theorem 4 are fulfilled and if $r=(n-1)q$, we have*

$$\begin{aligned}(2.29)\quad &\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right|\\ &\leq\frac{(b-a)^n}{2^{n+1}n!}\left[\frac{2nq-2n-q+1}{2(q-1)}\right]^{1-\frac{1}{q}}\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m\mu^\theta\\ &\quad\times\left\{\left[H\left((n-1)q+1;\mu^{-\frac{\alpha q}{2}}\right)\right]^{1/q}+\left[H\left((n-1)q+1;\mu^{\frac{\alpha q}{2}}\right)\right]^{1/q}\right\}.\end{aligned}$$

**Remark 1.** *Several interesting inequalities for m-logarithmically convex functions can be obtained by setting $\alpha=1$ in the results presented in this section. However, we leave the details to the interested reader.*

**Remark 2.** *We can get several interesting inequalities for logarithmically convex functions by setting $\alpha=1$ and $m=1$ in the results proved above. However, the details are left to the interested reader.*

## 3. Applications to Special Means

For positive real numbers $a>0$, $b>0$, we consider the following means

$$A(a,b)=\frac{a+b}{2},\ G(a,b)=\sqrt{ab},\ H(a,b)=\frac{2ab}{a+b},$$

$$I(a,b)=\begin{cases}\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}, & a\neq b,\\ a & a=b,\end{cases}$$

and

$$L_p(a,b)=\begin{cases}\left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{1/p}, & p\neq0,-1 \text{ and } a\neq b,\\ \frac{b-a}{\ln b-\ln a}, & p=-1 \text{ and } a\neq b,\\ I(a,b), & p=0 \text{ and } a\neq b,\\ a, & a=b.\end{cases}$$

It is well known that $A$, $G$, $H$, $L=L_{-1}$, $I=L_0$ and $L_p$ are called the arithmetic, geometric, harmonic, identric, exponential and generalized logarithmic means of positive real numbers $a$ and $b$.

In what follows we will use the above means and the established results of the previous section to obtain some interesting inequalities involving means.

**Theorem 5.** *Let $0<a<b\leq1$, $r<0$, $r\neq-1$ and $q\geq1$.*

(1) *If* $r \neq -2$, *then*

$$\begin{aligned}&\left|A\left(a^{r+1},b^{r+1}\right)-L_{r+1}^{r+1}(a,b)\right|\\&\leq(b-a)\left(\frac{1}{2}\right)^{3-\frac{2}{q}}|r+1|\left(\frac{1}{qr\left(\ln b-\ln a\right)}\right)^{1/q}\left\{b^{r/2}\left[b^{qr/2}-L\left(a^{qr/2},b^{qr/2}\right)\right]^{1/q}\right.\\&\left.+a^{r/2}\left[L\left(a^{qr/2},b^{qr/2}\right)-a^{qr/2}\right]^{1/q}\right\}.\end{aligned}$$

(2) *If* $r=-2$, *then*

$$\begin{aligned}&\left|\frac{1}{H(a,b)}-\frac{1}{L(a,b)}\right|\\&\leq(b-a)\left(\frac{1}{2}\right)^{3-\frac{1}{q}}\left(\frac{1}{q\left(\ln a-\ln b\right)}\right)^{1/q}\left\{b^{-1}\left[b^{-q}-L\left(a^{-q},b^{-q}\right)\right]^{1/q}\right.\\&\left.+a^{-1}\left[L\left(a^{-q},b^{-q}\right)-a^{-q}\right]^{1/q}\right\}.\end{aligned}$$

*Proof.* Let $f(x)=\frac{x^{r+1}}{r+1}$ for $0<x\leq 1$. Then $\left|f'(x)\right|=x^r$ and

$$\begin{aligned}&\ln\left|f'\left(\lambda x+(1-\lambda)y\right)\right|^q\\&\leq\lambda\ln\left|f'(x)\right|^q+(1-\lambda)\ln\left|f'(y)\right|^q\end{aligned}$$

for $x,\,y\in(0,1]$, $\lambda\in[0,1]$ and $q\geq 1$. This shows that $\left|f'(x)\right|^q=x^{rq}$ is logarithmically convex function on $(0,1]$ so that we have $(\alpha,m)=(1,1)$, $\mu=\left|\frac{f'(a)}{f'(b)}\right|$ and $\theta=\frac{1}{2}$.

Since $\left|f'(a)\right|>\left|f'(b)\right|=b^r\geq 1$, hence

$$\mu=\left|\frac{f'(a)}{f'(b)}\right|=\left(\frac{a}{b}\right)^r$$

and

$$\begin{aligned}&\left[f'\left(\frac{b}{m}\right)\right]^m\mu^\theta\left\{\left[F\left(1;\mu^{-\frac{\alpha q}{2}}\right)\right]^{1/q}+\left[F\left(1;\mu^{\frac{\alpha q}{2}}\right)\right]^{1/q}\right\}\\&=\sqrt{f'(b)f'(a)}\left\{\left[F\left(1;\mu^{-\frac{q}{2}}\right)\right]^{1/q}+\left[F\left(1;\mu^{\frac{q}{2}}\right)\right]^{1/q}\right\}\\&=\sqrt{a^rb^r}\left\{\left[F\left(1;\mu^{-\frac{q}{2}}\right)\right]^{1/q}+\left[F\left(1;\mu^{\frac{q}{2}}\right)\right]^{1/q}\right\}\\&=\left(\frac{2}{qr\left(\ln b-\ln a\right)}\right)^{1/q}\left\{b^{r/2}\left[b^{qr/2}-L\left(a^{qr/2},b^{qr/2}\right)\right]^{1/q}\right.\\&\left.+a^{r/2}\left[L\left(a^{qr/2},b^{qr/2}\right)-a^{qr/2}\right]^{1/q}\right\}.\end{aligned}$$

Substituting the above quantities in Corollary 2, we get the required results. □

**Remark 3.** *Many interesting inequalities of means can be obtained from the other results of Section 2, however, the details are left to the readers.*

## References

[1] M. W. Alomari, M. Darus and U. S. Kirmaci, Some inequalities of Hadamard type inequalities for $s$-convex functions, Acts Math. Sci. Ser. B Engl. Ed. 31 (2011), no. 4, 1643-1652.
[2] A. O. Akdemir and M. Tunç, On some inegral inequalities for $s$-logarithmically convex functions and their applications, arXiv:1212.1584 [math.FA].
[3] S. S. Dragomir, On some new inequalities of Hermite-Hadamard type for $m$-convex functions, Tamkang J. Math. 33 (2002) 45–55.
[4] S. S. Dragomir and S. Fitzpatric, The Hadamard inequalities for $s$-convex functions in the second sense, Demonstratio Math. 32 (1999), no. 4 687-696.
[5] S. S. Dragomir and C. E. M. Pearce, Selected Topics on Hermite-Hadamard Inequalities and Applications, RGMIA Monographs, Victoria University, 2000.
[6] S. S. Dragomir, R. P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, Appl. Math. Lett. 11 (1998) 91–95.
[7] J. Deng and J. R. Wang, Fractional Hermite-Hadamard inequalities for $(\alpha,m)$-logarithmically convex functions, J Inequal Appl 2013, 2013:364.

[8] W.-D. Jiang, D.-W. Niu, Y. Hua, and F. Qi, Generalizations of Hermite-Hadamard inequality to $n$-time differentiable functions which are $s$-convex in the second sense, Analysis (Munich) 32 (2012), 1001–1012.
[9] R.- F. Bai, F. Qi and B. -Y. Xi, Hermite-Hadamard type inequalities for the $m$- and $(\alpha, m)$-logarithmically convex functions, Filomat 27:1 (2013), 1-7.
[10] J. Hadamard, Étude sur les propriétés des fonctions entières et en particulier d'une fonction considerée par Riemann, J. Math Pures Appl., 58 (1893), 171–215.
[11] Ch. Hermite, Sur deux limites d'une intégrale définie, Mathesis 3 (1883), 82.
[12] D.-Y. Hwang, Some inequalities for $n$-times differentiable mappings and applications, Kyugpook Math. J. 43(2003), 335-343.
[13] U.S. Kırmacı, Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, Appl. Math. Comp., 147 (2004), 137-146.
[14] U.S. Kırmacı and M.E. Özdemir, On some inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, Appl. Math. Comp., 153 (2004), 361-368.
[15] U.S. Kırmacı, Improvement and further generalization of inequalities for differentiable mappings and applications, Comp and Math. with Appl., 55 (2008), 485-493.
[16] H. Kavurmaci and M. Tunç, On some inequalities for $s$-logarithmically convex functions in the second sense via fractional inegrals, arXiv:1212.1604 [math.FA].
[17] M. A. Latif and S. S. Dragomir, New inequalities of Hermite-Hadamard type for functions whose derivatives in absolute value are convex 'with applications to special means and to general quadrature formula, Acta Univ. M. Belii, ser. Math. (2013), 24–39.
[18] M. A. Latif and S. S. Dragomir, On Hermite-Hadamard type integral inequalities for $n$-times differentiable $(\alpha, m)$-logarithmically convex functions, RGMIA Research Report Collection, 17(2014), Article 14, 16 pp.
[19] M. A. Latif and S. S. Dragomir, New inequalities of Hermite-Hadamard type for $n$-times differentiable convex and concave functions with applications. (Accepted)
[20] M. A. Latif and S. S. Dragomir, On Hermite-Hadamard type integral inequalities for $n$-times differentiable log-preinvex functions. (Accepted)
[21] M. A. Latif and S. S. Dragomir,G eneralization of Hermite-Hadamard type inequalities for $n$-times differentiable functions which are $s$-preinvex in the second sense with applications. (Accepted)
[22] D. S. Mitrinović and I. B. Lacković, Hermite and convexity, Aequationes Math. 28 (1985), 229–232.
[23] C.E.M. Pearce and J. Pečarić, Inequalities for differentiable mappings with application to special means and quadrature formulae, Appl. Math. Lett., 13(2) (2000), 51–55.
[24] M. A. Noor, F. Qi and M. U. Awan, Some Hermite-Hadamard type inequalities for log-$h$-convex functions, Analysis 12/2013, 33(4):367–375. DOI:10.1524/anly.2013.1223
[25] M. Z. Sarikaya, A. Saglam and H. Yıldırım, On some Hadamard-type inequalities for $h$-convex functions, J Math Inequal, Vol. 2, No. 3 (2008), 335-341.
[26] M. Z. Sarikaya and N. Aktan, On the generalization of some integral inequalities and their applications Mathematical and Computer Modelling, Volume 54, Issues 9-10, November 2011, Pages 2175-2182.
[27] M. Z. Sarikaya, E. Set and M. E. Ozdemir, On some new inequalities of Hadamard type involving $h$-convex functions, Acta Mathematica Universitatis Comenianae, Vol. LXXIX, 2(2010), pp. 265-272.
[28] A. Saglam, M. Z. Sarikaya and H. Yildirim, Some new inequalities of Hermite-Hadamard's type, Kyungpook Mathematical Journal, 50(2010), 399-410.
[29] M. Tunç and A. O. Akdemir, Ostrowski type inequalities for $s$-logarithmically convex functions in the second sense with applications, arXiv:1301.3041 [math.CA].
[30] J. Wang, J. Deng, M. Fečkan, Exploring s-e-condition and applications to some Ostrowski type inequalities via Hadamard fractional integrals. Math. Slovaca (2013, in press).
[31] S.-H. Wang, B.-Y. Xi, F. Qi, Some new inequalities of Hermite-Hadamard type for $n$-time differentiable functions which are $m$-convex, Analysis (Munich) 32 (2012) 247–262.
[32] B.-Y. Xi, R.-F. Bai and F. Qi, Hermite-Hadamard type inequalities for the $m$- and $(\alpha, m)$-geometrically convex functions, Aequationes Math. 39 (2012), in press.
[33] B.-Y. Xi and F. Qi, Some integral inequalities of Hermite-Hadamard type for $s$-logarithmically convex functions. (Accepted)
[34] T.-Y. Zhang, A.-P. Ji, F. Qi, On integral inequalities of Hermite-Hadamard type for $s$-geometrically convex functions, Abstr. Appl. Anal., 2012 (2012).
[35] T.-Y. Zhang, M. Tunç, A.-P. Ji and B.-Y. Xi, Erratum to "On integral inequalities of Hermite-Hadamard type for $s$-geometrically convex functions", Abstr. Appl. Anal., Volume 2014, Article ID 294739, 5 pageshttp://dx.doi.org/10.1155/2014/294739.

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# Positive solutions for $p$-Laplacian fractional difference equation with a parameter *

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**Abstract**: In this paper, we consider a boundary value problem for fractional difference equation with $p$-Laplacian operator involving a parameter

$$\begin{cases}\Delta[\phi_p(\Delta_C^{\nu}u)](t)+\lambda^{p-1}f(t+\nu-1,u(t+\nu-1))=0, t\in[0,b-1]_{\mathbb{N}_0},\\ \Delta u(\nu-2)=\Delta_C^{\nu}u(0)=0, u(\nu+b)=\gamma u(\eta),\end{cases}$$

where $1<\nu\leq 2$ is a real number, $\phi_p(s)=|s|^{p-2}s$, $p>1$, $\gamma\in(0,1)$, $\eta\in(\nu,\nu+b)$, $\Delta_C^{\nu}$ denotes the discrete Caputo fractional difference of order $\nu$, $f:[\nu-1,\nu+b-2]_{\mathbb{N}_{\nu-1}}\times[0,+\infty)\to(0,+\infty)$ is a continuous function, $b\geq 3$ is an integer, $\lambda>0$ is a parameter. We study the existence of positive solutions to this problem by the properties of the Green function and Guo-Krasnosel'skii fixed point theorem in cones.

**Keywords**: boundary value problem; discrete fractional calculus; existence of solutions; $p$-Laplacian operator; Guo-Krasnosel'skii theorem.

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## 1 Introduction

In recent years, fractional differential equations have received increasing attention. With the development of computer, it is well known that discrete analogues of differential equations can be very useful, especially for using computer to simulate the behavior of solutions for certain dynamic equations. More recent works to the discrete fractional calculus can be find in [1–7] and references contained therein. For example, Y. Pan and Z. Han et al. [5] studied the the existence and nonexistence of positive solutions to a boundary value problem for fractional difference equation with a parameter

$$-\Delta^{\nu}y(t)=\lambda f(t+\nu-1,y(t+\nu-1)), t\in[0,b+1]_{\mathbb{N}_0},$$
$$y(\nu-2)=y(\nu+b+1)=0,$$

where $1<\nu\leq 2$ is a real number, $f:[\nu-1,\nu+b]_{\mathbb{N}_{\nu-1}}\times\mathbb{R}\to(0,+\infty)$ is a continuous function, $b\geq 2$ is an integer, $\lambda$ is a parameter. The eigenvalue intervals of boundary value problem to a nonlinear fractional difference equation are considered by the properties of the Green function and Guo-Krasnosel'skii fixed point theorem in cones, some sufficient conditions to the nonexistence of positive solutions for the boundary value problem are established.

Differential equations with $p$-Laplacian operator are applied in real life, especially in physics and engineering [8]. Some theories of fractional difference equations with $p$-Laplacian operator are just beginning to be investigated. W. Lv [9] investigated the following boundary value problem for fractional difference equation involving a $p$-Laplacian operator

$$\Delta[\phi_p(\Delta_C^{\alpha}u)](t)=f(t+\alpha-1,u(t+\alpha-1)), t\in[0,b]_{\mathbb{N}_0},$$
$$u(\alpha-2)=\beta_1 u(\alpha+b+1),$$

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$$\Delta u(\alpha-2)=\Delta u(\alpha-1)=\beta_2\Delta u(\alpha+b),$$

where $1<\alpha\leq 2$, $b\in\mathbb{N}_1$, $\beta_1\neq 1$, $\beta_2\neq 1$, $\Delta$ is the forward difference operator with step size 1, $\Delta_C^{\alpha}$ denotes the discrete Caputo fractional difference of order $\alpha$, $f:[\alpha-1,\alpha+b-1]_{\mathbb{N}_{\alpha-1}}\times\mathbb{R}\to\mathbb{R}$ is a continuous function, and $\phi_p$ is the $p$-Laplacian operator. Some existence and uniqueness results are obtained by using the Banach contraction mapping theorem.

In this paper, we discuss the following boundary value problem for fractional difference equation with $p$-Laplacian operator

$$\Delta[\phi_p(\Delta_C^{\nu}u)](t)+\lambda^{p-1}f(t+\nu-1,u(t+\nu-1))=0, t\in[0,b-1]_{\mathbb{N}_0}, \tag{1.1}$$

$$\Delta u(\nu-2)=\Delta_C^{\nu}u(0)=0, u(\nu+b)=\gamma u(\eta), \tag{1.2}$$

where $1<\nu\leq 2$ is a real number, $\gamma\in(0,1),\eta\in(\nu,\nu+b)$, $\Delta_C^{\nu}$ denotes the discrete Caputo fractional difference of order $\nu$, $f:[\nu-1,\nu+b-2]_{\mathbb{N}_{\nu-1}}\times[0,+\infty)\to(0,+\infty)$ is a continuous function, $b\geq 3$ is an integer, $\lambda>0$ is a parameter. $\phi_p$ is the $p$–Laplacian operator, that is, $\phi_p(s)=|s|^{p-2}s$, $p>1$. Obviously, $\phi_p$ is invertible and its inverse operator is $\phi_q$, where $q>1$ is a constant with $\frac{1}{p}+\frac{1}{q}=1$.

Our work presented in this article has the following features which are worth emphasizing.

($i$) As far as we know, there are not many results available concerning with three–point boundary value problem of fractional difference equation which $\Delta_C^{\nu}$ is the standard Caputo fractional difference.

($ii$) We consider the boundary value problem with $p$-Laplacian which arises in the modeling of different physical and natural phenomena.

($iii$) We investigate the intervals of parameter $\lambda$ for boundary value problem to a nonlinear fractional difference equation with $p$-Laplacian.

The plan of the paper is as follows. In Section 2, we shall present some definitions and lemmas in order to prove our main results, the corresponding Green function and some properties of the Green function. In Section 3, we shall deduce the existence of positive solutions to problem (1.1)–(1.2) by the properties of the Green function and Guo-Krasnosel'skii fixed point theorem in cones. In Section 4, we give some examples to illustrate the theorems.

## 2 Preliminaries

For the convenience of the reader, we give some necessary basic definitions and lemmas that will be important to us in what follows.

**Definition 2.1** ([6]) We define $t^{\underline{\nu}}:=\frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}$ for any $t$ and $\nu$, for which the right-hand side is defined. We also appeal to the convention that if $t+1-\nu$ is a pole of the Gamma function and $t+1$ is not a pole, then $t^{\underline{\nu}}=0$.

**Definition 2.2** ([7]) Assume $f:\mathbb{N}_a\to\mathbb{R}$ and $\nu>0$. Then the $\nu$-th fractional sum of $f$(based at $a$) at the point $t\in\mathbb{N}_{a+\nu}$ is defined by

$$\Delta_a^{-\nu}f(t):=\frac{1}{\Gamma(\nu)}\sum_{s=a}^{t-\nu}(t-s-1)^{\underline{\nu-1}}f(s).$$

Note that by our convention on delta sums we can extend the domain of $\Delta_a^{-\nu}f$ to $\mathbb{N}_{a+\nu-N}$, where $N$ is the unique positive integer satisfying $N-1<\nu\leq N$, by noting that

$$\Delta_a^{-\nu}f(t)=0, t\in\mathbb{N}_{a+\nu-N}^{a+\nu-1}.$$

**Definition 2.3** ([10]) The $\nu$-th Caputo fractional difference of a function $f:\mathbb{N}_a\to\mathbb{R}$, for $\nu>0$, $\nu\notin\mathbb{N}$, is defined by

$$\begin{aligned}\Delta_C^{\nu}f(t)&=\Delta^{-(n-\nu)}\Delta^n f(t)\\&=\tfrac{1}{\Gamma(n-\nu)}\Sigma_{s=a}^{t-n+\nu}(t-s-1)^{\underline{n-\nu-1}}\Delta^n f(s),\end{aligned}$$

for $t\in\mathbb{N}_{a+n-\nu}$, where $n$ is the smallest integer greater than or equal to $\nu$ and $\Delta^n$ is the $n$-th forward difference operator. If $\nu=n$, then $\Delta_C^{\nu}f(t)=\Delta^n f(t)$.

**Lemma 2.1** *([11]) Assume that $\nu > 0$ and $f$ is defined on $\mathbb{N}_a$. Then*

$$\Delta^{-\nu}\Delta_C^{\nu} f(t) = f(t) + C_0 + C_1 t + \cdots + C_{n-1} t^{\underline{n-1}},$$

*for some $C_i \in \mathbb{R}$, $i = 1, 2, \ldots, n-1$, and $n$ is the smallest integer greater than or equal to $\nu$.*

**Lemma 2.2** *([6]) Let $t$ and $\nu$ be any numbers for which $t^{\underline{\nu}}$ and $t^{\underline{\nu-1}}$ are defined. Then*

$$\Delta t^{\underline{\nu}} = \nu t^{\underline{\nu-1}}.$$

**Lemma 2.3** *([6]) For $t$ and $s$, for which both $(t-s-1)^{\underline{\nu}}$ and $(t-s-2)^{\underline{\nu}}$ are defined, we find that*

$$\Delta_s[(t-s-1)^{\underline{\nu}}] = -\nu(t-s-2)^{\underline{\nu-1}}.$$

**Lemma 2.4** *Let $f : [\nu-1, \nu+b-2]_{\mathbb{N}_{\nu-1}} \times [0, +\infty) \to (0, +\infty)$ be given. A function $u$ is a solution of the (1.1)–(1.2), if and only if it has the form*

$$\begin{aligned} u(t) = \quad & \lambda \textstyle\sum_{s=0}^{b} G(t,s)\phi_q\big(\sum_{\tau=0}^{s-1} f(\tau+\nu-1, u(\tau+\nu-1))\big) \\ & + \tfrac{\lambda\gamma}{1-\gamma} \textstyle\sum_{s=0}^{b} G(\eta,s)\phi_q\big(\sum_{\tau=0}^{s-1} f(\tau+\nu-1, u(\tau+\nu-1))\big), t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}, \end{aligned} \tag{2.2}$$

*where $G(t,s)$ is given by*

$$G(t,s) = \frac{1}{\Gamma(\nu)} \begin{cases} (\nu+b-s-1)^{\underline{\nu-1}} - (t-s-1)^{\underline{\nu-1}}, 0 \le s < t-\nu+1 \le b, \\ (\nu+b-s-1)^{\underline{\nu-1}}, 0 \le t-\nu+1 \le s \le b. \end{cases} \tag{2.3}$$

**Proof**. If $u(t)$ is a solution to (1.1)–(1.2). Then from (1.1), together with condition $\Delta_C^{\nu} u(0) = 0$, we find that

$$\begin{aligned} [\phi_p(\Delta_C^{\nu} u)](t) \quad &= \phi_p(\Delta_C^{\nu} u(0)) - \lambda^{p-1} \textstyle\sum_{s=0}^{t-1} f(s+\nu-1, u(s+\nu-1)) \\ &= -\lambda^{p-1} \textstyle\sum_{s=0}^{t-1} f(s+\nu-1, u(s+\nu-1)), t \in [0,b]_{\mathbb{N}_0}, \end{aligned}$$

so

$$\Delta_C^{\nu} u(t) = -\lambda\phi_q\bigg(\sum_{s=0}^{t-1} f(s+\nu-1, u(s+\nu-1))\bigg), t \in [0,b]_{\mathbb{N}_0},$$

in view of Lemma 2.1, we have

$$u(t) = -\frac{\lambda}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t-s-1)^{\underline{\nu-1}} \phi_q\bigg(\sum_{\tau=0}^{s-1} f(\tau+\nu-1, u(\tau+\nu-1))\bigg) + C_0 + C_1 t, t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}. \tag{2.6}$$

Furthermore, (2.3) implies that

$$\Delta u(t) = -\frac{\lambda}{\Gamma(\nu-1)} \sum_{s=0}^{t-(\nu-1)} (t-s-1)^{\underline{\nu-2}} \phi_q\bigg(\sum_{\tau=0}^{s-1} f(\tau+\nu-1, u(\tau+\nu-1))\bigg) + C_1, t \in [\nu-2, \nu+b-1]_{\mathbb{N}_{\nu-2}}.$$

By condition $\Delta u(\nu-2) = 0$, we can get that $C_1 = 0$. Then we obtain

$$u(t) = -\frac{\lambda}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t-s-1)^{\underline{\nu-1}} \phi_q\left(\sum_{\tau=0}^{s-1} f(\tau+\nu-1, u(\tau+\nu-1))\right) + C_0, t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}. \tag{2.8}$$

Now

$$u(\nu+b) = -\frac{\lambda}{\Gamma(\nu)} \sum_{s=0}^{b} (\nu+b-s-1)^{\underline{\nu-1}} \phi_q\left(\sum_{\tau=0}^{s-1} f(\tau+\nu-1, u(\tau+\nu-1))\right) + C_0,$$

$$\gamma u(\eta) = -\frac{\lambda\gamma}{\Gamma(\nu)}\sum_{s=0}^{\eta-\nu}(\eta-s-1)^{\underline{\nu-1}}\phi_q\left(\sum_{\tau=0}^{s-1}f(\tau+\nu-1,u(\tau+\nu-1))\right)+\gamma C_0,$$

by condition $u(\nu+b)=\gamma u(\eta)$, we obtain that

$$\begin{aligned} C_0 = \ & \tfrac{\lambda}{(1-\gamma)\Gamma(\nu)}\textstyle\sum_{s=0}^{b}(\nu+b-s-1)^{\underline{\nu-1}}\phi_q\left(\sum_{\tau=0}^{s-1}f(\tau+\nu-1,u(\tau+\nu-1))\right) \\ & -\tfrac{\lambda\gamma}{(1-\gamma)\Gamma(\nu)}\textstyle\sum_{s=0}^{\eta-\nu}(\eta-s-1)^{\underline{\nu-1}}\phi_q\left(\sum_{\tau=0}^{s-1}f(\tau+\nu-1,u(\tau+\nu-1))\right). \end{aligned}$$

Now, substitution of $C_0$ and $C_1$ into (2.6) gives

$$\begin{aligned} u(t) = \ & -\tfrac{\lambda}{\Gamma(\nu)}\textstyle\sum_{s=0}^{t-\nu}(t-s-1)^{\underline{\nu-1}}\phi_q\left(\sum_{\tau=0}^{s-1}f(\tau+\nu-1,u(\tau+\nu-1))\right) \\ & +\tfrac{\lambda}{(1-\gamma)\Gamma(\nu)}\textstyle\sum_{s=0}^{b}(\nu+b-s-1)^{\underline{\nu-1}}\phi_q\left(\sum_{\tau=0}^{s-1}f(\tau+\nu-1,u(\tau+\nu-1))\right) \\ & -\tfrac{\lambda\gamma}{(1-\gamma)\Gamma(\nu)}\textstyle\sum_{s=0}^{\eta-\nu}(\eta-s-1)^{\underline{\nu-1}}\phi_q\left(\sum_{\tau=0}^{s-1}f(\tau+\nu-1,u(\tau+\nu-1))\right), t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}, \end{aligned}$$

splitting the second sum in two parts on the basis of the following equality

$$\frac{1}{\Gamma(\nu)}+\frac{\gamma}{(1-\gamma)\Gamma(\nu)}=\frac{1}{(1-\gamma)\Gamma(\nu)},$$

therefore,

$$\begin{aligned} u(t) = \ & -\tfrac{\lambda}{\Gamma(\nu)}\textstyle\sum_{s=0}^{t-\nu}(t-s-1)^{\underline{\nu-1}}\phi_q\left(\sum_{\tau=0}^{s-1}f(\tau+\nu-1,u(\tau+\nu-1))\right) \\ & +\tfrac{\lambda}{\Gamma(\nu)}\textstyle\sum_{s=0}^{b}(\nu+b-s-1)^{\underline{\nu-1}}\phi_q\left(\sum_{\tau=0}^{s-1}f(\tau+\nu-1,u(\tau+\nu-1))\right) \\ & +\tfrac{\lambda\gamma}{(1-\gamma)\Gamma(\nu)}\textstyle\sum_{s=0}^{b}(\nu+b-s-1)^{\underline{\nu-1}}\phi_q\left(\sum_{\tau=0}^{s-1}f(\tau+\nu-1,u(\tau+\nu-1))\right) \\ & -\tfrac{\lambda\gamma}{(1-\gamma)\Gamma(\nu)}\textstyle\sum_{s=0}^{\eta-\nu}(\eta-s-1)^{\underline{\nu-1}}\phi_q\left(\sum_{\tau=0}^{s-1}f(\tau+\nu-1,u(\tau+\nu-1))\right), t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}, \end{aligned}$$

which is equivalent to (2.1) that

$$\begin{aligned} u(t) = \ & \lambda\textstyle\sum_{s=0}^{b}G(t,s)\phi_q\big(\sum_{\tau=0}^{s-1}f(\tau+\nu-1,u(\tau+\nu-1))\big) \\ & +\tfrac{\lambda\gamma}{1-\gamma}\textstyle\sum_{s=0}^{b}G(\eta,s)\phi_q\big(\sum_{\tau=0}^{s-1}f(\tau+\nu-1,u(\tau+\nu-1))\big), t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}. \end{aligned}$$

On the other hand, if the function $u(t)$ satisfies to (2.1), then $u(\nu+b)=\gamma u(\eta)$. What's more, function $u(t)$ defined by (2.2) can transform to (2.8) that

$$u(t)=-\frac{\lambda}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-s-1)^{\underline{\nu-1}}\phi_q\left(\sum_{\tau=0}^{s-1}f(\tau+\nu-1,u(\tau+\nu-1))\right)+C_0, t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}.$$

Then we find that

$$\Delta u(t)=-\frac{\lambda}{\Gamma(\nu-1)}\sum_{s=0}^{t-(\nu-1)}(t-s-1)^{\underline{\nu-2}}\phi_q\Big(\sum_{\tau=0}^{s-1}f(\tau+\nu-1,u(\tau+\nu-1))\Big), t\in[\nu-2,\nu+b-1]_{\mathbb{N}_{\nu-2}}, \tag{2.16}$$

and

$$\begin{aligned} \Delta_C^{\nu}u(t) &= -\Delta_C^{\nu}\left\{\Delta^{-\nu}\left[\lambda\phi_q\left(\textstyle\sum_{s=0}^{t-1}f(s+\nu-1,u(s+\nu-1))\right)\right]\right\} \\ &= -\phi_q\left(\textstyle\sum_{s=0}^{t-1}\lambda^{p-1}f(s+\nu-1,u(s+\nu-1))\right), t\in[0,b]_{\mathbb{N}_0}. \end{aligned} \tag{2.17}$$

From (2.16) and (2.17), we see that $\Delta u(\nu-2)=\Delta_C^{\nu}u(0)=0$.
From the above proofs, it is to say function $u(t)$ meets the boundary condition (1.2). Then we will proof that $u(t)$ satisfies the fractional difference equation (1.1). Taking $p$-Laplacian operators on sides of (2.17), we find that

$$[\phi_p(\Delta_C^{\nu}u)](t)=-\sum_{s=0}^{t-1}\lambda^{p-1}f(s+\nu-1,u(s+\nu-1)),t\in[0,b]_{\mathbb{N}_0}. \tag{2.18}$$

By equation (2.18), function $\Delta[\phi_p(\Delta_C^{\nu}u)](t)$ has the form

$$\Delta[\phi_p(\Delta_C^{\nu}u)](t)=-\lambda^{p-1}f(t+\nu-1,u(t+\nu-1)),t\in[0,b-1]_{\mathbb{N}_0},$$

which shows that if (1.1)–(1.2) has a solution, then it can be represented by (2.2) and that every function of the form (2.2) is a solution of (1.1)–(1.2), which completes the proof.

**Lemma 2.5** *([12]) Let $\nu$ be any positive real number and $a,b$ be two real numbers such that $\nu<a\le b$. Then the following are valid.*

*(i) $\frac{1}{x^{\underline{\nu}}}$ is a decreasing function for $x\in(0,+\infty)_{\mathbb{N}}$.*

*(ii) $\frac{(a-x)^{\underline{\nu}}}{(b-x)^{\underline{\nu}}}$ is a decreasing function for $x\in[0,a-\nu)_{\mathbb{N}}$.*

**Lemma 2.6** *The function $G(t,s)$ defined by (2.3) has the following properties:*

*1. $0\le G(t,s)\le G(s+\nu-1,s)$, for $t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}$ and $s\in[0,b]_{\mathbb{N}_0}$;*

*2. there exists a positive number $\kappa\in(0,1)$ such that*

$$\min_{t\in[\frac{(b+\nu)}{4},\frac{3(b+\nu)}{4}]}G(t,s)\ge\kappa\max_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}}G(t,s)=\kappa G(s+\nu-1,s),$$

*for $s\in[0,b]$.*

**Proof**. 1. For $0\le s<t-\nu+1\le b$, we get

$$(\nu+b-s-1)^{\underline{\nu-1}}\ge(t-s-1)^{\underline{\nu-1}},$$

which implies $G(t,s)\ge0$. For $0\le t-\nu+1\le s\le b$, clear $G(t,s)\ge0$.

On the other hand, case (1): $0\le s<t-\nu+1\le b$,

$$\Delta_tG(t,s)=-\frac{(\nu-1)(t-s-1)^{\underline{\nu-2}}}{\Gamma(\nu)}<0;$$

case (2): $0\le t-\nu+1\le s\le b$,

$$\Delta_tG(t,s)=0.$$

Combining the above two cases, we have the function $G(t,s)$ is non-increasing of $t$, thus

$$G(t,s)\le G(s+\nu-1,s),s\in[0,b]_{\mathbb{N}_0}.$$

2. For $s\ge t-\nu+1$ and $\frac{(b+\nu)}{4}\le t\le\frac{3(b+\nu)}{4}$, we have

$$\frac{G(t,s)}{G(s+\nu-1,s)}=1.$$

For $s<t-\nu+1$ and $\frac{(b+\nu)}{4}\le t\le\frac{3(b+\nu)}{4}$, we have that

$$\begin{aligned}\frac{G(t,s)}{G(s+\nu-1,s)}&=\frac{(\nu+b-s-1)^{\underline{\nu-1}}-(t-s-1)^{\underline{\nu-1}}}{(\nu+b-s-1)^{\underline{\nu-1}}}\\&=1-\frac{(t-s-1)^{\underline{\nu-1}}}{(\nu+b-s-1)^{\underline{\nu-1}}}\\&\ge1-\frac{(\frac{3(b+\nu)}{4}-s-1)^{\underline{\nu-1}}}{(\nu+b-s-1)^{\underline{\nu-1}}}.\end{aligned}$$

By Lemma 2.5 $(ii)$,

$$\frac{(\frac{3(b+\nu)}{4}-s-1)^{\underline{\nu-1}}}{(\nu+b-s-1)^{\underline{\nu-1}}}$$

is decreasing for $0 \leq s < \frac{3(b+\nu)}{4} - \nu + 1$. Hence

$$\frac{G(t,s)}{G(s+\nu-1,s)} \geq 1 - \frac{(\frac{3(b+\nu)}{4}-1)^{\underline{\nu-1}}}{(\nu+b-1)^{\underline{\nu-1}}},$$

which implies

$$\min_{t\in[\frac{(b+\nu)}{4},\frac{3(b+\nu)}{4}]} G(t,s) \geq \kappa G(s+\nu-1,s),$$

where $\kappa = 1 - \frac{(\frac{3(b+\nu)}{4}-1)^{\underline{\nu-1}}}{(\nu+b-1)^{\underline{\nu-1}}}$.

**Lemma 2.7** *([12]) Let $B$ be a Banach space and let $P \subseteq B$ be a cone. Assume that $\Omega_1$ and $\Omega_2$ are open subsets contained in $B$ such that $0 \in \Omega_1$ and $\overline{\Omega_1} \subseteq \Omega_2$. Assume, further, that $T : P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P$ is a completely continuous operator. If either*
*$(1)\|Tu\| \leq \|u\|$ for $u \in P \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|$ for $u \in P \cap \partial\Omega_2$; or*
*$(2)\|Tu\| \geq \|u\|$ for $u \in P \cap \partial\Omega_1$ and $\|Tu\| \leq \|u\|$ for $u \in P \cap \partial\Omega_2$.*
*Then $T$ has at least one fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.*

Define the Banach space $B$ by

$$B = \{u : [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}} \to \mathbb{R}\}$$

with norm $\|u\| = \max\{|u(t)|, t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}\}$.

Define the cone

$$P = \left\{ u \in B \mid u(t) \geq 0, t \in [\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}, \min_{t\in[\frac{(b+\nu)}{4},\frac{3(b+\nu)}{4}]_{\mathbb{N}_{\nu-2}}} u(t) \geq \sigma\|u\|, \sigma = \kappa(1-\gamma) \right\}. \tag{2.21}$$

From Lemma 2.4, we know that $u$ is a solution of (1.1)–(1.2) if and only if $u$ is a fixed point of the operator $T : B \to B$ defined by

$$\begin{aligned} Tu(t) = \quad & \lambda \textstyle\sum_{s=0}^{b} G(t,s)\phi_q\big(\sum_{\tau=0}^{s-1} f(\tau+\nu-1, u(\tau+\nu-1))\big) \\ & + \tfrac{\lambda\gamma}{1-\gamma} \textstyle\sum_{s=0}^{b} G(\eta,s)\phi_q\big(\sum_{\tau=0}^{s-1} f(\tau+\nu-1, u(\tau+\nu-1))\big), t \in [\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}. \end{aligned} \tag{2.22}$$

**Lemma 2.8** *Let $T$ be defined as in (2.22) and $P$ as in (2.21). Then $T : P \to P$ is completely continuous.*

**Proof**. Note that $T$ is a summation operator on a discrete finite set, so $T$ is trivially completely continuous. We have that

$$\|Tu\| \leq \frac{\lambda}{1-\gamma} \sum_{s=0}^{b} \frac{(\nu+b-s-1)^{\underline{\nu-1}}}{\Gamma(\nu)} \phi_q\left(\sum_{\tau=0}^{s-1} f(\tau+\nu-1, u(\tau+\nu-1))\right). \tag{2.23}$$

For all $u \in P$, it follows from (2.23) that

$$\begin{aligned}
\min_{t\in[\frac{(b+\nu)}{4},\frac{3(b+\nu)}{4}]}(Tu)(t) &= \min_{t\in[\frac{(b+\nu)}{4},\frac{3(b+\nu)}{4}]}\lambda\sum_{s=0}^{b}G(t,s)\phi_q\left(\sum_{\tau=0}^{s-1}f(\tau+\nu-1,u(\tau+\nu-1))\right)\\
&\quad+\frac{\lambda\gamma}{(1-\gamma)\Gamma(\nu)}\sum_{s=0}^{b}(\nu+b-s-1)^{\underline{\nu-1}}\phi_q\left(\sum_{\tau=0}^{s-1}f(\tau+\nu-1,u(\tau+\nu-1))\right)\\
&\quad-\frac{\lambda\gamma}{(1-\gamma)\Gamma(\nu)}\sum_{s=0}^{\eta-\nu}(\eta-s-1)^{\underline{\nu-1}}\phi_q\left(\sum_{\tau=0}^{s-1}f(\tau+\nu-1,u(\tau+\nu-1))\right)\\
&\geq \kappa\lambda\sum_{s=0}^{b}\max_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}}G(t,s)\phi_q\left(\sum_{\tau=0}^{s-1}f(\tau+\nu-1,u(\tau+\nu-1))\right)\\
&= \kappa(1-\gamma)\frac{\lambda}{1-\gamma}\sum_{s=0}^{b}G(s+\nu-1,s)\phi_q\left(\sum_{\tau=0}^{s-1}f(\tau+\nu-1,u(\tau+\nu-1))\right)\\
&\geq \sigma\|Tu\|.
\end{aligned}$$

It is obvious that $(Tu)(t)\geq 0$ whenever $u\in P$, thus, $T:P\to P$ as desired.

# 3 Existence of positive solutions

In this section, we will show the existence of positive solutions for boundary value problem (1.1)–(1.2). Let us put

$$\varphi(l)=\min\{f(t,u),(t,u)\in[\nu-1,\nu+b-2]_{\mathbb{N}_{\nu-1}}\times[0,l]\},$$

$$\psi(l)=\max\{f(t,u),(t,u)\in[\nu-1,\nu+b-2]_{\mathbb{N}_{\nu-1}}\times[0,l]\}.$$

(H1) The function $f(t,u)$ satisfies $\lim\limits_{u\to 0^+}\frac{\max_{t\in[\nu-1,\nu+b-2]_{\mathbb{N}_{\nu-1}}}f(t,u)}{u^{p-1}}=0$.

(H2) The function $f(t,u)$ satisfies $\lim\limits_{u\to+\infty}\frac{\min_{t\in[\nu-1,\nu+b-2]_{\mathbb{N}_{\nu-1}}}f(t,u)}{u^{p-1}}=+\infty$.

Set $l_0=\lceil\frac{(b+\nu)}{4}-\nu+1\rceil, l_1=\lfloor\frac{3(b+\nu)}{4}-\nu+1\rfloor$, $K=\max G(t,s)$, for $(t,s)\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}\times[0,b]_{\mathbb{N}_0}$. For convenience, we denote

$$A=\sum_{s=0}^{b}\phi_q(s), B=\sum_{s=l_0}^{l_1}\phi_q(s).$$

**Theorem 3.1** *If $f\in C([\nu-1,\nu+b-2]_{\mathbb{N}_{\nu-1}}\times[0,+\infty)\to(0,+\infty))$ and there exist two positive constants $\alpha_2>\alpha_1$ such that*

$$\phi_p(\delta_1\alpha_1)\leq\varphi(\alpha_1),\psi(\alpha_2)\leq\phi_p(\delta_2\alpha_2)$$

*hold, where $\delta_1,\delta_2$ are positive constants satisfying*

$$\frac{\delta_2 A}{1-\gamma}<\kappa\delta_1 B,$$

*then for each*

$$\lambda\in\left(\left(\kappa\delta_1 KB)\right)^{-1},\left(\frac{\delta_2 KA}{1-\gamma}\right)^{-1}\right),$$

*the problem (1.1)–(1.2) has at least one positive solution.*

**Proof**. Let $\Omega_1=\{u\in B:\|u\|\leq\alpha_1\}$. For any $u\in P$ with $\|u\|=\alpha_1$, we have $\phi_p(\delta_1\alpha_1)\leq\varphi(\alpha_1)\leq f(t+\nu-1,u(t+\nu-1))$ for $(t+\nu-1,u(t+\nu-1))\in[0,b-1]_{\mathbb{N}_0}\times[0,\alpha_1]$. We have

$$\begin{aligned}
|Tu(t)| &= \lambda \textstyle\sum_{s=0}^{b} G(t,s)\phi_q\big(\sum_{\tau=0}^{s-1} f(\tau+\nu-1,u(\tau+\nu-1))\big) \\
&\quad + \tfrac{\lambda\gamma}{1-\gamma}\textstyle\sum_{s=0}^{b} G(\eta,s)\phi_q\big(\sum_{\tau=0}^{s-1} f(\tau+\nu-1,u(\tau+\nu-1))\big) \\
&\geq \lambda \textstyle\sum_{s=l_0}^{l_1} G(t,s)\phi_q\big(\sum_{\tau=0}^{s-1} f(\tau+\nu-1,u(\tau+\nu-1))\big) \\
&\geq \kappa\lambda K \textstyle\sum_{s=l_0}^{l_1} \phi_q\big(\sum_{\tau=0}^{s-1} \varphi(\alpha_1)\big) \\
&\geq \kappa\lambda K \textstyle\sum_{s=l_0}^{l_1} \phi_q\big(\sum_{\tau=0}^{s-1} \phi_p(\delta_1\alpha_1)\big) \\
&= \alpha_1\delta_1\lambda\kappa K B \\
&\geq \alpha_1.
\end{aligned}$$

So,

$$\|Tu\| \geq \|u\|, u \in P \cap \partial\Omega_1. \tag{3.2}$$

On the other hand, let $\Omega_2 = \{u \in B : \|u\| \leq \alpha_2\}$. For any $u \in P$ with $\|u\| = \alpha_2$, we have $f(t+\nu-1,u(t+\nu-1)) \leq \psi(\alpha_2) \leq \phi_p(\delta_2\alpha_2)$ for $(t+\nu-1,u(t+\nu-1)) \in [0,b-1]_{\mathbb{N}_0} \times [0,\alpha_2]$. We have

$$\begin{aligned}
|Tu(t)| &= \lambda \textstyle\sum_{s=0}^{b} G(t,s)\phi_q\big(\sum_{\tau=0}^{s-1} f(\tau+\nu-1,u(\tau+\nu-1))\big) \\
&\quad + \tfrac{\lambda\gamma}{1-\gamma}\textstyle\sum_{s=0}^{b} G(\eta,s)\phi_q\big(\sum_{\tau=0}^{s-1} f(\tau+\nu-1,u(\tau+\nu-1))\big) \\
&\leq K\lambda \textstyle\sum_{s=0}^{b} \phi_q\left(\sum_{\tau=0}^{s-1} \psi(\alpha_2)\right) + \tfrac{\lambda\gamma}{1-\gamma} K \textstyle\sum_{s=0}^{b} \phi_q\left(\sum_{\tau=0}^{s-1} \psi(\alpha_2)\right) \\
&\leq K\tfrac{\lambda}{1-\gamma} \textstyle\sum_{s=0}^{b} \phi_q\left(\sum_{\tau=0}^{s-1} \phi_p(\delta_2\alpha_2)\right) \\
&= \delta_2\alpha_2\lambda\tfrac{K}{1-\gamma}A \\
&\leq \alpha_2.
\end{aligned}$$

Hence,

$$\|Tu\| \leq \|u\|, u \in P \cap \partial\Omega_2. \tag{3.4}$$

Consequently, from (3.2) and (3.4), we may invoke Lemma 2.6 to deduce that $T$ has a fixed point in the set $P \cap (\overline{\Omega_2} \setminus \Omega_1)$. Then the theorem is proved.

**Theorem 3.2** *Suppose that conditions (H1)–(H2) hold. If there exist a sufficient small positive constant $\delta_3$ and sufficient large $\delta_4$ such that*

$$\delta_3 \frac{A}{1-\gamma} < \kappa\delta_4 B$$

*holds, then for each*

$$\lambda \in \left(\left(\delta_4\kappa K B\right)^{-1}, \left(\frac{\delta_3 K A}{1-\gamma}\right)^{-1}\right),$$

*then problem (1.1)–(1.2) has at least one positive solution.*

**Proof**. Because of condition (H1), there exist $\beta_1 > 0$ and a sufficient small constant $\delta_3 > 0$ such that

$$f(t,u) \leq (\delta_3 u)^{p-1}, 0 < u \leq \beta_1. \tag{3.5}$$

So for $u \in P$ with $\|u\| = \beta_1$, by (2.22) and (3.5), we have for all $t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}$,

$$\begin{aligned} \|Tu\| &= \max_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}} \lambda \textstyle\sum_{s=0}^{b} G(t,s)\phi_q\big(\sum_{\tau=0}^{s-1} f(\tau+\nu-1, u(\tau+\nu-1))\big) \\ &\quad + \tfrac{\lambda\gamma}{1-\gamma} \textstyle\sum_{s=0}^{b} G(\eta,s)\phi_q\big(\sum_{\tau=0}^{s-1} f(\tau+\nu-1, u(\tau+\nu-1))\big) \\ &\le \lambda\|u\|\delta_3 KA + \|u\|\tfrac{\lambda\gamma}{1-\gamma}\delta_3 KA \\ &= \lambda\|u\|\delta_3 K\tfrac{1}{1-\gamma}A \\ &\le \|u\|. \end{aligned} \tag{3.6}$$

Thus, if we choose $\Omega_1 = \{u \in B : \|u\| \le \beta_1\}$, then (3.6) implies that

$$\|Tu\| \le \|u\|, u \in P \cap \partial\Omega_1. \tag{3.7}$$

On the other hand, condition (H2) implies that there exist a number $0 < \beta_1 < \beta_2$ and a sufficient large constant $\delta_4$ such that

$$f(t,u) \ge (\delta_4 u)^{p-1}, u \ge \beta_2. \tag{3.8}$$

And then we set $\beta_2^* = \frac{\beta_2}{\sigma} > \beta_2$. Then, $u \in P$ and $\|u\| = \beta_2^*$ implies $\min_{t\in[\frac{(b+\nu)}{4}, \frac{3(b+\nu)}{4}]} u(t) \ge \sigma\|u\| = \beta_2$, thus $u(t) \ge \beta_2$, for all $t \in [\frac{(b+\nu)}{4}, \frac{3(b+\nu)}{4}]$. Therefore, for all $t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}$, by (2.11) and (3.8), we have that

$$\begin{aligned} Tu(t) &= \lambda \textstyle\sum_{s=0}^{b} G(t,s)\phi_q\big(\sum_{\tau=0}^{s-1} f(\tau+\nu-1, u(\tau+\nu-1))\big) \\ &\quad + \tfrac{\lambda\gamma}{1-\gamma} \textstyle\sum_{s=0}^{b} G(\eta,s)\phi_q\big(\sum_{\tau=0}^{s-1} f(\tau+\nu-1, u(\tau+\nu-1))\big) \\ &\ge \lambda \textstyle\sum_{s=l_0}^{l_1} G(t,s)\phi_q\big(\sum_{\tau=0}^{s-1} f(\tau+\nu-1, u(\tau+\nu-1))\big) \\ &\ge \lambda\kappa \textstyle\sum_{s=l_0}^{l_1} K\phi_q\big(\sum_{\tau=0}^{s-1} f(\tau+\nu-1, u(\tau+\nu-1))\big) \\ &\ge \lambda\|u\|\delta_4\kappa KB \\ &\ge \|u\|. \end{aligned} \tag{3.9}$$

Hence, if we choose $\Omega_2 = \{u \in B : \|u\| \le \beta_2^*\}$, from (3.9) we have that

$$\|Tu\| \ge \|u\|, u \in P \cap \partial\Omega_2. \tag{3.10}$$

Consequently, from (3.7) and (3.10), we may invoke Lemma 2.6 to deduce that $T$ has a fixed point in the set $P \cap (\overline{\Omega_2} \setminus \Omega_1)$. Then the theorem is proved.

# 4 Examples

In this section, we will present some examples to illustrate main results.

**Example 4.1** Suppose that $\nu = \frac{3}{2}, b = 9, p = \frac{3}{2}$. Take $\gamma = 0.1, \alpha_1 = 2$ and $\alpha_2 = 80000$. Then $f(t,u) = t^2 + \sin u + 6$ and problem (1.1)–(1.2) becomes

$$\Delta[\phi_p(\Delta_C^{\frac{3}{2}}u)](t) + \lambda^{p-1}f(t,u) = 0, t \in [0,8]_{\mathbb{N}_0}, \tag{4.1}$$

$$\Delta u(\nu-2) = \Delta_C^{\frac{3}{2}}u(0) = 0, u(\nu+b) = 0.1u(\eta). \tag{4.2}$$

Make $\delta_1 = 15$ and $\delta_2 = \frac{1}{10}$. By calculation, we have $K = \max_{(t,s)\in[-\frac{1}{2},\frac{21}{2}]_{\mathbb{N}_0}\times[0,9]_{\mathbb{N}_0}} G(t,s) \approx 3.524$, $\sqrt{30} = \phi_p(\delta_2\alpha_1) < \varphi(\alpha_1) = 6$ and $88 = \psi(\alpha_2) < \phi_p(\delta_1\alpha_2) \approx 89$. Then $\delta_1 K\frac{1}{1-\gamma}\sum_{s=0}^{b}\phi_q(s) \approx$

$3.524 \times \frac{1}{9} \times 285 \approx 111.6$ and $\delta_2 \kappa K \sum_{s=\lceil \frac{(b+\nu)}{4} - \nu + 1 \rceil}^{\lfloor \frac{3(b+\nu)}{4} - \nu + 1 \rfloor} \phi_q(s) \approx 15 \times 0.0871 \times 3.524 \times 135 \approx 690$. So, the conditions of Theorem 3.1 are satisfied. Then the boundary value problem (4.1)–(4.2) has at least one positive solution for each $\lambda \in (0.001, 0.009)$.

**Example 4.2** Suppose that $\nu = \frac{3}{2}, b = 9, p = \frac{3}{2}$. Take $\gamma = 0.1, \delta_3 = \frac{1}{1000}$ and $\delta_4 = 2$. Then $f(t,u) = (t+1)u^{\frac{3}{2}}$, and problem (1.1)–(1.2) becomes

$$\Delta[\phi_p(\Delta_C^{\frac{3}{2}} u)](t) + \lambda^{p-1} f(t,u) = 0, t \in [0,8]_{\mathbb{N}_0}, \tag{4.3}$$

$$\Delta u(\nu - 2) = \Delta_C^{\frac{3}{2}} u(0) = 0, u(\nu + b) = 0.1u(\eta). \tag{4.4}$$

In addition, we have $K = \max_{(t,s) \in [-\frac{1}{2}, \frac{21}{2}]_{\mathbb{N}_0} \times [0,9]_{\mathbb{N}_0}} G(t,s) \approx 3.524$, $\lim\limits_{u \to 0^+} \frac{\max_{t \in [\nu-1, \nu+b-2]_{\mathbb{N}_{\nu-1}}} f(t,u)}{u^{p-1}} = 0$ and $\lim\limits_{u \to +\infty} \frac{\min_{t \in [\nu-1, \nu+b-2]_{\mathbb{N}_{\nu-1}}} f(t,u)}{u^{p-1}} = +\infty$. Then $\delta_3 K \frac{1}{1-\gamma} \sum_{s=0}^{b} \phi_q(s) \approx \frac{1}{1000} \times 3.524 \times \frac{10}{9} \times 285 \approx 1.116$ and $\delta_4 \kappa K \sum_{s=\lceil \frac{(b+\nu)}{4} - \nu + 1 \rceil}^{\lfloor \frac{3(b+\nu)}{4} - \nu + 1 \rfloor} \phi_q(s) \approx 2 \times 0.0871 \times 3.524 \times 135 \times \approx 90.90$. So, the conditions of Theorem 3.2 are satisfied. Then the boundary value problem (4.3)–(4.4) has at least one positive solution for each $\lambda \in (0.012, 0.896)$.

# References


[1] G. Wu, D. Baleanu, W. Luo, Mittag-Leffler function for discrete fractional modelling, Journal of King Saud University–Science, 28 (2016) 99–102.

[2] I. Area, J. Losada, J. Nieto, On quasi–periodic properties of fractional sums and fractional differences of periodic functions, Applied Mathematics and Computation, 273 (2016) 190–200.

[3] V. Tarasov, Discrete model of dislocations in fractional nonlocal elasticity, Journal of King Saud University–Science, 28 (2016) 33–36.

[4] Y. Chen, X. Tang, The difference between a class of discrete fractional and integer order boundary value problems, Communications in Nonlinear Science and Numerical Simulation, 19 (2014) 4025–4067.

[5] Y. Pan, Z. Han, D. Yang, The existence and nonexistence of positive solutions to a discrete fractional boundary value problem with a parameter, Applied Mathematics Letters, 36 (2014) 1–6.

[6] C. Goodrich, Existence and uniqueness of solutions to a fractional difference equation with nonlocal conditions, Computers and Mathematics with Applications, 61 (2011) 191–201.

[7] C. Goodrich, A. C. Peterson, Discrete fractional calculus, Springer International Publishing, Switzerland, 2015.

[8] H. Zhang, Q. Peng, Y. Wu, Wavelet inpainting based on $p$-Laplace operator, Acta Automatica Sinica, 33 (2003) 546–549.

[9] W. Lv, Solvability for discrete fractional boundary value problems with a $p$-Laplacian operator, Discrete Dynamics in Nature and Society, 2013 (2013) 1–8.

[10] T. Abdeljawad, On Riemann and Caputo fractional differences, Computers and Mathematics with Applications, 62 (2011) 1602–1611.

[11] W. Lv, Existence of solutions for discrete fractional boundary value problems with a $p$-Laplacian operator, Advances in Difference Equations, 2012 (2012) 1–10.

[12] F. Atici, P. Eloe, Two-point boundary value problems for finite fractional difference equations, Journal of Difference Equations and Applications, 17 (2011) 445–456.

# Weighted Composition Operators from analytic Morrey spaces into Zygmund spaces*

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**Abstract**

In this paper we characterize the boundedness and compactness of the weighted composition operator from the analytic Morrey spaces $\mathcal{L}^{2,\lambda}$ to the Zygmund space $\mathcal{Z}$, and the little analytic Morrey spaces $\mathcal{L}_0^{2,\lambda}$ to the little Zygmund space $\mathcal{Z}_0$, respectively.

**Keywords** Analytic Morrey space, Zygmund space; Weighted composition operator; Boundedness; Compactness

**2010 MR Subject Classification** 47B38, 30D99, 30H05

## 1 Introduction

Let $D = \{z : |z| < 1\}$ be the open unit disk in the complex plane and $H(D)$ denote the set of all analytic functions on $D$. Let $u, \varphi \in H(D)$, where $\varphi$ is an analytic self-map of $D$. Then the well-known *weighted composition operator* $uC_\varphi$ on $H(D)$ is defined by $uC_\varphi(f)(z) = u(z)\cdot(f\circ\varphi(z))$ for $f \in H(D)$ and $z \in D$. Weighted composition operators can be regarded as a generalization of multiplication operators $M_u$ and composition operators $C_\varphi$. In 2001, Ohno and Zhao studied the weighted composition operators on the classical Bloch space $\beta$ in [18], which has led many researchers to study this operator on other Banach spaces of analytic functions. The boundedness and compactness of it have been studied on various Banach spaces of analytic functions, such as Hardy, Bergman, BMOA, Bloch-type spaces, see, e.g. [4, 6, 11, 29].

For an arc $I \subset \partial\mathbb{D}$, let $|I| = \frac{1}{2\pi}\int_I |d\zeta|$ be the normalized arc length of $I$,

$$f_I = \frac{1}{|I|}\int_I f(\zeta)\frac{|d\zeta|}{2\pi}, f \in H(D),$$

and $S(I)$ be the Carleson box based on $I$ with

$$S(I) = \{z \in D : 1 - |I| \le |z| < 1, \frac{z}{|z|} \in I\}.$$

Clearly, if $I = \partial D$, then $S(I) = D$.

Let $\mathcal{L}^{2,\lambda}(D)$ represent the analytic *Morrey* spaces of all analytic functions $f \in H^2$ on $D$ such that

$$\sup_{I\subset\partial D}\big(\frac{1}{|I|^\lambda}\int_I |f(\zeta) - f_I|^2\frac{|d\zeta|}{2\pi}\big)^{1/2} < \infty,$$

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where $0 < \lambda \leq 1$ and the Hardy space $H^2$ consists of analytic functions $f$ in $D$ satisfying

$$\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta < \infty.$$

From Theorem 3.1 of [25] or Theorem 3.21 of [26], we can define the norm of function $f \in \mathcal{L}^{2,\lambda}(D)$ and its equivalent formula as follows

$$\begin{aligned} \|f\|_{\mathcal{L}^{2,\lambda}} &= |f(0)| + \sup_{I \subset \partial D} \big(\frac{1}{|I|^\lambda} \int_{S(I)} |f'(z)|^2 (1-|z|^2) dm(z)\big)^{1/2} \\ &\approx |f(0)| + \sup_{a \in D} \big((1-|a|^2)^{1-\lambda} \int_D |f'(z)|^2 (1-|\varphi_a(z)|^2) dm(z)\big)^{1/2}. \end{aligned}$$

Similarly to the relation between $BMOA$ space and $VMOA$ space, we have that $f \in \mathcal{L}_0^{2,\lambda}(D)$, the little analytic $Morrey$ spaces, if $f \in \mathcal{L}^{2,\lambda}(D)$ and

$$\lim_{|I| \to 0} \big(\frac{1}{|I|^\lambda} \int_I |f(\zeta) - f_I|^2 \frac{|d\zeta|}{2\pi}\big)^{1/2} = 0.$$

Clearly, $\mathcal{L}_0^{2,1}(D) = VMOA$. The following lemma gives equivalent conditions of $\mathcal{L}_0^{2,\lambda}$. The proof is similar to that of Theorem 6.3 in [10], we omit the details.

**Lemma 1.1** *Suppose that $0 < \lambda < 1$ and $f \in H(D)$. Let $a \in D$, $\varphi_a(z) = \dfrac{a-z}{1-\overline{a}z}$. Then the following statements are equivalent.*

*(i)* $f \in \mathcal{L}_0^{2,\lambda}(D)$;

*(ii)* $\lim\limits_{|a| \to 1} (1-|a|^2)^{1-\lambda} \displaystyle\int_D |f'(z)|^2 (1-|\varphi_a(z)|^2) dm(z) = 0$;

*(iii)* $\lim\limits_{|a| \to 1} (1-|a|^2)^{1-\lambda} \displaystyle\int_D |f'(z)|^2 \log \frac{1}{|\varphi_a(z)|} dm(z) = 0.$

It is known that $\mathcal{L}^{2,1}(D) = BMOA$ and if $0 < \lambda < 1$, $BMOA \subsetneq \mathcal{L}^{2,\lambda}(D)$. For more information on $BMOA$ and $VMOA$, see [10].

The Zygmund space $\mathcal{Z}$ consists of all analytic functions $f$ defined on $D$ such that

$$z(f) = \sup\{(1-|z|^2)|f''(z)| : z \in D\} < +\infty.$$

From a theorem of Zygmund (see [37, vol. I, p. 263] or [8, Theorem 5.3]), we see that $f \in \mathcal{Z}$ if and only if $f$ is continuous in the close unit disk $\overline{D} = \{z : |z| \leq 1\}$ and the boundary function $f(e^{i\theta})$ such that

$$\sup_{h>0,\theta} \frac{|f(e^{i(\theta+h)}) + f(e^{i(\theta-h)}) - 2f(e^{i\theta})|}{h} < \infty.$$

An analytic function $f \in H(D)$ is said to belong to the little Zymund space $\mathcal{Z}_0$ consists of all $f \in \mathcal{Z}$ satisfying $\lim_{|z| \to 1} (1-|z|^2)|f''(z)| = 0$. It can easily proved that $\mathcal{Z}$ is a Banach space under the norm

$$\|f\|_{\mathcal{Z}} = |f(0)| + |f'(0)| + z(f)$$

and the polynomials are norm-dense in closed subspace $\mathcal{Z}_0$ of $\mathcal{Z}$. For some other information on this space and some operators on it, see, for example, [12, 13, 15].

$Morrey$ space was initially introduced in 1938 by Morrey [17] to show that certain systems of partial differential equations (PDEs) had Hölder continuous solutions. In the past, $Morrey$ space has been studied heavily in different areas. For example, Adams and Xiao studied $Morrey$

spaces which is defined on Euclidean spaces $\mathbb{R}^n$ by potential theory and Hausdorff capacity in [1, 2]. Wang, Xiao [23] studied holomorphic Campanato spaces on the open unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$. Wang and Xiao [24] characterized the first and second preduals of the analytic $Morrey$ spaces $\mathcal{L}^{2,\lambda}$ on the unit disk. Xiao and Yuan [28] studied the analytic Campanato spaces (including the analytic $Morrey$ spaces) in terms of the Möbius mappings and the Littlewood-Paley forms. Xiao and Xu [27] studied the composition operators of $\mathcal{L}^{2,\lambda}$ spaces. Li, Liu and Lou[14] studied the Volterra-type operators on $\mathcal{L}^{2,\lambda}$ spaces. Zhuo and Ye [36] considered this operators from $\mathcal{L}^{2,\lambda}$ spaces to the classical Bloch space.

In 2006, the boundedness of composition operators on the Zygmund space $\mathcal{Z}$ was first studied by Choe, Koo, and Smith in [3]. Later, many researchers have studied composition operators and weighted composition operators acting on the Zygmund space $\mathcal{Z}$. Li and Stević in [12] studied the boundedness and compactness of the generalized composition operators on Zygmund spaces and Bloch type spaces. Ye and Hu in [34] characterized boundedness and compactness of weighted composition operators on the Zygmund space $\mathcal{Z}$. Esmaeili and Lindström in [9] studied weighted composition operators from Zygmund type spaces to Bloch type spaces and their essential norms. Sanatpour and Hassanlou in [20] gave the essential norms of this operators between Zygmund-type spaces and Bloch-type spaces. See also [7, 19, 22, 29, 30, 31, 32, 33, 35] for corresponding results for weighted composition operators from one Banach space of analytic functions to another. In this paper we consider the weighted composition operators from the analytic Morrey spaces $\mathcal{L}^{2,\lambda}$ to the Zygmund space $\mathcal{Z}$, and the little analytic Morrey spaces $\mathcal{L}_0^{2,\lambda}$ to the little Zygmund space $\mathcal{Z}_0$, respectively.

*Notations*: For two functions $F$ and $G$, if there is a constant $C > 0$ dependent only on indexes $p, \lambda...$ such that $F \leq CG$, then we say that $F \lesssim G$. Furthermore, denote that $F \approx G$ ($F$ is comparable with $G$) whenever $F \lesssim G \lesssim F$.

## 2 Auxiliary results

In order to prove the main results of this paper. we need some auxiliary results.

**Lemma 2.1** *Let* $0 < \lambda < 1$. *If* $f \in \mathcal{L}^{2,\lambda}$, *then*

*(i)* $|f(z)| \lesssim \dfrac{\|f\|_{\mathcal{L}^{2,\lambda}}}{(1-|z|^2)^{\frac{1-\lambda}{2}}}$ *for every* $z \in D$;
*(ii)* $|f'(z)| \lesssim \dfrac{\|f\|_{\mathcal{L}^{2,\lambda}}}{(1-|z|^2)^{\frac{3-\lambda}{2}}}$ *for every* $z \in D$;
*(iii)* $|f''(z)| \lesssim \dfrac{\|f\|_{\mathcal{L}^{2,\lambda}}}{(1-|z|^2)^{\frac{5-\lambda}{2}}}$ *for every* $z \in D$.

**Proof** (i) and (ii) are from Lemma 2.5 in [14]. For any $f \in \mathcal{L}^{2,\lambda}$. Fix $z \in D$ and let $\rho = \dfrac{1+|z|}{2}$, by the Cauchy integral formula, we obtain that

$$|f''(z)| = |\frac{1}{2\pi i}\int_{|\xi|=\rho} \frac{f'(\xi)}{(\xi-z)^2}\,d\xi| \leq \frac{\|f\|_{\mathcal{L}^{2,\lambda}}}{(1-\rho^2)^{\frac{3-\lambda}{2}}}\frac{1}{2\pi}\int_0^{2\pi}\frac{\rho\,d\theta}{|\rho e^{i\theta}-z|^2} = \frac{\|f\|_{\mathcal{L}^{2,\lambda}}}{(1-\rho^2)^{\frac{3-\lambda}{2}}}\frac{\rho}{\rho^2-|z|^2} \lesssim \frac{\|f\|_{\mathcal{L}^{2,\lambda}}}{(1-|z|^2)^{\frac{5-\lambda}{2}}}.$$

Hence (iii) holds.

**Lemma 2.2** *Let $0 < \lambda < 1$. If $f \in \mathcal{L}_0^{2,\lambda}$, then*

*(i)* $\lim\limits_{|z|\to 1}(1-|z|^2)^{\frac{3-\lambda}{2}}|f'(z)| = 0$;
*(ii)* $\lim\limits_{|z|\to 1}(1-|z|^2)^{\frac{1-\lambda}{2}}|f(z)| = 0$;
*(iii)* $\lim\limits_{|z|\to 1}(1-|z|^2)^{\frac{5-\lambda}{2}}|f''(z)| = 0$.

The proof of (i) is similar to that of Lemma 2.5 in [14], and we easily obtain (ii) and (iii) by (i). These details are omitted here.

**Lemma 2.3** *Suppose $uC_\varphi : \mathcal{L}_0^{2,\lambda} \to \mathcal{Z}_0$ is a bounded operator, then $uC_\varphi : \mathcal{L}^{2,\lambda} \to \mathcal{Z}$ is a bounded operator.*

The proof is similar to that of Lemma 2.3 in [33]. The details are omitted.

# 3 Boundedness of $uC_\varphi$

In this section we characterize the boundedness of the weighted composition operator $uC_\varphi$ from the analytic Morrey spaces $\mathcal{L}^{2,\lambda}$ to the Zygmund space $\mathcal{Z}$, and the little analytic Morrey spaces $\mathcal{L}_0^{2,\lambda}$ to the little Zygmund space $\mathcal{Z}_0$, respectively.

**Theorem 3.1** *Let $u$ be an analytic function on the unit disc $D$, and $\varphi$ an analytic self-map of $D$. Then $uC_\varphi$ is a bounded operator from the analytic Morrey spaces $\mathcal{L}^{2,\lambda}$ to the Zygmund space $\mathcal{Z}$ if and only if the following are satisfied:*

$$\sup_{z\in D}\frac{(1-|z|^2)|u''(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}} < \infty; \tag{3.1}$$

$$\sup_{z\in D}\frac{(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}} < \infty; \tag{3.2}$$

$$\sup_{z\in D}\frac{(1-|z|^2)|u(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{5-\lambda}{2}}} < \infty. \tag{3.3}$$

**Proof** Suppose $uC_\varphi$ is bounded from the analytic Morrey spaces $\mathcal{L}^{2,\lambda}$ to the Zygmund space $\mathcal{Z}$. Using functions $f(z) = 1$, $f(z) = z$ and and $f(z) = z^2$ in $\mathcal{L}^{2,\lambda}$, we have

$$u \in \mathcal{Z}, \tag{3.4}$$

$$\sup_{z\in D}(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)+\varphi(z)u''(z)| < +\infty, \tag{3.5}$$

and

$$\sup_{z\in D}(1-|z|^2)|4\varphi(z)\varphi'(z)u'(z)+\varphi^2(z)u''(z)+2u(z)(\varphi(z)\varphi''(z)+(\varphi'(z))^2)| < \infty.$$

Since $\varphi(z)$ is a self-map, we get

$$K_1 = \sup_{z\in D}(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)| < +\infty \tag{3.6}$$

and

$$K_2 = \sup_{z\in D}(1-|z|^2)|(\varphi'(z))^2u(z)| < +\infty. \tag{3.7}$$

Fix $a \in D$ with $|a| > \frac{1}{2}$, we take the test functions:

$$f_a(z) = \frac{1-|a|^2}{(1-\bar{a}z)^{\frac{3-\lambda}{2}}} - 2\frac{(1-|a|^2)^2}{(1-\bar{a}z)^{\frac{5-\lambda}{2}}} + \frac{(1-|a|^2)^3}{(1-\bar{a}z)^{\frac{7-\lambda}{2}}} \tag{3.8}$$

for $z \in D$. Then, arguing as the proof of Lemma 3.2 in [14] we obtain that $f_a \in \mathcal{L}^{2,\lambda}$ and $\sup_a \|f_a\|_{\mathcal{L}^{2,\lambda}} \lesssim 1$. Since $f_a(a) = 0,\ f_a'(a) = 0,\ f_a''(a) = \dfrac{2\bar{a}}{(1-|a|^2)^{\frac{5-\lambda}{2}}}$, it follows that for all $\lambda \in D$ with $|\varphi(\lambda)| > \frac{1}{2}$, we have

$$\begin{aligned}
\|f_a\|_{\mathcal{L}^{2,\lambda}} &\gtrsim \|uC_\varphi f_a\|_{\mathcal{Z}} \geq \sup_{z\in D}(1-|z|^2)|(uC_\varphi f_a)''(z)| \\
&= \sup_{z\in D}(1-|z|^2)|\big(2\varphi'(z)u'(z)+\varphi''(z)u(z)\big)f_a'(\varphi(z)) \\
&+ f_a''(\varphi(z))(\varphi'(z))^2u(z) + u''(z)f_a(\varphi(z))|.
\end{aligned}$$

Let $a = \varphi(\lambda)$, it follows that

$$\begin{aligned}
\|f_a\|_{\mathcal{L}^{2,\lambda}} &\gtrsim (1-|\lambda|^2)|\big(2\varphi'(\lambda)u'(\lambda)+\varphi''(\lambda)u(\lambda)\big)f'_{\varphi(\lambda)}(\varphi(\lambda)) \\
&+ f''_{\varphi(\lambda)}(\varphi(\lambda))(\varphi'(\lambda))^2u(\lambda) + u''(\lambda)f_{\varphi(\lambda)}(\varphi(\lambda))| \\
&= (1-|\lambda|^2)|(\varphi'(\lambda))^2u(\lambda)\frac{2\overline{\varphi(\lambda)}}{(1-|\varphi(\lambda)|^2)^{\frac{5-\lambda}{2}}}| \\
&\geq \frac{(1-|\lambda|^2)|\varphi'(\lambda))^2u(\lambda)|}{(1-|\varphi(\lambda)|^2)^{\frac{5-\lambda}{2}}}.
\end{aligned}$$

For $\forall\lambda \in D$ with $|\varphi(\lambda)| \leq \frac{1}{2}$, by (3.7), we have

$$\sup_{\lambda\in D}\frac{(1-|\lambda|^2)|\varphi'(\lambda))^2u(\lambda)|}{(1-|\varphi(\lambda)|^2)^{\frac{5-\lambda}{2}}} \leq (\frac{4}{3})^{\frac{5-\lambda}{2}}\sup_{\lambda\in D}(1-|\lambda|^2)|\varphi'(\lambda))^2u(\lambda)| < +\infty.$$

Hence (3.3) holds.

Next, we will show that (3.2) holds. Fix $a \in D$ with $|a| > \frac{1}{2}$, we take another test functions:

$$g_a(z) = \frac{1-|a|^2}{(1-\bar{a}z)^{\frac{3-\lambda}{2}}} - \frac{12-2\lambda}{7-\lambda}\frac{(1-|a|^2)^2}{(1-\bar{a}z)^{\frac{5-\lambda}{2}}} + \frac{5-\lambda}{7-\lambda}\frac{(1-|a|^2)^3}{(1-\bar{a}z)^{\frac{7-\lambda}{2}}} \tag{3.9}$$

for $z \in D$. Then $g_a \in \mathcal{L}^{2,\lambda}$ and $\sup_a \|g_a\|_{\mathcal{L}^{2,\lambda}} \lesssim 1$(see [14]). Since $g_a(a) = 0,\ \ g_a''(a) = 0,\ \ g_a'(a) = \dfrac{-2\bar{a}}{(7-\lambda)(1-|a|^2)^{\frac{3-\lambda}{2}}}$, it follows that for all $\lambda \in D$ with $|\varphi(\lambda)| > \frac{1}{2}$, we have

$$\begin{aligned}
\|g_a\|_{\mathcal{L}^{2,\lambda}} &\gtrsim \|uC_\varphi g_a\|_{\mathcal{Z}} \geq \sup_{z\in D}(1-|z|^2)|(uC_\varphi g_a)''(z)| \\
&= \sup_{z\in D}(1-|z|^2)|\big(2\varphi'(z)u'(z)+\varphi''(z)u(z)\big)g_a'(\varphi(z)) \\
&+ g_a''(\varphi(z))(\varphi'(z))^2u(z) + u''(z)g_a(\varphi(z))|.
\end{aligned}$$

Let $a=\varphi(\lambda)$, it follows that

$$\begin{aligned}\|g_a\|_{\mathcal{L}^{2,\lambda}} &\gtrsim (1-|\lambda|^2)|\big(2\varphi'(\lambda)u'(\lambda)+\varphi''(\lambda)u(\lambda)\big)g'_{\varphi(\lambda)}(\varphi(\lambda))\\ &+ g''_{\varphi(\lambda)}(\varphi(\lambda))(\varphi'(\lambda))^2u(\lambda)+u''(\lambda)g_{\varphi(\lambda)}(\varphi(\lambda))|\\ &= (1-|\lambda|^2)|\big(2\varphi'(\lambda)u'(\lambda)+\varphi''(\lambda)u(\lambda)\big)\frac{-2\overline{\varphi(\lambda)}}{(7-\lambda)(1-|\varphi(\lambda)|^2)^{\frac{3-\lambda}{2}}}|\\ &\geq \frac{1}{7-\lambda}\frac{(1-|\lambda|^2)|2\varphi'(\lambda)u'(\lambda)+\varphi''(\lambda)u(\lambda)|}{(1-|\varphi(\lambda)|^2)^{\frac{3-\lambda}{2}}}.\end{aligned}$$

For $\forall\lambda\in D$ with $|\varphi(\lambda)|\leq\frac{1}{2}$, by (3.6), we have

$$\sup_{\lambda\in D}\frac{(1-|\lambda|^2)|2\varphi'(\lambda)u'(\lambda)+\varphi''(\lambda)u(\lambda)|}{(1-|\varphi(\lambda)|^2)^{\frac{3-\lambda}{2}}}\leq(\frac{4}{3})^{\frac{3-\lambda}{2}}\sup_{\lambda\in D}(1-|\lambda|^2)|2\varphi'(\lambda)u'(\lambda)+\varphi''(\lambda)u(\lambda)|<+\infty.$$

Hence (3.2) holds.

Finally we will show (3.1) holds. Let

$$h_a(z)=\frac{1-|a|^2}{(1-\bar{a}z)^{\frac{3-\lambda}{2}}} \tag{3.10}$$

for $z\in D$. It is easily proved that $\sup_{\frac{1}{2}<|a|<1}\|h_a\|_{\mathcal{L}^{2,\lambda}}\lesssim 1$. Then,

$$\begin{aligned}\|h_a\|_{\mathcal{L}^{2,\lambda}} &\gtrsim \|uC_\varphi h_a\|_{\mathcal{Z}}\geq(1-|z|^2)|(uC_\varphi h_a)''(z)|\\ &\geq (1-|z|^2)|u''(z)h_a(\varphi(z))|-(1-|z|^2)|(2\varphi'(z)u'(z)+\varphi''(z)u(z))h'_a(\varphi(z))|\\ &- (1-|z|^2)|h''_a(\varphi(z))(\varphi'(z))^2u(z)|.\end{aligned}$$

Therefore, by Lemma 2.1, (3.2) and (3.3), we obtain that

$$\begin{aligned}\sup_{z\in D}(1-|z|^2)|u''(z)h_a(\varphi(z))| &\leq \sup_{z\in D}(1-|z|^2)|(2\varphi'(z)u'(z)+\varphi''(z)u(z))h'_a(\varphi(z))|\\ &+ \sup_{z\in D}(1-|z|^2)|h''_a(\varphi(z))(\varphi'(z))^2u(z)|+C\|h_a\|_{\mathcal{L}^{2,\lambda}}\\ &\lesssim \sup_{z\in D}\frac{(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}}\|h_a\|_{\mathcal{L}^{2,\lambda}}\\ &+ \sup_{z\in D}\frac{(1-|z|^2)|u(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{5-\lambda}{2}}}\|h_a\|_{\mathcal{L}^{2,\lambda}}+\|h_a\|_{\mathcal{L}^{2,\lambda}}<\infty.\end{aligned}$$

Let $a=\varphi(z)$, it follows that

$$\sup_{z\in D}(1-|z|^2)|u''(z)h_a(\varphi(z))|=\sup_{z\in D}\frac{(1-|z|^2)|u''(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}}<\infty.$$

For $\forall\lambda\in D$ with $|\varphi(\lambda)|\leq\frac{1}{2}$, by (3.4), we have

$$\sup_{\lambda\in D}\frac{(1-|\lambda|^2)|u''(\lambda)|}{(1-|\varphi(\lambda)|^2)^{\frac{1-\lambda}{2}}}=(\frac{4}{3})^{\frac{1-\lambda}{2}}\sup_{\lambda\in D}(1-|\lambda|^2)|u''(\lambda)|<\infty.$$

Hence (3.1) holds.

Conversely, suppose that (3.1), (3.2), and (3.2) hold. For $f \in \mathcal{L}^{2,\lambda}$, by Lemma 2.1, we have the following inequality:

$$\begin{aligned}
(1-|z|^2)|(uC_\varphi f)''(z)| &= (1-|z|^2)|\big(2\varphi'(z)u'(z)+\varphi''(z)u(z)\big)f'(\varphi(z)) \\
&+ f''(\varphi(z))(\varphi'(z))^2u(z)+u''(z)f(\varphi(z))| \\
&\le (1-|z|^2)|\big(2\varphi'(z)u'(z)+\varphi''(z)u(z)\big)f'(\varphi(z))| \\
&+ (1-|z|^2)|f''(\varphi(z))(\varphi'(z))^2u(z)|+(1-|z|^2)|u''(z)f(\varphi(z))| \\
&\lesssim \frac{(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}}\|f\|_{\mathcal{L}^{2,\lambda}} \\
&+ \frac{(1-|z|^2)|(\varphi'(z))^2u(z)|}{(1-|\varphi(z)|^2)^{\frac{5-\lambda}{2}}}\|f\|_{\mathcal{L}^{2,\lambda}}+\frac{(1-|z|^2)|u''(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}}\|f\|_{\mathcal{L}^{2,\lambda}} \\
&\lesssim \|f\|_{\mathcal{L}^{2,\lambda}},
\end{aligned}$$

and

$$\begin{aligned}
&|u(0)f(\varphi(0))|+|u'(0)f(\varphi(0))|+|u(0)f'(\varphi(0))\varphi'(0)| \\
&\le \big(\frac{|u(0)|+|u'(0)|}{(1-|\varphi(0)|^2)^{\frac{1-\lambda}{2}}}+\frac{|u(0)\varphi'(0)|}{(1-|\varphi(0)|^2)^{\frac{3-\lambda}{2}}}\big)\|f\|_{\mathcal{L}^{2,\lambda}}.
\end{aligned}$$

This shows that $uC_\varphi$ is bounded. This completes the proof of Theorem 3.1.

**Theorem 3.2** *Let $u$ be an analytic function on the unit disc $D$, and $\varphi$ an analytic self-map of $D$. Then $uC_\varphi$ is bounded from the little analytic Morrey spaces $\mathcal{L}_0^{2,\lambda}$ to the little Zygmund space $\mathcal{Z}_0$ if and only if $u \in \mathcal{Z}_0$, (3.1), (3.2) and (3.3) hold, and the following are satisfied:*

$$\lim_{|z|\to 1}(1-|z|^2)|u(z)(\varphi'(z))^2|=0; \tag{3.11}$$

$$\lim_{|z|\to 1}(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|=0. \tag{3.12}$$

**Proof** Suppose that $uC_\varphi$ is bounded from $\mathcal{L}_0^{2,\lambda}$ to $\mathcal{Z}_0$. Then $u=uC_\varphi 1 \in \mathcal{Z}_0$. Also $u\varphi = uC_\varphi z \in \mathcal{Z}_0$, thus

$$(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)+\varphi(z)u''(z)| \longrightarrow 0 \quad (|z|\to 1^-).$$

Since $|\varphi|\le 1$ and $u\in\mathcal{Z}_0$, we have $\lim\limits_{|z|\to}(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|=0$. Hence (3.12) holds.

Similarly, $uC_\varphi z^2 \in \mathcal{Z}_0$, then

$$(1-|z|^2)|4\varphi(z)\varphi'(z)u'(z)+\varphi^2(z)u''(z)+2u(z)(\varphi(z)\varphi''(z)+(\varphi'(z))^2)| \longrightarrow 0 \quad (|z|\to 1^-).$$

By (3.12), $|\varphi|\le 1$ and $u\in\mathcal{Z}_0$, we get that $\lim\limits_{|z|\to 1}(1-|z|^2)|u(z)(\varphi'(z))^2|=0$, i. e. that (3.11) holds.

On the other hand, by Lemma 2.3 and Theorem 3.1, we obtain that (3.1), (3.2) and (3.3) hold.

Conversely, let

$$M_1=\sup_{z\in D}\frac{(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}}<\infty;$$

$$M_2 = \sup_{z\in D} \frac{(1-|z|^2)|u(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{5-\lambda}{2}}} < \infty;$$

$$M_3 = \sup_{z\in D} \frac{(1-|z|^2)|u''(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}} < \infty.$$

For $\forall f \in \mathcal{L}_0^{2,\lambda}$, by Lemma 2.2, given $\epsilon > 0$ there is a $0 < \delta < 1$ such that $(1-|z|^2)^{\frac{3-\lambda}{2}}|f'(z)| < \frac{\epsilon}{3M_1}$, $(1-|z|^2)^{\frac{5-\lambda}{2}}|f''(z)| < \frac{\epsilon}{3M_2}$ and $(1-|z|^2)^{\frac{1-\lambda}{2}}|f(z)| < \frac{\epsilon}{3M_3}$ for all $z$ with $\delta < |z| < 1$.

If $|\varphi(z)| > \delta$, it follows that

$$\begin{aligned}
(1-|z|^2)|(uC_\varphi f)''(z)| &= (1-|z|^2)|\big(2\varphi'(z)u'(z)+\varphi''(z)u(z)\big)f'(\varphi(z)) \\
&+ f''(\varphi(z))(\varphi'(z))^2u(z)+u''(z)f(\varphi(z))| \\
&\le (1-|z|^2)|\big(2\varphi'(z)u'(z)+\varphi''(z)u(z)\big)f'(\varphi(z))| \\
&+ (1-|z|^2)|f''(\varphi(z))(\varphi'(z))^2u(z)| + (1-|z|^2)|u''(z)f(\varphi(z))| \\
&< \frac{(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}}\frac{\epsilon}{3M_1} \\
&+ \frac{(1-|z|^2)|(\varphi'(z))^2u(z)|}{(1-|\varphi(z)|^2)^{\frac{5-\lambda}{2}}}\frac{\epsilon}{3M_2} + \frac{(1-|z|^2)|u''(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}}\frac{\epsilon}{3M_3} \\
&< \epsilon,
\end{aligned}$$

We know that there exists a constant $K$ such that $|f(z)| \le K$, $|f'(z)| \le K$ and $|f''(z)| \le K$ for all $|z| \le \delta$.

If $|\varphi(z)| \le \delta$, it follows that

$$\begin{aligned}
(1-|z|^2)|(uC_\varphi f)''(z)| &= (1-|z|^2)|\big(2\varphi'(z)u'(z)+\varphi''(z)u(z)\big)f'(\varphi(z)) \\
&+ f''(\varphi(z))(\varphi'(z))^2u(z)+u''(z)f(\varphi(z))| \\
&\le K(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)| \\
&+ K(1-|z|^2)|(\varphi'(z))^2u(z)| + K(1-|z|^2)|u''(z)|.
\end{aligned}$$

Thus we conclude that $(1-|z|^2)|(uC_\varphi(f))''(z)| \to 0$ as $|z| \to 1^-$. Hence $uC_\varphi f \in \mathcal{Z}_0$ for all $f \in \mathcal{L}_0^{2,\lambda}$. On the other hand, $uC_\varphi$ is bounded from $\mathcal{L}^{2,\lambda}$ to $\mathcal{Z}$ by Theorem 3.1. Hence $uC_\varphi$ is a bounded operator from $\mathcal{L}_0^{2,\lambda}$ to $\mathcal{Z}_0$.

**Corollary 3.1** *Let $\varphi$ be an analytic self-map of $D$. Then $C_\varphi$ is a bounded operator from the analytic Morrey spaces $\mathcal{L}^{2,\lambda}$ to the Zygmund space $\mathcal{Z}$ if and only if the following are satisfied:*

$$\sup_{z\in D} \frac{(1-|z|^2)|(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{5-\lambda}{2}}} < \infty; \tag{3.13}$$

$$\sup_{z\in D}\frac{(1-|z|^2)|\varphi''(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}}<\infty. \tag{3.14}$$

**Corollary 3.2** *Let $\varphi$ be an analytic self-map of $D$. Then $C_\varphi$ is a bounded operator from the little analytic Morrey spaces $\mathcal{L}_0^{2,\lambda}$ to the little Zygmund space $\mathcal{Z}_0$ if and only if $\varphi\in\mathcal{Z}_0$, (3.13) and (3.14) holds.*

**Proof** By Theorem 3.2, $C_\varphi$ is a bounded operator from $\mathcal{L}_0^{2,\lambda}$ to $\mathcal{Z}_0$ if and only if $\varphi\in\mathcal{Z}_0$, $\lim_{|z|\to1^-}(1-|z|^2)|(\varphi'(z))^2|=0$, (3.13) and (3.14) hold. However, by (1.5) in [33], That $\varphi\in\mathcal{Z}_0$ implies that $\lim_{|z|\to1^-}(1-|z|^2)|(\varphi'(z))^2|=0$. Then, $C_\varphi$ is a bounded operator if and only if $\varphi\in\mathcal{Z}_0$, (3.13) and (3.14) hold.

In the formulation of lemma, we use the notation $M_u$ on $H(D)$ defined by $M_uf=uf$ for $f\in H(D)$.

**Corollary 3.3** *The pointwise multiplier $M_u:\mathcal{L}^{2,\lambda}\to\mathcal{Z}$ is a bounded operator if and only if $u=0$.*

# 4 Compactness of $uC_\varphi$

In order to prove the compactness of $uC_\varphi$, we require the following lemmas.

**Lemma 4.1** *Suppose that $uC_\varphi$ be a bounded operator from $\mathcal{L}^{2,\lambda}$ to $\mathcal{Z}$, then $uC_\varphi$ is compact if and only if for any bounded sequence $\{f_n\}$ in $\mathcal{L}^{2,\lambda}$ which converges to 0 uniformly on compact subsets of $D$. We have $\|uC_\varphi(f_n)\|_{\mathcal{Z}}\to0$ , as $n\to\infty$ .*

The proof is similar to that of Proposition 3.11 in [5] . The details are omitted.

**Lemma 4.2** *Let $U\subset\mathcal{Z}_0$. Then $U$ is compact if and only if it is closed, bounded and satisfies*

$$\lim_{|z|\to1}\sup_{f\in U}(1-|z|^2)|f''(z)|=0.$$

The proof is similar to that of Lemma 1 in [16], we omit it.

**Theorem 4.1** *Let $u$ be an analytic function on the unit disc $D$ and $\varphi$ an analytic self-map of $D$. Suppose that $uC_\varphi$ is a bounded operator from $\mathcal{L}^{2,\lambda}$ to $\mathcal{Z}$. Then $uC_\varphi$ is compact if and only if the following are satisfied:*

$$\lim_{|\varphi(z)|\to1}\frac{(1-|z|^2)|u''(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}}=0; \tag{4.1}$$

$$\lim_{|\varphi(z)|\to1}\frac{(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}}=0; \tag{4.2}$$

$$\lim_{|\varphi(z)|\to1}\frac{(1-|z|^2)|u(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{5-\lambda}{2}}}=0. \tag{4.3}$$

**Proof** Suppose that $uC_\varphi$ is compact from $\mathcal{L}^{2,\lambda}$ to the Zygmund space $\mathcal{Z}$. Let $\{z_n\}$ be a sequence in $D$ such that $|\varphi(z_n)| \to 1$ as $n \to \infty$. If such a sequence does not exist, then (4.1), (4.2) and (4.3) are automatically satisfied. Without loss of generality we may suppose that $|\varphi(z_n)| > \frac{1}{2}$ for all $n$. We take the test functions

$$f_n(z) = \frac{1-|\varphi(z_n)|^2}{(1-\overline{\varphi(z_n)}z)^{\frac{3-\lambda}{2}}} - \frac{2(1-|\varphi(z_n)|^2)^2}{(1-\overline{\varphi(z_n)}z)^{\frac{5-\lambda}{2}}} + \frac{(1-|\varphi(z_n)|^2)^3}{(1-\overline{\varphi(z_n)}z)^{\frac{7-\lambda}{2}}}. \tag{4.4}$$

By the proof of Theorem 3.1 we know that that $\sup_n \|f_n\|_{\mathcal{L}^{2,\lambda}} \le C < \infty$. Then $\{f_n\}$ is a bounded sequence in $\mathcal{L}^{2,\lambda}$ which converges to 0 uniformly on compact subsets of $D$. Then $\lim_{n\to\infty} \|uC_\varphi(f_n)\|_{\mathcal{Z}} = 0$ by Lemma 4.1. Note that $f_n(\varphi(z_n)) \equiv 0$, $f_n'(\varphi(z_n)) \equiv 0$ and $f_n''(\varphi(z_n)) = \dfrac{2\overline{\varphi(z_n)}}{(1-|\varphi(z_n)|^2)^{\frac{5-\lambda}{2}}}$.

It follows that

$$\begin{aligned}
&\|uC_\varphi f_n\|_{\mathcal{Z}} \ge (1-|z_n|^2)|(2u'(z_n)\varphi'(z_n) + \varphi''(z_n)u(z_n))f_n'(\varphi(z_n)) \\
&+u(z_n)f_n''(\varphi(z_n))(\varphi'(z_n))^2 + u''(z_n)f_n(\varphi(z_n))| \\
&= 2(1-|z_n|^2)|(\varphi'(z_n))^2 u(z_n)\frac{\overline{\varphi(z_n)}}{(1-|\varphi(z_n)|^2)^{\frac{5-\lambda}{2}}}| \\
&\ge \frac{(1-|z_n|^2)|u(z_n)(\varphi'(z_n))^2|}{(1-|\varphi(z_n)|^2)^{\frac{5-\lambda}{2}}}.
\end{aligned}$$

Then $\lim\limits_{n\to\infty} \dfrac{(1-|z_n|^2)|u(z_n)(\varphi'(z_n))^2|}{(1-|\varphi(z_n)|^2)^{\frac{5-\lambda}{2}}} = 0$. Thus (4.3) holds.

Next, let

$$g_n(z) = \frac{1-|\varphi(z_n)|^2}{(1-\overline{\varphi(z_n)}z)^{\frac{3-\lambda}{2}}} - \frac{12-2\lambda}{7-\lambda}\frac{(1-|\varphi(z_n)|^2)^2}{(1-\overline{\varphi(z_n)}z)^{\frac{5-\lambda}{2}}} + \frac{5-\lambda}{7-\lambda}\frac{(1-|\varphi(z_n)|^2)^3}{(1-\overline{\varphi(z_n)}z)^{\frac{7-\lambda}{2}}}. \tag{4.5}$$

We similarly obtain that $\{g_n\}$ is a bounded sequence in $\mathcal{L}^{2,\lambda}$ which converges to 0 uniformly on compact subsets of $D$. Then $\lim_{n\to\infty} \|uC_\varphi(g_n)\|_{\mathcal{Z}} = 0$ by Lemma 4.1. Note that $g_n(\varphi(z_n)) \equiv 0$, $g_n''(\varphi(z_n)) \equiv 0$ and $g_n'(\varphi(z_n)) = \dfrac{-2(\overline{\varphi(z_n)})^2}{(7-\lambda)(1-|\varphi(z_n)|^2)^{\frac{3-\lambda}{2}}}$. It follows that

$$\begin{aligned}
&\|uC_\varphi g_n\|_{\mathcal{Z}} \ge (1-|z_n|^2)|(2u'(z_n)\varphi'(z_n) + \varphi''(z_n)u(z_n))g_n'(\varphi(z_n)) \\
&+u(z_n)g_n''(\varphi(z_n))(\varphi'(z_n))^2 + u''(z_n)g_n(\varphi(z_n))| \\
&= (1-|z_n|^2)|2u'(z_n)\varphi'(z_n) + \varphi''(z_n)u(z_n)|\frac{2|\overline{\varphi(z_n)}|^2}{(7-\lambda)(1-|\varphi(z_n)|^2)^{\frac{3-\lambda}{2}}} \\
&\ge \frac{(1-|z_n|^2)|2u'(z_n)\varphi'(z_n) + \varphi''(z_n)u(z_n)|}{(7-\lambda)(1-|\varphi(z_n)|^2)^{\frac{3-\lambda}{2}}}.
\end{aligned}$$

Then $\lim\limits_{n\to\infty} \dfrac{(1-|z_n|^2)|2u'(z_n)\varphi'(z_n) + \varphi''(z_n)u(z_n)|}{(1-|\varphi(z_n)|^2)^{\frac{3-\lambda}{2}}} = 0$. Thus (4.2) holds.

Finally, let

$$h_n(z) = \frac{1-|\varphi(z_n)|^2}{(1-\overline{\varphi(z_n)}z)^{\frac{3-\lambda}{2}}}. \tag{4.6}$$

We know that $\{h_n\}$ is a bounded sequence in $\mathcal{L}^{2,\lambda}$ which converges to 0 uniformly on compact subsets of $D$. Then $\lim_{n\to\infty}\|uC_\varphi(h_n)\|_{\mathcal{Z}}=0$ by Lemma 4.1. Note that $h_n(\varphi(z_n))=\frac{1}{(1-|\varphi(z_n)|^2)^{\frac{1-\lambda}{2}}}$, $h_n'(\varphi(z_n))=\frac{3-\lambda}{2(1-|\varphi(z_n)|^2)^{\frac{3-\lambda}{2}}}$ and $h_n''(\varphi(z_n))=\frac{(3-\lambda)(5-\lambda)}{4(1-|\varphi(z_n)|^2)^{\frac{5-\lambda}{2}}}$.

It follows that

$$\begin{aligned}\|uC_\varphi h_n\|_{\mathcal{Z}} &\geq (1-|z_n|^2)|(2u'(z_n)\varphi'(z_n)+\varphi''(z_n)u(z_n))h_n'(\varphi(z_n))\\ &\quad +u(z_n)h_n''(\varphi(z_n))(\varphi'(z_n))^2+u''(z_n)h_n(\varphi(z_n))|.\end{aligned}$$

Then,

$$\begin{aligned}\frac{(1-|z_n|^2)|u''(z_n)|}{(1-|\varphi(z_n)|^2)^{\frac{1-\lambda}{2}}} &\leq \|uC_\varphi h_n\|_{\mathcal{Z}}+(1-|z_n|^2)|u(z_n)(\varphi'(z_n))^2|\frac{(3-\lambda)(5-\lambda)}{4(1-|\varphi(z_n)|^2)^{\frac{5-\lambda}{2}}}\\ &\quad +(1-|z_n|^2)|(2u'(z_n)\varphi'(z_n)+\varphi''(z_n)u(z_n))|\frac{3-\lambda}{2(1-|\varphi(z_n)|^2)^{\frac{3-\lambda}{2}}},\end{aligned}$$

hence (4.1) holds by (4.2) and (4.3). The proof of the necessary is completed.

Conversely, suppose that (4.1), (4.2), and (4.3) hold. Since $uC_\varphi$ is a bounded operator, by Theorem 3.1, we have

$$M_1=\sup_{z\in D}\frac{(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}}<\infty;$$

$$M_2=\sup_{z\in D}\frac{(1-|z|^2)|u(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{5-\lambda}{2}}}<\infty;$$

$$M_3=\sup_{z\in D}\frac{(1-|z|^2)|u''(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}}<\infty.$$

Let $\{f_n\}$ be a bounded sequence in $\mathcal{L}^{2,\lambda}$ with $\|f_n\|_{\mathcal{L}^{2,\lambda}}\leq 1$ and $f_n\to 0$ uniformly on compact subsets of $D$. We only prove $\lim_{n\to\infty}\|uC_\varphi(f_n)\|_{\mathcal{Z}}=0$ by Lemma 4.1. By the assumption, for any $\epsilon>0$, there is a constant $\delta$, $0<\delta<1$, such that $\delta<|\varphi(z)|<1$ implies

$$\frac{(1-|z|^2)|u(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{5-\lambda}{2}}}<\epsilon,\qquad \frac{(1-|z|^2)|u''(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}}<\epsilon,$$

and

$$\frac{(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}}<\epsilon.$$

Let $K=\{w\in D:|w|\leq\delta\}$. Noting that $K$ is a compact subset of $D$, we get that

$$\begin{aligned}
z(uC_\varphi f_n) &= \sup_{z\in D}(1-|z|^2)|(uC_\varphi f_n)''(z)| \\
&\le \sup_{z\in D}(1-|z|^2)|\big(2\varphi'(z)u'(z)+\varphi''(z)u(z)\big)f_n'(\varphi(z))| \\
&+ \sup_{z\in D}(1-|z|^2)|f_n''(\varphi(z))(\varphi'(z))^2u(z)| + \sup_{z\in D}(1-|z|^2)|u''(z)f_n(\varphi(z))| \\
&\lesssim 3\epsilon + \sup_{|\varphi(z)|\le\delta}(1-|z|^2)|\big(2\varphi'(z)u'(z)+\varphi''(z)u(z)\big)f_n'(\varphi(z))| \\
&+ \sup_{|\varphi(z)|\le\delta}(1-|z|^2)|f_n''(\varphi(z))(\varphi'(z))^2u(z)| + \sup_{|\varphi(z)|\le\delta}(1-|z|^2)|u''(z)f_n(\varphi(z))| \\
&\le 3\epsilon + M_1\sup_{w\in K}|f_n'(w)| + M_2\sup_{w\in K}|f_n''(w)| + M_3\sup_{w\in K}|f_n(w)|.
\end{aligned}$$

As $n\to\infty$,

$$\|uC_\varphi f_n\|_{\mathcal{Z}}\to 0.$$

Hence $uC_\varphi$ is compact. This completes the proof of Theorem 4.1.

**Theorem 4.2** *Let $u$ be an analytic function on the unit disc $D$, and $\varphi$ an analytic self-map of $D$. Then $uC_\varphi$ is compact from $\mathcal{L}_0^{2,\lambda}$ to $\mathcal{Z}_0$ if and only if the following are satisfied:*

$$\lim_{|z|\to 1}\frac{(1-|z|^2)|u''(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}}=0; \tag{4.7}$$

$$\lim_{|z|\to 1}\frac{(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}}=0; \tag{4.8}$$

$$\lim_{|z|\to 1}\frac{(1-|z|^2)|u(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{5-\lambda}{2}}}=0. \tag{4.9}$$

**Proof** Assume (4.7), (4.8), and (4.9) hold. From Theorem 4.2, we know that $uC_\varphi$ is bounded from $\mathcal{L}_0^{2,\lambda}$ to $\mathcal{Z}_0$. Suppose that $f\in\mathcal{L}_0^{2,\lambda}$ with $\|f\|_{\mathcal{L}^{2,\lambda}}\le 1$. We obtain that

$$\begin{aligned}
(1-|z|^2)|(uC_\varphi f)''(z)| &= (1-|z|^2)|\big(2\varphi'(z)u'(z)+\varphi''(z)u(z)\big)f'(\varphi(z)) \\
&+ f''(\varphi(z))(\varphi'(z))^2u(z)+u''(z)f(\varphi(z))| \\
&\le (1-|z|^2)|\big(2\varphi'(z)u'(z)+\varphi''(z)u(z)\big)f'(\varphi(z))| \\
&+ (1-|z|^2)|f''(\varphi(z))(\varphi'(z))^2u(z)| + (1-|z|^2)|u''(z)f(\varphi(z))| \\
&\lesssim \frac{(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}}\|f\|_{\mathcal{L}^{2,\lambda}} \\
&+ \frac{(1-|z|^2)|(\varphi'(z))^2u(z)|}{(1-|\varphi(z)|^2)^{\frac{5-\lambda}{2}}}\|f\|_{\mathcal{L}^{2,\lambda}} + \frac{(1-|z|^2)|u''(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}}\|f\|_{\mathcal{L}^{2,\lambda}},
\end{aligned}$$

thus

$$\sup\{|(1-|z|^2)(uC_\varphi f)''(z)| : f \in \mathcal{L}_0^{2,\lambda}, \|f\|_{\mathcal{L}^{2,\lambda}} \le 1\}$$

$$\lesssim \frac{(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}}$$

$$+\frac{(1-|z|^2)|(\varphi'(z))^2u(z)|}{(1-|\varphi(z)|^2)^{\frac{5-\lambda}{2}}}+\frac{(1-|z|^2)|u''(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}},$$

and it follows that

$$\lim_{|z|\to 1}\sup\{|(1-|z|^2)(uC_\varphi f)''(z)| : f \in \mathcal{L}_0^{2,\lambda}, \|f\|_{\mathcal{L}^{2,\lambda}} \le 1\} = 0,$$

hence $uC_\varphi : \mathcal{L}_0^{2,\lambda} \to \mathcal{Z}_0$ is compact by Lemma 4.2.

Conversely, suppose that $uC_\varphi : \mathcal{L}_0^{2,\lambda} \to \mathcal{Z}_0$ is compact.

First, it is obvious that $uC_\varphi : \mathcal{L}_0^{2,\lambda} \to \mathcal{Z}_0$ is bounded, then by Theorem 3.2, we have $u \in \mathcal{Z}_0$ and that (3.11) and (3.12) hold. On the other hand, by Lemma 4.2 we have

$$\lim_{|z|\to 1}\sup\{|(1-|z|^2)(uC_\varphi f)''(z)| : f \in \mathcal{L}_0^{2,\lambda}, \|f\|_{\mathcal{L}^{2,\lambda}} \le M\} = 0,$$

for some $M > 0$.

Next, noting that the proof of Theorem 3.1 and the fact that the functions given in (3.8) are in $\mathcal{L}_0^{2,\lambda}$ and have norms bounded independently of $a$, we obtain that

$$\lim_{|z|\to 1}\frac{(1-|z|^2)|u(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{5-\lambda}{2}}} = 0$$

for $|\varphi(z)| > \frac{1}{2}$. However, if $|\varphi(z)| \le \frac{1}{2}$, by (3.11), we easily have

$$\lim_{|z|\to 1}\frac{(1-|z|^2)|u(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{5-\lambda}{2}}}$$

$$\le (\frac{4}{3})^{\frac{5-\lambda}{2}}\lim_{|z|\to 1}(1-|z|^2)|u(z)(\varphi'(z))^2| = 0.$$

Thus (4.9) holds. Also, the second statement, that (4.8), is proved similarly. We omitted it here.

Similarly, noting that the functions given in (3.10) are in $\mathcal{L}_0^{2,\lambda}$ and have norms bounded independently of $a$, we obtain that

$$\lim_{|z|\to 1}\frac{(1-|z|^2)|u''(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}} \lesssim \lim_{|z|\to 1}\frac{(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}}$$

$$+\lim_{|z|\to 1}\frac{(1-|z|^2)|u(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{5-\lambda}{2}}}+\lim_{|z|\to 1}(1-|z|^2)|(uC_\varphi h_a)''(z)|,$$

for $|\varphi(z)| > \frac{1}{2}$. So by (4.8) and (4.9), it follows that

$$\lim_{|z|\to 1}\frac{(1-|z|^2)|u''(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}} = 0$$

for $|\varphi(z)| > \frac{1}{2}$. However, if $|\varphi(z)| \le \frac{1}{2}$, by $u \in \mathcal{Z}_0$, we easily have

$$\lim_{|z|\to 1}\frac{(1-|z|^2)|u''(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}} = \lim_{|z|\to 1}(\frac{4}{3})^{\frac{1-\lambda}{2}}(1-|z|^2)|u''(z)| = 0.$$

This completes the proof of Theorem 4.2.

**Corollary 4.1** *Let $\varphi$ be an analytic self-map of $D$. Then $C_\varphi$ is a compact operator from the analytic Morrey space $\mathcal{L}^{2,\lambda}$ to the Zygmund space $\mathcal{Z}$ if and only if $C_\varphi$ is bounded,*

$$\lim_{|\varphi(z)|\to 1}\frac{(1-|z|^2)|(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{5-\lambda}{2}}}=0 \quad \text{and} \quad \lim_{|\varphi(z)|\to 1}\frac{(1-|z|^2)|\varphi''(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}}=0.$$

**Corollary 4.2** *Let $\varphi$ be an analytic self-map of $D$. Then $C_\varphi$ is a compact operator from $\mathcal{L}_0^{2,\lambda}$ to $\mathcal{Z}_0$ if and only if*

$$\lim_{|z|\to 1}\frac{(1-|z|^2)|(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{5-\lambda}{2}}}=0$$

*and*

$$\lim_{|z|\to 1}\frac{(1-|z|^2)|\varphi''(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}}=0.$$

# References


[1] D. Adams and J. Xiao, Nonlinear potential analysis on Morrey spaces and their capacities, Indiana Univ. Math. J. 53 (2004) 1629-1663.

[2] D. Adams and J. Xiao, Morrey spaces in harmonic analysis, Ark. Mat.50 (2012) 201-230.

[3] B.R.Choe, H. Koo and W. Smith, *Composition operators on small spaces*, Integr. equ. oper. Theory, **56**(2006), 357-380.

[4] M.D. Contreras and A.G. Hernández-Díaz, *Weighted composition operators on Hardy spaces*, J. Math. Anal. Appl. **263**(2001), 224-233 .

[5] C.C. Cowen and B.D. Maccluer, *Composition Operator on Spaces of Analytic Functions*, CRC Press, Boca Raton, (1995).

[6] Z. Cuckovic and R. Zhao, *Weighted composition operators on the Bergman space*, J. London Math. Soc. **70**(2004), 499-511 .

[7] Z. Cuckovic and R. Zhao, *Weighted composition operators between different weighted Bergman spaces and different Hardy spaces*, Illinois J. Math. **51**(2007), 479-498.

[8] P. Duren, *Theory of $H^p$ Spaces*, Academic Press, New York, (1970).

[9] K. Esmaeili and M. Lindström, *Weighted composition operators between Zygmund type spaces and their essential norms*, Integr. Equ. Oper Theory, **75**(2013), 473-490.

[10] D. Girela, Analytic functions of bounded mean oscillation, Complex Function Spaces (Mekrijarvi, 1999), 61-170, Univ. Joensuu Dept. Rep. Ser., 4, Univ. Joensuu, Joensuu, 2001.

[11] J. Laitila, *Weighted composition operators on BMOA*, Comput. Methods Funct. Theory, **9**(2009), no.1, 27-46 .

[12] S. Li and S. Stević, *Gerneralized composition operators on the Zygmund spaces and Bloch type spaces*, J. Math Anal. Appl. **338**(2008), 1282-1295.

[13] S. Li and S. Stević, *Products of Volterra type operator and composition operator from $H^\infty$ and Bloch spaces to Zygmund spaces*, J. Math Anal. Appl. **345**(2008), no. 1, 40-52.

[14] P. Li, J. Liu and Z. Lou, Integral operators on analytic Morrey spaces, Sci China Math, 57 (2014) 1961-1974.

[15] S. Li and S. Stević, *Weighted composition operators from Zygmund spaces into Bloch spaces*, Appl. Math. Comput. **206**(2008), no. 2, 825-831 .

[16] K. Madigan and A. Matheson, *Compact composition operators on the Bloch space*, Trans, Amer. Math. soc. **347**(1995), 2679-2687.

[17] C.B. Morrey Jr., On the solutions of quasi-linear elliptic partial differential equations, Trans. Amer. Math. Soc. 43 (1938), 126-166.

[18] S. Ohnoand R. Zhao, *Weighted composition operators on the Bloch space*, Bull. Austral. Math. Soc. **63**(2001), no. 1, 177-185.

[19] S. Ohno, K. Stroethoff and R. Zha, *Weighted composition operators between Bloch-type spaces*, Rocky Mountain J. Math. **33**(2003), no. 1, 191-215.

[20] A. H. Sanatpour and M. Hassanlou, *Essential norms of weighted composition operators between Zygmund-type spaces and Bloch-type spaces*, Turk. J. Math. **38**(2014), 872-882.

[21] W. Smith, *Composition operators between Bergman and Hardy spaces*, Trans. Amer. Math. Soc. **348**(1996), 2331-2348.

[22] S. Stević, *Weighted composition operators from Bergman-Privalov-type spaces to weighted-type spaces on the unit ball*, Appl. Math. Comput. **217**(2010), no. 5, 1939-1943.

[23] J. Wang and J. Xiao, Holomorphic Campanato spaces on the unit Ball, arXiv:1405.6192v1 [math.CV].

[24] J. Wang and J. Xiao, Two predualities and three operators over analytic Campanato spaces, arXiv:1402.4377v1 [math.CV].

[25] H. Wulan and J. Zhou, $Q_K$ and Morrey type spaces. Ann Acad Sci Fenn Math. 38 (2013) 193-207.

[26] J. Xiao, Geometric $Q_p$ Functions, Frontiers in Mathematics. Birkhäauser Verlag, Basel, 2006.

[27] J. Xiao and W. Xu, Composition operators between analytic Campanato spaces, J. Geom. Anal, Doi 10.1007/s12220-012-9349-6.

[28] J. Xiao and C. Yuan, Analytic Campanato spaces and their compositions, arXiv: 1303.5032v2[math.CV].

[29] S. Ye, *A weighted composition operator between different weighted Bloch-type spaces*, Acta Math. Sinica (Chin. Ser.), **50**(2007), no. 4, 927-942.

[30] S. Ye, *Weighted composition operators from $F(p, q, s)$ into logarithmic Bloch space*, J. Korean Math. Soc. **45**(2008), no. 4, 977-991.

[31] S. Ye, *Weighted composition operators between the $\alpha$-Bloch spaces and the little logarithmic Bloch*, J. Comput. Anal. Appl. **11**(2009), no. 3, 443-450.

[32] S. Ye, *A weighted composition operators on the logarithmic Bloch space*, Bull. Korean Math. Soc. **47**(2010), no. 3, 527-540.

[33] S. Ye and Q. Hu, *Weighted composition operators on the Zygmund space*, Abstr. Appl. Anal.,**2012**, Article ID 462482, (2012), 18 pages.

[34] S. Ye and Z. Zhuo, *Weighted composition operators from Hardy to Zygmund type spaces*, Abstr. Appl. Anal., **2013**, Article ID 365286, (2013), 10 pages.

[35] S. Ye, *Weighted composition operators from Bloch spaces to Zygmund spaces*, J. Comput. Anal. Appl. **23**(2017), no. 2, 294-305.

[36] Z. Zhuo and S. Ye, Volterra-types from analytic Morrey spaces to Bloch space, J. Integral Equ. Appl. 272015 289-308.

[37] A. Zygmund, *Trigonometric Series*, Cambridge, (1959).

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# FOURIER SERIES OF FUNCTIONS INVOLVING EULER POLYNOMIALS

TAEKYUN KIM, DAE SAN KIM, GWAN-WOO JANG, AND JONGKYUM KWON

Abstract. Recently, T. Kim introduced Fourier series expansions of certain special polynomials and investigated some interesting identities and properties of these polynomials by using those Fourier series. In this paper, we consider three types of functions involving Euler polynomials and derive their Fourier series expansions. Moreover, we express each of them in terms of Benoulli functions.

## 1. Introduction

Let $E_m(x)$ be the Euler polynomials given by the generating function

$$\frac{2}{e^t+1}e^{xt}=\sum_{m=0}^{\infty}E_m(x)\frac{t^m}{m!},\quad (\text{see [1,2,5,7-11,16]}). \tag{1.1}$$

From this equation, we can derive the following relation.

$$E_0=1,(E+1)^n+E_n=\begin{cases}2, & \text{if } n=0,\\ 0, & \text{if } n\neq 0.\end{cases}$$

The Bernoulli polynomials $B_m(x)$ are defined by the generating function

$$\frac{t}{e^t-1}e^{xt}=\sum_{m=0}^{\infty}B_m(x)\frac{t^m}{m!},\quad (\text{see [1,2,5,9]}). \tag{1.2}$$

For any real number $x$, we let

$$< x >= x-[x]\in[0,1) \tag{1.3}$$

denote the fractional part of $x$.

Here we will consider the following three types of functions involving Euler polynomials and derive their Fourier series expansions. Further, we will express each of them in terms of Bernoulli functions $B_m(< x >)$.

(1) $\alpha_m(< x >)=\sum_{k=0}^{m}E_k(x)x^{m-k},(m\geq 1)$;
(2) $\beta_m(< x >)=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}E_k(x)x^{m-k},(m\geq 1)$;
(3) $\gamma_m(< x >)=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k(x)x^{m-k},(m\geq 2)$.

The reader may refer to any book (for example, see [13-15,17]), for elementary facts about Fourier analysis.

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As to $\gamma_m(< x >)$, we note that the polynomial identity (1.4) follows immediately from Theorems 4.2 and 4.3, which is in turn derived from the Fourier series expansion of $\gamma_m(< x >)$.

$$\begin{aligned}&\sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k(x)x^{m-k}\\&=-\frac{1}{m}\Big(\sum_{k=1}^{m}\frac{E_k}{k(m-k+1)}+\frac{1}{m(m+1)}-\frac{2}{m(m+1)}E_{m+1}\Big)\\&+\frac{1}{m}\sum_{s=1}^{m}\Big(\binom{m}{s}\frac{H_{m-1}-H_{m-s}}{m-s+1}(1-2E_{m-s+1})-\binom{m}{s}\sum_{l=s}^{m-1}\frac{E_{l-s+1}}{(l-s+1)(m-l)}\Big)B_s(x),\end{aligned}\tag{1.4}$$

where $H_m=\sum_{j=1}^{m}\frac{1}{j}$ are the harmonic numbers. The obvious polynomial identities can be derived also for $\alpha_m(< x >)$ and $\beta_m(< x >)$ from Theorems 2.1 and 2.2, and Theorems 3.1 and 3.2 , respectively.

## 2. The function $\alpha_m(< x >)$

Let $\alpha_m(x)=\sum_{k=0}^{m}E_k(x)x^{m-k},(m\geq 1)$. Then we consider the function $\alpha_m(< x >)=\sum_{k=0}^{m}E_k(< x >)< x >^{m-k}$,
defined on $(-\infty,\infty)$, which is periodic with period 1.

The Fourier series of $\beta_m(< x >)$ is $\sum_{n=-\infty}^{\infty}A_n^{(m)}e^{2\pi inx}$,
where

$$\begin{aligned}A_n^{(m)}&=\int_0^1\alpha_m(< x >)e^{-2\pi inx}dx\\&=\int_0^1\alpha_m(x)e^{-2\pi inx}dx.\end{aligned}\tag{2.1}$$

To proceed further, we note the following.

$$\begin{aligned}\alpha_m'(x)&=\sum_{k=0}^{m}\big(kE_{k-1}(x)x^{m-k}+(m-k)E_k(x)x^{m-k-1}\big)\\&=\sum_{k=1}^{m}kE_{k-1}(x)x^{m-k}+\sum_{k=0}^{m-1}(m-k)E_k(x)x^{m-k-1}\\&=\sum_{k=0}^{m-1}(k+1)E_k(x)x^{m-k-1}+\sum_{k=0}^{m-1}(m-k)E_k(x)x^{m-k-1}\\&=(m+1)\sum_{k=0}^{m-1}E_k(x)x^{m-1-k}\\&=(m+1)\alpha_{m-1}(x).\end{aligned}\tag{2.2}$$

So, $\alpha'_m(x) = (m+1)\alpha_{m-1}(x)$. From this, $\left(\frac{\alpha_{m+1}(x)}{m+2}\right)' = \alpha_m(x)$.

$$\int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\left(\alpha_{m+1}(1) - \alpha_{m+1}(0)\right). \tag{2.3}$$

$$\begin{aligned}\alpha_m(1) - \alpha_m(0) &= \sum_{k=0}^{m}\left(E_k(1) - E_k\delta_{m,k}\right)\\ &= \sum_{k=0}^{m}\left((-E_k + 2\delta_{k,0})\right) - \sum_{k=0}^{m} E_k\delta_{m,k}\\ &= -\sum_{k=0}^{m} E_k + 2 - E_m\end{aligned} \tag{2.4}$$

Thus

$$\alpha_m(1) = \alpha_m(0) \Longleftrightarrow \sum_{k=0}^{m} E_k = 2 - E_m. \tag{2.5}$$

Also,

$$\int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\left(-\sum_{k=0}^{m+1} E_k + 2 - E_{m+1}\right). \tag{2.6}$$

Now, we would like to determine the Fourier coefficients $A_n^{(m)}$.
$Case1 : n \neq 0$.

$$\begin{aligned}A_n^{(m)} &= \int_0^1 \alpha_m(x)e^{-2\pi inx}dx\\ &= -\frac{1}{2\pi in}\left[\alpha_m(x)e^{-2\pi inx}\right]_0^1 + \frac{1}{2\pi in}\int_0^1 \alpha'_m(x)e^{-2\pi inx}dx\\ &= -\frac{1}{2\pi in}\left(\alpha_m(1) - \alpha_m(0)\right) + \frac{m+1}{2\pi in}\int_0^1 \alpha_{m-1}(x)e^{-2\pi inx}dx\\ &= \frac{m+1}{2\pi in}A_n^{(m-1)} + \frac{1}{2\pi in}\left(\sum_{k=0}^{m} E_k - 2 + E_m\right)\\ &= \frac{m+1}{2\pi in}\left(\frac{m}{2\pi in}A_n^{(m-2)} + \frac{1}{2\pi in}\left(\sum_{k=0}^{m-1} E_k - 2 + E_{m-1}\right)\right) + \frac{1}{2\pi in}\left(\sum_{k=0}^{m} E_k - 2 + E_m\right)\\ &= \frac{(m+1)m}{(2\pi in)^2}A_n^{(m-2)} + \frac{m+1}{(2\pi in)^2}\left(\sum_{k=0}^{m-1} E_k - 2 + E_{m-1}\right) + \frac{1}{2\pi in}\left(\sum_{k=0}^{m} E_k - 2 + E_m\right)\\ &= \cdots\end{aligned}$$

$$\begin{aligned}
&= \frac{(m+1)_{m-1}}{(2\pi in)^{m-1}} A_n^{(1)} + \sum_{j=1}^{m-1} \frac{(m+1)_{j-1}}{(2\pi in)^j} \left( \sum_{k=0}^{m-j+1} E_k - 2 + E_{m-j+1} \right) \\
&= -\frac{(m+1)!}{(2\pi in)^m} + \sum_{j=1}^{m-1} \frac{(m+1)_{j-1}}{(2\pi in)^j} \left( \sum_{k=0}^{m-j+1} E_k - 2 + E_{m-j+1} \right) \\
&= \sum_{j=1}^{m} \frac{(m+1)_{j-1}}{(2\pi in)^j} \left( \sum_{k=0}^{m-j+1} E_k - 2 + E_{m-j+1} \right) \\
&= \frac{1}{m+2} \sum_{j=1}^{m} \frac{(m+2)_j}{(2\pi in)^j} \left( \sum_{k=0}^{m-j+1} E_k - 2 + E_{m-j+1} \right),
\end{aligned} \tag{2.7}$$

where $A_n^{(1)} = \int_0^1 \alpha_1(x) e^{-2\pi inx} dx = \int_0^1 (2x - \frac{1}{2}) e^{-2\pi inx} dx = -\frac{2}{2\pi in}$.
$Case 2 : n = 0$.

$$A_0^{(m)} = \int_0^1 \alpha_m(x) dx = \frac{1}{m+2} \left( -\sum_{k=0}^{m+1} E_k + 2 - E_{m+1} \right). \tag{2.8}$$

$\alpha_m(< x >), (m \geq 1)$ is piecewise $C^\infty$. Moreover, $\alpha_m(< x >)$ is continuous for those positive integers $m$ with $\sum_{k=0}^{m} E_k = 2 - E_m$ and discontinuous with jump discontinuities at integers for those positive integers $m$ with $\sum_{k=0}^{m} E_k \neq 2 - E_m$.

We recall the following facts about Bernoulli functions $B_n(< x >)$ :
(a) for $m \geq 2$,

$$B_m(< x >) = -m! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi inx}}{(2\pi in)^m}. \tag{2.9}$$

(b) for $m = 1$,

$$-\sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi inx}}{2\pi in} = \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases} \tag{2.10}$$

Assume first that $m$ is a positive integer with $\sum_{k=0}^{m} E_k = 2 - E_m$. Then $\alpha_m(1) = \alpha_m(0)$.
$\alpha_m(< x >)$ is piecewise $C^\infty$, and continuous. So the Fourier series of $\alpha_m(< x >)$ converges uniformly to $\alpha_m(< x >)$, and

$$\begin{aligned}\alpha_m(< x >) &= -\frac{1}{m+2}\left(\sum_{k=0}^{m+1} E_k - 2 + E_{m+1}\right)\\ &\quad + \frac{1}{m+2}\sum_{n=-\infty,n\neq 0}^{\infty}\left(\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi in)^j}\left(\sum_{k=0}^{m-j+1} E_k - 2 + E_{m-j+1}\right)\right)e^{2\pi inx}\\ &= -\frac{1}{m+2}\left(\sum_{k=0}^{m+1} E_k - 2 + E_{m+1}\right)\\ &\quad - \frac{1}{m+2}\sum_{j=1}^{m}\binom{m+2}{j}\left(\sum_{k=0}^{m-j+1} E_k - 2 + E_{m-j+1}\right)\\ &\quad \times\left(-j!\sum_{n=-\infty,n\neq 0}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^j}\right)\end{aligned} \tag{2.11}$$

$$\begin{aligned}&= -\frac{1}{m+2}\left(\sum_{k=0}^{m+1} E_k - 2 + E_{m+1}\right)\\ &- \frac{1}{m+2}\sum_{j=2}^{m}\binom{m+2}{j}\left(\sum_{k=0}^{m-j+1} E_k - 2 + E_{m-j+1}\right)B_j(< x >)\\ &- \frac{1}{m+2}\binom{m+2}{1}\left(\sum_{k=0}^{m} E_k - 2 + E_m\right)\cdot\begin{cases}B_1(< x >), & \text{for } x\notin\mathbb{Z},\\ 0, & \text{for } x\in\mathbb{Z}\end{cases},\end{aligned} \tag{2.12}$$

for all $x \in (-\infty, \infty)$.

Hence we obtain the following theorem.

**Theorem 2.1.** *Let $m$ be a positive integer with $\sum_{k=0}^{m} E_k = 2 - E_m$. Then we have the following.*
(a) $\sum_{k=0}^{m} E_k(< x >) < x >^{m-k}$ *has the Fourier series expansion*

$$\begin{aligned}&\sum_{k=0}^{m} E_k(< x >) < x >^{m-k}\\ &= -\frac{1}{m+2}\left(\sum_{k=0}^{m+1} E_k - 2 + E_{m+1}\right)\\ &\quad + \frac{1}{m+2}\sum_{n=-\infty,n\neq 0}^{\infty}\left(\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi in)^j}\left(\sum_{k=0}^{m-j+1} E_k - 2 + E_{m-j+1}\right)\right)e^{2\pi inx},\end{aligned}$$

*for all $x \in (-\infty, \infty)$, where the convergence is uniform.*

(b) $$\sum_{k=0}^{m} E_k(< x >) < x >^{m-k} = -\frac{1}{m+2}\sum_{j=0,j\neq 1}^{m}\binom{m+2}{j}\left(\sum_{k=0}^{m-j+1} E_k - 2 + E_{m-j+1}\right)B_j(< x >),$$

*for all $x \in (-\infty, \infty)$, where $B_k(< x >)$ is the Bernoulli function.*

Assume next that $m$ is a positive integer with $\sum_{k=0}^{m} E_k \neq 2 - E_m$. Then $\alpha_m(1) \neq \alpha_m(0)$. Hence $\alpha_m(< x >)$ is piecewise $C^{\infty}$ , and discontinuous with jump discontinuities at integers. The Fourier series of $\alpha_m(< x >)$ converges pointwise to $\alpha_m(< x >)$ , for $x \notin \mathbb{Z}$, and converges to

$$\begin{aligned}\frac{1}{2}\left(\alpha_m(0) + \alpha_m(1)\right) &= \alpha_m(0) - \frac{1}{2}\sum_{k=0}^{m} E_k + 1 - \frac{1}{2}E_m, \\ &= 1 - \frac{1}{2}\sum_{k=0}^{m-1} E_k,\end{aligned}$$

for $x \in \mathbb{Z}$.

Thus, we get the following theorem.

**Theorem 2.2.** *Let $m$ be a positive integer with $\sum_{k=0}^{m} E_k \neq 2 - E_m$. Then we have the following.*

$$\begin{aligned}&\text{(a)} - \frac{1}{m+2}\left(\sum_{k=0}^{m+1} E_k - 2 + E_{m+1}\right) \\ &+ \frac{1}{m+2}\sum_{n=-\infty, n\neq 0}^{\infty}\left(\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi i n)^j}\left(\sum_{k=0}^{m-j+1} E_k - 2 + E_{m-j+1}\right)\right)e^{2\pi i n x} \\ &= \begin{cases}\sum_{k=0}^{m} E_k(< x >) < x >^{m-k}, & \text{for } x \notin \mathbb{Z}, \\ 1 - \frac{1}{2}\sum_{k=0}^{m-1} E_k, & \text{for } x \in \mathbb{Z}.\end{cases}\end{aligned}$$

$$\begin{aligned}&\text{(b)} - \frac{1}{m+2}\sum_{j=0}^{m}\binom{m+2}{j}\left(\sum_{k=0}^{m-j+1} E_k - 2 + E_{m-j+1}\right)B_j(< x >) \\ &= \sum_{k=0}^{m} E_k(< x >) < x >^{m-k}, \ for\ x \notin \mathbb{Z}, \\ &- \frac{1}{m+2}\sum_{j=0, j\neq 1}^{m}\binom{m+2}{j}\left(\sum_{k=0}^{m-j+1} E_k - 2 + E_{m-j+1}\right)B_j(< x >) \\ &= 1 - \frac{1}{2}\sum_{k=0}^{m-1} E_k, \ for\ x \in \mathbb{Z}.\end{aligned}$$

Question: For what values of $m \geq 1$, does $\sum_{k=0}^{m} E_k = 2 - E_m$ hold ?

**Remark 2.3.** Another expression for $A_0^{(m)} = \int_0^1 \alpha_m(x)dx$ was obtained previously (see [3,4,6,12]) and is

$$\sum_{l=0}^{m-1}\sum_{j=1}^{m-l}\frac{(-1)^j\binom{m-l+1}{j}E_{l+j}}{(m-l+1)\binom{l+j}{l}} + \frac{4(-1)^{m+1}}{m+2}E_{m+1}. \tag{2.13}$$

So, we obtain the following identity.

$$\frac{1}{m+2}\left(-\sum_{k=0}^{m+1}E_k+2-E_{m+1}\right)$$
$$=\sum_{l=0}^{m-1}\sum_{j=1}^{m-l}\frac{(-1)^j\binom{m-l+1}{j}E_{l+j}}{(m-l+1)\binom{l+j}{l}}+\frac{4(-1)^{m+1}}{m+2}E_{m+1}.$$

## 3. The fuction $\beta_m(< x >)$

Let $\beta_m(x)=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}E_k(x)x^{m-k}$, $(m\geq 1)$. Then we will consider the function

$$\beta_m(< x >)=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}E_k(< x >)< x >^{m-k},$$

defined on $(-\infty,\infty)$, which is periodic with period 1.

The Fourier series of $\beta_m(< x >)$ is

$$\sum_{n=-\infty}^{\infty}B_n^{(m)}e^{2\pi inx},$$

where

$$B_n^{(m)}=\int_0^1\beta_m(< x >)e^{-2\pi inx}dx=\int_0^1\beta_m(x)e^{-2\pi inx}dx.$$

Before proceeding further, we observe the following:

$$\begin{aligned}\beta_m'(x)&=\sum_{k=0}^{m}\left\{\frac{k}{k!(m-k)!}E_{k-1}(x)x^{m-k}\right.\\&\left.+\frac{m-k}{k!(m-k)!}E_k(x)x^{m-k-1}\right\}\\&=\sum_{k=1}^{m}\frac{1}{(k-1)!(m-k)!}E_{k-1}(x)x^{m-k}\\&+\sum_{k=0}^{m-1}\frac{1}{k!(m-1-k)!}E_k(x)x^{m-1-k}\\&=\sum_{k=0}^{m-1}\frac{1}{k!(m-1-k)!}E_k(x)x^{m-1-k}\\&+\sum_{k=0}^{m-1}\frac{1}{k!(m-1-k)!}E_k(x)x^{m-1-k}\\&=2\beta_{m-1}(x).\end{aligned}\tag{3.1}$$

So, $\beta_m'(x)=2\beta_{m-1}(x)$. This implies that $\left(\frac{\beta_{m+1}(x)}{2}\right)'=\beta_m(x)$.

$$\int_0^1\beta_m(x)dx=\frac{1}{2}\Big(\beta_{m+1}(1)-\beta_{m+1}(0)\Big).\tag{3.2}$$

$$\begin{aligned}
&\beta_m(1)-\beta_m(0)\\
&=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}\big(E_k(1)-E_k(0)\delta_{m,k}\big)\\
&=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}\left\{(-E_k+2\delta_{k,0})\right\}.\\
&-\sum_{k=0}^{m}\frac{1}{k!(m-k)!}E_k\delta_{m,k}\\
&=-\sum_{k=0}^{m}\frac{E_k}{k!(m-k)!}+\frac{2}{m!}-\frac{E_m}{m!}.
\end{aligned}\tag{3.3}$$

So, $\int_0^1 \beta_m(x)dx=\frac{1}{2}\left(-\sum_{k=0}^{m+1}\frac{E_k}{k!(m+1-k)!}+\frac{2}{(m+1)!}-\frac{E_{m+1}}{(m+1)!}\right)$.
Also, $\beta_m(1)=\beta_m(0)\Leftrightarrow\sum_{k=0}^{m}\frac{E_k}{k!(m-k)!}=\frac{2}{m!}-\frac{E_m}{m!}$.

Now, we are going to determine the Fourier coefficients $B_n^{(m)}$.

Case 1:$n\neq 0$.

$$\begin{aligned}
&B_n^{(m)}=\int_0^1\beta_m(x)e^{-2\pi inx}dx\\
&=-\frac{1}{2\pi in}\Big[\beta_m(x)e^{-2\pi inx}\Big]_0^1+\frac{1}{2\pi in}\int_0^1\beta_m'(x)e^{-2\pi inx}dx\\
&=-\frac{1}{2\pi in}\Big(\beta_m(1)-\beta_m(0)\Big)+\frac{1}{\pi in}\int_0^1\beta_{m-1}(x)e^{-2\pi inx}dx\\
&=\frac{1}{\pi in}B_n^{(m-1)}-\frac{1}{2\pi in}\left(\beta_m(1)-\beta_m(0)\right)\\
&=\frac{1}{\pi in}\Big(\frac{1}{\pi in}B_n^{(m-2)}-\frac{1}{2\pi in}\left(\beta_{m-1}(1)-\beta_{m-1}(0)\right)\Big)-\frac{1}{2\pi in}\left(\beta_m(1)-\beta_m(0)\right)\\
&=\frac{1}{(\pi in)^2}B_n^{(m-2)}-\frac{2}{(2\pi in)^2}\left(\beta_{m-1}(1)-\beta_{m-1}(0)\right)-\frac{1}{2\pi in}\left(\beta_m(1)-\beta_m(0)\right)\\
&=\cdots\\
&=\frac{1}{(\pi in)^{m-1}}B_n^{(1)}-\sum_{j=1}^{m-1}\frac{2^{j-1}}{(2\pi in)^j}\left(\beta_{m-j+1}(1)-\beta_{m-j+1}(0)\right)\\
&=-\frac{1}{(\pi in)^m}+\sum_{j=1}^{m-1}\frac{2^{j-1}}{(2\pi in)^j}\left(\sum_{k=0}^{m-j+1}\frac{E_k}{k!(m-j-k+1)!}-\frac{2}{(m-j+1)!}+\frac{E_{m-j+1}}{(m-j+1)!}\right)\\
&=\sum_{j=1}^{m}\frac{2^{j-1}}{(2\pi in)^j}\left(\sum_{k=0}^{m-j+1}\frac{E_k}{k!(m-j-k+1)!}-\frac{2}{(m-j+1)!}+\frac{E_{m-j+1}}{(m-j+1)!}\right),
\end{aligned}\tag{3.4}$$

where $B_n^{(1)}=\int_0^1\beta_1(x)e^{-2\pi inx}dx=\int_0^1(2x-\frac{1}{2})e^{-2\pi inx}dx=-\frac{1}{\pi in}$.

Case 2: $n = 0$.

$$B_0^{(m)} = \int_0^1 \beta_m(x)dx = \frac{1}{2}\left(-\sum_{k=0}^{m+1}\frac{E_k}{k!(m+1-k)!} + \frac{2}{(m+1)!} - \frac{E_{m+1}}{(m+1)!}\right). \quad (3.5)$$

Let

$$\Omega_m = \beta_m(1) - \beta_m(0) = -\sum_{k=0}^{m}\frac{E_k}{k!(m-k)!} + \frac{2}{m!} - \frac{E_m}{m!},$$

for $m \geq 1$.

$\beta_m(< x >),\ (m \geq 1)$ is piecewise $C^\infty$. Moreover, $\beta_m(< x >)$ is continuous for those positive integers $m$ with $\Omega_m = 0$ and discontinuous with jump discontinuities at integers for those positive integers $m$ with $\Omega_m \neq 0$.

Assume first that $m$ is a positive integer with $\Omega_m = 0$. Then $\beta_m(1) = \beta_m(0)$. $\beta_m(< x >)$ is piecewise $C^\infty$, and continuous. So the Fourier series of $\beta_m(< x >)$ converges uniformly to $\beta_m(< x >)$, and

$$\begin{aligned}
&\beta_m(< x >)\\
&= \sum_{k=0}^{m}\frac{1}{k!(m-k)!}E_k(< x >) < x >^{m-k}\\
&= \frac{1}{2}\Omega_{m+1} - \sum_{n=-\infty,n\neq 0}^{\infty}\Big(\sum_{j=1}^{m}\frac{2^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}\Big)e^{2\pi inx}\\
&= \frac{1}{2}\Omega_{m+1} + \sum_{j=1}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1} \times \Big(-j!\sum_{n=-\infty,n\neq 0}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^j}\Big)\\
&= \frac{1}{2}\Omega_{m+1} + \sum_{j=2}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_k(< x >)\\
&+ \Omega_m \times \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z},\\ 0, & \text{for } x \in \mathbb{Z}, \end{cases}
\end{aligned}$$

for all $x \in (-\infty, \infty)$. (3.6)

Now, we obtain the following theorem.

**Theorem 3.1.** *For each positive integer $l$, let*

$$\Omega_l = -\sum_{k=0}^{l}\frac{E_k}{k!(l-k)!} + \frac{2}{l!} - \frac{E_l}{l!}.$$

*Assume that $\Omega_m = 0$, for a positive integer $m$. Then we have the following.*
(a) $\sum_{k=0}^{m}\frac{1}{k!(m-k)!}E_k(< x >) < x >^{m-k}$ *has the Fourier series expansion*

$$\sum_{k=0}^{m}\frac{1}{k!(m-k)!}E_k(< x >) < x >^{m-k} = \frac{1}{2}\Omega_{m+1} - \sum_{n=-\infty,n\neq 0}^{\infty}\Big(\sum_{j=1}^{m}\frac{2^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}\Big)e^{2\pi inx},$$

*for all* $x \in (-\infty, \infty)$. *Here the convergence is uniform.*

$$\text{(b)} \sum_{k=0}^{m} \frac{1}{k!(m-k)!} E_k(< x >) < x >^{m-k} = \sum_{j=0, j\neq 1}^{m} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_k(< x >),$$

*for all* $x \in (-\infty, \infty)$. *Here* $B_k(< x >)$ *is the Bernoulli function.*

Assume next that $m$ is a positive integer with $\Omega_m \neq 0$. Then, $\beta_m(1) \neq \beta_m(0)$. $\beta_m(< x >)$ is piecewise $C^\infty$ and discontinuous with jump discontinuities at integers. Thus the Fourier series of $\beta_m(< x >)$ converges pointwise to $\beta_m(< x >)$, for $x \notin \mathbb{Z}$, and converges to

$$\begin{aligned} &\frac{1}{2}(\beta_m(0) + \beta_m(1)) = \beta_m(0) + \frac{1}{2}\Omega_m \\ &= \frac{E_m}{m!} + \frac{1}{2}\left( -\sum_{k=0}^{m} \frac{E_k}{k!(m-k)!} + \frac{2}{m!} - \frac{E_m}{m!} \right) \\ &= \frac{1}{2}\left( \frac{2}{m!} - \sum_{k=0}^{m-1} \frac{E_k}{k!(m-k)!} \right). \end{aligned} \tag{3.7}$$

for $x \in \mathbb{Z}$.

So, we obtain the following theorem.

**Theorem 3.2.** *For each positive integer* $l$, *let*

$$\Omega_l = -\sum_{k=0}^{l} \frac{E_k}{k!(l-k)!} + \frac{2}{l!} - \frac{E_l}{l!}.$$

*Assume that* $\Omega_m \neq 0$, *for a positive integer* $m$. *Then we have the following.*

$$\begin{aligned} \text{(a)} &\frac{1}{2}\Omega_{m+1} - \sum_{n=-\infty, n\neq 0}^{\infty} \Big( \sum_{j=1}^{m} \frac{2^{j-1}}{(2\pi i n)^j} \Omega_{m-j+1} \Big) e^{2\pi i n x} \\ &= \begin{cases} \sum_{k=0}^{m} \frac{1}{k!(m-k)!} E_k(< x >) < x >^{m-k}, & \text{for } x \notin \mathbb{Z}, \\ \frac{E_m}{m!} + \frac{1}{2}\Omega_m, & \text{for } x \in \mathbb{Z}. \end{cases} \end{aligned}$$

*Here the convergence is pointwise.*

(b)

$$\begin{aligned} &\sum_{j=0}^{m} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_j(< x >) \\ &= \sum_{k=0}^{m} \frac{1}{k!(m-k)!} E_k(< x >) < x >^{m-k}, \quad \text{for } x \notin \mathbb{Z}, \\ &\sum_{j=0, j\neq 1}^{m} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_j(< x >) \\ &= \frac{E_m}{m!} + \frac{1}{2}\Omega_m, \qquad \text{for } x \in \mathbb{Z}. \end{aligned}$$

*Here* $B_k(< x >)$ *is the Bernoulli function.*

Question: For what values of $m \geq 1$, does $\sum_{k=0}^{m} \frac{E_k}{k!(m-k)!} = \frac{2}{m!} - \frac{E_m}{m!}$ hold ?

**Remark 3.3.** In a previous paper (see [3,4,6,12]), it was shown that

$$\int_0^1 \beta_m(x)dx = \sum_{l=0}^{m-1}\sum_{j=1}^{m-l} \frac{(-1)^j \binom{m+1}{l+j} E_{l+j}}{(m+1)!} + \frac{2(-1)^{m+1}E_{m+1}}{(m+1)!}. \tag{3.8}$$

Hence, we have the following identity.

$$\begin{aligned} &\frac{1}{2}\left(-\sum_{k=0}^{m+1}\frac{E_k}{k!(m+1-k)!} + \frac{2}{(m+1)!} - \frac{E_{m+1}}{(m+1)!}\right) \\ &= \sum_{l=0}^{m-1}\sum_{j=1}^{m-l} \frac{(-1)^j \binom{m+1}{l+j} E_{l+j}}{(m+1)!} + \frac{2(-1)^{m+1}E_{m+1}}{(m+1)!}. \end{aligned}$$

## 4. The fuction $\gamma_m(< x >)$

Let $\gamma_m(x) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} E_k(x)x^{m-k}, (m \geq 2)$. Then we will consider the function

$$\gamma_m(< x >) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} E_k(< x >) < x >^{m-k}, \tag{4.1}$$

defined on $(-\infty, \infty)$, which is periodic with period 1.

The Fourier series of $\gamma_m(< x >)$ is

$$\sum_{n=-\infty}^{\infty} C_n^{(m)} e^{2\pi i n x}, \tag{4.2}$$

where

$$C_n^{(m)} = \int_0^1 \gamma_m(< x >)e^{-2\pi i n x}dx = \int_0^1 \gamma_m(x)e^{-2\pi i n x}dx. \tag{4.3}$$

To proceed further, we observe the following.

$$\begin{aligned}
\gamma_m'(x) &= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \Big( kE_{k-1}(x)x^{m-k} + (m-k)E_k(x)x^{m-k-1} \Big) \\
&= \sum_{k=1}^{m-1} \frac{1}{m-k} E_{k-1}(x)x^{m-k} + \sum_{k=1}^{m-1} \frac{1}{k} E_k(x)x^{m-k-1} \\
&= \sum_{k=0}^{m-2} \frac{1}{m-k-1} E_k(x)x^{m-k-1} + \sum_{k=0}^{m-1} \frac{1}{k} E_k(x)x^{m-k-1} \\
&= \frac{1}{m-1}x^{m-1} + \sum_{k=1}^{m-2} \frac{1}{m-k-1} E_k(x)x^{m-k-1} + \frac{1}{m-1}E_{m-1}(x) \\
&\quad + \sum_{k=1}^{m-2} \frac{1}{k} E_k(x)x^{m-k-1} \\
&= (m-1) \sum_{k=1}^{m-2} \frac{1}{k(m-1-k)} E_k(x)x^{m-1-k} + \frac{1}{m-1} \Big( x^{m-1} + E_{m-1}(x) \Big) \\
&= (m-1)\gamma_{m-1}(x) + \frac{1}{m-1} \Big( x^{m-1} + E_{m-1}(x) \Big).
\end{aligned} \tag{4.4}$$

Thus,

$$\gamma_m'(x) = (m-1)\gamma_{m-1}(x) + \frac{1}{m-1} \Big( x^{m-1} + E_{m-1}(x) \Big).$$

From this, we have

$$\left( \frac{1}{m} \Big( \gamma_{m+1}(x) - \frac{1}{m(m+1)} x^{m+1} - \frac{1}{m(m+1)} E_{m+1}(x) \Big) \right)' = \gamma_m(x).$$

$$\begin{aligned}
&\int_0^1 \gamma_m(x)dx \\
&= \frac{1}{m} \Big[ \gamma_{m+1}(x) - \frac{1}{m(m+1)} x^{m+1} - \frac{1}{m(m+1)} E_{m+1}(x) \Big]_0^1 \\
&= \frac{1}{m} \Big( \gamma_{m+1}(1) - \gamma_{m+1}(0) - \frac{1}{m(m+1)} - \frac{1}{m(m+1)} \Big( E_{m+1}(1) - E_{m+1}(0) \Big) \Big) \\
&= \frac{1}{m} \Big( \gamma_{m+1}(1) - \gamma_{m+1}(0) - \frac{1}{m(m+1)} + \frac{2}{m(m+1)} E_{m+1} \Big).
\end{aligned} \tag{4.5}$$

$$\begin{aligned}
&\gamma_m(1) - \gamma_m(0) \\
&= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \big( E_k(1) - E_k(0)\delta_{m,k} \big) \\
&= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \Big( -E_k(0) + 2\delta_{k,0} \Big) - \sum_{k=1}^{m-1} \frac{1}{k(m-k)} E_k(0)\delta_{m,k} \\
&= -\sum_{k=1}^{m-1} \frac{E_k}{k(m-k)}.
\end{aligned} \tag{4.6}$$

Thus,

$$\gamma_m(1) = \gamma_m(0) \;\Leftrightarrow\; \sum_{k=1}^{m-1} \frac{E_k}{k(m-k)} = 0. \tag{4.7}$$

In addition,

$$\int_0^1 \gamma_m(x)dx = -\frac{1}{m}\left(\sum_{k=1}^{m} \frac{E_k}{k(m-k+1)} + \frac{1}{m(m+1)} - \frac{2}{m(m+1)}E_{m+1}\right). \tag{4.8}$$

Now, we would like to determine the Fourier coefficients $C_n^{(m)}$.
Case 1:$n \neq 0$.

$$\begin{aligned}
C_n^{(m)} &= \int_0^1 \gamma_m(x)e^{-2\pi inx}dx \\
&= -\frac{1}{2\pi in}\Big[\gamma_m(x)e^{-2\pi inx}\Big]_0^1 + \frac{1}{2\pi in}\int_0^1 \gamma_m'(x)e^{-2\pi inx}dx \\
&= -\frac{1}{2\pi in}\Big(\gamma_m(1) - \gamma_m(0)\Big) + \frac{m-1}{2\pi in}\int_0^1 \gamma_{m-1}(x)e^{-2\pi inx}dx \\
&+ \frac{1}{2\pi in(m-1)}\int_0^1 x^{m-1}e^{-2\pi inx}dx + \frac{1}{2\pi in(m-1)}\int_0^1 E_{m-1}(x)e^{-2\pi inx}dx \\
&= \frac{m-1}{2\pi in}C_n^{(m-1)} - \frac{1}{2\pi in}\Lambda_m - \frac{1}{2\pi in(m-1)}\Theta_m + \frac{2}{2\pi in(m-1)}\Phi_m,
\end{aligned} \tag{4.9}$$

where , for $l \geq 1$,

$$\int_0^1 E_l(x)e^{-2\pi inx}dx = \begin{cases} 2\sum_{k=1}^{l} \frac{(l)_{k-1}}{(2\pi in)^k}E_{l-k+1}, & \text{for } n \neq 0, \\ -\frac{2}{l+1}E_{l+1}, & \text{for } n = 0. \end{cases}$$

$$\int_0^1 x^l e^{-2\pi inx}dx = \begin{cases} -\sum_{k=1}^{l} \frac{(l)_{k-1}}{(2\pi in)^k}, & \text{for } n \neq 0, \\ \frac{1}{l+1}, & \text{for } n = 0. \end{cases}$$

Here, for $m \geq 2$,

$$\begin{aligned}
\Lambda_m &= \gamma_m(1) - \gamma_m(0) = -\sum_{k=1}^{m-1} \frac{E_k}{k(m-k)}, \\
\Theta_m &= \sum_{k=1}^{m-1} \frac{(m-1)_{k-1}}{(2\pi in)^k}, \\
\Phi_m &= \sum_{k=1}^{m-1} \frac{(m-1)_{k-1}}{(2\pi in)^k}E_{m-k}.
\end{aligned} \tag{4.10}$$

$$
\begin{aligned}
C_n^{(m)} &= \frac{m-1}{2\pi in}C_n^{(m-1)} - \frac{1}{2\pi in}\Lambda_m - \frac{1}{2\pi in(m-1)}\Theta_m + \frac{2}{2\pi in(m-1)}\Phi_m \\
&= \frac{m-1}{2\pi in}\Big(\frac{m-2}{2\pi in}C_n^{(m-2)} - \frac{1}{2\pi in}\Lambda_{m-1} - \frac{1}{2\pi in(m-2)}\Theta_{m-1} + \frac{2}{2\pi in(m-1)}\Phi_{m-1}\Big) \\
&- \frac{1}{2\pi in}\Lambda_m - \frac{1}{2\pi in(m-1)}\Theta_m + \frac{2}{2\pi in(m-1)}\Phi_m \\
&= \frac{(m-1)(m-2)}{(2\pi in)^2}C_n^{(m-2)} - \frac{m-1}{(2\pi in)^2}\Lambda_{m-1} - \frac{1}{2\pi in}\Lambda_m - \frac{m-1}{(2\pi in)^2(m-2)}\Theta_{m-1} \\
&- \frac{1}{(2\pi in)(m-1)}\Theta_m + \frac{2(m-1)}{(2\pi in)^2(m-2)}\Phi_{m-1} + \frac{2}{2\pi in(m-1)}\Phi_m \\
&= \cdots \\
&= \frac{(m-1)!}{(2\pi in)^{m-2}}C_n^{(2)} - \sum_{j=1}^{m-2}\frac{(m-1)_{j-1}}{(2\pi in)^j}\Lambda_{m-j+1} - \sum_{j=1}^{m-2}\frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Theta_{m-j+1} \\
&+ \sum_{j=1}^{m-2}\frac{2(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Phi_{m-j+1} \\
&= -\frac{1}{2}\frac{(m-1)!}{(2\pi in)^{m-1}} - \frac{2(m-1)!}{(2\pi in)^m} - \sum_{j=1}^{m-2}\frac{(m-1)_{j-1}}{(2\pi in)^j}\Lambda_{m-j+1} \\
&- \sum_{j=1}^{m-2}\frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Theta_{m-j+1} + \sum_{j=1}^{m-2}\frac{2(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Phi_{m-j+1} \\
&= -\sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j}\Lambda_{m-j+1} - \sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Theta_{m-j+1} \\
&+ \sum_{j=1}^{m-1}\frac{2(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Phi_{m-j+1},
\end{aligned}
\tag{4.11}
$$

where

$$
\begin{aligned}
&C_n^{(2)} = \int_0^1 \gamma_2(x)e^{-2\pi inx}dx = \int_0^1 (x^2 - \frac{1}{2}x)e^{-2\pi inx}dx = -\frac{1}{2}\frac{1}{2\pi in} - \frac{2}{(2\pi in)^2} \\
&\Lambda_2 = \frac{1}{2},\ \Theta_2 = \frac{1}{2\pi in},\ \Phi_2 = \frac{1}{2\pi in}\times(-\frac{1}{2}).
\end{aligned}
\tag{4.12}
$$

Before proceeding further, we note the following.

$$\begin{aligned}
&\sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j}\Lambda_{m-j+1}\\
&=-\sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j}\sum_{k=1}^{m-j}\frac{E_k}{k(m-j-k+1)}\\
&=-\frac{1}{m}\sum_{j=1}^{m-1}\sum_{k=1}^{m-j}\frac{(m)_j}{(2\pi in)^j k(m-j-k+1)}E_k\\
&=-\frac{1}{m}\sum_{s=1}^{m-1}\sum_{k=1}^{m-s}\frac{(m)_s}{(2\pi in)^s k(m-s-k+1)}E_k\\
&=-\frac{1}{m}\sum_{s=1}^{m}\sum_{l=s}^{m-1}\frac{(m)_s E_{l-s+1}}{(2\pi in)^s(l-s+1)(m-l)}.
\end{aligned}\tag{4.13}$$

$$\begin{aligned}
&\sum_{j=1}^{m-1}\frac{2(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Phi_{m-j+1}\\
&=\sum_{j=1}^{m-1}\frac{2(m-1)_{j-1}}{(2\pi in)^j(m-j)}\sum_{k=1}^{m-j}\frac{(m-j)_{k-1}}{(2\pi in)^k}E_{m-j-k+1}\\
&=\frac{2}{m}\sum_{j=1}^{m-1}\sum_{k=1}^{m-j}\frac{(m)_{j+k-1}}{(2\pi in)^{j+k}(m-j)}E_{m-j-k+1}\\
&=\frac{2}{m}\sum_{j=1}^{m-1}\frac{1}{m-j}\sum_{s=j+1}^{m}\frac{(m)_{s-1}}{(2\pi in)^s}E_{m-s+1}\\
&=\frac{2}{m}\sum_{s=2}^{m}\frac{(m)_{s-1}}{(2\pi in)^s}E_{m-s+1}\sum_{j=1}^{s-1}\frac{1}{m-j}\\
&=\frac{2}{m}\sum_{s=2}^{m}\frac{(m)_{s-1}}{(2\pi in)^s}E_{m-s+1}\left(H_{m-1}-H_{m-s}\right)\\
&=\frac{2}{m}\sum_{s=1}^{m}\frac{(m)_s}{(2\pi in)^s}\frac{E_{m-s+1}}{m-s+1}\left(H_{m-1}-H_{m-s}\right).
\end{aligned}\tag{4.14}$$

$$\begin{aligned}
&\sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Theta_{m-j+1}\\
&=\frac{1}{m}\sum_{s=1}^{m}\frac{(m)_s}{(2\pi in)^s}\frac{H_{m-1}-H_{m-s}}{m-s+1}.
\end{aligned}\tag{4.15}$$

Putting everything together, we have

$$\begin{aligned} C_n^{(m)} &= \frac{1}{m}\sum_{s=1}^{m}\sum_{l=s}^{m-1}\frac{(m)_s E_{l-s+1}}{(2\pi i n)^s (l-s+1)(m-l)} \\ &- \frac{1}{m}\sum_{s=1}^{m}\frac{(m)_s}{(2\pi i n)^s}\frac{H_{m-1}-H_{m-s}}{m-s+1} + \frac{2}{m}\sum_{s=1}^{m}\frac{(m)_s}{(2\pi i n)^s}\frac{E_{m-s+1}}{m-s+1}\left(H_{m-1}-H_{m-s}\right) \\ &= -\frac{1}{m}\sum_{s=1}^{m}\left(\frac{(m)_s}{(2\pi i n)^s}\frac{H_{m-1}-H_{m-s}}{m-s+1}(1-2E_{m-s+1}) - \sum_{l=s}^{m-1}\frac{(m)_s E_{l-s+1}}{(2\pi i n)^s (l-s+1)(m-l)}\right). \end{aligned} \tag{4.16}$$

Case 2: $n = 0$.

$$C_0^{(m)} = \int_0^1 \gamma_m(x)dx = -\frac{1}{m}\left(\sum_{k=1}^{m}\frac{E_k}{k(m-k+1)} + \frac{1}{m(m+1)} - \frac{2}{m(m+1)}E_{m+1}\right). \tag{4.17}$$

Question: For what values of $m \geq 1$, does $\sum_{k=0}^{m} E_k = 2 - E_m$ hold ?

**Remark 4.1.** In a previous paper (see [3,4,6,12]), it was shown that

$$\int_0^1 \gamma_m(x)dx = \frac{1}{m(m^2-1)}\sum_{l=1}^{m-1}\sum_{j=1}^{m-l}\frac{(-1)^j\binom{m+1}{l+j}E_{l+j}}{\binom{m-2}{l-1}} + \frac{2(-1)^{m+1}E_{m+1}}{m(m^2-1)}\sum_{l=1}^{m-1}\frac{(-1)^l}{\binom{m-2}{l-1}}. \tag{4.18}$$

So, we obtain the following identity.

$$\begin{aligned} &-\frac{1}{m}\left(\sum_{k=1}^{m}\frac{E_k}{k(m-k+1)} + \frac{1}{m(m+1)} - \frac{2}{m(m+1)}E_{m+1}\right) \\ &= \frac{1}{m(m^2-1)}\sum_{l=1}^{m-1}\sum_{j=1}^{m-l}\frac{(-1)^j\binom{m+1}{l+j}E_{l+j}}{\binom{m-2}{l-1}} + \frac{2(-1)^{m+1}E_{m+1}}{m(m^2-1)}\sum_{l=1}^{m-1}\frac{(-1)^l}{\binom{m-2}{l-1}}, \end{aligned} \tag{4.19}$$

for $m \geq 2$.

$\gamma_m(< x >)$, $(m \geq 2)$ is piecewise $C^\infty$. Moreover, $\gamma_m(< x >)$ is continuous for those integers $m \geq 2$ with and $\Lambda_m = 0$, and discontinuous with jump discontinuities at integers for those integers $\geq 2$ with $\Lambda_m \neq 0$.

Assume first that $\Lambda_m = 0$. Then $\gamma_m(1) = \gamma_m(0)$. $\gamma_m(< x >)$ is piecewise $C^\infty$ and continuous. So the Fourier series of $\gamma_m(< x >)$ converges uniformly to $\gamma_m(< x >)$, and

$$
\begin{aligned}
&\gamma_m(< x >)\\
&= -\frac{1}{m}\left(\sum_{k=1}^{m}\frac{E_k}{k(m-k+1)}+\frac{1}{m(m+1)}-\frac{2}{m(m+1)}E_{m+1}\right)\\
&-\sum_{n=-\infty,n\neq 0}^{\infty}\Big(\frac{1}{m}\sum_{s=1}^{m}\Big(\frac{(m)_s}{(2\pi in)^s}\frac{H_{m-1}-H_{m-s}}{m-s+1}(1-2E_{m-s+1})\\
&-\sum_{l=s}^{m-1}\frac{(m)_s E_{l-s+1}}{(2\pi in)^s(l-s+1)(m-l)}\Big)\Big)e^{2\pi inx}\\
&= -\frac{1}{m}\left(\sum_{k=1}^{m}\frac{E_k}{k(m-k+1)}+\frac{1}{m(m+1)}-\frac{2}{m(m+1)}E_{m+1}\right)\\
&+\frac{1}{m}\sum_{s=1}^{m}\Big(\binom{m}{s}\frac{H_{m-1}-H_{m-s}}{m-s+1}(1-2E_{m-s+1})-\binom{m}{s}\sum_{l=s}^{m-1}\frac{E_{l-s+1}}{(l-s+1)(m-l)}\Big)\\
&\times\Big(-s!\sum_{n=-\infty,n\neq 0}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^s}\Big)\\
&= -\frac{1}{m}\left(\sum_{k=1}^{m}\frac{E_k}{k(m-k+1)}+\frac{1}{m(m+1)}-\frac{2}{m(m+1)}E_{m+1}\right)\\
&+\frac{1}{m}\sum_{s=2}^{m}\Big(\binom{m}{s}\frac{H_{m-1}-H_{m-s}}{m-s+1}(1-2E_{m-s+1})-\binom{m}{s}\sum_{l=s}^{m-1}\frac{E_{l-s+1}}{(l-s+1)(m-l)}\Big)B_s(< x >)\\
&+\Big(-\sum_{l=1}^{m-1}\frac{E_l}{l(m-l)}\Big)\times\begin{cases}B_1(< x >), & \text{for } x\notin\mathbb{Z},\\ 0, & \text{for } x\in\mathbb{Z},\end{cases}
\end{aligned}
\tag{4.20}
$$

where $H_m=\sum_{k=1}^{m}\frac{1}{k}$.

Now, we get the following theorem.

**Theorem 4.2.** *Let $m$ be an integer $\geq 2$, with*

$$\Lambda_m=-\sum_{k=1}^{m-1}\frac{E_k}{k(m-k)}=0.$$

*Then we have the following.*
(a) $\sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k(<x>)<x>^{m-k}$ *has the Fourier series expansion*

$$\begin{aligned}
&\sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k(<x>)<x>^{m-k}\\
&=-\frac{1}{m}\Big(\sum_{k=1}^{m}\frac{E_k}{k(m-k+1)}+\frac{1}{m(m+1)}-\frac{2}{m(m+1)}E_{m+1}\Big)\\
&-\sum_{n=-\infty,n\neq 0}^{\infty}\Big(\frac{1}{m}\sum_{s=1}^{m}\Big(\frac{(m)_s}{(2\pi in)^s}\frac{H_{m-1}-H_{m-s}}{m-s+1}(1-2E_{m-s+1})\\
&-\sum_{l=s}^{m-1}\frac{(m)_sE_{l-s+1}}{(2\pi in)^s(l-s+1)(m-l)}\Big)\Big)e^{2\pi inx},
\end{aligned}$$

*for all* $x\in(-\infty,\infty)$, *where the convergence is uniform.*

(b)

$$\begin{aligned}
&\sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k(<x>)<x>^{m-k}\\
&=-\frac{1}{m}\Big(\sum_{k=1}^{m}\frac{E_k}{k(m-k+1)}+\frac{1}{m(m+1)}-\frac{2}{m(m+1)}E_{m+1}\Big)\\
&+\frac{1}{m}\sum_{s=2}^{m}\Big(\binom{m}{s}\frac{H_{m-1}-H_{m-s}}{m-s+1}(1-2E_{m-s+1})-\binom{m}{s}\sum_{l=s}^{m-1}\frac{E_{l-s+1}}{(l-s+1)(m-l)}\Big)B_s(<x>),
\end{aligned}$$

*for all* $x\in(-\infty,\infty)$. *Here* $B_k(<x>)$ *is the Bernoulli function.*

Assume next that $m$ is an integer $\geq 2$ with $\Lambda_m\neq 0$. Then, $\gamma_m(1)\neq\gamma_m(0)$. Hence $\gamma_m(<x>)$ is piecewise $C^\infty$ and discontinuous with jump discontinuities at integers. Thus the Fourier series of $\gamma_m(<x>)$ converges pointwise to $\gamma_m(<x>)$, for $x\notin\mathbb{Z}$, and converges to

$$\frac{1}{2}(\gamma_m(0)+\gamma_m(1))=\gamma_m(0)+\frac{1}{2}\Lambda_m=-\frac{1}{2}\sum_{k=1}^{m-1}\frac{E_k}{k(m-k)},$$

for $x\in\mathbb{Z}$.

Hence we obtain the following theorem.

**Theorem 4.3.** *Let* $m$ *be an integer* $\geq 2$, *with*

$$\Lambda_m=-\sum_{k=1}^{m-1}\frac{E_k}{k(m-k)}\neq 0.$$

*Then, we have the following.*

(a)

$$
\begin{aligned}
&-\frac{1}{m}\Big(\sum_{k=1}^{m}\frac{E_k}{k(m-k+1)}+\frac{1}{m(m+1)}-\frac{2}{m(m+1)}E_{m+1}\Big)\\
&-\sum_{n=-\infty,n\neq 0}^{\infty}\Big(\frac{1}{m}\sum_{s=1}^{m}\Big(\frac{(m)_s}{(2\pi in)^s}\frac{H_{m-1}-H_{m-s}}{m-s+1}(1-2E_{m-s+1})\\
&-\sum_{l=s}^{m-1}\frac{(m)_s E_{l-s+1}}{(2\pi in)^s(l-s+1)(m-l)}\Big)\Big)e^{2\pi inx}\\
&=\begin{cases}\sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k(<x>)<x>^{m-k}, & \text{for } x\notin\mathbb{Z},\\ -\frac{1}{2}\sum_{k=1}^{m-1}\frac{E_k}{k(m-k)}, & \text{for } x\in\mathbb{Z}.\end{cases}
\end{aligned}
$$

*Here the convergence is pointwise.*

(b)

$$
\begin{aligned}
&-\frac{1}{m}\Big(\sum_{k=1}^{m}\frac{E_k}{k(m-k+1)}+\frac{1}{m(m+1)}-\frac{2}{m(m+1)}E_{m+1}\Big)\\
&+\frac{1}{m}\sum_{s=1}^{m}\Big(\binom{m}{s}\frac{H_{m-1}-H_{m-s}}{m-s+1}(1-2E_{m-s+1})-\binom{m}{s}\sum_{l=s}^{m-1}\frac{E_{l-s+1}}{(l-s+1)(m-l)}\Big)B_s(<x>)\\
&=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}E_k(<x>)<x>^{m-k},\ \text{for } x\notin\mathbb{Z},
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{m}\Big(\sum_{k=1}^{m}\frac{E_k}{k(m-k+1)}+\frac{1}{m(m+1)}-\frac{2}{m(m+1)}E_{m+1}\Big)\\
&+\frac{1}{m}\sum_{s=2}^{m}\Big(\binom{m}{s}\frac{H_{m-1}-H_{m-s}}{m-s+1}(1-2E_{m-s+1})-\binom{m}{s}\sum_{l=s}^{m-1}\frac{E_{l-s+1}}{(l-s+1)(m-l)}\Big)B_s(<x>)\\
&=-\frac{1}{2}\sum_{k=1}^{m-1}\frac{E_k}{k(m-k)},\ \text{for } x\in\mathbb{Z}.
\end{aligned}
$$

Question: For what values of $m\geq 2$, does $\sum_{k=1}^{m-1}\frac{E_k}{k(m-k)}=0$ hold ?

## References


1. A. Bayad, T. Kim, *Higher recurrences for Apostol-Bernoulli-Euler numbers*, Russ. J. Math. Phys., **16**(2012), no.1, 1-10.
2. A. Bayad, T. Kim, *Identities involving values of Bernstein, q-Bernoulli, and q-Euler polynomials*, Russian J. Math. Phys., **18**(2011), no. 2, 133-143.
3. D.S. Kim, D. V. Dolgy, T. Kim, S.-H. Rim *Some Formulae for the Product of Two Bernoulli and Euler Polynomials,* Abstr. Appl. Anal. 2012, Art. ID 784307.
4. D.S. Kim, T. Kim, *Some identities of higher order Euler polynomials arising from Euler basis,* Integral Transforms Spec. Funct., **24(9)** (2013), 734-738.
5. D.S. Kim, T. Kim, Y.H. Lee, *Some arithmetic properties of Bernoulli and Euler nembers,* Adv. Stud. Contemp. Math., **22**(2010), no.4, 467-480.
6. D.S. Kim, T. Kim, T. Mansour, *Euler basis and the product of several Bernoulli and Euler polynomials,* Adv. Stud. Contemp. Math., **24**(2014), no.4, 535-547.

7. T. Kim, *Note on the Euler numbers and polynomials,* Adv. Stud. Contemp. Math. (Kyungshang), **17**(2008), 131–136.
8. T. Kim, *On the weighted q-Euler numbers and q-Berstein polynomials* , Adv. Stud. Contemp. Math., **22**(2012), no.1, 7-12.
9. T. Kim, *Some identities for the Bernoulli, the Euler and Genocchi numbers and polynomials*, Adv. Stud. Contemp. Math., **20**(2015), no.1, 23-28.
10. T. Kim, *Euler numbers and polynomials associated with zeta functions,* Abstr. Appl. Anal. 2008, Art. ID 581582, 11 pp.
11. T. Kim, J. Choi, Y. H. Kim, *A note on the values of Euler zeta functions at positive integers,* Adv. Stud. Contemp. Math. (Kyungshang), **22**(2012), 27–34.
12. T. Kim, D. S. Kim, D. V. Dolgy, S.-H. Rim, *Some identities on the Euler numbers arising from Euler basis polynomials,* ARS Combinatoria **109**(2013), 433–446.
13. T. Kim, D. S. Kim, S.-H. Rim, D. V. Dolgy, *Fourier series of higher-order Bernoulli functions and their applications,* J. Inequal. Appl. 2016, to appear.
14. J. E. Marsden, *Elementary classical analysis,* W. H. Freeman and Company, 1974.
15. B.H. Yadav, *Absolute convergence of Fourier series,* Thesis (Ph.D.)-Maharaja Sayajirao University of Baroda (India), 1964.
16. Y. Simsek, *Interpolation functions of the Eulerian type polynomials and numbers*, Adv. Stud. Contemp. Math. (Kyungshang), **23**(2013), no. 2, 301-307.
17. D. G. Zill, M. R. Cullen, *Advanced Engineering Mathematics,* Jones and Bartlett Publishers 2006.



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# Higher order generalization of Bernstein type operators defined by $(p,q)$-integers

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## Abstract

In this paper, we introduce the higher order generalization of Bernstein type operators defined by $(p,q)$-integers. We establish some approximation results for these new operators by using the modulus of continuity.

*Keywords and phrases*: $(p,q)$-integers; $(p,q)$-Bernstein operators; modulus of continuity; approximation theorems.

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## 1. Introduction and preliminaries

In 1912, S.N Bernstein [4] introduced the following sequence of operators $B_n : C[0,1] \to C[0,1]$ defined for any $n \in \mathbb{N}$ and for any $f \in C[0,1]$ such as

$$B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \ x \in [0,1]. \tag{1.1}$$

In approximation theory, $q$-type generalization of Bernstein polynomials was introduced by Lupaş [7].

For $f \in C[0,1]$, the generalized Bernstein polynomial based on the $q$-integers is defined by Phillips [15] as follows

$$B_{n,q}(f;x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-k-1} (1-q^s x) \ f\left(\frac{[k]_q}{[n]_q}\right), \ x \in [0,1]. \tag{1.2}$$

Recently, Mursaleen et al. [10] applied $(p,q)$-calculus in approximation theory and introduced first $(p,q)$-analogue of Bernstein operators and defined as:

$$B_{n,p,q}(f;x) = \frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n} f\left(\frac{[k]}{p^{k-n}[n]}\right) P_{n,k}(p,q;x), \ 0 < q < p \leq 1, \ x \in [0,1]$$

where

$$P_{n,k}(p,q;x) = p^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \\ 1 \end{bmatrix}_{p,q} x^k \prod_{s=0}^{n-k-1} (p^s - q^s x). \tag{1.3}$$

They have also introduced and studied approximation properties based on $(p,q)$-integers given as: $(p,q)$-Bernstein-Stancu operators [11], $(p,q)$-Bernstein-Shurer operators [14] and $(p,q)$-Bleimann-Butzer-Hahn operators [13]. In the sequel, some more articles on $(p,q)$-approximation have also been appeared, e.g. [1], [2], [3], [6], [9], [12] and [13].

We recall some basic properties of $(p,q)$-integers.

The $(p,q)$-integer $[n]_{p,q}$ is defined by

$$[n]_{p,q} = \frac{p^n - q^n}{p-q},\ n = 0,1,2,\cdots,\ 0 < q < p \leq 1.$$

The $(p,q)$-Binomial expansion is

$$(x+y)_{p,q}^n := (x+y)(px+qy)(p^2x+q^2y)\cdots(p^{n-1}x+q^{n-1}y)$$

and the $(p,q)$-binomial coefficients are defined by

$$\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} := \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}.$$

For $p = 1$, all the notions of $(p,q)$-calculus are reduced to $q$-calculus. For details on $(p,q)$-calculus and $q$-calculus, one can refer [5, 7]. In this paper we use the notation $[n]$ in place of $[n]_{p,q}$.

In [5], $(p,q)$-derivative of a function $f(x)$ is defined by

$$D_{p,q}f(x) = \frac{f(px) - f(qx)}{(p-q)x},\ x \neq 0, \tag{1.4}$$

and the formulae for the $(p,q)$-derivative for the product of two functions is given as

$$D_{p,q}(fg)(x) = f(px).D_{p,q}g(x) + \{D_{p,q}f(x)\}.g(qx), \tag{1.5}$$

also

$$D_{p,q}(fg)(x) = f(qx).D_{p,q}g(x) + \{D_{p,q}f(x)\}.g(px). \tag{1.6}$$

Let $r \in \mathbb{N} \cup \{0\}$ be a fixed number. For $f \in C^r[0,1]$ and $m \in \mathbb{N}$, we define $r^{th}$ order $(p,q)$-Bernstein type operators as follows:

$$B_{n,p,q}^{[r]}(f;x) = \frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n} P_{n,k}(p,q;x) \sum_{i=0}^{r} \frac{1}{i!} f^{(i)}\left(\frac{[k]}{p^{k-n}[n]}\right)\left(x - \frac{[k]}{p^{k-n}[n]}\right)^i \tag{1.7}$$

In this paper, using the moment estimates from [8], we give the estimates of the central moments for these operators. We also study some approximation properties of an $r^{th}$ order generalization of the operators defined by (1.7) using the techniques of the work on the higher order generalization of $q$-analogue [16]. Further, we study approximation properties and prove Voronovskaja type theorem for these operators.

If we put $p = 1$, then we get the moments for $q$-Bernstein operators [8] and the usual generalization higher order $q$-Bernstein operators [16], respectively.

## 2. Main results

We have the following elementary result.

**Proposition 2.1.** *For $n \geq 1,\ 0 < q < p \leq 1$*

$$D_{p,q}(1+x)_{p,q}^{n} = [n](1+qx)_{p,q}^{n-1}. \tag{2.1}$$

*Proof.* By applying simple calculation on $(p,q)$-analogue, we have

$$(1+px)_{p,q}^{n} = p^{n-1}(1+px)(1+qx)_{p,q}^{n-1}, (1+qx)_{p,q}^{n} = (p^{n-1}+q^{n}x)(1+qx)_{p,q}^{n-1}. \tag{2.2}$$

Applying $(p,q)$-derivative and result (2.2) we get the desired result. □

**Lemma 2.2.** *Let $B_{n,p,q}(f;x)$ be given by (1.7). Then for any $m \in \mathbb{N},\ x \in [0,1]$ and $0 < q < p \leq 1$ we have*

$$\begin{aligned} B_{n,p,q}\left((t-x)_{p,q}^{m+1};x\right) &= \frac{p^{m+n}x(1-x)}{[n]} D_{p,q}\left\{ B_{n,p,q}\left((t-\frac{x}{p})_{p,q}^{m};\frac{x}{p}\right)\right\} \\ &+ \frac{p^{m+n-1}[m]x(1-x)}{[n]} B_{n,p,q}\left((t-\frac{qx}{p})_{p,q}^{m-1};\frac{qx}{p}\right) \\ &+ \frac{[m](p^{n}-q^{n})x}{[n]} B_{n,p,q}\left((t-x)_{p,q}^{m};x\right). \end{aligned}$$

*Proof.* First of all by using (1.5) and Proposition 2.1, we have

$$\begin{aligned} &D_{p,q}\left(\frac{1}{p^{\frac{n(n-1)}{2}}}\sum_{k=0}^{n}\left(t-\frac{x}{p}\right)_{p,q}^{m} P_{n,k}(p,q;\frac{x}{p})\right) \\ &= \frac{1}{p^{\frac{n(n-1)}{2}}}\left(\sum_{k=0}^{n}(t-x)_{p,q}^{m} D_{p,q}\{P_{n,k}(p,q;\frac{x}{p})\} - \frac{[m]}{p}\sum_{k=0}^{n}\left(t-\frac{qx}{p}\right)_{p,q}^{m-1} P_{n,k}(p,q;\frac{qx}{p})\right). \end{aligned} \tag{2.3}$$

Now in the same way by using (1.5) and Proposition 2.1, we have

$$\begin{aligned} &D_{p,q}\left\{P_{n,k}\left(p,q;\tfrac{x}{p}\right)\right\} = D_{p,q}\left\{p^{\frac{k(k-1)}{2}}[k]\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}\left(\tfrac{x}{p}\right)^{k}\left(1-\tfrac{x}{p}\right)^{k}\right\} \\ &= p^{\frac{k(k-1)}{2}}\left([k]\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}\frac{1}{p^{k}}x^{k-1}\left(1-\frac{qx}{p}\right)_{p,q}^{n-k} - [n-k]\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}\frac{1}{p}x^{k}\left(1-\frac{qx}{p}\right)_{p,q}^{n-k-1}\right). \end{aligned} \tag{2.4}$$

Now by a simple calculation, we have

$$\left(1-\frac{qx}{p}\right)_{p,q}^{n-k} = \frac{1}{p^{n-k}}(p-qx)_{p,q}^{n-k+1} = \frac{1}{p^{n-k}}\frac{1}{(1-x)}(p^{n-k}-q^{n-k}x)(1-x)_{p,q}^{n-k} \tag{2.5}$$

$$\left(1-\frac{qx}{p}\right)_{p,q}^{n-k-1}=\frac{1}{p^{n-k-1}}\frac{1}{(1-x)}(1-x)_{p,q}^{n-k}. \tag{2.6}$$

From (2.4),(2.5) and (2.6), we get

$$D_{p,q}\left\{P_{n,k}\left(p,q;\frac{x}{p}\right)\right\}=\frac{P_{n,k}(p,q;x)}{p^nx(1-x)}\left([k](p^{n-k}-q^{n-k}x)-p^k[n-k]x\right),$$

which implies that

$$D_{p,q}\left\{P_{n,k}\left(p,q;\frac{x}{p}\right)\right\}=\frac{P_{n,k}(p,q;x)}{p^nx(1-x)}\left(p^{n-k}[k]-[n]x\right). \tag{2.7}$$

From (2.3), (2.7), we have

$$\begin{aligned}
&D_{p,q}\left(\sum_{k=0}^{n}\left(t-\tfrac{x}{p}\right)_{p,q}^{m}P_{n,k}(p,q;\tfrac{x}{p})\right)\\
&= -\frac{1}{p^{\frac{n(n-1)}{2}}}\frac{[m]}{p}\sum_{k=0}^{n}\left(t-\frac{qx}{p}\right)_{p,q}^{m-1}P_{n,k}(p,q;\frac{qx}{p})\\
&+ \frac{1}{p^{\frac{n(n-1)}{2}}}\frac{1}{p^nx(1-x)}\sum_{k=0}^{n}(t-x)_{p,q}^{m}P_{n,k}(p,q;x)(p^{n-k}[k]-[n]x)\\
&= -\frac{1}{p^{\frac{n(n-1)}{2}}}\frac{[m]}{p}\sum_{k=0}^{n}\left(t-\frac{qx}{p}\right)_{p,q}^{m-1}P_{n,k}(p,q;\frac{qx}{p})\\
&+ \frac{1}{p^{\frac{n(n-1)}{2}}}\frac{1}{p^nx(1-x)}\sum_{k=0}^{n}(t-x)_{p,q}^{m}P_{n,k}(p,q;x)\\
&\times \left(\frac{[n]}{p^m}(p^mt-q^mx)-\frac{[n]}{p^m}(p^m-q^m)x\right).
\end{aligned}$$

Hence we have

$$\begin{aligned}
&D_{p,q}\left\{B_{n,p,q}\left((t-\tfrac{x}{p})_{p,q}^{m};\tfrac{x}{p}\right)\right\}\\
&= -\frac{[m]}{p}B_{n,p,q}\left((t-\frac{qx}{p})_{p,q}^{m-1};\frac{qx}{p}\right)+\frac{[n]}{p^{m+n}x(1-x)}B_{n,p,q}\left((t-x)_{p,q}^{m+1};x\right)\\
&- \frac{[m](p^n-q^n)}{p^{m+n}(1-x)}B_{n,p,q}\left((t-x)_{p,q}^{m};x\right).
\end{aligned}$$

This complete the proof of Lemma 2.2. □

**Lemma 2.3.** *Let $B_{n,p,q}\left((t-x)_{p,q}^{m};x\right)$ be a polynomial in $x$ of degree less than or equal to $m$ and the minimum degree of $\frac{1}{[n]}$ is $\lfloor\frac{m+1}{2}\rfloor$. Then for any fixed $m\in\mathbb{N}$ and $x\in[0,1],\ 0<q<p\leq 1$ we have*

$$B_{n,p,q}\left((t-x)_{p,q}^{m};x\right)=\frac{x(1-x)}{[n]^{\lfloor\frac{m+1}{2}\rfloor}}\sum_{k=0}^{m-2}b_{k,m,n}(p,q)x^k, \tag{2.8}$$

*such that the coefficients $b_{k,m,n}(p,q)$ satisfy $\mid b_{k,m,n}(p,q)\mid\leq b_m,\ k=1,2,\cdots,m-2$ and $b_m$ does not depend on $x,t,p,q$; where $\lfloor a\rfloor$ is an integer part of $a\geq 0$.*

*Proof.* Clearly by Lemma 2.2 it is true for $m = 2$. Assuming it is true for $m$, then from the recurrence of Lemma 2.2 and equation (2.8) we easily get

$$B_{n,p,q}\left((t-x)_{p,q}^{m+1};x\right)$$

$$= \frac{x(1-x)}{[n]^{\lfloor\frac{m+2}{2}\rfloor}}\sum_{k=0}^{m-1} b_{k,m+1,n}(p,q)x^k,$$

where

$$\begin{aligned} b_{k,m+1,n}(p,q) &= \frac{1}{[n]^\alpha}\left(p^{m+n-k}[k]+p^{m+n-k-1}q^k\right)b_{k,m,n}(p,q) \\ &- \frac{1}{[n]^\alpha}\left(p^{m+n+1-k}[k-1]+[2]p^{m+n-k-1}q^{k-1}\right)b_{k-1,m,n}(p,q) \\ &+ \frac{1}{[n]^\alpha}[m](p^n-q^n)b_{k-1,m,n}(p,q)+[m]p^{m+n-k-1}q^k b_{k-1,m-1,n}(p,q) \\ &- [m]p^{m+n-k-1}q^k b_{k-2,m-1,n}(p,q). \end{aligned}$$

Clearly

$$\alpha = 1+\lfloor\frac{m+1}{2}\rfloor-\lfloor\frac{m+2}{2}\rfloor,\ 0\le k\le m-1,$$

which lead us that either $\alpha = 0$ or $\alpha = 1$.
Since $\mid b_{k,m,n}(p,q)\mid \le b_m$, for $k = m-1$, clearly we have

$$\begin{aligned} \mid b_{k,m+1,n}(p,q)\mid &\le \frac{1}{[n]^\alpha}\left(p^{n+1}[m-1]+p^nq^{m-1}\right)b_m+\frac{1}{[n]^\alpha}\left(p^{n+2}[m-2]+[2]p^nq^{m-2}\right)b_m \\ &+ \frac{1}{[n]^\alpha}[m](p^n-q^n)b_m+[m]p^nq^{m-1}b_{m-1} \\ &+ [m]p^nq^{m-1}b_{m-1} \\ &= \frac{1}{[n]^\alpha}\left(p[m-1]+q^{m-1}\right)b_m+\frac{1}{[n]^\alpha}\left(p^2[m-2]+[2]q^{m-2}\right)b_m \\ &+ \frac{1}{[n]^\alpha}[m]b_m+[m]q^{m-1}b_{m-1}+[m]q^{m-1}b_{m-1} \\ &= b_{m+1},\ k=1,2,\cdots m-1, \end{aligned}$$

and $b_m$ does not depend on $x,t,p,q$. This complete the proof. □

*Remark* 2.4. From the Lemma 2.3 we have

$$B_{n,p,q}\left((t-x)_{p,q}^m;x\right) = x(1-x)Q_{m-2},\ B_{n,p,q}\left((t-x)_{p,q}^m;x\right)\Big|_{x=0,1} = 0, \qquad (2.9)$$

where $Q_{m-2}$ is a polynomial of highest degree $m-2$.

From the Lemma 2.2 and Lemma 2.3 we have the following theorem.

**Theorem 2.5.** *Let $m \in \mathbb{N}$ and $0 < q < p \leq 1$. Then there exits a constant $C_m > 0$ such that for any $x \in [0,1]$, we have*

$$| B_{n,p,q}\left((t-x)^m_{p,q};x\right) | \leq C_m \frac{x(1-x)}{[n]^{\lfloor \frac{m+1}{2} \rfloor}}.$$

**Lemma 2.6.** *For any fixed $m \in \mathbb{N}$ and $x \in [0,1],\ 0 < q < p \leq 1$ we have*

$$(t-x)^m = \sum_{k=1}^{m} \gamma_{m,k}(p-q)^{m-k}x^{m-k}(t-x)^k_{p,q} = \sum_{k=1}^{m} \gamma_{m,k}\left(\frac{p^n-q^n}{[n]}\right)^{m-k} x^{m-k}(t-x)^k_{p,q} \tag{2.10}$$

*where*

$$\gamma_{m,k} = \begin{cases} \frac{\gamma_{m-1,k-1}}{p^{k-1}} - \frac{[k]\gamma_{m-1,k}}{p^k},\ k = 1,\cdots,m-1, & \gamma_{m,0} = 0 \\ 1, & k = m, \end{cases}$$

*the coefficients $\gamma_{m,k}$ satisfy $| \gamma_{m,k} | \leq \gamma_m,\ k = 1,\cdots,m$ and $\gamma_m$ does not depend on $x,t,p,q$.*

*Proof.* Inductively, for $m = 1$, it is obvious. For $m \geq 1$ the relation (2.10) holds. For $k = 1,\cdots,m$, we have

$$(t-x)^{m+1} = \sum_{k=1}^{m} \gamma_{m,k}(p-q)^{m-k}x^{m-k}(t-x)^k_{p,q}(t-x)^m, \tag{2.11}$$

We can write

$$t - x = \frac{1}{p^k}\left(p^k t - q^k x - (p-q)[k]_{p,q}x\right) \tag{2.12}$$

(2.11),(2.12) imply that,

$$\begin{aligned}
(t-x)^{m+1} &= \sum_{k=1}^{m} \gamma_{m,k}(p-q)^{m-k}x^{m-k}(t-x)^{k+1}_{p,q}\frac{1}{p^k} \\
&- \sum_{k=1}^{m} \gamma_{m,k}(p-q)^{m-k}x^{m-k}(t-x)^k_{p,q}\frac{1}{p^k}(p-q)[k]_{p,q}x \\
&= \frac{\gamma_{m,m}(t-x)^{m+1}_{p,q}}{p^m} + \sum_{k=2}^{m} \frac{1}{p^{k-1}}\gamma_{m,k-1}(p-q)^{m+1-k}x^{m+1-k}(t-x)^k_{p,q} \\
&- \frac{\gamma_{m,1}(p-q)^m x^m(t-x)}{p} - \sum_{k=2}^{m} \frac{1}{p^k}[k]\gamma_{m,k}(p-q)^{m+1-k}x^{m+1-k}(t-x)^k_{p,q} \\
&= \frac{\gamma_{m,m}(t-x)^{m+1}_{p,q}}{p^m} - \frac{\gamma_{m,1}(p-q)^m x^m(t-x)}{p} \\
&+ \sum_{k=2}^{m}\left(\frac{\gamma_{m,k-1}}{p^{k-1}} - \frac{[k]\gamma_{m,k}}{p^k}\right)(p-q)^{m+1-k}x^{m+1-k}(t-x)^k_{p,q} \\
&= \sum_{k=1}^{m+1} \gamma_{m+1,k}(p-q)^{m+1-k}x^{m+1-k}(t-x)^k_{p,q},
\end{aligned}$$

where

$$\gamma_{m+1,k} = \begin{cases} \frac{\gamma_{m,k-1}}{p^{k-1}} - \frac{[k]\gamma_{m,k}}{p^k}, \ k = 1, \cdots, m, & \gamma_{m,0} = 0 \\ \gamma_{m,m} = 1, & k = m+1. \end{cases}$$

□

**Theorem 2.7.** *Let $m \in \mathbb{N}$ and $0 < q < p \leq 1$. Then there exits a constant $E_m > 0$ such that for any $x \in [0,1]$, we have*

$$\mid B_{n,p,q}\left((t-x)^m; x\right) \mid \leq E_m \frac{x(1-x)}{[n]^{\lfloor \frac{m+1}{2} \rfloor}}.$$

*Proof.* From Lemma 2.6 we have

$$\begin{aligned} \mid B_{n,p,q}\left((t-x)^m, x\right) \mid &\leq \sum_{k=1}^{m} \mid \gamma_{m,k} \mid \left(\frac{p^n - q^n}{[n]}\right)^{m-k} \mid B_{n,p,q}\left((t-x)^k_{p,q}, x\right) \mid \\ &\leq \gamma_m \left( \mid B_{n,p,q}\left((t-x)^m_{p,q}, x\right) \mid + \sum_{k=1}^{m-1} \frac{1}{[n]^{m-k}} \mid B_{n,p,q}\left((t-x)^k_{p,q}, x\right) \mid \right) \end{aligned}$$

By using Theorem 2.5 we have

$$\begin{aligned} \mid B_{n,p,q}\left((t-x)^m, x\right) \mid &\leq \gamma_m \left( \mid B_{n,p,q}\left((t-x)^m_{p,q}, x\right) \mid + \sum_{k=1}^{m-1} \frac{1}{[n]^{m-k}} C_k \frac{x(1-x)}{[n]^{\lfloor \frac{k+1}{2} \rfloor}} \right) \\ &\leq \gamma_m \left( \mid B_{n,p,q}\left((t-x)^m_{p,q}, x\right) \mid + \frac{x(1-x)}{[n]^{1+\lfloor \frac{m}{2} \rfloor}} \sum_{k=1}^{m-1} C_k \right) \\ &\leq \gamma_m \left( C_m \frac{x(1-x)}{[n]^{\lfloor \frac{m+1}{2} \rfloor}} + \frac{x(1-x)}{[n]^{\lfloor \frac{m+1}{2} \rfloor}} \sum_{k=1}^{m-1} C_k \right) \\ &\leq \left( \gamma_m C_m + \sum_{k=1}^{m-1} C_k \right) \frac{x(1-x)}{[n]^{\lfloor \frac{m+1}{2} \rfloor}} \\ &= E_m \frac{x(1-x)}{[n]^{\lfloor \frac{m+1}{2} \rfloor}} \end{aligned}$$

□

**Corollary 2.8.** *Let $m \in \mathbb{N}$ and $0 < q < p \leq 1$. Then there exits a constant $K_m > 0$ such that for any $x \in [0,1]$, we have*

$$B_{n,p,q}\left((\mid t-x \mid)^m; x\right) \leq K_m \frac{x(1-x)}{[n]^{\frac{m}{2}}}. \tag{2.13}$$

*Proof.* For an even $m$, clearly we have

$$\begin{aligned} B_{n,p,q}\left((\mid t-x\mid)^m;x\right) &= B_{n,p,q}\left((t-x)^m;x\right) \\ &\leq E_m\frac{x(1-x)}{[n]^{\lfloor\frac{m+1}{2}\rfloor}} \\ &= K_m\frac{x(1-x)}{[n]^{\frac{m}{2}}} \end{aligned}$$

In case if $m$ is odd, say $m=2u+1$, we have
$B_{n,p,q}\left((\mid t-x\mid)^{2u+1};x\right)$

$$\begin{aligned} &\leq \sqrt{B_{n,p,q}\left((\mid t-x\mid)^{4u}_{p,q};x\right)}\sqrt{B_{n,p,q}\left((\mid t-x\mid)^2;x\right)} \\ &\leq \sqrt{E_{4u}\frac{x(1-x)}{[n]^{\lfloor\frac{4u+1}{2}\rfloor}}}\sqrt{E_2\frac{x(1-x)}{[n]^{\lfloor\frac{3}{2}\rfloor}}} \\ &= \sqrt{E_{4u}\frac{x(1-x)}{[n]^{\frac{2u}{2}}}}\sqrt{E_2\frac{x(1-x)}{[n]}} \\ &= K_{2u+1}\frac{x(1-x)}{[n]^{\frac{2u+1}{2}}}. \end{aligned}$$

This complete the proof. □

**Theorem 2.9.** *Let $B^{[r]}_{n,p,q}(f;x)$ be an operator from $C^r[0,1]\to C^r[0,1]$. Then for $0<q<p\leq 1$ there exits a constant $M(r)$ such that for every $f\in C^r[0,1]$, we have*

$$\parallel B^{[r]}_{n,p,q}(f;x)\parallel_{C[0,1]}\leq M(r)\sum_{i=0}^{r}\parallel f^{(i)}\parallel = M(r)\parallel f\parallel_{C^r[0,1]}. \tag{2.14}$$

*Proof.* Clearly $B^{[r]}_{n,p,q}(f;x)$ is continuous on $[0,1]$. From (1.7) we have

$$B^{[r]}_{n,p,q}(f;x) = \sum_{i=0}^{r}\frac{(-1)^i}{i!}B_{n,p,q}\left((t-x)^i f^{(i)}(t);x\right).$$

From the Corollary 2.8, we have

$$\begin{aligned} \mid B_{n,p,q}\left((t-x)^i f^{(i)}(t);x\right)\mid &\leq \parallel f^{(i)}\parallel B_{n,p,q}\left(\mid(t-x)\mid^i;x\right) \\ &\leq K_i\parallel f^{(i)}\parallel [n]^{-\frac{i}{2}}. \end{aligned}$$

Therefore

$$\begin{aligned} \parallel B^{[r]}_{n,p,q}(f;x)\parallel &\leq \sum_{i=0}^{r}\frac{(-1)^i}{i!}\parallel B_{n,p,q}\left((t-x)^i f^{(i)}(t);x\right)\parallel \\ &\leq M(r)\sum_{i=0}^{r}\parallel f^{(i)}\parallel. \end{aligned}$$

This complete the proof. □

## 3. Convergence properties of $B^{[r]}_{n,p,q}(f;x)$

The modulus of continuity of the derivative $f^{(r)}$ is given by

$$\omega\left(f^{(r)};t\right)=\sup\left\{\mid f^{(r)}(x)-f^{(r)}(y)\mid:\mid x-y\mid\leq t,\ x,y\in[0,1]\right\}. \tag{3.1}$$

**Theorem 3.1.** *Let $0<q<p\leq 1$ and $r\in\mathbb{N}\cup\{0\}$ be a fixed number. Then for $x\in[0,1],\ n\in\mathbb{N}$ there exits $D_r>0$ such that for every $f\in C^r[0,1]$ the following inequality holds*

$$\mid B^{[r]}_{n,p,q}(f;x)-f(x)\mid\leq D_r\frac{1}{[n]^{\frac{r}{2}}}\omega\left(f^{(r)};\frac{1}{\sqrt{[n]}}\right). \tag{3.2}$$

*Proof.* Let $r\in\mathbb{N}$. Then for $f\in C^r[0,1]$ at a given point $t\in[0,1]$, we have from the Taylor formula that

$$\begin{aligned} f(x) &= \sum_{i=0}^{r}\frac{f^{(i)}(t)}{i!}(x-t)^i+\frac{(x-t)^r}{((r-1)!)} \\ &\times \int_0^1(1-u)^{r-1}\left(f^{(r)}(t+u(x-t))-f^{(r)}(t)\right)\mathrm{d}u. \end{aligned}$$

On applying $B^{[r]}_{n,p,q}(f;x)$, we get

$$f(x)-B^{[r]}_{n,p,q}(f;x)=\sum_{k=0}^{n}\frac{(x-\frac{[k]}{p^{k-n}[n]})^r}{(r-1)!}\int_0^1(1-u)^{r-1}P_{n,k}(p,q;x)$$

$$\times\left[f^{(r)}\left(\frac{[k]}{p^{k-n}[n]}+u\left(x-\frac{[k]}{p^{k-n}[n]}\right)\right)-f^{(r)}\left(\frac{[k]}{p^{k-n}[n]}\right)\right]\mathrm{d}u. \tag{3.3}$$

Now from the definition and properties of modulus of continuity, we have

$$\left|f^{(r)}\left(\frac{[k]}{p^{k-n}[n]}+u\left(x-\frac{[k]}{p^{k-n}[n]}\right)\right)-f^{(r)}\left(\frac{[k]}{p^{k-n}[n]}\right)\right|\leq\omega\left(f^{(r)};u\left|x-\frac{[k]}{p^{k-n}[n]}\right|\right)$$

$$\omega\left(f^{(r)};u\left|x-\frac{[k]}{p^{k-n}[n]}\right|\right)\leq\left(\sqrt{[n]}\left|x-\frac{[k]}{p^{k-n}[n]}\right|+1\right)\omega\left(f^{(r)};\frac{1}{\sqrt{[n]}}\right). \tag{3.4}$$

□

Now for every $0\leq x\leq 1,\ 0<q<p\leq 1,\ k\in\mathbb{N}\cup\{0\},\ n\in\mathbb{N}$ and from (3.3) and (3.4), we get

$$\mid B^{[r]}_{n,p,q}(f;x)-f(x)\mid$$

$$\leq\frac{1}{r!}\omega\left(f^{(r)};\frac{1}{\sqrt{[n]}}\right)\sum_{k=0}^{n}\left|x-\frac{[k]}{p^{k-n}[n]}\right|^r\left(\sqrt{[n]}\left|x-\frac{[k]}{p^{k-n}[n]}\right|+1\right)P_{n,k}(p,q;x)$$

$$= \frac{1}{r!}\omega\left(f^{(r)};\frac{1}{\sqrt{[n]}}\right)\left(\sqrt{[n]}B_{n,p,q}\left(|\ x-t\ |^{r+1};x\right)+B_{n,p,q}\left(|\ x-t\ |^{r};x\right)\right). \quad (3.5)$$

Using (3.9) and (3.5) for $x\in[0,1]$, we have

$$\begin{aligned}|\ B^{[r]}_{n,p,q}(f;x)-f(x)\ | &\leq \frac{1}{r!}(K_{r+1}+K_r)\left(\frac{1}{\sqrt{[n]}}\right)^r\omega\left(f^{(r)};\frac{1}{\sqrt{[n]}}\right)\\ &= D_r\left(\frac{1}{\sqrt{[n]}}\right)^r\omega\left(f^{(r)};\frac{1}{\sqrt{[n]}}\right).\end{aligned}$$

In order to obtain the uniform convergence of $B^{[r]}_{n,p_n,q_n}(f;x)$ to a continuous function $f$, we take $q=q_n,\ p=p_n$ where $q_n\in(0,1)$ and $p_n\in(q_n,1]$ satisfying,

$$\lim_n p_n=1,\ \lim_n q_n=1. \quad (3.6)$$

**Corollary 3.2.** *Let $p=p_n,\ q=q_n,\ 0<q_n<p_n\leq 1$ satisfy (3.6) and $f\in C^r[0,1]$ for a fixed number $r\in\mathbb{N}\cup\{0\}$. Then*

$$\lim_{n\to\infty}[n]^{\frac{r}{2}}\ \|\ B^{[r]}_{n,k}(f)-f\ \|=0. \quad (3.7)$$

We say that (cf. [16]) a function $f\in C[0,1]$ belongs to $Lip_M(\alpha)$, $0<\alpha\leq 1$, provided

$$|\ f(x)-f(y)\ |\leq M\ |\ x-y\ |^{\alpha},\ (x,y\in[0,1]\text{ and }M>0). \quad (3.8)$$

**Corollary 3.3.** *Let $p=p_n,\ q=q_n,\ 0<q_n<p_n\leq 1$ satisfy (3.6) and $f\in C^r[0,1]$ for a fixed number $r\in\mathbb{N}\cup\{0\}$. If $f^{(r)}\in Lip_M(\alpha)$ then*

$$\|\ B^{[r]}_{n,p,q}(f)-f\ \|=O\left([n]^{-\frac{r+\alpha}{2}}\right). \quad (3.9)$$

*Proof.* From (3.2) and (3.8), we have

$$\|\ B^{[r]}_{n,p,q}(f)-f\ \|\leq D_rM\frac{1}{[n]^{\frac{r}{2}}}\frac{1}{[n]^{\frac{\alpha}{2}}}.$$

□

**Theorem 3.4.** *Let $0<q<p\leq 1$. Suppose that $f\in C^{r+2}[0,1]$, where $r\in\mathbb{N}\cup\{0\}$ is fixed then we have*

$$\begin{aligned}&\left|B^{[r]}_{n,p,q}(f;x)-f(x)-\frac{(-1)^rf^{(r+1)}(x)B_{n,p,q}\left((t-x)^{r+1};x\right)}{(r+1)!}\right.\\ -\ &\left.\frac{(-1)^rf^{(r+2)}(x)B_{n,p,q}\left((t-x)^{r+2};x\right)}{(r+2)!}\right|\\ \leq\ &(K_{r+2}+K_{r+4})\frac{x(1-x)}{[n]^{\frac{r}{2}+1}}\sum_{i=0}^{r}\frac{1}{i!(r+2-i)!}\omega\left(f^{(r+2-i)},[n]^{-\frac{1}{2}}\right).\end{aligned}$$

*Proof.* Let $f \in C^{r+2}[0,1]$ and $x \in [0,1]$ for a fixed number $r \in \mathbb{N} \cup \{0\}$ we have $f^{(i)} \in C^{r+2-i}[0,1],\ 0 \le i \le r$. Then by Taylor formula we can write

$$f^{(i)}(t) = \sum_{i=0}^{r+2-i} \frac{f^{(i+j)}(x)}{j!}(t-x)^j + R_{r+2-j}(f;t;x), \tag{3.10}$$

where

$$R_{r+2-i}(f;t;x) = \frac{f^{(r+2-i)}(\zeta_{p^{n-k-1}t}) - f^{(r+2-i)}(x)}{(r+2-i)!}(t-x)^{r+2-i},$$

and

$$| \zeta_t - x | < | t - x | .$$

Therefore from (1.7) and (3.10) we have

$$\begin{aligned} B^{[r]}_{n,p,q}(f;x) &= \sum_{k=0}^{n} P_{n,k}(p,q;x) \sum_{i=0}^{r} \frac{\left(x - \frac{[k]}{p^{k-n}[n]}\right)^i}{i!} \sum_{j=0}^{r+2-i} \frac{f^{(i+j)}(x)}{j!} \left(\frac{[k]}{p^{k-n}[n]} - x\right)^j \\ &+ \sum_{k=0}^{n} P_{n,k}(p,q;x) \sum_{i=0}^{r} \frac{\left(x - \frac{[k]}{p^{k-n}[n]}\right)^i}{i!} R_{r+2-i}(f;t;x) \\ &= I_1 + I_2, \ \{\text{where } t = \frac{[k]}{p^{k-n}[n]}\} \end{aligned}$$

Which implies that

$$\begin{aligned} &| B^{[r]}_{n,p,q}(f;x) - I_1 | \\ &\quad = | I_2 | \\ &\quad = \left| \sum_{k=0}^{n} P_{n,k}(p,q;x) \sum_{i=0}^{r} \frac{(-1)^i}{i!} \frac{f^{(r+2-i)}(\zeta_t) - f^{(r+2-i)}(x)}{(r+2-i)!}(t-x)^{r+2} \right| \\ &\quad = \left| B_{n,p,q}\left( \sum_{i=0}^{r} \frac{(-1)^i}{i!} \frac{f^{(r+2-i)}(\zeta_t) - f^{(r+2-i)}(x)}{(r+2-i)!}(t-x)^{r+2}; x \right) \right|. \end{aligned}$$

We use the well-known inequality

$$\omega(f, \lambda\delta) \le (1+\lambda^2)\omega(f,\delta),$$

$$\begin{aligned} | f^{(r+2-i)}(\zeta_t) - f^{(r+2-i)}(x) | &\le \omega\left(f^{(r+2-i)}, | \zeta_t - x |\right) \\ &\le \omega\left(f^{(r+2-i)}, | t - x |\right) \\ &\le \omega\left(f^{(r+2-i)}, [n]^{-\frac{1}{2}}\right)\left(1 + [n](t-x)^2\right). \end{aligned}$$

Hence

$$\begin{aligned} &| I_2 | \le \left| B_{n,p,q}\left( \sum_{i=0}^{r} \frac{(-1)^i}{i!} \frac{f^{(r+2-i)}(\zeta_t) - f^{(r+2-i)}(x)}{(r+2-i)!} \right| \ | t-x |^{r+2}; x \right) \\ &\le B_{n,p,q}\left( \sum_{i=0}^{r} \frac{1}{i!(r+2-i)!} \omega\left(f^{(r+2-i)}, [n]^{-\frac{1}{2}}\right)\left(1 + [n](t-x)^2\right) | t-x |^{r+2}; x \right) \end{aligned}$$

$= \sum_{i=0}^{r} \frac{1}{i!(r+2-i)!}\omega\left(f^{(r+2-i)}, [n]^{-\frac{1}{2}}\right)$
$\times (B_{n,p,q}(\mid t-x \mid^{r+2}; x) + [n]B_{n,p,q}(\mid t-x \mid^{r+4}; x))$
$\leq \sum_{i=0}^{r} \frac{1}{i!(r+2-i)!}\omega\left(f^{(r+2-i)}, [n]^{-\frac{1}{2}}\right)\left(K_{r+2}\frac{x(1-x)}{[n]^{\frac{r}{2}+1}} + K_{r+4}\frac{x(1-x)}{[n]^{\frac{r}{2}+1}}\right)$
$= (K_{r+2}+K_{r+4})\frac{x(1-x)}{[n]^{\frac{r}{2}+1}} \sum_{i=0}^{r} \frac{1}{i!(r+2-i)!}\omega\left(f^{(r+2-i)}, [n]^{-\frac{1}{2}}\right).$
Therefore

$$\mid B_{n,p,q}^{[r]}(f;x) - I_1 \mid \leq (K_{r+2}+K_{r+4})\frac{x(1-x)}{[n]^{\frac{r}{2}+1}} \sum_{i=0}^{r} \frac{1}{i!(r+2-i)!}\omega\left(f^{(r+2-i)}, [n]^{-\frac{1}{2}}\right).$$

Now we simplify for $I_1$

$$\begin{aligned}
I_1 &= \sum_{k=0}^{n} P_{n,k}(p,q;x) \sum_{i=0}^{r} \frac{(x-\frac{[k]}{p^{k-n}[n]})^i}{i!} \sum_{l=i}^{r+2} \frac{f^{(l)}(x)}{(l-i)!}\left(\frac{[k]}{p^{k-n}[n]} - x\right)^{l-i} \\
&= \sum_{k=0}^{n} P_{n,k}(p,q;x) \sum_{i=0}^{r} \frac{(-1)^i}{i!} \sum_{l=i}^{r} \frac{f^{(l)}(x)}{(l-i)!}\left(\frac{[k]}{p^{k-n}[n]} - x\right)^{l} \\
&+ \sum_{k=0}^{n} P_{n,k}(p,q;x) \sum_{i=0}^{r} \frac{(-1)^i}{i!} \frac{f^{(r+1)}(x)}{(r+1-i)!}\left(\frac{[k]}{p^{k-n}[n]} - x\right)^{r+1} \\
&+ \sum_{k=0}^{n} P_{n,k}(p,q;x) \sum_{i=0}^{r} \frac{(-1)^i}{i!} \frac{f^{(r+2)}(x)}{(r+2-i)!}\left(\frac{[k]}{p^{k-n}[n]} - x\right)^{r+2} \\
&= \sum_{k=0}^{n} P_{n,k}(p,q;x) \sum_{l=0}^{r} \frac{f^{(l)}(x)}{(l)!}\left(\frac{[k]}{p^{k-n}[n]} - x\right)^{l} \sum_{i=0}^{l} \binom{l}{i}(-1)^i \\
&+ \frac{f^{(r+1)}(x)}{(r+1)!} \sum_{k=0}^{n} P_{n,k}(p,q;x)\left(\frac{[k]}{p^{k-n}[n]} - x\right)^{r+1} \sum_{i=0}^{r} \binom{r+1}{i}(-1)^i \\
&+ \frac{f^{(r+2)}(x)}{(r+2)!} \sum_{k=0}^{n} P_{n,k}(p,q;x)\left(\frac{[k]}{p^{k-n}[n]} - x\right)^{r+2} \sum_{i=0}^{r} \binom{r+2}{i}(-1)^i.
\end{aligned}$$

For $n \in \mathbb{N},\ r \in \mathbb{N} \cup \{0\}$ we have

$$\sum_{i=0}^{r} \binom{r+1}{i}(-1)^i = (-1)^r,\ \sum_{i=0}^{r} \binom{r+2}{i}(-1)^i = (r+1)(-1)^r$$

Therefore

$$\begin{aligned}
I_1 &= f(x) + \frac{(-1)^r f^{(r+1)}(x) B_{n,p,q}\left((t-x)^{r+1}; x\right)}{(r+1)!} \\
&+ \frac{(-1)^r f^{(r+2)}(x) B_{n,p,q}\left((t-x)^{r+2}; x\right)}{(r+2)!}.
\end{aligned}$$

This complete the proof. □

**Corollary 3.5.** *Let* $p = p_n,\ q = q_n,\ 0 < q_n < p_n \leq 1$ *satisfy* (3.6) *and* $f \in C^2[0,1]$ *for a fixed number* $r \in \mathbb{N} \cup \{0\}$. *Then for every* $x \in [0,1]$ *we have*

$$\left| B^{[r]}_{n,p_n,q_n}(f;x) - f(x) - \frac{f''(x)}{2}\frac{x(1-x)}{[n]} \right| \leq K\frac{x(1-x)}{[n]}\omega\left(f'', [n]^{-\frac{1}{2}}\right),$$

*where* $K = \frac{K_2+K_4}{2}$. *Moreover,*

$$\lim_{n\to\infty}[n]\left(B_{n,p_n,q_n}(f;x) - f(x)\right) = \frac{x(1-x)}{2}f''(x)$$

*uniformly on* $[0,1]$.

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## References

[1] T. Acar, $(p,q)$-generalization of Szász–Mirakyan operators, Math. Meth. Appl. Sci., 39(10)(2016) 2685–2695.

[2] T. Acar, A. Aral1 and S. A. Mohiuddine, On Kantorovich modification of $(p,q)$-Baskakov operators, J. Ineq. Appl., 2016 (2016): 98.

[3] T. Acar, A. Aral and S. A. Mohiuddine, Approximation by bivariate $(p,q)$-Bernstein-Kantorovich operators, Iran. J. Sci. Technol. Trans. A Sci., DOI: 10.1007/s40995-016-0045-4.

[4] S.N. Bernstein, Démostration du théorèeme de Weierstrass fondée sur le calcul de probabilités, Comm. Soc. Math. Kharkow (2), 13 (1912/1913) 1-2.

[5] M.N. Hounkonnou, J. Désiré and B. Kyemba, $\mathcal{R}(p,q)$-calculus: differentiation and integration, SUT Jour. Math., 49(2) (2013) 145-167.

[6] Khalid Khan and D.K. Lobiyal, Bèzier curves based on Lupaş $(p,q)$-analogue of Bernstein functions, Jour. Comput. Appl. Math., DOI: 10.1016/j.cam.2016.12.016.

[7] A. Lupaş, A $q$-analogue of the Bernstein operator, Seminar on Numerical and Statistical Calculus, University of Cluj-Napoca, 9(1987) 85-92.

[8] N. Mahmudov, The moments for $q$-Bernstein operators in the case $0 < q < 1$. Numer Algor (2010) 53:439–450, DOI 10.1007/s11075-009-9312-1.

[9] M. Mursaleen, A. Alotaibi and K.J. Ansari, On a Kantorovich variant of $(p,q)$-Szász-Mirakjan operators, Jour. Function Spaces, Volume 2016, Article ID 1035253, 9 pages.

[10] M. Mursaleen, K.J. Ansari and A. Khan, On $(p,q)$-analogue of Bernstein operators, Appl. Math. Comput., 266(2015), 874-882 [Erratum: Appl. Math. Comput., 278 (2016) 70–71].

[11] M. Mursaleen, K.J. Ansari and A. Khan, Some approximation results by $(p,q)$-analogue of Bernstein-Stancu operators, Appl. Math. Comput., 264 (2015) 392-402 [Corrigendum: Appl. Math. Comput, 269 (2015) 744–746].

[12] M. Mursaleen, F. Khan and A. Khan, Approximation by $(p,q)$-Lorentz polynomials on a compact disk, Complex Analysis and Operator Theory, 10 (2016) 1725–1740.

[13] M. Mursaleen, Md. Nasiruzzaman, Asif Khan and K.J. Ansari, Some approximation results on Bleimann-Butzer-Hahn operators defined by $(p,q)$-integers, Filomat, 30:3 (2016) 639–648.

[14] M. Mursaleen, Md. Nasiuzzaman and Ashirbayev Nurgali, Some approximation results on Bernstein-Schurer operators defined by $(p,q)$-integers, Jou. Ineq. Appl., 2015 (2015): 249.

[15] G. M. Phillips, Bernstein polynomials based on the $q$-integers, Ann. Numer. Math., 4 (1997), 511–518.

[16] P. Sabancıgil, Higher order generalization of $q$-Bernstein operators, Jour. Comput. Analy. Appl., 12 (2010) 821-827.

# FOURIER SERIES OF FUNCTIONS INVOLVING GENOCCHI POLYNOMIALS

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ABSTRACT. We consider three types of functions involving Genocchi polynomials and derive their Fourier series expansions. In addition, we express each of them in terms of Bernoulli functions.

## 1. Introduction

Let $G_m(x)$ be the Genocchi polynomials given by the generating function

$$\frac{2t}{e^t+1}e^{xt}=\sum_{m=0}^{\infty}G_m(x)\frac{t^m}{m!},\quad (\text{see } [1,2,12-17,21]). \tag{1.1}$$

The first few Genocchi polynomials are as follows:

$$\begin{aligned}&G_0(x)=0,\ G_1(x)=1,\ G_2(x)=2x-1,\\ &G_3(x)=3x^2-3x,\ G_4(x)=4x^3-6x^2+1,\\ &G_5(x)=5x^4-10x^3+5x,\ G_6(x)=6x^5-15x^4+15x^2-3,\\ &G_7(x)=7x^6-21x^5+35x^3-21x.\end{aligned} \tag{1.2}$$

From the relation $G_m(x)=mE_{m-1}(x)(m\geq 1)$, we have

$$\begin{aligned}&\deg G_m(x)=m-1\ (m\geq 1),\ G_m=mE_{m-1}\ (m\geq 1),\\ &G_0=0,\ G_1=1,\ G_{2m+1}=0\ (m\geq 1),\ \text{and } G_{2m}\neq 0\ (m\geq 1).\end{aligned} \tag{1.3}$$

Moreover, we have

$$\begin{aligned}&\frac{d}{dx}G_m(x)=mG_{m-1}(x)\ (m\geq 1),\\ &G_m(x+1)+G_m(x)=2mx^{m-1}\ (m\geq 0).\end{aligned} \tag{1.4}$$

From these, we have

$$G_m(1)+G_m(0)=2\delta_{m,1},\quad (m\geq 0). \tag{1.5}$$

$$\begin{aligned}\int_0^1 G_m(x)dx&=\frac{1}{m+1}(G_{m+1}(1)-G_{m+1}(0))\\ &=\frac{2}{m+1}(-G_{m+1}(0)+\delta_{m,0})\\ &=\begin{cases}0, & \text{if}\quad m \text{ is even},\\ -\frac{2}{m+1}G_{m+1}, & \text{if}\quad m \text{ is odd}.\end{cases}\end{aligned} \tag{1.6}$$

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For any real number $x$, let $< x >= x - [x] \in [0,1)$ denote the fractional part of $x$. In this paper, we will study the Fourier series of the following three types of functions involving Genocchi polynomials $G_m(< x >)$.

(1) $\alpha_m(< x >) = \sum_{k=1}^{m} G_k(< x >) < x >^{m-k}, (m \geq 2)$;
(2) $\beta_m(< x >) = \sum_{k=1}^{m} \frac{1}{k!(m-k)!} G_k(< x >) < x >^{m-k}, (m \geq 2)$;
(3) $\gamma_m(< x >) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} G_k(< x >) < x >^{m-k}, (m \geq 2)$.

The reader may refer to any book (for example, see [6,18,22]) for elementary facts about Fourier analysis.

As to $\gamma_m(< x >)$, we note that the polynomial identity (1.7) follows immediately from Theorems 4.1 and 4.2, which can be derived in turn from the Fourier series expansion of $\gamma_m(< x >)$.

$$\begin{aligned}
&\sum_{k=1}^{m-1} \frac{1}{k(m-k)} G_k(x) x^{m-k} \\
&= -\frac{1}{m} \left( \sum_{k=1}^{m} \frac{G_k}{k(m-k+1)} - \frac{2}{m} - \frac{2}{m(m+1)} G_{m+1} \right) \\
&\quad + \frac{1}{m} \sum_{s=1}^{m-1} \binom{m}{s} \left( \frac{2}{m-s} - \frac{2G_{m-s+1}}{m-s+1} (H_{m-1} - H_{m-s}) - \sum_{l=s}^{m-1} \frac{G_{l-s+1}}{(l-s+1)(m-l)} \right) B_s(x).
\end{aligned} \tag{1.7}$$

The obvious polynomial identities can be derived also for $\alpha_m(< x >)$ and $\beta_m(< x >)$ from Theorems 2.1 and 2.2, and Theorems 3.1 and 3.2 , respectively. It is noteworthy that from the Fourier series expansion of the function $\sum_{k=1}^{m-1} \frac{1}{k(m-k)} B_k(< x >) B_{m-k}(< x >)$ we can derive a slightly different version of the well-known Miki's identity (see [3,5,19,20])

$$\begin{aligned}
&\sum_{k=1}^{m-1} \frac{1}{2k\,(2m-2k)} B_{2k} B_{2m-2k} \\
&= \frac{1}{m} \sum_{k=1}^{m} \frac{1}{2k} \binom{2m}{2k} B_{2k} B_{2m-2k} + \frac{1}{m} H_{2m-1} B_{2m}, \quad (m \geq 2).
\end{aligned} \tag{1.8}$$

In addition, we can derive the Faber-Pandharipande-Zagier identity (see [4])

$$\begin{aligned}
&\sum_{k=1}^{m-1} \frac{1}{2k\,(2m-2k)} \overline{B}_{2k} \overline{B}_{2m-2k} \\
&= \frac{1}{m} \sum_{k=1}^{m} \frac{1}{2k} \binom{2m}{2k} B_{2k} \overline{B}_{2m-2k} + \frac{1}{m} H_{2m-1} \overline{B}_{2m}, \quad (m \geq 2),
\end{aligned} \tag{1.9}$$

where $\overline{B}_m = \left( \frac{1-2^{m-1}}{2^{m-1}} \right) B_m = \left( 2^{1-m} - 1 \right) B_m = B_m\left(\frac{1}{2}\right)$, Some related works can be found in [1,7-11].

## 2. Fourier series of the first type of functions

In this section, we will study the Fourier series of first type of functions involving Genocchi polynomials. Let $\alpha_m(x) = \sum_{k=1}^{m} G_k(x) x^{m-k}$, $(m \geq 2)$. Note here that $\deg \alpha_m(x) = m-1$. Then we will consider the

function

$$\alpha_m(< x >) = \sum_{k=1}^{m} G_k(< x >) < x >^{m-k}, \ \ (m \geq 2). \tag{2.1}$$

defined on $(-\infty, -\infty)$ which is periodic of period 1. The Fourier series of $\alpha_m(< x >)$ is

$$\sum_{n=-\infty}^{\infty} A_n^{(m)} e^{2\pi i n x}, \tag{2.2}$$

where

$$\begin{aligned} A_n^{(m)} &= \int_0^1 \alpha_m(< x >) e^{-2\pi i n x} dx \\ &= \int_0^1 \alpha_m(x) e^{-2\pi i n x} dx. \end{aligned} \tag{2.3}$$

Before proceeding further, we first observe the following.

$$\begin{aligned} \alpha_m'(x) &= \sum_{k=1}^{m} (k G_{k-1}(x) x^{m-k} + (m-k) G_k(x) x^{m-k-1}) \\ &= \sum_{k=2}^{m} (k G_{k-1}(x) x^{m-k} + \sum_{k=1}^{m-1} (m-k) G_k(x) x^{m-k-1} \\ &= \sum_{k=1}^{m-1} (k+1) G_k(x) x^{m-k-1} + \sum_{k=1}^{m-1} (m-k) G_k(x) x^{m-k-1} \\ &= (m+1) \sum_{k=1}^{m-1} G_k(x) x^{m-1-k} \\ &= (m+1) \alpha_{m-1}(x). \end{aligned} \tag{2.4}$$

From this, we have $\left(\frac{\alpha_{m+1}(x)}{m+2}\right)' = \alpha_m(x)$. Then we have

$$\int_0^1 \alpha_m(x) dx = \frac{1}{m+2} (\alpha_{m+1}(1) - \alpha_{m+1}(0)), \tag{2.5}$$

$$\begin{aligned} \alpha_m(1) - \alpha_m(0) &= \sum_{k=1}^{m} (G_k(1) - G_k(0) \delta_{m,k}) \\ &= \sum_{k=1}^{m} (-G_k(0) + 2\delta_{k,1} - G_k(0) \delta_{m,k}) = -\sum_{k=1}^{m} G_k + 2 - G_m, \end{aligned} \tag{2.6}$$

$$\alpha_m(0) = \alpha_m(1) \Longleftrightarrow \sum_{k=1}^{m} G_k = 2 - G_m, \tag{2.7}$$

$$\int_0^1 \alpha_m(x) dx = \frac{1}{m+2} \left( -\sum_{k=1}^{m+1} G_k + 2 - G_{m+1} \right). \tag{2.8}$$

We are now ready to determine the Fourier coefficients $A_n^{(m)}$.
Case 1 : $n \neq 0$.

$$\begin{aligned}
A_n^{(m)} &= \int_0^1 \alpha_m(x)e^{-2\pi inx}dx \\
&= -\frac{1}{2\pi in}\left[\alpha_m(x)e^{-2\pi inx}\right]_0^1 + \frac{1}{2\pi in}\int_0^1 \alpha_m'(x)e^{-2\pi inx}dx \\
&= -\frac{1}{2\pi in}(\alpha_m(1)-\alpha_m(0)) + \frac{m+1}{2\pi in}\int_0^1 \alpha_{m-1}(x)e^{-2\pi inx}dx \\
&= \frac{m+1}{2\pi in}A_n^{(m-1)} + \frac{1}{2\pi in}\left(\sum_{k=1}^{m} G_k - 2 + G_m\right) \\
&= \frac{m+1}{2\pi in}\left(\frac{m}{2\pi in}A_n^{(m-2)} + \frac{1}{2\pi in}(\sum_{k=1}^{m-1} G_k - 2 + G_{m-1})\right) \\
&\quad + \frac{1}{2\pi in}\left(\sum_{k=1}^{m} G_k - 2 + G_m\right) \\
&= \frac{(m+1)m}{(2\pi in)^2}A_n^{(m-2)} + \frac{m+1}{(2\pi in)^2}\left(\sum_{k=1}^{m-1} G_k - 2 + G_{m-1}\right) \\
&\quad + \frac{1}{2\pi in}\left(\sum_{k=1}^{m} G_k - 2 + G_m\right) \\
&= \cdots \\
&= \frac{(m+1)_{m-2}}{(2\pi in)^{m-2}}A_n^{(2)} + \sum_{j=1}^{m-2}\frac{(m+1)_{j-1}}{(2\pi in)^j}\left(\sum_{k=1}^{m-j+1} G_k - 2 + G_{m-j+1}\right) \\
&= -\frac{3(m+1)_{m-2}}{(2\pi in)^{m-1}} + \sum_{j=1}^{m-2}\frac{(m+1)_{j-1}}{(2\pi in)^j}\left(\sum_{k=1}^{m-j+1} G_k - 2 + G_{m-j+1}\right) \\
&= \sum_{j=1}^{m-1}\frac{(m+1)_{j-1}}{(2\pi in)^j}\left(\sum_{k=1}^{m-j+1} G_k - 2 + G_{m-j+1}\right) \\
&= \frac{1}{m+2}\sum_{j=1}^{m-1}\frac{(m+1)_j}{(2\pi in)^j}\left(\sum_{k=1}^{m-j+1} G_k - 2 + G_{m-j+1}\right),
\end{aligned} \tag{2.9}$$

where

$$A_n^{(2)} = \int_0^1 (3x-1)e^{-2\pi inx}dx = -\frac{3}{2\pi in}. \tag{2.10}$$

Case 2: $n = 0$.

$$A_0^{(m)} = \int_0^1 \alpha_m(x)dx = -\frac{1}{m+2}(\sum_{k=1}^{m+1} G_k - 2 + G_{m+1}). \tag{2.11}$$

$\alpha_m(< x >), (m \geq 2)$ is piecewise $C^\infty$. Moreover, $\alpha_m(< x >)$ is continuous for those integers $m \geq 2$ with $\sum_{k=1}^{m} G_k = 2 - G_m$ and discontinuous with jump discontinuities at integers for those integers $m \geq 2$ with $\sum_{k=1}^{m} G_k \neq 2 - G_m$ .

We need the following facts about Bernoulli functions $B_m(< x >)$:

(a) for $m \geq 2$,

$$B_m(< x >) = -m! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi inx}}{(2\pi in)^m}. \tag{2.12}$$

(b) for $m = 1$,

$$-\sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi inx}}{2\pi in} = \begin{cases} B_1(< x >), & \text{for} \quad x \in \mathbb{Z}^c, \\ 0, & \text{for} \quad x \in \mathbb{Z}, \end{cases} \tag{2.13}$$

where $\mathbb{Z}^c = \mathbb{R} - \mathbb{Z}$. Assume first that $m \geq 2$ is an integer with $\sum_{k=1}^{m} G_k = 2 - G_m$. Then $\alpha_m(1) = \alpha_m(0)$. Thus $\alpha_m(< x >)$ is piecewise $C^\infty$, and continuous. So the Fourier series of $\alpha_m(< x >)$ converges uniformly to $\alpha_m(< x >)$, and

$$\begin{aligned}
&\alpha_m(< x >) \\
&= -\frac{1}{m+2}(\sum_{k=1}^{m+1} G_k - 2 + G_{m+1}) + \frac{1}{m+2} \sum_{n=-\infty, n\neq 0}^{\infty} \left( \sum_{j=1}^{m-1} \frac{(m+2)_j}{(2\pi in)^j} ( \sum_{k=1}^{m-j+1} G_k - 2 + G_{m-j+1}) \right) e^{2\pi inx} \\
&= -\frac{1}{m+2}(\sum_{k=1}^{m+1} G_k - 2 + G_{m+1}) \\
&\quad - \frac{1}{m+2} \sum_{j=1}^{m-1} \binom{m+2}{j} \left( \sum_{k=1}^{m-j+1} G_k - 2 + G_{m-j+1} \right) \left( -j! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi in}}{(2\pi in)^j} \right) \\
&= -\frac{1}{m+2} \left( \sum_{k=1}^{m+1} G_k - 2 + G_{m+1} \right) - \frac{1}{m+2} \sum_{j=2}^{m-1} \binom{m+2}{j} \left( \sum_{k=1}^{m-j+1} G_k - 2 + G_{m-j+1} \right) B_j(< x >) \\
&\quad - \left( \sum_{k=1}^{m} G_k - 2 + G_m) \right) \times \begin{cases} B_1(< x >), & \text{for} \quad x \in \mathbb{Z}^c, \\ 0, & \text{for} \quad x \in \mathbb{Z}, \end{cases}
\end{aligned} \tag{2.14}$$

for all $x \in (-\infty, \infty)$. Hence we obtain the following theorem.

**Theorem 2.1.** *Let $m \geq 2$ be an integer with $\sum_{k=1}^{m} G_k = 2 - G_m$. Then we have the following.*
*(a) $\sum_{k=1}^{m} G_k(< x >) < x >^{m-k}$ has the Fourier series expansion*

$$\begin{aligned}
&\sum_{k=1}^{m} G_k(< x >) < x >^{m-k} \\
&= -\frac{1}{m+2} \left( \sum_{k=1}^{m+1} G_k - 2 + G_{m+1} \right) \\
&\quad + \frac{1}{m+2} \sum_{n=-\infty, n\neq 0}^{\infty} \left( \sum_{j=1}^{m-1} \frac{(m+2)_j}{(2\pi in)^j} ( \sum_{k=1}^{m-j+1} G_k - 2 + G_{m-j+1}) \right) e^{2\pi inx},
\end{aligned} \tag{2.15}$$

*for all $x \in (-\infty, \infty)$. Here the convergence is uniform.*

*(b)*

$$\begin{aligned}&\sum_{k=1}^{m} G_k(< x >) < x >^{m-k} \\ &= -\frac{1}{m+2}\sum_{j=0,j\neq 1}^{m-1}\binom{m+2}{j}\left(\sum_{k=1}^{m-j+1} G_k - 2 + G_{m-j+1}\right) B_j(< x >),\end{aligned} \tag{2.16}$$

*for all $x \in (-\infty, \infty)$, where $B_j(< x >)$ is the Bernoulli function.*

Next, we assume that $m \geq 2$ is an integer with $\sum_{k=1}^{m} G_k \neq 2 - G_m$. Then $\alpha_m(1) \neq \alpha_m(0)$. Hence $\alpha_m(< x >)$ is piecewise $C^\infty$ and discontinuous with jump discontinuities at integers. The Fourier series of $\alpha_m(< x >)$ converges pointwise to $\alpha_m(< x >)$, for $x \in \mathbb{Z}^c$, and converges to

$$\begin{aligned}\frac{1}{2}(\alpha_m(0) + \alpha_m(1)) &= \alpha_m(0) - \frac{1}{2}\sum_{k=1}^{m} G_k + 1 - \frac{1}{2}G_m \\ &= 1 - \frac{1}{2}\sum_{k=1}^{m-1} G_k.\end{aligned} \tag{2.17}$$

Thus we get the following theorem.

**Theorem 2.2.** *Let $m \geq 2$ be an integer with $\sum_{k=1}^{m} G_k \neq 2 - G_m$. Then we have the following.*
*(a)*

$$\begin{aligned}&-\frac{1}{m+2}\left(\sum_{k=1}^{m+1} G_k - 2 + G_{m+1}\right) + \frac{1}{m+2}\sum_{n=-\infty, n\neq 0}^{\infty}\left(\sum_{j=1}^{m-1}\frac{(m+2)_j}{(2\pi i n)^j}\left(\sum_{k=1}^{m-j+1} G_k - 2 + G_{m-j+1}\right)\right) e^{2\pi i n x} \\ &= \begin{cases} \sum_{k=1}^{m} G_k(< x >) < x >^{m-k}, & \text{for} \quad x \in \mathbb{Z}^c, \\ 1 - \frac{1}{2}\sum_{k=1}^{m-1} G_k, & \text{for} \quad x \in \mathbb{Z}. \end{cases}\end{aligned} \tag{2.18}$$

*Here the convergence is pointwise.*
*(b)*

$$\begin{aligned}&-\frac{1}{m+2}\sum_{j=0}^{m-1}\binom{m+2}{j}\Big(\sum_{k=1}^{m-j+1} G_k - 2 + G_{m-j+1}\Big)B_j(< x >) \\ &= \sum_{k=1}^{m} G_k(< x >) < x >^{m-k}, \ \text{ for } x \in \mathbb{Z}^c \quad ; \\ &-\frac{1}{m+2}\sum_{j=0, j\neq 1}^{m-1}\binom{m+2}{j}\Big(\sum_{k=1}^{m-j+1} G_k - 2 + G_{m-j+1}\Big)B_j(< x >) \\ &= 1 - \frac{1}{2}\sum_{k=1}^{m-1} G_k, \ x \in \mathbb{Z}.\end{aligned} \tag{2.19}$$

**Question**: For what values of $m \geq 2$, does $\sum_{k=1}^{m} G_k = 2 - G_m$ hold?

## 3. Fourier series of the second type of functions

Let $\beta_m(x) = \sum_{k=1}^{m} \frac{1}{k!(m-k)!} G_k(x) x^{m-k}$, $(m \geq 2)$. Then, we consider the function

$$\beta_m(< x >) = \sum_{k=1}^{m} \frac{1}{k!(m-k)!} G_k(< x >) < x >^{m-k}, \tag{3.1}$$

defined on $(-\infty, -\infty)$ which is periodic with period 1. The Fourier series of $\beta_m(< x >)$ is

$$\sum_{n=-\infty}^{\infty} B_n^{(m)} e^{2\pi i n x}, \tag{3.2}$$

where

$$\begin{aligned} B_n^{(m)} &= \int_0^1 \beta_m(< x >) e^{-2\pi i n x} dx \\ &= \int_0^1 \beta_m(x) e^{-2\pi i n x} dx. \end{aligned} \tag{3.3}$$

Before proceeding further, we need the following.

$$\begin{aligned} \beta_m'(x) &= \sum_{k=1}^{m} \left\{ \frac{k}{k!(m-k)!} G_{k-1}(x) x^{m-k} + \frac{m-k}{k!(m-k)!} G_k(x) x^{m-k-1} \right\} \\ &= \sum_{k=2}^{m} \frac{1}{(k-1)!(m-k)!} G_{k-1}(x) x^{m-k} + \sum_{k=1}^{m-1} \frac{1}{k!(m-k-1)!} G_k(x) x^{m-k-1} \\ &= \sum_{k=1}^{m-1} \frac{1}{k!(m-1-k)!} G_k(x) x^{m-1-k} + \sum_{k=1}^{m-1} \frac{1}{k!(m-1-k)!} G_k(x) x^{m-1-k} \\ &= 2\beta_{m-1}(x). \end{aligned} \tag{3.4}$$

So, $\beta_m'(x) = 2\beta_{m-1}(x)$. From this, we see that

$$\left( \frac{\beta_{m+1}(x)}{2} \right)' = \beta_m(x) \tag{3.5}$$

and

$$\int_0^1 \beta_m(x) dx = \frac{1}{2} (\beta_{m+1}(1) - \beta_{m+1}(0)). \tag{3.6}$$

We also observe that

$$\begin{aligned} \beta_m(1) - \beta_m(0) &= \sum_{k=1}^{m} \frac{1}{k!(m-k)!} \left( G_k(1) - G_k(0)\delta_{m,k} \right) \\ &= \sum_{k=1}^{m} \frac{1}{k!(m-k)!} (-G_k(0) + 2\delta_{k,1}) - \sum_{k=1}^{m} \frac{G_k(0)\delta_{m,k}}{k!(m-k)!} \\ &= -\sum_{k=1}^{m} \frac{G_k}{k!(m-k)!} + \frac{2}{(m-1)!} - \frac{G_m}{m!}. \end{aligned} \tag{3.7}$$

We put

$$\Omega_m = \beta_m(1) - \beta_m(0) = -\sum_{k=1}^{m} \frac{G_k}{k!(m-k)!} + \frac{2}{(m-1)!} - \frac{G_m}{m!}, \tag{3.8}$$

for $m \geq 2$. Then

$$\beta_m(0) = \beta_m(1) \Longleftrightarrow \Omega_m = 0. \tag{3.9}$$

Moreover,

$$\begin{aligned} &\int_0^1 \beta_m(x)dx = \frac{1}{2}\Omega_{m+1} \\ &= -\frac{1}{2}\left\{ \sum_{k=1}^{m+1} \frac{G_k}{k!(m-k+1)!} - \frac{2}{m!} + \frac{G_{m+1}}{(m+1)!} \right\}. \end{aligned} \tag{3.10}$$

Now, we are going to determine the Fourier coefficients $B_n^{(m)}$.

Case 1: $n \neq 0$.

$$\begin{aligned} B_n^{(m)} &= \int_0^1 \beta_m(x)e^{-2\pi inx}dx \\ &= -\frac{1}{2\pi in}\left[\beta_m(x)e^{-2\pi inx}\right]_0^1 + \frac{1}{2\pi in}\int_0^1 \beta_m'(x)e^{-2\pi inx}dx \\ &= -\frac{1}{2\pi in}(\beta_m(1) - \beta_m(0)) + \frac{1}{\pi in}\int_0^1 \beta_{m-1}(x)e^{-2\pi inx}dx \\ &= \frac{1}{\pi in}B_n^{(m-1)} - \frac{1}{2\pi in}\Omega_m \\ &= \frac{1}{\pi in}\left(\frac{1}{\pi in}B_n^{(m-2)} - \frac{1}{2\pi in}\Omega_{m-1}\right) - \frac{1}{2\pi in}\Omega_m \\ &= \frac{1}{(\pi in)^2}B_n^{(m-2)} - \frac{2}{(2\pi in)^2}\Omega_{m-1} - \frac{1}{2\pi in}\Omega_m \\ &= \cdots \\ &= \frac{1}{(\pi in)^{m-2}}B_n^{(2)} - \sum_{j=1}^{m-2}\frac{2^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}, \end{aligned} \tag{3.11}$$

where

$$B_n^{(2)} = \int_0^1 \left(2x - \frac{1}{2}\right)e^{-2\pi inx}dx = -\frac{1}{\pi in}. \tag{3.12}$$

By (3.11) and (3.12), we get

$$\begin{aligned} B_n^{(m)} &= -\frac{1}{(\pi in)^{m-1}} - \sum_{j=1}^{m-2}\frac{2^{j-1}}{(2\pi in)^j}\Omega_{m-j+1} \\ &= -\sum_{j=1}^{m-1}\frac{2^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}. \end{aligned} \tag{3.13}$$

Case 2: $n = 0$.

$$B_0^{(m)} = \int_0^1 \beta_m(x)dx = \frac{1}{2}\Omega_{m+1}. \tag{3.14}$$

Here $\beta_m(< x >), (m \geq 2)$ is piecewise $C^\infty$. Moreover, $\beta_m(< x >)$ is continuous for those integers $m \geq 2$ with $\Omega_m = 0$ and discontinuous with jump discontinuities at integers for those integers $m \geq 2$ with $\Omega_m \neq 0$.

Assume first that $m \geq 2$ is an integer with $\Omega_m = 0$ . Then $\beta_m(0) = \beta_m(1)$. So $\beta_m(< x >)$ is piecewise $C^\infty$, and continuous. Hence the Fourier series of $\beta_m(< x >)$ converges uniformly to $\beta_m(< x >)$, and

$$\begin{aligned}
\beta_m(< x >) &= \sum_{k=1}^{m} \frac{1}{k!(m-k)!} G_k(< x >) < x >^{m-k} \\
&= \frac{1}{2}\Omega_{m+1} - \sum_{n=-\infty, n\neq 0}^{\infty} \left( \sum_{j=1}^{m-1} \frac{2^{j-1}}{(2\pi i n)^j} \Omega_{m-j+1} \right) e^{2\pi i n x} \\
&= \frac{1}{2}\Omega_{m+1} + \sum_{j=1}^{m-1} \frac{2^{j-1}}{j!} \Omega_{m-j+1} \left( -j! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^j} \right) \\
&= \frac{1}{2}\Omega_{m+1} + \sum_{j=2}^{m-1} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_j(< x >) \\
&\quad + \Omega_m \times \begin{cases} B_1(< x >), & \text{for} \quad x \in \mathbb{Z}^c, \\ 0, & \text{for} \quad x \in \mathbb{Z}, \end{cases}
\end{aligned} \tag{3.15}$$

for all $x \in (-\infty, \infty)$.

Thus we have the following theorem.

**Theorem 3.1.** *For each integer $l \geq 2$, let*

$$\Omega_l = -\sum_{k=1}^{l} \frac{G_k}{k!(l-k)!} + \frac{2}{(l-1)!} - \frac{G_l}{l!}. \tag{3.16}$$

*Assume that $\Omega_m = 0$, for an integer $m \geq 2$. Then we have the following.*

*(a) $\sum_{k=1}^{m} \frac{1}{k!(m-k)!} G_k(< x >) < x >^{m-k}$ has the Fourier series expansion*

$$\begin{aligned}
&\sum_{k=1}^{m} \frac{1}{k!(m-k)!} G_k(< x >) < x >^{m-k} \\
&= \frac{1}{2}\Omega_{m+1} - \sum_{n=-\infty, n\neq 0}^{\infty} \left( \sum_{j=1}^{m-1} \frac{2^{j-1}}{(2\pi i n)^j} \Omega_{m-j+1} \right) e^{2\pi i n x},
\end{aligned} \tag{3.17}$$

*for all $x \in (-\infty, \infty)$. Here the convergence is uniform.*

*(b)*

$$\begin{aligned}
&\sum_{k=1}^{m} \frac{1}{k!(m-k)!} G_k(< x >) < x >^{m-k} \\
&= \sum_{j=0, j\neq 1}^{m-1} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_j(< x >),
\end{aligned} \tag{3.18}$$

*for all $x \in (-\infty, \infty)$. Here $B_k(< x >)$ is the Bernoulli function.*

Assume next that $m \geq 2$ is an integer with $\Omega_m \neq 0$. Then $\beta_m(1) \neq \beta_m(0)$, and hence $\beta_m(< x >)$ is piecewise $C^\infty$ and discontinuous with jump discontinuities at integers. Thus the Fourier series of

$\beta_m(< x >)$ converges pointwise to $\beta_m(< x >)$, for $x \in \mathbb{Z}^c$, and converges to

$$\begin{aligned} &\frac{1}{2}(\beta_m(0)+\beta_m(1)) = \beta_m(0)+\frac{1}{2}\Omega_m \\ &= \frac{G_m}{m!}+\frac{1}{2}\left(-\sum_{k=1}^{m}\frac{G_k}{k!(m-k)!}+\frac{2}{(m-1)!}-\frac{G_m}{m!}\right) \\ &= \frac{1}{2}\left(\frac{2}{(m-1)!}-\sum_{k=1}^{m-1}\frac{G_k}{k!(m-k)!}\right), \end{aligned} \tag{3.19}$$

for $x \in \mathbb{Z}$. Hence we obtain the following theorem.

**Theorem 3.2.** *For each integer $l \geq 2$, let*

$$\Omega_l = -\sum_{k=1}^{l}\frac{G_k}{k!(l-k)!}+\frac{2}{(l-1)!}-\frac{G_l}{l!}. \tag{3.20}$$

*Assume that $\Omega_m \neq 0$, for an integer $m \geq 2$. Then we have the following.*
*(a)*

$$\begin{aligned} &\frac{1}{2}\Omega_{m+1}-\sum_{n=-\infty,n\neq 0}^{\infty}\left(\sum_{j=1}^{m-1}\frac{2^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}\right)e^{2\pi inx} \\ &= \begin{cases} \sum_{k=1}^{m}\frac{1}{k!(m-k)!}G_k(< x >) < x >^{m-k}, & \text{for} \quad x \in \mathbb{Z}^c, \\ \frac{G_m}{m!}+\frac{1}{2}\Omega_m, & \text{for} \quad x \in \mathbb{Z}. \end{cases} \end{aligned} \tag{3.21}$$

*Here the convergence is pointwise.*
*(b)*

$$\begin{aligned} &\sum_{j=0}^{m-1}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(< x >) \\ &= \sum_{k=1}^{m}\frac{1}{k!(m-k)!}G_k(< x >) < x >^{m-k}, \end{aligned} \tag{3.22}$$

*for $x \in \mathbb{Z}^c$;*

$$\begin{aligned} &\sum_{j=0,j\neq 1}^{m-1}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_k(< x >) \\ &= \frac{G_m}{m!}+\frac{1}{2}\Omega_m, \end{aligned} \tag{3.23}$$

*for $x \in \mathbb{Z}$. Here $B_j(< x >)$ is the Bernoulli function.*

Remark: For what values of $m \geq 2$, does $\sum_{k=1}^{m}\frac{G_m}{k!(m-k)!} = \frac{2}{(m-1)!}-\frac{G_m}{m!}$ hold?

## 4. Fourier series of the third type of functions

Let $\gamma_m(x) = \sum_{k=1}^{m-1}\frac{1}{k(m-k)}G_k(x)x^{m-k}$, $(m \geq 2)$. Then we will consider the function

$$\gamma_m(< x >) = \sum_{k=1}^{m-1}\frac{1}{k(m-k)}G_k(< x >) < x >^{m-k}, \tag{4.1}$$

defined on $(-\infty, -\infty)$ which is periodic of period 1. The Fourier series of $\gamma_m(< x >)$ is

$$\sum_{n=-\infty}^{\infty} C_n^{(m)} e^{2\pi i n x}, \tag{4.2}$$

where

$$C_n^{(m)} = \int_0^1 \gamma_m(< x >) e^{-2\pi i n x} dx = \int_0^1 \gamma_m(x) e^{-2\pi i n x} dx. \tag{4.3}$$

To proceed further, we need to observe the following.

$$\begin{aligned} \gamma_m'(x) &= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \left\{ k G_{k-1}(x) x^{m-k} + (m-k) G_k(x) x^{m-k-1} \right\} \\ &= \sum_{k=1}^{m-2} \frac{1}{m-k-1} G_k(x) x^{m-k-1} + \sum_{k=1}^{m-1} \frac{1}{k} G_k(x) x^{m-k-1} \\ &= (m-1) \sum_{k=1}^{m-2} \frac{1}{k(m-1-k)} G_k(x) x^{m-1-k} + \frac{1}{m-1} G_{m-1}(x) \\ &= (m-1)\gamma_{m-1}(x) + \frac{1}{m-1} G_{m-1}(x). \end{aligned} \tag{4.4}$$

Thus,

$$\gamma_m'(x) = (m-1)\gamma_{m-1}(x) + \frac{1}{m-1} G_{m-1}(x). \tag{4.5}$$

From this, we have

$$\left( \frac{1}{m} (\gamma_{m+1}(x) - \frac{1}{m(m+1)} G_{m+1}(x)) \right)' = \gamma_m(x) \tag{4.6}$$

and

$$\begin{aligned} &\int_0^1 \gamma_m(x) dx \\ &= \left[ \frac{1}{m} (\gamma_{m+1}(x) - \frac{1}{m(m+1)} G_{m+1}(x)) \right]_0^1 \\ &= \frac{1}{m} \left( (\gamma_{m+1}(1) - \gamma_{m+1}(0)) - \frac{1}{m(m+1)} (G_{m+1}(1) - G_{m+1}(0)) \right) \\ &= \frac{1}{m} \left( \gamma_{m+1}(1) - \gamma_{m+1}(0) + \frac{2}{m(m+1)} G_{m+1}(0) \right). \end{aligned} \tag{4.7}$$

Observe that

$$\begin{aligned} \gamma_m(1) - \gamma_m(0) &= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} (G_k(1) - G_k(0)\delta_{m,k}) \\ &= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} (-G_k(0) + 2\delta_{k,1}) - \sum_{k=1}^{m-1} \frac{1}{k(m-k)} G_k(0) \delta_{m,k} \\ &= -\sum_{k=1}^{m-1} \frac{G_k}{k(m-k)} + \frac{2}{m-1}. \end{aligned} \tag{4.8}$$

So, we have

$$\gamma_m(1) = \gamma_m(0) \Longleftrightarrow \sum_{k=1}^{m-1} \frac{G_k}{k(m-k)} = \frac{2}{m-1}. \tag{4.9}$$

Also,

$$\int_0^1 \gamma_m(x)dx = \frac{1}{m}\left(-\sum_{k=1}^{m} \frac{G_k}{k(m-k+1)} + \frac{2}{m} + \frac{2}{m(m+1)}G_{m+1}\right). \tag{4.10}$$

Now, we will determine the Fourier coefficients $C_n^{(m)}$.

Case 1: $n \neq 0$.

$$\begin{aligned}
C_n^{(m)} &= \int_0^1 \gamma_m(x)e^{-2\pi inx}dx \\
&= -\frac{1}{2\pi in}[\gamma_m(x)e^{-2\pi inx}]_0^1 + \frac{1}{2\pi in}\int_0^1 \gamma_m'(x)e^{-2\pi inx}dx \\
&= -\frac{1}{2\pi in}(\gamma_m(1)-\gamma_m(0)) + \frac{1}{2\pi in}\int_0^1 \left((m-1)\gamma_{m-1}(x) + \frac{1}{m-1}G_{m-1}(x)\right)e^{-2\pi inx}dx \\
&= -\frac{1}{2\pi in}(\gamma_m(1)-\gamma_m(0)) + \frac{m-1}{2\pi in}\int_0^1 \gamma_{m-1}(x)e^{-2\pi inx}dx \\
&\quad + \frac{1}{2\pi in(m-1)}\int_0^1 G_{m-1}(x)e^{-2\pi inx}dx \\
&= \frac{m-1}{2\pi in}C_n^{(m-1)} - \frac{1}{2\pi in}\Lambda_m + \frac{2}{2\pi in(m-1)}\Phi_m,
\end{aligned} \tag{4.11}$$

where

$$\Phi_m = \sum_{k=1}^{m-2} \frac{(m-1)_{k-1}}{(2\pi in)^k}G_{m-k}, \tag{4.12}$$

and one can show

$$\int_0^1 G_m(x)e^{-2\pi inx}dx \quad = \begin{cases} 2\Omega_{m+1}, & \text{for} \quad n \neq 0, \\ -2\frac{G_{m+1}}{m+1}, & \text{for} \quad n = 0. \end{cases} \tag{4.13}$$

$$
\begin{aligned}
C_n^{(m)} &= \frac{m-1}{2\pi in} C_n^{(m-1)} - \frac{1}{2\pi in}\Lambda_m + \frac{2}{2\pi in(m-1)}\Phi_m \\
&= \frac{m-1}{2\pi in}\left(\frac{m-2}{2\pi in} C_n^{(m-2)} - \frac{1}{2\pi in}\Lambda_{m-1} + \frac{2}{2\pi in(m-2)}\Phi_{m-1}\right) \\
&\quad - \frac{1}{2\pi in}\Lambda_m + \frac{2}{2\pi in(m-1)}\Phi_m \\
&= \frac{(m-1)(m-2)}{(2\pi in)^2} C_n^{(m-2)} - \frac{m-1}{(2\pi in)^2}\Lambda_{m-1} - \frac{1}{2\pi in}\Lambda_m \\
&\quad + \frac{2(m-1)}{(2\pi in)^2(m-2)}\Phi_{m-1} + \frac{2}{2\pi in(m-1)}\Phi_m \\
&= \cdots \\
&= \frac{(m-1)!}{(2\pi in)^{m-2}} C_n^{(2)} - \sum_{j=1}^{m-2} \frac{(m-1)_{j-1}}{(2\pi in)^j}\Lambda_{m-j+1} + \sum_{j=1}^{m-2} \frac{2(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Phi_{m-j+1} \\
&= -\sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi in)^j}\Lambda_{m-j+1} + \sum_{j=1}^{m-2} \frac{2(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Phi_{m-j+1} \\
&= -\frac{1}{m}\sum_{j=1}^{m-1} \frac{(m)_j}{(2\pi in)^j}\Lambda_{m-j+1} + \frac{1}{m}\sum_{j=1}^{m-2} \frac{2(m)_j}{(2\pi in)^j(m-j)}\Phi_{m-j+1},
\end{aligned}
\tag{4.14}
$$

where

$$
C_n^{(2)} = \int_0^1 x e^{-2\pi inx} dx = -\frac{1}{2\pi in}. \tag{4.15}
$$

In order to get a final expression for $C_n^{(m)}$, we need to observe the following.

$$
\begin{aligned}
&\sum_{j=1}^{m-2} \frac{2(m)_j}{(2\pi in)^j(m-j)}\Phi_{m-j+1} \\
&= \sum_{j=1}^{m-2} \frac{2(m)_j}{(2\pi in)^j(m-j)} \sum_{k=1}^{m-j-1} \frac{(m-j)_{k-1}}{(2\pi in)^k} G_{m-j-k+1} \\
&= \sum_{j=1}^{m-2}\sum_{k=1}^{m-j-1} \frac{2(m)_{j+k-1}}{(2\pi in)^{j+k}(m-j)} G_{m-j-k+1} \\
&= 2\sum_{j=1}^{m-2} \frac{1}{m-j} \sum_{s=j+1}^{m-1} \frac{(m)_{s-1}}{(2\pi in)^s} G_{m-s+1} \\
&= 2\sum_{s=2}^{m-1} \frac{(m)_{s-1}}{(2\pi in)^s} G_{m-s+1} \sum_{j=1}^{s-1} \frac{1}{m-j} \\
&= 2\sum_{s=1}^{m-1} \frac{(m)_s}{(2\pi in)^s} \frac{G_{m-s+1}}{m-s+1}(H_{m-1} - H_{m-s}),
\end{aligned}
\tag{4.16}
$$

and

$$\begin{aligned}
&\sum_{j=1}^{m-1}\frac{(m)_j}{(2\pi in)^j}\Lambda_{m-j+1}\\
&=\sum_{j=1}^{m-1}\frac{(m)_j}{(2\pi in)^j}\left\{-\sum_{k=1}^{m-j}\frac{G_k}{k(m-j-k+1)}+\frac{2}{m-j}\right\}\\
&=-\sum_{j=1}^{m-1}\sum_{k=1}^{m-j}\frac{(m)_jG_k}{(2\pi in)^jk(m-j-k+1)}+2\sum_{j=1}^{m-1}\frac{(m)_j}{(2\pi in)^j(m-j)}\\
&=-\sum_{s=1}^{m-1}\sum_{l=s}^{m-1}\frac{(m)_sG_{l-s+1}}{(2\pi in)^s(l-s+1)(m-l)}+2\sum_{s=1}^{m-1}\frac{(m)_s}{(2\pi in)^s(m-s)}.
\end{aligned}\tag{4.17}$$

Putting everything altogether,

$$\begin{aligned}
C_n^{(m)}&=\frac{1}{m}\sum_{s=1}^{m-1}\sum_{l=s}^{m-1}\frac{(m)_sG_{l-s+1}}{(2\pi in)^s(l-s+1)(m-l)}\\
&\quad-\frac{2}{m}\sum_{s=1}^{m-1}\frac{(m)_s}{(2\pi in)^s(m-s)}\\
&\quad+\frac{2}{m}\sum_{s=1}^{m-1}\frac{(m)_s}{(2\pi in)^s}\frac{G_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s})\\
&=-\frac{1}{m}\sum_{s=1}^{m-1}\frac{(m)_s}{(2\pi in)^s}\\
&\quad\times\left\{\frac{2}{m-s}-\frac{2G_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s})-\sum_{l=s}^{m-1}\frac{G_{l-s+1}}{(l-s+1)(m-l)}\right\}.
\end{aligned}\tag{4.18}$$

Case 2: $n=0$.

$$\begin{aligned}
C_0^{(m)}&=\int_0^1\gamma_m(x)dx\\
&=\frac{1}{m}\left(\Lambda_{m+1}+\frac{2}{m(m+1)}G_{m+1}\right)\\
&=\frac{1}{m}\left(-\sum_{k=1}^{m}\frac{G_k}{k(m-k+1)}+\frac{2}{m}+\frac{2}{m(m+1)}G_{m+1}\right).
\end{aligned}\tag{4.19}$$

$\gamma_m(<x>),(m\geq 2)$ is piecewise $C^\infty$. Moreover, $\gamma_m(<x>)$ is continuous for those integers $m\geq 2$ with $\Lambda_m=0$ and discontinuous with jump discontinuities at integers for those integers $\Lambda_m\neq 0$.

Assume first that $\Lambda_m = 0$ . Then $\gamma_m(0) = \gamma_m(1)$. So $\gamma_m(< x >)$ is piecewise $C^\infty$, and continuous. So the Fourier series of $\gamma_m(< x >)$ converges uniformly to $\gamma_m(< x >)$, and

$$\begin{aligned}
&\gamma_m(< x >)\\
&= -\frac{1}{m}\left(\sum_{k=1}^{m}\frac{G_k}{k(m-k+1)} - \frac{2}{m} - \frac{2}{m(m+1)}G_{m+1}\right)\\
&\quad - \frac{1}{m}\sum_{n=-\infty,n\neq 0}^{\infty}\left\{\sum_{s=1}^{m-1}\frac{(m)_s}{(2\pi in)^s}\left(\frac{2}{m-s} - \frac{2G_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s}) - \sum_{l=s}^{m-1}\frac{G_{l-s+1}}{(l-s+1)(m-l)}\right)\right\}e^{2\pi inx}\\
&= -\frac{1}{m}\left(\sum_{k=1}^{m}\frac{G_k}{k(m-k+1)} - \frac{2}{m} - \frac{2}{m(m+1)}G_{m+1}\right)\\
&\quad + \frac{1}{m}\sum_{s=1}^{m-1}\binom{m}{s}\left(\frac{2}{m-s} - \frac{2G_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s}) - \sum_{l=s}^{m-1}\frac{G_{l-s+1}}{(l-s+1)(m-l)}\right)\\
&\qquad \times\left(-s!\sum_{n=-\infty,n\neq 0}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^s}\right)\\
&= -\frac{1}{m}\left(\sum_{k=1}^{m}\frac{G_k}{k(m-k+1)} - \frac{2}{m} - \frac{2}{m(m+1)}G_{m+1}\right)\\
&\quad + \frac{1}{m}\sum_{s=2}^{m-1}\binom{m}{s}\left(\frac{2}{m-s} - \frac{2G_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s}) - \sum_{l=s}^{m-1}\frac{G_{l-s+1}}{(l-s+1)(m-l)}\right)B_s(< x >)\\
&\quad + \left(\frac{2}{m-1} - \sum_{l=1}^{m-1}\frac{G_l}{l(m-l)}\right)\times\begin{cases} B_1(< x >), & \text{for } x\in\mathbb{Z}^c,\\ 0, & \text{for } x\in\mathbb{Z},\end{cases}
\end{aligned} \tag{4.20}$$

for all $x \in (-\infty, \infty)$. Now, we obtain the following theorem.

**Theorem 4.1.** *Let $m \geq 2$ be an integer with $\Lambda_m = -\sum_{k=1}^{m-1}\frac{G_k}{k(m-k)} + \frac{2}{m-1} = 0$. Then we have the following.*

*(a) $\sum_{k=1}^{m-1}\frac{1}{k(m-k)}G_k(< x >) < x >^{m-k}$ has the Fourier expansion*

$$\begin{aligned}
&\sum_{k=1}^{m-1}\frac{1}{k(m-k)}G_k(< x >) < x >^{m-k}\\
&= -\frac{1}{m}\left(\sum_{k=1}^{m}\frac{G_k}{k(m-k+1)} - \frac{2}{m} - \frac{2}{m(m+1)}G_{m+1}\right)\\
&\quad - \frac{1}{m}\sum_{n=-\infty,n\neq 0}^{\infty}\left\{\sum_{s=1}^{m-1}\frac{(m)_s}{(2\pi in)^s}\left(\frac{2}{m-s} - \frac{2G_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s}) - \sum_{l=s}^{m-1}\frac{G_{l-s+1}}{(l-s+1)(m-l)}\right)\right\}e^{2\pi inx}
\end{aligned} \tag{4.21}$$

,

*for all $x \in (-\infty, \infty)$. Here the convergence is uniform.*

*(b)*

$$\begin{aligned}&\sum_{k=1}^{m-1}\frac{1}{k(m-k)}G_k(<x>)<x>^{m-k}\\&=-\frac{1}{m}\left(\sum_{k=1}^{m}\frac{G_k}{k(m-k+1)}-\frac{2}{m}-\frac{2}{m(m+1)}G_{m+1}\right)\\&\quad+\frac{1}{m}\sum_{s=2}^{m-1}\binom{m}{s}\left(\frac{2}{m-s}-\frac{2G_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s})-\sum_{l=s}^{m-1}\frac{G_{l-s+1}}{(l-s+1)(m-l)}\right)B_s(<x>)\end{aligned}\tag{4.22}$$

*for all $x\in(-\infty,\infty)$, where $B_s(<x>)$ is the Bernoulli function.*

Assume next that $m\geq 2$ is an integer with $\Lambda_m\neq 0$.Then $\gamma_m(0)\neq\gamma_m(1)$. $\gamma_m(<x>)$ is piecewise $C^\infty$ and discontinuous with jump discontinuities at integers.Thus the Fourier series of $\gamma_m(<x>)$ converges pointwise to $\gamma_m(<x>)$, for $x\in\mathbb{Z}^c$, and converges to

$$\begin{aligned}\frac{1}{2}(\gamma_m(0)+\gamma_m(1))&=\gamma_m(0)+\frac{1}{2}\Lambda_m\\&=\frac{1}{2}\left(-\sum_{k=1}^{m-1}\frac{G_k}{k(m-k)}+\frac{2}{m-1}\right),\end{aligned}\tag{4.23}$$

for $x\in\mathbb{Z}$. Hence we have the following theorem.

**Theorem 4.2.** *Let $m\geq 2$ be an integer with $\Lambda_m=-\sum_{k=1}^{m-1}\frac{G_k}{k(m-k)}+\frac{2}{m-1}\neq 0$. Then we have the following.*

*(a)*

$$\begin{aligned}&-\frac{1}{m}\left(\sum_{k=1}^{m}\frac{G_k}{k(m-k+1)}-\frac{2}{m}-\frac{2}{m(m+1)}G_{m+1}\right)\\&-\frac{1}{m}\sum_{n=-\infty,n\neq 0}^{\infty}\left\{\sum_{s=1}^{m-1}\frac{(m)_s}{(2\pi in)^s}\left(\frac{2}{m-s}-\frac{2G_{m-s+1}}{m-s+1}\right)(H_{m-1}-H_{m-s})-\sum_{l=s}^{m-1}\frac{G_{l-s+1}}{(l-s+1)(m-1)}\right\}e^{2\pi inx}\\&=\begin{cases}\sum_{k=1}^{m-1}\frac{1}{k(m-k)}G_k(<x>)<x>^{m-k}, & \text{for}\quad x\in\mathbb{Z}^c,\\ \frac{1}{2}\left(-\sum_{k=1}^{m-1}\frac{G_k}{k(m-k)}+\frac{2}{m-1}\right), & \text{for}\quad x\in\mathbb{Z}.\end{cases}\end{aligned}\tag{4.24}$$

*Here the convergence is uniform.*

*(b)*

$$\begin{aligned}&-\frac{1}{m}\left(\sum_{k=1}^{m}\frac{G_k}{k(m-k+1)}-\frac{2}{m}-\frac{2}{m(m+1)}G_{m+1}\right)\\&\quad+\frac{1}{m}\sum_{s=1}^{m-1}\binom{m}{s}\left(\frac{2}{m-s}-\frac{2G_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s})-\sum_{l=s}^{m-1}\frac{G_{l-s+1}}{(l-s+1)(m-l)}\right)B_s(<x>)\\&=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}G_k(<x>)<x>^{m-k},\end{aligned}\tag{4.25}$$

*for* $x \in \mathbb{Z}^c$ *and*

$$
\begin{aligned}
&-\frac{1}{m}\left(\sum_{k=1}^{m}\frac{G_k}{k(m-k+1)}-\frac{2}{m}-\frac{2}{m(m+1)}G_{m+1}\right)\\
&+\frac{1}{m}\sum_{s=2}^{m-1}\binom{m}{s}\left(\frac{2}{m-s}-\frac{2G_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s})-\sum_{l=s}^{m-1}\frac{G_{l-s+1}}{(l-s+1)(m-l)}\right)B_s(<x>) \quad (4.26)\\
&=\frac{1}{2}\left(-\sum_{k=1}^{m-1}\frac{G_k}{k(m-k)}+\frac{2}{m-1}\right),
\end{aligned}
$$

*for* $x \in \mathbb{Z}$.

**Question** For what values of $m \geq 2$, does $\sum_{k=1}^{m-1}\frac{G_k}{k(m-k)}=\frac{2}{m-1}$ hold?

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## References

1. S. Araci, E. Sen, and M. Acikgoz, *Theorems on Genocchi polynomials of higher order arising from Genocchi basis*, Taiwanese J. of Math., **18**(2014), no.2, 473-482.
2. M. Cenkci, M. Can, and V. Kurt, *q-Extensions of Genocchi Numbers*, J. Korean Math. Soc., **43**(2006), 183-198.
3. G. V. Dunne, C. Schubert, *Bernoulli number identities from quantum field theory and topological string theory,* Commun. Number Theory Phys., **7(2)**(2013), 225-249.
4. C. Faber, R. Pandharipande, *Hodge integrals and Gromov-Witten theory,* Invent. Math. **139(1)**(2000), 173-199.
5. I. M. Gessel, *On Miki's identities for Bernoulli numbers,* J. Number Theory **110(1)**(2005), 75-82.
6. L. C. Jang, T. Kim, D. J. Kang, *A note on the Fourier transform of fermionic p -adic integral on* $\mathbb{Z}_p$, J. Comput. Anal. Appl., **11(3)** (2009), 571-575.
7. D.S. Kim, T. Kim, *Identities arising from higher-order Daehee polynomial bases,* Open Math. **13**(2015), 196-208.
8. D.S. Kim, T. Kim, *Euler basis, identities, and their applications,* Int. J. Math. Math. Sci. 2012, Art. ID 343981.
9. D.S. Kim, T. Kim, *A note on higher-order Bernoulli polynomials,* J. Inequal. Appl. 2013, 2013:111.
10. D.S. Kim, T. Kim, *Bernoulli basis and the product of several Bernoulli polynomials,* Int. J. Math. Math. Sci. 2012, Art. ID 463659.
11. D.S. Kim, T. Kim, *Some identities of higher order Euler polynomials arising from Euler basis,* Integral Transforms Spec. Funct., **24(9)** (2013), 734-738.
12. T. Kim, *On the Multiple q-Genocchi and Euler Numbers*, Russ. J. Math. Phys., **15**(2008), 481-486.
13. T. Kim, *On the q-Extension of Euler and Genocchi Numbers*, J. Math. Anal. Appl. **326**(2007), 1458-1465.
14. T. Kim, *A note on the q-Genocchi Numbers and polynomials*, J. Inequalities and applications, **2007** Article ID 71452, 8pages,(2007).
15. T. Kim, *Some identities for the Bernoulli, the Euler and Genocchi numbers and polynomials*, Adv. Stud. Contemp. Math., **20**(2015), no.1, 23-28.
16. H. Liu, and W. Wang, *Some identities on the the Bernoulli, Euler and Genocchi poloynomials via power sums and alternate power sums*, Disc. Math., **309**(2009), 3346-3363.
17. Q. M. Luo, *Fouier expansions and integral representations for Genocchi poloynomials*, J. Integer Seq.,**12** (2009), Article 09.1.4.
18. J. E. Marsden, *Elementary classical analysis,* W. H. Freeman and Company, 1974.
19. H. Miki, *A relation between Bernoulli numbers,* J. Number Theory **10(3)**(1978), 297-302.
20. K. Shiratani, S. Yokoyama, *An application of p-adic convolutions,* Mem. Fac. Sci. Kyushu Univ. Ser. A **36(1)**(1982), 7383.
21. H. M. Srivastava, *Some generalizations and basic extensions of the Bernoulli, Euler and Genocchi polynomials*, Appl. Math. and Inf. Sci., **5**(2011), no. 3, 390-414.
22. D. G. Zill, M. R. Cullen, *Advanced Engineering Mathematics,* Jones and Bartlett Publishers 2006.

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# Lyapunov inequalities of quasi-Hamiltonian systems on time scales

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**Abstract** In this paper, we obtain several new Lyapunov-type inequalities for the following quasi-Hamiltonian systems

$$x^{\Delta}(t) = -W(t)x(\sigma(t)) - U(t)|y(t)|^{p-2}y(t), \;\; y^{\Delta}(t) = V(t)|x(\sigma(t))|^{q-2}x(\sigma(t)) + W^{T}(t)y(t)$$

on the time scale interval $[a,b]_{\mathbb{T}} \equiv [a,b] \cap \mathbb{T}$ for some $a, b \in \mathbb{T}$ $(\sigma(a) < b)$, where $U$ and $V$ are real $n \times n$ symmetric matrix-valued functions on $[a,b]_{\mathbb{T}}$ with $U$ being positive definite, $W$ is real $n \times n$ matrix-valued function on $[a,b]_{\mathbb{T}}$ with $I + \mu(t)W$ being invertible, and $x, y$ are real vector-valued functions on $[a,b]_{\mathbb{T}}$.

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## 1. Introduction

In 1990, Hilger [1] initiated the theory of time scales as a theory capable of treating continuous and discrete analysis in a consistent way, based on which some authors have studied some Lyapunov inequalities for dynamic equations on time scales (see [2-4]) during the last few years. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of real axis $\mathbb{R}$. On a time scale $\mathbb{T}$, the forward jump operator and the graininess function are defined

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \mu(t) = \sigma(t) - t,$$

respectively. For the notions used below we refer to [5,6] that provide some basic facts on time scales.

In this paper, we continue this line of investigation and study Lyapunov-type inequalities for the following quasi-Hamiltonian systems

$$x^{\Delta}(t) = -W(t)x(\sigma(t)) - U(t)|y(t)|^{p-2}y(t), \;\; y^{\Delta}(t) = V(t)|x(\sigma(t))|^{q-2}x(\sigma(t)) + W^{T}(t)y(t), \quad (1.1)$$

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on the time scale interval $[a,b]_{\mathbb{T}} \equiv [a,b] \cap \mathbb{T}$ for some $a, b \in \mathbb{T}$ $(\sigma(a) < b)$, where $p, q \in (0, +\infty)$ and $1/p + 1/q = 1$, $U$ and $V$ are real $n \times n$ symmetric matrix-valued functions on $[a,b]_{\mathbb{T}}$ with $U$ being positive definite, $W$ is real $n \times n$ matrix-valued function on $[a,b]_{\mathbb{T}}$ with $I + \mu(t)W$ being invertible, and $x, y$ are real vector-valued functions on $[a,b]_{\mathbb{T}}$.

When $n = 1$, (1.1) reduces to

$$x^{\Delta}(t) = \alpha(t)x(\sigma(t)) + \beta(t)|y(t)|^{p-2}y(t), \quad y^{\Delta}(t) = -\gamma(t)|x(\sigma(t))|^{q-2}x(\sigma(t)) - \alpha(t)y(t). \quad (1.2)$$

In 2011, Zhang et al. [7] obtained the following theorem.

**Theorem 1.1**[7] Suppose that $1 - \mu(t)\alpha(t) > 0$ and $\beta(t) \geq 0$ for any $t \in \mathbb{T}$ and $a$, $b \in \mathbb{T}^k$ with $\sigma(a) \leq b$. If (1.2) has a real solution $(x(t), y(t))$ satisfying

$$x(a) = 0 \text{ or } x(a)x(\sigma(a)) < 0, \ x(b) = 0 \text{ or } x(b)x(\sigma(b)) < 0, \ \max_{t\in[a,b]_{\mathbb{T}}} |x(t)| > 0,$$

then the following inequality holds:

$$\int_a^b |\alpha(t)| \triangle (t) + \Big( \int_a^{\sigma(b)} \beta(t) \triangle (t) \Big)^{\frac{1}{p}} \Big( \int_a^b \max\{\gamma(t), 0\} \triangle (t) \Big)^{\frac{1}{q}} \geq 2. \quad (1.3)$$

When $n = 1$ and $\mathbb{T} = \mathbb{R}$, Tiryaki et al. [8] obtained the following theorem.

**Theorem 1.2**[8] Suppose that $\beta(t) > 0$ for any $t \in \mathbb{R}$ and $a$, $b \in \mathbb{R}$ with $a < b$. If (1.2) has a real solution $(x(t), y(t))$ satisfying $x(a) = x(b) = 0$ and $\max_{t\in[a,b]} |x(t)| > 0$, then the following inequalities hold:

$$\int_a^b \frac{\max\{\gamma(t), 0\}}{h_a^{1-q}(t) + h_b^{1-q}(t)} dt \geq 1 \quad (1.4)$$

and

$$\int_a^b \max\{\gamma(t), 0\} 2^{q-2} \Big( \frac{1}{h_a(t)} + \frac{1}{h_b(t)} \Big)^{1-q} dt \geq 1, \quad (1.5)$$

where $h_a(t) = \int_a^t \beta(s) e^{-p\int_t^s \alpha(\tau)d\tau} ds$ and $h_b(t) = \int_t^b \beta(s) e^{-p\int_t^s \alpha(\tau)d\tau} ds$.

For some other related results on Lyapunov-type inequalities, see, e.g. [9-16] and the related references therein.

## 2. Preliminaries and some lemmas

For any $u \in \mathbb{R}^n$ and any $U \in \mathbb{R}^{n\times n}$ (the space of real $n \times n$ matrices), write

$$|u| = \sqrt{u^T u} \quad \text{and} \quad |U| = \max_{y\in\mathbb{R}^n - \{O\}} \frac{|Uy|}{|y|},$$

which are called the Euclidean norm of $u$ and the matrix norm of $U$ respectively, where $Q^T$ is the transpose of a $n \times m$ matrix $Q$. It follows from the definition that for any $y \in \mathbb{R}^n$ and any $U, V \in \mathbb{R}^{n\times n}$,

$$|Uy| \leq |U||y|, \quad |UV| \leq |U||V|.$$

Write $\mathbb{R}_s^{n\times n}=\{U\in\mathbb{R}^{n\times n}: U^T=U\}$. It is easy to show that for any $U\in\mathbb{R}_s^{n\times n}$,

$$|U|=\max_{\det|\lambda I-U|=0}|\lambda| \quad \text{and} \quad |U^2|=|U|^2,$$

where $\det|\lambda I-U|$ denotes determinant of the matrix $\lambda I-U$. An $U\in\mathbb{R}_s^{n\times n}$ is said to be positive definite (resp. semi-positive definite), written as $U>0$ (resp. $U\geq 0$), if $y^TUy>0$ (resp. $y^TUy\geq 0$) for all $y\in\mathbb{R}^n$ with $y\neq 0$. If $U$ is positive definite (resp. semi-positive definite), then there exists a unique positive definite matrix (resp. semi-positive definite matrix), written as $\sqrt{U}$, such that $[\sqrt{U}]^2=U$.

In this paper, we study Lyapunov-type inequalities of (1.1) which has some solution $(x(t),y(t))$ satisfying

$$x(a)=x(b)=0 \text{ and } \max_{t\in[a,b]_{\mathbb{T}}}|x(t)|>0. \tag{2.1}$$

We first introduce the following notions and lemmas.

The point $t\in\mathbb{T}$ is said to be left-dense (resp. left-scattered) if $\rho(t)=t$ (resp. $\rho(t)<t$). The point $t\in\mathbb{T}$ is said to be right-dense (resp. right-scattered) if $\sigma(t)=t$ (resp. $\sigma(t)>t$). If $\mathbb{T}$ has a left-scattered maximum $M$, then we define $\mathbb{T}^k=\mathbb{T}-\{M\}$, otherwise $\mathbb{T}^k=\mathbb{T}$.

A function $f:\mathbb{T}\longrightarrow\mathbb{R}$ is said to be rd-continuous provided that $f$ is continuous at right-dense points and has finite left-sided limits at left-dense points in $\mathbb{T}$. The set of all rd-continuous functions from $\mathbb{T}$ to $\mathbb{R}$ is denoted by $C_{rd}(\mathbb{T},\mathbb{R})$.

For a function $f:\mathbb{T}\longrightarrow\mathbb{R}$, the (delta) derivative $f^{\Delta}(t)$ at $t\in\mathbb{T}$ is defined to be the number ( if it exists), such that for given any $\varepsilon>0$, there is a neighborhood $U$ of $t$ with

$$|f(\sigma(t))-f(s)-f^{\Delta}(t)(\sigma(t)-s)|\leq\varepsilon|\sigma(t)-s|$$

for all $s\in U$. If the (delta) derivative $f^{\Delta}(t)$ exists for every $t\in\mathbb{T}^k$, then we say that $f$ is $\Delta-$ differentiable on $\mathbb{T}$.

**Definition 2.1**[5] Let $F,f\in C_{rd}(\mathbb{T},\mathbb{R})$. If $F^{\triangle}(t)=f(t)$ for all $t\in\mathbb{T}^k$, then we definite the Cauchy integral of $f$ by

$$\int_a^b f(t)\triangle t=F(b)-F(a) \text{ for any } a,b\in\mathbb{T}.$$

**Lemma 2.2**[5] (Holder's inequality) Let $a,b\in\mathbb{T}$ with $a\leq b$ and $f_1,f_2\in C_{rd}(\mathbb{T},\mathbb{R})$. Then

$$\int_a^b|f_1(t)f_2(t)|\triangle t\leq\Big(\int_a^b|f_1(t)|^p\triangle t\Big)^{\frac{1}{p}}\Big(\int_a^b|f_2(t)|^q\triangle t\Big)^{\frac{1}{q}},$$

where $p>1$ and $q=p/(p-1)$.

**Lemma 2.3**[5] Suppose that $W\in C_{rd}(\mathbb{T},\mathbb{R}^{n\times n})$ with $I+\mu(t)W(t)$ being invertible and $g\in C_{rd}(\mathbb{T},\mathbb{R}^n)$. Let $t_0\in\mathbb{T}$ and $x_0\in\mathbb{R}^n$. Then the initial value problem

$$x^{\Delta}(t)=-W(t)x(\sigma(t))+g(t),\ x(t_0)=x_0$$

has a unique solution

$$x(t)=e_{\ominus W}(t,t_0)x_0+\int_{t_0}^t e_{\ominus W}(t,\tau)g(\tau)\Delta\tau,$$

where $(\Theta A)(t) = -[I+\mu(t)A(t)]^{-1}A(t)$ for any $t \in \mathbb{T}^k$ and $e_{\Theta A}(t, t_0)$ is the unique matrix-valued solution of the initial value problem

$$\begin{cases} Y^{\Delta}(t) = (\Theta A)(t)Y(t), \\ Y(t_0) = I. \end{cases}$$

**Lemma 2.4**[5] Suppose that $A(t)$ and $B(t)$ are differentiable $n \times n$ matrix-valued functions. Then

$$(A(t)B(t))^{\Delta} = A^{\Delta}(t)B(\sigma(t)) + A(t)B^{\Delta}(t) = A(\sigma(t))B^{\Delta}(t) + A^{\Delta}(t)B(t).$$

**Lemma 2.5**[12] Let $a, b \in \mathbb{T}$ with $a \geq b$ and $x_1(t), x_2(t), \cdots, x_n(t)$ be $\Delta$-integrable on $[a, b]_{\mathbb{T}}$. Write $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))$. Then

$$\Big|\int_a^b x(t)\Delta t\Big| = \sqrt{\sum_{i=1}^{n}\Big(\int_a^b x_i(t)\Delta t\Big)^2} \leq \int_a^b \sqrt{\sum_{i=1}^{n} x_i^2(t)}\Delta t = \int_a^b |x(t)|\Delta t.$$

**Lemma 2.6**[12] Let $V, V_1 \in \mathbb{R}_s^{n\times n}$ with $V_1 \geq V$ (i.e., $V_1 - V \geq 0$) and $x \in \mathbb{R}^n$. Then $x^T V x \leq |V_1||x|^2$.

## 3. Main results and proofs

Write

$$\xi(t) = \begin{cases} (\int_a^t |e_{\Theta W}(t,s)|^p |U(s)|^{\frac{p(p-2)}{2}+1} |[\sqrt{U(s)}]^{-1}|^{p(p-2)}\Delta s)^{\frac{q}{p}}, & \text{if} \quad 1 < q < 2, \\ (\int_a^t |e_{\Theta W}(t,s)|^p |U(s)|\Delta s)^{\frac{q}{p}}, & \text{if} \quad q \geq 2, \end{cases} \tag{3.1}$$

and

$$\eta(t) = \begin{cases} (\int_t^b |e_{\Theta W}(t,s)|^p |U(s)|^{\frac{p(p-2)}{2}+1} |[\sqrt{U(s)}]^{-1}|^{p(p-2)}\Delta s)^{\frac{q}{p}}, & \text{if} \quad 1 < q < 2, \\ (\int_t^b |e_{\Theta W}(t,s)|^p |U(s)|\Delta s)^{\frac{q}{p}}, & \text{if} \quad q \geq 2. \end{cases} \tag{3.2}$$

**Theorem 3.1** Let $a, b \in \mathbb{T}$ with $\sigma(a) < b$ and $V_1 \in \mathbb{R}_s^{n\times n}$ with $V_1(t) \geq V(t)$. If (1.1) has a solution $(x(t), y(t))$ satisfying (2.1) on the interval $[a, b]_{\mathbb{T}}$, then the following inequality holds:

$$\int_a^b \frac{\xi(\sigma(t))\eta(\sigma(t))}{\xi(\sigma(t)) + \eta(\sigma(t))}|V_1(t)| \triangle t \geq 1. \tag{3.3}$$

**Proof** We claim that $y(t) \not\equiv 0$ ($t \in [a, b]_{\mathbb{T}}$). Indeed, if $y(t) \equiv 0$ ($t \in [a, b]_{\mathbb{T}}$), then the first equation of (1.1) reduces to

$$x^{\Delta}(t) = -W(t)x(\sigma(t)),\ x(a) = 0.$$

By Lemma 2.3, it follows $x(t) = e_{\Theta W}(t, a) \cdot x(a) = 0$, which is a contradiction with (2.1). Moreover, we have $y^T(t)U(t)y(t) \geq 0$ ($\not\equiv 0$) for $t \in [a, b]_{\mathbb{T}}$ since $U(t) > 0$.

Since $(x(t), y(t))$ satisfies the following equality

$$(y^T(t)x(t))^{\Delta} = (x^{\sigma}(t))^T V(t)|x^{\sigma}(t)|^{q-2}x^{\sigma}(t) - y^T(t)U(t)|y(t)|^{p-2}y(t), \tag{3.4}$$

where $x^\sigma(t) = x(\sigma(t))$. By integrating (3.4) from $a$ to $b$ and taking into account that $x(a) = x(b) = 0$, we see

$$\int_a^b |x^\sigma(t)|^{q-2}(x^\sigma(t))^T V(t)x^\sigma(t) \triangle t = \int_a^b |y(t)|^{p-2}y^T(t)U(t)y(t) \triangle t > 0. \tag{3.5}$$

For $t \in [a,b]_{\mathbb{T}}$, let $t_0 = a$ and $t_0 = b$ respectively, we obtain from Lemma 2.3 that

$$x(t) = -\int_a^t e_{\Theta W}(t,\tau)U(\tau)|y(\tau)|^{p-2}y(\tau)\Delta\tau = -\int_b^t e_{\Theta W}(t,\tau)U(\tau)|y(\tau)|^{p-2}y(\tau)\Delta\tau.$$

Which follows that for $t \in [a,b)_{\mathbb{T}}$,

$$x^\sigma(t) = -\int_a^{\sigma(t)} e_{\Theta W}(\sigma(t),\tau)U(\tau)|y(\tau)|^{p-2}y(\tau)\Delta\tau = \int_{\sigma(t)}^b e_{\Theta W}(\sigma(t),\tau)U(\tau)|y(\tau)|^{p-2}y(\tau)\Delta\tau.$$

**Case I**: Assume that $q \geq 2$. Then we have that for $a \leq \tau \leq \sigma(t) \leq b$ ,

$$\begin{aligned}
& |e_{\Theta W}(\sigma(t),\tau)U(\tau)|y(\tau)|^{p-2}y(\tau)| \\
\leq\ & |e_{\Theta W}(\sigma(t),\tau)||y(\tau)|^{p-2}|U(\tau)y(\tau)| \\
=\ & |e_{\Theta W}(\sigma(t),\tau)||y(\tau)|^{p-2}\{y^T(\tau)U^T(\tau)U(\tau)y(\tau)\}^{\frac{1}{2}} \\
\leq\ & |e_{\Theta W}(\sigma(t),\tau)||y(\tau)|^{p-2}\{|\sqrt{U(\tau)}y(\tau)||U(\tau)||\sqrt{U(\tau)}y(\tau)|\}^{\frac{1}{2}} \\
=\ & |e_{\Theta W}(\sigma(t),\tau)||y(\tau)|^{p-2}|U(\tau)|^{\frac{1}{2}}(y^T(\tau)U(\tau)y(\tau))^{\frac{1}{2}} \\
=\ & |e_{\Theta W}(\sigma(t),\tau)||y(\tau)|^{p-2}|U(\tau)|^{\frac{1}{2}}(y^T(\tau)U(\tau)y(\tau))^{\frac{1}{q}}(y^T(\tau)U(\tau)y(\tau))^{\frac{1}{2}-\frac{1}{q}} \\
=\ & |e_{\Theta W}(\sigma(t),\tau)||y(\tau)|^{p-2}|U(\tau)|^{\frac{1}{2}}(y^T(\tau)U(\tau)y(\tau))^{\frac{1}{q}}|\sqrt{U(\tau)}y(\tau)|^{2(\frac{1}{2}-\frac{1}{q})} \\
\leq\ & |e_{\Theta W}(\sigma(t),\tau)||U(\tau)|^{\frac{1}{2}}(y^T(\tau)U(\tau)y(\tau))^{\frac{1}{q}}|\sqrt{U(\tau)}|^{1-\frac{2}{q}}|y(\tau)|^{p-1-\frac{2}{q}} \\
\leq\ & |e_{\Theta W}(\sigma(t),\tau)||U(\tau)|^{1-\frac{1}{q}}(y^T(\tau)U(\tau)y(\tau))^{\frac{1}{q}}|y(\tau)|^{p-1-\frac{2}{q}}.
\end{aligned}$$

Combining Lemma 2.2 and Lemma 2.5 we obtain

$$\begin{aligned}
|x^\sigma(t)|^q &= \Big|\int_a^{\sigma(t)} e_{\Theta W}(\sigma(t),\tau)U(\tau)|y(\tau)|^{p-2}y(\tau)\Delta\tau\Big|^q \\
&\leq \Big(\int_a^{\sigma(t)} |e_{\Theta W}(\sigma(t),\tau)U(\tau)|y(\tau)|^{p-2}y(\tau)|\Delta\tau\Big)^q \\
&\leq \Big(\int_a^{\sigma(t)} |e_{\Theta W}(\sigma(t),\tau)||U(\tau)|^{1-\frac{1}{q}}(y^T(\tau)U(\tau)y(\tau))^{\frac{1}{q}}|y(\tau)|^{p-1-\frac{2}{q}}\Delta\tau\Big)^q \\
&\leq \Big(\int_a^{\sigma(t)} |e_{\Theta W}(\sigma(t),\tau)|^p|U(\tau)|\triangle\tau\Big)^{\frac{q}{p}}\Big(\int_a^{\sigma(t)} y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau\Big),
\end{aligned}$$

that is

$$|x^\sigma(t)|^q \leq \xi(\sigma(t))\int_a^{\sigma(t)} y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau. \tag{3.6}$$

Similarly, by letting $\eta(t)$ be as in (3.2), for $a \leq \sigma(t) \leq \tau \leq b$, we have

$$|x^\sigma(t)|^q \leq \eta(\sigma(t))\int_{\sigma(t)}^b y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau. \tag{3.7}$$

It follows from (3.6) and (3.7) that

$$\eta(\sigma(t))\xi(\sigma(t))\int_a^{\sigma(t)} y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau \geq |x^\sigma(t)|^q\eta(\sigma(t))$$

and

$$\eta(\sigma(t))\xi(\sigma(t))\int_{\sigma(t)}^{b} y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau \geq |x^\sigma(t)|^q\xi(\sigma(t)).$$

Thus

$$|x^\sigma(t)|^q \leq \frac{\xi(\sigma(t))\eta(\sigma(t))}{\xi(\sigma(t))+\eta(\sigma(t))}\int_a^b y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau.$$

By Lemma 2.6 and (3.5) we see

$$\begin{aligned}
\int_a^b |V_1(t)||x^\sigma(t)|^q\Delta t &\leq \int_a^b (|V_1(t)|\frac{\xi(\sigma(t))\eta(\sigma(t))}{\xi(\sigma(t))+\eta(\sigma(t))}\int_a^b y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau)\Delta t\\
&= \int_a^b |V_1(t)|\frac{\xi(\sigma(t))\eta(\sigma(t))}{\xi(\sigma(t))+\eta(\sigma(t))}\Delta t\int_a^b y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau\\
&= \int_a^b |V_1(t)|\frac{\xi(\sigma(t))\eta(\sigma(t))}{\xi(\sigma(t))+\eta(\sigma(t))}\Delta t\int_a^b |x^\sigma(t)|^{q-2}(x^\sigma(t))^T V(t)x^\sigma(t)\triangle t\\
&\leq \int_a^b |V_1(t)|\frac{\xi(\sigma(t))\eta(\sigma(t))}{\xi(\sigma(t))+\eta(\sigma(t))}\Delta t\int_a^b |V_1(t)||x^\sigma(t)|^q\Delta t.
\end{aligned}$$

Since

$$\int_a^b |V_1(t)||x^\sigma(t)|^q\Delta t \geq \int_a^b |x^\sigma(t)|^{q-2}(x^\sigma(t))^T V(t)x^\sigma(t)\triangle t = \int_a^b |y(t)|^{p-2}y^T(t)U(t)y(t)\Delta t > 0,$$

we get

$$\int_a^b \frac{\xi(\sigma(t))\eta(\sigma(t))}{\xi(\sigma(t))+\eta(\sigma(t))}|V_1(t)|\triangle t \geq 1.$$

This completes the proof of Case I.

**Case II**: Assume that $1 < q < 2$. Then $p > 2$. Note that for $a \leq \tau \leq \sigma(t) \leq b$ ,

$$\begin{aligned}
& |e_{\Theta W}(\sigma(t),\tau)U(\tau)|y(\tau)|^{p-2}y(\tau)|\\
\leq\ & |e_{\Theta W}(\sigma(t),\tau)||y(\tau)|^{p-2}|U(\tau)y(\tau)|\\
=\ & |e_{\Theta W}(\sigma(t),\tau)||y(\tau)|^{p-2}\{y^T(\tau)U^T(\tau)U(\tau)y(\tau)\}^{\frac{1}{2}}\\
=\ & |e_{\Theta W}(\sigma(t),\tau)||y(\tau)|^{p-2}\{(\sqrt{U(\tau)}y(\tau))^T U(\tau)\sqrt{U(\tau)}y(\tau)\}^{\frac{1}{2}}\\
\leq\ & |e_{\Theta W}(\sigma(t),\tau)||(\sqrt{U(\tau)})^{-1}\sqrt{U(\tau)}y(\tau)|^{p-2}\{|\sqrt{U(\tau)}y(\tau)||U(\tau)||\sqrt{U(\tau)}y(\tau)|\}^{\frac{1}{2}}\\
\leq\ & |e_{\Theta W}(\sigma(t),\tau)||(\sqrt{U(\tau)})^{-1}|^{p-2}|\sqrt{U(\tau)}y(\tau)|^{p-2}|U(\tau)|^{\frac{1}{2}}|\sqrt{U(\tau)}y(\tau)|\\
=\ & |e_{\Theta W}(\sigma(t),\tau)||(\sqrt{U(\tau)})^{-1}|^{p-2}|U(\tau)|^{\frac{1}{2}}|\sqrt{U(\tau)}y(\tau)|^{p-1}\\
=\ & |e_{\Theta W}(\sigma(t),\tau)||(\sqrt{U(\tau)})^{-1}|^{p-2}|U(\tau)|^{\frac{1}{2}}|\sqrt{U(\tau)}y(\tau)|^{\frac{2}{q}}|\sqrt{U(\tau)}y(\tau)|^{p-1-\frac{2}{q}}\\
\leq\ & |e_{\Theta W}(\sigma(t),\tau)||(\sqrt{U(\tau)})^{-1}|^{p-2}|U(\tau)|^{\frac{1}{2}}|\sqrt{U(\tau)}y(\tau)|^{\frac{2}{q}}|\sqrt{U(\tau)}|^{\frac{(p-1)(p-2)}{p}}|y(\tau)|^{p-1-\frac{2}{q}}\\
=\ & |e_{\Theta W}(\sigma(t),\tau)||(\sqrt{U(\tau)})^{-1}|^{p-2}|U(\tau)|^{\frac{1}{2}}(y^T(\tau)U(\tau)y(\tau))^{\frac{1}{q}}|\sqrt{U(\tau)}|^{\frac{(p-1)(p-2)}{p}}|y(\tau)|^{p-1-\frac{2}{q}}.
\end{aligned}$$

Then we obtain

$$\begin{aligned}
|x^{\sigma}(t)|^q &= \Big|\int_a^{\sigma(t)} e_{\Theta W}(\sigma(t),\tau)U(\tau)|y(\tau)|^{p-2}y(\tau)\Delta\tau\Big|^q \\
&\le \Big(\int_a^{\sigma(t)} |e_{\Theta A}(\sigma(t),\tau)U(\tau)|y(\tau)|^{p-2}y(\tau)|\Delta\tau\Big)^q \\
&\le \Big(\int_a^{\sigma(t)} |e_{\Theta W}(\sigma(t),\tau)||(\sqrt{U(\tau)})^{-1}|^{p-2}|U(\tau)|^{\frac{1}{2}} \\
&\quad \times (y^T(\tau)U(\tau)y(\tau))^{\frac{1}{q}}|\sqrt{U(\tau)}|^{\frac{(p-1)(p-2)}{p}}|y(\tau)|^{p-1-\frac{2}{q}}\Delta\tau\Big)^q \\
&\le \Big(\int_a^{\sigma(t)} |e_{\Theta W}(\sigma(t),\tau)||(\sqrt{U(\tau)})^{-1}|^{p-2}|\sqrt{U(\tau)}|^{\frac{(p-1)(p-2)}{p}+1} \\
&\quad \times (y^T(\tau)U(\tau)y(\tau))^{\frac{1}{q}}|y(\tau)|^{p-1-\frac{2}{q}}\Delta\tau\Big)^q \\
&\le \Big(\int_a^{\sigma(t)} |e_{\Theta W}(\sigma(t),\tau)|^p|(\sqrt{U(\tau)})^{-1}|^{p(p-2)}|U(\tau)|^{\frac{p(p-2)}{2}+1}\triangle\tau\Big)^{\frac{q}{p}} \\
&\quad \times\Big(\int_a^{\sigma(t)} y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau\Big).
\end{aligned}$$

That is

$$|x^{\sigma}(t)|^q \le \xi(\sigma(t))\int_a^{\sigma(t)} y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau. \tag{3.8}$$

Similarly, by letting $\eta(t)$ be as in (3.2), for $a \le \sigma(t) \le \tau \le b$, we have

$$|x^{\sigma}(t)|^q \le \eta(\sigma(t))\int_{\sigma(t)}^b y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau. \tag{3.9}$$

The rest of the proof is similar to the Case I, we have

$$\int_a^b \frac{\xi(\sigma(t))\eta(\sigma(t))}{\xi(\sigma(t))+\eta(\sigma(t))}|V_1(t)|\triangle t \ge 1.$$

This completes the proof of Theorem 3.1

**Corollary 3.2** Let $a, b \in \mathbb{T}$ with $\sigma(a) < b$ and $V_1 \in \mathbb{R}_s^{n\times n}$ with $V_1(t) \ge V(t)$. If (1.1) has a solution $(x(t), y(t))$ satisfying (2.1) on the interval $[a,b]_{\mathbb{T}}$, then the following inequality holds:

$$\int_a^b (\xi(\sigma(t))+\eta(\sigma(t)))|V_1(t)|\triangle t \ge 4. \tag{3.10}$$

**Proof** Note

$$\frac{\xi(\sigma(t))\eta(\sigma(t))}{\xi(\sigma(t))+\eta(\sigma(t))} \le \frac{\xi(\sigma(t))+\eta(\sigma(t))}{4}.$$

It follows from (3.3) that

$$\int_a^b \frac{\xi(\sigma(t))+\eta(\sigma(t))}{4}|V_1(t)|\triangle t \ge 1.$$

That is

$$\int_a^b (\xi(\sigma(t))+\eta(\sigma(t)))|V_1(t)|\triangle t \ge 4.$$

This completes the proof of Corollary 3.2.

**Corollary 3.3** Let $a, b \in \mathbb{T}$ with $\sigma(a) < b$ and $V_1 \in \mathbb{R}_s^{n\times n}$ with $V_1(t) \geq V(t)$. If (1.1) has a solution $(x(t), y(t))$ satisfying (2.1) on the interval $[a, b]_{\mathbb{T}}$, then the following inequality holds:

$$\int_a^b (\xi(\sigma(t))\eta(\sigma(t)))^{\frac{1}{2}}|V_1(t)| \triangle t \geq 2. \tag{3.11}$$

**Proof** Note

$$\xi(\sigma(t)) + \eta(\sigma(t)) \geq 2(\xi(\sigma(t))\eta(\sigma(t)))^{\frac{1}{2}}.$$

It follows from (3.3) that

$$\int_a^b \frac{(\xi(\sigma(t))\eta(\sigma(t)))^{\frac{1}{2}}}{2}|V_1(t)| \triangle t \geq 1.$$

That is

$$\int_a^b (\xi(\sigma(t))\eta(\sigma(t)))^{\frac{1}{2}}|V_1(t)| \triangle t \geq 2.$$

This completes the proof of Corollary 3.3.

**Theorem 3.4** Let $a, b \in \mathbb{T}$ with $\sigma(a) < b$ and $V_1 \in \mathbb{R}_s^{n\times n}$ with $V_1(t) \geq V(t)$. If (1.1) has a solution $(x(t), y(t))$ satisfying (2.1) on the interval $[a, b]_{\mathbb{T}}$, then there exists an $c \in (a, b)$ such that

$$\int_a^{\sigma(c)} \xi(\sigma(t))|V_1(t)| \triangle t \geq 1 \quad \text{and} \quad \int_c^b \eta(\sigma(t))|V_1(t)| \triangle t \geq 1. \tag{3.12}$$

**Proof** Let

$$F(t) = \int_a^t \xi(\sigma(s))|V_1(s)| \triangle s - \int_t^b \eta(\sigma(s))|V_1(s)| \triangle s.$$

Then we have $F(a) < 0$ and $F(b) > 0$. Hence we can choose an $c \in (a, b)$ such that $F(c) \leq 0$ and $F(\sigma(c)) \geq 0$, that is

$$\int_a^c \xi(\sigma(s))|V_1(s)| \triangle s \leq \int_c^b \eta(\sigma(s))|V_1(s)| \triangle s \tag{3.13}$$

and

$$\int_a^{\sigma(c)} \xi(\sigma(s))|V_1(s)| \triangle s \geq \int_{\sigma(c)}^b \eta(\sigma(s))|V_1(s)| \triangle s. \tag{3.14}$$

From (3.6) and (3.8), we have

$$|V_1(t)||x^\sigma(t)|^q \leq \xi(\sigma(t))|V_1(t)| \int_a^{\sigma(t)} y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau. \tag{3.15}$$

Note that for $a \leq \tau \leq \sigma(t) \leq \sigma(c) \leq b$ . Integrating (3.15) from $a$ to $\sigma(c)$, we obtain

$$\begin{aligned}\int_a^{\sigma(c)} |V_1(t)||x^\sigma(t)|^q \triangle t &\leq \int_a^{\sigma(c)} \xi(\sigma(t))|V_1(t)|\Big(\int_a^{\sigma(t)} y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau\Big) \triangle t \\ &\leq \int_a^c \xi(\sigma(t))|V_1(t)| \triangle t \int_a^{\sigma(c)} y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau. \\ &+\xi(\sigma(c))|V_1(c)|(\sigma(c) - c)\int_a^{\sigma(c)} y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau.\end{aligned}$$

$$= \int_a^{\sigma(c)} \xi(\sigma(t))|V_1(t)| \triangle t \int_a^{\sigma(c)} y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau.$$

Similarly, for $a \le \sigma(c) \le \sigma(t) \le \tau \le b$, we can obtain from (3.7),(3.9) and (3.14) that

$$\int_{\sigma(c)}^b |V_1(t)||x^\sigma(t)|^q \triangle t \le \int_{\sigma(c)}^b \eta(\sigma(t))|V_1(t)| \triangle t \int_{\sigma(c)}^b y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau$$

$$\le \int_a^{\sigma(c)} \xi(\sigma(t))|V_1(t)| \triangle t \int_{\sigma(c)}^b y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau.$$

These yield

$$\int_a^b |V_1(t)||x^\sigma(t)|^q \triangle t \le \int_a^{\sigma(c)} \xi(\sigma(t))|V_1(t)| \triangle t \int_a^b y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau$$

$$= \int_a^{\sigma(c)} \xi(\sigma(t))|V_1(t)| \triangle t \int_a^b |x^\sigma(t)|^{q-2}(x^\sigma(t))^T V(t)x^\sigma(t) \triangle t$$

$$\le \int_a^{\sigma(c)} \xi(\sigma(t))|V_1(t)| \triangle t \int_a^b |V_1(t)||x^\sigma(t)|^q \triangle t.$$

Since

$$\int_a^b |V_1(t)||x^\sigma(t)|^q\Delta t \ge \int_a^b |x^\sigma(t)|^{q-2}(x^\sigma(t))^T V(t)x^\sigma(t) \triangle t = \int_a^b |y(t)|^{p-2}y^T(t)U(t)y(t)\Delta t > 0,$$

we obtain $\int_a^{\sigma(c)} \xi(\sigma(t))|V_1(t)| \triangle t \ge 1$.

Next, we have from (3.7) and (3.9) that

$$|x^\sigma(t)|^q|V_1(t)| \le \eta(\sigma(t))|V_1(t)| \int_{\sigma(t)}^b y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau. \tag{3.16}$$

Integrating (3.16) from $c$ to $b$, we obtain that for $a \le c \le t \le \sigma(t) \le \tau \le b$,

$$\int_c^b |V_1(t)||x^\sigma(t)|^q \triangle t \le \int_c^b \eta(\sigma(t))|V_1(t)|\Big(\int_{\sigma(t)}^b y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau\Big) \triangle t$$

$$\le \int_c^b \eta(\sigma(t))|V_1(t)| \triangle t \int_{\sigma(c)}^b y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau.$$

Similarly, for $a \le \tau \le \sigma(t) \le \sigma(c) \le b$, we can obtain

$$\int_a^c |V_1(t)||x^\sigma(t)|^q \triangle t \le \int_a^c \xi(\sigma(t))|V_1(t)| \triangle t \int_a^{\sigma(c)} y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau$$

$$\le \int_c^b \eta(\sigma(t))|V_1(t)| \triangle t \int_a^{\sigma(c)} y^T(\tau)U(\tau)y(\tau)|y(\tau)|^{p-2}\Delta\tau.$$

These yield

$$\int_a^b |V_1(t)||x^\sigma(t)|^q \triangle t \le \int_c^b \eta(\sigma(t))|V_1(t)| \triangle t \int_a^b y^T(t)U(t)y(t)|y(t)|^{p-2}\Delta t$$

$$= \int_c^b \eta(\sigma(t))|V_1(t)| \triangle t \int_a^b |x^\sigma(t)|^{q-2}(x^\sigma(t))^T V(t)x^\sigma(t) \triangle t$$

$$\leq \int_c^b \eta(\sigma(t))|V_1(t)| \triangle t \int_a^b |V_1(t)||x^\sigma(t)|^q \triangle t.$$

Thus, we also obtain $\int_c^b \eta(\sigma(t))|V_1(t)| \triangle t \geq 1$. This completes the proof of Theorem 3.4.

**Theorem 3.5** Let $a, b \in \mathbb{T}$ with $\sigma(a) < b$ and $V_1 \in \mathbb{R}_s^{n\times n}$ with $V_1(t) \geq V(t)$. If (1.1) has a solution $(x(t), y(t))$ satisfying (2.1) on the interval $[a, b]_{\mathbb{T}}$, then the following inequalities hold:

$$\int_a^b |W(t)| \triangle t + \Big(\int_a^b |U(t)| \triangle t\Big)^{\frac{1}{p}} \Big(\int_a^b |V_1(t)| \triangle t\Big)^{\frac{1}{q}} \geq 2, \quad \text{if } q \geq 2,$$

$$\int_a^b |W(t)| \triangle t + \Big(\int_a^b |U(t)|^{\frac{p(p-2)}{2}+1} |(\sqrt{U(t)})^{-1}|^{p(p-2)} \triangle t\Big)^{\frac{1}{p}} \Big(\int_a^b |V_1(t)| \triangle t\Big)^{\frac{1}{q}} \geq 2, \quad \text{if } 1 < q < 2.$$

**Proof** From the proof of Theorem 3.1, we have

$$\int_a^b |y(t)|^{p-2} y^T(t) U(t) y(t) \triangle t = \int_a^b |x^\sigma(t)|^{q-2} (x^\sigma(t))^T V(t) x^\sigma(t) \triangle t.$$

It follows from the first equation of (1.1) that for all $a \leq t \leq b$,

$$x(t) = \int_a^t (-W(\tau)x^\sigma(\tau) - U(\tau)|y(\tau)|^{p-2}y(\tau)) \triangle \tau,$$

$$x(t) = \int_t^b (W(\tau)x^\sigma(\tau) + U(\tau)|y(\tau)|^{p-2}y(\tau)) \triangle \tau.$$

**Case I**: Assume that $q \geq 2$. We have

$$\begin{aligned}
|x(t)| &= \Big|\int_a^t (-W(\tau)x^\sigma(\tau) - U(\tau)|y(\tau)|^{p-2}y(\tau)) \triangle \tau\Big| \\
&\leq \int_a^t |W(\tau)x^\sigma(\tau) + U(\tau)|y(\tau)|^{p-2}y(\tau)| \triangle \tau \\
&\leq \int_a^t |W(\tau)x^\sigma(\tau)| \triangle \tau + \int_a^t |U(\tau)|y(\tau)|^{p-2}y(\tau)| \triangle \tau \\
&\leq \int_a^t |W(\tau)||x^\sigma(\tau)| \triangle \tau + \int_a^t |U(\tau)|^{1-\frac{1}{q}} (y^T(\tau)U(\tau)y(\tau))^{\frac{1}{q}} |y(\tau)|^{p-1-\frac{2}{q}} \triangle \tau.
\end{aligned}$$

Similarly, we have

$$|x(t)| \leq \int_t^b |W(\tau)||x^\sigma(\tau)| \triangle \tau + \int_t^b |U(\tau)|^{1-\frac{1}{q}} (y^T(\tau)U(\tau)y(\tau))^{\frac{1}{q}} |y(\tau)|^{p-1-\frac{2}{q}} \triangle \tau.$$

Then from Lemma 2.2 and Lemma 2.6, we obtain

$$\begin{aligned}
|x(t)| &\leq \frac{1}{2}\Big[\int_a^b |W(t)||x^\sigma(t)| \triangle t + \int_a^b |U(t)|^{1-\frac{1}{q}} (y^T(t)U(t)y(t))^{\frac{1}{q}} |y(t)|^{p-1-\frac{2}{q}} \triangle t\Big] \\
&\leq \frac{1}{2}\Big[\int_a^b |W(t)||x^\sigma(t)| \triangle t + \Big(\int_a^b |U(t)| \triangle t\Big)^{\frac{1}{p}} \Big(\int_a^b y^T(t)U(t)y(t)|y(t)|^{p-2} \triangle t\Big)^{\frac{1}{q}}\Big] \\
&= \frac{1}{2}\Big[\int_a^b |W(t)||x^\sigma(t)| \triangle t + \Big(\int_a^b |U(t)| \triangle t\Big)^{\frac{1}{p}} \Big(\int_a^b |x^\sigma(t)|^{q-2}(x^\sigma(t))^T V(t) x^\sigma(t) \triangle t\Big)^{\frac{1}{q}}\Big] \\
&\leq \frac{1}{2}\Big[\int_a^b |W(t)||x^\sigma(t)| \triangle t + \Big(\int_a^b |U(t)| \triangle t\Big)^{\frac{1}{p}} \Big(\int_a^b |V_1(t)||x^\sigma(t)|^q \triangle t\Big)^{\frac{1}{q}}\Big].
\end{aligned}$$

Denote $M = \max_{a\le t\le b}|x(t)| > 0$, then

$$M \le \frac{1}{2}\Big[\int_a^b |W(t)|M \triangle t + \Big(\int_a^b |U(t)| \triangle t\Big)^{\frac{1}{p}}\Big(\int_a^b |V_1(t)|M^q \triangle t\Big)^{\frac{1}{q}}\Big].$$

Thus

$$\int_a^b |W(t)| \triangle t + \Big(\int_a^b |U(t)| \triangle t\Big)^{\frac{1}{p}}\Big(\int_a^b |V_1(t)| \triangle t\Big)^{\frac{1}{q}} \ge 2.$$

**Case II**: Assume that $1 < q < 2$. Then $p \ge 2$ and

$$\begin{aligned}
|x(t)| &\le \int_a^t |W(\tau)x^\sigma(\tau) + U(\tau)|y(\tau)|^{p-2}y(\tau)| \triangle \tau \\
&\le \int_a^t |W(\tau)x^\sigma(\tau)| \triangle \tau + \int_a^t |U(\tau)|y(\tau)|^{p-2}y(\tau)| \triangle \tau \\
&\le \int_a^t |W(\tau)||x^\sigma(\tau)| \triangle \tau + \int_a^t |(\sqrt{U(\tau)})^{-1}|^{p-2}|U(\tau)|^{\frac{1}{2}} \\
&\quad \times (y^T(\tau)U(\tau)y(\tau))^{\frac{1}{q}}|\sqrt{U(\tau)}|^{\frac{(p-1)(p-2)}{p}}|y(\tau)|^{p-1-\frac{2}{q}} \triangle \tau.
\end{aligned}$$

and

$$\begin{aligned}
|x(t)| &\le \int_t^b |W(\tau)||x^\sigma(\tau)| \triangle \tau + \int_t^b |(\sqrt{U(\tau)})^{-1}|^{p-2}|U(\tau)|^{\frac{1}{2}} \\
&\quad \times (y^T(\tau)U(\tau)y(\tau))^{\frac{1}{q}}|\sqrt{U(\tau)}|^{\frac{(p-1)(p-2)}{p}}|y(\tau)|^{p-1-\frac{2}{q}} \triangle \tau.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
|x(t)| &\le \frac{1}{2}\Big[\int_a^b |W(t)||x^\sigma(t)| \triangle t + \int_a^b |(\sqrt{U(t)})^{-1}|^{p-2}|U(t)|^{\frac{1}{2}} \\
&\quad \times (y^T(t)U(t)y(t))^{\frac{1}{q}}|\sqrt{U(t)}|^{\frac{(p-1)(p-2)}{p}}|y(t)|^{p-1-\frac{2}{q}} \triangle t\Big] \\
&\le \frac{1}{2}\Big[\int_a^b |W(t)||x^\sigma(t)| \triangle t + \Big(\int_a^b (|(\sqrt{U(t)})^{-1}|^{p-2}|\sqrt{U(t)}|^{\frac{(p-1)(p-2)}{p}+1})^p \triangle t\Big)^{\frac{1}{p}} \\
&\quad \times \Big(\int_a^b ((y^T(t)U(t)y(t))^{\frac{1}{q}}|y(t)|^{p-1-\frac{2}{q}})^q \triangle t\Big)^{\frac{1}{q}}\Big] \\
&= \frac{1}{2}\Big[\int_a^b |W(t)||x^\sigma(t)| \triangle t + \Big(\int_a^b |U(t)|^{\frac{p(p-2)}{2}+1}|(\sqrt{U(t)})^{-1}|^{p(p-2)} \triangle t\Big)^{\frac{1}{p}} \\
&\quad \times \Big(\int_a^b |x^\sigma(t)|^{q-2}(x^\sigma(t))^T V(t)x^\sigma(t) \triangle t\Big)^{\frac{1}{q}}\Big] \\
&\le \frac{1}{2}\Big[\int_a^b |W(t)||x^\sigma(t)| \triangle t + \Big(\int_a^b |U(t)|^{\frac{p(p-2)}{2}+1}|(\sqrt{U(t)})^{-1}|^{p(p-2)} \triangle t\Big)^{\frac{1}{p}} \\
&\quad \times \Big(\int_a^b |V_1(t)||x^\sigma(t)|^q \triangle t\Big)^{\frac{1}{q}}\Big].
\end{aligned}$$

Similarly, we also have

$$\int_a^b |W(t)| \triangle t + \Big(\int_a^b |U(t)|^{\frac{p(p-2)}{2}+1}|(\sqrt{U(t)})^{-1}|^{p(p-2)} \triangle t\Big)^{\frac{1}{p}}\Big(\int_a^b |V_1(t)| \triangle t\Big)^{\frac{1}{q}} \ge 2.$$

This completes the proof of Theorem 3.5.

# REFERENCES

[1] S. Hilger, Analysis on measure chains − a unified approach to continuous and discrete calculus, Results Math., 18(1990): 18-56.

[2] L. Jiang, Z. Zhou, Lyapunov inequality for linear Hamiltonian systems on time scales, J. Math. Anal. Appl. 310(2005)579-593.

[3] F. Wong, S. Yu, C. Yeh, W. Lian, Lyapunov's inequality on time scales, Appl. Math. Lett. 19(2006)1293-1299.

[4] X. Liu, M. Tang, Lyapunov-type inequality for higher order difference equations, Appl. Math. Comput. 232(2014)666-669.

[5] M. Bohner, A. Peterson, Dynamic Equations on Time Scales: An Introduction with Applications, Birkhauser, Boston, 2001.

[6] M. Bohner, A. Peterson, Advances in Dynamic Equations on Time Scales, Birkhauser, Boston, 2003.

[7] Q. Zhang, X. He, J. Jiang, On Lyapunov-type inequalities for nonlinear dynamic systems on time scales, Comput. Math. Appl. 62(2011): 4028-4038.

[8] A. Tiryaki, D. Cakmak, M. F. Aktas, Lyapunov-type inequalities for a certain class of nonlinear systems, Comput. Math. Appl. 64(2012): 1804-1811.

[9] G. Sh. Guseinov, B. Kaymakcalan, Lyapunov inequalities for discrete linear Hamiltonian systems, Comput. Math. Appl. 45(2003): 1399-1416.

[10] X. He, Q. Zhang, X. Tang, On inequalities of Lyapunov for linear Hamiltonian systems on time scales, J. Math. Anal. Appl. 381(2011): 695-705.

[11] X. Tang, M. Zhang, Lyapunov inequalities and stability for linear Hamiltonian systems, J. Differential Equations. 252(2012): 358-381.

[12] J. Liu, T. Sun, X. Kong, Q. He, Lyapunov inequalities of linear Hamiltonian systems on time scales, J. Comput. Anal. Appl. 21(2016)1160-1169.

[13] R. P. Agarwal, M. Bohner, P. Rehak, Half-linear dynamic equations, Nonlinear Anal. Appl. (2003)1-56.

[14] R. P. Agarwal, A. Ozbekler, Lyapunov type inequalities for even order differential equations with mixed nonlinearities, J. Inequal. Appl. 2015(2015)142, 10 pages.

[15] R. P. Agarwal, A. Ozbekler, Disconjugacy via Lyapunov and Valée–Poussin type inequalities for forced differential equations, Appl. Math. Comput., 265(2015)456-468.

[16] R. P. Agarwal, A. Ozbekler, Lyapunov type inequalities for second order sub and super–half–linear differential equations, Dynam. Sys. Appl. 24(2015)211-220.

# A new three-step iterative method for a countable family of pseudo-contractive mappings in Hilbert spaces

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**Abstract:** In this paper, we propose a new three-step iterative method for a countable family of pseudo-contractive mappings in a real Hilbert space. We also prove the strong convergence of the proposed iterative algorithm under appropriate conditions.

**Key words:** pseudo-contractive mapping; iterative method; fixed point; strong convergence

**AMS subject classification (2000):** 47H09

## 1 Introduction

In this paper, we assume that $H$ is a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$, $C$ is a nonempty closed convex subset of $H$ and $T : C \to C$ is a self-mapping of $C$. $\mathcal{F}(T)$ denotes the fixed point set of the mapping $T$. Recall that $T$ is called a $k$-strictly pseudo-contractive mapping if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall\, x, y \in C, \tag{1.1}$$

and $T$ is called a pseudo-contractive mapping if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \quad \forall\, x, y \in C. \tag{1.2}$$

It is obvious that $k = 0$, then the mapping $T$ is nonexpansive, that is

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall\, x, y \in C. \tag{1.3}$$

Finding the fixed points of nonexpansive mappings is an important topic in the theory of nonexpansive mappings and has wide applications in a number of applied areas, such as the convex feasibility problem [1, 2], the split feasibility problem [3], image recovery and signal

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processing [4]. After that, as an important generalization of nonexpansive mappings, strictly pseudo-contractive mappings become one of the most interesting studied class of nonexpansive mappings. In fact, strictly pseudo-contractive mappings have more powerful application than nonexpansive mappings do such as in solving inverse problem [5].

Iterative methods for nonexpansive mappings have been extensively investigated (see *e.g.*, [6–16, 31–33] and the references contained therein). However, iterative methods for strictly pseudo-contractive mappings are far less developed than those for nonexpansive mappings and the reason is probably that the second term appearing on the right hand side of (1.1) impedes the convergence analysis for iterative algorithms used to find a fixed point of the strictly pseudo-contractive mapping $T$.

The most general iterative algorithm for nonexpansive mappings studied by many authors is Mann's iteration algorithm [18] which is as following:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad n \geq 0, \tag{1.4}$$

where $x_0 \in C$ is chosen arbitrarily and $\{\alpha_n\}$ is a real sequence in $(0,1)$. Under the following additional assumptions: (i) $\lim_{n\to\infty} \alpha_n = 0$ and (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$, the sequence $\{x_n\}$ generated by (1.4) is generally referred to as Mann's iteration algorithm in the light of [18]. The Mann's iteration algorithm dose not generally converge to a fixed point of $T$ even the fixed point exists. For example, $C$ is a nonempty closed convex and bounded subset of a real Hilbert space, $T : C \to C$ is nonexpansive, one can only prove that the sequence generated by Mann's iteration algorithm (1.4) with the assumptions (i) and (ii) is an approximate fixed point sequence, that is, $\|x_n - Tx_n\| \to 0$ as $n \to \infty$. In [19], Reich proved that if $X$ is a uniformly convex Banach space with a Fréchet differentiable norm and if $\{\alpha_n\}$ is chosen such that $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ defined by (1.4) converges weakly to a fixed point of $T$. To get the sequence $\{x_n\}$ to converge strongly to a fixed point of $T$ (when such a fixed point exists), some type of compactness condition must be additionally imposed either on $C$ (*e.g.*, $C$ is compact) or on $T$ (*e.g.*, $T$ is demicompact or $T$ is semicompact, see [20, 21]).

The first convergence result for $k$-strictly pseudo-contractive mappings was proposed by Browder and Petryshyn [22] in 1967. They proved that if the sequence $\{x_n\}$ is generated by the following:

$$x_{n+1} = \alpha x_n + (1-\alpha)Tx_n, \quad n \geq 0, \tag{1.5}$$

for any starting point $x_0 \in C$ and $\alpha$ is a constant such that $k < \alpha < 1$, then the sequence $\{x_n\}$ converges weakly to a fixed point of $k$-strictly pseudo-contractive mapping $T$. In [23], Marino and Xu extended the result of Browder and Petryshyn [22] to Mann's iteration algorithm (1.4), they proved that the sequence $\{x_n\}$ generated by (1.4) converges weakly to a fixed point of $k$-strictly pseudo-contractive mapping $T$ for the conditions that $k < \alpha_n < 1$ for all $n$ and $\sum_{n=0}^{\infty}(\alpha_n - k)(1-\alpha_n) = \infty$.

However, the well known strong convergence result for pseudo-contractive mappings is Ishikawa's iteration algorithm which was proved by Ishikawa [24] in 1974 and it is more general than that of Mann's iteration algorithm (1.4) in some sense. More precisely, he got the following theorem.

**Theorem 1.1** ([24]) *Let $C$ be a convex compact subset of a Hilbert space $H$ and let $T : C \to C$ be a Lipschitz pseudo-contractive mapping. For any $x_1 \in C$, suppose the sequence $\{x_n\}$ is defined iteratively for each $n \geq 1$ by*

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T x_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \end{cases} \tag{1.6}$$

*where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences of positive number that satisfy the following there conditions:* (i) $0 \leq \alpha_n \leq \beta_n \leq 1$; (ii) $\lim_{n\to\infty} \beta_n = 0$; (iii) $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$. *Then the sequence $\{x_n\}$ converges strongly to a fixed point of $T$.*

In 2001, Chidume and Mutangadura [25] gave an example to show that the Mann's iteration algorithm (1.4) failed to be convergent to a fixed point of Lipschitz pseudo-contractive mappings. In order to obtain a strong convergence result for pseudo-contractive mappings without the compactness assumption, Zhou [26] established the hybrid Ishikawa algorithm for Lipschitz pseudo-contractive mappings as following:

**Theorem 1.2** ([26]) *Let $C$ be a closed convex subset of a real Hilbert space $H$ and let $T : C \to C$ be a Lipschitz pseudo-contraction such that $\mathcal{F}(T) \neq \phi$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $(0,1)$ satisfying the conditions:* (i) $\alpha_n \leq \beta_n, \forall n \geq 0$; (ii) $\liminf_{n\to\infty} \alpha_n > 0$; (iii) $\limsup_{n\to\infty} \alpha_n \leq \alpha < \dfrac{1}{\sqrt{1+L^2}+1}$, $n \geq 0$, *where $L \geq 1$ is the Lipschitzian constant of $T$. Let a sequence $\{x_n\}$ be generated by*

$$\begin{cases} x_0 \in C, \\ y_n = (1-\alpha_n)x_n + \alpha_n T x_n, \\ z_n = (1-\beta_n)x_n + \beta_n T y_n, \\ C_n = \{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 - \alpha_n\beta_n(1 - 2\alpha_n - L^2\alpha_n^2)\|x_n - Tx_n\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n \geq 0. \end{cases} \tag{1.7}$$

*Then, $\{x_n\}$ converges strongly to a fixed point $v$ of $T$, where $v = P_{\mathcal{F}(T)}(x_0)$.*

We observe that the iterative algorithm (1.7) generates a sequence $\{x_n\}$ by projecting $x_0$ on to the intersection of the suitably constructed closed convex sets $C_n$ and $Q_n$. Recently, Yao *et al.* [27] introduced the hybrid iterative algorithm which just involved one closed convex set for pseudo-contractive mappings in Hilbert spaces as follows:

Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $T : C \to C$ be a pseudo-contractive mapping. Let $\{\alpha_n\}$ be a sequence in $(0, 1)$. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, define a sequence $\{x_n\}$ of $C$ as follows:

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n T x_n, \\ C_{n+1} = \{z \in C_n : \|\alpha_n(I-T)y_n\|^2 \le 2\alpha_n\langle x_n - z, (I-T)y_n\rangle\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \in N. \end{cases} \tag{1.8}$$

**Theorem 1.3** ( [27]) *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $T : C \to C$ be a L-Lipschitz pseudo-contractive mapping such that $\mathcal{F}(T) \neq \phi$. Assume the sequence $\alpha_n \in [a, b]$ for some $a, b \in (0, \frac{1}{L+1})$. Then the sequence $\{x_n\}$ generated by (1.8) converges strongly $P_{\mathcal{F}(T)}(x_0)$.*

In [28], Tang *et al.* proposed the hybrid algorithm (1.8) to the Ishikawa's iteration algorithm (1.6) and got the following result.

Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $T : C \to C$ be a pseudo-contractive mapping. Let $\{\alpha_n\}$, $\{\beta_n\}$ be two sequences in $[0, 1]$. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, define a sequence $\{x_n\}$ of $C$ as follows:

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n T z_n, \\ z_n = (1-\beta_n)x_n + \beta_n T x_n, \\ C_{n+1} = \{z \in C_n : \|\alpha_n(I-T)y_n\|^2 \le 2\alpha_n\langle x_n - z, (I-T)y_n\rangle \\ \qquad\qquad + 2\alpha_n\beta_n L\|x_n - Tx_n\| \cdot \|y_n - x_n + \alpha_n(I-T)y_n\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \ge 1. \end{cases} \tag{1.9}$$

**Theorem 1.4** ( [28]) *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $T : C \to C$ be a L-Lipschitz pseudo-contractive mapping with $L \ge 1$ such that $\mathcal{F}(T) \neq \phi$. Assume the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $(0, 1)$ satisfying:* (i) $b \le \alpha_n < \alpha_n(L+1)(1+\beta_n L) < a < 1$, *for some $a, b \in (0, 1)$;* (ii) $\lim_{n\to\infty} \beta_n = 0$. *Then the sequence $\{x_n\}$ generated by (1.9) converges strongly $P_{\mathcal{F}(T)}(x_0)$.*

Recently, Zegeye *et al.* [29] generalized Ishikawa's iteration algorithm (1.6) to a common fixed point of a finite family of Lipschitz pseudo-contractive mappings and obtained the following theorem.

**Theorem 1.5** ( [29]) *Let $C$ be a nonempty, closed convex subset of a real Hilbert space $H$. Let $T_i : C \to C$, $i = 1, 2, \cdots, N$, be a finite family of Lipschitz pseudo-contractive mappings with Lipschitzian constants $L_i$, for $i = 1, 2, \cdots, N$, respectively. Assume that the interior of $\mathcal{F} := \bigcap_{i=1}^{N} \mathcal{F}(T_i)$ is nonempty. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0 \in C$ by*

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T_n x_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_n y_n, \end{cases} \tag{1.10}$$

*where* $T_n := T_{n(mod\ N)}$ *and* $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ *satisfying the following conditions:* (i) $\alpha_n \leq \beta_n, \forall\ n \geq 0$*;* (ii) $\liminf_{n\to\infty} \alpha_n = \alpha > 0$*;* (iii) $\sup_{n\geq 1} \beta_n \leq \beta < \dfrac{1}{\sqrt{1+L^2}+1}$ *for* $L := \max\{L_i : i = 1, 2, \cdots, N\}$*. Then,* $\{x_n\}$ *converges strongly to a common fixed point of* $\{T_1, T_2, \cdots, T_N\}$*.*

In [30], Cheng *et al.* extended the algorithm (1.10) to a countable family of pseudo-contractive mappings and gave a three-step iterative method, which is as follows:

**Theorem 1.6** ( [30]) *Let* $C$ *be a nonempty, closed convex subset of a real Hilbert space* $H$*, let* $\{T_n\}_{n=1}^{\infty} : C \to C$ *be a countable family of uniformly closed and uniformly Lipschitz pseudo-contractive mappings with Lipschitzian constants* $L_n$*, let* $L := \sup_{n\geq 1} L_n$*. Assume that the interior of* $\mathcal{F} := \bigcap_{n=1}^{\infty} \mathcal{F}(T_n)$ *is nonempty. Let* $\{x_n\}$ *be a sequence generated from an arbitrary* $x_0 \in C$ *by the following algorithm:*

$$\begin{cases} z_n = (1-\gamma_n)x_n + \gamma_n T_n x_n, \\ y_n = (1-\beta_n)x_n + \beta_n T_n z_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_n y_n, \end{cases} \tag{1.11}$$

*where* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$ *satisfying the following conditions:* (i) $\alpha_n \leq \beta_n \leq \gamma_n, \forall\ n \geq 0$*;* (ii) $\liminf_{n\to\infty} \alpha_n = \alpha > 0$*;* (iii) $\sup_{n\geq 1} \gamma_n \leq \gamma$ *with* $\gamma^3 L^4 + 2\gamma^2 L^3 + \gamma^2 L^2 + \gamma L^2 + 2\gamma < 1$*. Then,* $\{x_n\}$ *converges strongly to* $x^* \in \mathcal{F}$*.*

**Remark 1.1** *The condition* (iii) *of the Theorem 1.6 is not correct, it is replaced by* $\sup_{n\geq 1} \gamma_n \leq \gamma$ *with* $\gamma^3 L^4 + 2\gamma^2 L^3 + \gamma^2 L^2 + 2\gamma L^2 + 2\gamma < 1$*.*

Motivated and inspired by the above works, in this paper, we propose a new three-step iterative method for a countable family of pseudo-contractive mappings in Hilbert spaces and prove its strong convergence theorem under appropriate conditions.

## 2 Preliminaries

In this section, we recall some definitions and useful results which will be used in the next section.

**Definition 2.1** *Let* $C$ *be a subset of a real Hilbert space* $H$*.*

(1) *A mapping* $T : C \to H$ *is said to be* $L$*-Lipschitz if there exists* $L > 0$ *such that*

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall\ x,\ y \in C.$$

*When* $L = 1$*,* $T$ *is nonexpansive. If* $L < 1$*,* $T$ *is called a contraction. It is easy to see that every contractive mapping is nonexpansive and every nonexpansive mapping is Lipschitz.*

(2) *A countable family of mappings* $\{T_n\}_{n=1}^{\infty} : C \to H$ *is said to be uniformly Lipschitz with Lipschitzian constants* $L_n > 0$, $n \geq 1$, *if there exists* $0 < L := \sup_{n\geq 1} L_n$ *such that*

$$\|T_n x - T_n y\| \leq L\|x - y\|, \quad \forall\ x,\ y \in C,\ n \geq 1.$$

(3) *A countable family of mappings* $\{T_n\}_{n=1}^{\infty} : C \to H$ *is said to be uniformly closed if* $x_n \to x^*$ *and* $\|x_n - T_n x_x\| \to 0$ *imply* $x^* \in \bigcap_{n=1}^{\infty} \mathcal{F}(T_n)$.

**Definition 2.2** *A mapping* $T$ *with domain* $\mathcal{D}(T)$ *and range* $\mathcal{R}(T)$ *in a real Hilbert space* $H$ *is said to be monotone if the inequality*

$$\|x - y\| \leq \|x - y + s(Tx - Ty)\|$$

*holds for every* $x,\ y \in \mathcal{D}(T)$ *and for all* $s > 0$.

We observe that

$$\begin{aligned} T \text{ is monotone} &\Leftrightarrow \langle Tx - Ty, x - y\rangle \geq 0 \\ &\Leftrightarrow \|(I - T)x - (I - T)y\|^2 \leq \|x - y\|^2 + \|Tx - Ty\|^2 \\ &\Leftrightarrow \|Ax - Ay\|^2 \leq \|x - y\|^2 + \|(I - A)x - (I - A)y\|^2,\ A := I - T \\ &\Leftrightarrow A \text{ is pseudo-contractive.} \end{aligned}$$

Furthermore, a zero of $T$ is a fixed point of $A$, that is,

$$x \in \mathcal{N}(T) := \{x \in \mathcal{D}(T) : Tx = 0\} \Leftrightarrow x \in \mathcal{F}(A) := \{x \in \mathcal{D}(A) : Ax = x\}.$$

**Lemma 2.1** *Let* $H$ *be a real Hilbert space. Then for* $\alpha \in [0, 1]$ *the following equality*

$$\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2$$

*holds for all* $x, y \in H$.

**Lemma 2.2** *If the sequences* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1)$ *satisfying the following conditions:*

(i) $\beta_n \leq \gamma_n, \forall\ n \geq 1$,

(ii) $(1 - \alpha)\gamma + \alpha\beta(\gamma^2 L^2 + 2\gamma - 1) < 0$,

*where* $\alpha = \liminf_{n\to\infty} \alpha_n$, $\beta = \liminf_{n\to\infty} \beta_n$, $\gamma \geq \sup_{n\geq 1} \gamma_n$, *and* $L > 0$ *is a constant. Then, we have*

$$\alpha > 0, \beta > 0 \text{ and } (1 - \alpha_n)\gamma_n + \alpha_n\beta_n(\gamma_n^2 L^2 + 2\gamma_n - 1) < 0.$$

**Proof.** On one hand, it is obvious that $\alpha > 0$, $\beta > 0$ and $\gamma^2 L^2 + 2\gamma - 1 < 0$ because of $(1-\alpha)\gamma+\alpha\beta(\gamma^2 L^2+2\gamma-1) < 0$. And we get that $(1-\alpha_n)\gamma \leq (1-\alpha)\gamma$ and $\alpha_n\beta_n(\gamma^2 L^2+2\gamma-1) \leq \alpha\beta(\gamma^2 L^2 + 2\gamma - 1)$. Then

$$(1 - \alpha_n)\gamma + \alpha_n\beta_n(\gamma^2 L^2 + 2\gamma - 1) \leq (1 - \alpha)\gamma + \alpha\beta(\gamma^2 L^2 + 2\gamma - 1) < 0.$$

On the other hand, it is easy to know that $(1-\alpha_n)\gamma_n \leq (1-\alpha_n)\gamma$ and $\alpha_n\beta_n(\gamma_n^2+2\gamma_n-1) \leq \alpha_n\beta_n(\gamma^2L^2+2\gamma-1)$. We can obtain

$$(1-\alpha_n)\gamma_n + \alpha_n\beta_n(\gamma_n^2+2\gamma_n-1) \leq (1-\alpha_n)\gamma + \alpha_n\beta_n(\gamma^2L^2+2\gamma-1) < 0.$$

Hence $(1-\alpha_n)\gamma_n + \alpha_n\beta_n(\gamma_n^2L^2+2\gamma_n-1) < 0$. □

## 3 The main result

**Theorem 3.1** *Let $C$ be a nonempty, closed convex subset of a real Hilbert space $H$, let $\{T_n\}_{n=1}^{\infty}: C \to C$ be a countable family of uniformly closed and uniformly Lipschitz pseudo-contractive mappings with Lipschitzian constants $L_n$, let $L := \sup_{n\geq 1} L_n$. Assume that the interior of $\mathcal{F} := \bigcap_{n=1}^{\infty}\mathcal{F}(T_n)$ is nonempty. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_1 \in C$ by*

$$\begin{cases} z_n = (1-\gamma_n)x_n + \gamma_n T_n x_n, \\ y_n = (1-\beta_n)x_n + \beta_n T_n z_n, \\ x_{n+1} = (1-\alpha_n)z_n + \alpha_n y_n, \end{cases} \tag{3.1}$$

*where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$ satisfying the following conditions:*

(i) $\beta_n \leq \gamma_n, \forall\ n \geq 1$,

(ii) $(1-\alpha)\gamma + \alpha\beta(\gamma^2L^2+2\gamma-1) < 0$,

*where $\alpha = \liminf_{n\to\infty}\alpha_n$, $\beta = \liminf_{n\to\infty}\beta_n$ and $\gamma \geq \sup_{n\geq 1}\gamma_n$. Then, $\{x_n\}$ converges strongly to $x^* \in \mathcal{F}$.*

**Proof.** Take $p \in \mathcal{F}$ arbitrarily. By (3.1) and Lemma 2.1, we have

$$\begin{aligned} \|x_{n+1}-p\|^2 &= \|(1-\alpha_n)z_n + \alpha_n y_n - p\|^2 \\ &= \|(1-\alpha_n)(z_n-p) + \alpha_n(y_n-p)\|^2 \\ &= (1-\alpha_n)\|z_n-p\|^2 + \alpha_n\|y_n-p\|^2 - \alpha_n(1-\alpha_n)\|z_n-y_n\|^2 \\ &\leq (1-\alpha_n)\|z_n-p\|^2 + \alpha_n\|y_n-p\|^2, \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} \|z_n-p\|^2 &= \|(1-\gamma_n)x_n + \gamma_n T_n x_n - p\|^2 \\ &= \|(1-\gamma_n)(x_n-p) + \gamma_n(T_n x_n - p)\|^2 \\ &= (1-\gamma_n)\|x_n-p\|^2 + \gamma_n\|T_n x_n - p\|^2 - \gamma_n(1-\gamma_n)\|x_n - T_n x_n\|^2 \\ &\leq (1-\gamma_n)\|x_n-p\|^2 + \gamma_n(\|x_n-p\|^2 + \|x_n - T_n x_n\|^2) \\ &\quad - \gamma_n(1-\gamma_n)\|x_n - T_n x_n\|^2 \\ &= \|x_n-p\|^2 + \gamma_n^2\|x_n - T_n x_n\|^2, \end{aligned} \tag{3.3}$$

where the inequality is based on that $\{T_n\}_{n=1}^{\infty}$ is a countable family of pseudo-contractive mappings. Similarly, we can get

$$\begin{aligned}\|y_n-p\|^2 &= \|(1-\beta_n)x_n+\beta_nT_nz_n-p\|^2\\ &= \|(1-\beta_n)(x_n-p)+\beta_n(T_nz_n-p)\|^2\\ &= (1-\beta_n)\|x_n-p\|^2+\beta_n\|T_nz_n-p\|^2-\beta_n(1-\beta_n)\|x_n-T_nz_n\|^2\\ &\leq (1-\beta_n)\|x_n-p\|^2+\beta_n(\|z_n-p\|^2+\|z_n-T_nz_n\|^2)\\ &\quad -\beta_n(1-\beta_n)\|x_n-T_nz_n\|^2. \end{aligned}\tag{3.4}$$

In addition, using (3.1), we have that

$$\begin{aligned}\|z_n-T_nz_n\|^2 &= \|(1-\gamma_n)x_n+\gamma_nT_nx_n-T_nz_n\|^2\\ &= \|(1-\gamma_n)(x_n-T_nz_n)+\gamma_n(T_nx_n-T_nz_n)\|^2\\ &= (1-\gamma_n)\|x_n-T_nz_n\|^2+\gamma_n\|T_nx_n-T_nz_n\|^2-\gamma_n(1-\gamma_n)\|x_n-T_nx_n\|^2\\ &\leq (1-\gamma_n)\|x_n-T_nz_n\|^2+\gamma_nL^2\|x_n-z_n\|^2-\gamma_n(1-\gamma_n)\|x_n-T_nx_n\|^2\\ &= (1-\gamma_n)\|x_n-T_nz_n\|^2+\gamma_nL^2\|\gamma_n(x_n-T_nx_n)\|^2-\gamma_n(1-\gamma_n)\|x_n-T_nx_n\|^2\\ &= (1-\gamma_n)\|x_n-T_nz_n\|^2+\gamma_n(\gamma_n^2L^2+\gamma_n-1)\|x_n-T_nx_n\|^2, \end{aligned}\tag{3.5}$$

where the inequality is based on that $\{T_n\}_{n=1}^{\infty}$ is a countable family of uniformly Lipschitz mappings. Substituting (3.3) and (3.5) into (3.4), we obtain that

$$\begin{aligned}\|y_n-p\|^2 &\leq (1-\beta_n)\|x_n-p\|^2+\beta_n\Big(\|x_n-p\|^2+\gamma_n^2\|x_n-T_nx_n\|^2\Big)\\ &\quad +\beta_n\Big((1-\gamma_n)\|x_n-T_nz_n\|^2+\gamma_n(\gamma_n^2L^2+\gamma_n-1)\|x_n-T_nx_n\|^2\Big)\\ &\quad -\beta_n(1-\beta_n)\|x_n-T_nz_n\|^2\\ &= \|x_n-p\|^2+\beta_n\gamma_n(\gamma_n^2L^2+2\gamma_n-1)\|x_n-T_nx_n\|^2+\beta_n(\beta_n-\gamma_n)\|x_n-T_nz_n\|^2\\ &\leq \|x_n-p\|^2+\beta_n\gamma_n(\gamma_n^2L^2+2\gamma_n-1)\|x_n-T_nx_n\|^2, \end{aligned}\tag{3.6}$$

where the last inequality is based on the condition (i). Therefore, substituting (3.3) and (3.6) into (3.2), we get

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leq (1-\alpha_n)\Big(\|x_n-p\|^2+\gamma_n^2\|x_n-T_nx_n\|^2\Big)\\ &\quad +\alpha_n\Big(\|x_n-p\|^2+\beta_n\gamma_n(\gamma_n^2L^2+2\gamma_n-1)\|x_n-T_nx_n\|^2\Big)\\ &= \|x_n-p\|^2+\Big((1-\alpha_n)\gamma_n^2+\alpha_n\beta_n\gamma_n(\gamma_n^2L^2+2\gamma_n-1)\Big)\|x_n-T_nx_n\|^2. \end{aligned}\tag{3.7}$$

According to the conditions and Lemma 2.2, inequality (3.7) implies that

$$\|x_{n+1}-p\|^2\leq\|x_n-p\|^2.\tag{3.8}$$

It is obvious that $\lim_{n\to\infty}\|x_n-p\|$ exists, then $\{\|x_n-p\|\}$ is bounded. This implies that $\{x_n\}$, $\{T_nx_n\}$, $\{z_n\}$, $\{T_nz_n\}$ and $\{y_n\}$ are also bounded.

Furthermore, we have that

$$\|x_n-p\|^2=\|x_n-x_{n+1}\|^2+\|x_{n+1}-p\|^2+2\langle x_{n+1}-p,x_n-x_{n+1}\rangle.$$

This implies

$$\langle x_{n+1}-p,x_n-x_{n+1}\rangle+\frac{1}{2}\|x_n-x_{n+1}\|^2=\frac{1}{2}(\|x_n-p\|^2-\|x_{n+1}-p\|^2). \tag{3.9}$$

Moreover, since the interior of $\mathcal{F}$ is nonempty, then there exists $p^*\in\mathcal{F}$ and $r>0$ such that $p^*+rh\in\mathcal{F}$ whenever $\|h\|\le 1$. Thus, from (3.8), we have

$$\begin{aligned}0 &\le \langle x_{n+1}-(p^*+rh),x_n-x_{n+1}\rangle+\frac{1}{2}\|x_n-x_{n+1}\|^2\\ &= \frac{1}{2}(\|x_n-(p^*+rh)\|^2-\|x_{n+1}-(p^*+rh)\|^2).\end{aligned} \tag{3.10}$$

From (3.9) and (3.10), we obtain that

$$\begin{aligned}r\langle h,x_n-x_{n+1}\rangle &\le \langle x_{n+1}-p^*,x_n-x_{n+1}\rangle+\frac{1}{2}\|x_n-x_{n+1}\|^2\\ &= \frac{1}{2}(\|x_n-p^*\|^2-\|x_{n+1}-p^*\|^2).\end{aligned} \tag{3.11}$$

Since $h$ with $\|h\|\le 1$ is arbitrary, we can take $h=\dfrac{x_n-x_{n+1}}{\|x_n-x_{n+1}\|}$ with $\|h\|=1$, then

$$\|x_n-x_{n+1}\|\le\frac{1}{2r}(\|x_n-p^*\|^2-\|x_{n+1}-p^*\|^2).$$

So, for $n>m$, we can get

$$\begin{aligned}\|x_m-x_n\| &= \|(x_m-x_{m+1})+(x_{m+1}-x_{m+2})+\cdots+(x_{n-1}-x_n)\|\\ &\le \sum_{i=m}^{n-1}\|x_i-x_{i+1}\|\\ &\le \sum_{i=m}^{n-1}\frac{1}{2r}(\|x_i-p^*\|^2-\|x_{i+1}-p^*\|^2)\\ &= \frac{1}{2r}(\|x_m-p^*\|^2-\|x_n-p^*\|^2).\end{aligned}$$

From (3.8), we know that $\{\|x_n-p^*\|^2\}$ converges. Therefore, $\{x_n\}$ is a Cauchy sequence. Since $C$ is closed subset of Hilbert space $H$, then there exists $x^*\in C$ such that

$$x_n\to x^*\in C. \tag{3.12}$$

Furthermore, from the conditions and Lemma 2.2, we have

$$0 < \beta\Big((\alpha-1)\gamma+\alpha\beta(1-2\gamma-\gamma^2L^2)\Big)$$

$$\begin{aligned} &\leq \gamma_n\Big((\alpha_n-1)\gamma_n+\alpha_n\beta_n(1-2\gamma_n-\gamma_n^2L^2)\Big)\\ &= (\alpha_n-1)\gamma_n^2+\alpha_n\beta_n\gamma_n(1-2\gamma_n-\gamma_n^2L^2). \end{aligned} \tag{3.13}$$

Then, by (3.7) and (3.13), we conclude that

$$\begin{aligned} &\Big((\alpha-1)\gamma\beta+\alpha\beta^2(1-2\gamma-\gamma^2L^2)\Big)\sum_{n=1}^{\infty}\|x_n-T_nx_n\|^2\\ \leq &\sum_{n=1}^{\infty}\Big((\alpha_n-1)\gamma_n^2+\alpha_n\beta_n\gamma_n(1-2\gamma_n-\gamma_n^2L^2)\Big)\|x_n-T_nx_n\|^2\\ \leq &\sum_{n=1}^{\infty}(\|x_n-p\|^2-\|x_{n+1}-p\|^2)<\infty, \end{aligned}$$

from which it follows that

$$\lim_{n\to\infty}\|x_n-T_nx_n\|=0. \tag{3.14}$$

Since $\{T_n\}_{n=1}^{\infty}$ are uniformly closed mappings, then from (3.12) and (3.14), we can obtain

$$x^*\in\bigcap_{n=1}^{\infty}\mathcal{F}(T_n)=\mathcal{F}.$$

The proof is complete. □

**Remark 3.1** *We now give an example of a countable family of uniformly closed and uniformly Lipschitz pseudo-contractive mappings with the interior of the common fixed points nonempty. This example comes from* [30]. *Suppose that* $H:=R$ *and* $C:=[-1,1]\in H$. *Let* $\{T_n\}_{n=1}^{\infty}:C\to C$ *be defined by*

$$T_nx:=\begin{cases}x, & x\in[-1,0),\\ (\dfrac{1}{2^n}+\dfrac{1}{2})x, & x\in[0,1].\end{cases}$$

*Then* $\mathcal{F}:=\bigcap_{n=1}^{\infty}\mathcal{F}(T_n)=[-1,0]$, *and hence the interior of the common fixed points is nonempty. Moreover, it is easy to show that* $\{T_n\}_{n=1}^{\infty}$ *is a countable family of uniformly closed and uniformly Lipschitz pseudo-contractive mappings with Lipschitz constant* $L:=\sup_{n\geq1}L_n=2$.

*For this example, we can let* $\alpha_n=\dfrac{3}{4}+\dfrac{1}{n+4}$, $\beta_n=\dfrac{1}{10}+\dfrac{1}{n+40}$ *and* $\gamma_n=\dfrac{3}{20}-\dfrac{1}{n+40}$ *for* $n\geq1$. *Then* $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}\subset(0,1)$ *and* $\beta_n\leq\gamma_n,\forall\ n\geq1$. *Furthermore,* $\alpha=\liminf_{n\to\infty}\alpha_n=\dfrac{3}{4}$, $\beta=\liminf_{n\to\infty}\beta_n=\dfrac{1}{10}$, $\sup_{n\geq1}\gamma_n\leq\dfrac{3}{20}$, *and*

$$(1-\alpha)\gamma+\alpha\beta(\gamma^2L^2+2\gamma-1)=(1-\frac{3}{4})\times\frac{3}{20}+\frac{3}{4}\times\frac{1}{10}\times\Big((\frac{3}{20})^2\times2^2+2\times\frac{3}{20}-1\Big)=-\frac{33}{4000}<0.$$

*It satisfies all conditions in Theorem 3.1. Hence, from Theorem 3.1, we can obtain the sequence* $\{x_n\}$ *generated by (3.1) and staring with an arbitrary* $x_1\in C$ *will converges strongly to a common fixed point of* $\{T_n\}_{n=1}^{\infty}$.

# 4 Applications

If in Theorem 3.1, we consider a finite family of Lipschitz pseudo-contractive mappings, then we have the following result.

**Theorem 4.1** *Let $C$ be a nonempty, closed convex subset of a real Hilbert space $H$, let $\{T_i\}_{i=1}^N : C \to C$ be a finite family of uniformly closed and Lipschitz pseudo-contractive mappings with Lipschitzian constants $L_i$, for $i = 1, 2, \cdots, N$, respectively. Assume that the interior of $\mathcal{F} := \bigcap_{i=1}^N \mathcal{F}(T_i)$ is nonempty. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_1 \in C$ by*

$$\begin{cases} z_n = (1-\gamma_n)x_n + \gamma_n T_n x_n, \\ y_n = (1-\beta_n)x_n + \beta_n T_n z_n, \\ x_{n+1} = (1-\alpha_n)z_n + \alpha_n y_n, \end{cases} \tag{4.1}$$

*where $T_n := T_{n(mod\ N)}$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$ satisfying the following conditions:*

(i) $\beta_n \leq \gamma_n, \forall\ n \geq 1$,

(ii) $(1-\alpha)\gamma + \alpha\beta(\gamma^2 L^2 + 2\gamma - 1) < 0$,

*where $\alpha = \liminf_{n\to\infty} \alpha_n$, $\beta = \liminf_{n\to\infty} \beta_n$ and $\gamma \geq \sup_{n\geq 1} \gamma_n$, for $L := \max\{L_i : i = 1, 2, \cdots, N\}$. Then, $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2, \cdots, T_N\}$.*

If in Theorem 3.1, we consider a single Lipschitz pseudo-contractive mapping, then we may add a condition that is $\sum_{n=1}^{\infty} \gamma_n = \infty$.

**Theorem 4.2** *Let $C$ be a nonempty, closed convex subset of a real Hilbert space $H$, let $T : C \to C$ be a Lipschitz pseudo-contractive mapping with Lipschitzian constant $L$. Assume that the interior of $\mathcal{F}(T)$ is nonempty. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_1 \in C$ by*

$$\begin{cases} z_n = (1-\gamma_n)x_n + \gamma_n T x_n, \\ y_n = (1-\beta_n)x_n + \beta_n T z_n, \\ x_{n+1} = (1-\alpha_n)z_n + \alpha_n y_n, \end{cases} \tag{4.2}$$

*where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$ satisfying the following conditions:*

(i) $\beta_n \leq \gamma_n, \forall\ n \geq 1$,

(ii) $\sum_{n=1}^{\infty} \gamma_n = \infty$,

(iii) $(1-\alpha)\gamma + \alpha\beta(\gamma^2 L^2 + 2\gamma - 1) < 0$,

*where $\alpha = \liminf_{n\to\infty} \alpha_n$, $\beta = \liminf_{n\to\infty} \beta_n$ and $\gamma \geq \sup_{n\geq 1} \gamma_n$. Then, $\{x_n\}$ converges strongly to a fixed point of $T$.*

**Proof.** Following the method of the proof of Theorem 3.1, we also obtain that

$$\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2 + \Big((1-\alpha_n)\gamma_n^2 + \alpha_n\beta_n\gamma_n(\gamma_n^2 L^2 + 2\gamma_n - 1)\Big)\|x_n - Tx_n\|^2,$$

and $x_n \to x^* \in C$. Now, from Lemma 2.2, we get

$$\begin{aligned}
&\Big((\alpha-1)\gamma+\alpha\beta(1-2\gamma-\gamma^2L^2)\Big)\sum_{n=1}^{\infty}\gamma_n\|x_n-Tx_n\|^2\\
\leq\ &\sum_{n=1}^{\infty}\gamma_n\Big((\alpha_n-1)\gamma_n+\alpha_n\beta_n(1-2\gamma_n-\gamma_n^2L^2)\Big)\|x_n-Tx_n\|^2\\
=\ &\sum_{n=1}^{\infty}\Big((\alpha_n-1)\gamma_n^2+\alpha_n\beta_n\gamma_n(1-2\gamma_n-\gamma_n^2L^2)\Big)\|x_n-Tx_n\|^2\\
\leq\ &\sum_{n=1}^{\infty}(\|x_n-p\|^2-\|x_{n+1}-p\|^2)<\infty,
\end{aligned}$$

from which it follows that

$$\liminf_{n\to\infty}\|x_n-Tx_n\|=0,$$

and hence there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\lim_{n\to\infty}\|x_{n_k}-Tx_{n_k}\|=0.$$

Thus, $x_{n_k}\to x^*$ and the continuity of $T$ imply that $x^*=Tx^*$ and hence $x^*\in\mathcal{F}(T)$. □

Now, we prove a convergence theorem for a countable family of monotone mappings.

**Theorem 4.3** *Let $H$ be a real Hilbert space, let $\{T_n\}_{n=1}^{\infty}: H\to H$ be a countable family of uniformly Lipschitz monotone mappings with Lipschitzian constants $L_n$, let $L:=\sup_{n\geq1}L_n$. And if $x_n\to x^*$ and $\|T_nx_n\|\to0$, then $x^*\in\bigcap_{n=1}^{\infty}\mathcal{N}(T_n)$. Assume that the interior of $\mathcal{N}:=\bigcap_{n=1}^{\infty}\mathcal{N}(T_n)$ is nonempty. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_1\in C$ by*

$$\begin{cases} z_n=x_n-\gamma_nT_nx_n,\\ y_n=x_n-\beta_n(x_n-z_n)-\beta_nT_nz_n,\\ x_{n+1}=(1-\alpha_n)z_n+\alpha_ny_n, \end{cases}\tag{4.3}$$

*where $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}\subset(0,1)$ satisfying the following conditions:*

(i) $\beta_n\leq\gamma_n,\forall\ n\geq1$,

(ii) $(1-\alpha)\gamma+\alpha\beta(\gamma^2L^2+2\gamma-1)<0$,

*where $\alpha=\liminf_{n\to\infty}\alpha_n$, $\beta=\liminf_{n\to\infty}\beta_n$ and $\gamma\geq\sup_{n\geq1}\gamma_n$. Then, $\{x_n\}$ converges strongly to $x^*\in\mathcal{N}$.*

**Proof.** Since $T_n$ is monotone if and only if $A_n:=I-T_n$ is pseudo-contractive and $\bigcap_{n=1}^{\infty}\mathcal{F}(A_n)=\bigcap_{n=1}^{\infty}\mathcal{N}(T_n)\neq\emptyset$, then the conclusion follows from Theorem 3.1. □

If in Theorem 4.3, we consider a finite family of monotone mappings and a single monotone mapping, respectively, then we get the following corollaries.

**Corollary 4.1** *Let $H$ be a real Hilbert space, let $\{T_i\}_{i=1}^N : H \to H$ be a finite family of Lipschitz monotone mappings with Lipschitzian constants $L_i$, for $i = 1, 2, \cdots, N$, respectively. And if $x_n \to x^*$ and $\|T_n x_n\| \to 0$, then $x^* \in \bigcap_{i=1}^N \mathcal{N}(T_i)$. Assume that the interior of $\mathcal{N} := \bigcap_{i=1}^N \mathcal{N}(T_i)$ is nonempty. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_1 \in C$ by*

$$\begin{cases} z_n = x_n - \gamma_n T_n x_n, \\ y_n = x_n - \beta_n(x_n - z_n) - \beta_n T_n z_n, \\ x_{n+1} = (1 - \alpha_n) z_n + \alpha_n y_n, \end{cases} \tag{4.4}$$

*where $T_n := T_{n(mod\ N)}$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1)$ satisfying the following conditions:*

(i) $\beta_n \leq \gamma_n, \forall\ n \geq 1$,

(ii) $(1 - \alpha)\gamma + \alpha\beta(\gamma^2 L^2 + 2\gamma - 1) < 0$,

*where $\alpha = \liminf_{n\to\infty} \alpha_n$, $\beta = \liminf_{n\to\infty} \beta_n$ and $\gamma \geq \sup_{n\geq 1} \gamma_n$. for $L := \max\{L_i : i = 1, 2, \cdots, N\}$. Then, $\{x_n\}$ converges strongly to a common zero point of $\{T_1, T_2, \cdots, T_N\}$.*

**Corollary 4.2** *Let $H$ be a real Hilbert space, let $T : H \to H$ be a Lipschitz monotone mapping with Lipschitzian constant $L$. Assume that the interior of $\mathcal{N}(T)$ is nonempty. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_1 \in C$ by*

$$\begin{cases} z_n = x_n - \gamma_n T x_n, \\ y_n = x_n - \beta_n(x_n - z_n) - \beta_n T z_n, \\ x_{n+1} = (1 - \alpha_n) z_n + \alpha_n y_n, \end{cases} \tag{4.5}$$

*where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1)$ satisfying the following conditions:*

(i) $\beta_n \leq \gamma_n, \forall\ n \geq 1$,

(ii) $\sum_{n=1}^{\infty} \gamma_n = \infty$,

(iii) $(1 - \alpha)\gamma + \alpha\beta(\gamma^2 L^2 + 2\gamma - 1) < 0$,

*where $\alpha = \liminf_{n\to\infty} \alpha_n$, $\beta = \liminf_{n\to\infty} \beta_n$ and $\gamma \geq \sup_{n\geq 1} \gamma_n$. Then, $\{x_n\}$ converges strongly to a zero point of $T$.*

## Competing interests

The authors declare that they have no competing interests regarding the publication of this article.

## Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

## References

[1] D. Butnariu, Y. Censor, P. Gurfil, E. Hadar, On the behavior of subgradient projections methods for convex feasibility problem in Euclidean spaces, SIAM J. Optim. 19 (2) (2008) 786-807.

[2] S. Maruster, C. Popirlan, On the Mann-type iteration and the convex feasibility problem, J. Comput. Appl. Math. 212 (2008) 390-396.

[3] H.K. Xu, A variable Krasnosel'skii-Mann algorithm and the multiple-set split feasibility problem, Inverse Problems 22 (2006) 2021-2034.

[4] D. Youla, Mathematical theory of image restoration by the method of convex projections, in: H. Stark (Ed.), Image Recovery Theory and Applications, Academic Press, Orlando, 1987, pp. 29-77.

[5] O. Scherzer, Convergence criteria of iterative methods based on Landweber iteration for solving nonlinear problems, J. Math. Anal. Appl. 194 (1991) 911-933.

[6] H. Bauschke, The approximation of fixed points of compositions of nonexpansive mappings in Hilbert spaces, J. Math. Anal. Appl. 202 (1996) 150-159.

[7] D.P. Wu, S.S. Chang, G.X. Yuan, Approximation of common fixed points for a family of finite nonexpansive mappings in Banach space, Nonlinear Anal. 63 (2005) 987-999.

[8] B. Halpern, Fixed points of nonexpanding maps, Bull. Amer. Math. Soc. 73 (1967) 957-961.

[9] N. Shioji, W. Takahashi, Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces, Proc. Amer. Math. Soc. 125 (1997) 3641-3645.

[10] R. Wittmann, Approximation of fixed points of nonexpansive mappings, Arch. Math. 58 (1992) 486-491.

[11] K. Aoyama, Y. Kimura, W. Takahashi, M. Toyoda, Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space, Nonlinear Anal. 67 (2007) 2350-2360.

[12] S.Y. Matsushita, W. Takahashi, A strong convergence theorem for relatively nonexpansive mappings in a Banach space, J. Approx. Theory 134 (2005) 257-266.

[13] S.S. Chang, Viscosity approximation methods for a finite family of nonexpansive mappings in Banach spaces, J. Math. Anal. Appl. 323 (2006) 1402-1416.

[14] H. Zegeye, N. Shahzad, Viscosity approximation methods for a common fixed point of a family of quasi-nonexpansive mappings, Nonlinear Anal. 68 (7) (2008) 2005-2012.

[15] C.E. Chidume, C.O. Chidume, Iterative approximation of fixed points of nonexpansive mappings, J. Math. Anal. Appl. 318 (2006) 288-295.

[16] T. Suzuki, Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces, Proc. Amer. Math. Soc. 135 (2007) 99-106.

[17] W. Takahashi, Y. Takeuchi, R. Kubota, Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces, J. Math. Anal. Appl. 341 (1) (2008) 276-286.

[18] W.R. Mann, Mean value methods in iteration, Proc. Amer. Math. Soc. 4 (1953) 506C510.

[19] S. Reich, Weak convergence theorems for nonexpansive mappings in Banach spaces, J. Math. Anal. Appl. 67 (1979) 274C276.

[20] C.E. Chidume, On approximation of fixed points of nonexpansive mappings, Houston J. Math. 7 (1981) 345-355.

[21] W.A. Kirk, Locally nonexpansive mappings in Banach spaces, in: Lecture Notes in Math., vol. 886, Springer, Berlin, 1981, pp. 178-198.

[22] F.E. Browder, W.V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert spaces, J. Math. Anal. Appl. 20 (1967) 197-228.

[23] G. Marino, H.K. Xu, Weak and strong convergence theorems for strict pseudo-contractions in Hilbert Spaces, J. Math. Anal. Appl. 329 (2007) 336C346.

[24] S. Ishikawa, Fixed points by a new iteration method, Proc. Amer. Math. Soc. 44 (1) (1974) 147-150.

[25] C.E. Chidume, S.A. Mutangadura, An example on the Mann iteration method for Lipschitz pseudocontractions, Proc. Amer. Math. Soc. 129 (8) (2001) 2359-2363.

[26] H. Zhou, Convergence theorems of fixed points for Lipschitz pseudo-contractions in Hilbert spaces, J. Math. Anal. Appl. 343 (2008) 546-556.

[27] Y.H. Yao, Y.C. Liou, G. Marino, A hybrid algorithm for pseudo-contractive mappings, Nonlinear Anal. 71 (2009) 4997C5002.

[28] Y.C. Tang, J.G. Peng, L.W. Liu, Strong convergence theorem for pseudo-contractive mappings in Hilbert spaces, Nonlinear Anal. 74 (2011) 380-385.

[29] H. Zegeye, N. Shahzad, M.A. Alghamdi, Convergence of Ishikawas iteration method for pseudocontractive mappings, Nonlinear Anal. 74 (2011) 7304-7311.

[30] Q.Q. Cheng, Y.F. Su, J.L. Zhang, Convergence theorems of a three-step iteration method for a countable family of pseudocontractive mappings, Fixed Point Theory and Applications 2013, 2013: 100.

[31] N. Huang, C. Ma, A new extragradient-like method for solving variational inequality problems, Fixed Point Theory and Applications 2012, 2012: 223.

[32] Y. Ke, C. Ma, A new relaxed extragradient-like algorithm for approaching common solutions of generalized mixed equilibrium problems, a more general system of variational inequalities and a fixed point problem, Fixed Point Theory and Applications 2013, 2013: 126.

[33] Y. Ke, C. Ma, The convergence analysis of the projection methods for a system of generalized relaxed cocoercive variational inequalities in Hilbert spaces, Fixed Point Theory and Applications 2013, 2013: 182.

# Harmonic analysis in the product of commutative hypercomplex systems

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## Abstract

The main aim of this paper is to give integral representations for strongly negative definite functions defined on the product hypercomplex systems. Harmonic properties for strongly negative definite functions are investigated. We construct a Lèvy measure on the product hypercomplex systems, then we study the conditions that guarantee the existence of some integrations having an integrand parts as a function of the constructed kernel. Finally, we give a Lèvy - Khinchin type formula for strongly negative definite functions defined on the product hypercomplex systems.

**Keywords.** Lèvy – Khinchin; Hypercomplex; Negative definite.

**2010 Mathematics subject classification.** 43A35; 43A65; 43A25.

## 1. Introduction.

The integral representation of negative definite functions is defined as Lèvy-Khinchin formula. This was established by Lèvy-Khinchin in1930's for $G = \mathbb{R}$. Many author's paid attention to generalize this result in different spaces. It had been extended by Hunt [4] to Lie groups, Parthasarathy et al [8] to locally compact commutative groups, Berg et al [3] to comutative semigroups with identical involution and by Lasser [6] for commutative hypergroups. The main aim of this paper is devoted to find the integral representations for strongly negative definite functions defined on the product dual hypercomplex system. Let $Q$ be a commutative separable locally compact metric space of points $p, q, r, \dots$; $B(Q)$ is the $\sigma$ –algebra of Borel subsets on $Q$ and $B_0(Q)$ is the subring of $B(Q)$, which consists of sets with compact closure. We denote by $C(Q)$ the space of continuous functions on $Q$; $C_b(Q), C_\infty(Q)$ and $C_0(Q)$ consists of bounded, tending to zero at infinity and compactly supported functions from $C(Q)$, respectivly. For a fixed $r \in Q,\ B \in B(Q)$, we will denote by $c(A, B; r)$ a commutative Borel structure measure in $A \in B(Q)$. The hypercomplex system $L_1(Q, dm)$ is the Banach algebra of functions on $Q$ with respect to the multiplicative measure $m$ and convolution " $*$ " defined for any $\phi * \psi \in L_1(Q, dm)$ by:

$$(\phi * \psi)(r) = \int_Q \phi(p)dp \int_Q \psi(q)dq\ c(E_p, E_q; r)$$

$$= \int_Q \int_Q \phi(p)\psi(q)\ c(p,q;r)\ dm(p)dm(q)$$

$$= \int_Q \int_Q \phi(p)\psi(q)\ dm_r(p,q)$$

The space $C_\infty(Q)$ is a Banach space with norm

$$||.||_\infty = \sup_{r\in Q}|(.)(r)|$$

We will denote by $\mathcal{M}(Q)$, the space of Radon measure on $Q$, i.e. the space of continuous linear functionals defined on $C_0(Q)$. Let $\mathcal{M}_b(Q) = (\ C_\infty(Q))'$denote the space of bounded Radon measures with norm

$$||\mu||_\infty = \sup\{|\mu(f)|; f \in C_\infty, |f| \leq 1\}$$

The topology of simple convergence on functions from  in the space of Radon measures, is called vague topology.

## 2. Strongly Negative Definite Functions.

A hypercomplex system $L_1(Q, dm)$ may or may not have a unity. In this paper we will concern our efforts on hypercomplex system with unity. A normal hypercomplex system contain a basis unity if there exists $e \in Q$ such that $e^* = e$ and

$$c(A, B; e) = m(A^* \cap B), \quad A, B \in B(Q).$$

A nonzero measurable and bounded almost everywhere function $Q \ni r \to \chi(r) \in \mathbb{C}$ is said to be a character of the hypercomplex system $L_1(Q, dm)$if for all $A, B \in B_0(Q)$ we have

$$\int_Q c(A, B; r)\ \chi(r)dm(r) = \chi(A)\chi(B)$$

and

$$\int_C \chi(r)\ dm(r) = \chi(C), \ \ C \in B_0(Q).$$

We will denote by $X_h$ the set of all bounded Hermitian characters, i.e.

$$X_h := \{\chi \in C_b(Q); \int_Q c(A,B;r)\,\chi(r)dm(r) = \chi(A)\chi(B),\ \overline{\chi(r)} = \chi(r^*)\}$$

A continuous bounded function $\psi: Q \to \mathbb{C}$ is called negative definite if for any $r_1, r_2, \dots, r_n \in Q;\ c_1, c_2, \dots, c_n \in \mathbb{C}$ and $n \in \mathbb{N}$ we have:

$$\sum_{i,j=1}^{n}[\psi(r_i) + \overline{\psi(r_j)} - (R_{r^*_j}\psi)(r_i)]\,c_i\overline{c_j} \geq 0,$$

and a continuous bounded function $\varphi: Q \to \mathbb{C}$ is called positive definite if for any $r_1, r_2, \dots, r_n \in Q;\ c_1, c_2, \dots, c_n \in \mathbb{C}$ and $n \in \mathbb{N}$ we have:

$$\sum_{i,j=1}^{n}(R_{r^*_j}\varphi)(r_i)\,c_i\overline{c_j} \geq 0,$$

where $R_r\ (r \in Q)$, denote the generalized translation operators on$L_1(Q, dm)$.

As pointed out of [1], every positive definite function $\varphi \in P(Q)$ admits a unique representation in the integral form

$$\varphi(r) = \hat{\mu}(\chi) = \int_{X_h}\chi(r)d\,\mu(\chi),\ \ \chi \in X_h, \tag{2.1}$$

where $\mu$ is a finite nonnegative regular measure on the space $X_h$. Conversely, each function have the integral form (1.1) belongs to the set of all positive definite function $P(Q)$.

Let $Q_1$ and $Q_2$ be two commutative separable locally compact metric spaces, with identities $e_1$ and $e_2$ respectively, and suppose $A$ be a non empty subset of $L_1(Q_1) \times L_1(Q_2)$, then the strongly positive definite function will be defined as follows:

**Definition 2.1.** A locally bounded continuous measurable function $\Phi \in A$ is called strongly positive definite, if there exists two positive definite functions $\varphi_1 \in P(Q_1)$ and $\varphi_2 \in P(Q_2)$ and a Radon measure $\mu \in \mathcal{M}_+(Q_1 \times Q_2)$, such that

$$\hat{\mu}(\chi,\tau) = \begin{cases} \varphi_1(\chi)+\varphi_2(\tau), & (\chi,\tau) \in A \\ 0, & (\chi,\tau) \notin A\,. \end{cases} \tag{2.2}$$

A locally bounded continuous measurable function $\Psi \in A$ is called strongly negative definite, if $\Psi(e_1, e_2) \geq 0$and $\exp(-t\Psi)$ is strongly positive definite in $A$ for each $t > 0$.

Clearly each strongly positive (negative) definite function is positive (negative) definite but the converse implication does not hold. Negative definiteness is an analogue of one half of Schoenberg's duality result, It is not known for which hypercomplex system, negative definiteness implies strong negative definiteness. The following Lemma is in fact, an adaption of

whatever done for hypergroups [7], we will not repeat the proof, wherever the proof for hypergroups can be applied to the hypercomplex with necessary modification.

**Lemma 2.2.** The sum and the point-wise limit of strongly negative definite functions on hypercomplex are also strongly negative definite.

**Theorem 2.3.** A function $\Psi: Q \to \mathbb{C}$ is strongly negative definite if and only if the following conditions are satisfied:

(i) $\Psi(e_1, e_2) \geq 0$, $\Psi$ is continuous bounded function ;
(ii) $\overline{\Psi(\boldsymbol{r})} = \Psi(\boldsymbol{r}^*)$ for each $\boldsymbol{r} \in Q_1 \times Q_2$ ;
(iii) if for any $\boldsymbol{r}_1, \boldsymbol{r}_2, \dots, \boldsymbol{r}_n \in Q_1 \times Q_2$ *and* $c_1, c_2, \dots, c_n \in \mathbb{C}$ with $\sum_{i=1}^n c_i = 0$ and $\boldsymbol{r}_i = (r_1^i, r_1^i) \in Q_1 \times Q_2$ , we have

$$\sum_{i,j=1}^n (R_{\boldsymbol{r}^*{}_j}\, \Psi)(\boldsymbol{r}_i)\, c_i \overline{c_j} \leq 0.$$

**Proof.** Suppose that the function $\Psi$ is strongly negative definite. From the above definition of strongly negative definite functions, it is clear that $\Psi$ satisfies (i) and (ii). Let $\boldsymbol{r}_1, \boldsymbol{r}_2, \dots, \boldsymbol{r}_n \in Q_1 \times Q_2$ *and* $c_1, c_2, \dots, c_n \in \mathbb{C}$ with $\sum_{i=1}^n c_i = 0$ . Since, every strongly negative definite function is negative definite, so

$$\begin{aligned} 0 &\leq \sum_{i,j=1}^n [\Psi(\boldsymbol{r}_i) + \overline{\Psi(\boldsymbol{r}_j)} - (R_{\boldsymbol{r}^*{}_j}\, \Psi)(\boldsymbol{r}_i)]\, c_i \overline{c_j} \\ &= \overline{(\sum_{j=1}^n c_j)}\ \sum_{i=1}^n [\Psi(\boldsymbol{r}_i)]\, c_i + (\sum_{i=1}^n c_i)\ \overline{\sum_{j=1}^n [\Psi(\boldsymbol{r}_j)]} \\ &\quad - \sum_{i,j=1}^n [(R_{\boldsymbol{r}^*{}_j}\, \Psi)(\boldsymbol{r}_i)]\, c_i \overline{c_j} \\ &= -\sum_{i,j=1}^n [(R_{\boldsymbol{r}^*{}_j}\, \Psi)(\boldsymbol{r}_i)]\, c_i \overline{c_j} \end{aligned}$$

Conversely, suppose that $\Psi$ satisfies the above conditions. Let $e, \boldsymbol{r}_1, \boldsymbol{r}_2, \dots, \boldsymbol{r}_n \in Q_1 \times Q_2$ *and* $c_1, c_2, \dots, c_n \in \mathbb{C}$ with $\sum_{i=1}^n c_i = 0$. From (iii) we have

$$\begin{aligned} 0 &\geq \sum_{i,j=0}^n [(R_{\boldsymbol{r}^*{}_j}\, \Psi)(\boldsymbol{r}_i)]\, c_i \overline{c_j} \\ &= \sum_{i,j=1}^n [(R_{\boldsymbol{r}^*{}_j}\, \Psi)(\boldsymbol{r}_i)]\, c_i \overline{c_j} + \overline{c_0} \sum_{i=1}^n [\Psi(\boldsymbol{r}_i)]\, c_i + c_0 \overline{\sum_{j=1}^n [\Psi(\boldsymbol{r}_j)]\, c_j} + \Psi(\boldsymbol{e})|c_0|^2 \\ &= \sum_{i,j=1}^n [\Psi(\boldsymbol{r}_i) + \overline{\Psi(\boldsymbol{r}_j)} - (R_{\boldsymbol{r}^*{}_j}\, \Psi)(\boldsymbol{r}_i)]\, c_i \overline{c_j} + \Psi(\boldsymbol{e})|c_0|^2 \end{aligned}$$

This implies

$$\sum_{i,j=1}^n [\Psi(\boldsymbol{r}_i) + \overline{\Psi(\boldsymbol{r}_j)} - (R_{\boldsymbol{r}^*{}_j}\, \Psi)(\boldsymbol{r}_i)]\, c_i \overline{c_j} \geq \Psi(\boldsymbol{e})|c_0|^2 \geq 0$$

**Corollary 2.4.** For any functions $\Phi, \Psi$ on the product $Q_1 \times Q_2$ we have:

(i) If $\Psi$ belongs to the set of strongly negative definite function on $Q_1 \times Q_2$, then the function $\boldsymbol{r} \to \Psi(\boldsymbol{r}) - \Psi(e_1, e_2)$ is also strongly negative definite function.

(ii) If $\Phi$ belongs to the set of strongly positive definite function on $Q_1 \times Q_2$, then the function $\boldsymbol{r} \to \Phi(\boldsymbol{r}) - \Phi(e_1, e_2)$ is also strongly positive definite function.

**Proof.** Let $\boldsymbol{r}_1, \boldsymbol{r}_2, \dots, \boldsymbol{r}_n \in Q_1 \times Q_2$ $and$ $c_1, c_2, \dots, c_n \in \mathbb{C}$ with $\sum_{i=1}^{n} c_i = 0$. Then we have

$$\sum_{i,j=0}^{n} [R_{\boldsymbol{r}^*{}_j}(\Psi(\boldsymbol{r}_i) - \Psi(e_1, e_2))]\, c_i \overline{c_j} = \sum_{i,j=0}^{n} (R_{\boldsymbol{r}^*{}_j}\Psi)(\boldsymbol{r}_i) c_i \overline{c_j} - \Psi(e_1, e_2) \mid \sum_{i=1}^{n} c_i \mid^2$$

$$= \sum_{i,j=0}^{n} (R_{\boldsymbol{r}^*{}_j}\Psi)(\boldsymbol{r}_i) c_i \overline{c_j} \le 0$$

This proves the strongly negative definiteness of $\Psi(\boldsymbol{r}) - \Psi(e_1, e_2)$. Similarly, let $\boldsymbol{r}_1, \boldsymbol{r}_2, \dots, \boldsymbol{r}_n \in Q_1 \times Q_2$ $and$ $c_1, c_2, \dots, c_n \in \mathbb{C}$ with $\sum_{i=1}^{n} c_i = 0$. Then we find

$$\sum_{i,j=0}^{n} [R_{\boldsymbol{r}^*{}_j}(\Phi(e_1, e_2) - \Phi(\boldsymbol{r}_i))]\, c_i \overline{c_j} = -\sum_{i,j=0}^{n} (R_{\boldsymbol{r}^*{}_j}\Phi)(\boldsymbol{r}_i) c_i \overline{c_j} - \Phi(e_1, e_2) \mid \sum_{i=1}^{n} c_i \mid^2$$

$$= -\sum_{i,j=0}^{n} (R_{\boldsymbol{r}^*{}_j}\Phi)(\boldsymbol{r}_i) c_i \overline{c_j} \le 0$$

Because $\Phi$ belongs to the set of strongly positive definite functions, hence (ii).

**Theorem 2.5.** For every strongly negative definite function $\Psi$ on the product $Q_1 \times Q_2$ with $\Psi(e_1, e_2) \ge 0$, the function $\frac{1}{\Psi}$ is strongly positive definite function on the product $Q_1 \times Q_2$.

**Proof.** Suppose $\Psi$ strongly negative definite function on the product $Q_1 \times Q_2$, so $\exp(-t\Psi)$ is strongly positive definite on the product $Q_1 \times Q_2$. This implies

$$|\exp(-t\Psi)| \le |\exp(-t\Psi(e_1, e_2))| \quad \text{for all } t > 0.$$

It follows, for all $(\chi, \tau) \in \widehat{Q_1 \times Q_2}$ we have

$$\frac{1}{\Psi(\chi, \tau)} = \int_0^{\infty} \exp\big(-t\Psi(\chi, \tau)\big)\, dt = \int_0^{\infty} \widehat{\mu_t}(\chi, \tau) dt$$

Where $\mu_t$ is the corresponding measure for $\exp(-t\Psi)$. Moreover, applying Lèvy continuity Theorem, there exists a measure $\upsilon \in \mathcal{M}_+(Q_1 \times Q_2)$such that

$$\upsilon(\chi,\tau) \coloneqq \hat{\upsilon}(\chi,\tau) = \int_0^\infty \widehat{\mu_t}(\chi,\tau)dt$$

and

$$\upsilon(e_1\,,e_2) = \frac{1}{\Psi(e_1\,,e_2)} < \infty$$

Consequently, $\upsilon \in \mathcal{M}_+^b(Q_1 \times Q_2)$. This implies the required to prove.

## 3. Construction of Lèvy measure.

Let $L_1(Q_1 \times Q_2)$ denote a commutative normal hypercomplex system with the product basis $Q_1 \times Q_2$ and basis unity $\boldsymbol{e} = (e_1\,,e_2)$. A family of bounded Radon measures $(\mu_t)_{t>0}$ will be called a convolution semigroup on $Q_1 \times Q_2$ if it satisfies the following items:

(i) $\mu_t(Q_1 \times Q_2) \leq 1,$ for each $t > 0$;
(ii) $\mu_{t_1} * \mu_{t_2} = \mu_{t_1+t_2}$ for each $t_1, t_2 > 0$;
(iii) $\lim_{t\to 0} \mu_t = \in_{\boldsymbol{e}}$, with respect to the vague topology on $\mu \in \mathcal{M}^b(Q_1 \times Q_2)$.

**Theorem 3.1.** For any strongly negative definite function $\Psi$ on $Q_1 \times Q_2$, there exists a unique convolution semigroup on $Q_1 \times Q_2$ such that $\Psi$ is associated to $(\mu_t)_{t>0}$.

**Proof.** Firstly, we will prove that, for $(\chi,\tau) \in \widehat{Q_1 \times Q_2}$, the function $t \to \hat{\mu}_t(\chi,\tau)$ is continuous. As pointed out of Ursohn's lemma [9], there exists $f \in C_c(Q_1 \times Q_2)$ that satisfies $f(\boldsymbol{e}) = 1\ and\ 0 \leq f < 1$. Applying the above conditions for the convolution semigroup on $Q_1 \times Q_2$, we have:

$$1 = f(\boldsymbol{e}) = \lim_{t\to 0} <\mu_t, f> \leq \liminf_{t\to 0} \mu_t(Q_1 \times Q_2) \leq \limsup_{t\to 0} \mu_t(Q_1 \times Q_2) \leq 1$$

and this shows that

$$\lim_{t\to 0} \mu_t = \in_{\boldsymbol{e}} \qquad (\text{ in the Bernolli topology}).$$

As pointed out of [2], for each $t_1, t_2 > 0$, we have

$$|\hat{\mu}_t(\chi,\tau) - \hat{\mu}_{t_0}(\chi,\tau)| \leq |\hat{\mu}_{|t-t_0|}(\chi,\tau) - 1|$$

the right hand side tends to zero uniformally on compact subset of $\widehat{Q_1 \times Q_2}$, so

$$\lim_{t\to 0} \mu_t = \mu_{t_0} \qquad (\text{ in the Bernolli topology}).$$

Secondly, from the definition of strongly negative definite function, there exists a unique determined measures $\mu_t \in \mathcal{M}^b(Q_1 \times Q_2), t > 0,$ such that $\hat{\mu}_t(\chi) = \exp(-t\Psi)$ It is clear that,

the family $(\mu_t)_{t>0}$ satisfies conditions (i) and (ii). The boundedness of the function $\Psi$on compact subsets of $\widehat{Q_1 \times Q_2}$ implies that

$$\lim_{t\to 0} \hat{\mu}_t(\chi) = \lim_{t\to 0} \exp(-t\Psi) = 1.$$

From [5], there exists a multiplicative measure $\widehat{m}$ on the dual $\widehat{Q_1 \times Q_2}$, such that for every $f \in C_0(Q_1 \times Q_2)$ and $\varepsilon > 0$, there exits $g \in C_0(\widehat{Q_1 \times Q_2})$such that $\widehat{Q_1 \times Q_2}\,||f - \tilde{g}|| < \varepsilon$ and

$$|\mu_t(f) - \varepsilon_{\boldsymbol{e}}(f)| \le 2\varepsilon + \int_{\widehat{Q_1 \times Q_2}} |g(\chi,\tau)|\,|\hat{\mu}_t(\chi,\tau) - 1| d\widehat{m}(\chi,\tau)$$

this implies (iii).

Let $S$ denote the set of probability and symmetric measures on $Q_1 \times Q_2$ with compact support, i.e.

$$S = \{\sigma;\ \sigma \in \mathcal{M}^1(Q_1 \times Q_2) \cap \mathcal{M}^c(Q_1 \times Q_2), \sigma(\chi,\tau) = \tilde{\sigma}(\chi,\tau)\}$$

Let $(\mu_t)_{t>0}$ be a convolution semigroup on $Q_1 \times Q_2$ and $\Psi: \widehat{Q_1 \times Q_2} \to \mathbb{C}$ the strongly negative definite function associated to $(\mu_t)_{t>0}$. Applying the same technique of [2] for the hypercomplex system instead of semigroups, we can see that, the net $(\frac{1}{t}\mu_t|\widehat{Q_1 \times Q_2}\setminus\{\boldsymbol{e}\})_{t>0}$ of positive measures on $\widehat{Q_1 \times Q_2}\setminus\{\boldsymbol{e}\}$converges vaguely as $t \to 0$ to a measure $\mu$ on $\widehat{Q_1 \times Q_2}\setminus\{\boldsymbol{e}\}$, and for every $\sigma \in S$, the function $\Psi * \sigma - \Psi$ is continuous strongly positive definite on $\widehat{Q_1 \times Q_2}$ and the positive bounded measure $\mu_\sigma$ on $Q_1 \times Q_2$ whose Fourier transform is $\Psi * \sigma - \Psi$ satisfies

$$(1 - \tilde{\sigma})\mu = \mu_\sigma|Q_1 \times Q_2 \setminus\{\boldsymbol{e}\}. \tag{3.1}$$

The positive measure $\mu$ on $Q_1 \times Q_2 \setminus\{\boldsymbol{e}\}$ defined by (3.1) is called the strong Lèvy measure for the convolution semigroup $(\mu_t)_{t>0}$ on $Q_1 \times Q_2$.

**Theorem 3.2.** Let $\mu$ denote the Lèvy measure for the convolution semigroup $(\mu_t)_{t>0}$ on $Q_1 \times Q_2$. Then

$$\int_{Q_1 \times Q_2 \setminus\{\boldsymbol{e}\}} (1 - Re(\chi,\tau)(r))d\,\mu(\chi,\tau) < \infty,\ \ (\chi,\tau) \in \widehat{Q_1 \times Q_2}. \tag{3.2}$$

**Proof.** For $(\chi,\tau) \in \widehat{Q_1 \times Q_2}$, let $\sigma = \frac{1}{2}(\epsilon_{(\chi,\tau)} + \epsilon_{\overline{(\chi,\tau)}}) \in S$; then $\tilde{\sigma} = Re(\chi,\tau)(r)$, substituting in (3.2) we get

$$\int_{Q_1 \times Q_2 \setminus\{\boldsymbol{e}\}} (1 - Re(\chi,\tau)(r))d\,\mu(\chi,\tau) = \int_{Q_1 \times Q_2 \setminus\{\boldsymbol{e}\}} (1 - \tilde{\sigma}(r))d\,\mu(\chi,\tau) = \mu_\sigma|Q_1 \times Q_2 \setminus\{\boldsymbol{e}\} < \infty.$$

# 4. Integral representation theorem.

A continuous function $h: Q_1 \times Q_2 \to \mathbb{R}$ is called homomorphism if it satisfies $h(\boldsymbol{r}^*) = -h(\boldsymbol{r})$ and $R_{\boldsymbol{r}}h(\boldsymbol{s}) = h(\boldsymbol{r}) + h(\boldsymbol{s}),\ \boldsymbol{r}, \boldsymbol{s} \in Q_1 \times Q_2$. Clearly, if $h: Q_1 \times Q_2 \to \mathbb{R}$ is a homomorphism, then the function $\Psi = ih$ is strongly negative definite. A continuous function $q: Q_1 \times Q_2 \to \mathbb{R}$ is called a quadratic form, if it satisfies

$$R_{\boldsymbol{r}}q(\boldsymbol{s}) + R_{\boldsymbol{r}^*}q(\boldsymbol{s}) = 2q(\boldsymbol{r}) + 2q(\boldsymbol{s}),\ \boldsymbol{r}, \boldsymbol{s} \in Q_1 \times Q_2 . \tag{4.1}$$

**Theorem 4.1.** Let $\Psi$ be a strongly negative definite function associated the convolution semigroup $(\mu_t)_{t>0}$ on $Q_1 \times Q_2$. If the Lèvy measure $\mu$ of $(\mu_t)_{t>0}$ is symmetric, then $Im\Psi$ is a homomorphism.

**Proof.** As remarked in [2], a continuous function $f: Q_1 \times \widehat{Q_2} \to \mathbb{R}$ which satisfies $f(e_1, e_2) = 0$ is a homomorphism if and only if $f * \upsilon - f = 0$ for all $\upsilon \in S$. Since, $\check{\mu} = \mu$ is equivalent to $\check{\mu}_\sigma = \mu_\sigma$ for each $\sigma \in S$. So, $Im\Psi * \upsilon - Im\Psi = 0$ for each $\sigma \in S$, hence, then $Im\Psi$ is a homomorphism. In particular, we have $i\, Im\Psi$ is strongly negative definite.

**Lemma 4.2.** For every positive definite symmetric measure $\mu$ on the product $Q_1 \times Q_2 \setminus \{\boldsymbol{e}\}$ such that

$$\int_{Q_1 \times Q_2 \setminus \{\boldsymbol{e}\}} (1 - Re(\chi, \tau)(\boldsymbol{r}))\, d\,\mu(\boldsymbol{r}) < \infty,\ \ (\chi, \tau) \in \widehat{Q_1 \times Q_2}. \tag{4.2}$$

The function $\Psi_\mu: \widehat{Q_1 \times Q_2} \to \mathbb{C}$ defined by

$$\Psi_\mu := \int_{Q_1 \times Q_2 \setminus \{\boldsymbol{e}\}} (1 - Re(\chi, \tau)(\boldsymbol{r}))\, d\,\mu(\boldsymbol{r}) < \infty,\ \ (\chi, \tau) \in \widehat{Q_1 \times Q_2}, \tag{4.3}$$

is strongly negative definite function.

**Proof.** To prove the function $\Psi_\mu$ is strongly negative definite, we will sufficiently prove that the measure $\mu$ is strong Lèvy measure for $\Psi_\mu$. For $f \in C_C^+(\widehat{Q_1 \times Q_2})$ such that $f(\bar{\chi}) = f(\chi)$ and $\int f(\chi)\, dx = 1$, Applying Fubini's Theorem we get

$$\begin{aligned}(\Psi_\mu * f)(\chi) &= \int_{\widehat{Q_1 \times Q_2}} \big(R_\rho f\big)(\chi)\Psi_\mu(\rho)d\rho \\ &= \int_{\widehat{Q_1 \times Q_2}} f(\rho) \int_{Q_1 \times Q_2 \setminus \{(e_1, e_2)\}} [1 - Re\chi(r)\rho(r)]\, d\,\mu(r) \\ &= \int_{Q_1 \times Q_2 \setminus \{\boldsymbol{e}\}} \big[1 - Re\chi(r)\tilde{f}(r)\big]\, d\,\mu(r)\end{aligned} \tag{4.4}$$

Specially, for $\chi = 1$, we have

$$\int_{Q_1 \times Q_2 \setminus \{\boldsymbol{e}\}} \big[1 - \tilde{f}(r)\big]\, d\,\mu(r) = \int f(\rho)\, \Psi_\mu(\rho) d\rho$$

Clearly, $dv(r) = \left[1 - \tilde{f}(r)\right]d\mu(r)$ is positive definite measure on $Q_1 \times Q_2 \backslash \{\boldsymbol{e}\}$, so can be considered as positive definite measure on $Q_1 \times Q_2$. This implies

$$\hat{v}(\chi) = Re\hat{v}(\chi) = \int_{Q_1 \times Q_2 \backslash \{\boldsymbol{e}\}} Re\chi(r)\left[1 - \tilde{f}(r)\right]d\,\mu(r) \quad \text{for } \chi \in \widehat{Q_1 \times Q_2}\,.$$

Putting $f = \sigma$ in (4.4) implies that

$$\Psi_\mu * \sigma(\chi) - \Psi_\mu(\chi) = \int_{Q_1 \times Q_2 \backslash \{\boldsymbol{e}\}} Re\chi(r)[1 - \tilde{\sigma}(r)]d\,\mu(r)$$

So, $\Psi_\mu * \sigma - \Psi_\mu$ is the Fourier transform of the measure $[1 - \tilde{\sigma}(r)]|\mu$, this implies $\mu$ is the Lèvy measure of $\Psi_\mu$.

**Theorem 4.3.(*Main Result*)** Let $\Psi: \widehat{Q_1 \times Q_2} \to \mathbb{C}$ be a strongly negative definite function associated the convolution semigroup $(\mu_t)_{t>0}$ with a symmetric positive Lèvy measure $\mu$ such that

$$\int_{Q_1 \times Q_2 \backslash \{\boldsymbol{e}\}} (1 - Re(\chi,\tau)(r))d\,\mu(r) < \infty, \ \ (\chi,\tau) \in \widehat{Q_1 \times Q_2}\,,$$

Then $\Psi$ admits the integral representation

$$\Psi(\chi,\tau) = \Psi(e) + iIm\Psi + q(\chi,\tau)$$

$$+ \int_{Q_1 \times Q_2 \backslash \{\boldsymbol{e}\}} (1 - Re(\chi,\tau)(r))d\,\mu(r) < \infty, \ \ (\chi,\tau) \in \widehat{Q_1 \times Q_2}\,,$$

where

$$q(\chi,\tau) = \lim_{n\to\infty}\left[\frac{(R^n_{(\chi,\tau)}\Psi)(\chi,\tau)}{4n^2} + \frac{(R^n_{\overline{(\chi,\tau)}}\Psi)(\chi,\tau)}{2n}\right].$$

**Proof.** Regarding Theorem 4.1, the symmetries of the measure $\mu$ implies $h = Im\Psi$ is a homomorphism and $ih$ belongs to the space of strongly negative definite functions on $\widehat{Q_1 \times Q_2}$. Hence, the function $\Psi - CI$ belongs to the space of strongly negative definite functions on $\widehat{Q_1 \times Q_2}$ associated Lèvy measure $\mu$, where $C = \Psi(\boldsymbol{e})$. This implies the function $\Psi^{\#} = \Psi - CI - ih$ belongs to the space of strongly negative definite functions on $\widehat{Q_1 \times Q_2}$ associated Lèvy measure $\mu$. By virtue of the argument of Theorem 3.2, the integral

$$\Psi_\mu := \int_{Q_1 \times Q_2 \backslash \{\boldsymbol{e}\}} (1 - Re(\chi,\tau)(\boldsymbol{r}))d\,\mu(\boldsymbol{r})$$

is finite for all $(\chi,\tau) \in \widehat{Q_1 \times Q_2}$. Observing Lemma 4.2, we get that, the function $q = \Psi^{\#} - \Psi_\mu$ is a real valued symmetric function with $q(\boldsymbol{e}) = 0$. As remarked in [3], for $\sigma \in S$ we have

$$\Psi^{\#} * \sigma - \Psi^{\#} = \Psi * \sigma - \Psi$$

and

$$\Psi_\mu * \sigma - \Psi_\mu = \int_{Q_1 \times Q_2 \setminus \{\boldsymbol{e}\}} Re\chi(r)[1 - \tilde{\sigma}(r)]d\,\mu(r) \tag{4.5}$$

Applying (3.1) and (4.5), we get

$$q * \sigma - q = (\Psi^{\#} - \Psi_\mu) * \sigma - (\Psi^{\#} - \Psi_\mu) = \hat{\mu}_\sigma(\{\boldsymbol{e}\}) \geq 0 \tag{4.6}$$

As pointed in [2], (4.6) implies that the function q is a nonnegative quadratic form on $\widehat{Q_1 \times Q_2}$. Recalling the integral

$$\Psi_\mu \coloneqq \int_{Q_1 \times Q_2 \setminus \{\boldsymbol{e}\}} (1 - Re(\chi,\tau)(\boldsymbol{r}))d\,\mu(\boldsymbol{r})$$

By Lemma 4.2 the function $\Psi_\mu$ is strongly negative definite. Since every quadratic form satisfies the following relation[2]

$$\lim_{n\to\infty}\left[\frac{(R^n_{(\chi,\tau)}q)(\chi,\tau)}{4n^2}\right] = q(\chi,\tau) - \frac{1}{2}(R_{\overline{(\chi,\tau)}}q)(\chi,\tau)$$

So

$$\begin{aligned} & q(\chi,\tau) - \tfrac{1}{2}(R_{\overline{(\chi,\tau)}}q)(\chi,\tau) \\ &= \lim_{n\to\infty}\left[\frac{(R^n_{(\chi,\tau)}\Psi)(\chi,\tau)}{4n^2}\right] - \lim_{n\to\infty}\left[\frac{(R^n_{(\chi,\tau)}\Psi_\mu)(\chi,\tau)}{4n^2}\right] \\ &= \lim_{n\to\infty}\left[\frac{(R^n_{(\chi,\tau)}\Psi)(\chi,\tau)}{4n^2}\right] - \lim_{n\to\infty}\frac{1}{4n^2}\int_{Q_1 \times Q_2 \setminus \{\boldsymbol{e}\}} (1 - Re((\chi,\tau)(\boldsymbol{r}))^{2n})d\,\mu(\boldsymbol{r}) \end{aligned} \tag{4.7}$$

Since the product $Q_1 \times Q_2$ is locally compact, then for every compact K of $\widehat{Q_1 \times Q_2}$, there exists a constant $M_K \geq 0$, a nieghbourhood $N_K$ of **e** and a finite subset $S_K$ of K such that for every element $r \in N_K$ we have

$$\sup_{\mathrm{r}}\{1 - Re(\chi,\tau)(\boldsymbol{r});\ (\chi,\tau) \in K\} \quad \leq M_K \sup_{\mathrm{r}}\{1 - Re(\chi,\tau)(\boldsymbol{r});\ (\chi,\tau) \in S_K\}.$$

If $(\chi,\tau)(\boldsymbol{r}) \neq 0$, let $(\chi,\tau)(\boldsymbol{r}) = \rho\exp(i\vartheta)$ for some $0 < \rho \leq 1$ and $-\pi \leq \vartheta \leq \pi$. Then for

$n \in \mathbb{N}$ the ratio $\frac{\sin(n\vartheta)}{n\vartheta}$ is bounded a way from $Q_1 \times Q_2$ on $[\frac{\pi}{2}, \pi]$, this implies the existence of a positive constant $C \geq 0$ such that

$$\frac{1}{4n^2}(1-\cos(2n\vartheta)) = \frac{1}{2}\left[\frac{\sin(n\vartheta)}{n\vartheta}\right]^2\left[\frac{\vartheta}{\sin(n\vartheta)}\right]^2\left[\frac{1-\cos(2\vartheta)}{2}\right]$$

$$\leq C(1-\cos(2\vartheta))$$

Also, we have

$$\frac{1-\rho^{2n}}{4n^2} \leq \frac{1-\rho}{2n} \leq \frac{1-\rho^2}{2}$$

These gives

$$\frac{1}{4n^2}\left(1-Re\left((\chi,\tau)(\boldsymbol{r})\right)^{2n}\right) = \frac{1}{4n^2}(1-\rho^{2n}) + \frac{\rho^{2n}}{4n^2}(1-\cos(2n\vartheta))$$

$$\leq \tfrac{1-\rho^2}{2} + C\rho^{2n}(1-\cos(2\vartheta))$$

$$\leq \frac{1-\rho^2}{2} + C\rho^2(1-\cos(2\vartheta))$$

$$\leq \frac{1-\rho^2}{2} + C(1-Re((\chi,\tau)(\boldsymbol{r}))^2)$$

Applying the dominated convergence theorem gives

$$\frac{1}{4n^2}\int_{Q_1\times Q_2\setminus\{e\}}(1-Re((\chi,\tau)(\boldsymbol{r}))^{2n})d\,\mu(\boldsymbol{r}) = 0$$

Substituting in (4.7) gives

$$q(\chi,\tau) = \tfrac{1}{2}(R_{\overline{(\chi,\tau)}}q)(\chi,\tau) + \lim_{n\to\infty}\left[\tfrac{(R^n_{(\chi,\tau)}\Psi)(\chi,\tau)}{4n^2}\right] \tag{4.8}$$

Observing that

$$(R_{\overline{(\chi,\tau)}}q)(\chi,\tau) = \lim_{n\to\infty}\left[\frac{(R^n_{\overline{(\chi,\tau)}}q)(\chi,\tau)}{2n}\right]$$

$$= \lim_{n\to\infty}\left[\frac{(R^n_{\overline{(\chi,\tau)}}\Psi)(\chi,\tau)}{2n}\right] - \lim_{n\to\infty}\frac{1}{2n}\int\limits_{Q_1\times Q_2\setminus\{\boldsymbol{e}\}}(1-|(\chi,\tau)(\boldsymbol{r})|^{2n})d\,\mu(\boldsymbol{r})$$

But

$$\frac{1}{2n}(1-|(\chi,\tau)(\boldsymbol{r})|^{2n}) \le 1-|(\chi,\tau)(\boldsymbol{r})|^2$$

Applying the dominated convergence theorem again gives

$$\lim_{n\to\infty}\frac{1}{2n}\int\limits_{Q_1\times Q_2\setminus\{\boldsymbol{e}\}}(1-|(\chi,\tau)(\boldsymbol{r})|^{2n})d\,\mu(\boldsymbol{r}) = 0$$

and so

$$(R_{\overline{(\chi,\tau)}}q)(\chi,\tau) = \lim_{n\to\infty}\left[\frac{(R^n_{\overline{(\chi,\tau)}}\Psi)(\chi,\tau)}{2n}\right]$$

This complete the proof of the Theorem.

## 5. Conclusion

In this paper integral representations for strongly negative definite functions defined on the product hypercomplex systems is given. Harmonic properties for strongly negative definite functions are investigated. We construct a Lèvy measure on the product hypercomplex systems, then we study the conditions that guarantee the existence of some integrations having an integrand parts as a function of the constructed kernel. Finally, we give a Lèvy - Khinchin type formula for strongly negative definite functions defined on the product hypercomplex systems.

## 6. Competing Interests

"The authors declare that they have no competing interests."

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## References

[1] Ju. M. Berezanskii and A. A. Kalyuhnyi, *Harmonic analysis in hypercomplex systems*, Kive, Naukova Dumka, (1992).

[2] C. Berg and G. Forst, *Potential theory on locally compact abelian groups*, Springer-Verlag, Berlin-Heidelberg-New York (1975).

[3] C. Berg, J.P.R. Christensen and P. Ressel, *Harmonic analysis on semigroups.Theory of positive definite and related functions*, Graduated texts inMath., 100, Springer-Verlag, Berlin-Heidelberg-New-York (1984).

[4] G. A. Hunt, *Semigroups of measures on Lie groups*, Trans. Amer. Math. Soc, 81(1956), 264-293.

[5] R.I. Jewett, *Spaces with an abstract convolution of measures*, Adv. in Math., 18 (1975), 1-101.

[6] R. Lasser, *On the Lèvy Hincin formula for commutative hypergroups*, Lecture notes in Math, 1064(1984), 298-308.

[7] A. S. Okb El Bab and H. A. Ghany, *Harmonic analysis on hypergroups*, American Institute of Physics Conf. Proc. 1309(2010), 312.

[8] K.R. Parthasarathy, *Probability measures on metric spaces*, Academic Press, New York-London, (1967).

[9] W. Rudin, Real and complex analysis, McGraw-Hill Book Co, New York, (1974).

# Nonlinear delay fractional difference equations with applications on discrete fractional Lotka–Volterra competition model

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**Abstract.** The existence and uniqueness of solutions for nonlinear delay fractional difference equations are investigated in this paper. We prove the main results by employing the theorems of Krasnoselskii's Fixed Point and Arzela–Ascoli. As an application of the main theorem, we provide an existence result on the discrete fractional Lotka–Volterra model.

**Keywords.** Existence and uniqueness; Fractional difference equations; Krasnoselskii Fixed Point Theorem; Arzela-Ascoli's Theorem; Discrete fractional Lotka–Volterra model.

**AMS subject classification:** 34A08, 34A12, 39A12.

## 1 Introduction

Fractional differential equations have received a special attention during the last decades since it has been found that these type of equations provide an excellent instruments for the description of memory and hereditary properties of various materials and processes [1, 2, 3]. The problem of the existence of solutions for fractional differential equations, in particular, has been considered in several recent papers; ( see Refs. [4, 5, 6, 7, 8] and the references therein).

For the development of the theory of fractional difference equations, which is the discrete counterpart of fractional differential equations, still there exists less interest among researchers. In fact the progress of the theory of fractional difference equations is still in its early stages. Indeed, some mathematicians have recently taken the lead to develop the qualitative properties of fractional difference equations. We name here for instance Atici et. al. [9, 10, 11, 12, 13] who developed the transform methods, properties of initial value problems and studied applications of these equations on the tumor growth, Abdeljawad et. al. [14, 15, 16, 17, 18] who investigated the properties of Riemann and Caputo's fractional sum and difference operators, Anastassiou [19, 20] who defined a Caputo like discrete fractional difference and studied some discrete fractional inequalities, Goodrich [21, 22, 23] who established sufficient conditions for the existence of solutions for initial and boundary value problems of discrete fractional equations and Chen et. al. [24, 25, 26] who studied the stability of certain fractional difference equations. In [27, 28], Wu and Baleanu provided some applied results concerning with certain real life problems described by discrete fractional equations. For further details on these achievements, we recommend the reader to consult the new publications [29, 30].

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Obviously, the existence and uniqueness of solutions are essentially significant concept for differential equations. To the best of authors' knowledge, there are no results concerning with the existence and uniqueness of solutions for nonlinear delay fractional difference equations. The objective of this paper is to cover this gap and study the existence and uniqueness problem for equations of the form

$$\begin{cases} {}^{c}\nabla_0^{\alpha} x(t) = f(t, x(t), x(t-\tau)),\ t \in \mathbb{N}_0 = \{0, 1, 2, \ldots\},\ \tau \geq 0, \\ x(t) = \phi(t),\ t \in [-\tau, -\tau+1, \ldots, 0], \end{cases} \tag{1}$$

where $f : \mathbb{N}_0 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and ${}^{c}\nabla_0^{\alpha}$ denotes the Caputo's fractional difference of order $\alpha \in (0, 1)$. To prove our main results, we employ the Krasnoselskii Fixed Point Theorem and the Arzela-Ascoli's Theorem. As an application of the main theorem, we provide an existence result on the discrete fractional Lotka–Volterra model.

## 2 Preliminaries

Throughout this paper, we will make use of the following notations, definitions and known results of discrete fractional calculus [29]. For any $\alpha, t \in \mathbb{R}$, the $\alpha$ rising function is defined by

$$t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)},\ t \in \mathbb{R}\backslash\{\ldots, -2, -1, 0\},\ 0^{\overline{\alpha}} = 0, \tag{2}$$

where $\Gamma$ is the well known Gamma function satisfying $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$.

**Definition 1.** *Let $x : \mathbb{N}_0 \to \mathbb{R}$, $\rho(s) = s - 1$, $\alpha \in \mathbb{R}^+$ and $\mu > -1$. Then*

1. *The nabla difference of $x$ is defined by*

$$\nabla x(t) = x(t) - x(t-1),\ \ t \in \mathbb{N}_1 = \{1, 2, \ldots\}.$$

2. *The Riemann–Liouville's sum operator of $x$ of order $\alpha > 0$ is defined by*

$$\nabla_0^{-\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t - \rho(s))^{\overline{\alpha-1}} x(s),\ \ t \in \mathbb{N}_1. \tag{3}$$

3. *The Riemann–Liouville's difference operator of $x$ of order $0 < \alpha < 1$ is defined by*

$${}^{c}\nabla_0^{\alpha} x(t) = \nabla_0^{-(1-\alpha)} \nabla x(t) = \frac{1}{\Gamma(1-\alpha)} \sum_{s=1}^{t} \big(t - \rho(s)\big)^{\overline{-\alpha}} \nabla x(s),\ \ t \in \mathbb{N}_1. \tag{4}$$

4. *The power rule is defined by*

$$\nabla_0^{-\alpha} t^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} (t)^{\overline{\alpha+\mu}},\ \ t \in \mathbb{N}_0. \tag{5}$$

**Lemma 1.** [40] *$x(t)$ denotes a solution of equation (1) if and only if it admits the following representation*

$$x(t) = \phi(0) + \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t - \rho(s))^{\overline{\alpha-1}} f(s, x(s), x(s-\tau)),\ \ t \in \mathbb{N}_0, \tag{6}$$

*and $x(t) = \phi(t),\ t \in [-\tau, -\tau+1, \ldots, 0]$.*

The space $l_\infty$ denotes the set of real bounded sequences with respect to the usual supremum norm. We recall that $l_\infty$ is a Banach space.

**Definition 2.** *A set $D$ of sequences in $l_\infty$ is uniformly Cauchy if for every $\varepsilon > 0$, there exists an integer $N$ such that $|x(t) - x(s)| < \varepsilon$ whenever $t, s > N$ for any $x = \{x(n)\}$ in $D$.*

The following discrete version of Arzela–Ascoli's Theorem has a crucial role in the proof of our main theorem.

**Theorem 1.** *(Arzela–Ascoli's Theorem) A bounded, uniformly Cauchy subset $D$ of $l_\infty$ is relatively compact.*

The proof of the main theorem is achieved by employing the following fixed point theorem.

**Theorem 2.** [31] *(Krasnoselskii Fixed Point Theorem) Let $D$ be a nonempty, closed, convex and bounded subset of a Banach space $(X, \|x\|)$. Suppose that $A : X \to X$ and $B : D \to X$ are two operators such that*

(i) *$A$ is a contraction.*

(ii) *$B$ is continuous and $B(D)$ resides in a compact subset of $X$,*

(iii) *for any $x, y \in D$, $Ax + By \in D$.*

*Then the operator equation $Ax + Bx = x$ has a solution $x \in D$.*

## 3 Main results

We prove our main results under the following assumptions:

(I) $f(t, x(t), y(t)) = f_1(t, x(t)) + f_2(t, x(t), y(t))$, where $f_i$ are Lipschitz functions with Lipschitz constants $L_{f_i},\ i = 1, 2$.

(II) $|f_1(t, x(t))| \le M_1|x(t)|,\ \ |f_2(t, x(t), y(t))| \le M_2|x(t)| \times |y(t)|$ for any positive numbers $M_1$ and $M_2$.

Let $B(\mathbb{N}_{-\tau}, \mathbb{R})$ denote the set of all bounded functions (sequences). Define the set

$$D = \{x :\ x \in B(\mathbb{N}_{-\tau}, \mathbb{R}),\ |x| \le r,\ \ t \in \mathbb{N}_0\},$$

where $r$ satisfies

$$|\phi(0)| + \frac{M_1 r + M_2 r^2}{\Gamma(\alpha)} \le r.$$

Define the operators $F_1$ and $F_2$ by

$$F_1 x(t) = \phi(0) + \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t - \rho(s))^{\overline{\alpha-1}} f_1(s, x(s)),$$

and

$$F_2 x(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t - \rho(s))^{\overline{\alpha-1}} f_2(s, x(s), x(s-\tau)).$$

It is clear that $x(t)$ is a solution of (1) it it is a fixed point of the operator $Fx = F_1 x + F_2 x$.

**Theorem 3.** *Let conditions* (I)–(II) *hold. Then, equation (1) has a solution in the set* $D$ *provided that* $\frac{L_{f_1}C(\alpha)}{\Gamma(\alpha)} < 1$ *and* $|\phi(0)| + \frac{\left(M_1 r + M_2 r^2\right)C(\alpha)}{\Gamma(\alpha)} \leq r$.

*Proof.* From the assumptions on the set $D$, one can easily see that $D$ is a nonempty, closed, convex and bounded set.

**Step.1:** We prove that $F_1$ is contractive. We can easily see that

$$\begin{aligned}
\left|F_1x(t) - F_1y(t)\right| &= \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\left|f_1(s,x(s)) - f_1(s,y(s))\right| \\
&\leq \frac{L_{f_1}}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}|x(s)-y(s)| \\
&\leq \frac{L_{f_1}}{\Gamma(\alpha)}\|x-y\|\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}. \qquad (7)
\end{aligned}$$

By virtue of (2), (3), (5) and since $(t-0)^{\overline{0}} = 1$, one can see that

$$\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}(t-0)^{\overline{0}} = \Gamma(\alpha)\nabla_0^{-\alpha}(t-0)^{\overline{0}} = \frac{\Gamma(t+\alpha)}{\alpha\Gamma(t)}.$$

Therefore, (7) becomes

$$\left|F_1x(t) - F_1y(t)\right| \leq \frac{L_{f_1}C(\alpha)}{\Gamma(\alpha)}\|x-y\|, \;\; t < T_1,$$

where $C(\alpha) = \frac{\Gamma(T_1+\alpha)}{\alpha\Gamma(T_1)}$ is a positive constant depending on the order $\alpha$. By the assumption $\frac{L_{f_1}C(\alpha)}{\Gamma(\alpha)} < 1$, we conclude that $F_1$ is contractive. Furthermore, we obtain for $x \in D$

$$\begin{aligned}
\left|F_1x(t) + F_2x(t)\right| &\leq |\phi(0)| + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\left|f_1(s,x(s)) + f_2(s,x(s),x(s-\tau))\right| \\
&\leq |\phi(0)| + \frac{M_1\|x\| + M_2\|x\|^2}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}} \\
&\leq |\phi(0)| + \frac{\left(M_1 r + M_2 r^2\right)C(\alpha)}{\Gamma(\alpha)},
\end{aligned}$$

which implies that $F_1x + F_2x \in D$. For $x \in D$, we also get

$$|F_2x(t)| \leq \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\left|f_2(s,x(s),x(s-\tau))\right| \leq \frac{\left(M_2 r^2\right)C(\alpha)}{\Gamma(\alpha)} \leq r,$$

which implies that $F_2(D) \subset D$.

**Step.2:** We prove that $F_2$ is continuous. Let a sequence $x_n$ converge to $x$. Taking the norm of $F_2x_n(t) - F_2x(t)$, we have

$$\begin{aligned}\big|F_2x_n(t) - F_2x(t)\big| &\leq \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\big|f_2(s,x_n(s),x_n(s-\tau)) - f_2(s,x(s),x(s-\tau))\big| \\ &\leq \frac{L_{f_2}}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\Big(|x_n(s)-x(s)| - \big|x_n(s-\tau)) - x(s-\tau)\big|\Big) \\ &\leq \frac{2L_{f_2}}{\Gamma(\alpha)}\|x_n - x\|\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}} = \frac{\big(2L_{f_2}\big)C(\alpha)}{\Gamma(\alpha)}\|x_n - x\|.\end{aligned}$$

From the above discussion, we conclude that whenever $x_n \to x$, $Fx_n \to Fx$. This proves the continuity of $F_2$. To prove that $F_2(D)$ resides in a relatively compact subset of $l_\infty$, we let $t_1 \leq t_2 \leq H$ to get

$$\begin{aligned}\big|F_2x(t_2) - F_2x(t_1)\big| &\leq \frac{1}{\Gamma(\alpha)}\Big|\sum_{s=1}^{t_2}(t_2-\rho(s))^{\overline{\alpha-1}}f_2(s,x(s),x(s-\tau)) \\ &- \sum_{s=1}^{t_1}(t_1-\rho(s))^{\overline{\alpha-1}}f_2(s,x(s),x(s-\tau))\Big| \\ &\leq \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t_1}\Big|(t_2-\rho(s))^{\overline{\alpha-1}} - (t_1-\rho(s))^{\overline{\alpha-1}}\Big||f_2(s,x(s),x(s-\tau))| \\ &+ \frac{1}{\Gamma(\alpha)}\sum_{s=t_1+1}^{t_2}\big|(t_2-\rho(s))^{\overline{\alpha-1}}\big||f_2(s,x(s),x(s-\tau))|.\end{aligned}$$

Upon employing condition (II), we obtain

$$\begin{aligned}\big|F_2x(t_2) - F_2x(t_1)\big| &\leq M_2r^2\Big[\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t_1}(t_2-\rho(s))^{\overline{\alpha-1}} - \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t_1}(t_1-\rho(s))^{\overline{\alpha-1}} \\ &+ \frac{1}{\Gamma(\alpha)}\sum_{s=t_1+1}^{t_2}(t_2-\rho(s))^{\overline{\alpha-1}}\Big].\end{aligned}$$

By using (3), we get

$$\big|F_2x(t_2) - F_2x(t_1)\big| \leq M_2r^2\Big[\nabla_0^{-\alpha}(t_2-0)^{\overline{0}} - \nabla_0^{-\alpha}(t_1-0)^{\overline{0}} + \nabla_{t_1}^{-\alpha}(t_2-t_1)^{\overline{0}}\Big].$$

From (5), it follows that

$$\big|F_2x(t_2) - F_2x(t_1)\big| \leq \frac{M_2r^2}{\Gamma(\alpha+1)}\Big[t_2^{\overline{0}} - t_1^{\overline{0}} + (t_2-t_1)^{\overline{0}}\Big].$$

This implies that $F_2$ is bounded and uniformly subset of $l^\infty$. Thus, by virtue of the Discrete Arzela Ascoli's Theorem 1, we conclude that $F_2$ is relatively compact.

**Step.3:** It remains to show that for any $x, y \in D$, we have $F_1x(t) + F_2y(t) \in D$. If $x = F_1x(t) + F_2y(t)$, then we have

$$\begin{aligned}
|x(t)| &\leq \big|F_1x(t) + F_2y(t)\big| \leq |\phi(0)| + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\big|f_1(s,x(s)) + f_2(s,y(s),y(s-\tau))\big| \\
&\leq |\phi(0)| + \frac{M_1\|x\| + M_2\|x\|^2}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}} \\
&\leq |\phi(0)| + \frac{\big(M_1r + M_2r^2\big)C(\alpha)}{\Gamma(\alpha)},
\end{aligned}$$

which implies that $x(t) \in D$.

By employing the Krasnoselskii Fixed Point Theorem, we conclude that there exists $x \in D$ such that $x = Fx = F_1x + F_2x$ which is a fixed point of $F$. Hence, equation (1) has at least one solution in $D$.

□

# 4 Applications

The Lotka–Volterra model has been extensively investigated through different approaches [32, 33, 34, 35, 36, 37]. However, all the above mentioned papers studied the integer order Lotka–Volterra model. In spite of the fact that the study of population and medical models of fractional order has been initiated in [12, 38, 39], there is no literature achieved in the direction of discrete fractional Lotka–Volterra model. Therefore, in this section, we employ Theorem 3 to prove an existence and uniqueness result for the solutions of this model.

For a bounded sequence $g$ on $\mathbb{N}$, we define $g^+$ and $g^-$ as follows

$$g^+ = \sup_{t\in\mathbb{N}} g(t) \quad \text{and} \quad g^- = \inf_{t\in\mathbb{N}} g(t).$$

Let $f(t, x(t), x(t-\tau)) = x(t)\big(\gamma(t) - \beta(t)x(t-\tau)\big)$ in equation (1), then we have the following discrete fractional Lotka–Volterra model:

$$\begin{cases} {}^c\nabla_0^\alpha x(t) = x(t)\big(\gamma(t) - \beta(t)x(t-\tau)\big),\ t \in \mathbb{N}_0 \\ x(t) = \phi(t),\ t \in [-\tau, -\tau+1, \ldots, 0],\ 0 < \alpha < 1, \end{cases} \tag{8}$$

where the coefficients $\gamma$ and $\beta$ satisfy the boundedness relations

$$\gamma^- \leq \gamma(t) \leq \gamma^+, \ \ \beta^- \leq \beta(t) \leq \beta^+,$$

which are medically and biologically feasible. Model (8) represents the interspecific competition in single species with $\tau$ denotes the maturity time period.

Denote

$$\overline{f}_1(t, x(t)) = x(t)\gamma(t), \quad \overline{f}_2(t, x(t), x(t-\tau)) = -\beta(t)x(t)x(t-\tau).$$

It follows that the functions $\overline{f}_1$ and $\overline{f}_2$ satisfy the conditions

(III) $|\overline{f}_1(t,x(t))| \leq \gamma^+|x(t)|,\ \ |\overline{f}_2(t,x(t),x(t-\tau))| \leq \beta^+|x(t)| \times |x(t-\tau)|.$

(IV) $\overline{f}_i$ are Lipschitz functions with Lipschitz constants $\overline{L}_{f_i},\ i=1,2.$

The solution of model (8) has the form

$$x(t) = \phi(0) + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}x(s)\Big(\gamma(s)-\beta(s)x(s-\tau)\Big),\ \ t\in\mathbb{N}_0, \tag{9}$$

and $x(t)=\phi(t),\ \ t\in[-\tau,-\tau+1,\ldots,0]$. Define a function $G$ by

$$Gx(t) = G_1x(t) + G_2x(t),$$

where

$$G_1x(t) = \phi(0) + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}x(s)\gamma(s),$$

and

$$G_2x(t) = -\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}x(s)\beta(s)x(s-\tau).$$

One can easily employ the same arguments used in the proof of Theorem 3 to complete the proof of the following theorem for equation (8).

**Theorem 4.** *Let conditions* (III)–(IV) *hold. Then, the model (8) has a solution in the set $D$ provided that* $\frac{L_{f_1}C(\alpha)}{\Gamma(\alpha)} < 1$ *and* $|\phi(0)| + \frac{\left(\gamma^+r+\beta^+r^2\right)C(\alpha)}{\Gamma(\alpha)} \leq r$.

**Remark 1.** *The above result can be extended to $n$ species competitive Lotka–Volterra system of the form*

$$\begin{cases} \nabla_0^\alpha x_i(t) = x_i(t)\big(\gamma_i(t) - \sum_{j=1}^n \beta_{ij}(t)x_j(t-\tau_{ij})\big),\ t\in\mathbb{N}_0,\ i=1,2,\ldots,n. \\ x_i(t)=\phi_i(t),\ t\in[-\tau_i,-\tau_i+1,\ldots,0],\ 0<\alpha<1,\ \tau_i=\max_{1\leq j\leq n}\tau_{ij}, \end{cases} \tag{10}$$

*where* $\gamma^- \leq \gamma_i(t) \leq \gamma^+,\ \ \beta^- \leq \beta_{ij}(t) \leq \beta^+$.

**Remark 2.** *Results of this paper can be carried out for the equation*

$$\begin{cases} \nabla_0^\alpha x(t) = f(t,x(t),x(t-\tau)),\ t\in\mathbb{N}_2=\{2,3,\ldots\},\ \tau\geq 0, \\ x(t)=\phi(t),\ t\in[-\tau,-\tau+1,\ldots,1], \end{cases} \tag{11}$$

*where $f:\mathbb{N}_0\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $\nabla_0^\alpha$ denotes the Riemann–Liouville's fractional difference of order $\alpha\in(0,1)$. The solution of equation (11) has the form*

$$x(t) = \frac{t^{\overline{\alpha-1}}}{\Gamma(\alpha)}\phi(1) + \frac{1}{\Gamma(\alpha)}\sum_{s=2}^{t}(t-\rho(s))^{\overline{\alpha-1}}f(s,x(s),x(s-\tau)). \tag{12}$$

# 5 Conclusion

A comprehensive literature survey on the predator–prey type Lotka–Volterra model reveals that a considerable amount of work has already been done by many esteemed researchers during the last century. However the concept of the model related to fractional time derivatives is an original one.

The fractional Lotka–Volterra equation is obtained from the classical equations by replacing the first order time derivative by fractional derivative of order $\alpha \in (0, 1)$. One of the most significant outcomes of this evolution equation is the generation of fractional Brownian motions.

It has been discernible that the discrete analogue of ordinary differential equations has tremendous applications in computational analysis and computer simulations. Motivated by this reality, the study of the discrete analogue of fractional differential equations has become pressing and compulsory.

In this paper, we studied the existence and uniqueness of solutions for nonlinear delay fractional difference equations. The main theorem is proved with the help the Krasnoselskii fixed point theorem and the Arzela–Ascoli's Theorem. Prior to the main result, we set forth some notations and definitions which enriched the knowledge of discrete fractional calculus. To demonstrate the applicability of the main theorem, we provide an existence result for the discrete fractional Lotka–Volterra model.

It is to be noted that the analysis carried out in this paper is based on the use of nabla rather than delta operators. Indeed, unlike the delta operator the range of nabla fractional sum and difference operators depends only of the starting point and independent of the order $\alpha$. This provides exceptional ability to treat skilfully different circumstances throughout the proofs. The delta approach can be obtained from nabla operator through the implementation of the dual identities discussed in [41].

# References

[1] I. Podlubny: Fractional Differential Equations, vol. 198 of Mathematics in Science and Engineering, Academic Press, San Diego, Calif, USA, 1999.

[2] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo: Theory and Applications of Fractional Differential Equations, Elsevier, North Holland,2006.

[3] K. V. Miller, B. Ross: An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley–Interscience, New York, 1993.

[4] A. B. Abdulla, M. Al–Refai, A. Al–Rawashdeh: On the existence and uniqueness of solutions for a class of non-linear fractional boundary value problems, J. King Saud Univ. 28 (1) (2016), 103–110.

[5] Z-Dong Mei, J-Gen Peng, J-Huai Gao: Existence and uniqueness of solutions for nonlinear general fractional differential equations in Banach spaces, Indag. Math. 26 (4) (2015), 669–678.

[6] B. Zhu, L. Liu, Y. Wu: Existence and uniqueness of global mild solutions for a class of nonlinear fractional reaction–diffusion equations with delay, Comput. Math. Appl. In Press, Corrected Proof, Available online 12 February 2016

[7] Y. Zhou, F. Jiao, J. Li: Existence and uniqueness for fractional neutral differential equations with infinite delay, Nonlinear Anal. 71 (7–8) (2009), 3249–3256.

[8] N. Li, C. Wang: New existence results of positive solution for a class of nonlinear fractional differential equations, Acta Math. Sinica 33 (3)(2013), 847–854.

[9] F. M. Atici, P. W. Eloe: A transform method in discrete fractional calculus, Int. J. Difference Equ. 2 (2) (2007), 165–176.

[10] F. M. Atici, P. W. Eloe: Initial value problems in discrete fractional calculus, Proc. Amer. Math. Soc. 137 (3) (2009), 981–989. 12 Advances in Difference Equations

[11] F. M. Atici, P. W. Eloe: Discrete fractional calculus with the nabla operator, Electron. J. Qual. Theory Differ. Equ. vol. 2009, no. 3, pp. 1–12, 2009.

[12] F. M. Atici, S. Şengül: Modeling with Factorial Difference Equations, J. Math. Anal. Appl. 369 (1) (2010), 1–9.

[13] F. M. Atici, P. W. Eloe: Linear systems of fractional nabla difference equations, Rocky Mountain J. Math. 41 (2011), 353–370.

[14] T. Abdeljawad: On Riemann and Caputo fractional differences, Comput. Math. Appl. 62 (2011), 1602–1611.

[15] T. Abdeljawad, F. M. Atici: On the definitions of nabla fractional operators, Abstr. Appl. Anal. 2012, (2012), 13 pages.

[16] T. Abdeljawad, D. Baleanu: Fractional differences and integration by Parts, J. Comput. Anal. Appl. 13(2011), no. 5, 574–582

[17] T. Abdeljawad, D. Baleanu, F. Jarad, R. P. Agarwal: Fractional sums and differences with binomial coefficients, Discrete Dyn. Nat. Soc. Volume 2013, Article ID 104173, 6 pages.

[18] T. Abdeljawad, J. O. Alzabut,: The $q$–fractional analogue for Gronwall–type inequality, J. Funct. Spaces Appl. 2013, Art. ID 543839.

[19] G. A. Anastassiou: Discrete fractional calculus and inequalities, http://arxiv.org/abs/0911.3370.

[20] G. A. Anastassiou: Nabla discrete fractional calculus and nabla inequalities, Math. Comput. Modell. 51 (2010), 562–571.

[21] C. S. Goodrich: Existence of a positive solution to a system of discrete fractional boundary value problems, App. Math. Comput. 217 (9) (2011), 4740–4753.

[22] C. S. Goodrich: Existence and uniqueness of solutions to a fractional difference equation with nonlocal conditions, Comput. Math. Appl. 61 (2) (2011), 191–202.

[23] C. S. Goodrich: On a discrete fractional three–point boundary value problem, J. Differ. Equ. Appl. 18 (3) (2012), 397–415.

[24] F. Chen: Fixed points and asymptotic stability of nonlinear fractional difference equations, Electron. J. Qual. Theory Differ. Equ. 39 (2011), 1-?18.

[25] F. Chen, X. Luo, Y. Zhou: Existence results for nonlinear fractional order difference equation, Adv. Differ. Equ. 2011 (2011), Article ID 713201.

[26] F. Chen, Y. Zhou: Existence and Ulam stability of solutions for discrete fractional boundary value problem, Discrete Dyn. Nat. Soc. Volume 2013 (2013), Article ID 459161.

[27] G. C. Wu, D. Baleanu: Discrete fractional logistic map and its chaos, Nonlinear Dyn. 75 (2014) 283–287.

[28] G. C. Wu, D. Baleanu: Chaos synchronization of the discrete fractional logistic map, Sign. Proc. 102 (2014) 96–99.

[29] L. Erbe, C. S. Goodrich, B. Jia, A. Peterson: Survey of the qualitative properties of fractional difference operators: monotonicity, convexity, and asymptotic behavior of solutions, Adv. Differ. Equ. 2016 2016:43.

[30] C. Goodrich, A. Peterson: Discrete Fractional Calculus, Springer 2015.

[31] T. A. Burton: A fixed point theorem of Krasnoselskii fixed point theorem, Appl. Math. Lett. 11 (1998), 85–88.

[32] T. Faria: Sharp conditions for global stability of Lotka–Volterra systems with distributed delays, J. Differ. Equ. 246 (11) (2009), 4391–4404.

[33] F. Capone, R. D. Luca, S. Rionero: On the stability of non–autonomous perturbed Lotka–Volterra models, Appl. Math. Comp.219 (12) (2013), 6868–6881.

[34] J. X. Li, J. R. Yan: Persistence for Lotka–volterra patch–system with time delay, Nonlinear Anal. RWA 9 (2) (2008), 490–499.

[35] J. X. Li, J. R. Yan: Permanence and extinction for a non linear diffusive predator–prey system, Nonlinear Anal. 71 (1?2) (2009), 399–417.

[36] S. S. Chen, J. P. Shi, J. J. Wei: A note on Hopf bifurcations in a delayed diffusive Lotka–Volterra predator–prey system, Comput. Math. Appl. 62 (5) (2011), 2240–2245.

[37] Z. J. Liu, L. S. Chen: Periodic solution of neutral Lotka–volterra system with periodic delays, J. Math. Anal. Appl. 324 (1) (2006), 435-?451.

[38] C. N. Angstmann, B. I. Henry, A.V. McGann: A fractional–order infectivity SIR model, Physica A, 452 (15) (2016), 86–93.

[39] R. Khoshsiar Ghaziani, J. Alidousti, A. Bayati Eshkaftaki: Stability and dynamics of a fractional order Leslie–Gower prey–predator model, Appl. Math. Model. 40 (3) (2016), 2075–2086.

[40] T. Abdeljawad: On delta and nabla Caputo fractional differences and dual identities, Discrete Dyn. Nat. Soc. Volume 2013 (2013), Article ID 406910, 12 page.

[41] T. Abdeljawad: Dual identities in fractional difference calculus within Riemann, Adv. Differ. Equ. 2013: 36.

# Some sharp results on NLC-operators in $G_p$-metric spaces

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**Abstract**: In this paper we generalize, complement and improve some recent results on NLC-operators established in $G_p$-metric spaces. Several examples are given to support our theoretical approach.

**Keywords**: $G_p$-metric space, NLC-operator, supporting sequence, $G_p$-complete, fixed point

## 1 Introduction and preliminaries

Partial metric space and $G$-metric space are two different generalized metric spaces. In 1994 Matthews [13] introduced partial metric space as follows:

**Definition 1.1**. Let $X$ be a nonempty set. A partial metric is a mapping $p : X^2 \to [0, +\infty)$ which satisfies that

(p1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$, for all $x, y \in X$;

(p2) $p(x, x) \leq p(x, y)$, for all $x, y \in X$;

(p3) $p(x, y) = p(y, x)$, for all $x, y \in X$;

(p4) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$, for all $x, y, z \in X$.

Then the pair $(X, p)$ is called a partial metric space.

It is clear that each (standard) metric space is a partial metric space, while on the contrary it does not hold, in general. In recent years, many authors have obtained lots of fixed point results in partial metric spaces, for example, see [12], [13], [15], [17], [21] and the references therein.

On the other hand, in 2006 Mustafa and Sims [14] introduced another kind of generalized metric space, so-called $G$-metric space as follows:

**Definition 1.2**. Let $X$ be a nonempty set. A mapping $G : X^3 \to [0, +\infty)$ is called $G$-metric if it satisfies the following conditions:

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(G1) $x = y = z \Leftrightarrow G(x, y, z) = 0$ for all $x, y, z \in X$;

(G2) $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$;

(G3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $z \neq y$;

(G4) $G(x, y, z) = G(P\{x, y, z\})$, where $P$ is a permutation of $x, y, z \in X$ (symmetry in all three variables);

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$ (rectangle inequality).

Then the pair $(X, G)$ is called a $G$-metric space.

Based on this notion, many fixed point results under different contractive conditions have been obtained (see [1], [7]-[10], [14], and the references therein).

In 2011 Zand and Nezhad [23] introduced a concept as a generalization of both partial metric space and $G$-metric space as follows:

**Definition 1.3**. Let $X$ be a nonempty set. A mapping $G_p : X^3 \to [0, +\infty)$ is called a $G_p$-metric if the following conditions are satisfied:

($G_p1$) $x = y = z$ if $G_p(x, y, z) = G_p(x, x, x) = G_p(y, y, y) = G_p(z, z, z)$ for all $x, y, z \in X$;

($G_p2$) $G_p(x, x, x) \leq G_p(x, x, y) \leq G_p(x, y, z)$ for all $x, y, z \in X$;

($G_p3$) $G_p(x, y, z) = G_p(P\{x, y, z\})$, where $P$ is a permutation of $x, y, z \in X$ (symmetry in all three variables);

($G_p4$) $G_p(x, y, z) \leq G_p(x, a, a) + G_p(a, y, z) - G_p(a, a, a)$, for all $x, y, z, a \in X$ (rectangle inequality).

Then the pair $(X, G_p)$ is called a $G_p$-metric space.

**Remark 1.4.** It is worth mentioning that authors in [2], [3], [5], [19] and [23] used ($G_p2$) while in [6], [18] and [20] authors used the following condition:

($G_p2'$) $G_p(x, x, x) \leq G_p(x, x, y) \leq G_p(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$.

In the former case $(X, G_p)$ is a symmetric $G_p$-metric space, that is., $G_p(x, x, y) = G_p(x, y, y)$ for all $x, y \in X$. However, in the latter case this does not hold.

Otherwise, each symmetric $G$-metric space is symmetric $G_p$-metric space, but the converse is not true (see Example 1 from [23]) as well as each $G$-metric space is $G_p$-metric space in the sense of [18]. However, the claim from [23] (page 87, lines 6-,7-) that each $G$-metric space is also $G_p$-metric space is false (see [18], page 79). In addition, It is noteworthy that Example 3 in [23] is symmetric $G$-metric space, and hence it is $G_p$-metric space. It is also clear that Definition 6 (because ($G_p2$)) in [23] is superfluous.

First our important result in this section is the following:

**Proposition 1.5.** Every $G_p$-metric space $(X, G_p)$ in the sense of [18] defines a metric space $\left(X, d_{G_p}\right)$ as follows:

$$d_{G_p}(x, y) = G_p(x, y, y) + G_p(x, x, y) - G_p(x, x, x) - G_p(y, y, y), \text{ for all } x, y \in X.$$

**Proof.** Using ($G_p2$), we have $d_{G_p}(x, y) \geq 0$ for all $x, y \in X$. Also, if $x = y$, then $d_{G_p}(x, y) = 0$. Conversely, let $d_{G_p}(x, y) = 0$, then

$$G_p(x, y, y) + G_p(x, x, y) - G_p(x, x, x) - G_p(y, y, y) = 0,$$

that is.,

$$[G_p(x,x,y) - G_p(x,x,x)] + [G_p(x,y,y) - G_p(y,y,y)] = 0,$$

or equivalently, $G_p(x,x,y) = G_p(x,x,x)$ and $G_p(x,y,y) = G_p(y,y,y)$. Further, on account of ($G_p4$) it implies that $G_p(x,y,y) \leq 2G_p(x,x,y) - G_p(x,x,x) = G_p(x,x,y)$. Similarly it follows that $G_p(x,x,y) \leq G_p(x,y,y)$ for all $x,y \in X$. Then

$$G_p(x,y,x) = G_p(x,x,x) = G_p(y,y,y),$$

thus by ($G_p1$) it gives $x = y$.

It is obvious that $d_{G_p}(x,y) = d_{G_p}(y,x)$ for all $x,y \in X$.

Finally, we shall prove that

$$d_{G_p}(x,z) \leq d_{G_p}(x,y) + d_{G_p}(y,z),$$

for all $x,y,z \in X$, or equivalently,

$$\begin{aligned} &G_p(x,x,z) + G_p(x,z,z) - G_p(x,x,x) - G_p(z,z,z) \\ \leq\ &G_p(x,x,y) + G_p(x,y,y) - G_p(x,x,x) - G_p(y,y,y) \\ &+ G_p(y,y,z) + G_p(y,z,z) - G_p(y,y,y) - G_p(z,z,z), \end{aligned}$$

that is.,

$$\begin{aligned} &G_p(x,x,z) + G_p(x,z,z) \\ \leq\ &G_p(x,x,y) + G_p(x,y,y) - G_p(y,y,y) + G_p(y,y,z) + G_p(y,z,z) - G_p(y,y,y). \end{aligned}$$

Notice that

$$G_p(x,x,z) = G_p(z,x,x) \leq G_p(z,y,y) + G_p(y,x,x) - G_p(y,y,y)$$

and

$$G_p(x,z,z) \leq G_p(x,y,y) + G_p(y,z,z) - G_p(y,y,y),$$

so the proof is completed. □

**Remark 1.6.** Our proof of this proposition is more detailed than one of [23].

Further, we announce the following definition with valid approaches which complements Definition 1.9 from [18].

**Definition 1.7.** Let $(X, G_p)$ be a $G_p$-metric space and $\{x_n\}$ a sequence in $X$. Then

(1) $\{x_n\}_{n\in\mathbb{N}}$ is called $G_p$-convergent to a point $x \in X$ if $\lim_{n,m\to\infty} G_p(x,x_n,x_m) = G_p(x,x,x)$. In this case, we write $x_n \to x$ as $n \to \infty$;

(2) $\{x_n\}$ is called a $G_p$-Cauchy sequence if $\lim_{n,m\to\infty} G_p(x_n,x_m,x_m) = r \in \mathbb{R}$. Particularly, $\{x_n\}$ is called 0-Cauchy sequence if $r = 0$;

(3) $(X, G_p)$ is called $G_p$-complete if for every $G_p$-Cauchy sequence $\{x_n\}$ in $X$ is $G_p$-convergent to $x \in X$.

Now, we give the following conclusion which corrects Proposition 4 of [23]:

**Proposition 1.8.** Let $(X, G_p)$ be a symmetric $G_p$-metric space. Then for a sequence $\{x_n\} \subseteq X$ and a point $x \in X$ the following are equivalent:

(1) $\{x_n\}$ is $G_p$-convergent to $x$;

(2) $G_p(x_n, x_{n,}x) \to G_p(x, x, x)$ as $n \to \infty$;

(3) $G_p(x_n, x, x) \to G_p(x, x, x)$ as $n \to \infty$.

**Proof.** Since $(X, G_p)$ is symmetric $G_p$-metric space, then (2) is equivalent to (3). Taking $m = n$ in (1), we speculate that (1) implies (2), thus, (1) implies (3). For the converse we have that

$$\begin{aligned}
&G_p(x, x_n, x_m) - G_p(x, x, x) \\
&= G_p(x_n, x_m, x) - G_p(x, x, x) \\
&\leq G_p(x_n, x, x) + G_p(x, x_m, x) - G_p(x, x, x) - G_p(x, x, x) \\
&= [G_p(x_n, x, x) - G_p(x, x, x)] + [G_p(x_m, x, x) - G_p(x, x, x)] \\
&\to 0 + 0 = 0, \text{ as } n, m \to \infty,
\end{aligned}$$

then (3) implies (1). We complete the proof. □

Next we generalize Lemma 1.10 from [2] (see also [3], [5], [6], [18], [20]), that is., we announce the following assertion:

**Proposition 1.9.** Let $(X, G_p)$ be a $G_p$-metric space in the sense of [18]. Then

(A) if $G_p(x, y, z) = 0$, then $x = y = z$;

(B) if $x \neq y$, then $G_p(x, y, y) > 0$.

**Proof.** (A) If $x \neq y \neq z \neq x$, then (A) is an immediate consequence of ($G_p2'$) and ($G_p1$). If for instance, $x \neq y = z$, then $G_p(x, y, z) = G_p(x, y, y) = 0$. In this case, we get $G_p(x, x, x) = G_p(x, x, y) = G_p(y, y, y) = 0$. Indeed, by ($G_p4$) it follows that

$$G_p(x, x, y) \leq G_p(x, y, y) + G_p(y, x, y) - G_p(y, y, y) \leq 2G_p(x, y, y) = 0.$$

Since $G_p(x, x, x) \leq G_p(x, x, y)$ and $G_p(y, y, y) \leq G_p(x, y, y)$ hold for all $x, y \in X$, then we arrive at

$$G_p(x, y, y) = G_p(x, x, y) = G_p(x, x, x) = G_p(y, y, y) = 0,$$

so by ($G_p1$), we obtain the desired result. □

(B) Let $G_p(x, y, y) = 0$. Now, based on the proof of (A) when $x \neq y = z$, we claim that $x = y$. A contradiction.

# 2 Auxiliary results

In the sequel, let $(X, G_p)$ be a $G_p$-metric space in the sense of [18]. First of all, we introduce the following notion:

**Definition 2.1.** Let $(X, G_p)$ be a $G_p$-metric space, $\alpha \in (0, 1)$ a constant and $T : X \to X$ a mapping. We say that $T$ is an NLC-operator on $X$ if for each $x \in X$ there is some

$n(x) \in \mathbb{N}$ such that for each $y \in X$ it holds

$$G_p\left(T^{n(x)}x, T^{n(x)}x, T^{n(x)}y\right) \leq \max\left\{\alpha G_p(x,x,y), G_p(x,x,x)\right\}. \tag{2.1}$$

For an NLC-operator $T$ and $x \in X$ we define supporting sequence at $x$ as a sequence $\{s_k\}_{k\in\mathbb{N}\cup\{0\}}$ where $s_0 = 0$ and $s_{k+1} = s_k + n(T^{s_k}x), k \in \mathbb{N}\cup\{0\}$. Also set $J_T(X) = \{x \in X : T^m x = T^{m+1}x \text{ for some } m \in \mathbb{N}\}$.

**Remark 2.2.** (i) Condition (2.1) implies that for any $i \geq s_k$, it is valid that

$$G_p\left(T^{s_k}x, T^{s_k}x, T^i x\right) \leq \max\left\{\alpha G_p\left(T^{s_{k-1}}x, T^{s_{k-1}}x, T^j x\right), G_p\left(T^{s_{k-1}}x, T^{s_{k-1}}x, T^{s_{k-1}}x\right)\right\}, \tag{2.2}$$

where $j = i - s_k + s_{k-1} \geq s_{k-1}$, and specially that

$$G_p\left(T^{s_k}x, T^{s_k}x, T^{s_k}x\right) \leq G_p\left(T^{s_{k-1}}x, T^{s_{k-1}}x, T^{s_{k-1}}x\right). \tag{2.3}$$

Now, fix $x \in X\backslash J_T(X)$. For $k \in \mathbb{N}$ and $i \geq s_k$ use (2.2), repeatedly fix integers $l_j \geq s_j, 0 \leq j < k$ and $t_1, t_2, ..., t_k \in \{0,1\}$ such that $l_k := i$, then

$$G_p\left(T^{s_j}x, T^{s_j}x, T^{l_j}x\right) \leq \alpha^{t_j} \cdot G_p\left(T^{s_{j-1}}x, T^{s_{j-1}}x, T^{l_{j-1}}x\right)$$

for all $0 \leq j \leq k$, where

$$t_j = \begin{cases} 1, & \text{if } s_{j-1} < l_{j-1}, \\ 0, & \text{if } s_{j-1} = l_{j-1}. \end{cases}$$

Let us recall $(l_0, l_1, ..., l_{k-1})$ and $(t_1, t_2, ..., t_k)$ as the $(k,l)$-descent and $(k,i)$-signature at $x$, respectively.

Further put $r_{k,i} =: k - h_{k,i}$, where $h_{k,i}$ is a number of zeroes in $(k,i)$-signature at $x$.

We shall say that $x$ is Type 1 if there are sequences of positive integers $\{k_m\}_{m\in\mathbb{N}\cup\{0\}}$ and $\{i_m\}_{m\in\mathbb{N}\cup\{0\}}$, one of them is strictly increasing such that for all $m \in \mathbb{N}\cup\{0\}$ we have that $i_m \geq s_m$ and $r_{k_m,i_m} < r_{k_{m+1},i_{m+1}}$.

We shall say that $x$ is Type 2 if $x$ is not Type 1, i.e., there are $k_0, B \in \mathbb{N}$ such that for all $k \geq k_0$ and $i \geq s_k$ it holds $r_{k,i} < B$.

(ii) In the framework of $G$-metric spaces, condition (2.1) becomes

$$G_p\left(T^{n(x)}x, T^{n(x)}x, T^{n(x)}y\right) \leq \alpha G_p(x,x,y), \tag{2.3'}$$

hence, it is iterate contractive condition of Sehgal-Guseman type in this framework (see [11], [16]).

**Lemma 2.3.** Let $T$ be an NLC-operator on $G_p$-metric space $(X, G_p), x \notin J_T(X)$, and let $\{s_k\}_{k\in\mathbb{N}\cup\{0\}}$ be a supporting sequence at $x$. Then

(a) if $(l_0, l_1, ..., l_{k-1})$ is $(k, i_0)$-descent at $x$, then

$$G_p\left(T^{s_k}x, T^{s_k}x, T^{i_0}x\right) \leq \alpha^{r_{k,i_0}} \cdot G_p\left(x, x, T^{l_0}x\right),$$
$$G_p\left(T^{s_k}x, T^{s_k}x, T^{i_0}x\right) \leq G_p\left(T^{s_j}x, T^{s_j}x, T^{l_j}x\right)$$

for all $0 \leq j \leq k$, where $l_k := i_0$;

(b) if $P \subseteq \{0,1,...,k-1\}$ and $r_{k,i_0} < \text{card}P$ (cardP is the number of elements of $P$), then for some $j_0 \in P$ it holds

$$G_p\left(T^{s_k}x, T^{s_k}x, T^{i_0}x\right) \leq G_p\left(T^{s_{j_0}}x, T^{s_{j_0}}x, T^{s_{j_0}}x\right).$$

**Proof.** Using the definition of $r_{k,i}$, (a) is obvious. To prove (b), under the hypothesis, the set $\{j+1 : j \in P\}$ is subset of $\{1,2,...,k\}$ with $\text{card}(P) > r_{k,i_0}$, so there is some $j_0 \in P$ with $t_{j_0+1} = 0$. Then

$$\begin{aligned} G_p\left(T^{s_{j_0+1}}x, T^{s_{j_0+1}}x, T^{i_0}x\right) &\leq G_p\left(T^{l_{j_0+1}}x, T^{l_{j_0+1}}x, T^{i_0+1}x\right) \\ &\leq \alpha^{t_{j_0}+1} G_p\left(T^{s_{j_0}}x, T^{s_{j_0}}x, T^{l_{j_0}}x\right) \\ &= G_p\left(T^{s_{j_0}}x, T^{s_{j_0}}x, T^{s_{j_0}}x\right), \end{aligned}$$

whereof (a) and $s_{j_0} = l_{j_0}$ have been used. □

**Lemma 2.4.** Let $T$ be an NLC-operator on $G_p$-metric space $(X, G_p)$ and $x \in X$, then there is some $M_x > 0$ such that for all $i \geq 0$ it satisfies that

$$G_p\left(x, x, T^i x\right) \leq M_x, \tag{2.4}$$

and so $G_p\left(T^j x, T^j x, T^i x\right) \leq 3M_x$, for each $i, j \in \mathbb{N} \cup \{0\}$.

**Proof.** If $x \in J_T(X)$, then this is obvious. Thus, let $x \notin J_T(X)$ and set

$$b(x) = G_p(x,x,x) + G_p(x,x,Tx) + \cdots + G_p\left(x,x,T^{n(x)}x\right).$$

Let us prove by induction that

$$G_p\left(x, x, T^i x\right) \leq \frac{1}{1-\alpha} b(x), \text{ for all } i \in \mathbb{N}.$$

Obviously (2.4) is true for $0 \leq k \leq n(x)$. Now assume that the same is valid for some $k \geq n(x)$. Then

$$\begin{aligned} &G_p\left(x, x, T^{k+1}x\right) \leq G_p\left(x, x, T^{n(x)}x\right) + G_p\left(T^{n(x)}x, T^{n(x)}x, T^{k+1}x\right) \\ &\leq G_p\left(x, x, T^{n(x)}x\right) + \max\left\{\alpha G_p\left(x, x, T^{k+1-n(x)}x\right), G_p(x,x,x)\right\} \\ &\leq G_p\left(x, x, T^{n(x)}x\right) + G_p(x,x,x) + \frac{\alpha}{1-\alpha} b(x) \\ &\leq b(x) + \frac{\alpha}{1-\alpha} b(x) \\ &= \frac{1}{1-\alpha} b(x), \end{aligned}$$

so (2.4) is proved with $M_x = \frac{1}{1-\alpha} b(x)$.

Further, we have

$$G_p\left(T^i x, T^j x, T^j x\right) \leq G_p\left(T^i x, x, x\right) + 2G_p\left(T^j x, x, x\right) \leq 3M_x,$$

for all $i, j \in \mathbb{N} \cup \{0\}$. □

**Lemma 2.5.** Let $T$ be an NLC-operator on $G_p$-metric space $(X, G_p)$ and $x \in X \backslash J_T(X)$. If $x$ is Type 1, then $\lim_{i,j\to\infty} G_p(T^i x, T^i x, T^j x) = 0$.

**Proof.** Fix $m \in \mathbb{N} \cup \{0\}$. If $(l_0, ..., l_{k_m-1})$ is $(s_{k_m}, i_m)$-descent, then by (a) of Lemma 2.3 we have

$$G_p\left(T^{s_{k_m}} x, T^{s_{k_m}} x, T^{i_m} x\right) \leq \alpha^{r_{k_m,i_m}} \cdot G_p\left(x, x, T^{l_0} x\right) \leq \alpha^{r_{k_m,i_m}} M_x.$$

In view of $\lim_{m\to\infty} r_{k_m,i_m} = \infty$, it follows that

$$\lim_{m\to\infty} G_p\left(T^{s_{k_m}} x, T^{s_{k_m}} x, T^{i_m} x\right) = \lim_{m\to\infty} G_p\left(T^{s_{k_m}} x, T^{s_{k_m}} x, T^{s_{k_m}} x\right) = 0.$$

For given $\varepsilon > 0$, choose $m_0 \in \mathbb{N}$ such that $\alpha^{m_0} M_x < \varepsilon$ and $G_p(T^{s_{k_m}} x, T^{s_{k_m}} x, T^{s_{k_m}} x) < \varepsilon$ for all $m \geq m_0$. Let $r_{k_{2m_0},i} \geq m_0$. Then

$$G_p\left(T^{s_{k_{2m_0}}} x, T^{s_{k_{2m_0}}} x, T^i x\right) \leq \alpha^{r_{k_{2m_0},i}} \cdot M_x \leq \alpha^{m_0} M_x < \varepsilon.$$

Now suppose that $r_{k_{2m_0},i} < m_0$. For $P_i = \{k_{m_0}, ..., k_{2m_0-1}\} \subseteq \{0, 1, ..., k_{2m_0} - 1\}$, we have $\mathrm{card}(P) > r_{k_{2m_0},i}$, so by Lemma 2.3, there exists some $m_0 \leq j \leq 2m_0 - 1$ such that

$$G_p\left(T^{s_{k_{2m_0}}} x, T^{s_{k_{2m_0}}} x, T^i x\right) \leq G_p\left(T^{s_{k_j}} x, T^{s_{k_j}} x, T^{s_{k_j}} x, \right) < \varepsilon$$

for each $i \geq s_{k_{2m_0}}$.

Accordingly, if $i, j \geq s_{k_{2m_0}}$, then

$$\begin{aligned} G_p\left(T^i x, T^i x, T^j x\right) &\leq G_p\left(T^{s_{k_{2m_0}}} x, T^{s_{k_{2m_0}}} x, T^j x\right) + G_p\left(T^i x, T^i x, T^{s_{k_{2m_0}}}\right) \\ &\leq G_p\left(T^{s_{k_{2m_0}}} x, T^{s_{k_{2m_0}}} x, T^j x\right) + 2G_p\left(T^i x, T^{s_{k_{2m_0}}} x, T^{s_{k_{2m_0}}} x\right) \\ &< 3\varepsilon. \end{aligned}$$

Therefore, we prove that $\lim_{i,j\to\infty} G_p(T^i x, T^i x, T^j x) = 0$. □

**Lemma 2.6.** Let $T$ be an NLC-operator and $x \in X \backslash J_T(X)$. If $x$ is Type 2, then the sequence $\{T^i x\}_{i\in\mathbb{N}\cup\{0\}}$ is $G_p$-Cauchy.

**Proof.** By (2.3), it is easy to see that $\{G_p(T^{s_k} x, T^{s_k} x, T^{s_k} x)\}_{k\in\mathbb{N}\cup\{0\}}$ is a nonincreasing sequence, where $\{s_k\}_{k\in\mathbb{N}\cup\{0\}}$ is a supporting sequence at $x$. Then there exists

$$r_x := \lim_k G_p(T^{s_k} x, T^{s_k} x, T^{s_k} x) = \inf_k \{G_p(T^{s_k} x, T^{s_k} x, T^{s_k} x)\}$$

such that it is finite.

At first let us prove that for any $\varepsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$ and $i \geq s_m$, one has

$$G_p\left(T^{s_m} x, T^{s_m} x, T^i x\right) \in (r_x - \varepsilon, r_x + \varepsilon). \tag{2.5}$$

Since $x$ is Type 2 there are $k_0, B \in \mathbb{N}$ such that for all $k \geq k_0$ and all $i \geq s_k$, there holds $r_{k,i} < B$. Let $\varepsilon > 0$, take $m_1 \geq k_0$ such that for all $m \geq m_1$,

$$G_p(T^{s_m} x, T^{s_m} x, T^{s_m} x) \in (r_x - \varepsilon, r_x + \varepsilon). \tag{2.6}$$

Let $m \geq m_1+B$ and $i \geq s_m$ be arbitrary. For $P = \{m_1, ..., m_1 + B - 1\} \subseteq \{0, 1, ..., m-1\}$, we have card$P \geq B > r_{m,i}$, then there exists $m_1 \leq j \leq m_1 + B - 1$ such that

$$r_x - \varepsilon < G_p\left(T^{s_m}x, T^{s_m}x, T^{s_m}x\right) \leq G_p\left(T^{s_m}x, T^{s_m}x, T^{i}x\right) \leq G_p\left(T^{s_j}x, T^{s_j}x, T^{s_j}x\right) < r_x + \varepsilon.$$

So we get (2.5).

Now let us prove that for any $\varepsilon > 0$, there is $k^* \in \mathbb{N}$ such that for all $i, j \geq k^*$,

$$G_p\left(T^{i}x, T^{i}x, T^{j}x\right) < r_x + \varepsilon. \tag{2.7}$$

Indeed, for any $\varepsilon > 0$, consider $m_0$ as in (2.5) and let $i, j \geq s_{m_0}$ be arbitrary. Then

$$\begin{aligned} G_p\left(T^{i}x, T^{i}x, T^{j}x\right) &\leq G_p\left(T^{s_{m_0}}x, T^{s_{m_0}}x, T^{j}x\right) + G_p\left(T^{i}x, T^{i}x, T^{s_{m_0}}x\right) \\ &\quad - G_p\left(T^{s_{m_0}}x, T^{s_{m_0}}x, T^{s_{m_0}}x\right) \\ &< r_x + \varepsilon + 2G_p\left(T^{s_{m_0}}x, T^{s_{m_0}}x, T^{i}x\right) - 2G_p\left(T^{s_{m_0}}x, T^{s_{m_0}}x, T^{s_{m_0}}x\right) \\ &< r_x + \varepsilon + 2\left(r_x + \varepsilon\right) - 2\left(r_x - \varepsilon\right) \\ &= r_x + 5\varepsilon. \end{aligned}$$

To prove $\lim_{i,j\to\infty} G_p\left(T^{i}x, T^{i}x, T^{j}x\right) = r_x$, we only need to show that for any $\varepsilon > 0$, there exists $\widetilde{k} \in \mathbb{N}$ such that for all $i \geq \widetilde{k}$, ones always have

$$r_x - \varepsilon < G_p\left(T^{i}x, T^{i}x, T^{j}x\right). \tag{2.8}$$

Suppose on the contrary, that for any $k$ there is some $i_0 \geq k$ satisfying

$$G_p\left(T^{i_0}x, T^{i_0}x, T^{i_0}x\right) \leq r_x - \varepsilon.$$

Put $z := T^{i_0}x$. Obviously, $x \notin J_T(X)$ implies that $z \notin J_T(X)$. If $z$ is Type 1, then by Lemma 2.5 it follows that

$$0 = \lim_{i,j} G_p\left(T^{i}z, T^{i}z, T^{j}z\right) = \lim_{i,j} G_p\left(T^{i}x, T^{i}x, T^{j}x\right) = r_x,$$

so $\{T^{i}x\}_{i\in\mathbb{N}\cup\{0\}}$ is 0-Cauchy sequence.

Now suppose that $z$ is Type 2, and let $\{q_m\}_{m\in\mathbb{N}\cup\{0\}}$ be a supporting sequence at $z$. Then, for each $m \in \mathbb{N}\cup\{0\}$,

$$G_p\left(T^{q_m}z, T^{q_m}z, T^{q_m}z\right) \leq G_p\left(z, z, z\right) \leq r_x - \varepsilon,$$

so

$$r_z = \lim_{m} G_p\left(T^{q_m}z, T^{q_m}z, T^{q_m}z\right) \leq r_x - \varepsilon.$$

Note that $r_z < \frac{r_x+r_z}{2}$, then for $j_0 \in \mathbb{N}$, one obtain that

$$G_p\left(T^{j}z, T^{j}z, T^{j}z\right) < \frac{r_x + r_z}{2}, \text{ for all } j \geq j_0.$$

As $\lim_{m\to\infty} G_p\left(T^{s_m}x, T^{s_m}x, T^{s_m}x\right) = r_x$, then there is some $m \geq i_0 + j_0$ such that

$$G_p\left(T^j x, T^j x, T^j x\right) > \frac{r_x + r_z}{2},$$

which is impossible, so (2.8) is satisfied. Now, for $s_m - i_0 \geq m - i_0 \geq j_0$, we claim that

$$\frac{r_x + r_z}{2} < G_p\left(T^{s_m}x, T^{s_m}x, T^{s_m}x\right) = G_p\left(T^{s_m - i_0}x, T^{s_m - i_0}x, T^{s_m - i_0}x\right) < \frac{r_x + r_z}{2}.$$

In the end, from

$$r_x - \varepsilon < G_p\left(T^i x, T^i x, T^i x\right) \leq G_p\left(T^i x, T^i x, T^j x\right),$$

it follows that $\{T^i x\}_{i\in\mathbb{N}\cup\{0\}}$ is a $G_p$-Cauchy sequence. □

**Lemma 2.7.** Let $T : X \to X$ be an operator on $G_p$-metric space $(X, G_p)$. Suppose that $x \in X$ is a point such that $T^k x = x$ holds for some positive integer $k$, and there is $y \in X$ such that

$$G_p(y, y, y) = \lim_i G_p\left(y, T^i x, T^i x\right) = \lim_{i,j} G_p\left(T^i x, T^i x, T^j x\right), \tag{2.9}$$

then $Tx = x$.

**Proof.** Since $T^{k\cdot i}x = x$, then for any $i \in \mathbb{N}\cup\{0\}$, we have that

$$G_p(y, y, y) = \lim_i G_p\left(y, T^{ki}x, T^{ki}x\right) = G_p(y, x, x)$$

and

$$G_p(y, y, y) = \lim_i G_p\left(T^{ki}x, T^{ki}x, T^{ki}x\right) = G_p(x, x, x),$$

so $y = x$. Now (2.9) implies that

$$G_p(x, x, x) = \lim_i G_p\left(x, T^{ki+1}x, T^{ki+1}x\right) = G_p(x, Tx, Tx)$$

and

$$G_p(x, x, x) = \lim_i G_p\left(T^{ki+1}x, T^{ki+1}x, T^{ki+1}x\right) = G_p(Tx, Tx, Tx).$$

Thus, $Tx = x$. □

# 3 Main results

Both results in this section generalize many existing results in the literature (see [12, Theorem 3.1] and [4, Lemmas 3.-5, Theorems 1 and 2]). Firstly, we announce our first result for NLC-operator in $G_p$-complete $G_p$-metric space as follows.

**Proposition 3.1.** Let $T$ be an NLC-operator on $G_p$-complete $G_p$-metric space $(X, G_p)$, then

(1) for each $x \in X$, the sequence $\{T^i x\}_{i\in\mathbb{N}\cup\{0\}}$ $G_p$-converges to some $v_x \in X$;

(2) for all $x, y \in X$, one has

$$G_p(v_y, v_y, v_x) = \max\left\{G_p(v_x, v_x, v_x), G_p(v_y, v_y, v_y)\right\}.$$

**Proof.** Since $(X, G_p)$ is $G_p$-complete, then for each $x \in X$, the existence of $v_x$ is assured by Lemma 2.5 and Lemma 2.6. Let us prove (2). Let $x, y \in X$ and $G_p(v_y, v_y, v_y) \geq G_p(v_x, v_x, v_x)$. If $G_p(v_y, v_y, v_y) = 0$, then $v_x = v_y$ and the claim is clear. Thus, assume that $G_p(v_y, v_y, v_y) > 0$.

For any $0 < \varepsilon < \frac{1-\alpha}{2(1+\alpha)} G_p(v_y, v_y, v_y)$, there is some $m_0 \in \mathbb{N}$ such that for all $i, j \geq m_0$, we have

$$\begin{aligned}\max\{&G_p(T^i y, v_y, v_y) - G_p(T^i y, T^i y, T^i y), |G_p(T^i y, T^i y, T^i y) - G_p(v_y, v_y, v_y)|,\\ &G_p(T^i y, T^i y, v_y) - G_p(v_y, v_y, v_y)\} < \varepsilon\end{aligned}$$

and

$$\max\left\{G_p\left(v_x, v_x, T^j x\right) - G_p\left(v_x, v_x, v_x\right), G_p\left(v_x, T^j x, T^j x\right) - G_p\left(T^j x, T^j x, T^j x\right)\right\} < \varepsilon.$$

For $i, j \geq m_0$, we have

$$\begin{aligned}G_p\left(T^i y, T^i y, T^i x\right) &\leq G_p\left(T^i y, T^i y, v_y\right) - G_p\left(v_y, v_y, v_y\right)\\ &\quad + G_p\left(v_y, v_y, v_x\right) + G_p\left(v_x, v_x, T^i x\right) - G_p\left(v_x, v_x, v_x\right)\\ &< 2\varepsilon + G_p\left(v_y, v_y, v_x\right)\end{aligned}$$

and

$$\begin{aligned}G_p\left(v_y, v_y, v_x\right) &\leq G_p\left(v_y, v_y, T^i y\right) - G_p\left(T^i y, T^i y, T^i y\right) + G_p\left(v_x, T^j x, T^j x\right)\\ &\quad - G_p\left(T^j y, T^j y, T^j y\right) + G_p\left(T^i y, T^i y, T^j y\right)\\ &< 2\varepsilon + G_p\left(T^i y, T^i y, T^j x\right).\end{aligned}$$

For any $i_0 \geq m_0$ and $i_1 := n\left(T^{i_0} y\right)$, we get

$$\begin{aligned}G_p\left(v_y, v_y, v_x\right) - 2\varepsilon &\leq G_p\left(T^{i_0+i_1} y, T^{i_0+i_1} y, T^{i_0+i_1} x\right)\\ &\leq \max\left\{\alpha G_p\left(T^{i_0} y, T^{i_0} y, T^{i_0} x\right), G_p\left(T^{i_0} y, T^{i_0} y, T^{i_0} y\right)\right\}\\ &< \max\left\{\alpha\left(2\varepsilon + G_p\left(v_y, v_y, v_x\right), \varepsilon + G_p\left(v_y, v_y, v_y\right)\right)\right\}.\end{aligned}$$

If

$$G_p\left(v_y, v_y, v_x\right) - 2\varepsilon < 2\alpha\varepsilon + \alpha G_p\left(v_y, v_y, v_x\right),$$

then

$$G_p\left(v_y, v_y, v_x\right) < 2\varepsilon\left(1 + \alpha\right) < G_p\left(v_y, v_y, v_x\right).$$

This is a contradiction. As a consequence, we deduce that

$$G_p\left(v_y, v_y, v_x\right) < 3\varepsilon + G_p\left(v_y, v_y, v_y\right),$$

so

$$G_p\left(v_y, v_y, v_x\right) \leq G_p\left(v_y, v_y, v_y\right).$$

Finally, by ($G_p2$), we speculate that

$$G_p(v_y, v_y, v_x) = G_p(v_y, v_y, v_x) = \max\{G_p(v_x, v_x, v_x), G_p(v_y, v_y, v_y)\}.$$

□

Now, we announce our second result in the framework of $G_p$-complete $G_p$-metric spaces.

**Theorem 3.2.** Let $T$ be an NLC-opertor on $G_p$-complete $G_p$-metric space $(X, G_p)$, then there is a fixed point $z \in X$ of $T$ such that $G_p(z,z,z) = \inf\{G_p(v_x, v_x, v_x) : x \in X\}$.

**Proof.** For $x \in X$, put $r_x := G_p(v_x, v_x, v_x) = \lim_{k\to\infty} G_p(T^{s_k}x, T^{s_k}x, T^{s_k}x)$ for $\{s_k\}_{k\in\mathbb{N}\cup\{0\}}$ which is the supporting sequence at $x$. Let $I := \inf\{r_x : x \in X\}$. For $m \geq 1$, take $x_m \in X$ such that for all $i, j \in \mathbb{N}\cup\{0\}$, it holds

$$G_p\left(Tx_m^i, Tx_m^i, Tx_m^j\right) \in \left(I - \frac{1}{m}, I + \frac{1}{m}\right). \tag{3.1}$$

At first we shall prove that $\lim_{m,k\to\infty} G_p(x_m, x_m, x_k) = I$. For $m, k \geq 2$, let $C_{m,k} > 0$ and

$$G_p\left(T^j x_m, T^j x_m, T^i x_k\right) < C_{m,k}, \quad i, j \in \mathbb{N}\cup\{0\}.$$

Fix $m, k \geq 2$ and let $\{s_q\}_{q\in\mathbb{N}\cup\{0\}}$ be the supporting sequence at $x_m$ and let $l \geq 1$ be an integer such that $\alpha^l \cdot C_{m,k} < \frac{1}{k+m}$. Then

$$\begin{aligned}
&G_p(x_m, x_m, x_k)\\
&\leq G_p(x_m, x_m, T^{s_l}x_m) - G_p(T^{s_l}x_m, T^{s_l}x_m, T^{s_l}x_m) + G_p(T^{s_l}x_m, T^{s_l}x_m, x_k)\\
&\leq G_p(x_m, x_m, T^{s_l}x_m) - G_p(T^{s_l}x_m, T^{s_l}x_m, T^{s_l}x_m)\\
&\quad + G_p(x_k, T^{s_l}x_k, T^{s_l}x_k) - G_p(T^{s_l}x_k, T^{s_l}x_k, T^{s_l}x_k) + G_p(T^{s_l}x_m, T^{s_l}x_m, T^{s_l}x_k).
\end{aligned}$$

Denote

$$\begin{aligned}
A_{m,k} &:= G_p(x_m, x_m, T^{s_l}x_m) - G_p(T^{s_l}x_m, T^{s_l}x_m, T^{s_l}x_m) < \frac{2}{m},\\
D_{m,k} &:= G_p(x_k, T^{s_l}x_k, T^{s_l}x_k) - G_p(T^{s_l}x_k, T^{s_l}x_k, T^{s_l}x_k) < \frac{2}{k}.
\end{aligned}$$

At first, assume that

$$G_p(T^{s_l}x_m, T^{s_l}x_m, T^{s_l}x_k) > G_p\left(T^i x_m, T^i x_m, T^i x_m\right) \text{ for all } i \in \{0, 1, ..., s_l\}.$$

Then by (1.2), it is clear that

$$\begin{aligned}
&G_p(T^{s_{j-1}}x_m, T^{s_{l-1}}x_m, T^{s_{l-1}}x_k) \leq \alpha G_p(T^{s_{l_1}}x_m, T^{s_{l_1}}x_m, T^{s_{l_1}}x_k)\\
&\leq \alpha^2 G_p(T^{s_{l-2}}x_m, T^{s_{l-2}}x_m, T^{s_{l-2}}x_k) \leq ...\\
&\leq \alpha^l G_p(x_m, x_m, x_k) \leq \alpha^l \cdot C_{m,k} < \frac{1}{k+n}.
\end{aligned}$$

If $G_p(T^{s_l}x_m, T^{s_l}x_m, T^{s_l}x_k) \leq G_p(T^i x_m, T^i x_m, T^i x_k)$ for some $i \in \{0, ..., s_l\}$, then by (3.1), $G_p(T^{s_l}x_m, T^{s_l}x_m, T^{s_l}x_k) < I + \frac{1}{m}$, so

$$\begin{aligned}
G_p(x_m, x_m, x_k) &\leq A_{m,k} + D_{m,k} + G_p(T^{s_l}x_m, T^{s_l}x_m, T^{s_l}x_k)\\
&< \frac{2}{m} + \frac{2}{k} + I + \frac{1}{m}.
\end{aligned}$$

From the above consideration and

$$I - \frac{1}{m} < G_p(x_m, x_m, x_m) \leq G_p(x_m, x_m, x_k),$$

it follows that

$$\lim_{m,k\to\infty} G_p(x_m, x_m, x_m) = I.$$

Now that $(X, G_p)$ is $G_p$-complete, there is some $u \in X$ such that

$$I = \lim_{m,k\to\infty} G_p(x_m, x_m, x_k) = \lim_{m\to\infty} G_p(x_m, x_m, u) = \lim_{m\to\infty} G_p(x_m, u, u) = G_p(u, u, u).$$

It is easy to see that

$$G_p\left(T^{n(u)}u, T^{n(u)}u, T^{n(u)}u\right) = G_p(u, u, u) = I.$$

Now we shall prove that $T^{n(u)}u = u$.

From

$$\begin{aligned} I = G_p(u, u, u) &\leq G_p\left(u, u, T^j x_m\right) \\ &\leq G_p(u, u, x_m) + G_p\left(x_m, x_m, T^j x_m\right) - G_p(x_m, x_m, x_m) \\ &\leq G_p(u, u, x_m) + I + \frac{1}{m} - \left(I - \frac{1}{m}\right) \\ &= G_p(u, u, x_m) + \frac{2}{m}, \end{aligned}$$

it means that

$$\lim_{m\to\infty} G_p\left(u, u, T^j x_m\right) = I, \quad j \in \mathbb{N}.$$

On the other hand,

$$\begin{aligned} I = G_p\left(T^{n(u)}u, T^{n(u)}u, T^{n(u)}u\right) &\leq G_p\left(T^{n(u)}u, T^{n(u)}u, T^{n(u)}x_m\right) \\ &\leq \max\left\{\alpha G_p(u, u, x_m), G_p(u, u, u)\right\}, \end{aligned}$$

which implies that

$$\lim_{m\to\infty} G_p\left(T^{n(u)}u, T^{n(u)}u, T^{n(u)}x_m\right) = I.$$

Now that

$$\begin{aligned} I &\leq G_p\left(u, T^{n(u)}u, T^{n(u)}u\right) \\ &\leq G_p\left(u, T^{n(u)}x_m, T^{n(u)}x_m\right) + G_p\left(T^{n(u)}u, T^{n(u)}u, T^{n(u)}x_m\right) \\ &\quad - G_p\left(T^{n(u)}x_m, T^{n(u)}x_m, T^{n(u)}x_m\right) \\ &\leq 2G_p\left(u, u, T^{n(u)}, x_m\right) - G_p(u, u, u) + G_p\left(T^{n(u)}u, T^{n(u)}u, T^{n(u)}x_m\right) - I + \frac{1}{m} \\ &\to I, \quad \text{as } m \to \infty, \end{aligned}$$

so

$$G_p\left(u, T^{n(u)}u, T^{n(u)}u\right) = I = G_p(u, u, u) = G_p\left(T^{n(u)}u, T^{n(u)}u, T^{n(u)}u\right),$$

and $T^{n(u)}u = u$. Finally, by utilizing Lemma 2.7, the remaining proof is valid. □

# 4 Some examples

Now, we give four examples to support our theoretical approach.

**Example 4.1.** Let $X = \{0, 1, 2\}$ be a set and $G_p : X^3 \to [0, +\infty)$ a mapping satisfying

$$\begin{aligned}
&G_p(x, x, x) = \frac{1}{2} \text{ for all } x \in X,\\
&G_p(0, 0, 1) = G_p(0, 1, 0) = G_p(1, 0, 0) = 1,\\
&G_p(0, 1, 1) = G_p(1, 0, 1) = G_p(1, 1, 0) = 1,\\
&G_p(1, 2, 2) = G_p(2, 1, 2) = G_p(2, 2, 1) = 3,\\
&G_p(0, 0, 2) = G_p(0, 2, 0) = G_p(2, 0, 0) = 3,\\
&G_p(0, 2, 2) = G_p(2, 0, 2) = G_p(2, 2, 0) = 3,\\
&G_p(1, 1, 2) = G_p(1, 2, 1) = G_p(2, 1, 1) = 3.1,\\
&G_p(0, 1, 2) = G_p(0, 2, 1) = G_p(1, 0, 2) = 3.2.
\end{aligned}$$

Then $G_p$ is an asymmetric $G_p$-metric as $G_p(1, 2, 2) \neq G_p(1, 1, 2)$. Further, $(X, G_p)$ is a $G_p$-metric space in the sense of [18]. Let $T : X \to X$ be defined by $T0 = T1 = 0, T2 = 1$. We shall prove that $T$ is an NLC-operator where $\alpha = \frac{1}{2}$, while $n(x) = 1$ for all $x \in X$. Indeed, we need to check

$$G_p(Tx, Tx, Ty) \leq \max\left\{\frac{1}{2}G_p(x, x, y), G_p(x, x, x)\right\} \tag{4.1}$$

for all $x, y \in X$ into nine cases as follows:

$$\begin{aligned}
&(1)\ \ x = 0,\ y = 0 \Longrightarrow \frac{1}{2} = G_p(T0, T0, T0) \leq \max\left\{\frac{1}{2}G_p(0, 0, 0), G_p(0, 0, 0)\right\} = \frac{1}{2},\\
&(2)\ \ x = 0,\ y = 1 \Longrightarrow \frac{1}{2} = G_p(T0, T0, T1) \leq \max\left\{\frac{1}{2}G_p(0, 0, 1), G_p(0, 0, 0)\right\} = \frac{1}{2},\\
&(3)\ \ x = 0,\ y = 2 \Longrightarrow 1 = G_p(T0, T0, T2) \leq \max\left\{\frac{1}{2}G_p(0, 0, 2), G_p(0, 0, 0)\right\} = \frac{3}{2},\\
&(4)\ \ x = 1,\ y = 0 \Longrightarrow \frac{1}{2} = G_p(T1, T1, T0) \leq \max\left\{\frac{1}{2}G_p(1, 1, 0), G_p(1, 1, 1)\right\} = \frac{1}{2},\\
&(5)\ \ x = 1,\ y = 1 \Longrightarrow \frac{1}{2} = G_p(T1, T1, T1) \leq \max\left\{\frac{1}{2}G_p(1, 1, 1), G_p(1, 1, 1)\right\} = \frac{1}{2},\\
&(6)\ \ x = 1,\ y = 2 \Longrightarrow 1 = G_p(T1, T1, T2) \leq \max\left\{\frac{1}{2}G_p(1, 1, 2), G_p(1, 1, 1)\right\} = \frac{3.1}{2},\\
&(7)\ \ x = 2,\ y = 0 \Longrightarrow 1 = G_p(T2, T2, T0) \leq \max\left\{\frac{1}{2}G_p(2, 2, 0), G_p(2, 2, 2)\right\} = \frac{3}{2},\\
&(8)\ \ x = 2,\ y = 1 \Longrightarrow 1 = G_p(T2, T2, T1) \leq \max\left\{\frac{1}{2}G_p(2, 2, 1), G_p(2, 2, 2)\right\} = \frac{3}{2},\\
&(9)\ \ x = 2,\ y = 2 \Longrightarrow \frac{1}{2} = G_p(T2, T2, T2) \leq \max\left\{\frac{1}{2}G_p(2, 2, 2), G_p(2, 2, 2)\right\} = \frac{1}{2}.
\end{aligned}$$

Hence all cases show that (4.1) is satisfied and then both Proposition 3.1 and Theorem 3.2 are true. □

**Example 4.2.** Let $(X, G_p)$ be a $G_p$-metric space where $X = [0, +\infty)$ and $G_p(x, y, z) = \max\{x, y, x\}$. Define $T : X \to X$ by $Tx = \frac{x^2}{3(1+x)}$. We shall prove that $T$ is an NLC-operator, that is, for each $x \in X$, there is $n(x)$ such that for each $y \in X$,

$$G_p\left(T^{n(x)}x, T^{n(x)}x, T^{n(x)}y\right) \leq \max\left\{\alpha G_p(x, x, y), G_p(x, x, x)\right\},$$

where $\alpha \in (0, 1)$. Let $\alpha = \frac{1}{3}$. It is easy to see that $n(x) = 1$. Indeed, we shall check that

$$\max\left\{\frac{x^2}{3(1+x)}, \frac{x^2}{3(1+x)}, \frac{y^2}{3(1+y)}\right\} \leq \max\left\{\frac{1}{3}\max\{x, x, y\}, \max\{x, x, x\}\right\}, \quad (4.2)$$

for all $x, y \in [0, +\infty)$.

Consider the following three possible cases.

(i) $y \leq x$. In this case (4.2) becomes:

$$\frac{x^2}{3(1+x)} \leq \max\left\{\frac{1}{3}x, x\right\} = x, \quad (4.3)$$

which is true for any $x \in [0, +\infty)$.

(ii) $\frac{y}{3} \leq x \leq y$. In this case (4.2) becomes:

$$\frac{y^2}{3(1+y)} \leq \max\left\{\frac{1}{3}y, x\right\} = x. \quad (4.4)$$

By virtue of $\frac{y^2}{3(1+y)} = \frac{y}{1+y} \cdot \frac{y}{3} \leq \frac{y}{3} \leq x$, it follows that (4.4) holds.

(iii) $0 \leq x \leq \frac{y}{3}$. Because of $x \leq y$, (4.2) becomes:

$$\frac{y^2}{3(1+y)} \leq \max\left\{\frac{1}{3}y, x\right\} = \frac{1}{3}y. \quad (4.5)$$

Obviously, (4.5) holds for each $y \in [0, +\infty)$.

Hence, (4.2) holds for all $x, y \in [0, +\infty)$, that is., all the conditions of Proposition 3.1 and Theorem 3.2 are satisfied and $T$ has a fixed point (which is $x = 0$). □

**Example 4.3.** Let $X = \{a, b\}$ be a set with $G_p$-metric defined by

$$G_p(a, a, a) = 0, \quad G_p(a, a, b) = G_p(a, b, a) = G_p(b, a, a) = 1,$$
$$G_p(b, b, b) = G_p(a, b, b) = G_p(b, a, b) = G_p(b, b, a) = 2.$$

Since $G_p(a, a, b) \neq G_p(a, b, b)$, we get that $(X, G_p)$ is an asymmetric $G_p$-metric space. Also, we have that for all $x, y \in X$,

$$d_{G_p}(x, y) = G_p(x, x, y) + G_p(x, y, y) - G_p(x, x, x) - G_p(y, y, y) = \begin{cases} 1, & x \neq y, \\ 0, & x = y, \end{cases}$$

is a (standard) metric on $X$. □

**Example 4.4.** Let $X = \{a, b\}$ be a set with $G_p$-metric defined by

$$G_p(a, a, a) = 0, \quad G_p(a, a, b) = G_p(a, b, a) = G_p(b, a, a) = 1 = G_p(b, b, b),$$
$$G_p(a, b, b) = G_p(b, a, b) = G_p(b, b, a) = 2.$$

It ensures us that the sequence $\{x_n = a\}$ converges to $a$. However, conditions (2) and (3) of Proposition 1.8 are not equivalent. Indeed,

$$G_p(x_n, x_n, b) = G_p(a, a, b) \to G_p(b, b, b) \quad (n \to \infty),$$

while

$$G_p(x_n, b, b) = G_p(a, b, b) \nrightarrow G_p(b, b, b) \quad (n \to \infty).$$

Thus, $G_p(\cdot, \cdot, )$ may not be continuous in the sense that $x_n \to x,\ y_n \to y$ and $z_n \to z$ implies $G_p(x_n, y_n, z_n) \to G_p(x, y, z)$. In fact, we take $x_n = y_n = a$ and $z_n = b$ for all $n \in \mathbb{N}$. Further, it is easy to check that $x_n \to b,\ y_n \to a$ and $z_n \to b$ but $G_p(x_n, y_n, z_n) \nrightarrow G_p(b, a, b)$, this is because $G_p(x_n, y_n, z_n) = G_p(a, a, b) = 1 \neq 2 = G_p(b, a, b)$. □

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# References

[1] M. Abbas, T. Nazir, P. Vetro, *Common fixed point results for three maps in G-metric spaces*, Filomat, **25**(4) (2011), 1-7.

[2] H. Aydi, E. Karapinar, P. Salimi, *Some fixed point results in $G_p$-metric spaces*, J. Appl. Math., Vol. 2012, Article ID 891713, 16 pages.

[3] M. A. Barakat, A. M. Zidan, *A common fixed point theorem for weak contractive maps in $G_p$-metric spaces*, J. Egyptian Math. Soc., **23** (2014), 309-314.

[4] C. Di Bari, *Common fixed points for self-mappings on partial metric spaces*, Fixed Point Theory Appl., **2012**, 2012: 140.

[5] N. Bilgili, E. Karapinar, P. Salimi, *Fixed point theorems for generalized contractions on $G_p$-metric spaces*, J. Inequal. Appl., **2013**, 2013: 39.

[6] Lj. Ćirić, S. M. Alsulami, V. Parvaneh, R. Roshan, *Some fixed point results in ordered $G_p$-metric spaces*, Fixed Point Theory Appl., **2013**, 2013: 317.

[7] Lj. Gajić, M. Stojaković, *On fixed point results for Matkowski type of mappings in G-metric spaces*, Filomat, **29**(10) (2015), 2301-2309.

[8] Lj. Gajić, Z. Lozanov-Crvenković, *On mappings with contractive iterate at point in generalized metric spaces*, Fixed Point Theory Appl., Volume 2010, Article ID 458086, 16 pages.

[9] Lj. Gajić, Z. Lozanov-Crvenković, *A fixed point result for mappings with contractive iterate at a point in G-metric spaces*, Filomat, **25**(2) (2011), 53-58.

[10] Lj. Gajić, M. Stojaković, *On mappings with $\varphi$-contractive iterate at a point on generalized metric spaces*, Fixed Point Theory Appl., **2014**, 2014: 46.

[11] L. F. Guseman, *Fixed point mappings with a contractive iterate at a point*, Proc. Amer. Math. Soc., **26** (1970), 615-618

[12] D. Ilić, V. Pavlović, V. Rakočević, *Fixed points of mappings with contractive iterate at a point in partial metric spaces*, Fixed Point Theory Appl., **2013**, 2013: 335.

[13] S. G. Matthews, *Partial metric topology*, Proc. 8th Summer Conference on General Topology and Applications, Ann New York Acad Sci, **728** (1994), 183-197.

[14] Z. Mustafa, B. Sims, *A new approach to generalized metric spaces*, J. Nonlinear Conv. Anal., **7**(2) (2006), 289-297.

[15] S. Oltra, O. Valero, *Banach fixed point theorem for partial metric spaces*, Rendinconti dell'Istituto di Matematica dell'Universitá di Trieste, **36**(1-2) (2004), 17-26.

[16] V. M. Sehgal, *A fixed point theorem for mappings with a contractive iterate*, Proc. Amer. Math. Soc., **23** (1969), 631-634.

[17] O. Valero, *On Banach fixed point theorems for partial metric spaces*, Appl. General. Topol., **6**(2) (2005), 229-240.

[18] V. Paravneh, J. R. Roshan, Z. Kadelburg, *On generalized weakly $G_p$-contractive mappings in ordered $G_p$-metric spaces*, Gulf J. Math., **1** (2013), 78-97.

[19] V. Paravneh, P. Salimi, P. Vetro, A. D. Nezhad, S. Radenović, *Fixed point results for $GP_(\Lambda, \Theta)$-contractive mappings*, J. Nonlinear Sci. Appl., **7** (2014), 150-159.

[20] V. Popa, A. M. Patriciu, *Two general fixed point theorems for a sequence of mappings satisfying implicit relations in $G_p$-metric spaces*, Appl. General. Topol., **16**(2) (2015), 225-231.

[21] S. Romaguera, *A Kirk type characterizations of a completeness for partial metric spaces*, Fixed Point Theory Appl., **2011**, 2011: 4.

[22] P. Salimi, P. Vetro, *A result of Suzuki type in partial G-metric spaces*, Acta Mathematica Scientia, **34B**(2) (2014), 274-284.

[23] M. R. A. Zand, A. D. Nezhad, *A generalization of partial metric spaces*, J. Contemporary Appl. Math., **1**(1) (2011), 86-93.

# Most general Self Adjoint Operator Chebyshev-Grüss Inequalities

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**Abstract**

We demonstrate here most general self adjoint operator Chebyshev-Grüss type inequalities to all cases. We finish with applications.

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**Key Words and Phrases:** Self adjoint operator, Hilbert space, Chebyshev-Grüss inequalities.

## 1 Motivation

Here we mention the following interesting and motivating results.

**Theorem 1** *(Čebyšev, 1882, [2]). Let $f, g : [a,b] \to \mathbb{R}$ absolutely continuous functions. If $f', g' \in L_\infty([a,b])$, then*

$$\left| \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)\right| \tag{1}$$
$$\leq \frac{1}{12}(b-a)^2 \|f'\|_\infty \|g'\|_\infty .$$

Also we mention

**Theorem 2** *(Grüss, 1935, [6]). Let $f, g$ integrable functions from $[a,b]$ into $\mathbb{R}$, such that $m \leq f(x) \leq M$, $\rho \leq g(x) \leq \sigma$, for all $x \in [a,b]$, where $m, M, \rho, \sigma \in \mathbb{R}$. Then*

$$\left| \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)\right| \tag{2}$$
$$\leq \frac{1}{4}(M-m)(\sigma-\rho).$$

## 2 Background

Let $A$ be a selfadjoint linear operator on a complex Hilbert space $(H;\langle\cdot,\cdot\rangle)$. The Gelfand map establishes a $*-$isometrically isomorphism $\Phi$ between the set $C(Sp(A))$ of all continuous functions defind on the spectrum of $A$, denoted $Sp(A)$, and the $C^*$-algebra $C^*(A)$ generated by $A$ and the identity operator $1_H$ on $H$ as follows (see e.g. [5, p. 3]):

For any $f,g\in C(Sp(A))$ and any $\alpha,\beta\in\mathbb{C}$ we have

(i) $\Phi(\alpha f+\beta g)=\alpha\Phi(f)+\beta\Phi(g)$;

(ii) $\Phi(fg)=\Phi(f)\Phi(g)$ (the operation composition is on the right) and $\Phi\left(\overline{f}\right)=(\Phi(f))^*$;

(iii) $\|\Phi(f)\|=\|f\|:=\sup_{t\in Sp(A)}|f(t)|$;

(iv) $\Phi(f_0)=1_H$ and $\Phi(f_1)=A$, where $f_0(t)=1$ and $f_1(t)=t$, for $t\in Sp(A)$.

With this notation we define

$$f(A):=\Phi(f),\text{ for all }f\in C(Sp(A)),$$

and we call it the continuous functional calculus for a selfadjoint operator $A$.

If $A$ is a selfadjoint operator and $f$ is a real valued continuous function on $Sp(A)$ then $f(t)\geq 0$ for any $t\in Sp(A)$ implies that $f(A)\geq 0$, i.e. $f(A)$ is a positive operator on $H$. Moreover, if both $f$ and $g$ are real valued functions on $Sp(A)$ then the following important property holds:

(P) $f(t)\geq g(t)$ for any $t\in Sp(A)$, implies that $f(A)\geq g(A)$ in the operator order of $B(H)$.

Equivalently, we use (see [4], pp. 7-8):

Let $U$ be a selfadjoint operator on the complex Hilbert space $(H,\langle\cdot,\cdot\rangle)$ with the spectrum $Sp(U)$ included in the interval $[m,M]$ for some real numbers $m<M$ and $\{E_\lambda\}_\lambda$ be its spectral family.

Then for any continuous function $f:[m,M]\to\mathbb{C}$, it is well known that we have the following spectral representation in terms of the Riemann-Stieljes integral:

$$\langle f(U)x,y\rangle=\int_{m-0}^{M}f(\lambda)\,d(\langle E_\lambda x,y\rangle),\tag{3}$$

for any $x,y\in H$. The function $g_{x,y}(\lambda):=\langle E_\lambda x,y\rangle$ is of bounded variation on the interval $[m,M]$, and

$$g_{x,y}(m-0)=0\ \text{ and }\ g_{x,y}(M)=\langle x,y\rangle,$$

for any $x,y\in H$. Furthermore, it is known that $g_x(\lambda):=\langle E_\lambda x,x\rangle$ is increasing and right continuous on $[m,M]$.

In this article we will be using a lot the formula

$$\langle f(U)x, x\rangle = \int_{m-0}^{M} f(\lambda)\, d\left(\langle E_\lambda x, x\rangle\right), \quad \forall\ x \in H. \tag{4}$$

As a symbol we can write

$$f(U) = \int_{m-0}^{M} f(\lambda)\, dE_\lambda. \tag{5}$$

Above, $m = \min\{\lambda | \lambda \in Sp(U)\} := \min Sp(U)$, $M = \max\{\lambda | \lambda \in Sp(U)\} := \max Sp(U)$. The projections $\{E_\lambda\}_{\lambda\in\mathbb{R}}$, are called the spectral family of $A$, with the properties:

(a) $E_\lambda \leq E_{\lambda'}$ for $\lambda \leq \lambda'$;

(b) $E_{m-0} = 0_H$ (zero operator), $E_M = 1_H$ (identity operator) and $E_{\lambda+0} = E_\lambda$ for all $\lambda \in \mathbb{R}$.

Furthermore

$$E_\lambda := \varphi_\lambda(U), \quad \forall\ \lambda \in \mathbb{R}, \tag{6}$$

is a projection which reduces $U$, with

$$\varphi_\lambda(s) := \begin{cases} 1, & \text{for } -\infty < s \leq \lambda, \\ 0, & \text{for } \lambda < s < +\infty. \end{cases}$$

The spectral family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ determines uniquely the self-adjoint operator $U$ and vice versa.

For more on the topic see [7], pp. 256-266, and for more detalis see there pp. 157-266. See also [3].

Some more basics are given (we follow [4], pp. 1-5):

Let $(H; \langle\cdot,\cdot\rangle)$ be a Hilbert space over $\mathbb{C}$. A bounded linear operator $A$ defined on $H$ is selfjoint, i.e., $A = A^*$, iff $\langle Ax, x\rangle \in \mathbb{R}$, $\forall\ x \in H$, and if $A$ is selfadjoint, then

$$\|A\| = \sup_{x\in H:\|x\|=1} |\langle Ax, x\rangle|. \tag{7}$$

Let $A, B$ be selfadjoint operators on $H$. Then $A \leq B$ iff $\langle Ax, x\rangle \leq \langle Bx, x\rangle$, $\forall\ x \in H$.

In particular, $A$ is called positive if $A \geq 0$.

Denote by

$$\mathcal{P} := \left\{ \varphi(s) := \sum_{k=0}^{n} \alpha_k s^k | n \geq 0,\ \alpha_k \in \mathbb{C},\ 0 \leq k \leq n \right\}. \tag{8}$$

If $A \in \mathcal{B}(H)$ (the Banach algebra of all bounded linear operators defined on $H$, i.e. from $H$ into itself) is selfadjoint, and $\varphi(s) \in \mathcal{P}$ has real coefficients, then $\varphi(A)$ is selfadjoint, and

$$\|\varphi(A)\| = \max\{|\varphi(\lambda)|, \lambda \in Sp(A)\}. \tag{9}$$

If $\varphi$ is any function defined on $\mathbb{R}$ we define

$$\|\varphi\|_A := \sup\{|\varphi(\lambda)|, \lambda \in Sp(A)\}. \tag{10}$$

If $A$ is selfadjoint operator on Hilbert space $H$ and $\varphi$ is continuous and given that $\varphi(A)$ is selfadjoint, then $\|\varphi(A)\| = \|\varphi\|_A$. And if $\varphi$ is a continuous real valued function so it is $|\varphi|$, then $\varphi(A)$ and $|\varphi|(A) = |\varphi(A)|$ are selfadjoint operators (by [4], p. 4, Theorem 7).

Hence it holds

$$\||\varphi(A)|\| = \||\varphi|\|_A = \sup\{||\varphi(\lambda)||, \lambda \in Sp(A)\}$$

$$= \sup\{|\varphi(\lambda)|, \lambda \in Sp(A)\} = \|\varphi\|_A = \|\varphi(A)\|,$$

that is

$$\||\varphi(A)|\| = \|\varphi(A)\|. \tag{11}$$

For a selfadjoint operator $A \in \mathcal{B}(H)$ which is positive, there exists a unique positive selfadjoint operator $B := \sqrt{A} \in \mathcal{B}(H)$ such that $B^2 = A$, that is $\left(\sqrt{A}\right)^2 = A$. We call $B$ the square root of $A$.

Let $A \in \mathcal{B}(H)$, then $A^*A$ is selfadjoint and positive. Define the "operator absolute value" $|A| := \sqrt{A^*A}$. If $A = A^*$, then $|A| = \sqrt{A^2}$.

For a continuous real valued function $\varphi$ we observe the following:

$$|\varphi(A)| \text{ (the functional absolute value) } = \int_{m-0}^{M} |\varphi(\lambda)| \, dE_\lambda =$$

$$\int_{m-0}^{M} \sqrt{(\varphi(\lambda))^2} dE_\lambda = \sqrt{(\varphi(A))^2} = |\varphi(A)| \text{ (operator absolute value)},$$

where $A$ is a selfadjoint operator.

That is we have

$$|\varphi(A)| \text{ (functional absolute value) } = |\varphi(A)| \text{ (operator absolute value)}. \tag{12}$$

Let $A, B \in \mathcal{B}(H)$, then

$$\|AB\| \leq \|A\| \|B\|, \tag{13}$$

by Banach algebra property.

# 3 Main Results

Next we present most general Chebyshev-Grüss type operator inequalities based on Theorem 26.9 of [1], p. 404.

Then we specialize them for $n = 1$.

We give

**Theorem 3** *Let $n \in \mathbb{N}$ and $f_1, f_2 \in C^n([a,b])$ with $[m,M] \subset (a,b)$, $m < M$; $g \in C^1([a,b])$ and $g^{-1} \in C^n([a,b])$. Here $A$ is a selfadjoint linear operator on the Hilbert space $H$ with spectrum $Sp(A) \subseteq [m,M]$. We consider any $x \in H$ : $\|x\| = 1$.*

*Then*

$$\langle (\Delta(f_1, f_2; g))(A)\, x, x\rangle :=$$

$$\left| \langle f_1(A) f_2(A) x, x\rangle - \langle f_1(A) x, x\rangle \cdot \langle f_2(A) x, x\rangle - \frac{1}{2(M-m)} \left\{ \sum_{k=1}^{n-1} \frac{1}{k!} \cdot \right.\right.$$

$$\left\{ \left[ \left\langle \left( f_2(A) \int_{m-0}^{M} \left( \int_m^M \left(f_1 \circ g^{-1}\right)^{(k)} (g(t)) (g(\lambda) - g(t))^k \, dt \right) dE_\lambda \right) x, x \right\rangle - \right.$$

$$\langle f_2(A) x, x\rangle \int_{m-0}^{M} \left( \int_m^M \left(f_1 \circ g^{-1}\right)^{(k)} (g(t)) (g(\lambda) - g(t))^k \, dt \right) d\langle E_\lambda x, x\rangle \Big] +$$

$$\left[ \left\langle \left( f_1(A) \int_{m-0}^{M} \left( \int_m^M \left(f_2 \circ g^{-1}\right)^{(k)} (g(t)) (g(\lambda) - g(t))^k \, dt \right) dE_\lambda \right) x, x \right\rangle - \right.$$

$$\left.\left.\left. \langle f_1(A) x, x\rangle \int_{m-0}^{M} \left( \int_m^M \left(f_2 \circ g^{-1}\right)^{(k)} (g(t)) (g(\lambda) - g(t))^k \, dt \right) d\langle E_\lambda x, x\rangle \right] \right\} \right\} \Bigg|$$

$$\leq \frac{\|g\|_{\infty,[m,M]}^{n-1} \|g'\|_{\infty,[m,M]}}{(n+1)!(M-m)} \left[ \|f_2(A)\| \left\| \left(f_1 \circ g^{-1}\right)^{(n)} \circ g \right\|_{\infty,[m,M]} + \right.$$

$$\left. \|f_1(A)\| \left\| \left(f_2 \circ g^{-1}\right)^{(n)} \circ g \right\|_{\infty,[m,M]} \right] \left[ \left\| (M 1_H - A)^{n+1} \right\| + \left\| (A - m 1_H)^{n+1} \right\| \right]. \tag{14}$$

**Proof.** Call $l_i = f_i \circ g^{-1}$, $i = 1,2$. Then $l_i, l_i', ..., l_i^{(n)}$ are continuous from $g([a,b])$ into $f_i([a,b])$, $i = 1,2$. Hence $\left(f_i \circ g^{-1}\right)^{(n)} \circ g \in C([a,b])$, $i = 1,2$. Here $\{E_\lambda\}_\lambda$ is the spectral family of $A$.

Next we use Theorem 26.9 of [1], p. 404. We have that $(i = 1,2)$

$$f_i(\lambda) = \frac{1}{M-m} \int_m^M f_i(t)\, dt +$$

$$\frac{1}{(M-m)} \left\{ \sum_{k=1}^{n-1} \frac{1}{k!} \int_m^M \left(f_i \circ g^{-1}\right)^{(k)} (g(t)) (g(\lambda) - g(t))^k \, dt \right\} +$$

$$+ \frac{1}{(n-1)!(M-m)} \int_m^M (g(\lambda) - g(t))^{n-1} \left(f_i \circ g^{-1}\right)^{(n)} (g(t)) g'(t) K(t,\lambda)\, dt, \tag{15}$$

$\forall\ \lambda \in [m,M]$,

where

$$K(t,\lambda) := \begin{cases} t-m, & m \le t \le \lambda \le M, \\ t-M, & m \le \lambda < t \le M. \end{cases} \tag{16}$$

By applying the spectral representation theorem on (15), i.e. integrating against $E_\lambda$ over $[m,M]$, see (4), we obtain:

$$f_i(A) = \left(\frac{1}{M-m}\int_m^M f_i(t)\,dt\right) 1_H +$$

$$\frac{1}{(M-m)}\left\{\sum_{k=1}^{n-1}\frac{1}{k!}\int_{m-0}^{M}\left(\int_m^M \left(f_i\circ g^{-1}\right)^{(k)}(g(t))\,(g(\lambda)-g(t))^k\,dt\right)dE_\lambda\right\}$$

$$+\frac{1}{(n-1)!\,(M-m)}\cdot$$

$$\int_{m-0}^{M}\left(\int_m^M (g(\lambda)-g(t))^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(g(t))\,g'(t)\,K(t,\lambda)\,dt\right)dE_\lambda, \tag{17}$$

$i=1,2$.

We notice that

$$f_1(A)\,f_2(A) = f_2(A)\,f_1(A), \tag{18}$$

to be used next.

Hence it holds

$$f_2(A)\,f_1(A) = \left(\frac{1}{M-m}\int_m^M f_1(t)\,dt\right) f_2(A) + \tag{19}$$

$$\frac{1}{(M-m)}\left\{\sum_{k=1}^{n-1}\frac{1}{k!}f_2(A)\int_{m-0}^{M}\left(\int_m^M \left(f_1\circ g^{-1}\right)^{(k)}(g(t))\,(g(\lambda)-g(t))^k\,dt\right)dE_\lambda\right\}$$

$$+\frac{1}{(n-1)!\,(M-m)}f_2(A)\cdot$$

$$\int_{m-0}^{M}\left(\int_m^M (g(\lambda)-g(t))^{n-1}\left(f_1\circ g^{-1}\right)^{(n)}(g(t))\,g'(t)\,K(t,\lambda)\,dt\right)dE_\lambda,$$

and

$$f_1(A)\,f_2(A) = \left(\frac{1}{M-m}\int_m^M f_2(t)\,dt\right) f_1(A) +$$

$$\frac{1}{(M-m)}\left\{\sum_{k=1}^{n-1}\frac{1}{k!}f_1(A)\int_{m-0}^{M}\left(\int_m^M \left(f_2\circ g^{-1}\right)^{(k)}(g(t))\,(g(\lambda)-g(t))^k\,dt\right)dE_\lambda\right\}$$

$$+\frac{1}{(n-1)!\,(M-m)}f_1(A)\cdot$$

$$\int_{m-0}^{M}\left(\int_{m}^{M}\left(g\left(\lambda\right)-g\left(t\right)\right)^{n-1}\left(f_2\circ g^{-1}\right)^{(n)}\left(g\left(t\right)\right)g'\left(t\right)K\left(t,\lambda\right)dt\right)dE_{\lambda}. \quad (20)$$

Here from now on we consider $x\in H:\left\|x\right\|=1$; immediately we get

$$\int_{m-0}^{M}d\left\langle E_{\lambda}x,x\right\rangle =1.$$

Then it holds $(i=1,2)$

$$\left\langle f_i\left(A\right)x,x\right\rangle =\left(\frac{1}{M-m}\int_{m}^{M}f_i\left(t\right)dt\right)+ \quad (21)$$

$$\frac{1}{\left(M-m\right)}\left\{\sum_{k=1}^{n-1}\frac{1}{k!}\int_{m-0}^{M}\left(\int_{m}^{M}\left(f_i\circ g^{-1}\right)^{(k)}\left(g\left(t\right)\right)\left(g\left(\lambda\right)-g\left(t\right)\right)^{k}dt\right)d\left\langle E_{\lambda}x,x\right\rangle\right\}$$

$$+\frac{1}{\left(n-1\right)!\left(M-m\right)}\cdot$$

$$\int_{m-0}^{M}\left(\int_{m}^{M}\left(g\left(\lambda\right)-g\left(t\right)\right)^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}\left(g\left(t\right)\right)g'\left(t\right)K\left(t,\lambda\right)dt\right)d\left\langle E_{\lambda}x,x\right\rangle .$$

It follows that

$$\left\langle f_2\left(A\right)x,x\right\rangle \left\langle f_1\left(A\right)x,x\right\rangle =\left(\frac{1}{M-m}\int_{m}^{M}f_1\left(t\right)dt\right)\left\langle f_2\left(A\right)x,x\right\rangle +\frac{1}{\left(M-m\right)}\cdot$$

$$\left\{\sum_{k=1}^{n-1}\frac{1}{k!}\left\langle f_2\left(A\right)x,x\right\rangle \int_{m-0}^{M}\left(\int_{m}^{M}\left(f_1\circ g^{-1}\right)^{(k)}\left(g\left(t\right)\right)\left(g\left(\lambda\right)-g\left(t\right)\right)^{k}dt\right)d\left\langle E_{\lambda}x,x\right\rangle\right\}$$

$$+\frac{1}{\left(n-1\right)!\left(M-m\right)}\left\langle f_2\left(A\right)x,x\right\rangle \cdot \quad (22)$$

$$\int_{m-0}^{M}\left(\int_{m}^{M}\left(g\left(\lambda\right)-g\left(t\right)\right)^{n-1}\left(f_1\circ g^{-1}\right)^{(n)}\left(g\left(t\right)\right)g'\left(t\right)K\left(t,\lambda\right)dt\right)d\left\langle E_{\lambda}x,x\right\rangle ,$$

and

$$\left\langle f_1\left(A\right)x,x\right\rangle \left\langle f_2\left(A\right)x,x\right\rangle =\left(\frac{1}{M-m}\int_{m}^{M}f_2\left(t\right)dt\right)\left\langle f_1\left(A\right)x,x\right\rangle +\frac{1}{\left(M-m\right)}\cdot$$

$$\left\{\sum_{k=1}^{n-1}\frac{1}{k!}\left\langle f_1\left(A\right)x,x\right\rangle \int_{m-0}^{M}\left(\int_{m}^{M}\left(f_2\circ g^{-1}\right)^{(k)}\left(g\left(t\right)\right)\left(g\left(\lambda\right)-g\left(t\right)\right)^{k}dt\right)d\left\langle E_{\lambda}x,x\right\rangle\right\}$$

$$+\frac{1}{\left(n-1\right)!\left(M-m\right)}\left\langle f_1\left(A\right)x,x\right\rangle \cdot \quad (23)$$

$$\int_{m-0}^{M}\left(\int_{m}^{M}\left(g\left(\lambda\right)-g\left(t\right)\right)^{n-1}\left(f_{2}\circ g^{-1}\right)^{(n)}\left(g\left(t\right)\right)g'\left(t\right)K\left(t,\lambda\right)dt\right)d\left\langle E_{\lambda}x,x\right\rangle .$$

Furthermore we obtain

$$\left\langle f_{1}\left(A\right)f_{2}\left(A\right)x,x\right\rangle =\left(\frac{1}{M-m}\int_{m}^{M}f_{1}\left(t\right)dt\right)\left\langle f_{2}\left(A\right)x,x\right\rangle +\frac{1}{\left(M-m\right)}\cdot$$

$$\left\{\sum_{k=1}^{n-1}\frac{1}{k!}\left\langle\left(f_{2}\left(A\right)\int_{m-0}^{M}\left(\int_{m}^{M}\left(f_{1}\circ g^{-1}\right)^{(k)}\left(g\left(t\right)\right)\left(g\left(\lambda\right)-g\left(t\right)\right)^{k}dt\right)dE_{\lambda}\right)x,x\right\rangle\right\}$$

$$+\frac{1}{\left(n-1\right)!\left(M-m\right)}\cdot \tag{24}$$

$$\left\langle\left(f_{2}\left(A\right)\int_{m-0}^{M}\left(\int_{m}^{M}\left(g\left(\lambda\right)-g\left(t\right)\right)^{n-1}\left(f_{1}\circ g^{-1}\right)^{(n)}\left(g\left(t\right)\right)g'\left(t\right)K\left(t,\lambda\right)dt\right)dE_{\lambda}\right)x,x\right\rangle ,$$

and

$$\left\langle f_{1}\left(A\right)f_{2}\left(A\right)x,x\right\rangle =\left(\frac{1}{M-m}\int_{m}^{M}f_{2}\left(t\right)dt\right)\left\langle f_{1}\left(A\right)x,x\right\rangle +\frac{1}{\left(M-m\right)}\cdot$$

$$\left\{\sum_{k=1}^{n-1}\frac{1}{k!}\left\langle\left(f_{1}\left(A\right)\int_{m-0}^{M}\left(\int_{m}^{M}\left(f_{2}\circ g^{-1}\right)^{(k)}\left(g\left(t\right)\right)\left(g\left(\lambda\right)-g\left(t\right)\right)^{k}dt\right)dE_{\lambda}\right)x,x\right\rangle\right\}$$

$$+\frac{1}{\left(n-1\right)!\left(M-m\right)}\cdot \tag{25}$$

$$\left\langle\left(f_{1}\left(A\right)\int_{m-0}^{M}\left(\int_{m}^{M}\left(g\left(\lambda\right)-g\left(t\right)\right)^{n-1}\left(f_{2}\circ g^{-1}\right)^{(n)}\left(g\left(t\right)\right)g'\left(t\right)K\left(t,\lambda\right)dt\right)dE_{\lambda}\right)x,x\right\rangle .$$

By (24)-(22) we obtain

$$E:=\left\langle f_{1}\left(A\right)f_{2}\left(A\right)x,x\right\rangle -\left\langle f_{1}\left(A\right)x,x\right\rangle \left\langle f_{2}\left(A\right)x,x\right\rangle =\frac{1}{\left(M-m\right)}\cdot$$

$$\left\{\sum_{k=1}^{n-1}\frac{1}{k!}\left[\left\langle\left(f_{2}\left(A\right)\int_{m-0}^{M}\left(\int_{m}^{M}\left(f_{1}\circ g^{-1}\right)^{(k)}\left(g\left(t\right)\right)\left(g\left(\lambda\right)-g\left(t\right)\right)^{k}dt\right)dE_{\lambda}\right)x,x\right\rangle\right.\right.$$

$$\left.\left.-\left\langle f_{2}\left(A\right)x,x\right\rangle\int_{m-0}^{M}\left(\int_{m}^{M}\left(f_{1}\circ g^{-1}\right)^{(k)}\left(g\left(t\right)\right)\left(g\left(\lambda\right)-g\left(t\right)\right)^{k}dt\right)d\left\langle E_{\lambda}x,x\right\rangle\right]\right\}$$

$$+\frac{1}{\left(n-1\right)!\left(M-m\right)}\cdot$$

$$\left[\left\langle\left(f_{2}\left(A\right)\int_{m-0}^{M}\left(\int_{m}^{M}\left(g\left(\lambda\right)-g\left(t\right)\right)^{n-1}\left(f_{1}\circ g^{-1}\right)^{(n)}\left(g\left(t\right)\right)g'\left(t\right)K\left(t,\lambda\right)dt\right)dE_{\lambda}\right)x,x\right\rangle\right. \tag{26}$$

$$- \langle f_2(A)x,x\rangle \cdot$$

$$\int_{m-0}^{M}\left(\int_{m}^{M}(g(\lambda)-g(t))^{n-1}\left(f_1\circ g^{-1}\right)^{(n)}(g(t))\,g'(t)\,K(t,\lambda)\,dt\right)d\langle E_\lambda x,x\rangle\Bigg],$$

and by (25)-(23) we have

$$E=\frac{1}{(M-m)}\cdot$$

$$\left\{\sum_{k=1}^{n-1}\frac{1}{k!}\left[\left\langle\left(f_1(A)\int_{m-0}^{M}\left(\int_{m}^{M}\left(f_2\circ g^{-1}\right)^{(k)}(g(t))(g(\lambda)-g(t))^{k}\,dt\right)dE_\lambda\right)x,x\right\rangle\right.\right.$$

$$\left.\left.-\langle f_1(A)x,x\rangle\int_{m-0}^{M}\left(\int_{m}^{M}\left(f_2\circ g^{-1}\right)^{(k)}(g(t))(g(\lambda)-g(t))^{k}\,dt\right)d\langle E_\lambda x,x\rangle\right]\right\}$$

$$+\frac{1}{(n-1)!\,(M-m)}\cdot$$

$$\left[\left\langle\left(f_1(A)\int_{m-0}^{M}\left(\int_{m}^{M}(g(\lambda)-g(t))^{n-1}\left(f_2\circ g^{-1}\right)^{(n)}(g(t))\,g'(t)\,K(t,\lambda)\,dt\right)dE_\lambda\right)x,x\right\rangle\right.$$

$$-\langle f_1(A)x,x\rangle\cdot$$

$$\left.\int_{m-0}^{M}\left(\int_{m}^{M}(g(\lambda)-g(t))^{n-1}\left(f_2\circ g^{-1}\right)^{(n)}(g(t))\,g'(t)\,K(t,\lambda)\,dt\right)d\langle E_\lambda x,x\rangle\right]. \tag{27}$$

Consequently, by adding (26) and (27), we get that

$$2E=\frac{1}{(M-m)}\cdot \tag{28}$$

$$\left\{\sum_{k=1}^{n-1}\frac{1}{k!}\left\{\left[\left\langle\left(f_2(A)\int_{m-0}^{M}\left(\int_{m}^{M}\left(f_1\circ g^{-1}\right)^{(k)}(g(t))(g(\lambda)-g(t))^{k}\,dt\right)dE_\lambda\right)x,x\right\rangle\right.\right.\right.$$

$$\left.\left.-\langle f_2(A)x,x\rangle\int_{m-0}^{M}\left(\int_{m}^{M}\left(f_1\circ g^{-1}\right)^{(k)}(g(t))(g(\lambda)-g(t))^{k}\,dt\right)d\langle E_\lambda x,x\rangle\right]+\right.$$

$$\left[\left\langle\left(f_1(A)\int_{m-0}^{M}\left(\int_{m}^{M}\left(f_2\circ g^{-1}\right)^{(k)}(g(t))(g(\lambda)-g(t))^{k}\,dt\right)dE_\lambda\right)x,x\right\rangle\right.$$

$$\left.\left.\left.-\langle f_1(A)x,x\rangle\int_{m-0}^{M}\left(\int_{m}^{M}\left(f_2\circ g^{-1}\right)^{(k)}(g(t))(g(\lambda)-g(t))^{k}\,dt\right)d\langle E_\lambda x,x\rangle\right]\right\}\right\}$$

$$+\frac{1}{(n-1)!\,(M-m)}\cdot$$

$$\left\{\left[\left\langle\left(f_2(A)\int_{m-0}^{M}\left(\int_{m}^{M}(g(\lambda)-g(t))^{n-1}\left(f_1\circ g^{-1}\right)^{(n)}(g(t))\,g'(t)\,K(t,\lambda)\,dt\right)dE_\lambda\right)x,x\right\rangle\right.\right.$$

$$- \langle f_2(A)x,x\rangle \cdot$$

$$\int_{m-0}^{M}\left(\int_m^M (g(\lambda)-g(t))^{n-1}\left(f_1\circ g^{-1}\right)^{(n)}(g(t))\,g'(t)\,K(t,\lambda)\,dt\right)d\langle E_\lambda x,x\rangle\Bigg] +$$

$$\Bigg[\left\langle\left(f_1(A)\int_{m-0}^{M}\left(\int_m^M (g(\lambda)-g(t))^{n-1}\left(f_2\circ g^{-1}\right)^{(n)}(g(t))\,g'(t)\,K(t,\lambda)\,dt\right)dE_\lambda\right)x,x\right\rangle$$

$$- \langle f_1(A)x,x\rangle \cdot$$

$$\int_{m-0}^{M}\left(\int_m^M (g(\lambda)-g(t))^{n-1}\left(f_2\circ g^{-1}\right)^{(n)}(g(t))\,g'(t)\,K(t,\lambda)\,dt\right)d\langle E_\lambda x,x\rangle\Bigg]\Bigg\}.$$

We find that

$$\langle f_1(A)f_2(A)x,x\rangle - \langle f_1(A)x,x\rangle\langle f_2(A)x,x\rangle - \frac{1}{2(M-m)}\cdot$$

$$\Bigg\{\sum_{k=1}^{n-1}\frac{1}{k!}\Bigg\{\Bigg[\left\langle\left(f_2(A)\int_{m-0}^{M}\left(\int_m^M \left(f_1\circ g^{-1}\right)^{(k)}(g(t))\,(g(\lambda)-g(t))^k\,dt\right)dE_\lambda\right)x,x\right\rangle$$

$$- \langle f_2(A)x,x\rangle\int_{m-0}^{M}\left(\int_m^M \left(f_1\circ g^{-1}\right)^{(k)}(g(t))\,(g(\lambda)-g(t))^k\,dt\right)d\langle E_\lambda x,x\rangle\Bigg] +$$

$$\Bigg[\left\langle\left(f_1(A)\int_{m-0}^{M}\left(\int_m^M \left(f_2\circ g^{-1}\right)^{(k)}(g(t))\,(g(\lambda)-g(t))^k\,dt\right)dE_\lambda\right)x,x\right\rangle$$

$$- \langle f_1(A)x,x\rangle\int_{m-0}^{M}\left(\int_m^M \left(f_2\circ g^{-1}\right)^{(k)}(g(t))\,(g(\lambda)-g(t))^k\,dt\right)d\langle E_\lambda x,x\rangle\Bigg]\Bigg\}\Bigg\}$$

$$= \frac{1}{2(n-1)!\,(M-m)}\cdot$$

$$\Bigg\{\Bigg[\left\langle\left(f_2(A)\int_{m-0}^{M}\left(\int_m^M (g(\lambda)-g(t))^{n-1}\left(f_1\circ g^{-1}\right)^{(n)}(g(t))\,g'(t)\,K(t,\lambda)\,dt\right)dE_\lambda\right)x,x\right\rangle$$

$$- \langle f_2(A)x,x\rangle\cdot$$

$$\int_{m-0}^{M}\left(\int_m^M (g(\lambda)-g(t))^{n-1}\left(f_1\circ g^{-1}\right)^{(n)}(g(t))\,g'(t)\,K(t,\lambda)\,dt\right)d\langle E_\lambda x,x\rangle\Bigg] +$$

$$\Bigg[\left\langle\left(f_1(A)\int_{m-0}^{M}\left(\int_m^M (g(\lambda)-g(t))^{n-1}\left(f_2\circ g^{-1}\right)^{(n)}(g(t))\,g'(t)\,K(t,\lambda)\,dt\right)dE_\lambda\right)x,x\right\rangle$$

$$- \langle f_1(A)x,x\rangle\cdot$$

$$\int_{m-0}^{M}\left(\int_m^M (g(\lambda)-g(t))^{n-1}\left(f_2\circ g^{-1}\right)^{(n)}(g(t))\,g'(t)\,K(t,\lambda)\,dt\right)d\langle E_\lambda x,x\rangle\Bigg]\Bigg\} =: R. \tag{29}$$

Hence we have

$$|R| \leq \frac{1}{2\left(n-1\right)!\left(M-m\right)}\cdot$$

$$\left\{\left[\left|\left\langle\left(f_2\left(A\right)\int_{m-0}^{M}\left(\int_m^M\left(g\left(\lambda\right)-g\left(t\right)\right)^{n-1}\left(f_1\circ g^{-1}\right)^{(n)}\left(g\left(t\right)\right)g'\left(t\right)K\left(t,\lambda\right)dt\right)dE_\lambda\right)x,x\right\rangle\right|\right.$$

$$+\left|\left\langle f_2\left(A\right)x,x\right\rangle\right|$$

$$\left.\left|\int_{m-0}^{M}\left(\int_m^M\left(g\left(\lambda\right)-g\left(t\right)\right)^{n-1}\left(f_1\circ g^{-1}\right)^{(n)}\left(g\left(t\right)\right)g'\left(t\right)K\left(t,\lambda\right)dt\right)d\left\langle E_\lambda x,x\right\rangle\right|\right]+$$

$$\left[\left|\left\langle\left(f_1\left(A\right)\int_{m-0}^{M}\left(\int_m^M\left(g\left(\lambda\right)-g\left(t\right)\right)^{n-1}\left(f_2\circ g^{-1}\right)^{(n)}\left(g\left(t\right)\right)g'\left(t\right)K\left(t,\lambda\right)dt\right)dE_\lambda\right)x,x\right\rangle\right|\right.$$

$$+\left|\left\langle f_1\left(A\right)x,x\right\rangle\right|\cdot$$

$$\left.\left.\left|\int_{m-0}^{M}\left(\int_m^M\left(g\left(\lambda\right)-g\left(t\right)\right)^{n-1}\left(f_2\circ g^{-1}\right)^{(n)}\left(g\left(t\right)\right)g'\left(t\right)K\left(t,\lambda\right)dt\right)d\left\langle E_\lambda x,x\right\rangle\right|\right]\right\} \tag{30}$$

(here notice that

$$\left|\int_m^M\left(g\left(\lambda\right)-g\left(t\right)\right)^{n-1}\left(f_1\circ g^{-1}\right)^{(n)}\left(g\left(t\right)\right)g'\left(t\right)K\left(t,\lambda\right)dt\right|\leq$$

$$\int_m^M\left|g\left(\lambda\right)-g\left(t\right)\right|^{n-1}\left|\left(f_1\circ g^{-1}\right)^{(n)}\left(g\left(t\right)\right)\right|\left|g'\left(t\right)\right|\left|K\left(t,\lambda\right)\right|dt\leq$$

$$\left(\int_m^M\left|\lambda-t\right|^{n-1}\left|K\left(t,\lambda\right)\right|dt\right)\left\|g\right\|_\infty^{n-1}\left\|\left(f_1\circ g^{-1}\right)^{(n)}\circ g\right\|_\infty\left\|g'\right\|_\infty= \tag{31}$$

$$\frac{\left\|g\right\|_\infty^{n-1}\left\|g'\right\|_\infty\left\|\left(f_1\circ g^{-1}\right)^{(n)}\circ g\right\|_\infty}{n\left(n+1\right)}\left[\left(M-\lambda\right)^{n+1}+\left(\lambda-m\right)^{n+1}\right])$$

$$\leq\frac{1}{2\left(n-1\right)!\left(M-m\right)}\cdot$$

$$\left\{\left[\left\|f_2\left(A\right)\int_{m-0}^{M}\left(\int_m^M\left(g\left(\lambda\right)-g\left(t\right)\right)^{n-1}\left(f_1\circ g^{-1}\right)^{(n)}\left(g\left(t\right)\right)g'\left(t\right)K\left(t,\lambda\right)dt\right)dE_\lambda\right\|\right.$$

$$\left.+\left\|f_1\left(A\right)\int_{m-0}^{M}\left(\int_m^M\left(g\left(\lambda\right)-g\left(t\right)\right)^{n-1}\left(f_2\circ g^{-1}\right)^{(n)}\left(g\left(t\right)\right)g'\left(t\right)K\left(t,\lambda\right)dt\right)dE_\lambda\right\|\right]$$

$$+\left[\left\|f_2\left(A\right)\right\|\frac{\left\|g\right\|_\infty^{n-1}\left\|g'\right\|_\infty\left\|\left(f_1\circ g^{-1}\right)^{(n)}\circ g\right\|_\infty}{n\left(n+1\right)}\cdot\right.$$

$$\left[\left\langle (M1_H - A)^{n+1} x, x\right\rangle + \left\langle (A - m1_H)^{n+1} x, x\right\rangle\right] +$$

$$\|f_1(A)\| \frac{\|g\|_\infty^{n-1} \|g'\|_\infty \left\|\left(f_2 \circ g^{-1}\right)^{(n)} \circ g\right\|_\infty}{n(n+1)}.$$

$$\left[\left\langle (M1_H - A)^{n+1} x, x\right\rangle + \left\langle (A - m1_H)^{n+1} x, x\right\rangle\right]\Bigg\} \tag{32}$$

$$\leq \frac{1}{2(n-1)!(M-m)}.$$

$$\left\{\|f_2(A)\| \left\| \int_{m-0}^{M} \left( \int_m^M (g(\lambda) - g(t))^{n-1} \left(f_1 \circ g^{-1}\right)^{(n)} (g(t)) g'(t) K(t,\lambda)\, dt \right) dE_\lambda \right\| + \right.$$

$$\left. \|f_1(A)\| \left\| \int_{m-0}^{M} \left( \int_m^M (g(\lambda) - g(t))^{n-1} \left(f_2 \circ g^{-1}\right)^{(n)} (g(t)) g'(t) K(t,\lambda)\, dt \right) dE_\lambda \right\| \right]$$

$$+ \left[ \frac{\|f_2(A)\| \|g\|_\infty^{n-1} \|g'\|_\infty \left\|\left(f_1 \circ g^{-1}\right)^{(n)} \circ g\right\|_\infty}{n(n+1)} + \right.$$

$$\left. \frac{\|f_1(A)\| \|g\|_\infty^{n-1} \|g'\|_\infty \left\|\left(f_2 \circ g^{-1}\right)^{(n)} \circ g\right\|_\infty}{n(n+1)} \right].$$

$$\left[\left\langle (M1_H - A)^{n+1} x, x\right\rangle + \left\langle (A - m1_H)^{n+1} x, x\right\rangle\right]\Bigg\} =: (\xi). \tag{33}$$

Notice here that

$$\left\| \int_{m-0}^{M} \left( \int_m^M (g(\lambda) - g(t))^{n-1} \left(f_1 \circ g^{-1}\right)^{(n)} (g(t)) g'(t) K(t,\lambda)\, dt \right) dE_\lambda \right\| =$$

$$\sup_{\|x\|=1} \left| \int_{m-0}^{M} \left( \int_m^M (g(\lambda) - g(t))^{n-1} \left(f_1 \circ g^{-1}\right)^{(n)} (g(t)) g'(t) K(t,\lambda)\, dt \right) d\langle E_\lambda x, x\rangle \right|$$

$$\leq \frac{\|g\|_\infty^{n-1} \|g'\|_\infty \left\|\left(f_1 \circ g^{-1}\right)^{(n)} \circ g\right\|_\infty}{n(n+1)}.$$

$$\left[\left\|(M1_H - A)^{n+1}\right\| + \left\|(A - m1_H)^{n+1}\right\|\right]. \tag{34}$$

A similar estimate to (34) holds for $f_2$.

Hence we obtain by (33), (34) that

$$(\xi) \leq \frac{1}{(n-1)!(M-m)} \left[ \|f_2(A)\| \frac{\|g\|_\infty^{n-1} \|g'\|_\infty \left\|\left(f_1 \circ g^{-1}\right)^{(n)} \circ g\right\|_\infty}{n(n+1)} \right.$$

$$+\left\|f_1(A)\right\|\frac{\|g\|_\infty^{n-1}\|g'\|_\infty\left\|\left(f_2\circ g^{-1}\right)^{(n)}\circ g\right\|_\infty}{n(n+1)}\Bigg]\cdot$$

$$\left[\left\|(M1_H-A)^{n+1}\right\|+\left\|(A-m1_H)^{n+1}\right\|\right]= \tag{35}$$

$$\frac{\|g\|_\infty^{n-1}\|g'\|_\infty}{(n+1)!(M-m)}\cdot$$

$$\left[\left\|f_2(A)\right\|\left\|\left(f_1\circ g^{-1}\right)^{(n)}\circ g\right\|_\infty+\left\|f_1(A)\right\|\left\|\left(f_2\circ g^{-1}\right)^{(n)}\circ g\right\|_\infty\right]\cdot$$

$$\left[\left\|(M1_H-A)^{n+1}\right\|+\left\|(A-m1_H)^{n+1}\right\|\right].$$

We have proved that

$$|R|\leq\frac{\|g\|_\infty^{n-1}\|g'\|_\infty}{(n+1)!(M-m)}\cdot$$

$$\left[\left\|f_2(A)\right\|\left\|\left(f_1\circ g^{-1}\right)^{(n)}\circ g\right\|_\infty+\left\|f_1(A)\right\|\left\|\left(f_2\circ g^{-1}\right)^{(n)}\circ g\right\|_\infty\right]\cdot \tag{36}$$

$$\left[\left\|(M1_H-A)^{n+1}\right\|+\left\|(A-m1_H)^{n+1}\right\|\right],$$

that is proving the claim.

Above it is $\|\cdot\|_\infty=\|\cdot\|_{\infty,[m,M]}$. ■

We give

**Corollary 4** *($n=1$ case of Theorem 3) For every $x\in H:\|x\|=1$, we obtain that*

$$\left|\langle f_1(A)f_2(A)x,x\rangle-\langle f_1(A)x,x\rangle\langle f_2(A)x,x\rangle\right|\leq\frac{\|g'\|_{\infty,[m,M]}}{2(M-m)}\cdot$$

$$\left[\left\|f_2(A)\right\|\left\|\left(f_1\circ g^{-1}\right)'\circ g\right\|_{\infty,[m,M]}+\left\|f_1(A)\right\|\left\|\left(f_2\circ g^{-1}\right)'\circ g\right\|_{\infty,[m,M]}\right]\cdot$$

$$\left[\left\|(M1_H-A)^2\right\|+\left\|(A-m1_H)^2\right\|\right]. \tag{37}$$

We present

**Theorem 5** *Here all as in Theorem 3. Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\langle(\Delta(f_1,f_2;g))(A)x,x\rangle\leq$$

$$\frac{\|g\|_{\infty,[m,M]}^{n-1}\|g'\|_{\infty,[m,M]}}{(n-1)!(M-m)}\left(\frac{\Gamma(p(n-1)+1)\Gamma(p+1)}{\Gamma(pn+2)}\right)^{\frac{1}{p}}\cdot$$

$$\left[\left\|f_2(A)\right\|\left\|\left(f_1\circ g^{-1}\right)^{(n)}\circ g\right\|_{q,[m,M]}+\left\|f_1(A)\right\|\left\|\left(f_2\circ g^{-1}\right)^{(n)}\circ g\right\|_{q,[m,M]}\right]\cdot$$

$$\left[\left\|(M1_H-A)^{n+\frac{1}{p}}\right\|+\left\|(A-m1_H)^{n+\frac{1}{p}}\right\|\right], \tag{38}$$

*where $\Gamma$ is the gamma function.*

**Proof.** We observe that

$$\left|\int_m^M (g(\lambda)-g(t))^{n-1}\left(f_1\circ g^{-1}\right)^{(n)}(g(t))\,g'(t)\,K(t,\lambda)\,dt\right|\leq$$

$$\int_m^M |g(\lambda)-g(t)|^{n-1}\left|\left(f_1\circ g^{-1}\right)^{(n)}(g(t))\right||g'(t)|\,|K(t,\lambda)|\,dt\leq$$

$$\|g\|_{\infty,[m,M]}^{n-1}\|g'\|_{\infty,[m,M]}\int_m^M|\lambda-t|^{n-1}\left|\left(f_1\circ g^{-1}\right)^{(n)}(g(t))\right||K(t,\lambda)|\,dt= \tag{39}$$

$$\|g\|_{\infty,[m,M]}^{n-1}\|g'\|_{\infty,[m,M]}\left[\int_m^\lambda(\lambda-t)^{n-1}(t-m)\left|\left(f_1\circ g^{-1}\right)^{(n)}(g(t))\right|dt+\right.$$

$$\left.\int_\lambda^M(M-t)(t-\lambda)^{n-1}\left|\left(f_1\circ g^{-1}\right)^{(n)}(g(t))\right|dt\right]\leq$$

$$\|g\|_{\infty,[m,M]}^{n-1}\|g'\|_{\infty,[m,M]}\left[\left(\int_m^\lambda(\lambda-t)^{(p(n-1)+1)-1}(t-m)^{(p+1)-1}\,dt\right)^{\frac{1}{p}}+\right.$$

$$\left.\left(\int_\lambda^M(M-t)^{(p+1)-1}(t-\lambda)^{(p(n-1)+1)-1}\,dt\right)^{\frac{1}{p}}\right]\left\|\left(f_1\circ g^{-1}\right)^{(n)}\circ g\right\|_{q,[m,M]}=$$

$$\|g\|_{\infty,[m,M]}^{n-1}\|g'\|_{\infty,[m,M]}\left\|\left(f_1\circ g^{-1}\right)^{(n)}\circ g\right\|_{q,[m,M]}\cdot$$

$$\left(\frac{\Gamma(p(n-1)+1)\,\Gamma(p+1)}{\Gamma(pn+2)}\right)^{\frac{1}{p}}\left[(\lambda-m)^{n+\frac{1}{p}}+(M-\lambda)^{n+\frac{1}{p}}\right], \tag{40}$$

$\forall\ \lambda\in[m,M]$.

So we got so far

$$\left|\int_m^M (g(\lambda)-g(t))^{n-1}\left(f_1\circ g^{-1}\right)^{(n)}(g(t))\,g'(t)\,K(t,\lambda)\,dt\right|\leq$$

$$\|g\|_{\infty,[m,M]}^{n-1}\|g'\|_{\infty,[m,M]}\left\|\left(f_1\circ g^{-1}\right)^{(n)}\circ g\right\|_{q,[m,M]}\cdot$$

$$\left(\frac{\Gamma(p(n-1)+1)\,\Gamma(p+1)}{\Gamma(pn+2)}\right)^{\frac{1}{p}}\left[(M-\lambda)^{n+\frac{1}{p}}+(\lambda-m)^{n+\frac{1}{p}}\right], \tag{41}$$

$\forall\ \lambda\in[m,M]$.

Hence it holds

$$\left|\int_{m-0}^M\left(\int_m^M (g(\lambda)-g(t))^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(g(t))\,g'(t)\,K(t,\lambda)\,dt\right)d\langle E_\lambda x,x\rangle\right|\leq$$

$$\|g\|_{\infty,[m,M]}^{n-1} \|g'\|_{\infty,[m,M]} \left\| \left(f_i \circ g^{-1}\right)^{(n)} \circ g \right\|_{q,[m,M]} \cdot$$

$$\left( \frac{\Gamma\left(p\left(n-1\right)+1\right)\Gamma\left(p+1\right)}{\Gamma\left(pn+2\right)} \right)^{\frac{1}{p}} \left[ \left\| \left(M1_H - A\right)^{n+\frac{1}{p}} \right\| + \left\| \left(A - m1_H\right)^{n+\frac{1}{p}} \right\| \right], \tag{42}$$

for $i = 1, 2$.

Thus we derive

$$\left\| \int_{m-0}^{M} \left( \int_{m}^{M} \left(g\left(\lambda\right) - g\left(t\right)\right)^{n-1} \left(f_i \circ g^{-1}\right)^{(n)} \left(g\left(t\right)\right) g'\left(t\right) K\left(t,\lambda\right) dt \right) dE_\lambda \right\| \leq$$

$$\|g\|_{\infty,[m,M]}^{n-1} \|g'\|_{\infty,[m,M]} \left\| \left(f_i \circ g^{-1}\right)^{(n)} \circ g \right\|_{q,[m,M]} \cdot$$

$$\left( \frac{\Gamma\left(p\left(n-1\right)+1\right)\Gamma\left(p+1\right)}{\Gamma\left(pn+2\right)} \right)^{\frac{1}{p}} \left[ \left\| \left(M1_H - A\right)^{n+\frac{1}{p}} \right\| + \left\| \left(A - m1_H\right)^{n+\frac{1}{p}} \right\| \right], \tag{43}$$

for $i = 1, 2$.

Next we use (42) and (43).

Acting as in the proof of Theorem 3 we find that

$$|R| \overset{(30)}{\leq} \frac{1}{2\left(n-1\right)!\left(M-m\right)} \cdot$$

$$\left\{ 2 \left[ \left\| f_2\left(A\right) \right\| \|g\|_{\infty,[m,M]}^{n-1} \|g'\|_{\infty,[m,M]} \left\| \left(f_1 \circ g^{-1}\right)^{(n)} \circ g \right\|_{q,[m,M]} \cdot \right. \right.$$

$$\left( \frac{\Gamma\left(p\left(n-1\right)+1\right)\Gamma\left(p+1\right)}{\Gamma\left(pn+2\right)} \right)^{\frac{1}{p}} \left[ \left\| \left(M1_H - A\right)^{n+\frac{1}{p}} \right\| + \left\| \left(A - m1_H\right)^{n+\frac{1}{p}} \right\| \right] \Bigg] + \tag{44}$$

$$2 \left[ \left\| f_1\left(A\right) \right\| \|g\|_{\infty,[m,M]}^{n-1} \|g'\|_{\infty,[m,M]} \left\| \left(f_2 \circ g^{-1}\right)^{(n)} \circ g \right\|_{q,[m,M]} \cdot \right.$$

$$\left. \left( \frac{\Gamma\left(p\left(n-1\right)+1\right)\Gamma\left(p+1\right)}{\Gamma\left(pn+2\right)} \right)^{\frac{1}{p}} \left[ \left\| \left(M1_H - A\right)^{n+\frac{1}{p}} \right\| + \left\| \left(A - m1_H\right)^{n+\frac{1}{p}} \right\| \right] \right] \Bigg\} =$$

$$\frac{1}{\left(n-1\right)!\left(M-m\right)} \left\{ \|g\|_{\infty,[m,M]}^{n-1} \|g'\|_{\infty,[m,M]} \left( \frac{\Gamma\left(p\left(n-1\right)+1\right)\Gamma\left(p+1\right)}{\Gamma\left(pn+2\right)} \right)^{\frac{1}{p}} \cdot \right.$$

$$\left[ \left\| f_2\left(A\right) \right\| \left\| \left(f_1 \circ g^{-1}\right)^{(n)} \circ g \right\|_{q,[m,M]} + \left\| f_1\left(A\right) \right\| \left\| \left(f_2 \circ g^{-1}\right)^{(n)} \circ g \right\|_{q,[m,M]} \right] \cdot$$

$$\left. \left[ \left\| \left(M1_H - A\right)^{n+\frac{1}{p}} \right\| + \left\| \left(A - m1_H\right)^{n+\frac{1}{p}} \right\| \right] \right\}, \tag{45}$$

proving the claim. ■

We give for $n = 1$:

**Corollary 6** *(to Theorem 5) It holds*

$$|\langle f_1(A) f_2(A) x, x\rangle - \langle f_1(A) x, x\rangle \langle f_2(A) x, x\rangle| \leq \frac{\|g'\|_{\infty,[m,M]}}{(M-m)(p+1)^{\frac{1}{p}}} \cdot$$

$$\left[\|f_2(A)\| \left\|\left(f_1 \circ g^{-1}\right)' \circ g\right\|_{q,[m,M]} + \|f_1(A)\| \left\|\left(f_2 \circ g^{-1}\right)' \circ g\right\|_{q,[m,M]}\right] \cdot$$

$$\left[\left\|(M1_H - A)^{1+\frac{1}{p}}\right\| + \left\|(A - m1_H)^{1+\frac{1}{p}}\right\|\right]. \tag{46}$$

We continue with

**Theorem 7** *All as in Theorem 3. Then*

$$\langle(\Delta(f_1, f_2; g))(A) x, x\rangle \leq \frac{(M-m)^{n-1}}{(n-1)!} \|g\|^{n-1}_{\infty,[m,M]} \|g'\|_{\infty,[m,M]} \cdot$$

$$\left[\|f_1(A)\| \left\|\left(f_2 \circ g^{-1}\right)^{(n)} \circ g\right\|_{1,[m,M]} + \|f_2(A)\| \left\|\left(f_1 \circ g^{-1}\right)^{(n)} \circ g\right\|_{1,[m,M]}\right]. \tag{47}$$

**Proof.** We observe that

$$\left|\int_m^M (g(\lambda) - g(t))^{n-1} \left(f_i \circ g^{-1}\right)^{(n)} (g(t)) g'(t) K(t, \lambda)\, dt\right| \leq$$

$$\int_m^M |g(\lambda) - g(t)|^{n-1} |g'(t)| |K(t, \lambda)| \left|\left(f_i \circ g^{-1}\right)^{(n)} (g(t))\right| dt \leq$$

$$\|g\|^{n-1}_{\infty,[m,M]} \|g'\|_{\infty,[m,M]} (M-m)^n \left(\int_m^M \left|\left(f_i \circ g^{-1}\right)^{(n)} (g(t))\right| dt\right) =$$

$$\|g\|^{n-1}_{\infty,[m,M]} \|g'\|_{\infty,[m,M]} (M-m)^n \left\|\left(f_i \circ g^{-1}\right)^{(n)} \circ g\right\|_{1,[m,M]}, \quad i = 1, 2. \tag{48}$$

Hence it holds $(i = 1, 2)$

$$\left|\int_{m-0}^M \left(\int_m^M (g(\lambda) - g(t))^{n-1} \left(f_i \circ g^{-1}\right)^{(n)} (g(t)) g'(t) K(t, \lambda)\, dt\right) d\langle E_\lambda x, x\rangle\right| \leq$$

$$\|g\|^{n-1}_{\infty,[m,M]} \|g'\|_{\infty,[m,M]} (M-m)^n \left\|\left(f_i \circ g^{-1}\right)^{(n)} \circ g\right\|_{1,[m,M]}, \tag{49}$$

the last is valid since

$$\int_{m-0}^M d\langle E_\lambda x, x\rangle = 1, \text{ for } x \in H : \|x\| = 1. \tag{50}$$

Therefore it holds

$$\left\| \int_{m-0}^{M} \left( \int_{m}^{M} \left( g\left( \lambda \right) - g\left( t \right) \right)^{n-1} \left( f_i \circ g^{-1} \right)^{(n)} \left( g\left( t \right) \right) g'\left( t \right) K\left( t, \lambda \right) dt \right) dE_\lambda \right\| \leq$$

$$\left\| g \right\|_{\infty,[m,M]}^{n-1} \left\| g' \right\|_{\infty,[m,M]} \left( M - m \right)^n \left\| \left( f_i \circ g^{-1} \right)^{(n)} \circ g \right\|_{1,[m,M]}, \tag{51}$$

for $i = 1, 2$.

Acting as in the proof of Theorem 3 we find that

$$\left| R \right| \overset{\text{(by (30), (49), (51))}}{\leq} \frac{1}{2\left( n-1 \right)! \left( M - m \right)} \cdot$$

$$\left\{ 2 \left\| f_2\left( A \right) \right\| \left\| g \right\|_{\infty,[m,M]}^{n-1} \left\| g' \right\|_{\infty,[m,M]} \left( M - m \right)^n \left\| \left( f_1 \circ g^{-1} \right)^{(n)} \circ g \right\|_{1,[m,M]} + \right.$$

$$\left. 2 \left\| f_1\left( A \right) \right\| \left\| g \right\|_{\infty,[m,M]}^{n-1} \left\| g' \right\|_{\infty,[m,M]} \left( M - m \right)^n \left\| \left( f_2 \circ g^{-1} \right)^{(n)} \circ g \right\|_{1,[m,M]} \right\} = \tag{52}$$

$$\frac{\left( M - m \right)^{n-1}}{\left( n-1 \right)!} \left\| g \right\|_{\infty,[m,M]}^{n-1} \left\| g' \right\|_{\infty,[m,M]} \cdot$$

$$\left[ \left\| f_2\left( A \right) \right\| \left\| \left( f_1 \circ g^{-1} \right)^{(n)} \circ g \right\|_{1,[m,M]} + \left\| f_1\left( A \right) \right\| \left\| \left( f_2 \circ g^{-1} \right)^{(n)} \circ g \right\|_{1,[m,M]} \right],$$

proving the claim. ■

We finish this section with

**Corollary 8** *(to Theorem 7, $n = 1$) It holds*

$$\left| \left\langle f_1\left( A \right) f_2\left( A \right) x, x \right\rangle - \left\langle f_1\left( A \right) x, x \right\rangle \left\langle f_2\left( A \right) x, x \right\rangle \right| \leq \left\| g' \right\|_{\infty,[m,M]} \cdot$$

$$\left[ \left\| f_1\left( A \right) \right\| \left\| \left( f_2 \circ g^{-1} \right)' \circ g \right\|_{1,[m,M]} + \left\| f_2\left( A \right) \right\| \left\| \left( f_1 \circ g^{-1} \right)' \circ g \right\|_{1,[m,M]} \right]. \tag{53}$$

# 4 Applications

We give

**Theorem 9** *Let $f_1, f_2 \in C'\left( [a, b] \right)$ with $[m, M] \subset (a, b)$, $m < M$. Here $A$ is a selfadjoint linear operator on the Hilbert space $H$ with spectrum $Sp\left( A \right) \subseteq [m, M]$. We consider any $x \in H : \left\| x \right\| = 1$, and $\rho > 0 : M < \ln \rho$.*

*Then*

$$\left| \left\langle f_1\left( A \right) f_2\left( A \right) x, x \right\rangle - \left\langle f_1\left( A \right) x, x \right\rangle \left\langle f_2\left( A \right) x, x \right\rangle \right| \leq \frac{e^M}{2\left( M - m \right) \rho}$$

$$\left[\left\|f_2(A)\right\|\left\|\left(f_1\circ\ln\rho t\right)'\circ\frac{e^t}{\rho}\right\|_{\infty,[m,M]}+\left\|f_1(A)\right\|\left\|\left(f_2\circ\ln\rho t\right)'\circ\frac{e^t}{\rho}\right\|_{\infty,[m,M]}\right]$$
$$\left[\left\|\left(M1_H-A\right)^2\right\|+\left\|\left(A-m1_H\right)^2\right\|\right]. \tag{54}$$

**Proof.** Apply Corollary 4 for $g(t)=\frac{e^t}{\rho}$. ■

We continue with

**Theorem 10** *All as in Theorem 9. Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\left|\left\langle f_1(A)f_2(A)x,x\right\rangle-\left\langle f_1(A)x,x\right\rangle\left\langle f_2(A)x,x\right\rangle\right|\leq\frac{e^M}{(M-m)(p+1)^{\frac{1}{p}}\rho}$$

$$\left[\left\|f_2(A)\right\|\left\|\left(f_1\circ\ln\rho t\right)'\circ\frac{e^t}{\rho}\right\|_{q,[m,M]}+\left\|f_1(A)\right\|\left\|\left(f_2\circ\ln\rho t\right)'\circ\frac{e^t}{\rho}\right\|_{q,[m,M]}\right]$$
$$\left[\left\|\left(M1_H-A\right)^{1+\frac{1}{p}}\right\|+\left\|\left(A-m1_H\right)^{1+\frac{1}{p}}\right\|\right]. \tag{55}$$

**Proof.** Use of Corollary 6 and $g(t)=\frac{e^t}{\rho}$; $\rho>0$, $M<\ln\rho$. ■

We finish article with

**Theorem 11** *Here all as in Theorem 9. Then*

$$\left|\left\langle f_1(A)f_2(A)x,x\right\rangle-\left\langle f_1(A)x,x\right\rangle\left\langle f_2(A)x,x\right\rangle\right|\leq\frac{e^M}{\rho} \tag{56}$$

$$\left[\left\|f_1(A)\right\|\left\|\left(f_2\circ\ln\rho t\right)'\circ\frac{e^t}{\rho}\right\|_{1,[m,M]}+\left\|f_2(A)\right\|\left\|\left(f_1\circ\ln\rho t\right)'\circ\frac{e^t}{\rho}\right\|_{1,[m,M]}\right].$$

**Proof.** Use of Corollary 8. ■

# References


[1] G.A. Anastassiou, *Intelligent Mathematics: Computational Analysis*, Springer, New York, Heidelberg, 2011.

[2] P.L. Čebyšev, *Sur les expressions approximatives des intégrales définies par les autres proses entre les mêmes limites*, Proc. Math. Soc. Charkov, 2 (1882), 93-98.

[3] S.S. Dragomir, *Inequalities for functions of selfadjoint operators on Hilbert Spaces*, ajmaa.org/RGMIA/monographs/InFuncOp.pdf, 2011.

[4] S. Dragomir, *Operator inequalities of Ostrowski and Trapezoidal type*, Springer, New York, 2012.

[5] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.

[6] G. Grüss, *Über das Maximum des absoluten Betrages von* $\left[\left(\frac{1}{b-a}\right)\int_a^b f(x)\,g(x)\,dx - \left(\frac{1}{(b-a)^2}\int_a^b f(x)\,dx\int_a^b g(x)\,dx\right)\right]$, Math. Z., 39 (1935), 215-226.

[7] G. Helmberg, *Introduction to Spectral Theory in Hilbert Space*, John Wiley & Sons, Inc., New York, 1969.

# FOURIER SERIES OF SUMS OF PRODUCTS OF POLY-BERNOULLI AND GENOCCHI FUNCTIONS AND THEIR APPLICATIONS

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Abstract. We derive Fourier series expansions of three types of sums of products of poly-Bernoulli and Genocchi functions. In addition, we express each of them in terms of Bernoulli functions.

## 1. Introduction

For any integer $r$, the poly-Bernoulli polynomials $\mathbb{B}_m^{(r)}(x)$ of index $r$ are given by the generating function

$$\frac{Li_r(1-e^{-t})}{e^t-1}e^{xt}=\sum_{m=0}^{\infty}\mathbb{B}_m^{(r)}(x)\frac{t^m}{m!},\quad (\text{see } [1-4,7-10,12,13,17,18]), \tag{1.1}$$

where $Li_r(x)=\sum_{m=0}^{\infty}\frac{x^m}{m^r}$ is the $r$-th polylogarithmic function for $r\geq 1$ and a rational function for $r\leq 0$. We observe here that

$$\frac{d}{dx}\left(Li_{r+1}(x)\right)=\frac{1}{x}Li_r(x). \tag{1.2}$$

As to poly-Bernoulli polynomials, we note the following:

$$\begin{aligned}&\frac{d}{dx}\left(\mathbb{B}_m^{(r)}(x)\right)=m\mathbb{B}_{m-1}^{(r)}(x),\ \ (m\geq 1).\\ &\mathbb{B}_m^{(1)}(x)=B_m(x),\ \mathbb{B}_0^{(r)}(x)=1,\ \mathbb{B}_m^{(0)}(x)=x^m,\\ &\mathbb{B}_m^{(0)}=\delta_{m,0},\ \mathbb{B}_m^{(r+1)}(1)-\mathbb{B}_m^{(r+1)}(0)=\mathbb{B}_{m-1}^{(r)},\ (m\geq 1).\end{aligned} \tag{1.3}$$

The Genocchi polynomials $G_m(x)$ are given by the generating function

$$\frac{2t}{e^t+1}e^{xt}=\sum_{m=0}^{\infty}G_m(x)\frac{t^m}{m!},\quad (\text{see } [14-16]). \tag{1.4}$$

The first few Genocchi polynomials are as follows:

$$\begin{aligned}&G_0(x)=0,\ G_1(x)=1,\ G_2(x)=2x-1,\\ &G_3(x)=3x^2-3x,\ G_4(x)=4x^3-6x^2+1,\\ &G_5(x)=5x^4-10x^3+5x,\ G_6(x)=6x^5-15x^4+15x^2-3,\\ &G_7(x)=7x^6-21x^5+35x^3-21x.\end{aligned} \tag{1.5}$$

From the relation $G_m(x)=mE_{m-1}(x)(m\geq 1)$, we have

$$\begin{aligned}&\deg G_m(x)=m-1\ (m\geq 1),\ G_m=mE_{m-1}\ (m\geq 1),\\ &G_0=0,\ G_1=1,\ G_{2m+1}=0\ (m\geq 1),\ \text{and } G_{2m}\neq 0\ (m\geq 1).\end{aligned} \tag{1.6}$$

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In addition,

$$\begin{aligned} &\frac{d}{dx}G_m(x) = mG_{m-1}(x)\ (m \geq 1), \\ &G_m(x+1) + G_m(x) = 2mx^{m-1}\ (m \geq 0). \end{aligned} \tag{1.7}$$

From these, we also have

$$G_m(1) + G_m(0) = 2\delta_{m,1}, \quad (m \geq 0). \tag{1.8}$$

$$\begin{aligned} \int_0^1 G_m(x)dx &= \frac{1}{m+1}(G_{m+1}(1) - G_{m+1}(0)) \\ &= \frac{2}{m+1}(-G_{m+1}(0) + \delta_{m,0}) \\ &= \begin{cases} 0, & \text{if} \quad m \text{ is even}, \\ -\frac{2}{m+1}G_{m+1}, & \text{if} \quad m \text{ is odd}. \end{cases} \end{aligned} \tag{1.9}$$

For any real number $x$, let $< x >= x - [x] \in [0,1)$ denote the fractional part of $x$. In this paper, we will study the Fourier series of the following three types of sums of products of poly-Bernoulli and Genocchi functions:

(1) $\alpha_m(< x >) = \sum_{k=0}^{m-1} \mathbb{B}_k^{(r+1)}(< x >)G_{m-k}(< x >), (m \geq 2)$;

(2) $\beta_m(< x >) = \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!}\mathbb{B}_k^{(r+1)}(< x >)G_{m-k}(< x >), (m \geq 2)$;

(3) $\gamma_m(< x >) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)}\mathbb{B}_k^{(r+1)}(< x >)G_{m-k}(< x >), (m \geq 2)$.

For some elementary facts about Fourier analysis, the reader may refer to [20,22]. As to $\gamma_m(< x >)$, we note that the polynomial identity (1.10) follows immediately from (4.21) and (4.25), which is derived in turn from the Fourier series expansion of $\gamma_m(< x >)$.

$$\begin{aligned} &\sum_{k=1}^{m-1} \frac{1}{k(m-k)}\mathbb{B}_k^{(r+1)}(x)G_{m-k}(x) \\ &= \frac{1}{m}\left(\Lambda_{m+1} + \frac{2G_{m+1}}{m(m+1)}\right) \\ &\quad + \frac{1}{m}\sum_{s=1}^{m-1}\binom{m}{s}\left(\Lambda_{m-s+1} - \frac{2G_{m-s+1}}{m-s+1}(H_{m-1} - H_{m-s})\right)B_s(x). \end{aligned} \tag{1.10}$$

The obvious polynomial identities can be derived also for $\alpha_m(< x >)$ and $\beta_m(< x >)$ from (2.19) and (2.23), and (3.16) and (3.20), respectively. It is worth noting that from the Fourier series expansion of the function $\sum_{k=1}^{m-1} \frac{1}{k(m-k)}B_k(< x >)B_{m-k}(< x >)$ we can derive the following polynomial identity:

$$\begin{aligned} &\sum_{k=1}^{m-1} \frac{1}{k\,(m-k)}B_k\,(x)\,B_{m-k}\,(x) \\ =&\frac{2}{m^2}\left(B_m + \frac{1}{2}\right) + \frac{2}{m}\sum_{k=1}^{m-2}\frac{1}{m-k}\binom{m}{k}B_{m-k}B_k\,(x) + \frac{2}{m}H_{m-1}B_m\,(x)\,, \quad (m \geq 2). \end{aligned} \tag{1.11}$$

From (1.11), we can derive the following slightly different version of the well-known Miki's identity (see [5,21])

$$\begin{aligned}&\sum_{k=1}^{m-1}\frac{1}{2k\,(2m-2k)}B_{2k}B_{2m-2k} \\ =&\frac{1}{m}\sum_{k=1}^{m}\frac{1}{2k}\binom{2m}{2k}B_{2k}B_{2m-2k}+\frac{1}{m}H_{2m-1}B_{2m},\quad (m\geq 2)\,.\end{aligned} \tag{1.12}$$

Also, from (1.11) and with $\overline{B}_m=\left(\frac{1-2^{m-1}}{2^{m-1}}\right)B_m=\left(2^{1-m}-1\right)B_m=B_m\left(\frac{1}{2}\right)$, we have

$$\begin{aligned}&\sum_{k=1}^{m-1}\frac{1}{2k\,(2m-2k)}\overline{B}_{2k}\overline{B}_{2m-2k} \\ =&\frac{1}{m}\sum_{k=1}^{m}\frac{1}{2k}\binom{2m}{2k}B_{2k}\overline{B}_{2m-2k}+\frac{1}{m}H_{2m-1}\overline{B}_{2m},\quad (m\geq 2)\,,\end{aligned} \tag{1.13}$$

which is the Faber-Pandharipande-Zagier identity (see [6]). Some related works can be found in [11,19].

## 2. Fourier series of functions of the first type

In this section, we will study the Fourier series of first type of sums of products of poly-Bernoulli and Genocchi functions.

$$\alpha_m(x)=\sum_{k=0}^{m-1}\mathbb{B}_k^{(r+1)}(x)G_{m-k}(x),\ \ (m\geq 2). \tag{2.1}$$

Note here that $\deg\alpha_m(x)=m-1$. We now consider the function

$$\alpha_m(<x>)=\sum_{k=0}^{m-1}\mathbb{B}_k^{(r+1)}(<x>)G_{m-k}(<x>),\ \ (m\geq 2), \tag{2.2}$$

defined on $(-\infty,-\infty)$, which is periodic of period 1. The Fourier series of $\alpha_m(<x>)$ is

$$\sum_{n=-\infty}^{\infty}A_n^{(m)}e^{2\pi inx}, \tag{2.3}$$

where

$$\begin{aligned}A_n^{(m)}&=\int_0^1\alpha_m(<x>)e^{-2\pi inx}dx \\ &=\int_0^1\alpha_m(x)e^{-2\pi inx}dx.\end{aligned} \tag{2.4}$$

Before proceeding further, we need to observe the following.

$$\begin{aligned}
\alpha_m'(x) &= \sum_{k=0}^{m-1}(k\mathbb{B}_{k-1}^{(r+1)}(x)G_{m-k}(x)+(m-k)\mathbb{B}_k^{(r+1)}(x)G_{m-k-1}(x)) \\
&= \sum_{k=1}^{m-1}k\mathbb{B}_{k-1}^{(r+1)}(x)G_{m-k}(x)+\sum_{k=0}^{m-2}(m-k)\mathbb{B}_k^{(r+1)}(x)G_{m-k-1}(x) \\
&= \sum_{k=0}^{m-2}(k+1)\mathbb{B}_k^{(r+1)}(x)G_{m-1-k}(x)+\sum_{k=0}^{m-2}(m-k)\mathbb{B}_k^{(r+1)}(x)G_{m-1-k}(x) \\
&= (m+1)\sum_{k=0}^{m-2}\mathbb{B}_k^{(r+1)}(x)G_{m-1-k}(x) \\
&= (m+1)\alpha_{m-1}(x).
\end{aligned} \tag{2.5}$$

So, $\alpha_m'(x)=(m+1)\alpha_{m-1}(x)$, and hence

$$\left(\frac{\alpha_{m+1}(x)}{m+2}\right)' = \alpha_m(x), \tag{2.6}$$

and

$$\int_0^1 \alpha_m(x)dx = \frac{1}{m+2}(\alpha_{m+1}(1)-\alpha_{m+1}(0)). \tag{2.7}$$

For $m\geq 2$,

$$\begin{aligned}
\Delta_m &= \alpha_m(1)-\alpha_m(0) \\
&= \sum_{k=0}^{m-1}\left(\mathbb{B}_k^{(r+1)}(1)G_{m-k}(1)-\mathbb{B}_k^{(r+1)}G_{m-k}\right) \\
&= \mathbb{B}_0^{(r+1)}(1)G_m(1)-\mathbb{B}_0^{(r+1)}G_m+\sum_{k=1}^{m-1}\left(\mathbb{B}_k^{(r+1)}(1)G_{m-k}(1)-\mathbb{B}_k^{(r+1)}G_{m-k}\right) \\
&= -2G_m+2\delta_{m,1}+\sum_{k=1}^{m-1}\left(\left(\mathbb{B}_k^{(r+1)}+\mathbb{B}_{k-1}^{(r)}\right)(-G_{m-k}+2\delta_{m-1,k})-\mathbb{B}_k^{(r+1)}G_{m-k}\right) \\
&= -2G_m+\sum_{k=1}^{m-1}\left(-2\mathbb{B}_k^{(r+1)}G_{m-k}+2\mathbb{B}_k^{(r+1)}\delta_{m-1,k}-\mathbb{B}_{k-1}^{(r)}G_{m-k}+2\mathbb{B}_{k-1}^{(r)}\delta_{m-1,k}\right) \\
&= -2G_m-2\sum_{k=1}^{m-1}\mathbb{B}_k^{(r+1)}G_{m-k}+2\mathbb{B}_{m-1}^{(r+1)}-\sum_{k=1}^{m-1}\mathbb{B}_{k-1}^{(r)}G_{m-k}+2\mathbb{B}_{m-2}^{(r)} \\
&= -2\sum_{k=0}^{m-2}\mathbb{B}_k^{(r+1)}G_{m-k}-\sum_{k=0}^{m-2}\mathbb{B}_k^{(r)}G_{m-k-1}+2\mathbb{B}_{m-2}^{(r)}.
\end{aligned} \tag{2.8}$$

$$\alpha_m(0)=\alpha_m(1) \Longleftrightarrow \Delta_m = 0. \tag{2.9}$$

$$\int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\Delta_{m+1}. \tag{2.10}$$

Now, we are going to determine the Fourier coefficients $A_n^{(m)}$.
Case 1 : $n \neq 0$.

$$\begin{aligned}
A_n^{(m)} &= \int_0^1 \alpha_m(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n} \left[\alpha_m(x) e^{-2\pi i n x}\right]_0^1 + \frac{1}{2\pi i n} \int_0^1 \alpha_m'(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n} (\alpha_m(1) - \alpha_m(0)) + \frac{m+1}{2\pi i n} \int_0^1 \alpha_{m-1}(x) e^{-2\pi i n x} dx \\
&= \frac{m+1}{2\pi i n} A_n^{(m-1)} - \frac{1}{2\pi i n} \Delta_m \\
&= \frac{m+1}{2\pi i n} \left( \frac{m}{2\pi i n} A_n^{(m-2)} - \frac{1}{2\pi i n} \Delta_{m-1} \right) - \frac{1}{2\pi i n} \Delta_m \\
&= \frac{(m+1)m}{(2\pi i n)^2} A_n^{(m-2)} - \frac{m+1}{(2\pi i n)^2} \Delta_{m-1} - \frac{1}{2\pi i n} \Delta_m \\
&= \cdots \\
&= \frac{(m+1)_{m-1}}{(2\pi i n)^{m-1}} A_n^{(1)} - \sum_{j=1}^{m-1} \frac{(m+1)_{j-1}}{(2\pi i n)^j} \Delta_{m-j+1} \\
&= -\sum_{j=1}^{m-1} \frac{(m+1)_{j-1}}{(2\pi i n)^j} \Delta_{m-j+1} \\
&= -\frac{1}{m+2} \sum_{j=1}^{m-1} \frac{(m+2)_j}{(2\pi i n)^j} \Delta_{m-j+1},
\end{aligned} \tag{2.11}$$

where

$$A_n^{(1)} = \int_0^1 \alpha_1(x) e^{-2\pi i n x} dx = \int_0^1 e^{-2\pi i n x} dx = 0. \tag{2.12}$$

Case 2: $n = 0$.

$$A_0^{(m)} = \int_0^1 \alpha_m(x) dx = \frac{1}{m+2} \Delta_{m+1}. \tag{2.13}$$

Here we recall the following facts about Bernoulli functions $B_m(< x >)$:
(a) for $m \geq 2$,

$$B_m(< x >) = -m! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^m}. \tag{2.14}$$

(b) for $m = 1$,

$$-\sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{2\pi i n} = \begin{cases} B_1(< x >), & \text{for} \quad x \in \mathbb{Z}^c, \\ 0, & \text{for} \quad x \in \mathbb{Z}, \end{cases} \tag{2.15}$$

where $\mathbb{Z}^c = \mathbb{R} - \mathbb{Z}$. $\alpha_m(< x >), (m \geq 2)$ is piecewise $C^\infty$. Moreover, $\alpha_m(< x >)$ is continuous for those integers $m \geq 2$ with $\Delta_m = 0$ and discontinuous with jump discontinuities at integers for those integers $m \geq 2$ with $\Delta_m \neq 0$. Assume first that $m$ is an integer $\geq 2$ with $\Delta_m = 0$. Then $\alpha_m(0) = \alpha_m(1)$. $\alpha_m(< x >)$ is piecewise $C^\infty$, and continuous. Thus, the Fourier series of $\alpha_m(< x >)$ converges uniformly

to $\alpha_m(< x >)$, and

$$\begin{aligned}
&\alpha_m(< x >)\\
&= \frac{1}{m+2}\Delta_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m-1}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}\right)e^{2\pi inx}\\
&= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2}\sum_{j=1}^{m-1}\binom{m+2}{j}\Delta_{m-j+1}\left(-j!\sum_{n=-\infty, n\neq 0}^{\infty}\frac{e^{2\pi in}}{(2\pi in)^j}\right) \qquad (2.16)\\
&= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2}\sum_{j=2}^{m-1}\binom{m+2}{j}\Delta_{m-j+1}B_j(< x >)\\
&\quad + \frac{1}{m+2}\binom{m+2}{1}\Delta_m \times \begin{cases} B_1(< x >), & \text{for} \quad x \in \mathbb{Z}^c,\\ 0, & \text{for} \quad x \in \mathbb{Z}. \end{cases}
\end{aligned}$$

Now, we can state our first theorem.

**Theorem 2.1.** *For each integer $l \geq 2$, let*

$$\Delta_l = -2\sum_{k=0}^{l-2}\mathbb{B}_k^{(r+1)}G_{l-k} - \sum_{k=0}^{l-2}\mathbb{B}_k^{(r)}G_{l-k-1} + 2\mathbb{B}_{l-2}^{(r)}. \qquad (2.17)$$

*Assume that $\Delta_m = 0$, for an integer $m \geq 2$. Then we have the following.*

*(a) $\sum_{k=0}^{m-1}\mathbb{B}_k^{(r+1)}(< x >)G_{m-k}(< x >)$ has the Fourier series expansion*

$$\begin{aligned}
&\sum_{k=0}^{m-1}\mathbb{B}_k^{(r+1)}(< x >)G_{m-k}(< x >)\\
&= \frac{1}{m+2}\Delta_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m-1}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}\right)e^{2\pi inx},
\end{aligned} \qquad (2.18)$$

*for all $x \in (-\infty, \infty)$, where the convergence is uniform.*

*(b)*

$$\begin{aligned}
&\sum_{k=0}^{m-1}\mathbb{B}_k^{(r+1)}(< x >)G_{m-k}(< x >)\\
&= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2}\sum_{j=2}^{m-1}\binom{m+2}{j}\Delta_{m-j+1}B_j(< x >),
\end{aligned} \qquad (2.19)$$

*for all $x \in (-\infty, \infty)$. Here $B_j(< x >)$ is the Bernoulli function.*

Assume next that $m \geq 2$ is an integer with $\Delta_m \neq 0$. Then $\alpha_m(0) \neq \alpha_m(1)$. So $\alpha_m(< x >)$ is piecewise $C^\infty$ and discontinuous with jump discontinuities at integers. The Fourier series of $\alpha_m(< x >)$ converges pointwise to $\alpha_m(< x >)$, for $x \in \mathbb{Z}^c$, and converges to

$$\frac{1}{2}(\alpha_m(0) + \alpha_m(1)) = \alpha_m(0) + \frac{1}{2}\Delta_m, \qquad (2.20)$$

for $x \in \mathbb{Z}$. Next, we can state our second theorem.

**Theorem 2.2.** *For each integer $l \geq 2$, let*

$$\Delta_l = -2\sum_{k=0}^{l-2} \mathbb{B}_k^{(r+1)} G_{l-k} - \sum_{k=0}^{l-2} \mathbb{B}_k^{(r)} G_{l-k-1} + 2\mathbb{B}_{l-2}^{(r)}. \tag{2.21}$$

*Assume that $\Delta_m \neq 0$, for an integer $\geq 2$. Then we have the following.*

*(a)*

$$\begin{aligned} &\frac{1}{m+2}\Delta_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \left( -\frac{1}{m+2} \sum_{j=1}^{m-1} \frac{(m+2)_j}{(2\pi i n)^j} \Delta_{m-j+1} \right) e^{2\pi i n x} \\ &= \begin{cases} \sum_{k=0}^{m-1} \mathbb{B}_k^{(r+1)}(< x >) G_{m-k}(< x >), & \text{for} \quad x \in \mathbb{Z}^c, \\ \sum_{k=0}^{m-1} \mathbb{B}_k^{(r+1)} G_{m-k} + \frac{1}{2}\Delta_m, & \text{for} \quad x \in \mathbb{Z}. \end{cases} \end{aligned} \tag{2.22}$$

*(b)*

$$\begin{aligned} &\frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2} \sum_{j=1}^{m-1} \binom{m+2}{j} \Delta_{m-j+1} B_j(< x >) \\ &= \sum_{k=0}^{m-1} \mathbb{B}_k^{(r+1)}(< x >) G_{m-k}(< x >), \quad \text{for } x \in \mathbb{Z}^c; \end{aligned} \tag{2.23}$$

$$\begin{aligned} &\frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2} \sum_{j=1}^{m-1} \binom{m+2}{j} \Delta_{m-j+1} B_j(< x >) \\ &\sum_{k=0}^{m-1} \mathbb{B}_k^{(r+1)} G_{m-k} + \frac{1}{2}\Delta_m, \ x \in \mathbb{Z}. \end{aligned} \tag{2.24}$$

## 3. Fourier series of functions of the second type

Let $\beta_m(x) = \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} \mathbb{B}_k^{(r+1)}(x) G_{m-k}(x), \ (m \geq 2)$. Observe that

$$\begin{aligned} \beta_m'(x) &= \sum_{k=0}^{m-1} \left\{ \frac{k}{k!(m-k)!} \mathbb{B}_{k-1}^{(r+1)}(x) G_{m-k}(x) + \frac{m-k}{k!(m-k)!} \mathbb{B}_k^{(r+1)}(x) G_{m-k-1}(x) \right\} \\ &= \sum_{k=1}^{m-1} \frac{1}{(k-1)!(m-k)!} \mathbb{B}_{k-1}^{(r+1)}(x) G_{m-k}(x) + \sum_{k=0}^{m-2} \frac{1}{k!(m-k-1)!} \mathbb{B}_k^{(r+1)}(x) G_{m-k-1}(x) \\ &= 2\sum_{k=0}^{m-2} \frac{1}{k!(m-1-k)!} \mathbb{B}_k^{(r+1)}(x) G_{m-1-k}(x) \\ &= 2\beta_{m-1}(x). \end{aligned} \tag{3.1}$$

From this, we have

$$\left( \frac{\beta_{m+1}(x)}{2} \right)' = \beta_m(x), \tag{3.2}$$

and

$$\int_0^1 \beta_m(x)dx = \frac{1}{2}(\beta_{m+1}(1) - \beta_{m+1}(0)). \tag{3.3}$$

For $m \geq 2$, we have

$$\begin{aligned}
\Omega_m &= \Omega_m(r) = \beta_m(1) - \beta_m(0) \\
&= \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} \left(\mathbb{B}_k^{(r+1)}(1)G_{m-k}(1) - \mathbb{B}_k^{(r+1)}G_{m-k}\right) \\
&= \frac{1}{m!} \left(\mathbb{B}_0^{(r+1)}(1)G_m(1) - \mathbb{B}_0^{(r+1)}G_m\right) + \sum_{k=1}^{m-1} \frac{1}{k!(m-k)!} \left(\mathbb{B}_k^{(r+1)}(1)G_{m-k}(1) - \mathbb{B}_k^{(r+1)}G_{m-k}\right) \\
&= \frac{1}{m!} (-2G_m + 2\delta_{m,1}) + \sum_{k=1}^{m-1} \frac{1}{k!(m-k)!} \left(\left(\mathbb{B}_k^{(r+1)} + \mathbb{B}_{k-1}^{(r)}\right)(-G_{m-k} + 2\delta_{m-1,k}) - \mathbb{B}_k^{(r+1)}G_{m-k}\right) \\
&= -\frac{2}{m!}G_m + \sum_{k=1}^{m-1} \frac{1}{k!(m-k)!} \left(-2\mathbb{B}_k^{(r+1)}G_{m-k} + 2\mathbb{B}_k^{(r+1)}\delta_{m-1,k} - \mathbb{B}_{k-1}^{(r)}G_{m-k} + 2\mathbb{B}_{k-1}^{(r)}\delta_{m-1,k}\right) \\
&= -\frac{2}{m!}G_m - 2\sum_{k=1}^{m-1} \frac{\mathbb{B}_k^{(r+1)}G_{m-k}}{k!(m-k)!} + 2\frac{\mathbb{B}_{m-1}^{(r+1)}}{(m-1)!} - \sum_{k=1}^{m-1} \frac{\mathbb{B}_{k-1}^{(r)}G_{m-k}}{k!(m-k)!} + 2\frac{\mathbb{B}_{m-2}^{(r)}}{(m-1)!} \\
&= -2\sum_{k=0}^{m-2} \frac{\mathbb{B}_k^{(r+1)}G_{m-k}}{k!(m-k)!} - \sum_{k=1}^{m-1} \frac{\mathbb{B}_{k-1}^{(r)}G_{m-k}}{k!(m-k)!} + 2\frac{\mathbb{B}_{m-2}^{(r)}}{(m-1)!}.
\end{aligned} \tag{3.4}$$

Then

$$\beta_m(0) = \beta_m(1) \Longleftrightarrow \Omega_m = 0. \tag{3.5}$$

Also,

$$\int_0^1 \beta_m(x)dx = \frac{1}{2}\Omega_{m+1}. \tag{3.6}$$

Now, we are going to consider the function

$$\beta_m(< x >) = \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!}\mathbb{B}_k^{(r+1)}(< x >)G_{m-k}(< x >),\ (m \geq 2), \tag{3.7}$$

defined on $(-\infty, \infty)$, which is periodic with period 1. The Fourier series of $\beta_m(< x >)$ is

$$\sum_{k=-\infty}^{\infty} B_n^{(m)} e^{2\pi inx}, \tag{3.8}$$

where

$$\begin{aligned}
B_n^{(m)} &= \int_0^1 \beta_m(< x >)e^{-2\pi inx}dx \\
&= \int_0^1 \beta_m(x)e^{-2\pi inx}dx.
\end{aligned} \tag{3.9}$$

We are now going to determine the Fourier coefficients $B_n^{(m)}$.

Case 1: $n \neq 0$.

$$\begin{aligned}
B_n^{(m)} &= \int_0^1 \beta_m(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n} \left[\beta_m(x) e^{-2\pi i n x}\right]_0^1 + \frac{1}{2\pi i n} \int_0^1 \beta_m'(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n} (\beta_m(1) - \beta_m(0)) + \frac{2}{2\pi i n} \int_0^1 \beta_{m-1}(x) e^{-2\pi i n x} dx \\
&= \frac{2}{2\pi i n} B_n^{(m-1)} - \frac{1}{2\pi i n} \Omega_m \\
&= \frac{2}{2\pi i n} \left( \frac{2}{2\pi i n} B_n^{(m-2)} - \frac{1}{2\pi i n} \Omega_{m-1} \right) - \frac{1}{2\pi i n} \Omega_m \\
&= \left( \frac{2}{2\pi i n} \right)^2 B_n^{(m-2)} - \frac{2}{(2\pi i n)^2} \Omega_{m-1} - \frac{1}{2\pi i n} \Omega_m \\
&= \cdots \\
&= \left( \frac{2}{2\pi i n} \right)^{m-1} B_n^{(1)} - \sum_{j=1}^{m-1} \frac{2^{j-1}}{(2\pi i n)^j} \Omega_{m-j+1} \\
&= -\sum_{j=1}^{m-1} \frac{2^{j-1}}{(2\pi i n)^j} \Omega_{m-j+1},
\end{aligned} \tag{3.10}$$

where

$$B_n^{(1)} = \int_0^1 \beta_1(x) e^{-2\pi i n x} dx = \int_0^1 e^{-2\pi i n x} dx = 0. \tag{3.11}$$

Case 2: $n = 0$.

$$B_0^{(m)} = \int_0^1 \beta_m(x) dx = \frac{1}{2} \Omega_{m+1}. \tag{3.12}$$

$\beta_m(< x >), (m \geq 2)$ is piecewise $C^\infty$. Moreover, $\beta_m(< x >)$ is continuous for those integers $m \geq 2$ with $\Omega_m = 0$ and discontinuous with jump discontinuities at integers for those integers $m \geq 2$ with $\Omega_m \neq 0$.

Assume first that $\Omega_m = 0$, for an integer $m \geq 2$. Then $\beta_m(0) = \beta_m(1)$. $\beta_m(< x >)$ is piecewise $C^\infty$, and continuous. Thus the Fourier series of $\beta_m(< x >)$ converges uniformly to $\beta_m(< x >)$, and

$$\begin{aligned}
\beta_m(< x >) &= \frac{1}{2} \Omega_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \left( -\sum_{j=1}^{m-1} \frac{2^{j-1}}{(2\pi i n)^j} \Omega_{m-j+1} \right) e^{2\pi i n x} \\
&= \frac{1}{2} \Omega_{m+1} + \sum_{j=1}^{m-1} \frac{2^{j-1}}{j!} \Omega_{m-j+1} \left( -j! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^j} \right) \\
&= \frac{1}{2} \Omega_{m+1} + \sum_{j=2}^{m-1} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_j(< x >) + \Omega_m \times \begin{cases} B_1(< x >), & \text{for} \quad x \in \mathbb{Z}^c, \\ 0, & \text{for} \quad x \in \mathbb{Z}. \end{cases}
\end{aligned} \tag{3.13}$$

Now, we are ready to state our first theorem.

**Theorem 3.1.** *For each integer $l \geq 2$, let*

$$\Omega_l = -2\sum_{k=0}^{l-2} \frac{\mathbb{B}_k^{(r+1)} G_{l-k}}{k!(l-k)!} - \sum_{k=1}^{l-1} \frac{\mathbb{B}_{k-1}^{(r)} G_{l-k}}{k!(l-k)!} + 2\frac{\mathbb{B}_{l-2}^{(r)}}{(l-1)!}. \tag{3.14}$$

*Assume that $\Omega_m = 0$, for an integer $m \geq 2$. Then we have the following.*

*(a) $\sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} \mathbb{B}_k^{(r+1)}(< x >) G_{m-k}(< x >)$ has the Fourier series expansion*

$$\begin{aligned} &\sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} \mathbb{B}_k^{(r+1)}(< x >) G_{m-k}(< x >) \\ &= \frac{1}{2}\Omega_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \left( -\sum_{j=1}^{m-1} \frac{2^{j-1}}{(2\pi i n)^j} \Omega_{m-j+1} \right) e^{2\pi i n x}, \end{aligned} \tag{3.15}$$

*for all $x \in (-\infty, \infty)$, where the convergence is uniform.*

*(b)*

$$\begin{aligned} &\sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} \mathbb{B}_k^{(r+1)}(< x >) G_{m-k}(< x >) \\ &= \frac{1}{2}\Omega_{m+1} + \sum_{j=2}^{m-1} \frac{2^{j-1}}{j!} \Omega_{m-j+1} B_j(< x >), \end{aligned} \tag{3.16}$$

*for all $x \in (-\infty, \infty)$, where $B_j(< x >)$ is the Bernoulli function.*

Assume next that $\Omega_m \neq 0$, for an integer $m \geq 2$. Then $\beta_m(0) \neq \beta_m(1)$. Thus $\beta_m(< x >)$ is piecewise $C^\infty$ and discontinuous with jump discontinuities at integers. The Fourier series of $\beta_m(< x >)$ converges pointwise to $\beta_m(< x >)$, for $x \in \mathbb{Z}^c$, and converges to

$$\begin{aligned} &\frac{1}{2}(\beta_m(0) + \beta_m(1)) = \beta_m(0) + \frac{1}{2}\Omega_m \\ &\sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} \mathbb{B}_k^{(r+1)} G_{m-k} + \frac{1}{2}\Omega_m, \end{aligned} \tag{3.17}$$

for $x \in \mathbb{Z}$. We can now state our second theorem.

**Theorem 3.2.** *For each integer $l \geq 2$, let*

$$\Omega_l = -2\sum_{k=0}^{l-2} \frac{\mathbb{B}_k^{(r+1)} G_{l-k}}{k!(l-k)!} - \sum_{k=1}^{l-1} \frac{\mathbb{B}_{k-1}^{(r)} G_{l-k}}{k!(l-k)!} + 2\frac{\mathbb{B}_{l-2}^{(r)}}{(l-1)!}. \tag{3.18}$$

*Assume that $\Omega_m \neq 0$, for an integer $m \geq 2$. Then we have the following.*

*(a)*

$$\begin{aligned} &\frac{1}{2}\Omega_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \left( -\sum_{j=1}^{m-1} \frac{2^{j-1}}{(2\pi i n)^j} \Omega_{m-j+1} \right) e^{2\pi i n x} \\ &= \begin{cases} \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} \mathbb{B}_k^{(r+1)}(< x >) G_{m-k}(< x >), & \text{for} \quad x \in \mathbb{Z}^c, \\ \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} \mathbb{B}_k^{(r+1)} G_{m-k} + \frac{1}{2}\Omega_m, & \text{for} \quad x \in \mathbb{Z}. \end{cases} \end{aligned} \tag{3.19}$$

*Here the convergence is pointwise.*

*(b)*

$$\begin{aligned}&\frac{1}{2}\Omega_{m+1}+\sum_{j=1}^{m-1}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(<x>)\\&=\sum_{k=0}^{m-1}\frac{1}{k!(m-k)!}\mathbb{B}_k^{(r+1)}(<x>)G_{m-k}(<x>),\end{aligned}\tag{3.20}$$

*for $x\in\mathbb{Z}^c$;*

$$\begin{aligned}&\frac{1}{2}\Omega_{m+1}+\sum_{j=2}^{m-1}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(<x>)\\&=\sum_{k=0}^{m-1}\frac{1}{k!(m-k)!}\mathbb{B}_k^{(r+1)}G_{m-k}+\frac{1}{2}\Omega_m,\end{aligned}\tag{3.21}$$

*for $x\in\mathbb{Z}$. Here $B_j(<x>)$ is the Bernoulli function.*

## 4. Fourier series of functions of the third type

Let $\gamma_m(x)=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{B}_k^{(r+1)}(x)G_{m-k}(x),\ (m\geq 2)$. We observe the following.

$$\begin{aligned}\gamma_m'(x)&=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\left\{k\mathbb{B}_{k-1}^{(r+1)}(x)G_{m-k}(x)+(m-k)\mathbb{B}_k^{(r+1)}(x)G_{m-k-1}(x)\right\}\\&=\sum_{k=0}^{m-2}\frac{1}{m-k-1}\mathbb{B}_k^{(r+1)}(x)G_{m-k-1}(x)+\sum_{k=1}^{m-1}\frac{1}{k}\mathbb{B}_k^{(r+1)}(x)G_{m-k-1}(x)\\&=\frac{1}{m-1}G_{m-1}(x)+\sum_{k=1}^{m-2}\frac{1}{m-1-k}\mathbb{B}_k^{(r+1)}(x)G_{m-1-k}(x)+\sum_{k=1}^{m-2}\frac{1}{k}\mathbb{B}_k^{(r+1)}(x)G_{m-1-k}(x)\\&=\frac{1}{m-1}G_{m-1}(x)+(m-1)\sum_{k=1}^{m-2}\frac{1}{k(m-1-k)}\mathbb{B}_k^{(r+1)}(x)G_{m-1-k}(x)\\&=\frac{1}{m-1}G_{m-1}(x)+(m-1)\gamma_{m-1}(x).\end{aligned}\tag{4.1}$$

From this, we see that

$$\left(\frac{1}{m}(\gamma_{m+1}(x)-\frac{1}{m(m+1)}G_{m+1}(x))\right)'=\gamma_m(x)\tag{4.2}$$

and

$$\begin{aligned}
&\int_0^1 \gamma_m(x)dx \\
&= \frac{1}{m}\left[\gamma_{m+1}(x) - \frac{1}{m(m+1)}G_{m+1}(x)\right]_0^1 \\
&= \frac{1}{m}\left(\gamma_{m+1}(1) - \gamma_{m+1}(0) - \frac{1}{m(m+1)}(G_{m+1}(1) - G_{m+1}(0))\right) \\
&= \frac{1}{m}\left(\gamma_{m+1}(1) - \gamma_{m+1}(0) - \frac{1}{m(m+1)}(-2G_{m+1}(0) + 2\delta_{m,0})\right) \\
&= \frac{1}{m}\left(\gamma_{m+1}(1) - \gamma_{m+1}(0) + \frac{2G_{m+1}}{m(m+1)}\right).
\end{aligned} \tag{4.3}$$

For $m \geq 2$, we let

$$\begin{aligned}
&\Lambda_m = \Lambda_m(r) = \gamma_m(1) - \gamma_m(0) \\
&= \sum_{k=1}^{m-1} \frac{1}{k(m-k)}\left(\mathbb{B}_k^{(r+1)}(1)G_{m-k}(1) - \mathbb{B}_k^{(r+1)}G_{m-k}\right) \\
&= \sum_{k=1}^{m-1} \frac{1}{k(m-k)}\left(\left(\mathbb{B}_k^{(r+1)} + \mathbb{B}_{k-1}^{(r)}\right)(-G_{m-k} + 2\delta_{m-1,k}) - \mathbb{B}_k^{(r+1)}G_{m-k}\right) \\
&= \sum_{k=1}^{m-1} \frac{1}{k(m-k)}\left(-2\mathbb{B}_k^{(r+1)}G_{m-k} + 2\mathbb{B}_k^{(r+1)}\delta_{m-1,k} - \mathbb{B}_{k-1}^{(r)}G_{m-k} + 2\mathbb{B}_{k-1}^{(r)}\delta_{m-1,k}\right) \\
&= -2\sum_{k=1}^{m-1} \frac{1}{k(m-k)}\mathbb{B}_k^{(r+1)}G_{m-k} + \frac{2}{m-1}\mathbb{B}_{m-1}^{(r+1)} - \sum_{k=1}^{m-1} \frac{1}{k(m-k)}\mathbb{B}_{k-1}^{(r)}G_{m-k} + \frac{2}{m-1}\mathbb{B}_{m-2}^{(r)}.
\end{aligned} \tag{4.4}$$

So,

$$\gamma_m(1) = \gamma_m(0) \Longleftrightarrow \Lambda_m = 0. \tag{4.5}$$

Also,

$$\int_0^1 \gamma_m(x)dx = \frac{1}{m}\left(\Lambda_{m+1} + \frac{2}{m(m+1)}G_{m+1}\right). \tag{4.6}$$

We are now going to consider

$$\gamma_m(< x >) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)}\mathbb{B}_k^{(r+1)}(< x >)G_{m-k}(< x >), \tag{4.7}$$

defined on $(-\infty, \infty)$ , which is periodic with period 1. The Fourier series of $\gamma_m(< x >)$ is

$$\sum_{n=-\infty}^{\infty} C_n^{(m)} e^{2\pi i n x}, \tag{4.8}$$

where

$$C_n^{(m)} = \int_0^1 \gamma_m(< x >)e^{-2\pi i n x}dx = \int_0^1 \gamma_m(x)e^{-2\pi i n x}dx. \tag{4.9}$$

Now, we want to determine the Fourier coefficients $C_n^{(m)}$.

Case 1: $n \neq 0$.

$$\begin{aligned}
C_n^{(m)} &= \int_0^1 \gamma_m(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}[\gamma_m(x)e^{-2\pi i n x}]_0^1 + \frac{1}{2\pi i n}\int_0^1 \gamma_m'(x)e^{-2\pi i n x}dx \\
&= -\frac{1}{2\pi i n}(\gamma_m(1)-\gamma_m(0)) + \frac{1}{2\pi i n}\int_0^1 \left(\frac{1}{m-1}G_{m-1}(x) + (m-1)\gamma_{m-1}(x)\right)e^{-2\pi i n x}dx \\
&= -\frac{1}{2\pi i n}\Lambda_m + \frac{m-1}{2\pi i n}C_n^{(m-1)} + \frac{1}{2\pi i n(m-1)}\int_0^1 G_{m-1}(x)e^{-2\pi i n x}dx \\
&= -\frac{1}{2\pi i n}\Lambda_m + \frac{m-1}{2\pi i n}C_n^{(m-1)} + \frac{1}{2\pi i n(m-1)}\Phi_m,
\end{aligned} \tag{4.10}$$

where

$$\Phi_m = \sum_{k=1}^{m-2} \frac{(m-1)_{k-1}}{(2\pi i n)^k} G_{m-k}, \tag{4.11}$$

and one can show

$$\int_0^1 G_l(x)e^{-2\pi i n x}dx = \begin{cases} \sum_{k=1}^{l-1} \frac{2(l)_{k-1}}{(2\pi i n)^k} G_{l-k+1}, & \text{for} \quad n \neq 0, \\ -\frac{2G_{l+1}}{l+1}, & \text{for} \quad n = 0. \end{cases} \tag{4.12}$$

We observe that

$$\begin{aligned}
C_n^{(m)} &= \frac{m-1}{2\pi i n}C_n^{(m-1)} - \frac{1}{2\pi i n}\Lambda_m + \frac{2}{2\pi i n(m-1)}\Phi_m \\
&= \frac{m-1}{2\pi i n}\left(\frac{m-2}{2\pi i n}C_n^{(m-2)} - \frac{1}{2\pi i n}\Lambda_{m-1} + \frac{2}{2\pi i n(m-2)}\Phi_{m-1}\right) \\
&\quad - \frac{1}{2\pi i n}\Lambda_m + \frac{2}{2\pi i n(m-1)}\Phi_m \\
&= \frac{(m-1)(m-2)}{(2\pi i n)^2}C_n^{(m-2)} - \frac{m-1}{(2\pi i n)^2}\Lambda_{m-1} - \frac{1}{2\pi i n}\Lambda_m \\
&\quad + \frac{2(m-1)}{(2\pi i n)^2(m-2)}\Phi_{m-1} + \frac{2}{2\pi i n(m-1)}\Phi_m \\
&= \cdots \\
&= \frac{(m-1)_{m-2}}{(2\pi i n)^{m-2}}C_n^{(2)} - \sum_{j=1}^{m-2}\frac{(m-1)_{j-1}}{(2\pi i n)^j}\Lambda_{m-j+1} + \sum_{j=1}^{m-2}\frac{2(m-1)_{j-1}}{(2\pi i n)^j(m-j)}\Phi_{m-j+1} \\
&= -\frac{(m-1)!}{(2\pi i n)^{m-1}}\Lambda_2 - \sum_{j=1}^{m-2}\frac{(m-1)_{j-1}}{(2\pi i n)^j}\Lambda_{m-j+1} + \sum_{j=1}^{m-2}\frac{2(m-1)_{j-1}}{(2\pi i n)^j(m-j)}\Phi_{m-j+1} \\
&= -\frac{1}{m}\sum_{j=1}^{m-1}\frac{(m)_j}{(2\pi i n)^j}\Lambda_{m-j+1} + \frac{1}{m}\sum_{j=1}^{m-2}\frac{2(m)_j}{(2\pi i n)^j(m-j)}\Phi_{m-j+1},
\end{aligned} \tag{4.13}$$

where

$$\begin{aligned} C_n^{(2)} &= \int_0^1 \gamma_2(x)e^{-2\pi inx}dx \\ &= -\frac{1}{2\pi in}\left[\gamma_2(x)e^{-2\pi inx}\right]_0^1 + \frac{1}{2\pi in}\int_0^1 \gamma_2'(x)e^{-2\pi inx}dx \\ &= -\frac{1}{2\pi in}(\gamma_2(1)-\gamma_2(0)) = -\frac{1}{2\pi in}\Lambda_2. \end{aligned} \tag{4.14}$$

In order to get a final expression for $C_n^{(m)}$, we observe the following.

$$\begin{aligned} &\sum_{j=1}^{m-2}\frac{2(m)_j}{(2\pi in)^j(m-j)}\Phi_{m-j+1} \\ &= \sum_{j=1}^{m-2}\frac{2(m)_j}{(2\pi in)^j(m-j)}\sum_{k=1}^{m-j-1}\frac{(m-j)_{k-1}}{(2\pi in)^k}G_{m-j-k+1} \\ &= \sum_{j=1}^{m-2}\sum_{k=1}^{m-j-1}\frac{2(m)_{j+k-1}}{(2\pi in)^{j+k}(m-j)}G_{m-j-k+1} \\ &= 2\sum_{j=1}^{m-2}\frac{1}{m-j}\sum_{s=j+1}^{m-1}\frac{(m)_{s-1}}{(2\pi in)^s}G_{m-s+1} \\ &= 2\sum_{s=2}^{m-1}\frac{(m)_{s-1}}{(2\pi in)^s}G_{m-s+1}\sum_{j=1}^{s-1}\frac{1}{m-j} \\ &= 2\sum_{s=1}^{m-1}\frac{(m)_s}{(2\pi in)^s}\frac{G_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s}). \end{aligned} \tag{4.15}$$

Putting everything altogether, we have

$$\begin{aligned} C_n^{(m)} &= -\frac{1}{m}\sum_{s=1}^{m-1}\frac{(m)_s}{(2\pi in)^s}\Lambda_{m-s+1} + \frac{2}{m}\sum_{s=1}^{m-1}\frac{(m)_s}{(2\pi in)^s}\frac{G_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s}) \\ &= -\frac{1}{m}\sum_{s=1}^{m-1}\frac{(m)_s}{(2\pi in)^s}\left\{\Lambda_{m-s+1} - \frac{2G_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s})\right\}. \end{aligned} \tag{4.16}$$

Case 2: $n = 0$.

$$C_0^{(m)} = \int_0^1 \gamma_m(x)dx = \frac{1}{m}\left(\Lambda_{m+1} + \frac{2G_{m+1}}{m(m+1)}\right). \tag{4.17}$$

$\gamma_m(< x >), (m \geq 2)$ is piecewise $C^\infty$. In addition, $\gamma_m(< x >)$ is continuous for those integers $m \geq 2$ with $\Lambda_m = 0$, and discontinuous with jump discontinuities at integers for those integers $\Lambda_m \neq 0$.

Assume first that $\Lambda_m = 0$ . Then $\gamma_m(0) = \gamma_m(1)$. $\gamma_m(< x >)$ is piecewise $C^\infty$, and continuous. So the Fourier series of $\gamma_m(< x >)$ converges uniformly to $\gamma_m(< x >)$, and

$$\begin{aligned}
&\gamma_m(< x >) \\
&= \frac{1}{m}\left(\Lambda_{m+1} + \frac{2G_{m+1}}{m(m+1)}\right) \\
&\quad - \frac{1}{m}\sum_{n=-\infty, n\neq 0}^{\infty}\left\{\sum_{s=1}^{m-1}\frac{(m)_s}{(2\pi i n)^s}\left(\Lambda_{m-s+1} - \frac{2G_{m-s+1}}{m-s+1}(H_{m-1} - H_{m-s})\right)\right\} e^{2\pi i n x} \\
&= \frac{1}{m}\left(\Lambda_{m+1} + \frac{2G_{m+1}}{m(m+1)}\right) \\
&\quad + \frac{1}{m}\sum_{s=1}^{m-1}\binom{m}{s}\left(\Lambda_{m-s+1} - \frac{2G_{m-s+1}}{m-s+1}(H_{m-1} - H_{m-s})\right) \\
&\times \left(-s!\sum_{n=-\infty, n\neq 0}^{\infty}\frac{e^{2\pi i n x}}{(2\pi i n)^s}\right) \\
&= \frac{1}{m}\left(\Lambda_{m+1} + \frac{2G_{m+1}}{m(m+1)}\right) \\
&\quad + \frac{1}{m}\sum_{s=2}^{m-1}\binom{m}{s}\left(\Lambda_{m-s+1} - \frac{2G_{m-s+1}}{m-s+1}(H_{m-1} - H_{m-s})\right) B_s(< x >) \\
&\quad + \Lambda_m \times \begin{cases} B_1(< x >), & \text{for} \quad x \in \mathbb{Z}^c, \\ 0, & \text{for} \quad x \in \mathbb{Z}. \end{cases}
\end{aligned} \tag{4.18}$$

Now, we can state our first result.

**Theorem 4.1.** *For each integer $l \geq 2$, let*

$$\begin{aligned}
\Lambda_l = &-2\sum_{k=1}^{l-1}\frac{1}{k(l-k)}\mathbb{B}_k^{(r+1)}G_{l-k} + \frac{2}{l-1}\mathbb{B}_{l-1}^{(r+1)} \\
&- \sum_{k=1}^{l-1}\frac{1}{k(l-k)}\mathbb{B}_{k-1}^{(r)}G_{l-k} + \frac{2}{l-1}\mathbb{B}_{l-2}^{(r)}.
\end{aligned} \tag{4.19}$$

*Assume that $\Lambda_m = 0$, for an integer $m \geq 2$. Then we have the following.*

*(a) $\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{B}_k^{(r+1)}(< x >)G_{m-k}(< x >)$ has the Fourier series expansion*

$$\begin{aligned}
&\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{B}_k^{(r+1)}(< x >)G_{m-k}(< x >) \\
&= \frac{1}{m}\left(\Lambda_{m+1} + \frac{2G_{m+1}}{m(m+1)}\right) \\
&\quad - \frac{1}{m}\sum_{n=-\infty, n\neq 0}^{\infty}\left\{\sum_{s=1}^{m-1}\frac{(m)_s}{(2\pi i n)^s}\left(\Lambda_{m-s+1} - \frac{2G_{m-s+1}}{m-s+1}(H_{m-1} - H_{m-s})\right)\right\} e^{2\pi i n x},
\end{aligned} \tag{4.20}$$

*for all $x \in (-\infty, \infty)$. Here the convergence is uniform.*

*(b)*

$$\begin{aligned}&\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{B}_k^{(r+1)}(<x>)G_{m-k}(<x>)\\&=\frac{1}{m}\left(\Lambda_{m+1}+\frac{2}{m(m+1)}G_{m+1}\right)\\&\quad+\frac{1}{m}\sum_{s=2}^{m-1}\binom{m}{s}\left(\Lambda_{m-s+1}-\frac{2G_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s})\right)B_s(<x>)\end{aligned}\tag{4.21}$$

*for all $x\in(-\infty,\infty)$. Here $B_s(<x>)$ is the Bernoulli function.*

Assume next that $m\geq 2$ is an integer with $\Lambda_m\neq 0$. Then $\gamma_m(0)\neq\gamma_m(1)$. $\gamma_m(<x>)$ is piecewise $C^\infty$ and discontinuous with jump discontinuities at integers. Thus the Fourier series of $\gamma_m(<x>)$ converges pointwise to $\gamma_m(<x>)$, for $x\in\mathbb{Z}^c$, and converges to

$$\begin{aligned}\frac{1}{2}(\gamma_m(0)+\gamma_m(1))&=\gamma_m(0)+\frac{1}{2}\Lambda_m\\&=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{B}_k^{(r+1)}G_{m-k}+\frac{1}{2}\Lambda_m,\end{aligned}\tag{4.22}$$

for $x\in\mathbb{Z}$. Next, we can state our second result.

**Theorem 4.2.** *For each integer $l\geq 2$, let*

$$\Lambda_l=-2\sum_{k=1}^{l-1}\frac{1}{k(l-k)}\mathbb{B}_k^{(r+1)}G_{l-k}+\frac{2}{l-1}\mathbb{B}_{l-1}^{(r+1)}-\sum_{k=1}^{l-1}\frac{1}{k(l-k)}\mathbb{B}_{k-1}^{(r)}G_{l-k}+\frac{2}{l-1}\mathbb{B}_{l-2}^{(r)}.\tag{4.23}$$

*Assume that $\Lambda_m\neq 0$, for an integer $m\geq 2$. Then we have the following.*

*(a)*

$$\begin{aligned}&\frac{1}{m}\left(\Lambda_{m+1}+\frac{2G_{m+1}}{m(m+1)}\right)\\&\quad-\frac{1}{m}\sum_{n=-\infty,n\neq 0}^{\infty}\left(\sum_{s=1}^{m-1}\frac{(m)_s}{(2\pi in)^s}\left(\Lambda_{m-s+1}-\frac{2G_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s})\right)\right)e^{2\pi inx}\\&=\begin{cases}\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{B}_k^{(r+1)}(<x>)G_{m-k}(<x>), & \text{for}\quad x\in\mathbb{Z}^c,\\ \sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{B}_k^{(r+1)}G_{m-k}+\frac{1}{2}\Lambda_m, & \text{for}\quad x\in\mathbb{Z}.\end{cases}\end{aligned}\tag{4.24}$$

*Here the convergence is pointwise.*

*(b)*

$$\begin{aligned}&\frac{1}{m}\left(\Lambda_{m+1}+\frac{2G_{m+1}}{m(m+1)}\right)\\&\quad+\frac{1}{m}\sum_{s=1}^{m-1}\binom{m}{s}\left(\Lambda_{m-s+1}-\frac{2G_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s})\right)B_s(<x>)\\&=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{B}_k^{(r+1)}(<x>)G_{m-k}(<x>),\end{aligned}\tag{4.25}$$

*for* $x \in \mathbb{Z}^c$ *and*

$$\begin{aligned}
&\frac{1}{m}\left(\Lambda_{m+1}+\frac{2G_{m+1}}{m(m+1)}\right)\\
&\quad+\frac{1}{m}\sum_{s=2}^{m-1}\binom{m}{s}\left(\Lambda_{m-s+1}-\frac{2G_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s})\right)B_s(<x>)\\
&\sum_{s=1}^{m-1}\frac{1}{k(m-k)}\mathbb{B}_k^{(r+1)}G_{m-k}+\frac{1}{2}\Lambda_m,
\end{aligned} \tag{4.26}$$

*for* $x \in \mathbb{Z}$.

## References


1. T. Arakawa, M. Kaneko, *On poly-Bernoulli numbers,* Comment. Math. Univ. St. Paul. **48**(1999), no. 2, 159–167.
2. A. Bayad, Y. Hamahata, *Multiple polylogarithms and multi-poly-Bernoulli polynomials,* Funct. Approx. Comment. Math.**46**(2012), part 1, 45–61.
3. D. V. Dolgy, D. S. Kim, T. Kim, T. Mansour, *Degenerate poly-Bernoulli polynomials of the second kind,* J. Comput. Anal. Appl. **21**(2016), no.5, 954–966.
4. D. V. Dolgy, D. S. Kim, T. Kim,T. Mansour, *Degenerate poly-Cauchy polynomials,* Appl. Math. Comput. **269**(2015), 637–646.
5. G. V. Dunne, C. Schubert, *Bernoulli number identities from quantum field theory and topological string theory,* Commun. Number Theory Phys. **7(2)**(2013), 225-249.
6. C. Faber, R. Pandharipande, *Hodge integrals and Gromov-Witten theory,* Invent. Math. **139(1)**(2000), 173-199.
7. M. Kaneko, *Poly-Bernoulli numbers,* J. Theor. Nombres Bordeaux **9**(1997), no. 1, 221–228.
8. D. S. Kim, T. Kim, *A note on degenerate poly-Bernoulli numbers and polynomials,* Adv. Difference Equ. 2015, 2015:258, 8 pp.
9. D. S. Kim, T. Kim, *A note on poly-Bernoulli and higher-order poly-Bernoulli polynomials,* Russ. J. Math. Phys., **22(1)** (2015), 26–33.
10. D. S. Kim, T. Kim, T. Mansour, J.-J. Seo, *Fully degenerate poly-Bernoulli polynomials with a q parameter,* Filomat **30**(2016), no.4, 1029–1035.
11. D.S. Kim, T. Kim, *A note on higher-order Bernoulli polynomials,* J. Inequal. Appl. 2013, 2013:111.
12. D. S. Kim, T. Kim, *Higher-order Bernoulli and poly-Bernoulli mixed type polynomials,* Georgian Math. J. **22**(2015), no. 2, 265–272.
13. D. S. Kim, T. Kim, H. I. Kwon, T. Mansour, *Degenerate poly-Bernoulli polynomials with umbral calculus viewpoint,* J. Inequal. Appl. 2015, 2015:228, 13 pp.
14. T. Kim, *On the Multiple q-Genocchi and Euler Numbers*, Russ. J. Math. Phys. **15**(2008), 481-486.
15. T. Kim, *A note on the q-Genocchi Numbers and polynomials*, J. Inequalities and applications, **2007** Article ID 71452, 8pages,(2007).
16. T. Kim, *Some identities for the Bernoulli, the Euler and Genocchi numbers and polynomials*, Adv. Stud. Contemp. Math. **20**(2015), no.1, 23-28.
17. T. Kim, D. S. Kim, *Fully degenerate poly-Bernoulli numbers and polynomials,* Open Math. **14**(2016), 545–556.
18. T. Kim, D. S. Kim, T. Komatsu, S.-H. Lee,*Higher-order Daehee of the second kind and poly-Cauchy of the second kind mixed-type polynomials,* J. Nonlinear Convex Anal. **16**(2015), no. 10, 1993–2015.
19. T. Kim, D.S. Kim, S.-H. Rim, D.-V. Dolgy, *Fourier series of higher-order Bernoulli functions and their applications,* J. Inequalities and Applications **2017** (2017), 2017:8 Pages.
20. J. E. Marsden, *Elementary classical analysis,* W. H. Freeman and Company, 1974.
21. K. Shiratani, S. Yokoyama, *An application of p-adic convolutions,* Mem. Fac. Sci. Kyushu Univ. Ser. A **36(1)**(1982), 7383.
22. D. G. Zill, M. R. Cullen, *Advanced Engineering Mathematics,* Jones and Bartlett Publishers 2006.

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# Convergence of the Newton-HSS Method under the Lipschitz Condition with the L-average

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**Abstract:** Under the hypothesis that the Jacobian matrix satisfies the center Lipschitz condition with the L-average, we prove the local convergence of the Newton-HSS method, which is used to solve large sparse systems of nonlinear equations with positive definite Jacobian matrices at the solution points. Numerical results are given to examine its feasibility and effectiveness.

**Keywords:** Large sparse systems; Nonlinear equations; Newton-HSS method; Center Lipschtiz condition with the L-average.

**AMS classifications:** 65F10, 65F50, 65W05,

# 1 Introduction

In this paper, we consider the following system of nonlinear equations

$$F(x) = 0, \tag{1.1}$$

where $F : \mathbb{D} \subset \mathbb{C}^n \to \mathbb{C}^n$ is nonlinear and continuously differentiable, $\mathbb{D}$ is an open convex subset of the n-dimensional complex linear space $\mathbb{C}^n$. The Jacobian matrix $F^{'}(x) \in \mathbb{C}^{n\times n}$ is sparse, nonsymmetric and positive definite. This kind of nonlinear equations can be derived in many areas of scientific computing and engineering applications [1, 2, 3].

The most classic and important iterative method for the system of nonlinear equations (1.1) is Newton's method [14, 15], which can be formulated as

$$x_{k+1} = x_k - F^{'}(x_k)^{-1}F(x_k), \quad k = 0, 1, \cdots,$$

where $x_0 \in \mathbb{D}$ is a given initial vector. Obviously, at the $k$-th iteration step, it is necessary to solve the so-called Newton equation

$$F^{'}(x_k)s_k = -F(x_k), \tag{1.2}$$

which is the dominant task in implementations of the Newton method, then get the $k+1$-th iterative vector $x_{k+1} = x_k + s_k$. Bai and Guo [5], Guo and Duff [10], first used the HSS iteration [4] to solve approximately the Newton equation (1.2), and used inexact Newton method [8] as the outer solver, presented the Newton-HSS method for solving

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the system of nonlinear equations (1.1), and gave the convergence theorems under the Lipschtiz continuous conditions. The following is HSS iterative method, which is used to solve non-Hermitian positive-definite linear system $Ax = b$ [4].

**Algorithm 1.1. HSS**

**1.** *Given an initial guess $x_0 \in \mathbb{C}^n$, and positive constant $\alpha$.*

**2.** *Split the linear matrix $A$ into its Hermitian part $H$ and skew-Hermitian part $S$*

$$H = \frac{1}{2}(A + A^*) \ \ and \ \ S = \frac{1}{2}(A - A^*).$$

**3.** *For $k = 0, 1, 2, \cdots$, compute $x_{k+1}$ using the following iteration scheme until $\{x_k\}$ satisfies the stopping criterion:*

$$\begin{cases} (\alpha I + H)x_{k+\frac{1}{2}} = (\alpha I - S)x_k + b, \\ (\alpha I + S)x_{k+1} = (\alpha I - H)x_{k+\frac{1}{2}} + b, \end{cases} \tag{1.3}$$

*where $I$ denotes the identity matrix.*

Recently, using HSS method as the inner solver for solving Newton equations (1.2), and the modified Newton method as the outer solver, Chen et al. have proposed the modified Newton-HSS method for the system of nonlinear equations (1.1). They have proved the convergence theorems under Hölder continuous condition, which is weaker than the usual Lipschtiz condition. When using Newton's method to solve equations (1.1), Guo [11] studied its semi-local convergence property, which is as brief as Newton-Kantorovich theorem [9, 12], under the hypothese that the derivative satisfies center Lipschtiz condition with the L-average, which is weaker than Hölder condition and Lipschtiz condition, and has gotten a lot of attention and been extensively studied [13, 17, 18].

The following conditions were introduced by Wang in [13], named the center Lipschitz condition with the L-average.

**Definition 1.1.** *Let $Y$ be a Banach space and let $x_* \in \mathbb{C}^n$. Let $G$ be a mapping from $\mathbb{C}^n$ to $Y$. Then $G$ is said to satisfy the center Lipschitz condition with the L-average on $B(x_*, r)$ if*

$$\|G(x) - G(x_*)\| \leq \int_0^{\|x-x_*\|} L(u)du \quad for \quad each \quad x \in B(x_*, r);$$

In this paper, motivated by the idea of [11], the main work is to study the local convergence theorem of the Newton-HSS method under the hypothese that the derivative satisfies the center Lipschitz condition with the L-average. The organization of the paper is as follows. In Section 2, we introduce the Newton-HSS iterative method. In Section 3, we first give some lemmas which are useful for our main result, then present the new local convergence theorems under the hypothese that the derivative satisfies the center Lipschitz condition with the L-average. An numerical example is given to illustrate the applications of the results in our paper in Section 4. Finally, in Section 5, some conclusions are given.

## 2 Newton-HSS iteration

For Jacobian matrix $F^{'}(x)$, let

$$H(x) = \frac{1}{2}(F^{'}(x) + F^{'}(x)^{*})$$

be its Hermitian part,

$$S(x) = \frac{1}{2}(F^{'}(x) - F^{'}(x)^{*})$$

be the skew-Hermitian part, the following is the Newton-HSS method [5, 10].

**Algorithm 2.1. Newton-HSS**
**1.** *Given an initial guess $x_0$, positive constants $\alpha$ and tol, and positive integer sequence $\{l_k\}_{k=0}^{\infty}$;*
**2.** *for $k = 0, 1, \cdots$ until $\|F(x_k)\| \leq tol\|F(x_0)\|$ do:*

**2.1.** *Set $d_{k,0} = 0$;*
**2.2.** *for $l = 0, 1, \cdots, l_k - 1$, apply Algorithm HSS to the linear system (1.2):*

$$\begin{cases}(\alpha I + H(x_k))d_{k,l+\frac{1}{2}} = (\alpha I - S(x_k))d_{k,l} - F(x_k),\\ (\alpha I + S(x_k))d_{k,l+1} = (\alpha I - H(x_k))d_{k,l+\frac{1}{2}} - F(x_k),\end{cases}$$

*and obtain $d_{k,l_k}$ such that*

$$\|F(x_k) + F^{'}(x_k)d_{k,l_k}\| \leq \eta_k\|F(x_k)\| \qquad \eta_k \in [0, 1). \tag{2.1}$$

**2.3.** *Set $x_{k+1} = x_k + d_{k,l_k}$.*

Denote

$$\begin{aligned} B(\alpha; x) &= \frac{1}{2\alpha}(\alpha I + H(x))(\alpha I + S(x)),\\ C(\alpha; x) &= \frac{1}{2\alpha}(\alpha I - H(x))(\alpha I - S(x)),\\ T(\alpha; x) &= (\alpha I + S(x))^{-1}(\alpha I - H(x))(\alpha I + H(x))^{-1}(\alpha I - S(x)).\end{aligned} \tag{2.2}$$

Thus we have the following formulas

$$\begin{aligned} T(\alpha; x) &= B(\alpha; x)^{-1}C(\alpha; x),\\ F^{'}(x) &= B(\alpha; x) - C(\alpha; x),\\ F^{'}(x)^{-1} &= (I - T(\alpha; x))^{-1}B(\alpha; x)^{-1}.\end{aligned} \tag{2.3}$$

From the Newton-HSS method we can get [10]

$$x_{k+1} = x_k - (I - T_k^{l_k})F^{'}(x_k)^{-1}F(x_k), \tag{2.4}$$

here $T_k := T(\alpha; x_k)$.

# 3 Local convergence theorem under the center Lipschitz condition with the L-average

In this section, we establish a new local convergence theorem for the Newton-HSS method under the assumption that the derivative satisfies the center Lipschitz condition with the L-average, which is weaker than Hölder condition and Lipschtiz condition. Firstly, we give the assumption.

**Assumption 3.1.** *Let $F : \mathbb{D} \subset \mathbb{C}^n \to \mathbb{C}^n$ be G-differentiable on an open neighborhood $\mathbb{N}_0 \subset \mathbb{D}$ of a point $x_* \in \mathbb{D}$ at which $F'(x)$ is continuous, positive definite, and $F(x_*) = 0$. Assume the following conditions hold for all $x \in B(x_*, r) \subset \mathbb{N}_0$, where $B(x_*, r)$ denotes an open ball centered at $x_*$ with radius $r$:*
*(A1) (The Bounded Condition) there exist positive constants $\beta$, $\gamma$ and $\delta$ such that*

$$max\{\|H(x_*)\|, \|S(x_*)\|\} \leq \beta, \ \ \|F^{'}(x_*)^{-1}\| \leq \gamma.$$

*(A2) (The Center Lipschitz Condition with the L-average) there exist positive integrable functions $L_h(u)$ and $L_s(u)$ such that,*

$$\|H(x) - H(x_*)\| \leq \int_0^{\rho(x)} L_h(u)du,$$
$$\|S(x) - S(x_*)\| \leq \int_0^{\rho(x)} L_s(u)du,$$

*here $\rho(x) = \|x - x_*\|$.*

Let $L(u) = L_h(u) + L_s(u)$, thus $L(u)$ is a positive valued integrable function on $[0, +\infty)$. Before giving the main theorem, we list a series of useful lemmas as follows for our purpose. Lemma 3.1 is taken from [8], and we will give a proof of Lemma 3.2.

**Lemma 3.1.** *Define $\chi(t) = \frac{1}{t}\int_0^t L(u)(t-u)du, \ \ t \geq 0$, Then $\chi$ is increasing on $[0, +\infty)$.*

**Lemma 3.2.** *Under Assumption 3.1, if $\gamma \int_0^r L(u)du < 1$, then for $x \in B(x_*, r) \subset \mathbb{N}_0$, $F'(x)^{-1}$ exists, and*

(1) $\|F'(x) - F'(x_*)\| \leq \int_0^{\rho(x)} L(u)du,$

(2) $\|F'(x)^{-1}\| \leq \dfrac{\gamma}{1 - \gamma\int_0^{\rho(x)} L(u)du},$

(3) $\|F(x)\| \leq (\dfrac{1}{\rho(x)}\int_0^{\rho(x)} L(u)(\rho(x) - u)du + 2\beta)\|x - x_*\|,$

(4) $\|x - x_* - F'(x)^{-1}F(x)\|$
$\leq \dfrac{\gamma}{1 - \gamma\int_0^{\rho(x)} L(u)du}(\dfrac{1}{\rho(x)}\int_0^{\rho(x)} L(u)(\rho(x) - u)du + \int_0^{\rho(x)} L(u)du)\|x - x_*\|.$

*Proof.* By Assumption 3.1, $F'(x) = H(x) + S(x)$, perturbation lemma [14], and condition $\gamma \int_0^r L(u)du < 1$, it is easy to get (1) and (2).

For (3), from integral mean-value theorem and Assumption 3.1, we first have

$$\begin{aligned} &\|F(x) - F(x_*) - F'(x_*)(x - x_*)\| \\ =&\| \int_0^1 (F'(x + t(x - x_*)) - F'(x_*))dt(x - x_*)\| \\ \leq&\int_0^1 \int_0^{t\rho(x)} L(u)dudt\|x - x_*\| \\ =&\frac{1}{\rho(x)} \int_0^{\rho(x)} L(u)(\rho(x) - u)du\|x - x_*\|, \end{aligned} \tag{3.5}$$

thus, together with $\|F'(x_*)\| = \|H(x_*) + S(x_*)\| \leq 2\beta$, we have

$$\begin{aligned} \|F(x)\| &\leq \|F(x) - F(x_*) - F'(x_*)(x - x_*)\| + \|F'(x_*)(x - x_*)\| \\ &\leq (\frac{1}{\rho(x)} \int_0^{\rho(x)} L(u)(\rho(x) - u)du + 2\beta)\|x - x_*\|. \end{aligned}$$

For (4), by integral mean-value theorem, Assumption 3.1, and (3.5), we can get

$$\begin{aligned} &\|x - x_* - F'(x)^{-1}F(x)\| \\ =&\| - F'(x)^{-1}(F(x) - F(x_*) - F'(x_*)(x - x_*) + F'(x_*)(x - x_*) - F'(x)(x - x_*))\| \\ \leq&\|F'(x)^{-1}\|(\|F(x) - F(x_*) - F'(x_*)(x - x_*)\| + \|F'(x_*) - F'(x)\|\|x - x_*\|) \\ \leq&\frac{\gamma}{1 - \gamma \int_0^{\rho(x)} L(u)du}(\frac{1}{\rho(x)} \int_0^{\rho(x)} L(u)(\rho(x) - u)du + \int_0^{\rho(x)} L(u)du)\|x - x_*\|. \end{aligned}$$

□

Then we can give the following local convergence theorem of the Newton-HSS method under the center Lipschtiz condition with the L-average .

**Theorem 3.1.** *Assume that Assumption 3.1 holds with $r \in (0, r_*)$, here $r_*$ is defined by $r_* := \min\{r_1, r_2\}$, where $r_1$ and $r_2$ satisfy*

$$\int_0^{r_1} L(u)du = 2(\alpha + \beta)\big(\sqrt{\frac{\alpha\tau\theta}{(2 + \tau\theta)\gamma(\alpha + \beta)^2} + 1} - 1\big), \tag{3.6}$$

$$\frac{1}{r_2} \int_0^{r_2} L(u)(2r_2 - u)du = \frac{1 - 2\beta\gamma((\tau + 1)\theta)^{l_*}}{2\gamma}, \tag{3.7}$$

*and with $l_* = \liminf_{k\to\infty} l_k$ satisfying*

$$l_* > \lfloor \frac{\ln 2\beta\gamma}{\ln((\tau + 1)\theta)} \rfloor, \tag{3.8}$$

*where the symbol $\lfloor\cdot\rfloor$ is used to denote the smallest integer no less than the corresponding real number, $\tau \in (0, (1-\theta)/\theta)$, and*

$$\theta \equiv \theta(\alpha; x_*) = \|T(\alpha; x_*)\| \leq \max_{\lambda\in\sigma(H(x_*))} \frac{|\alpha-\lambda|}{\alpha+\lambda} \equiv \sigma(\alpha; x_*) < 1.$$

*Then, for any $x_0 \in B(x_*, r)$, and any sequence $\{l_k\}_{k=0}^{\infty}$, the iteration sequence $\{x_k\}_{k=0}^{\infty}$ generated by Algorithm Newton-HSS is well defined and converges to $x_*$. Moreover, it holds that*

$$\limsup_{k\to\infty} \|x_k - x_*\|^{\frac{1}{k}} \leq g(r_*; l_*),$$

*here,*

$$g(t,l) := \frac{\gamma}{1-\gamma\int_0^t L(u)du}\left(\frac{2}{t}\int_0^t L(u)(t-u)du + \int_0^t L(u)du + 2\beta\gamma((\tau+1)\theta)^l\right).$$

*Proof.* First of all, we will show the following estimate about the iterative matrix $T(\alpha; x)$ of the linear solver: if $x \in B(x_*, r)$, then

$$\|T(\alpha; x)\| < (\tau+1)\theta < 1.$$

In fact, from the definition of $B(\alpha; x)$ in (2.2) and Assumption 3.1, denote $\rho(x) = \|x - x_*\|$, we can get

$$\begin{aligned}
&\|B(\alpha; x) - B(\alpha; x_*)\| \\
\leq&\frac{1}{2}\|H(x) - H(x_*) + S(x) - S(x_*)\| + \frac{1}{2\alpha}\|H(x)S(x) - H(x_*)S(x_*)\| \\
\leq&\frac{1}{2}\int_0^{\rho(x)} L(u)du \\
&+ \frac{1}{2\alpha}\|(H(x) - H(x_*) + H(x_*))(S(x) - S(x_*)) + (H(x) - H(x_*))S(x_*)\| \\
\leq&\frac{1}{2}\int_0^{\rho(x)} L(u)du + \frac{1}{2\alpha}[(\int_0^{\rho(x)} L_h(u)du + \beta)\int_0^{\rho(x)} L_s(u)du + \beta\int_0^{\rho(x)} L_h(u)du] \\
\leq&\frac{1}{2}\int_0^{\rho(x)} L(u)du + \frac{1}{2\alpha}(\frac{(\int_0^{\rho(x)} L(u)du)^2}{4} + \beta\int_0^{\rho(x)} L(u)du) \\
\leq&\frac{1}{8\alpha}(\int_0^{\rho(x)} L(u)du)^2 + \frac{\alpha+\beta}{2\alpha}\int_0^{\rho(x)} L(u)du.
\end{aligned} \tag{3.9}$$

Similarly, we have

$$\|C(\alpha; x) - C(\alpha; x_*)\| \leq \frac{1}{8\alpha}(\int_0^{\rho(x)} L(u)du)^2 + \frac{\alpha+\beta}{2\alpha}\int_0^{\rho(x)} L(u)du. \tag{3.10}$$

Then from (2.3), it follows that

$$\|B(\alpha; x_*)^{-1}\| = \|(I - T(\alpha; x_*))F'(x_*)^{-1}\| \leq (1+\theta)\gamma < 2\gamma.$$

Therefore from (3.6), we obtain

$$\begin{aligned}
&\|I - B(\alpha; x_*)^{-1}B(\alpha; x)\| \\
&\leq \|B(\alpha; x_*)^{-1}\| \cdot \|B(\alpha; x) - B(\alpha; x_*)\| \\
&\leq \gamma \frac{(\int_0^{\rho(x)} L(u)du)^2 + 4(\alpha+\beta)\int_0^{\rho(x)} L(u)du}{4\alpha} \\
&< 1.
\end{aligned}$$

Hence using the perturbation lemma, we get $B(\alpha; x)^{-1}$ exists, and

$$\begin{aligned}
&\|B(\alpha; x)^{-1}\| \\
&\leq \frac{\|B(\alpha; x_*)^{-1}\|}{1 - \|I - B(\alpha; x_*)^{-1}B(\alpha; x)\|} \\
&\leq \frac{8\alpha\gamma}{4\alpha - \gamma[(\int_0^{\rho(x)} L(u)du)^2 + 4(\alpha+\beta)\int_0^{\rho(x)} L(u)du]}.
\end{aligned} \tag{3.11}$$

Hence, together with (3.6), (3.9)-(3.11), the estimate about the gap between inner iterative matrix $T(\alpha; x)$ and $T(\alpha; x_0)$ is obtained as follows:

$$\begin{aligned}
&\|T(\alpha; x) - T(\alpha; x_*)\| \\
&= \|B(\alpha; x)^{-1}(C(\alpha; x) - C(\alpha; x_*)) - B(\alpha; x)^{-1}(B(\alpha; x) - B(\alpha; x_*))B(\alpha; x_*)^{-1}C(\alpha; x_*)\| \\
&\leq \frac{2\gamma[(\int_0^{\rho(x)} L(u)du)^2 + 4(\alpha+\beta)\int_0^{\rho(x)} L(u)du]}{4\alpha - \gamma[(\int_0^{\rho(x)} L(u)du)^2 + 4(\alpha+\beta)\int_0^{\rho(x)} L(u)du]} \\
&< \tau\theta.
\end{aligned}$$

Consequently,

$$\|T(\alpha; x)\| \leq \|T(\alpha; x) - T(\alpha; x_*)\| + \|T(\alpha; x_*)\| < (\tau+1)\theta < 1. \tag{3.12}$$

Next, we turn to estimate the error about the Newton-HSS iteration sequence $\{x_k\}$ defined by (2.4). Clearly, from $\int_0^{\rho(x)} L(u)du < \frac{1}{\rho(x)}\int_0^{\rho(x)} L(u)(2\rho(x) - u)du$ and Lemma 3.2, it holds that $\gamma\int_0^{\rho(x)} L(u)du < 1$, hence, using Lemma 3.1, Lemma 3.2, (3.7), (3.8)

and (3.12), we obtain

$$\begin{aligned}
&\|x_{k+1}-x_*\|\\
=&\|x_k-x_*-F'(x_k)^{-1}F(x_k)+T_k^{l_k}F(x_k)^{-1}F(x_k)\|\\
\le&\|x_k-x_*-F'(x_k)^{-1}F(x_k)\|+\|T_k^{l_k}\|\|F(x_k)^{-1}\|\|F(x_k)\|\\
\le&\frac{\gamma}{1-\gamma\int_0^{\rho(x_k)}L(u)du}(\frac{1}{\rho(x_k)}\int_0^{\rho(x_k)}L(u)(\rho(x_k)-u)du+\int_0^{\rho(x_k)}L(u)du)\|x_k-x_*\|\\
&+\frac{((\tau+1)\theta)^{l_k}\gamma}{1-\gamma\int_0^{\rho(x_k)}L(u)du}(\frac{1}{\rho(x_k)}\int_0^{\rho(x_k)}L(u)(\rho(x_k)-u)du+2\beta)\|x_k-x_*\|\\
\le&\frac{\gamma}{1-\gamma\int_0^{\rho(x_k)}L(u)du}(\frac{2}{\rho(x_k)}\int_0^{\rho(x_k)}L(u)(\rho(x_k)-u)du+\int_0^{\rho(x_k)}L(u)du\\
&+2\beta\gamma((\tau+1)\theta)^{l_k})\|x_k-x_*\|\\
:=&g(\rho(x_k);l_k)\|x_k-x_*\|\\
\le&g(r_*,l_*)\|x_k-x_*\|\\
\le&g(r_2,l_*)\|x_k-x_*\|\\
<&\|x_k-x_*\|,
\end{aligned}$$

when $x_k\in B(x_*,r_*)$, here, we have used the notation

$$g(t,l):=\frac{\gamma}{1-\gamma\int_0^t L(u)du}(\frac{2}{t}\int_0^t L(u)(t-u)du+\int_0^t L(u)du+2\beta\gamma((\tau+1)\theta)^l).$$

Thus, we can further prove that $\{x_k\}\subset B(x_*,r)$ with the estimates

$$\|x_{k+1}-x_*\|\le g(r_*,l_*)\|x_k-x_*\|,\quad k=0,1,2,\cdots. \tag{3.13}$$

In fact, for $k=0$ we have $\|x_0-x_*\|<r$, as $x_0\in B(x_*,r)$. Together with $g(r_*,l_*)<1$, it follows from (3.13) that

$$\|x_1-x_*\|\le g(r_*,l_*)\|x_0-x_*\|<r,$$

hence, $x_1\in B(x_*,r)$. Suppose that $x_k\in B(x_*,r)$, then using (3.13) again, we have

$$\|x_{k+1}-x_*\|\le g(r_*,l_*)\|x_k-x_*\|<r,$$

hence, $x_{k+1}\in B(x_*,r)$. Moreover, we have

$$\|x_k-x_*\|\le g(r_*,l_*)\|x_k-x_*\|\le g(r_*,l_*)^{k+1}\|x_0-x_*\|.$$

Now the proof is complete. □

**Remark 1.** *If we assume the integrable functions $L_h(u)$ and $L_s(u)$ in (A2) are positive constants $L_h$ and $L_s$, respectively, then the center Lipschtiz condition with the L-average becomes usual Lipschtiz condition. If both integrable functions are $\frac{L}{p}u^{p-1}$, where $0 < p < 1$, $L$ is a positive constant, then the center Lipschtiz condition with the L-average becomes usual Hölder condition. Therefore, Theorem 3.1 is an extension of Theorem 3.2 in [10]. If the outer solver, Newton method, is changed to the modified Newton method, then 3.1 is an extension of Theorem 3.1 in [7].*

# 4 Application

In this section, we apply the main result on a two-demensional nonlinear convection-diffusion equation.

**Example 1.** Consider the following two-dimensional nonlinear convection-diffusion equation

$$\begin{cases} -(u_{xx} + u_{yy}) + q_1 u_x + q_2 u_y = u^c, & (x, y) \in \Omega, \\ u(x, y) = 0, & on\ (x, y) \in \partial\Omega, \end{cases} \tag{4.14}$$

where $c$ is a rational number, $\Omega = (0, 1) \times (0, 1)$, $\partial\Omega$ is the boundary. $q_1$ and $q_2$ are positive constants used to measure magnitudes of the convective terms. By applying the centered finite difference scheme on the equidistant discretization grid with the stepsize $h = 1/(N + 1)$, the system of nonlinear equations (1.1) is obtained with the following form

$$F(x) = Mx + h^2\phi(x) = 0,$$

where $N$ is a prescribed positive integer,

$$M = (T_x \otimes I + I \otimes T_y),$$
$$\phi(x) = (x_1^c, x_2^c, \cdots, x_n^c)^T,$$

with $T_x = tridiag(-1 - Re_1, 2, -1 + Re_1)$, $T_y = tridiag(-1 - Re_2, 2, -1 + Re_2)$, here, $Re_j = \frac{1}{2}q_j h$, $j = 1, 2$, $Re = \max\{Re_1, Re_2\}$ is the mesh Reynolds number, $\otimes$ is the Kronecker product, and $n = N \times N$.

Obviously, $x_* = 0$ is a solution of (4.14), and it is easy to get $F'(x) = M + ch^2 diag(x_1^{c-1}, x_2^{c-1}, \cdots, x_n^{c-1})$. Hence $F'(x_*) = M$. Moreover, we have

$$\|F'(x) - F'(x_*)\| \le ch^2\|x - x_*\|^{c-1} = \int_0^{\rho(x)} L(u)du, \quad x \in B(x_*, r) \subset \Omega,$$

where $\|\cdot\|$ denotes the 2-norm, $L(u) = c(c - 1)h^2u^{c-2}$. Hence, the center Lipschtiz condition with the L average is satisfied.

Thus we can obtain the convergence result of nonlinear equation (4.14).

**Corollary 4.1.** *Consider (4.14), define $r_* := \min\{r_1, r_2\}$, where $r_1$ and $r_2$ satisfy*

$$r_1^{c-1} = \frac{2(\alpha + \beta)}{ch^2}\Big(\sqrt{\frac{\alpha\tau\theta}{(2 + \tau\theta)\gamma(\alpha + \beta)^2} + 1} - 1\Big),$$

$$r_2^{c-1} = \frac{1 - 2\beta\gamma((\tau+1)\theta)^{l_*}}{2(c+1)\gamma h^2},$$

*and* $l_* = \liminf_{k\to\infty} l_k$ *satisfies*

$$l_* > \lfloor \frac{\ln 2\beta\gamma}{\ln((\tau+1)\theta)} \rfloor,$$

*where the symbol* $\lfloor\cdot\rfloor$ *is used to denote the smallest integer no less than the corresponding real number,* $\tau \in (0, (1-\theta)/\theta)$*, and*

$$\theta \equiv \theta(\alpha; x_*) = \|T(\alpha; x_*)\| \leq \max_{\lambda\in\sigma(H(x_*))} \frac{|\alpha - \lambda|}{\alpha + \lambda} \equiv \sigma(\alpha; x_*) < 1.$$

*Then, for any* $x_0 \in B(x_*, r)$*, and any sequence* $\{l_k\}_{k=0}^{\infty}$*, the iteration sequence* $\{x_k\}_{k=0}^{\infty}$ *generated by Algorithm Newton-HSS is well defined and converges to* $x_*$*. Moreover, it holds that*

$$\limsup_{k\to\infty} \|x_k - x_*\|^{\frac{1}{k}} \leq g(r_*; l_*),$$

*here,*

$$g(t, l) := \frac{\gamma}{1 - c\gamma h^2 t^{c-1}}((c+2)h^2 t^{c-1} + 2\beta\gamma((\tau+1)\theta)^l).$$

**Remark 2.** *For equation (4.14), if* $c = 4/3$ *or* $c = 3/2$*, then equation (4.14) becomes the equation studied in [6] and [7], respectively, hence satisfies Hölder condition. If* $c = 2$*, thus* $L(u) = 2h^2$ *is a positive constant, hence equation (4.14) satisfies Lipschtiz condition.*

Now we consider the numerical results of the corollary. We choose $c = 2$. In the following computation, the stopping criterion for the outer Newton method is set to be

$$\frac{\|F(x_k)\|_2}{\|F(x_0)\|_2} \leq 10^{-10},$$

and the prescribed tolerance for controlling the accuracy of the HSS iteration is set to be $\eta_k = \eta$. Let the initial guess $x_0 = 1$, then parameters $\beta$, $\gamma$, can be estimated from Assumption 3.1. Take positive constants $q_1 = q$, $q_2 = 1/h$, and adopt experimentally optimal parameter $\alpha$, which yields the smallest value of $\|x_{k+1} - x_*\|$, then $\theta$ and $\tau$ can be estimated from the definition of $\|T(\alpha; x_0)\|$ and the estimation of $\|T(\alpha; x)\|$, respectively, and the Newton-HSS method is examined for different problem size $n = N \times N$, different quantity $q = q_1$ and different tolerance $\eta$, from the values of $\|x_{k+1} - x_*\|$, $\frac{\|x_{k+1}-x_*\|}{\|x_k-x_*\|}$. We list the numerical results in Tables 4.2 and 4.3.

Table 4.1. The optimal value $\alpha$ for the Newton-HSS method

| N | q=600 | | | q=800 | | | q=1000 | | |
|---|---|---|---|---|---|---|---|---|---|
| | $\eta = 0.1$ | $\eta = 0.2$ | $\eta = 0.4$ | $\eta = 0.1$ | $\eta = 0.2$ | $\eta = 0.4$ | $\eta = 0.1$ | $\eta = 0.2$ | $\eta = 0.4$ |
| 30 | 0.9 | 3.1 | 1.8 | 1.3 | 3.3 | 1 | 0.6 | 0.6 | 1.1 |
| 40 | 1 | 3.3 | 2.1 | 1 | 0.7 | 1.5 | 1 | 3.9 | 0.6 |
| 50 | 3.7 | 2.1 | 4.9 | 0.7 | 3.1 | 3.4 | 0.7 | 0.5 | 2.9 |

Table 4.2. Values of $\|x_{k+1} - x_*\|$ for different $N$ and $q$ ($\eta = 0.1$)

| k | N=30 | | | N=40 | | | N=50 | | |
|---|---|---|---|---|---|---|---|---|---|
| | q=600 | q=800 | q=1000 | q=600 | q=800 | q=1000 | q=600 | q=800 | q=1000 |
| 1 | 1.5115 | 1.8882 | 3.1749 | 2.1595 | 1.1569 | 3.1537 | 1.6414 | 2.1067 | 2.0363 |
| 2 | 0.1171 | 0.2152 | 0.2880 | 0.1665 | 0.1154 | 0.2779 | 0.0877 | 0.1133 | 0.1477 |
| 3 | 0.0087 | 0.0156 | 0.0268 | 0.0116 | 0.0085 | 0.0219 | 0.0075 | 0.0079 | 0.0121 |
| 4 | 6.49e-04 | 0.0011 | 0.0026 | 8.30e-04 | 8.90e-04 | 0.0021 | 5.03e-04 | 7.05e-04 | 9.03e-04 |
| 5 | 6.05e-05 | 9.33e-05 | 2.61e-04 | 5.97e-05 | 7.90e-05 | 1.83e-04 | 4.16e-05 | 5.63e-05 | 7.18e-05 |
| 6 | 5.80e-06 | 7.70e-06 | 2.21e-05 | 4.11e-06 | 5.98e-06 | 1.30e-05 | 3.55e-06 | 5.03e-06 | 6.28e-06 |
| 7 | 5.26e-07 | 5.21e-07 | 1.53e-06 | 3.12e-07 | 4.52e-07 | 7.06e-07 | 2.73e-07 | 3.32e-07 | 6.01e-07 |
| 8 | 3.23e-08 | 3.86e-08 | 1.58e-07 | 2.35e-08 | 3.14e-08 | 4.35e-08 | 1.96e-08 | 2.54e-08 | 5.40e-08 |
| 9 | 2.51e-09 | 3.36e-09 | 1.32e-08 | 1.80e-09 | 2.64e-09 | 3.81e-09 | 1.34e-09 | 2.10e-09 | 4.50e-09 |
| 10 | 1.46e-10 | 2.14e-10 | 9.05e-10 | 1.73e-10 | 2.51e-10 | 3.10e-10 | 8.91e-11 | 1.81e-10 | 3.59e-10 |

In Table 4.2, we present the values of $\|x_{k+1} - x_*\|$, corresponding to the problem size $N = 30$, 40 and 50, and parameter $q = 600$, 800 and 1000, respectively, for the inner tolerance $\eta = 0.1$. From the table, we can see that the sequence $\{x_k\}$ generated by the Newton-HSS method converges to the solution $x_*$ in all these situations.

Table 4.3. Values of $\frac{\|x_{k+1}-x_*\|}{\|x_k-x_*\|}$ for different $N$ and $q$ ($\eta = 0.1$)

| k | N=30 | | | N=40 | | | N=50 | | |
|---|---|---|---|---|---|---|---|---|---|
| | q=600 | q=800 | q=1000 | q=600 | q=800 | q=1000 | q=600 | q=800 | q=1000 |
| 1 | 0.0504 | 0.0629 | 0.1058 | 0.0540 | 0.0392 | 0.0788 | 0.0328 | 0.0421 | 0.0407 |
| 2 | 0.0775 | 0.0114 | 0.0907 | 0.0771 | 0.0736 | 0.0881 | 0.0534 | 0.0538 | 0.0725 |
| 3 | 0.0741 | 0.0726 | 0.0931 | 0.0699 | 0.0736 | 0.0788 | 0.0858 | 0.0693 | 0.0819 |
| 4 | 0.0742 | 0.0729 | 0.0957 | 0.0713 | 0.1048 | 0.0966 | 0.0669 | 0.0898 | 0.0747 |
| 5 | 0.0933 | 0.0820 | 0.1017 | 0.0720 | 0.0888 | 0.0863 | 0.0827 | 0.0799 | 0.0795 |
| 6 | 0.0959 | 0.0825 | 0.0848 | 0.0688 | 0.0756 | 0.0712 | 0.0854 | 0.0893 | 0.0874 |
| 7 | 0.0906 | 0.0676 | 0.0689 | 0.0759 | 0.0757 | 0.0542 | 0.0768 | 0.0659 | 0.0958 |
| 8 | 0.0614 | 0.0742 | 0.1037 | 0.0752 | 0.0694 | 0.0616 | 0.0718 | 0.0767 | 0.0898 |
| 9 | 0.0778 | 0.0869 | 0.0835 | 0.0766 | 0.0840 | 0.0876 | 0.0686 | 0.0826 | 0.0833 |
| 10 | 0.0582 | 0.0636 | 0.0686 | 0.0961 | 0.0953 | 0.0815 | 0.0663 | 0.0860 | 0.0799 |

In Table 4.3, we present the values of $\frac{\|x_{k+1}-x_*\|}{\|x_k-x_*\|}$ corresponding to the problem size $N = 30$, 40 and 50, and parameter $q = 600$, 800 and 1000, respectively, for the inner tolerance $\eta = 0.1$. From the table, we can observe that all the values of $g(r_*, l_*)$ are smaller than 0.11.

# 5 Conclusion

The Newton-HSS method is a considerable method for solving large sparse nonlinear systems with non-Hermitian positive definite Jacobian matrices. In this paper, Under the hypothesis that the Jacobian matrix satisfies the center Lipschtiz condition with the L-average, which is weaker than H*ö*lder condition and Lipschtiz condition, we establish the local convergence theorem for the Newton-HSS method. Finally, a numerical example is given to confirm the concrete applications of the results of our paper.

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# References

[1] O. Axelsson, G. F. Carey, On the numerical solution of two-point singularly perturbed boundary value problems, *Comput. Methods Appl. Mech. Eng.*, 50, 217–229(1985).

[2] O. Axelsson, M. Nikolova, Avoiding slave points in an adaptive refinement procedure for convection-diffusion problems in 2D, *Computing*, 61, 331–357(1998).

[3] Z.-Z. Bai, A class of two-stage iterative methods for systems of weakly nonlinear equations, *Numer. Algor.*, 14, 295–319(1997).

[4] Z.-Z. Bai, G. H. Golub, M. K. Ng, Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems, *SIAM J. Matrix Anal. Appl.*, 24, 603–626(2003).

[5] Z.-Z. Bai, X.-P. Guo, On Newton-HSS method for systems of nonlinear equations with positive-definite Jacobian matrices, *J. Comput. Math.*, 28, 235–260(2010).

[6] M.-H. Chen, R.-F. Lin, Q.-B. Wu, Convergence analysis of the modified Newton-HSS method under the Hölder continuous condition, *J. Comput. Appl. Math.*, 264, 115–130(2014).

[7] M.-H. Chen, Q.-B. Wu, R.-F. Lin, Semilocal convergence analysis for the Modified Newton-HSS method under the Hölder condition, *Numer. Algor.*, 3, 1-19(2016).

[8] R. S. Dembo, S. C. Eisenstat, T. Steihaug, Inexact Newton methods, *SIAM J. Numer. Anal.*, 19, 400–408(1982).

[9] O. P. Ferreira, B. F. Svaiter, Kantorovich's majorants principle for Newton's method, *Comput. Optim. Appl.*, 42, 213–229(2009).

[10] X.-P. Guo, I. S. Duff, Semilocal and global convergence of the Newton-HSS method for systems of nonlinear equations, *Numer. Linear Algebra Appl.*, 18, 299–315(2011).

[11] X.-P. Guo, On the convergence of Newton's method in Banach space, *Journal of Zhejiang University(Sciences Edition)*, 27(5), 484–492(2000).

[12] L. V. Kantorovich, G. P. Akilov, *Functional Analysis in Normed Spaces*, Oxford, Pergamon, 1964.

[13] C. Li, K. F. NG, Majorizing functions and convergence of the Gauss-Newton method for convex composite optimation, *SIAM J. OPTIM.*, 18(2), 613–642(2007).

[14] J. M. Ortega, W. C. Rheinbolt, *Iterative Solution of Nonlinear Equations in Several Variables*, Academic Press, NewYork, 1970.

[15] W. C. Rheinboldt, *Methods of solving systems of nonlinear equations (2nd edn)*, SIAM, Philadelphia, 1998.

[16] W.-P. Shen, C. Li, Kantorovich-type convergence criterion for inexact Newton methods, *Appl. Numer. Math.*, 59, 1599–1611(2009).

[17] X.-H. Wang, Convergence of Newton's method and inverse function theorem in Banach space, *Math. Comput.*, 68, 169–185(1999).

[18] X.-H. Wang, Convergence of Newton's method and uniqueness of the solution of equations in Banach space, *IMA J. Numer. Anal.*, 20, 123–134(2000).

# Oscillation for Fractional Neutral Functional Differential Systems*

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**Abstract**

In this paper, we discuss the linear autonomous system of neutral delay differential equations with Riemann–Liouville fractional derivative

$$D^{\alpha}\left[x(t)+\sum_{j=1}^{l}P_j x(t-\tau_j)\right]+\sum_{i=1}^{n}Q_i x(t-\delta_i)=0$$

where $D^{\alpha}x(t)=[D^{\alpha_1}x_1(t),D^{\alpha_2}x_2(t),...,D^{\alpha_m}x_m(t)]^T$ is Riemann–Liouville fractional derivative, the coefficients $P_j(j=1,2,...,l)$ and $Q_i(i=1,2,...,n)$ are real $m\times m$ matrices and the delays $\tau_j(j=1,2,...,l)$ and $\delta_i(i-1,2,...,n)$ are non-negative real numbers. Sufficient conditions for all solutions of the given equation to be oscillatory are obtained by using fractional calculus and Laplace transform.

**Key words and phrases:** Fractional neutral differential equations; Riemann–Liouville derivative; Oscillation; Laplace transform.
**AMS (MOS) Subject Classifications**: 15A60; 26A33; 34A30; 34K11; 44A10.

## 1 Introduction

Fractional differential equations have gained considerable importance due to their application in various disciplines, such as physics, mechanics, chemistry, engineering, etc. In the recent years, there has been a significant development in ordinary and partial differential equations involving fractional derivatives, see the monographs of Podlubny[1], Kilbas et al.[2], Diethelm[3], Zhou[4, 5], the recent papers[6, 7, 8, 9] and the references therein.

On the other hand, the objective of oscillation theory is to acquire as much information as possible about the qualitative properties of solutions of differential equations. Oscillation theory of functional differential equations with integer derivative has been developed in the past thirty years. The several monographs by Ladde et al.[10], Györi and Ladas[11], Gopalsamy[12], Erbe et al.[13], Agarwal et al.[14] summarize a lot of important works in this area.

However, to the best of our knowledge, there are few results on oscillation for fractional differential equations. Recently, Grace, Agarwal and Wong, et al.[15], Bolat[16], Duan, Wang and Fu[17], Harikrishnan, Prakash and Nieto[18] investigated oscillation and forced oscillation of fractional-order delay differential equations.

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In this paper, we discuss the neutral functional differential equations with Riemann–Liouville fractional derivative

$$D^{\alpha}\Big[x(t)+\sum_{j=1}^{l}P_jx(t-\tau_j)\Big]+\sum_{i=1}^{n}Q_ix(t-\delta_i)=0, \tag{E}$$

where $x(t)=[x_1(t),x_2(t),...,x_m(t)]^T$, $D^{\alpha}x(t)=[D^{\alpha_1}x_1(t),D^{\alpha_2}x_2(t),...,D^{\alpha_m}x_m(t)]^T$ is Riemann–Liouville fractional derivative of order $0<\alpha,\alpha_r<1$, $\alpha_r=p_r/q_r$, $p_r,q_r$ are co-prime, for $r=1,2,...,m$, and $P_j,Q_i\in\mathbb{R}^{m\times m}$, $j=1,2,...,l$, $i=1,2,...,n$, the delays $\tau_j(j=1,2,...,l)$ and $\delta_i(i-1,2,...,n)$ are non-negative real numbers.

Our aim is to establish sufficient conditions for oscillation of the system (E). In the next section, we introduce some useful preliminaries. In section 3, we obtain various sufficient conditions for oscillation of all solutions to the system (E) by using fractional calculus and Laplace transform.

## 2 Preliminaries

In this section, we introduce preliminary facts which are used throughout this paper.

**Definition 2.1** *[2] Let $[a,b](-\infty<a<b<\infty)$ be a finite interval and let $AC[a,b]$ be the space of functions $f$ which are absolutely continuous on $[a,b]$. It is known [see Kolmogorov and Fomin ([16], p.338) that $AC[a,b]$ coincides with the space of primitives of Lebesgue summable functions:*

$$f(x)\in AC[a,b]\ \Rightarrow f(x)=c+\int_a^x\psi(t)dt\ (\psi(t)\in L(a,b)).$$

**Definition 2.2** *[2] The fractional integral of order $\alpha$ with the lower limit zero for a function $f$ is defined as*

$$(I^{\alpha}f)(t)=\frac{1}{\Gamma(\alpha)}\int_0^t\frac{f(s)}{(t-s)^{1-\alpha}}ds,\ t>0,\ 0<\alpha<1,$$

*provided the right side is point-wise defined on $[0,b]$, where $\Gamma(\cdot)$ is the gamma function.*

**Definition 2.3** *[2] Riemann-Liouville derivative of order $\alpha$ with the lower limit zero for a function $f$ can be written as*

$$(D^{\alpha}f)(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t\frac{f(s)}{(t-s)^{\alpha}}ds,\ t>0,\ 0<\alpha<1.$$

Firstly, we consider the fractional delay differential systems

$$D^{\alpha}x(t)+\sum_{i=1}^{n}P_ix(t-\tau_i)=0,\quad t\geq 0, \tag{1}$$

where $x(t)=[x_1(t),x_2(t),...,x_m(t)]^T$, $D^{\alpha}x(t)=[D^{\alpha_1}x_1(t),D^{\alpha_2}x_2(t),...,D^{\alpha_m}x_m(t)]^T$ is Riemann–Liouville fractional derivative of order $0<\alpha,\alpha_i<1$, $\alpha_j=p_j/q_j$, $p_j,q_j$ are odd numbers, for $i=1,2,...,m$, and $P_i\in\mathbb{R}^{m\times m}$, $\tau_i\in[0,\infty)$ for $i=1,2,...,n$.

Without loss of generality we will assume the coefficients of $P_i$ of (1) are all nonzero and that $\tau_1=\max\{\tau_1,...,\tau_n\}$.

**Definition 2.4** *By a solution of (1) in $[0,\infty)$ with initial function $\varphi\in AC[-\tau_1,0]$ we mean a function $x\in AC[-\tau_1,\infty)$ such that $x(t)=\varphi(t)$, $t\in[-\tau_1,0]$, $(D^{\alpha}x)(t)$ exists and $x(t)$ satisfies (1) in $[0,\infty)$. A solution $x(t)=[x_1(t),...,x_m(t)]^T$ of system (1) is said to oscillate if every component $x_i(t)$ of the solution has arbitrarily large zeros. Otherwise the solution is called non-oscillatory.*

We recall some facts about Laplace transforms. If $X(s)$ is the Laplace transform of $x(t)$,

$$X(s)=(Lx)(s)=\int_0^{\infty}e^{-st}x(t)dt,$$

then the abscissa of convergence of $X(s)$ is defined by

$$b = \inf\{\delta \in \mathbb{R} : X(\delta) \ exists\}.$$

Then $X(s)$ is analytic for $Re(s) > b$.

We call a function $x(t)$ to be eventually positive if there exists a $c \geq 0$ such that $x_c(t) > 0$ for all $t > 0$, where $x_c(t) = x(t + c)$.

For any $m$-dimensional vector $x = (x_1, x_2, ..., x_m)^T \in \mathbb{R}^m$, $\|x\|$ denotes its norm. For any $m \times m$ real matrix $A$, the associated matrix norm is then defined by $\|A\| = \max_{\|x\|=1} \|Ax\|$. Denote $\mu(P_i)$ is the logarithmic norm with $\mu(P_i) = \max_{\|u\|=1}(P_i u, u)$.

**Lemma 2.5** *[2] Let $(LD^\alpha x)(s)$ is the Laplace transform of the Riemann–Liouville fractional derivative of order $\alpha$ with the lower limit 0 for a function $x$, and $X(s)$ is the Laplace transform of $x(t) \in AC[0, b]$, for any $b > 0$, and the following estimate*

$$|x(t)| \leq Ae^{p_0 t} \quad (t > b > 0)$$

*holds for constants $A > 0$ and $p_0 > 0$. Then the relation*

$$(LD^\alpha x)(s) = s^\alpha BX(s) - (I^{1-\alpha} x)(0), \ 0 < \alpha < 1$$

*is valid for $Re(s) > p_0$, where*

$$X(s) = [X_1(s), X_2(s), ..., X_m(s)]^T, \quad s^\alpha = [s^{\alpha_1}, s^{\alpha_2}, ..., s^{\alpha_m}]$$

$$(I^{1-\alpha} x)(0) = \left( (I^{1-\alpha_1} x_1)(0), (I^{1-\alpha_2} x_2)(0)], ..., (I^{1-\alpha_m} x_m)(0) \right)^T,$$

$$B = [B_1, B_2, ..., B_m]^T \ B_i = (b_{ij})_{m\times m}$$

$$b_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

**Lemma 2.6** *[19] If $X(s)$ is the Laplace transform of a non-negative function $x(t)$ and has abscissa of convergence $b > -\infty$, then $X(s)$ has a singularity at the point $s = b$.*

**Lemma 2.7** *[20] Let $v, w : [0, \infty) \to [0, \infty)$ be continuous functions. If $w(\cdot)$ is nondecreasing and there are constants $a > 0$ and $0 < \beta < 1$ such that*

$$v(t) \leq w(t) + a \int_0^t \frac{v(s)}{(t - s)^\beta} ds,$$

*then there exists a constant $k = k(\beta)$ such that*

$$v(t) \leq w(t) + ka \int_0^t \frac{w(s)}{(t - s)^\beta} ds$$

*for every $t \in [0, \infty)$.*

# 3 Main Results

In this section, we present our main results.

**Lemma 3.1** *For any $c \in \mathbb{R}$, the Laplace transform $X_c(s)$ of $x_c(t)$ exists and has the same abscissa of convergence as $X(s)$.*

**Proof.** Given that

$$\begin{aligned} X_c(s) &= \int_0^\infty e^{-st}x_c(t)dt = \int_0^\infty e^{-st}x(t+c)dt = e^{sc}\int_c^\infty e^{-st}x(t)dt \\ &= e^{sc}[X(s) - \int_0^c e^{-st}x(t)dt]. \end{aligned}$$

Since the last integral defines an entire function of the complex variable $s$, therefore $X(s)$ and $X_c(s)$ converge or diverge for the same values of $s$, and have their singularities at the same points. This completes the proof. □

**Lemma 3.2** *The solution of equation (1) has an exponent estimate*

$$x(t) = o(e^{q_0 t}) \quad (t > b > 0)$$

*for constant* $q_0 > 0$.

**Proof.** Taking Riemann-Liouville integral of equation (1), we get

$$\begin{aligned} x(t) &= \frac{t^{\alpha-1}}{\Gamma(\alpha)}Bx_0 - \sum_{i=1}^n P_i \int_0^t \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}Bx(s-\tau_i)ds \\ &= \frac{t^{\alpha-1}}{\Gamma(\alpha)}Bx_0 - \sum_{i=1}^n P_i F_i(t), \end{aligned} \tag{2}$$

where

$$x_0 = (I^{1-\alpha}x)(0),$$

$$F_i(t) = \int_0^t \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}Bx(s-\tau_i)ds, \quad \frac{t^{\alpha-1}}{\Gamma(\alpha)} = \left[\frac{t^{\alpha_1-1}}{\Gamma(\alpha_1)}, \frac{t^{\alpha_2-1}}{\Gamma(\alpha_2)}, ..., \frac{t^{\alpha_m-1}}{\Gamma(\alpha_m)}\right].$$

As $AC[-\tau_1, 0]$ is the Banach space with the norm $\|\varphi\|_{AC} = [\|\varphi_1\|_{AC}, \|\varphi_2\|_{AC}, ..., \|\varphi_m\|_{AC}]^T$. Then we have

$$\begin{aligned} \|F_i(t)\| &\le \int_0^t \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}B\|x(s-\tau_i)\|ds \\ &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}B \max_{s-\tau_i\le\eta\le s}\|x(\eta)\|ds \\ &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}B \max_{s-\tau_1\le\eta\le s}\|x(\eta)\|ds, \end{aligned}$$

or

$$\|F_i(t)\| \le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}B\left(\max_{s-\tau_1\le\eta\le s}\|x(\eta)\| + \|\varphi\|_{AC}\right)ds. \tag{3}$$

From (3), it follows that

$$\begin{aligned} \|x(t)\| &\le \frac{b^{\alpha-1}}{\Gamma(\alpha)}B|x_0| + \sum_{i=1}^n \frac{\|P_i\|}{\Gamma(\alpha)}\left[\int_0^t (t-s)^{\alpha-1}B\left(\max_{s-\tau_1\le\eta\le s}\|x(\eta)\| + \|\varphi\|_{AC}\right)ds\right] \\ &\le \frac{b^{\alpha-1}}{\Gamma(\alpha)}B|x_0| + \frac{np}{\Gamma(\alpha)}\frac{t^\alpha}{\alpha}B\|\varphi\|_{AC} + \frac{np}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}B\max_{s-\tau_1\le\eta\le s}\|x(\eta)\|ds, \end{aligned}$$

where $p = \max\{\|P_i\|\}$, for $i = 1, 2, ..., n$.

Next we introduce a nondecreasing function $m(t)$ as

$$m(t) = \frac{b^{\alpha-1}}{\Gamma(\alpha)}B|x_0| + \frac{np}{\Gamma(\alpha)}\frac{t^\alpha}{\alpha}B\|\|\varphi\|_{AC}\|.$$

By Lemma 2.7, there exists a number $\alpha_0$ in $\{\alpha_i\}$ such that

$$\|x(t)\| \leq \max_{t-\tau_1 \leq s \leq t} \|x(s)\| \leq m(t) + \frac{knp}{\Gamma(\alpha_0)} \int_0^t (t-s)^{\alpha_0 - 1} m(s) ds \leq m(t) \left(1 + \frac{knp}{\alpha_0 \Gamma(\alpha_0)} t^{\alpha_0}\right). \tag{4}$$

Obviously, from (4) we infer that $x(t)$ has an exponent estimate. The proof is complete. □

**Theorem 3.3** *If the characteristic equation*

$$\det\left(\lambda^\alpha B + \sum_{i=1}^n P_i e^{-\lambda \tau_i}\right) = 0 \tag{5}$$

*has no real roots, then every solution of (1) is oscillatory, where* $\lambda^\alpha = [\lambda^{\alpha_1}, \lambda^{\alpha_2}, ..., \lambda^{\alpha_m}]$.

**Proof.** For the sake of contradiction, let us assume that (5) has no real roots and that (1) has a non-oscillatory solution $x(t) = [x_1(t), ..., x_m(t)]^T$. This means that one of the components of $x(t)$ is non-oscillatory. Without loss of generality we assume that the component $x_1(t)$ is eventually positive, such that for some $c \geq 0$, $x_c(t) > 0$ for $t \geq 0$. As (1) is autonomous, it follows by Lemma 3.1 that $X_1(s)$ and $X_c(s)$ have the same convergence. Then we assume that $x_1(t) > 0$ for $t \geq -\tau_1$. Taking Laplace transform of both sides of (1), we obtain

$$s^\alpha B X(s) - (I^{1-\alpha} x)(0) + \sum_{i=1}^n P_i \int_0^\infty e^{-st} x(t - \tau_i) dt = 0,$$

i.e.

$$s^\alpha B X(s) - (I^{1-\alpha} x)(0) + \sum_{i=1}^n P_i e^{-s\tau_i} X(s) + \sum_{i=1}^n P_i e^{-s\tau_i} \int_{-\tau_i}^0 e^{-st} x(t) dt = 0.$$

Hence

$$(s^\alpha B + \sum_{i=1}^n P_i e^{-s\tau_i}) X(s) = (I^{1-\alpha} x)(0) - \sum_{i=1}^n P_i e^{-s\tau_i} \int_{-\tau_i}^0 e^{-st} x(t) dt. \tag{6}$$

Let

$$F(s) = s^\alpha B + \sum_{i=1}^n P_i e^{-s\tau_i},\ x_0 = (I^{1-\alpha} x)(0),$$

$$\Phi(s) = x_0 - \sum_{i=1}^n P_i e^{-s\tau_i} \int_{-\tau_i}^0 e^{-st} x(t) dt.$$

Then, from (6) we get

$$F(s) X(s) = \Phi(s),\ Re(s) > b. \tag{7}$$

Since $\det[F(s)] = 0$ has no real roots, $\det[F(s)] > 0,\ s \in \mathbb{R}$. By Cramer's rule, we have

$$X_1(s) = \frac{\det[D(s)]}{\det[F(s)]},\ Re(s) > b, \tag{8}$$

where

$$D(s) = \begin{pmatrix} \Phi_1(s) & F_{12}(s) & \cdots & F_{1m}(s) \\ \vdots & \vdots & & \vdots \\ \Phi_m(s) & F_{m2}(s) & \cdots & F_{mm}(s) \end{pmatrix},$$

$\Phi_i(s)$ is the $i$th component of the vector $\Phi(s)$ and $F_{ij}(s)$ is the $(i,j)$th component of the matrix $F(s)$. Clearly, for all $i, j = 1, 2, ..., m$ the functions $\Phi_i(s)$ and $F_{ij}(s)$ are entire and hence $\det[D(s)]$ and $\det[F(s)]$ are also entire functions.

Since $\det[F(s)] > 0$ for $s \in \mathbb{R}$, so $\det[D(s)]/\det[F(s)]$ holds for $s \in \mathbb{R}$ and thus (8) becomes

$$X_1(s) = \frac{\det[D(s)]}{\det[F(s)]},\ s \in \mathbb{R}. \tag{9}$$

As $x_1(t) > 0$, it follows that $X_1(s) > 0$ for all $s \in \mathbb{R}$ and, by $\det[F(s)] > 0$ $s \in \mathbb{R}$ and (9), $\det[D(s)] > 0$ $s \in \mathbb{R}$. Now one can see from the definitions of $D(s)$, $F(s)$ and $\Phi(s)$ that there exist positive constants $M, \beta$, and $s_0$ such that

$$\det[D(s)] \leq Me^{-\beta s} \quad for\ s \leq -s_0. \tag{10}$$

Since $\det[F(s)]$ is a continuous function in the variables $s, e^{-s\tau_1}, ..., e^{-s\tau_n}$, and $\det[F(s)] > 0,\ s \in \mathbb{R}$, it follows that there exists a positive number $m_0$ such that

$$\det[F(s)] \geq m_0 \quad for\ s \in \mathbb{R}. \tag{11}$$

From (9), (10) and (11), it follows that

$$X_1(s) = \int_0^\infty e^{-st}x_1(t)dt \geq \int_T^\infty e^{-st}x_1(t)dt \geq e^{-sT}\int_T^\infty x_1(t)dt > 0$$

and so

$$0 < \int_T^\infty x_1(t)dt \leq \frac{M}{m_0}e^{s(T-\beta)} \to 0 \quad as\ s \to -\infty.$$

This implies that $x_1(t) \equiv 0$ for $t \geq T$, which is a contradiction. The proof is complete. □

In Theorem 3.3, the characteristic equation (5) plays an important role in the investigation of the oscillation of equation (1). However, to determine whether (5) has a real root, is quite an issue in itself. In the following we derive some sufficient conditions for the oscillation of equation (1) which can easily be applied.

Before proceeding for it, we need the following lemma which is interesting in its own right.

**Lemma 3.4** *Assume that*
$P_i \in \mathbb{R}^{m\times m}$ , $\tau_i \geq 0$ *for* $i = 1, 2, ..., n$, *and* $\bar{\alpha} = \min\{\alpha_j\}$, *for* $j = 1, 2, ..., m$ *with*

$$\sum_{i=1}^n \mu(-P_i)e^{-\lambda\tau_i} < 0 \quad for \quad \lambda \in \mathbb{R} \tag{12}$$

*and*

$$\inf_{\lambda<0}\left[\frac{1}{\lambda^{\bar{\alpha}}}\sum_{i=1}^n \mu(-P_i)e^{-\lambda\tau_i}\right] > 1. \tag{13}$$

*Then every solution of (1) oscillates.*

**Proof.** Assume, for the sake of contraction, that (1) has a non-oscillatory solution. Then, by Theorem 3.3, the characteristic equation (5) has a real root $\lambda_0$. In consequence, there exists a vector $u \in \mathbb{R}^n$ with $\|u\| = 1$ such that

$$\left(\lambda_0^\alpha B + \sum_{i=1}^n P_i e^{-\lambda_0\tau_i}\right)u = 0,$$

i.e.

$$\lambda_0^\alpha Bu = -\sum_{i=1}^n P_i e^{-\lambda_0\tau_i}u.$$

Hence

$$\begin{aligned}\lambda_0^{\bar{\alpha}} &= (\lambda_0^{\bar{\alpha}}u, u) \leq (\lambda_0^\alpha Bu, u) = (-\sum_{i=1}^n P_i e^{-\lambda_0\tau_i}u, u)\\ &= (-\sum_{i=1}^n P_i u, u)e^{-\lambda_0\tau_i} \leq \sum_{i=1}^n \mu(-P_i)e^{-\lambda_0\tau_i}.\end{aligned}$$

Then by (12), $\lambda_0 < 0$ such that

$$\left[\frac{1}{\lambda_0^{\bar{\alpha}}}\sum_{i=1}^n \mu(-P_i)e^{-\lambda_0\tau_i}\right] \leq 1 \ \text{ or } \ -\lambda_0 \geq \left[\sum_{i=1}^n -\mu(-P_i)\right]^{\frac{1}{\bar{\alpha}}}. \tag{14}$$

This contradicts (13) and completes the proof. □

**Theorem 3.5** *Assume that for each* $i = 1, 2, ..., n$,

$$P_i \in \mathbb{R}^{m\times m}, \quad \tau_i \geq 0 \quad \text{and} \quad \mu(-P_i) \leq 0.$$

*Then each of the following two conditions is sufficient for the oscillation of all solutions of (1):*

(i) $\displaystyle\sum_{i=1}^{n} -\mu(-P_i)\tau_i \Big[\sum_{i=1}^{n} -\mu(-P_i)\Big]^{\frac{1-\bar{\alpha}}{\bar{\alpha}}} > \frac{1}{e}$;

(ii) $\displaystyle\Big[\prod_{i=1}^{n}(-\mu(-P_i))\Big]^{\frac{1}{n}} \sum_{i=1}^{n} \tau_i \Big[\sum_{i=1}^{n} -\mu(-P_i)\Big]^{\frac{1-\bar{\alpha}}{\bar{\alpha}}} > \frac{1}{e}$.

**Proof.** We employ Lemma 3.4. As $\mu(-P_i) \leq 0$, (12) is satisfied and so it suffices to establish (13). First, assume that (i) holds. Then, by using the inequality $e^x \geq ex$, we see that for all $\lambda < 0$,

$$\begin{aligned}
\frac{1}{\lambda^{\bar{\alpha}}} \sum_{i=1}^{n} \mu(-P_i) e^{-\lambda\tau_i} &\geq \frac{1}{\lambda^{\bar{\alpha}}} \sum_{i=1}^{n} \mu(-P_i) e(-\lambda\tau_i) \\
&= -e\frac{1}{\lambda^{\bar{\alpha}}} \sum_{i=1}^{n} \mu(-P_i)\tau_i \lambda^{\bar{\alpha}} (-\lambda)^{1-\bar{\alpha}} \\
&\geq e\sum_{i=1}^{n} -\mu(-P_i)\tau_i \Big[\sum_{i=1}^{n} -\mu(-P_i)\Big]^{\frac{1-\bar{\alpha}}{\bar{\alpha}}},
\end{aligned}$$

which, together with (i), implies that (13) holds. Next, assume that (ii) holds. Then, by using the arithmetic mean–geometric mean inequality we find that for all $\lambda < 0$,

$$\begin{aligned}
\frac{1}{\lambda^{\bar{\alpha}}} \sum_{i=1}^{n} \mu(-P_i) e^{-\lambda\tau_i} &= -\frac{1}{\lambda^{\bar{\alpha}}} \sum_{i=1}^{n} -\mu(-P_i) e^{-\lambda\tau_i} \\
&\geq -\frac{1}{\lambda^{\bar{\alpha}}} n \Big[\prod_{i=1}^{n} -\mu(-P_i) e^{-\lambda\tau_i}\Big]^{\frac{1}{n}} \\
&= -\frac{1}{\lambda^{\bar{\alpha}}} n \Big[\prod_{i=1}^{n} -\mu(-P_i)\Big]^{\frac{1}{n}} \exp\Big(-\frac{1}{n}\lambda \sum_{i=1}^{n} \tau_i\Big) \\
&\geq -\frac{1}{\lambda^{\bar{\alpha}}} \Big[\prod_{i=1}^{n} -\mu(-P_i)\Big]^{\frac{1}{n}} e\Big(-\lambda \sum_{i=1}^{n} \tau_i\Big) \\
&= \Big[\prod_{i=1}^{n} -\mu(-P_i)\Big]^{\frac{1}{n}} e(-\lambda)^{1-\bar{\alpha}} \sum_{i=1}^{n} \tau_i \\
&\geq e\Big[\prod_{i=1}^{n}(-\mu(-P_i))\Big]^{\frac{1}{n}} \sum_{i=1}^{n} \tau_i \Big[\sum_{i=1}^{n} -\mu(-P_i)\Big]^{\frac{1-\bar{\alpha}}{\bar{\alpha}}}.
\end{aligned}$$

From this and (ii) it follows that (13) holds. The proof is complete. □

As a special case of the delay differential system with one delay,

$$(D^{\alpha}x)(t) + Px(t-\tau) = 0, \tag{15}$$

where

$$P \in \mathbb{R}^{m\times m} \quad \text{and} \quad \tau \geq 0,$$

the conditions (i) and (ii) coincide and each reduces to

$$[-\mu(-P)]^{\frac{1}{\bar{\alpha}}}\tau > \frac{1}{e}. \tag{16}$$

Note that (16) is sharp in the sense that the lower bound $1/e$ cannot be improved. Moreover, when $P$ is a scalar, (16) is a sufficient condition for the oscillation of all solutions to equation (15).

For the delay differential system (15), we also have the following explicit sufficient condition for the oscillation of all solutions.

**Theorem 3.6** *Assume that*

$$P \in \mathbb{R}^{m\times m} \quad \text{and} \quad \tau \geq 0.$$

*If $P$ has no real eigenvalues in the interval $(-\infty, 1/(e\tau)^{\bar{\alpha}}]$ (when $\tau = 0$, replace $1/(e\tau)^{\bar{\alpha}}$ by $+\infty$), then every solution of (15) oscillates.*

**Proof.** For $\tau = 0$, this result follows immediately from Theorem 3.1. So assume $\tau > 0$. Note that the characteristic equation $\det(\lambda^\alpha B + Pe^{-\lambda\tau}) = 0$ has a real root $\lambda_0$, that is, $\det(\lambda_0^\alpha e^{\lambda_0\tau}B + P) = 0$ if and only if $\mu_0^\alpha = -\lambda_0^\alpha e^{\lambda_0\tau}$ is a real eigenvalue of $P$. For convenience, we take one element $\lambda_0^{\alpha_i}$ of $\lambda_0^\alpha$:

$$\mu_0^{\alpha_i} = -\lambda_0^{\alpha_i} e^{\lambda_0\tau}. \tag{17}$$

Observe that (17) holds if $\lambda_0^{\alpha_i} + \mu_0^{\alpha_i} e^{-\lambda_0\tau} = 0$, that is, the equation $\lambda^{\alpha_i} + \mu_0^{\alpha_i} e^{-\lambda\tau} = 0$ has a real root. If $\mu_0 \leq 1/e\tau$, then the eigenvalue $\mu_0^\alpha$ of $P$ should lie in the interval $(-\infty, 1/(e\tau)^{\bar{\alpha}}]$. The proof is complete. □

**Definition 3.7 ([13])** *We say that (1) is oscillatory, globally in the delays, if for all $\tau_i \geq 0$ for $i = 1, 2, ..., n$, every solution of (1) oscillates.*

The following corollary is an immediate consequence of Theorem 3.6.

**Corollary 3.8** *Equation (15) is oscillatory globally in the delay $\tau$ if $P$ has no real eigenvalues.*

Next, we consider the linear autonomous system of neutral delay differential equations

$$D^\alpha\left[x(t) + \sum_{j=1}^{l} P_j x(t-\tau_j)\right] + \sum_{i=1}^{n} Q_i x(t-\delta_i) = 0, \tag{18}$$

where the coefficients $P_j$ and $Q_i$ are real $m \times m$ matrices and the delays $\tau_j$ and $\delta_i$ are non-negative real numbers. Associated with (18), the characteristic equation is

$$det\left(\lambda^\alpha B + \lambda^\alpha \sum_{j=1}^{l} P_j e^{-\lambda\tau_j} + \sum_{i=1}^{n} Q_i e^{-\lambda\delta_i}\right) = 0. \tag{19}$$

**Lemma 3.9** *If $lp < 1$, then the solution of equation (18) has an exponent estimate*

$$\|x(t)\| \leq A_0 e^{b_0 t} \qquad (t > b > 0)$$

*for constants $A_0 > 0$ and $b_0 > 0$.*

**Proof.** Let $\delta = \max\{\delta_i\}$, $j = 1, 2, ..., l$, $\bar{\delta} = \max\{\tau_1, \delta\}$, $q = \max\{\|Q_i\|\}$ $i = 1, 2, ..., n$, and take $x_0 = (I^{1-\alpha}x)(0) + \sum_{j=1}^{l} P_j(I^{1-\alpha}x)(-\tau_j)$ with $x(t) \in AC[0, b]$. Then there exists a constant $M$ such that $\|x(t)\| \leq M$. Then, for $t > b$,

$$\begin{aligned}
\|x(t)\| &\leq c\|x_0\| + lp\|x(t-\tau_j)\| + \frac{nq}{\Gamma(\alpha_0)}\int_0^t (t-s)^{\alpha_0-1}\|x(s-\delta_i)\|ds \\
&\leq c\|x_0\| + lp \max_{t-\tau_i\leq s\leq t}\|x(s)\| + \frac{nq}{\Gamma(\alpha_0)}\int_0^t (t-s)^{\alpha_0-1}\max_{s-\delta_i\leq\eta\leq s}\|x(\eta)\|ds \\
&\leq c\|x_0\| + lp(M + \|\varphi\|_{AC} + \max_{b\leq s\leq t}\|x(s)\|) \\
&\quad + \frac{nq}{\Gamma(\alpha_0)}\int_0^b (t-s)^{\alpha_0-1}\max_{s-\bar{\delta}\leq\eta\leq b}\|x(\eta)\|ds + \frac{nq}{\Gamma(\alpha_0)}\int_b^t (t-s)^{\alpha_0-1}\max_{b\leq\eta\leq s}\|x(\eta)\|ds
\end{aligned}$$

$$\begin{aligned} &\leq c\|x_0\| + (M + \|\varphi\|_{AC})\left(lp + \frac{nq}{\Gamma(\alpha_0)}\frac{t^{\alpha_0}}{\alpha_0}\right) + lp \max_{b\leq s\leq t}\|x(s)\| \\ &\quad + \frac{nq}{\Gamma(\alpha_0)}\int_b^t (t-s)^{\alpha_0-1}\max_{b\leq\eta\leq s}\|x(\eta)\|ds. \end{aligned}$$

Set

$$a = c\|x_0\| + (M + \|\varphi\|_{AC})\left(lp + \frac{nq}{\Gamma(\alpha_0)}\frac{t^{\alpha_0}}{\alpha_0}\right),$$

which yields

$$\max_{b\leq s\leq t}\|x(s)\| \leq \frac{1}{1-lp}\left(a + \frac{nq}{\Gamma(\alpha_0)}\int_b^t (t-s)^{\alpha_0-1}\max_{b\leq\eta\leq s}\|x(\eta)\|ds\right),$$

Consequently, by Lemma 3.2, we obtain

$$\begin{aligned} \|x(t)\| &\leq \max_{b\leq s\leq t}\|x(s)\| \leq \frac{a(t)}{1-lp} + \frac{1}{(1-lp)^2}\frac{knq}{\Gamma(\alpha_0)}\int_0^t (t-s)^{\alpha_0-1}a(s)ds \\ &\leq \frac{a(t)}{1-lp}\left(1 + \frac{1}{1-lp}\frac{knq}{\alpha_0\Gamma(\alpha_0)}t^{\alpha_0}\right). \end{aligned}$$

The proof is complete. □

A slight modification in the proof of Theorem 3.3 shows that the following result is also true.

**Theorem 3.10** *Assume that for $j = 1, 2, ..., l$ and $i = 1, 2, ..., n$,*

$$P_j, Q_i \in \mathbb{R}^{m\times m}, \quad \tau_j \in (0,\infty) \quad \text{and} \quad \delta_i \in [0,\infty).$$

*If the characteristic equation (19) has no real roots, then every solution of (18) oscillates.*

**Proof.** If we modify the functions $F(s)$ and $\Phi(s)$, defined in Theorem 3.3, as

$$F(s) = s^\alpha B + s^\alpha B\sum_{j=1}^{l} P_j e^{-s\tau_j} + \sum_{i=1}^{n} Q_i e^{-s\delta_i},$$

$$\Phi(s) = x_0 - \sum_{i=1}^{n} Q_i e^{-s\delta_i}\int_{-\delta_i}^{0} e^{-st}x(t)dt - s^\alpha B\sum_{j=1}^{l} P_j e^{-s\tau_j}\int_{\tau_j}^{0} e^{-st}x(t)dt + \sum_{j=1}^{l} P_j(I^{1-\alpha}x)(-\tau_j).$$

Then, following the method of proof for Theorem 3.3, one can complete the proof. □

# References


[1] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[2] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science B.V., Amsterdam, 2006.

[3] K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer-Verlag, Berlin, 2010.

[4] Y. Zhou, *Basic Theory of Fractional Differential Equations*, World Scientific, Singapore, 2014.

[5] Y. Zhou, *Fractional Evolution Equations and Inclusions: Analysis and Control*, Academic Press, 2016.

[6] Y. Zhou, L. Peng, On the time-fractional Navier-Stokes equations, *Comput. Math. Appl.* **73** (2017) 874-891.

[7] Y. Zhou, L. Peng, Weak solutions of the time-fractional Navier-Stokes equations and optimal control, *Comput. Math. Appl.* **73** (2017) 1016-1027.

[8] Y. Zhou, L. Zhang, Existence and multiplicity results of homoclinic solutions for fractional Hamiltonian systems, *Comput. Math. Appl.* **73** (2017) 1325-1345.

[9] Y. Zhou, V. Vijayakumar, R. Murugesu, Controllability for fractional evolution inclusions without compactness, *Evol. Equ. Control The.* **4** (2015) 507-524.

[10] G.S. Ladde, V. Lakshmikantham, B.G. Zhang, *Oscillation Theory of Differential Equations with Deviation Arguments*, Dekker, New York, 1989.

[11] I. Györi, G. Ladas, *Oscillation Theory of Delay Differential Equations with Applications*, Clarendon, Oxford, 1991.

[12] K. Gopalsamy, *Stability and Oscillation in Delay Differential Equations of Population Dynamics*, Kluwer Academic, Boston, 1992.

[13] L.H. Erbe, Q.K. Kong, B.G. Zhang, *Oscillation Theory for Functional Differential Equations*, Marcel Dekker Inc., New York, 1995.

[14] R.P. Agarwal, M. Bohner, W.T. Li, *Nonoscillation and Oscillation: Theory for Functional Differential Equations*, Marcel Dekker Inc., 2004.

[15] S. Grace, R. Agarwal, P. Wong, et al., On the oscillation of fractional differential equations,*Fract. Calc. Appl. Anal.* **15** (2012) 222-231.

[16] Y. Bolat, On the oscillation of fractional-order delay differential equations with constant coefficients, *Commun. Nonlinear Sci. Numer. Simul.* **19** (2014) 3988-3993.

[17] J.S. Duan, Z. Wang, S.Z. Fu, The zeros of the solutions of the fractional oscillation equation, *Fract. Calc. Appl. Anal.* **17** (2014) 10-22.

[18] S. Harikrishnan, P. Prakash, J.J. Nieto, Forced oscillation of solutions of a nonlinear fractional partial differential equation, *Appl. Math. Comput.* **254** (2015) 14-19.

[19] V. S. Vladimirov, *Equations of Mathematical Physics*, Nauka, Moscow, 1981.

[20] D. Henry, *Geometric Theory of Semilinear Parabolic Partial Differential Equations*, Springer, Berlin, Germany, 1989.

# FIXED POINT RESULTS FOR A PAIR OF MULTI DOMINATED MAPPINGS ON A SMALLEST SUBSET IN $K$-SEQUENTIALLY DISLOCATED QUASI METRIC SPACE WITH AN APPLICATION

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Abstract. The aim of this paper is to establish fixed point results for semi $\alpha_*$-dominated multivalued pair of mappings satisfying generalized locally $\alpha_*$-$\psi$-type contractive conditions for a pair of multivalued dominated mappings in complete dislocated quasi metric space. Applications have been given and an example has been constructed to demonstrate the novelty of our results.

## 1. Introduction and preliminaries

Let $H : Z \to Z$ be a mapping. A point $x \in Z$ is said to be a fixed point of $Z$ if $x = Zx$. Fixed point results are a tool to estimate the unique solution of nonlinear functional equations. Many results appeared in literature related to the fixed point of mappings which are contractive on the whole domain. It may happen that $H : Z \to Z$ is not a contraction but is a contraction on a subset of $Z$. It is possible for one to get fixed point for such mappings if they satisfiy certain conditions. It has been shown the existence of fixed point for such mappings that fulfill certain conditions on a closed ball by Beg et al. [8] (see also [3, 4, 5, 14, 24, 25, 26, 27]).

Many authors established fixed point theorems in complete dislocated metric spaces. The idea of dislocated topologies has useful applications in the context of logic programming semantics (see [11]). Dislocated metric space (metric-like space) (see [17]) is a generalization of partial metric space (see [16]). Furthermore, dislocated quasi metric space (quasi-metric-like space) (see [8, 21, 30, 31]) generalized the idea of dislocated metric space and quasi-partial metric space (see [18, 25]).

Nadler [20] initiated the study of fixed point theorems for the multivalued mappings (see also [7]). Several results on multivalued mappings have been observed [1, 10, 19, 29]. Asl et al. [6] gave the idea of $\alpha_*$-$\psi$ contractive multifunctions, $\alpha_*$-admissible mapping and got some fixed point conclusions for these multifunctions (see also [2, 12]).

In this paper, we evaluate some fixed point results for$\alpha_*$-$\psi$-contractive type multivalued $\alpha_*$-dominated mappings in a closed ball in left(right) $K$-sequentially complete dislocated quasi metric space. Moreover, we give examples of multivalued mappings which are $\alpha_*$-dominated but not $\alpha_*$-admissible. We give the following definitions and results which will be needed in the sequel

**Definition 1.1.** [30] Let $X$ be a nonempty set and let $d_q : X \times X \to [0, \infty)$ be a function, called a dislocated quasi metric (or simply $d_q$-metric) if the following conditions hold for any $x, y, z \in X$ :

(i) If $d_q(x, y) = d_q(y, x) = 0$, then $x = y$;

(ii) $d_q(x, y) \leq d_q(x, z) + d_q(z, y)$.

The pair $(X, d_q)$ is called a dislocated quasi metric space.

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It is clear that if $d_q(x,y) = d_q(y,x) = 0$, then from (i), $x = y$. But if $x = y$, $d_q(x,y)$ may not be 0. It is observed that if $d_q(x,y) = d_q(y,x)$ for all $x, y \in X$, then $(X, d_q)$ becomes a dislocated metric space (metric-like space) $(X, d_l)$. For $x \in X$ and $\varepsilon > 0$, $B_{d_q}(x,\varepsilon) = \{y \in X : d_q(x,y) < \varepsilon$ and $d_q(y,x) < \varepsilon\}$ and $\overline{B_{d_q}(x,\varepsilon)} = \{y \in X : d_q(x,y) \leq \varepsilon$ and $d_q(y,x) \leq \varepsilon\}$ are an open ball and a closed ball in $(X, d_q)$, respectively. Also $B_{d_l}(x,\varepsilon) = \{y \in X : d_q(x,y) \leq \varepsilon\}$ is a closed ball in $(X, d_l)$.

**Example 1.2.** [8] Let $X = R^+ \cup \{0\}$ and $d_q(x,y) = x + \max\{x,y\}$ for any $x, y \in X$.

(i) If $d_q(x,y) = d_q(y,x) = 0$, then $x+\max\{x,y\} = y+\max\{y,x\} = 0$, which implies that $x = y = 0$.

(ii) Case 1: If $x \geq y$, then $d_q(x,y) = x + \max\{x,y\} = 2x$. Let $z \in X$. If $z \leq x$, then

$$\begin{aligned} d_q(x,z) + d_q(z,y) &= x + \max\{x,z\} + z + \max\{z,y\} \\ &= x + x + z + \max\{z,y\} \geq 2x = d_q(x,y). \end{aligned}$$

If $z > x$, then $d_q(x,z) + d_q(z,y) = x + z + z + z \geq 2x = d_q(x,y)$.

Case 2: If $x < y$, then $d_q(x,y) = x + y$. If $z \geq y$, then $d_q(x,z) + d_q(z,y) = x + z + z + z \geq x + y = d_q(x,y)$. If $z < y$, then $d_q(x,z) + d_q(z,y) = x + \max\{x,z\} + z + y \geq x + y = d_q(x,y)$.

Hence both the conditions of Definition 1.1 hold and so $d_q(x,y) = x+\max\{x,y\}$ defines a dislocated quasi metric on $X$.

**Definition 1.3.** [8] Let $(X, d_q)$ be a dislocated quasi metric space.

(a) A sequence $\{x_n\}$ in $(X, d_q)$ is called left (resp., right) $K$-Cauchy if for all $\varepsilon > 0$, there exists $n_0 \in N$ such that for all $n > m \geq n_0$ (resp., for all $m > n \geq n_0$), $d_q(x_m, x_n) < \varepsilon$.

(b) A sequence $\{x_n\}$ is called dislocated quasi-converges (for short, $d_q$-converges) to $x$ if $\lim_{n\to\infty} d_q(x_n, x) = \lim_{n\to\infty} d_q(x, x_n) = 0$ or for any $\varepsilon > 0$, there exists $n_0 \in N$ such that for all $n > n_0$, $d_q(x, x_n) < \varepsilon$ and $d_q(x_n, x) < \varepsilon$. In this case, $x$ is called a $d_q$-limit of $\{x_n\}$.

(c) $(X, d_q)$ is called left (resp., right) $K$-sequentially complete if every left (resp., right) $K$-Cauchy sequence in $X$ converges to a point $x \in X$ such that $d_q(x,x) = 0$.

**Definition 1.4.** Let $(X, d_q)$ be a dislocated quasi metric space. Let $K$ be a nonempty subset of $X$ and $x \in X$. An element $y_0 \in K$ is called a best approximation in $K$ if

$$\begin{aligned} d_q(x,K) &= d_q(x,y_0), \text{ where } d_q(x,K) = \inf_{y\in K} d_q(x,y), \\ d_q(K,x) &= d_q(y_0,x), \text{ where } d_q(K,x) = \inf_{y\in K} d_q(y,x). \end{aligned}$$

If each $x \in X$ has at least one best approximation in $K$, then $K$ is called a proximinal set.

We denote by $P(X)$ the set of all proximinal subsets of $X$.

**Definition 1.5.** [22] Let $(S,T) : X \to P(X)$ and $\beta : X \times X \to [0, +\infty)$ be a function. We say that the pair $(S,T)$ is $\beta_\star$-admissible if for all $x, y \in X$,

$$\beta(x,y) \geq 1 \text{ implies } \beta_\star(Tx, Sy) \geq 1 \text{ and } \beta_\star(Tx, Sy) \geq 1.$$

Again the pair $(S,T)$ is said to be $\beta$-admissible if $\beta(x,y) \geq 1$ implies $\beta(a,b) \geq 1$ for all $a \in Sx$ and $b \in Ty$.

Let $\Psi$ denote the family of all nondecreasing functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$ for all $t > 0$, where $\psi^n$ is the $n^{th}$ iterate of $\psi$. If $\psi \in \Psi$, then $\psi(t) < t$ for all $t > 0$.

**Definition 1.6.** [22] Let $(X, d)$ be a complete metric space and $\beta : X \times X \to [0, +\infty)$ be a mapping. Let $(S,T) : X \to P(X)$ be a multifunction and $\psi \in \Psi$. We say that the pair $(S,T)$ is a $\beta_\star$-$\psi$ contractive multifunction whenever

$$\begin{aligned} \beta_\star(Tx, Sy)H(Tx, Sy) &\leq \psi(d(x,y)) \ \forall \ x, y \in X, \\ \beta_\star(Sx, Ty)H(Sx, Ty) &\leq \psi(d(x,y)) \ \forall \ x, y \in X, \end{aligned}$$

where $\beta_\star(Tx, Sy) = \inf\{\beta(a, b) : a \in Tx,\ b \in Sy\}$.

**Definition 1.7.** [27] Let $(X, d_l)$ be a dislocated metric space, $S : X \to P(X)$ be a multivalued mapping and $\alpha : X \times X \to [0, +\infty)$. Let $A \subseteq X$. Then we say that $S$ is semi $\alpha_*$-admissible on $A$, whenever $\alpha(x, y) \geq 1$ implies $\alpha_\star(Sx, Sy) \geq 1$ for all $x \in A$, where $\alpha_*(Sx, Sy) = \inf\{\alpha(a, b) : a \in Sx, b \in Sx\}$. If $A = X$, then we say that $S$ is $\alpha_*$-admissible on $X$.

**Definition 1.8.** Let $(X, d_l)$ be a dislocated metric space, $S, T : X \to P(X)$ be multivalued mappings and $\alpha : X \times X \to [0, +\infty)$. Let $A \subseteq X$. Then we say that $S$ is semi $\alpha_*$-dominated on $A$, whenever $\alpha_*(x, Sx) \geq 1$ for all $x \in A$, where $\alpha_*(x, Sx) = \inf\{\alpha(x, b) : b \in Sx\}$. If $A = X$, then we say that $S$ is $\alpha_*$-dominated on $X$.

**Definition 1.9.** [23] The function $H_{d_q} : P(X) \times P(X) \to X$, defined by

$$H_{d_q}(A, B) = \max\{\sup_{a \in A} d_q(a, B),\ \sup_{b \in B} d_q(A, b)\},$$

is called a dislocated quasi Hausdorff metric on $P(X)$. Also $(P(X), H_{d_q})$ is known as a dislocated quasi Hausdorff metric space.

**Lemma 1.10.** [23] *Let $(X, d_q)$ be a dislocated quasi metric space. Let $(P(X), H_{d_q})$ be a dislocated quasi Hausdorff metric space on $P(X)$. Then, for all $A, B \in P(X)$ and for each $a \in A$, there exists $b_a \in B$ such that $H_{d_q}(A, B) \geq d_q(a, b_a)$ and $H_{d_q}(B, A) \geq d_q(b_a, a)$, where $d_q(a, B) = d_q(a, b_a)$ and $d_q(B, a) = d_q(b_a, a)$.*

**Example 1.11.** Let $X = \mathbb{R}$. Define the mapping $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 \text{ if } x > y \\ \frac{1}{2} \text{ otherwise.} \end{cases}$$

Define the multivalued mappings $S, T : X \to P(X)$ by

$$Sx = \{[x - 4, x - 3] \text{ if } x \in X\},$$

$$Ty = \{[y - 2, y - 1] \text{ if } y \in X\}.$$

Suppose $x = 3$ and $y = 2$. Since $3 > 2$, $\alpha(3, 2) \geq 1$. Now $\alpha_\star(S3, T2) = \inf\{\alpha(a, b) : a \in S3, b \in T2\} = \frac{1}{2} \ngeq 1$, which means that $\alpha_\star(S3, T2) < 1$, that is, the pair $(S, T)$ is not $\alpha_\star$-admissible. Also $\alpha_\star(S3, S2) \ngeq 1$ and $\alpha_\star(T3, T2) \ngeq 1$. This implies that $S$ and $T$ are not $\alpha_\star$-admissible individually. Since $\alpha_\star(x, Sx) = \inf\{\alpha(x, b) : b \in Sx\} \geq 1$ for all $x \in X$, $S$ is an $\alpha_\star$-dominated mapping. Similarly, $\alpha_\star(y, Ty) = \inf\{\alpha(y, b) : b \in Ty\} \geq 1$. Hence it is clear that $S$ and $T$ are $\alpha_\star$-dominated but not $\alpha_\star$-admissible.

**Lemma 1.12.** [23] *Every closed ball $Y$ in a left (right) $K$-sequentially complete dislocated quasi metric space $X$ is left (right) $K$-sequentially complete.*

## 2. Main results

Let $(X, d_q)$ be a dislocated quasi metric space, $x_0 \in X$ and $S, T : X \to P(X)$ be multifunctions on $X$. Let $x_1 \in Sx_0$ be an element such that $d_q(x_0, Sx_0) = d_q(x_0, x_1)$. Let $x_2 \in Tx_1$ be such that $d_q(x_1, Tx_1) = d_q(x_1, x_2)$. Let $x_3 \in Sx_2$ be such that $d_q(x_2, Sx_2) = d_q(x_2, x_3)$.Continuing this process, we construct a sequence $\{x_n\}$ of points in $X$ such that $x_{2n+1} \in Sx_{2n}$ and $x_{2n+2} \in Tx_{2n+1}$, where $n = 0, 1, 2, ....$ Also $d_q(x_{2n}, Sx_{2n}) = d_q(x_{2n}, x_{2n+1})$ and $d_q(x_{2n+1}, Tx_{2n+1}) = d_q(x_{2n+1}, x_{2n+2})$. We denote this iterative sequence by $\{TS(x_n)\}$. We say that $\{TS(x_n)\}$ is a sequence in $X$ generated by $x_0$.

**Theorem 2.1.** *Let $(X, d_q)$ be a left (right) $K$-sequentially complete dislocated quasi metric space. Suppose there exists a function $\alpha : X \times X \to [0, \infty)$. Let $r > 0$, $x_0 \in \overline{B_{d_q}(x_0, r)}$ and $S, T : X \to P(X)$ be semi $\alpha_*$-dominated mappings on $\overline{B_{d_q}(x_0, r)}$. Assume that, for some $\psi \in \Psi$ and $D_q(x, y) = \max\{d_q(x, y), d_q(x, Sx), d_q(y, Ty)\}$, the following hold:*

$$\max\{\alpha_*(x, Sx)H_{dq}(Sx, Ty), \alpha_*(y, Ty)H_{dq}(Ty, Sx)\} \leq \min\{\psi(D_q(x, y)), \psi(D_q(y, x))\} \tag{2.1}$$

*for all $x, y \in \overline{B_{d_q}(x_0, r)} \cap \{TS(x_n)\}$ with either $\alpha(x, y) \geq 1$ or $\alpha(y, x) \geq 1$ whenever $x \in Sy$, and*

$$\sum_{i=0}^{n} \max\{\psi^i(d_q(x_1, x_0), \psi^i(d_q(x_0, x_1))\} \leq r \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{2.2}$$

*Then $\{TS(x_n)\}$ is a sequence in $\overline{B_{d_q}(x_0, r)}$ and $\{TS(x_n)\} \to x^* \in \overline{B_{d_q}(x_0, r)}$. Also if the inequality (2.1) holds for $x^*$ and either $\alpha(x_n, x^*) \geq 1$ or $\alpha(x^*, x_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, then $S$ and $T$ have a common fixed point $x^*$ in $\overline{B_{d_q}(x_0, r)}$ and $d_q(x^*, x^*) = 0$.*

*Proof.* Consider a sequence $\{TS(x_n)\}$ generated by $x_0$. Then, we have $x_{2n+1} \in Sx_{2n}$ and $x_{2n+2} \in Tx_{2n+1}$, where $n = 0, 1, 2, ....$ Also $d_q(x_{2n}, Sx_{2n}) = d_q(x_{2n}, x_{2n+1})$, $d_q(x_{2n+1}, Tx_{2n+1}) = d_q(x_{2n+1}, x_{2n+2})$. By Lemma 1.10, we have

$$d_q(x_{2n}, x_{2n+1}) \leq H_{d_q}(Tx_{2n-1}, Sx_{2n}), \tag{2.3}$$

$$d_q(x_{2n+1}, x_{2n+2}) \leq H_{d_q}(Sx_{2n}, Tx_{2n+1}) \tag{2.4}$$

for all $n = 1, 2, ....$ From (2.2), we have

$$\max\{d_q(x_1, x_0), d_q(x_0, x_1)\} \leq \sum_{i=0}^{j} \max\{\psi^i(d_q(x_1, x_0), \psi^i(d_q(x_0, x_1))\} \leq r.$$

It follows that, $d_q(x_1, x_0) \leq r$ and $d_q(x_0, x_1) \leq r$. Hence we have

$$x_1 \in \overline{B_{d_q}(x_0, r)}.$$

Let $x_2, \cdots, x_j \in \overline{B_{d_q}(x_0, r)}$ for some $j \in \mathbb{N}$. Since $S, T : X \to P(X)$ are semi $\alpha_*$-dominated mappings on $\overline{B_{d_q}(x_0, r)}$, $\alpha_*(x_{2i}, Sx_{2i}) \geq 1$ and $\alpha_*(x_{2i+1}, Tx_{2i+1}) \geq 1$. Since $\alpha_*(x_{2i}, Sx_{2i}) \geq 1$, $\inf\{\alpha(x_{2i}, b) : b \in Sx_{2i}\} \geq 1$. Also $x_{2i+1} \in Sx_{2i}$ and so $\alpha(x_{2i}, x_{2i+1}) \geq 1$. Now by (2.3), we obtain

$$\begin{aligned}
d_q(x_{2i+1}, x_{2i+2}) &\leq H_{d_q}(Sx_{2i}, Tx_{2i+1}) \leq \max\{\alpha_*(x_{2i}, Sx_{2i})H_{d_q}(Sx_{2i}, Tx_{2i+1}), \\
&\quad \alpha_*(x_{2i+1}, Tx_{2i+1})H_{d_q}(Tx_{2i+1}, Sx_{2i})\} \\
&\leq \min\{\psi(D_q(x_{2i}, x_{2i+1})), \psi(D_q(x_{2i+1}, x_{2i}))\} \leq \psi(D_q(x_{2i}, x_{2i+1})) \\
&\leq \psi(\max\{d_q(x_{2i}, x_{2i+1}), d_q(x_{2i}, Sx_{2i}), d_q(x_{2i+1}, Tx_{2i+1})\}) \\
&\leq \psi(\max\{d_q(x_{2i}, x_{2i+1}), d_q(x_{2i}, x_{2i+1}), d_q(x_{2i+1}, x_{2i+2})\}) \\
&\leq \psi(\max\{d_q(x_{2i}, x_{2i+1}), d_q(x_{2i+1}, x_{2i+2})\}).
\end{aligned}$$

If $\max\{d_q(x_{2i}, x_{2i+1}), d_q(x_{2i+1}, x_{2i+2})\} = d_q(x_{2i+1}, x_{2i+2})$, then $d_q(x_{2i+1}, x_{2i+2}) \leq \psi(d_q(x_{2i+1}, x_{2i+2}))$, which contradicts to the fact that $\psi(t) < t$ for all $t > 0$. So $\max\{d_q(x_{2i}, x_{2i+1}), d_q(x_{2i+1}, x_{2i+2})\} = d_q(x_{2i}, x_{2i+1})$. Hence we obtain

$$d_q(x_{2i+1}, x_{2i+2}) \leq \psi(d_q(x_{2i}, x_{2i+1})). \tag{2.5}$$

Since $\alpha_*(x_{2i-1}, Tx_{2i-1}) \geq 1$ and $x_{2i} \in Tx_{2i-1}$, $\alpha(x_{2i-1}, x_{2i}) \geq 1$. Now by (2.4), we have

$$\begin{aligned} d_q(x_{2i}, x_{2i+1}) &\leq H_{d_q}(Tx_{2i-1}, Sx_{2i}) \leq \max\{\alpha_*(x_{2i}, Sx_{2i})H_{d_q}(Sx_{2i}, Tx_{2i-1}), \\ &\quad \alpha_*(x_{2i-1}, Tx_{2i-1})H_{d_q}(Tx_{2i-1}, Sx_{2i})\} \\ &\leq \min\{\psi(D_q(x_{2i}, x_{2i-1})), \psi(D_q(x_{2i-1}, x_{2i}))\} \leq \psi(D_q(x_{2i}, x_{2i-1})) \\ &\leq \psi(\max\{d_q(x_{2i}, x_{2i-1}), d_q(x_{2i}, Sx_{2i}), d_q(x_{2i-1}, Tx_{2i-1})\}) \\ &\leq \psi(\max\{d_q(x_{2i}, x_{2i-1}), d_q(x_{2i}, x_{2i+1}), d_q(x_{2i-1}, x_{2i})\}) \\ &\leq \psi(\max\{d_q(x_{2i}, x_{2i-1}), d_q(x_{2i}, x_{2i+1}), d_q(x_{2i-1}, x_{2i})\}). \end{aligned}$$

If $\max\{d_q(x_{2i}, x_{2i-1}), d_q(x_{2i}, x_{2i+1}), d_q(x_{2i-1}, x_{2i})\} = d_q(x_{2i}, x_{2i+1})$, then $d_q(x_{2i}, x_{2i+1}) \leq \psi(d_q(x_{2i}, x_{2i+1}))$, which contradicts to the fact that $\psi(t) < t$ for all $t > 0$. Hence we obtain

$$d_q(x_{2i}, x_{2i+1}) \leq \psi(\max\{d_q(x_{2i}, x_{2i-1}), d_q(x_{2i-1}, x_{2i})\}).$$

If $\max\{d_q(x_{2i}, x_{2i-1}), d_q(x_{2i-1}, x_{2i})\} = d_q(x_{2i-1}, x_{2i})$, then

$$d_q(x_{2i}, x_{2i+1})) \leq \psi(d_q(x_{2i-1}, x_{2i})).$$

Since $\psi$ is a nondecreasing function,

$$\psi(d_q(x_{2i}, x_{2i+1})) \leq \psi^2(d_q(x_{2i-1}, x_{2i})).$$

By (2.5), we obtain

$$d_q(x_{2i+1}, x_{2i+2}) \leq \psi^2(d_q(x_{2i-1}, x_{2i})). \tag{2.6}$$

If $\max\{d_q(x_{2i}, x_{2i-1}), d_q(x_{2i-1}, x_{2i})\} = d_q(x_{2i}, x_{2i-1})$, then

$$d_q(x_{2i+1}, x_{2i+2}) \leq \psi^2(d_q(x_{2i}, x_{2i-1})) \tag{2.7}$$

By (2.6) and (2.7), we obtain

$$d_q(x_{2i+1}, x_{2i+2}) \leq \max\{\psi^2(d_q(x_{2i}, x_{2i-1})), \psi^2(d_q(x_{2i-1}, x_{2i}))\}.$$

Continuing in this way, we obtain

$$d_q(x_{2i+1}, x_{2i+2}) \leq \max\{\psi^{2i+1}(d_q(x_1, x_0)), \psi^{2i+1}(d_q(x_0, x_1))\} \tag{2.8}$$

Similarly, we have

$$d_q(x_{2i}, x_{2i+1}) \leq \max\{\psi^{2i}(d_q(x_1, x_0)), \psi^{2i}(d_q(x_0, x_1))\}. \tag{2.9}$$

By (2.8) and (2.9), we obtain

$$d_q(x_j, x_{j+1}) \leq \max\{\psi^j(d_q(x_1, x_0)), \psi^j(d_q(x_0, x_1))\} \text{ for some } j \in \mathbb{N}. \tag{2.10}$$

By Lemma 1.10 and (2.1), we have

$$\begin{aligned} d_q(x_{2i+2}, x_{2i+1}) &\leq H_{d_q}(Tx_{2i+1}, Sx_{2i}) \leq \max\{\alpha_*(x_{2i}, Sx_{2i})H_{d_q}(Sx_{2i}, Tx_{2i+1}), \\ &\quad \alpha_*(x_{2i+1}, Tx_{2i+1})H_{d_q}(Tx_{2i+1}, Sx_{2i})\} \\ &\leq \min\{\psi(D_q(x_{2i}, x_{2i+1})), \psi(D_q(x_{2i+1}, x_{2i}))\}. \end{aligned}$$

By the same reasoning as in the proof of (**??**), we have

$$d_q(x_{j+1}, x_j) \leq \max\{\psi^j(d_q(x_1, x_0)), \psi^j(d_q(x_0, x_1))\} \text{ for some } j \in \mathbb{N}. \tag{2.10}$$

Now

$$\begin{aligned} d_q(x_0, x_{j+1}) &\leq d_q(x_0, x_1) + ... + d_q(x_j, x_{j+1}) \\ &\leq d_q(x_0, x_1) + ... + \max\{\psi^j(d_q(x_1, x_0)), \psi^j(d_q(x_0, x_1))\} \\ &\leq \sum_{i=0}^{j} \max\{\psi^i(d_q(x_1, x_0), \psi^i(d_q(x_0, x_1))\} \leq r. \end{aligned} \tag{2.11}$$

Also

$$\begin{aligned} d_q(x_{j+1}, x_0) &\leq d_q(x_{j+1}, x_j) + ... + d_q(x_1, x_0) \\ &\leq \max\{\psi^j(d_q(x_1, x_0)), \psi^j(d_q(x_0, x_1))\} + ... + d_q(x_1, x_0) \\ &\leq \sum_{i=0}^{j} \max\{\psi^i(d_q(x_1, x_0), \psi^i(d_q(x_0, x_1))\} \leq r. \end{aligned} \tag{2.12}$$

By (2.11) and (2.12), we have $x_{j+1} \in \overline{B_{dq}(x_0, r)}$. Hence by mathematical induction $x_n \in \overline{B_{dq}(x_0, r)}$ for all $n \in \mathbb{N}$. Therefore, $\{TS(x_n)\}$ is a sequence in $\overline{B_{d_q}(x_0, r)}$. Since $S, T : X \to P(X)$ are semi $\alpha_*$-dominated mappings on $\overline{B_{d_q}(x_0, r)}$, $\alpha_*(x_n, Sx_n) \geq 1$ and $\alpha_*(x_n, Tx_n) \geq 1$ for all $n \in \mathbb{N}$. Now (2.8) and (2.9) can be written as

$$d_q(x_n, x_{n+1}) \leq \max\{\psi^n(d_q(x_1, x_0)), \psi^n(d_q(x_0, x_1))\} \text{ for all } n \in \mathbb{N}. \tag{2.13}$$

$$d_q(x_{n+1}, x_n) \leq \max\{\psi^n(d_q(x_1, x_0)), \psi^n(d_q(x_0, x_1))\} \text{ for all } n \in \mathbb{N}. \tag{2.14}$$

Fix $\varepsilon > 0$ and let $k_1(\varepsilon) \in \mathbb{N}$ such that $\sum\limits_{k \geq k_1(\varepsilon)} \max\{\psi^k(d_q(x_1, x_0)), \psi^k(d_q(x_0, x_1))\} < \varepsilon$. Let $n, m \in \mathbb{N}$ with $m > n > k_1(\varepsilon)$. Then we obtain

$$\begin{aligned} d_q(x_n, x_m) &\leq \sum_{k=n}^{m-1} d_q(x_k, x_{k+1}) \\ &\leq \sum_{k=n}^{m-1} \max\{\psi^k(d_q(x_1, x_0)), \psi^k(d_q(x_0, x_1))\} \text{ by (2.13)}, \\ d_q(x_n, x_m) &\leq \sum_{k \geq k_1(\varepsilon)} \max\{\psi^n(d_q(x_1, x_0)), \psi^n(d_q(x_0, x_1))\} < \varepsilon. \end{aligned}$$

Thus we obtain that $\{TS(x_n)\}$ is a left $K$-Cauchy sequence in $(\overline{B_{d_q}(x_0, r)}, d_q)$.

Similarly, by (2.14), we have

$$d_q(x_m, x_n) \leq \sum_{k=n}^{m-1} d_q(x_{k+1}, x_k) < \varepsilon.$$

Hence $\{TS(x_n)\}$ is a right $K$-Cauchy sequence in $(\overline{B_{d_q}(x_0, r)}, d_q)$. Since every closed ball in left(right) $K$-sequentially complete dislocated quasi metric space is left(right) $K$-sequentially complete, there exists $x^* \in \overline{B_{dq}(x_0, r)}$ such that $\{TS(x_n)\} \to x^*$, that is,

$$\lim_{n \to \infty} d_q(x_n, x^*) = \lim_{n \to \infty} d_q(x^*, x_n) = 0. \tag{2.15}$$

Now

$$\begin{aligned} d_q(x^*, Tx^*) &\leq d_q(x^*, x_{2n+1}) + d_q(x_{2n+1}, Tx^*) \\ &\leq d_q(x^*, x_{2n+1}) + H_{d_q}(Sx_{2n}, Tx^*) \text{ by Lemma 1.10.} \end{aligned} \tag{2.16}$$

Since $\alpha_*(x^*, Tx^*) \geq 1$, $\alpha_*(x_{2n}, Sx_{2n}) \geq 1$ and $\alpha(x_{2n}, x^*) \geq 1$, we obtain

$$\begin{aligned} H_{d_q}(Sx_{2n}, Tx^*) &\leq \max\{\alpha_*(x_{2n}, Sx_{2n})H_{d_q}(Sx_{2n}, Tx^*), \alpha_*(x^*, Tx^*)H_{dq}(Tx^*, Sx_{2n})\} \\ &\leq \min\{\psi(D_q(x_{2n}, x^*)), \psi(D_q(x^*, x_{2n}))\} \\ &\leq \psi(\max\{d_q(x_{2n}, x^*), d_q(x_{2n}, x_{2n+1}), d_q(x^*, Tx^*)\}) \\ &\leq \psi(\max\{d_q(x_{2n}, x^*), d_q(x_{2n}, x^*) + d_q(x^*, x_{2n+1}), d_q(x^*, Tx^*)\}). \end{aligned} \tag{2.17}$$

By (2.16) and (2.17), we have

$$d_q(x^*, Tx^*) \leq d_q(x^*, x_{2n+1}) + \psi(\max\{d_q(x_{2n}, x^*), d_q(x_{2n}, x^*) + d_q(x^*, x_{2n+1}), d_q(x^*, Tx^*)\}).$$

Letting $n \to \infty$, and by (2.15), we obtain $d_q(x^*, Tx^*) \leq \psi(d_q(x^*, Tx^*))$ and hence $d_q(x^*, Tx^*) = 0$. Now

$$\begin{aligned} d_q(Tx^*, x^*) &\leq d_q(Tx^*, x_{2n+1}) + d_q(x_{2n+1}, x^*) \\ &\leq H_{d_q}(Tx^*, Sx_{2n}) + d_q(x_{2n+1}, x^*), \text{ by Lemma 1.10.} \end{aligned}$$

By using a similar argument, we obtain $d_q(Tx^*, x^*) = 0$ or $x^* \in Tx^*$.

Similarly, by Lemma 1.10, (2.15) and

$$d_q(x^*, Sx^*) \leq d_q(x^*, x_{2n+2}) + d_q(x_{2n+2}, Sx^*),$$

we can show that $d_q(x^*, Sx^*) = 0$ and $x^* \in Sx^*$.

Similarly, $d_q(Sx^*, x^*) = 0$. Hence $S$ and $T$ have a common fixed point $x^*$ in $\overline{B_{d_q}(x_0, r)}$. Now

$$d_q(x^*, x^*) \leq d_q(x^*, Tx^*) + d_q(Tx^*, x^*) \leq 0.$$

This implies that $d_q(x^*, x^*) = 0$. □

**Corollary 2.2.** *Let $(X, d_q)$ be a left (right) $K$-sequentially complete dislocated quasi metric space. Suppose that there exists a function $\alpha : X \times X \to [0, \infty)$. Let $r > 0$, $x_0 \in \overline{B_{d_q}(x_0, r)}$ and $S : X \to P(X)$ be a semi $\alpha_*$-dominated mapping on $\overline{B_{d_q}(x_0, r)}$. Assume that, for some $\psi \in \Psi$ and $D_q(x, y) = \max\{d_q(x, y), d_q(x, Sx), d_q(y, Sy)\}$, the following hold:*

$$\max\{\alpha_*(x, Sx)H_{dq}(Sx, Sy),\ \alpha_*(y, Sy)H_{dq}(Sy, Sx)\} \leq \min\{\psi(D_q(x, y)), \psi(D_q(y, x))\} \tag{2.18}$$

*for all $x, y \in \overline{B_{d_q}(x_0, r)} \cap \{S(x_n)\}$ with either $\alpha(x, y) \geq 1$ or $\alpha(y, x) \geq 1$, and*

$$\sum_{i=0}^{n} \max\{\psi^i(d_q(x_1, x_0), \psi^i(d_q(x_0, x_1))\} \leq r \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

*Then $\{S(x_n)\}$ is a sequence in $\overline{B_{d_q}(x_0, r)}$ and $\{S(x_n)\} \to x^* \in \overline{B_{d_q}(x_0, r)}$. Also, if (2.18) holds for $x^*$ and either $\alpha(x_n, x^*) \geq 1$ or $\alpha(x^*, x_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, then $S$ has a fixed point $x^*$ in $\overline{B_{d_q}(x_0, r)}$ and $d_q(x^*, x^*) = 0$.*

**Corollary 2.3.** *Let $(X, d_l)$ be a complete dislocated metric space. Suppose that there exists a function $\alpha : X \times X \to [0, \infty)$. Let $r > 0$, $x_0 \in \overline{B_{d_l}(x_0, r)}$ and $S, T : X \to P(X)$ be semi $\alpha_*$-dominated mappings on $\overline{B_{d_l}(x_0, r)}$. Assume that, for some $\psi \in \Psi$ and $D_l(x, y) = \max\{d_l(x, y), d_l(x, Sx), d_l(y, Ty)\}$, the following hold:*

$$\max\{\alpha_*(x, Sx)H_{d_l}(Sx, Ty),\ \alpha_*(y, Ty)H_{d_l}(Sx, Ty)\} \leq \psi(D_l(x, y)) \tag{2.19}$$

*for all $x, y \in \overline{B_{d_l}(x_0, r)} \cap \{TS(x_n)\}$ with either $\alpha(x, y) \geq 1$ or $\alpha(y, x) \geq 1$, and*

$$\sum_{i=0}^{n} \psi^i(d_l(x_0, x_1)) \leq r \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

*Then* $\{TS(x_n)\}$ *is a sequence in* $\overline{B_{d_l}(x_0,r)}$ *and* $\{TS(x_n)\}\to x^*\in\overline{B_{d_l}(x_0,r)}$. *Also if* (2.19) *holds for* $x^*$ *and either* $\alpha(x_n,x^*)\geq 1$ *or* $\alpha(x^*,x_n)\geq 1$ *for all* $n\in\mathbb{N}\cup\{0\}$, *then* $S$ *and* $T$ *have a common fixed point* $x^*$ *in* $\overline{B_{d_l}(x_0,r)}$ *and* $d_q(x^*,x^*)=0$.

**Corollary 2.4.** *Let* $(X,d_l)$ *be a complete dislocated metric space. Suppose that there exists a function* $\alpha:X\times X\to[0,\infty)$. *Let* $r>0$, $x_0\in\overline{B_{d_l}(x_0,r)}$ *and* $S:X\to P(X)$ *be a semi* $\alpha_*$*-dominated mapping on* $\overline{B_{d_l}(x_0,r)}$. *Assume that, for some* $\psi\in\Psi$ *and* $D_l(x,y)=\max\{d_l(x,y),d_l(x,Sx),d_l(y,Sy)\}$, *the following hold:*

$$\max\{\alpha_*(x,Sx)H_{d_l}(Sx,Sy),\ \alpha_*(y,Sy)H_{d_l}(Sx,Sy)\}\leq\psi(D_l(x,y)) \tag{2.20}$$

*for all* $x,y\in\overline{B_{d_l}(x_0,r)}\cap\{S(x_n)\}$ *with either* $\alpha(x,y)\geq 1$ *or* $\alpha(y,x)\geq 1$, *and*

$$\sum_{i=0}^{n}\psi^i(d_l(x_0,x_1))\leq r \text{ for all } n\in\mathbb{N}\cup\{0\}.$$

*Then* $\{S(x_n)\}$ *is a sequence in* $\overline{B_{d_l}(x_0,r)}$ *and* $\{S(x_n)\}\to x^*\in\overline{B_{d_l}(x_0,r)}$. *Also, if* (2.20) *holds for* $x^*$ *and either* $\alpha(x_n,x^*)\geq 1$ *or* $\alpha(x^*,x_n)\geq 1$ *for all* $n\in\mathbb{N}\cup\{0\}$, *then* $S$ *has a fixed point* $x^*$ *in* $\overline{B_{d_l}(x_0,r)}$ *and* $d_q(x^*,x^*)=0$.

Let $X$ be a nonempty set , $\preceq$ a partial order on $X$ and $A\subseteq X$. We say that $a\preceq B$ whenever for all $b\in B$, we have $a\preceq b$. A mapping $S:X\to P(X)$ is said to be semi dominated on $A$ if $a\preceq Sa$ for each $a\in A\subseteq X$. If $A=X$, then $S:X\to P(X)$ is said to be dominated.

**Corollary 2.5.** *Let* $(X,\preceq,d_q)$ *be a left (right)* $K$*-sequentially ordered complete dislocated quasi metric space. Let* $r>0$, $x_0\in\overline{B_{d_q}(x_0,r)}$ *and* $S,T:X\to P(X)$ *be semi dominated mappings on* $\overline{B_{d_q}(x_0,r)}$. *Assume that, for some* $\psi\in\Psi$ *and* $D_q(x,y)=\max\{d_q(x,y),d_q(x,Sx),d_q(y,Ty)\}$, *the following hold:*

$$\max\{H_{dq}(Sx,Ty),H_{dq}(Ty,Sx)\}\leq\min\{\psi(D_q(x,y)),\psi(D_q(y,x))\} \tag{2.21}$$

*for all* $x,y\in\overline{B_{d_q}(x_0,r)}\cap\{TS(x_n)\}$ *with either* $x\preceq y$ *or* $y\preceq x$, *and*

$$\sum_{i=0}^{n}\max\{\psi^i(d_q(x_1,x_0),\psi^i(d_q(x_0,x_1))\}\leq r \text{ for all } n\in\mathbb{N}\cup\{0\}. \tag{2.22}$$

*Then* $\{TS(x_n)\}$ *is a sequence in* $\overline{B_{d_q}(x_0,r)}$ *and* $\{TS(x_n)\}\to x^*\in\overline{B_{d_q}(x_0,r)}$. *Also if* (2.21) *holds for* $x^*$ *and either* $x_n\preceq x^*$ *or* $x^*\preceq x_n$ *for all* $n\in\mathbb{N}\cup\{0\}$, *then* $S$ *and* $T$ *have a common fixed point* $x^*$ *in* $\overline{B_{d_q}(x_0,r)}$ *and* $d_q(x^*,x^*)=0$.

*Proof.* Let $\alpha:X\times X\to[0,+\infty)$ be a function defined by $\alpha(x,y)=1$ for all $x\in\overline{B_{d_q}(x_0,r)}$ with either $x\preceq y$ or $y\preceq x$, and $\alpha(x,y)=0$ for all other elements $x,y\in X$. Since $S$ and $T$ are the semi dominated mappings on $\overline{B_{d_q}(x_0,r)}$, $x\preceq Sx$ and $x\preceq Tx$ for all $x\in\overline{B_{d_q}(x_0,r)}$. This implies that $x\preceq b$ for all $b\in Sx$ and $x\preceq c$ for all $c\in Tx$. So $\alpha(x,b)=1$ for all $b\in Sx$ and $\alpha(x,c)=1$ for all $c\in Tx$. This implies that $\inf\{\alpha(x,y):y\in Sx\}=1$ and $\inf\{\alpha(x,y):y\in Tx\}=1$. Hence $\alpha_*(x,Sx)=1$ and $\alpha_*(x,Tx)=1$ for all $x\in\overline{B_{d_q}(x_0,r)}$. So $S,T:X\to P(X)$ are semi $\alpha_*$-dominated mappings on $\overline{B_{d_q}(x_0,r)}$. Moreover, (2.21) can be written as

$$\max\{\alpha_*(x,Sx)H_{dq}(Sx,Ty),\ \alpha_*(y,Ty)H_{dq}(Ty,Sx)\}\leq\min\{\psi(D_q(x,y)),\psi(D_q(y,x))\}$$

for all $x,y$ in $\overline{B_{d_q}(x_0,r)}\cap\{TS(x_n)\}$ with either $\alpha(x,y)\geq 1$ or $\alpha(y,x)\geq 1$. Also, (2.22) holds. Then by Theorem 2.1, $\{TS(x_n)\}$ is a sequence in $\overline{B_{d_q}(x_0,r)}$ and $\{TS(x_n)\}\to x^*\in\overline{B_{d_q}(x_0,r)}$. Now, $x_n,x^*\in\overline{B_{d_q}(x_0,r)}$ and either $x_n\preceq x^*$ or $x^*\preceq x_n$ implies that either $\alpha(x_n,x^*)\geq 1$ or $\alpha(x^*,x_n)\geq 1$.

So all the conditions of Theorem 2.1 are satisfied. Hence by Theorem 2.1, $S$ and $T$ have a common fixed point $x^*$ in $\overline{B_{d_q}(x_0,r)}$ and $d_q(x^*,x^*)=0$. □

**Example 2.6.** Let $X=Q^+\cup\{0\}$ and let $d_l: X\times X\to X$ be a complete dislocated quasi metric on $X$ defined by

$$d_q(x,y)=x+y \text{ for all } x,y\in X.$$

Define multivalued mappings $S,T: X\times X\to P(X)$ by,

$$Sx=\begin{cases} [\frac{x}{3},\frac{2}{3}x] \text{ if } x\in[0,1]\cap X \\ [x,x+1] \text{ if } x\in(1,\infty)\cap X \end{cases}$$

and

$$Tx=\begin{cases} [\frac{x}{4},\frac{3}{4}x] \text{ if } x\in[0,1]\cap X \\ [x+1,x+3] \text{ if } x\in(1,\infty)\cap X. \end{cases}$$

Considering $x_0=1$, $r=8$, we get $\overline{B_{d_q}(x_0,r)}=[0,7]\cap X$. Now $d_q(x_0,Sx_0)=d_q(1,S1)=d_q(1,\frac{1}{3})=\frac{4}{3}$. So we obtain a sequence $\{TS(x_n)\}=\{1,\frac{1}{12},\frac{1}{144},\frac{1}{1728},....\}$ in $X$ generated by $x_0$. Also $\overline{B_{d_q}(x_0,r)}\cap\{TS(x_n)\}=\{1,\frac{1}{12},\frac{1}{144},...\}$. Let $\psi(t)=\frac{4t}{5}$ and

$$\alpha(x,y)=\begin{cases} 1 \text{ if } x,y\in[0,1] \\ \frac{3}{2} \quad \text{otherwise.} \end{cases}$$

Now if $x,y\notin\overline{B_{d_q}(x_0,r)}\cap\{TS(x_n)\}$, then we have the following cases.

Case 1. If $\max\{\alpha_*(x,Sx)H_{d_q}(Sx,Ty),\ \alpha_*(y,Ty)H_{d_q}(Ty,Sx)\}=\alpha_*(x,Sx)H_{d_q}(Sx,Ty)$, then for $x=2$ and $y=3$, we have

$$\alpha_*(2,S2)Hd_q(S2,T3)=\frac{3}{2}(8)>\psi(D_q(x,y))=\frac{28}{5}.$$

Case 2. If $\max\{\alpha_*(x,Sx)H_{d_q}(Sx,Ty),\ \alpha_*(y,Ty)H_{dq}(Ty,Sx)\}=\alpha_*(y,Ty)H_{dq}(Ty,Sx)$, then for $x=2$and $y=3$, we have

$$\alpha_*(3,T3)Hd_q(T3,S2)=\frac{3}{2}(8)>\psi(D_q(y,x))=\frac{28}{5}.$$

So the contractive condition does not hold on thewhole space $X$.

Now, for all $x,y\in\overline{B_{d_q}(x_0,r)}\cap\{TS(x_n)\}$, we have the following.

Case 3. If $\max\{\alpha_*(x,Sx)H_{d_q}(Sx,Ty),\ \alpha_*(y,Ty)H_{d_q}(Ty,Sx)\}=\alpha_*(x,Sx)H_{d_q}(Sx,Ty)$, then we have

$$\begin{aligned}
\alpha_*(x,Sx)H_{d_q}(Sx,Ty) &= 1[\max\{\sup_{a\in Sx}d_q(a,Ty),\sup_{b\in Ty}d_q(Sx,b)\}] \\
&= \max\{\sup_{a\in Sx}d_q(a,[\frac{y}{4},\frac{3y}{4}]),\sup_{b\in Ty}d_q([\frac{x}{3},\frac{2x}{3}],b)\} \\
&= \max\{d_q(\frac{2x}{3},[\frac{y}{4},\frac{3y}{4}]),d_q([\frac{x}{3},\frac{2x}{3}],\frac{3y}{4})\} \\
&= \max\{d_q(\frac{2x}{3},\frac{y}{4}),d_q(\frac{x}{3},\frac{3y}{4})\} \\
&= \max\{\frac{2x}{3}+\frac{y}{4},\frac{x}{3}+\frac{3y}{4}\} \\
&\leq \psi(\max\{x+y,\frac{4x}{3},\frac{5y}{4}\})=\psi(D_q(x,y)).
\end{aligned}$$

Case 4. If $\max\{\alpha_*(x,Sx)H_{d_q}(Sx,Ty),\ \alpha_*(y,Ty)H_{d_q}(Ty,Sx)\} = \alpha_*(y,Ty)H_{d_q}(Ty,Sx)$, then we have

$$\begin{aligned}
\alpha_*(y,Ty)H_{d_q}(Ty,Sx) &= 1[\max\{\sup_{b\in Ty} d_q(Sx,b),\ \sup_{a\in Sx} d_q(a,Ty)\}] \\
&= \max\{\sup_{b\in Ty} d_q([\frac{x}{3},\frac{2x}{3}],b),\ \sup_{a\in Sx} d_q(a,[\frac{y}{4},\frac{3y}{4}])\} \\
&= \max\{d_q([\frac{x}{3},\frac{2x}{3}],\frac{3y}{4}), d_q(\frac{2x}{3},[\frac{y}{4},\frac{3y}{4}])\} \\
&= \max\{d_q(\frac{x}{3},\frac{3y}{4}), d_q(\frac{2x}{3},\frac{y}{4})\} \\
&= \max\{\frac{x}{3}+\frac{3y}{4},\frac{2x}{3}+\frac{y}{4}\} \\
&\leq \psi(\max\{y+x,\frac{5y}{4},\frac{4x}{3}\}) = \psi(D_q(y,x)).
\end{aligned}$$

So the contractive condition holds on $\overline{B_{d_q}(x_0,r)} \cap \{TS(x_n)\}$. Also

$$\sum_{i=0}^{n} \max\{\psi^i(d_q(x_1,x_0),\psi^i(d_q(x_0,x_1))\} = \frac{4}{3}\sum_{i=0}^{n}(\frac{4}{5})^i < 8 = r.$$

Hence all the conditions of Theorem 2.1 are satisfied. Now we have $\{TS(x_n)\}$ is a sequence in $\overline{B_{d_q}(x_0,r)}$ and $\{TS(x_n)\} \to 0 \in \overline{B_{d_q}(x_0,r)}$. Also $\alpha(x_n,0) \geq 1$ or $\alpha(0,x_n) \geq 1$ for all $n \in \mathbb{N}\cup\{0\}$. Moreover, 0 is a common fixed point of $S$ and $T$.

## 3. Fixed point results for graphic contractions

In this section, we present an application of Theorem 2.1 in graph theory. Jachymski [15] proved the contraction principle for mappings on a metric space with a graph. Let $(X,d)$ be a metric space and $\triangle$ represent the diagonal of the cartesian product $X\times X$. Assume that $G$ is a directed graph and $V(G)$ is the set of vertices along with $X$ and the set $E(G)$ denotes the edges of $X$ included all loops, i.e., $E(G)\supseteq\triangle$. If $G$ has no parallel edges, then we can unify $G$ with pair $(V(G),E(G))$. Furthermore, we consider $G$ as a weighted graph (see [15]) which showing to each edge the distance between its vertices. If $l$ and $m$ are the vertices in a graph $G$, then a path in $G$ from $l$ to $m$ of length $N$ ($N\in\mathbb{N}$) is a sequence $\{x_i\}_{i=o}^{N}$ of $N+1$ vertices such that $l_o = l$, $l_N = m$ and $(l_{n-1},l_n)\in E(G)$ where $i = 1,2,\cdots N$. Hussain et al. [13] established fixed points for $\psi$-graphic contraction with an application to integral equations. A graph $G$ is connected if there is a path between any two vertices (for more details, see [9, 14, 28]).

**Definition 3.1.** Let $X$ be a nonempty set and $G=(V(G),E(G))$ be a graph such that $V(G)=X$. Then $S: X\to CB(X)$ is said to be semi graph dominated on $A\subseteq X$ if, for each $x\in A$, $(x,y)\in E(G)$ for all $y\in Sx$. If $A=X$, then we say that $S$ is graph dominated on $X$.

**Theorem 3.2.** *Let $(X,d_q)$ be a complete dislocated quasi metric space endowed with a graph $G$. Let $r>0$, $x_0\in\overline{B_{d_q}(x_0,r)}$, $S,T: X\to P(X)$ mappings and $\{TS(x_n)\}$ be a sequence in $X$ generated by $x_0$. Assume that the following hold:*

*(i) $S$ and $T$ are semi graph dominated on $\overline{B_{d_q}(x_0,r)}$;*

*(ii) there exists $\psi\in\Psi$ and $D_q(x,y)=\max\{d_q(x,y),d_q(x,Sx),d_q(y,Ty)\}$ such that*

$$\max\left\{H_{d_q}(Sx,Ty),H_{d_q}(Ty,Sx)\right\} \leq \min\{\psi(D_q(x,y)),\psi(D_q(y,x))\} \tag{3.1}$$

*for all $x,y\in\overline{B_{d_q}(x_0,r)}\cap\{TS(x_n)\}$ with $(x,y)\in E(G)$ or $(y,x)\in E(G)$;*

*(iii) $\sum_{i=0}^{n}\max\{\psi^i(d_q(x_1,x_0),\psi^i(d_q(x_0,x_1))\}\leq r$ for all $n\in\mathbb{N}\cup\{0\}$.*

*Then* $\{TS(x_n)\}$ *is a sequence in* $\overline{B_{d_q}(x_0,r)}$ *and* $\{TS(x_n)\} \to x^*$. *Also if* $(x_n,x^*) \in E(G)$ *or* $(x^*,x_n) \in E(G)$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* (3.1) *holds for* $x^*$, *then* $S$ *and* $T$ *have a common fixed point* $x^*$ *in* $\overline{B_{d_q}(x_0,r)}$.

*Proof.* Define $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } x \in \overline{B_{d_q}(x_0,r)},\ (x,y) \in E(G) \text{ or } (y,x) \in E(G) \\ 0, & \text{otherwise.} \end{cases}$$

Since $S$ and $T$ are semi graph dominated on $\overline{B_{d_q}(x_0,r)}$, for $x \in \overline{B_{d_q}(x_0,r)}$, $(x,y) \in E(G)$ for all $y \in Sx$ and $(x,y) \in E(G)$ for all $y \in Tx$. So $\alpha(x,y) = 1$ for all $y \in Sx$ and $\alpha(x,y) = 1$ for all $y \in Tx$. This implies that $\inf\{\alpha(x,y) : y \in Sx\} = 1$ and $\inf\{\alpha(x,y) : y \in Tx\} = 1$. Hence $\alpha_*(x,Sx) = 1$, $\alpha_*(x,Tx) = 1$ for all $x \in \overline{B_{d_q}(x_0,r)}$. So $S,T : X \to P(X)$ are semi $\alpha_*$-dominated mappings on $\overline{B_{d_q}(x_0,r)}$. Moreover, (3.1) can be written as

$$\max\{\alpha_*(x,Sx)H_{dq}(Sx,Ty),\ \alpha_*(y,Ty)H_{dq}(Ty,Sx)\} \le \min\{\psi(D_q(x,y)),\psi(D_q(y,x))\}$$

for all $x,y$ in $\overline{B_{d_q}(x_0,r)} \cap \{TS(x_n)\}$ with either $\alpha(x,y) \ge 1$ or $\alpha(y,x) \ge 1$. Also (iii) holds. Then, by Theorem 2.1, we have $\{TS(x_n)\}$ is a sequence in $\overline{B_{d_q}(x_0,r)}$ and $\{TS(x_n)\} \to x^* \in \overline{B_{d_q}(x_0,r)}$. Now $x_n, x^* \in \overline{B_{d_q}(x_0,r)}$ and either $(x_n,x^*) \in E(G)$ or $(x^*,x_n) \in E(G)$ implies that either $\alpha(x_n,x^*) \ge 1$ or $\alpha(x^*,x_n) \ge 1$. So all the conditions of Theorem 2.1 are satisfied. Hence by Theorem 2.1, $S$ and $T$ have a common fixed point $x^*$ in $\overline{B_{d_q}(x_0,r)}$ and $d_q(x^*,x^*) = 0$. □

## References

[1] A. Ahmed, N. Abdou, Common fixed point results for multivalued mappings with some examples, J. Nonlinear Sci. Appl. **9** (2016), 787–798.

[2] M. Ali, T. Kamran, E. Karapınar, Further discussion on modified multivalued $\alpha^*$-$\psi$-contractive type mapping, Filomat **29** (2015), 1893–1900.

[3] M. Arshad, A. Azam, M. Abbas, A. Shoaib, Fixed point results of dominated mappings on a closed ball in ordered partial metric spaces without continuity, Politehn. Univ. Bucharest Sci. Bull., Ser. A Appl. Math. Phys. **76** (2014). 123–134.

[4] M. Arshad, A. Shoaib, M. Abbas, A. Azam, Fixed points of a pair of Kannan type mappings on a closed ball in ordered partial metric spaces, Miskolc Math. Notes **14** (2013), 769–784.

[5] M. Arshad, A. Shoaib, P. Vetro, Common fixed points of a pair of Hardy-Rogers type mappings on a closed ball in ordered dislocated metric spaces, J. Funct. Spaces **2013** (2013), Article ID 63818.

[6] J. H. Asl, S. Rezapour, N. Shahzad, On fixed points of $\alpha$-$\psi$ contractive multifunctions, Fixed Point Theory Appl. **2012**, 2012:212.

[7] A. Azam, M. Arshad, Fixed points of a sequence of locally contractive multi-valued maps, Comp. Math. Appl. **57** (2009), 96–100.

[8] I. Beg, M. Arshad, A. Shoaib, Fixed point on a closed ball in ordered dislocated metric space, Fixed Point Theory **16** (2015), 195–206.

[9] F. Bojor, Fixed point theorems for Reich type contraction on metric spaces with a graph, Nonlinear Anal. **75** (2012), 3895–3901.

[10] M. F. Bota, C. Chifu, E. Karapinar, Fixed point theorem for generalized $(\alpha_* - \psi)$ Ciric-type contractive multivalued operator in $b$-metric spaces, J. Nonlinear Sci. Appl. **9** (2016), 1165–1177.

[11] P. Hitzler, A. K. Seda, Dislocated topologies, J. Electrical Engineering **51** (2000), No. 12, 3–7.

[12] N. Hussain, J. Ahmad, A. Azam, Generalized fixed point theorems for multi-valued $\alpha$-$\psi$-contractive mappings, J. Inequal. Appl. **2014**, 2014:348.

[13] N. Hussain, S. Al-Mezel, P. Salimi, Fixed points for $\psi$-graphic contractions with application to integral equations, Abstr. Appl. Anal. **2013** (2013), Article ID 575869.

[14] N. Hussain, M. Arshad, A. Shoaib, Fahimuddin, Common fixed point results for $\alpha$-$\psi$-contractions on a metric space endowed with graph, J. Inequal. Appl. **2014**, 2014:136.

[15] J. Jachymski, The contraction principle for mappings on a metric space with a graph, Proc. Am. . Math. Soc. **136** (2008), 1359–1373.
[16] S. G. Matthews, Partial metric topology, in: Proc. 8th Summer Conference on General Topology and Applications, in: Ann. New York Acad. Sci. **728** (1994), pp. 183–197.
[17] E. Karapınar, H. Piri, H. H. Alsulami, Fixed points of modified $F$-contractive mappings in complete metric-like spaces, J. Funct. Spaces **2015** (2015), Article ID 270971.
[18] E. Karapınar, İ.M. Erhan, A. Öztürk, Fixed point theorems on quasi-partial metric spaces, Math. Comp. Modelling **57** (2013), 2442–2448.
[19] M. S. Khan, Common fixed point theorems for multivalued mappings, Pacific J. Math. **95** (1981), 337–347.
[20] Jr. Nadler, Multi-valued contraction mappings. Pacific J. Math. **30)** (1969), 475–478.
[21] M. Sarwar, M. U. Rahman, G. Ali, Some fixed point results in dislocated quasi metric (dq-metric) spaces, J. Inequal. Appl. **2014**, 2014:278.
[22] T. Senapati, L. K. Dey, Common fixed point theorems for multivalued $\beta_*$-$\psi$-contractive mappings, Thai J. Math. (in press).
[23] A. Shoaib, $\alpha$-$\eta$ dominated mappings and related common fixed point results in closed ball, J. Concrete Appl. Math. **13** (2015), 152–170.
[24] A. Shoaib, Fixed point results for $\alpha_*$-$\psi$-multivalued mappings on an intersection of a closed ball and a sequence endowed with a graph, Bull. Math. Anal. Appl. (in press).
[25] A. Shoaib, M. Arshad, J. Ahmad, Fixed point results of locally cotractive mappings in ordered quasi-partial metric spaces, Scientific World J. **2013** (2013), Article ID 194897.
[26] A. Shoaib, M. Arshad, M. A. Kutbi, Common fixed points of a pair of Hardy Rogers type mappings on a closed ball in ordered partial metric spaces, J. Comput. Anal. Appl. **17** (2014), 255–264.
[27] A. Shoaib, A. Hussain, M. Arshad, A. Azam, Fixed point results for $\alpha_*$-$\psi$-Ciric type multivalued mappings on an intersection of a closed ball and a sequence with graph, J. Math. Anal. **7** (2016), 41–50.
[28] J. Tiammee, S. Suantai, Coincidence point theorems for graph-preserving multi-valued mappings, Fixed Point Theory Appl. **2014**, 2014:70.
[29] B. R. Wadkar, R. Bhardawaj, B. K.Singh, A common fixed point theorem in dislocared metric space, Int. J. Engg Research Devolpment **10** (2014), 14–17.
[30] F. M. Zeyada, G. H. Hassan, M. A. Ahmed, A generalization of a fixed point theorem due to Hitzler and Seda in dislocated quasi-metric spaces, Arabian J. Sci. Eng. A **31** (2006), 111–114.
[31] L. Zhu, C. Zhu, C. Chen, Z. Stojanović, Multidimensional fixed points for generalized Ψ-quasi-contractions in quasi-metric-like spaces, J. Inequal. Appl. **2014**, 2014:27.

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# Global existence and blow-up of solutions to strongly damped wave equations with nonlinear degenerate damping and source terms *

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**Abstract** This paper deals with the initial-boundary value problem of a class of strongly damped wave equations with nonlinear degenerate damping and source terms. By potential well theory, the global existence of weak solutions is proved if the initial data enter into the stable set. By Nakao inequality, the asymptotic behavior is obtained. Moreover, by establishing a new second order differential inequality, we prove a finite-time blow-up result under arbitrary positive initial energy.

**Keywords** strong damped wave equations; global existence; asymptotic behavior; blow-up

**AMS Classification (2010):** 35L35,35B40,35B44.

## 1 Introduction

In this paper, we are concerned with the following initial boundary value problem:

$$u_{tt} - \Delta u_{tt} - \Delta u_t - \Delta u + (|u|^k u)_t = |u|^q u, \ (x,t) \in \Omega \times (0,T), \tag{1.1}$$

$$u(x,t) = 0, \ x \in \partial\Omega, t \in (0,T), \tag{1.2}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \ x \in \Omega, \tag{1.3}$$

where $\Omega$ is a bounded domain in $R^n$ with sufficiently smooth boundary $\partial\Omega$.

Evolution equation (1.1) is a simple prototype of the more general equation

$$\frac{\partial^2}{\partial t^2}(\triangle u - u) + \frac{\partial}{\partial t}(\triangle u - g(x,u)) + \triangle u + f(x,u) = 0, \tag{1.4}$$

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which describes ion acoustic waves in a plasma taking account of strong nonlinear dissipation and nonlinear sources [1, 2], where $f(x,u)$ and $g(x,u)$ describe the distribution of the sources of bound charges and the 'sinks' of free charges, respectively. Korpusov [2] proved that for any initial data in $H_0^1(\Omega)$ the problem (1.4), (1.2), (1.3) has a local strong generalized solution and obtained sufficient conditions for the blow-up of a solution in finite time provided that

$$\int_\Omega [|\nabla u_0|^2 + |\nabla u_1|^2 + |u_1|^2]dx - 2\int_\Omega \int_0^{u_0(x)} f(x,s)dsdx < 0. \tag{1.5}$$

Korpusov [3] gave sufficient conditions for finite-time blow-up of solutions of the following abstract Cauchy problem for a formally hyperbolic equation with double non-linearity

$$A\frac{d^2u}{dt^2} + \frac{du}{dt}(A_0u + \sum_{j=1}^{n} A_j(u)) + H_f'(u) = F_f'(u).$$

As far as we know, there is little information about the equation (1.4).

It is worth noting here that if the damping terms $(|u|^k u)_t$ is absent, the problem (1.1)-(1.3) is studied extensively in the literature. For example, Shang [4, 5] investigated the existence, uniqueness, asymptotic behavior, and the blow up phenomenon of the solutions under some specific assumptions on $f$ for the fourth-order wave equation

$$u_{tt} - \Delta u - \Delta u_t - \Delta u_{tt} = f(u). \tag{1.6}$$

For the initial boundary value problem of equation (1.6), Zhang and Hu [6] proved the existence and the stability of global weak solution. Xie and Zhong [7] established the existence of global attractors. Xu et al. [8] investigated the asymptotic behavior of solutions by using the multiplier method. For more results on the long-time behaviors of global strong solutions of the initial boundary value problem of equation (1.6), the reader is referred to [9, 10, 11, 12, 13, 14, 15, 16].

We mention also that, recently, in the absence of dispersive term and strong damped term ( i.e. the terms $\Delta u_{tt}$ and $-\Delta u_t$ are absence), equation (1.1) can be take the more general form

$$u_{tt} - \Delta u + |u|^k \partial j(u_t) = |u|^q u. \tag{1.7}$$

Under suitable conditions on $j$ and the parameters in equation (1.7), Barbuet al. [17] established the existence and uniqueness of global weak solutions. They also obtained a nonexistence result of global solutions with negative initial energy on condition that $q$ is greater than the critical value. In [18], the same authors established the blow-up result for the generalized solution with additional regularity for equation (1.7) under more restrictions on $j$ provided that $q$ is greater than the critical value and the initial energy is negative. A special case of equation (1.7) is the following polynomial-damped wave equation

$$u_{tt} - \Delta u + |u|^k |u_t|^{m-1} u_t = |u|^q u, \tag{1.8}$$

which has been studied extensively in the literature. We refer the reader to [19, 20, 21, 22, 23, 24, 25, 26, 27] and the references therein. But much less work is known for the initial boundary problem for a wave equation with degenerate damping term $(|u|^k u)_t$, dispersive term $\Delta u_{tt}$, strong damping term $\Delta u_t$ and nonlinear source term $|u|^q u$.

In this paper, we prove the global existence of weak solutions for problem (1.1)-(1.3) by potential well theory [28] if the initial data enter into the stable set. By Nakao's inequality [29], the asymptotic behavior is obtained, and this method is different from that in [22, 26]. Moreover, we will extend the results in [2, 3] that the sufficient conditions for finite-time blow-up of solutions with negative initial energy to that with positive initial energy. We will establish a different differential inequality and prove a finite-time blow-up result under arbitrary positive initial energy. This different differential inequality and blow-up result are, to the best of our knowledge, new in the literature. The concavity method of Levine [30] is one of the most powerful methods for proving finite time blow-up of the solutions to nonlinear wave equations. The main idea of Levine is to replace the investigations of the equation with the study of ordinary differential inequality

$$\phi\phi'' - \alpha(\phi')^2 + \beta\phi^2 \geq 0, \alpha > 1, \beta \geq 0.$$

Later on, the generalization of this inequality

$$\phi\phi'' - \alpha(\phi')^2 + \beta\phi^2 + \gamma\phi\phi' \geq 0, \alpha > 1, \beta \geq 0, \gamma \geq 0,$$

was obtained in [31]. On the other hand, as pointed in [2], Levine's method as presented in [30, 31] cannot be used here due to the term $-\Delta u_t + (|u|^k u)_t$, so a different differential inequality was used to prove Theorem 3 in [3] for negative initial energy case (i.e. in the case of (1.5))

$$\phi\phi'' - \alpha(\phi')^2 + \beta\phi^2 + \gamma\phi^{2+q_1} \geq 0, \alpha > 1, \beta \geq 0, \gamma \geq 0, q_1 \geq 0.$$

But the above inequality cannot be used to the positive initial energy case. In this paper, we will establish a new differential inequality and prove a finite-time blow-up result under arbitrary positive initial energy.

This article is organized as follows. In Section 2, we are concerned with some notations and state our main results. Following the potential well theory introduced by [28], we get global existence in Section 3. Section 3 gives also an asymptotic stability results of the problem (1.1)-(1.3). In section 4, it is shown that the weak solution of the problem (1.1) -(1.3) blow-up in the case of positive initial energy $E(0) > 0$ and $q > k$.

## 2 Preliminaries

In this section we present some notations and state our main results. We use the standard Lebesgue space $L^p(\Omega)(1 \leq p \leq \infty)$ and Sobolev space $H_0^1(\Omega)$. We denote by $||u||_p$ the $L^p(\Omega)$

norm and by $||\nabla \cdot||$ the norm in $H_0^1(\Omega)$. Moreover, for later use we denote by $(\cdot,\cdot)$ the duality paining between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. In this paper, we will always assume that

$$0 < q < \infty, \quad if\ n = 1,\ 2; \quad 0 < q < \frac{4}{n-2}, \quad if\ n \geq 3, \tag{2.1}$$

and then the Sobolev imbedding $H_0^1(\Omega) \hookrightarrow L^{q+2}(\Omega)$ holds. Furthermore, we denote $C_*$ is the embedding constant, that is

$$||\nabla u|| \leq C_* ||u||_{q+2}.$$

The constants $C_i$ $(i = 1, 2, \cdots)$ used throughout this paper are positive generic constants, which may be different in various occurrences.

Now, we give the definition of a weak solution to problem (1.1)-(1.3).

A weak solution to the initial boundary value problem (1.1)-(1.3) over $\Omega \times [0, T)$ is a function $u \in L^\infty(0, T; H_0^1(\Omega))$ with $u_t, u_{tt} \in L^\infty(0, T; H_0^1(\Omega))$ such that

$$(u_{tt}, \varphi) + (\nabla u_{tt}, \nabla \varphi) + (\nabla u, \nabla \varphi) + (\nabla u_t, \nabla \varphi) + ((|u|^k u)_t, \varphi) = \int_\Omega u|u|^q \varphi dx,$$

for all test functions $\varphi \in H_0^1(\Omega)$ and for almost all $t \in [0, T)$ and

$$u(x, 0) = u_0 \in H_0^1(\Omega), u_t(x, 0) = u_1 \in H_0^1(\Omega).$$

We introduce the following functionals:

$$I(t) = I(u) = ||\nabla u||^2 - ||u||_{q+2}^{q+2}, \tag{2.2}$$

$$J(t) = J(u) = \frac{1}{2}||\nabla u||^2 - \frac{1}{q+2}||u||_{q+2}^{q+2}, \tag{2.3}$$

$$E(t) = E(u) = \frac{1}{2}||u_t||^2 + \frac{1}{2}||\nabla u_t||^2 + J(u), \tag{2.4}$$

and the level

$$d = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \max_{\lambda \geq 0} J(\lambda u).$$

Moreover the value $d$ is shown to be the Mountain Pass level associated to the underlying Dirichlet problem $-\Delta u = |u|^q u$ in $\Omega$, $u = 0$ on $\partial\Omega$ [32].

**Remark 2.1** By multiplying Equation (1.1) by $u_t$, integrating over $\Omega$, and using integration by parts, we get

$$E'(t) = -||\nabla u_t||^2 - (k+1)\int_\Omega |u|^k |u_t|^2 dx \leq 0, \quad for\ t \geq 0. \tag{2.5}$$

Therefore,

$$E(t) \leq E(0), \quad for\ t \geq 0. \tag{2.6}$$

We can now define the stable set [32, 33, 34]

$$W = \{u \in H_0^1 | I(u) > 0, J(u) < d\}$$

and one can easily see that the stable set can also be defined by

$$W = \{(\lambda, E) \in [0, +\infty) \times R : 0 < g(\lambda) \leq E < d, 0 < \lambda < \lambda_0\},$$

where $g(\lambda) = \frac{1}{2}\lambda^2 - C_*^{q+2}\frac{\lambda^{q+2}}{q+2}$, $\lambda_0 = C_*^{-\frac{q+2}{q}}$ is the absolute maximum point of $g$, and finally $d = g(\lambda_0) = (\frac{1}{2} - \frac{1}{q+2})\lambda_0^2 > 0$.

In order to get the energy decay of the solution, we introduce the following set

$$W_1 = \{u \in H_0^1 | I(u) > 0, J(u) < E_1\},$$

where $\lambda_1 = ((q+2)C_*^{q+2})^{-\frac{1}{q}}, E_1 = (\frac{1}{2} - \frac{1}{q+2})\lambda_1^2$. Obviously, $d > E_1$ and $W_1 \subset W$.

Our main results read as follows. The first result is concerned with the global existence of weak solutions to the problem (1.1)-(1.3). Namely, we have the following theorem.

**Theorem 2.1** Let $k > 1,\ if\ n = 1,\ 2;\ k < \frac{2n}{n-2},\ if\ n \geq 3;\ u_0, u_1 \in H_0^1(\Omega)$, assuming that $k > q, E(0) < d$ and $u_0 \in W$, then the problem (1.1)-(1.3) admits a global weak solution $u$ and $u(\cdot) \in W$ for $t \geq 0$.

The second result is about the asymptotic stability results of the weak solutions.

**Theorem 2.2** Under the assumptions of Theorem 2.1, $k > q$ and $u_0 \in W_1$, and $||\nabla u_0|| < \lambda_1, E(0) < E_1$, then there exist positive constant $\alpha$ such that the energy $E(t)$ satisfies the energy estimates

$$E(t) \leq E(0)\exp\{-\alpha[t-1]^+\} \qquad for\ large\ t,$$

where $[t-1]^+ = \max\{t-1, 0\}$.

**Remark 2.2** Let us mention that the special polynomial form of the dissipation and source terms in (1.1)-(1.3) is not essential. The results can be extend to the case of more general nonlinearities under suitable assumptions.

Our final result provides a finite time blowup property of the weak solutions to problem (1.1)-(1.3).

**Theorem 2.3** Assume that $k < q$. If the initial data are such that

$$E(0) > 0, \tag{2.7}$$

$$(\nabla u_0, \nabla u_1) + (u_0, u_1) > 0, \tag{2.8}$$

and

$$\begin{aligned}(\phi'(0))^2 - \tfrac{\beta_0}{\alpha_0 - 1}\phi^2(0) + \tfrac{2\gamma_0}{(\alpha_0-1)\delta_1'}\phi^{2\alpha_0+\delta_1'(1-\alpha_0)}(0) \\ + \tfrac{2\delta_0}{(\alpha_0-1)\delta_2'}\phi^{2\alpha_0+\delta_2'(1-\alpha_0)}(0) = B_0 > 0,\end{aligned} \tag{2.9}$$

where $\alpha_0 = 2 + \frac{k}{2}$, $\beta_0 = \frac{2}{q-k}$, $\gamma_0 = \frac{2(k+1)^2 C_0^{2(k+1)}}{q-k}$, $\delta_0 = (q+2)E(0)$, $\delta_1' = \frac{2\alpha_0 - 2 - k}{\alpha_0 - 1}$, $\delta_2' = \frac{2\alpha_0 - 1}{\alpha_0 - 1}$ and $C_0$ is embedding constant from $H_0^1$ to $L^{2(k+1)}$, then there exists

$$T_\infty \leq \phi^{1-\alpha_0}(0)A^{-1} \quad such\ that \quad \lim_{t \to T_\infty^-} \phi(t) = \infty,$$

where $A^2 = (\alpha_0 - 1)^2 \phi^{-2\alpha_0}(0)B_0$ and $\phi(t) = \frac{1}{2}||\nabla u||^2 + \frac{1}{2}||u||^2$.

Finally, we state the local existence result of problem (1.1)-(1.3).

**Theorem2.4** Let $u_0, u_1 \in H_0^1(\Omega)$, then problem (1.1)-(1.3) has a unique weak solution on $[0, T_0)$ for some $T_0 > 0$, and we have either $T_0 = +\infty$ or $T_0 < +\infty$ and

$$\lim_{t \to T_0 +} sup[||u_t||^2_{H_0^1(\Omega)} + ||u||^2_{H_0^1(\Omega)}] = +\infty.$$

This lemma can be established by combining the arguments of Theorem 8.1 (or Theorem 6.8) and Example 9.5 in [3], Theorem 2 in [2] and [35], so we omit it.

## 3 Global existence and asymptotic stability of the solutions

In this section we study the existence and asymptotic stability of global solutions for problem (1.1)-(1.3). We start by the following lemma.

**Lemma 3.1** Suppose that $u$ is the solution of problem (1.1)-(1.3), and $u_0, u_1 \in H_0^1$, if $u_0 \in W$ and $E(0) < d$, then $u(t)$ remains inside the set $\in W$ for any $t \geq 0$.

The proof is similar to that of Lemma 2.2 in [33], so we omit it.

**Proof of Theorem 2.1.** By Lemma 3.1, we have $u(t) \in W$ for all $t \in [0, T_0)$, then $I(u) > 0, J(u) < d$ for all $t \in [0, T_0)$. Therefore,

$$(\frac{1}{2} - \frac{1}{q+2})||u_m||^{q+2}_{q+2} = \frac{1}{2}||\nabla u||^2 - \frac{1}{q+2}||u||^{q+2}_{q+2} - \frac{1}{2}I(u) \leq J(u) < d, \tag{3.1}$$

then

$$||u||^{q+2}_{q+2} < d. \tag{3.2}$$

By the energy equation (2.6), definition of $J(u)$ and (3.1), we arrive

$$\frac{1}{2}||u_t||^2 + \frac{1}{2}||\nabla u_t||^2 + \frac{1}{2}||\nabla u||^2 \leq E(0) + \frac{1}{q+2}||u||^{q+2}_{q+2} \leq Cd, \quad for\ 0 \leq t < T_0, \tag{3.3}$$

It follows from (3.3) and from a standard continuous argument that the local solution $u$ furnished by Theorem 2.4 can be extended to the whole interval $[0, +\infty)$, that is, $u$ is a global solution. Finally from Lemma 3.1 we get $u \in W$ for $t \in [0, \infty)$.

In order to get the energy decay of the solution, we prepare the following lemma.

**Lemma 3.2**[29] Let $\varphi(x)$ be a nonnegative and non-increasing function defined on $[0, \infty)$, satisfying

$$\varphi^{1+r}(t) \leq k_0(\varphi(t) - \varphi(t+1)), \qquad t \in [0, T],$$

for $k_0 > 1$, and $r \geq 0$. Then we have, for each $t \in [0, T]$,

$$\varphi(t) \leq \varphi(0) \exp(-\alpha[t-1]^+), \qquad if\ r = 0,$$
$$\varphi(t) \leq (\varphi(0)^{-r} + k_0 r[t-1]^+)^{-\frac{1}{r}}, \qquad if\ r > 0,$$

where $[t-1]^+ = \max\{t-1, 0\}$, and $\alpha = \ln(\frac{k_0}{k_0-1})$.

Adapting the idea of Vitillaro[34], we have the following lemma.

**Lemma 3.3** Suppose that $u$ is the solution of problem (1.1)-(1.3), and $u_0, u_1 \in H_0^1$, if $u_0 \in W_1$ and $||\nabla u_0|| < \lambda_1, E(0) < E_1$, then $u(t)$ remains inside the set $W_1$ and $||\nabla u|| < \lambda_1, E(t) < E_1$ for any $t \geq 0$.

**Lemma 3.4** Under the condition of Theorem 2.2. and $q > 0$, then, for $t \geq 0$,

$$||\nabla u||^2 \geq 2||u||_{q+2}^{q+2}. \tag{3.4}$$

$$E(t) \geq \frac{q+1}{2(q+2)}||\nabla u||^2 \geq \frac{q+1}{q+2}||u||_{q+2}^{q+2}, \tag{3.5}$$

**Proof** By the definition $E(t)$ and embedding theorem, we have

$$\begin{aligned} E(t) &\geq \tfrac{1}{2}||\nabla u||^2 - \tfrac{1}{q+2}||u||_{q+2}^{q+2} \geq \tfrac{1}{2}||\nabla u||^2 - ||u||_{q+2}^{q+2} \\ &\geq \tfrac{1}{2}||\nabla u||^2 - C_*^{q+2}||\nabla u||^{q+2} = g_1(||\nabla u||). \end{aligned} \tag{3.6}$$

where $g_1(\lambda) = \frac{1}{2}\lambda^2 - C_*^{q+2}\lambda^{q+2}$. Note that $g_1(\lambda)$ has the maximum at $\lambda_1 = ((q+2)C_*^{q+2})^{-\frac{1}{q}}$ and the maximum value $g_1(\lambda_1) = E_1$. We see that $g_1(\lambda)$ is increasing in $(0, \lambda_1)$, decreasing in $(\lambda_1, +\infty)$ and $g_1(\lambda) \to -\infty$ as $\lambda \to \infty$. Since $||\nabla u_0||^2 < \lambda_1, E(0) < E_1$,then $||\nabla u||^2 < \lambda_1$, for any $t \geq 0$, so $g_1(||\nabla u||) \geq 0$. By (3.6), we have

$$\begin{aligned} ||\nabla u||^2 - ||u||_{q+2}^{q+2} &= \tfrac{1}{2}||\nabla u||^2 + (\tfrac{1}{2}||\nabla u||^2 - ||u||_{q+2}^{q+2}) \\ &\geq \tfrac{1}{2}||\nabla u||^2 + g_1(||\nabla u||), \end{aligned}$$

then (3.4) holds since $g_1(||\nabla u||) > 0$. Furthermore, we have

$$E(t) \geq \frac{1}{2}||\nabla u||^2 - \frac{1}{q+2}||u||_{q+2}^{q+2} \geq \frac{q+1}{2(q+2)}||\nabla u||^2.$$

So (3.5) hold.

**Proof of Theorem 2.2** From (2.5), we know that and $E(t)$ is nonincreasing. Setting $F(t) = \sqrt{E(t) - E(t+1)}$, then we have

$$F^2(t) = \int_t^{t+1} [||\nabla u_s||^2 + (k+1)\int_\Omega |u|^k |u_s|^2] ds \geq \int_t^{t+1} ||\nabla u_s||^2 ds, \tag{3.7}$$

$$F^2(t) \geq (k+1)\int_t^{t+1}\int_\Omega |u|^k |u_s|^2 ds. \tag{3.8}$$

Applying the mean value theorem in (3.7), there exists $t_1 \in [t, t+\frac{1}{4}]$ and $t_2 \in [t+\frac{3}{4}, t+1]$ such that

$$||\nabla u_t(t_i)||^2 \leq 2F^2(t), \qquad i = 1, 2. \tag{3.9}$$

Multiplying $u$ in (1.1), intergrating over $[t_1, t_2] \times \Omega$, we have

$$\begin{aligned}
&\int_{t_1}^{t_2} [||\nabla u||^2 - ||u||_{q+2}^{q+2}]dt \\
&= -\int_\Omega u_t u|_{t=t_1}^{t=t_2} dx - \int_\Omega \nabla u_t \nabla u|_{t=t_1}^{t=t_2} dx + \int_{t_1}^{t_2} (||u_t||^2 + ||\nabla u_t||^2)dt \\
&- \int_{t_1}^{t_2}\int_\Omega \nabla u \nabla u_t dx dt - (k+1)\int_{t_1}^{t_2}\int_\Omega |u|^k u u_t dx dt \\
&= \sum_{i=1}^{5} M_i.
\end{aligned} \tag{3.10}$$

Now we estimate the terms of the right-hand side of (3.10). By Hölder inequality, Poincaré inequality, (3.9), Lemma 3.4, the fact that the $E(t)$ is non-increasing, and Young inequality with $\varepsilon > 0$, we have

$$\begin{aligned}
|M_1| &= |-\int_\Omega u_t u|_{t=t_1}^{t=t_2} dx| \leq \sum_{i=1}^{2} ||u_t(t_i)||||u(t_i)|| \\
&\leq \sum_{i=1}^{2} C_2 ||u_t(t_i)||||\nabla u(t_i)|| \leq C_3 E^{\frac{1}{2}}(t) F(t) \\
&\leq C_1(\varepsilon) F^2(t) + \varepsilon E(t), \qquad (3.11) \\
|M_2| &= |-\int_\Omega \nabla u_t \nabla u|_{t=t_1}^{t=t_2} dx| \\
&\leq \sum_{i=1}^{2} ||\nabla u_t(t_i)||||\nabla u(t_i)|| \leq C_4 E^{\frac{1}{2}}(t) F(t) \\
&\leq C_2(\varepsilon) F^2(t) + \varepsilon E(t). \qquad (3.12)
\end{aligned}$$

By Poincaré inequality and (3.7), we have

$$|M_3| = |\int_{t_1}^{t_2} (||u_t||^2 + ||\nabla u_t||^2)dt| \leq C_5 \int_t^{t+1} ||\nabla u_s||^2 ds \leq C_5 F^2(t). \tag{3.13}$$

From Lemma 3.4 and the fact that the $E(t)$ is non-increasing, we arrive

$$\begin{aligned}
|M_4| = |\int_{t_1}^{t_2} \nabla u \nabla u_s ds| &\leq \int_t^{t+1} [C_3(\varepsilon)||\nabla u_s||^2 + \varepsilon \tfrac{2(q+2)}{q+1} ||\nabla u||^2] ds \\
&\leq C_3(\varepsilon) F^2(t) + \varepsilon E(t).
\end{aligned} \tag{3.14}$$

According to Hölder inequality, embedding theorem, the assumption $k > q$ and the Lemma3.2, the fact that the $E(t)$ is non-increasing and Young inequality, we have

$$\begin{aligned}
|M_5| &\leq (k+1)\int_{t_1}^{t_2}\int_{\Omega}|u|^{k+1}|u_t|dxdt \\
&= (k+1)\int_{t_1}^{t_2}\int_{\Omega}|u|^{\frac{k}{2}}|u_t||u|^{\frac{k+2}{2}}dxdt \\
&\leq (k+1)(\int_{t_1}^{t_2}\int_{\Omega}|u|^{k}|u_t|^2dxdt)^{\frac{1}{2}}(\int_{t_1}^{t_2}\int_{\Omega}|u|^{k+2}dxdt)^{\frac{1}{2}} \\
&\leq C_7F(t)(\int_{t_1}^{t_2}||u||_{k+2}^{k+2}dt)^{\frac{1}{2}} \\
&\leq C_9F(t)(\int_{t_1}^{t_2}||\nabla u||^{k}||\nabla u||^2dt)^{\frac{1}{2}} \\
&\leq C_{10}F(t)(\int_{t_1}^{t_2}E(t)dt)^{\frac{1}{2}} \\
&\leq C_4(\varepsilon)F^2(t)+\varepsilon E(t). \qquad (3.15)
\end{aligned}$$

Substituting (3.11)-(3.15) into (3.10), we get the estimate

$$\int_{t_1}^{t_2}[||\nabla u||^2-||u||_{q+2}^{q+2}]dt \leq C_5(\varepsilon)F^2(t)+4\varepsilon E(t). \qquad (3.16)$$

On the other hand, it follows from the definition of $E(t)$ and (3.5) in Lemma 3.4 that

$$\begin{aligned}
E(t) &= \frac{1}{2}(||\nabla u||^2-||u||_{q+2}^{q+2})+\frac{q}{2(q+2)}||u||_{q+2}^{q+2}+\frac{1}{2}||u_t||^2+\frac{1}{2}||\nabla u_t||^2 \\
&\leq \frac{1}{2}(||\nabla u||^2-||u||_{q+2}^{q+2})+\frac{1}{2}||u_t||^2+\frac{1}{2}||\nabla u_t||^2+\frac{q}{2(q+1)}E(t). \qquad (3.17)
\end{aligned}$$

Then we have

$$\frac{q+2}{2(q+1)}E(t) \leq \frac{1}{2}(||\nabla u||^2-||u||_{q+2}^{q+2})+\frac{1}{2}||u_t||^2+\frac{1}{2}||\nabla u_t||^2. \qquad (3.18)$$

Therefore, by (3.18), (3.16) and (3.13), we arrive that

$$\begin{aligned}
\int_{t_1}^{t_2}E(s)ds &\leq \frac{2(q+1)}{q+2}\int_{t_1}^{t_2}(||\nabla u||^2-||u||_{q+2}^{q+2})ds+\frac{q+2}{2(q+1)}\int_{t_1}^{t_2}(||u_t||^2+||\nabla u_t||^2)ds \\
&\leq C_6(\varepsilon)F^2(t)+4\varepsilon\frac{2(q+1)}{q+2}E(t). \qquad (3.19)
\end{aligned}$$

Since $E(t)$ is non-increasing, we can choose $t_3 \in [t_1, t_2]$ such that

$$E(t_3) \leq C\int_{t_1}^{t_2}E(s)ds. \qquad (3.20)$$

Then, using (2.5), $t_3 < t+1$, it follows from (3.20) and the fact that $E(t)$ is non-increasing that

$$\begin{aligned}
E(t) &= E(t+1)+\int_t^{t+1}||\nabla u_s||^2ds+(k+1)\int_t^{t+1}\int_{\Omega}|u|^k|u_s|^2ds \\
&\leq E(t_3)+\int_t^{t+1}||\nabla u_s||^2ds+(k+1)\int_t^{t+1}\int_{\Omega}|u|^k|u_s|^2ds \\
&\leq C\int_{t_1}^{t_2}E(s)ds+\int_t^{t+1}||\nabla u_s||^2ds+(k+1)\int_t^{t+1}\int_{\Omega}|u|^k|u_s|^2ds. \qquad (3.21)
\end{aligned}$$

Combining (3.21) with (3.19), (3.7) and (3.8), we have

$$E(t) \leq C_7(\varepsilon)F^2(t) + 4\varepsilon\frac{2(q+1)}{q+2}E(t). \tag{3.22}$$

Choosing $\varepsilon$ sufficiently small, (3.22) leads to

$$E(t) \leq C_8F^2(t),$$

where $C_8 = 2(1 + 2C_6 + \frac{C_7^2}{2})$.

Since $E(t)$ is nonincreasing, using Lemma 3.2, we conclude that for each $t \in [0, \infty)$,

$$E(t) \leq E(0)\exp(-\alpha[t-1]^+)$$

where $[t-1]^+ = \max\{t-1, 0\}$, and $\alpha = \ln(\frac{C_8}{C_8-1})$. Then the exponential decay of the energy is obtained. The proof of Theorem 2.2 is completed.

# 4 Blowup of the solutions

In this section our aim is to establish sufficient condition for blow-up of solutions to problem (1.1)-(1.3). We assume that $k < q$ and $u$ be a weak solution to the problem (1.1)-(1.3) on the interval $[0, T]$. We note that the Levine energy method [30] is one of basic methods for studying the blow-up phenomenon. The role of the differential inequality [30]

$$\phi\phi'' - \alpha(\phi')^2 + \beta\phi^2 \geq 0, \alpha > 1, \beta \geq 0,$$

in the standard Levine method is known. The generalization of this inequality

$$\phi\phi'' - \alpha(\phi')^2 + \beta\phi^2 + \gamma\phi\phi' \geq 0, \alpha > 1, \beta \geq 0, \gamma \geq 0,$$

was obtained in [31]. On the other hand, a somewhat different differential inequality was used to prove Theorem 3 in [3]

$$\phi\phi'' - \alpha(\phi')^2 + \beta\phi^2 + \gamma\phi^{2+q_1} \geq 0, \alpha > 1, \beta \geq 0, \gamma \geq 0, q_1 \geq 0.$$

Now, we consider our main differential inequality

$$\phi\phi'' - \alpha(\phi')^2 + \beta\phi^2 + \gamma\phi^{2+l} + \delta\phi \geq 0, \tag{4.1}$$

where $\phi(t) \in C^2([0, T])$ and

$$\alpha > 1, \beta \geq 0, \gamma \geq 0, \delta \geq 0, l \geq 0. \tag{4.2}$$

**Lemma 4.1** Suppose $\phi(t) \in C^2([0, T])$. Let conditions (4.1) and (4.2) be satisfied and moreover the following conditions hold:

$$2\alpha > 2 + l, \tag{4.3}$$

$$\phi(t) \geq 0, \phi'(0) > 0, \phi(0) > 0, \tag{4.4}$$

$$(\phi'(0))^2 - \tfrac{\beta}{\alpha-1}\phi^2(0) + \tfrac{2\gamma}{(\alpha-1)\delta_1}\phi^{2\alpha+\delta_1(1-\alpha)}(0) + \tfrac{2\delta}{(\alpha-1)\delta_2}\phi^{2\alpha+\delta_2(1-\alpha)}(0) = B > 0, \tag{4.5}$$

where $\delta_1 = \frac{2\alpha-2-l}{\alpha-1}$ and $\delta_2 = \frac{2\alpha-1}{\alpha-1}$, then any solution to the differential inequality (4.1) satisfi es the condition

$$T_\infty \leq \phi^{1-\alpha}(0)A^{-1}, \lim_{t\to T_\infty^-} \phi(t) = \infty$$

where $A^2 = (\alpha-1)^2\phi^{-2\alpha}(0)B$.

**Proof** Condition (4.4) imply the existence of a time $t_1 > 0$ for which the inequality $\phi'(t) > 0$ for $t \in [0, t_1)$ holds. Hence, $\phi(t) > \phi(0) > 0$ for $t \in [0, t_1)$. Dividing both sides of (4.1) by $\phi^{1+\alpha}$ we obtain

$$\phi^{-\alpha}\phi'' - \alpha\phi^{-(1+\alpha)}(\phi')^2 + \beta\phi^{1-\alpha} + \gamma\phi^{1+l-\alpha} + \delta\phi^{-\alpha} \geq 0. \tag{4.6}$$

Noting $\frac{d^2}{dt^2}\phi^{1-\alpha} = (1-\alpha)\phi^{-\alpha}\phi'' - \alpha(1-\alpha)\phi^{-(1+\alpha)}(\phi')^2$, from (4.6) it is easy to derive the inequality

$$\tfrac{1}{1-\alpha}\tfrac{d^2}{dt^2}\phi^{1-\alpha} + \beta\phi^{1-\alpha} + \gamma\phi^{1+l-\alpha} + \delta\phi^{-\alpha} \geq 0. \tag{4.7}$$

We introduce the new function

$$\psi = \phi^{1-\alpha}, \tag{4.8}$$

then we obtain

$$\tfrac{1}{1-\alpha}\psi'' + \beta\psi + \gamma\psi^{\frac{1+l-\alpha}{1-\alpha}} + \delta\psi^{\frac{-\alpha}{1-\alpha}} \geq 0. \tag{4.9}$$

Note now that by (4.8), we have

$$\psi'(t) = (1-\alpha)\phi'\phi^{-\alpha}, \tag{4.10}$$

so in view of $\phi'(t) > 0$ and $\alpha > 1$, it follows that

$$\psi'(t) \leq 0. \tag{4.11}$$

Now multiplying (4.9) by $\psi'(t)$ we obtain

$$\tfrac{1}{1-\alpha}\psi'(t)\psi'' + \beta\psi'(t)\psi + \gamma\psi'(t)\psi^{\frac{\alpha-1-l}{\alpha-1}} + \delta\psi'(t)\psi^{\frac{\alpha}{\alpha-1}} \leq 0,$$

since $\alpha > 1$, which gives us

$$\psi'(t)\psi'' \geq \beta(\alpha-1)\psi'(t)\psi + \gamma(\alpha-1)\psi'(t)\psi^{\frac{\alpha-1-l}{\alpha-1}} + \delta(\alpha-1)\psi'(t)\psi^{\frac{\alpha}{\alpha-1}}.$$

Hence

$$\tfrac{1}{2}\tfrac{d}{dt}(\psi'(t))^2 \geq \tfrac{\beta(\alpha-1)}{2}\tfrac{d}{dt}\psi^2 + \tfrac{\gamma(\alpha-1)}{\delta_1}\tfrac{d}{dt}\psi^{\delta_1} + \tfrac{\delta(\alpha-1)}{\delta_2}\tfrac{d}{dt}\psi^{\delta_2}, \tag{4.12}$$

where $\delta_1 = 1 + \frac{\alpha-1-l}{\alpha-1} = \frac{2\alpha-2-l}{\alpha-1}$, $\delta_2 = 1 + \frac{\alpha}{\alpha-1} = \frac{2\alpha-1}{\alpha-1}$. Since $2\alpha > 2 + l$ and $\alpha > 1$, it follows that $\delta_1 > 0$ and $\delta_2 > 0$.

Integrating (4.12), we obtain that

$$\tfrac{1}{2}(\psi'(t))^2 \geq \tfrac{1}{2}(\psi'(0))^2 + \tfrac{\beta(\alpha-1)}{2}(\psi^2 - \psi^2(0)) + \tfrac{\gamma(\alpha-1)}{\delta_1}(\psi^{\delta_1} - \psi^{\delta_1}(0)) + \tfrac{\delta(\alpha-1)}{\delta_2}(\psi^{\delta_2} - \psi^{\delta_2}(0)),$$

then, we obtain the following equivalent inequality

$$(\psi'(t))^2 \geq A^2 + \beta(\alpha-1)\psi^2 + \tfrac{2\gamma(\alpha-1)}{\delta_1}\psi^{\delta_1} + \tfrac{2\delta(\alpha-1)}{\delta_2}\psi^{\delta_2} \geq A^2, \tag{4.13}$$

where

$$A^2 = (\psi'(0))^2 - \beta(\alpha-1)\psi^2(0) - \tfrac{2\gamma(\alpha-1)}{\delta_1}\psi^{\delta_1}(0) - \tfrac{2\delta(\alpha-1)}{\delta_2}\psi^{\delta_2}(0).$$

From (4.8), (4.10) and the initial condition (4.5) we get

$$\begin{aligned} A^2 &= (1-\alpha)^2\phi^{-2\alpha}(0)(\phi'(0))^2 - \beta(\alpha-1)\phi^{2(1-\alpha)}(0) \\ &\quad -\tfrac{2\gamma(\alpha-1)}{\delta_1}\phi^{\delta_1(1-\alpha)}(0) - \tfrac{2\delta(\alpha-1)}{\delta_2}\phi^{\delta_2(1-\alpha)}(0) \\ &= (\alpha-1)^2\phi^{-2\alpha}(0)B > 0. \end{aligned}$$

A further analysis of inequality (4.13) yields

$$|\psi'| \geq A > 0, \forall t \in [0, t_0), \tag{4.14}$$

where $t_0$ is the life time of the solution. Since $\psi' < 0$ by (4.11), hence,

$$\psi' \leq -A < 0. \tag{4.15}$$

Integrating the inequality (4.15) we obtain

$$\psi(t) \leq \psi(0) - At,$$

in view of (4.8) and $\alpha > 1$, therefore

$$\phi^{1-\alpha}(t) \leq \phi^{1-\alpha}(0) - At.$$

As a result, we obtain the lower estimate

$$\phi(t) \geq (\phi^{1-\alpha}(0) - At)^{\frac{-1}{\alpha-1}},$$

which implies that $T \neq +\infty$, since otherwise there exists $T_\infty < \phi^{1-\alpha}(0)A^{-1}$ such that $\lim_{t\to T_\infty^-} \phi(t) = \infty$. Then the proof is completed.

**Proof of Theorem 2.3** We introduce the following notations

$$\phi(t) = \frac{1}{2}||\nabla u||^2 + \frac{1}{2}||u||^2, G(t) = ||\nabla u_t||^2 + ||u_t||^2.$$

If we multiply the equation (1.1) by $u$, we obtain the following equality:

$$\phi'' - G + (\nabla u_t, \nabla u) + ((|u|^k u)_t, u) + ||\nabla u||^2 = ||u||_{q+2}^{q+2}. \tag{4.16}$$

The third and fourth terms in (4.16) can be estimated as follows:

$$|(\nabla u_t, \nabla u)| \leq \tfrac{\epsilon}{2}||\nabla u_t||^2 + \tfrac{1}{2\epsilon}||\nabla u||^2 \leq \tfrac{\epsilon}{2}G + \tfrac{1}{\epsilon}\phi \tag{4.17}$$

and

$$|((|u|^k u)_t, u)| \leq (k+1)||u_t||||u||_{2(k+1)}^{k+1}$$
$$\leq \tfrac{\epsilon}{2}||u_t||^2 + \tfrac{(k+1)^2 C_0^{2(k+1)}}{2\epsilon}||\nabla u||^{2(k+1)} \leq \tfrac{\epsilon}{2}G + \tfrac{(k+1)^2 C_0^{2(k+1)}}{\epsilon}\phi^{k+1} \tag{4.18}$$

for any $\epsilon > 0$.

Hence, by the estimate (4.17) and (4.18), equality (4.16) yields

$$\phi'' - G + \epsilon G + \tfrac{1}{\epsilon}\phi + \tfrac{(k+1)^2 C_0^{2(k+1)}}{\epsilon}\phi^{k+1} + ||\nabla u||^2 \geq ||u||_{q+2}^{q+2}. \tag{4.19}$$

Integrating (2.5) with respect to $t$ gives

$$E(t) = \frac{1}{2}||u_t||^2 + \frac{1}{2}||\nabla u_t||^2 + \frac{1}{2}||\nabla u||^2 - \frac{1}{q+2}||u||_{q+2}^{q+2} \leq E(0).$$

Then we get

$$||u||_{q+2}^{q+2} \geq \frac{q+2}{2}G + \frac{q+2}{2}||\nabla u||^2 - (q+2)E(0). \tag{4.20}$$

Then, (4.20) and (4.19) yield

$$\phi'' - \tfrac{q+4}{2}G + \epsilon G + \tfrac{1}{\epsilon}\phi + \tfrac{(k+1)^2 C_0^{2(k+1)}}{\epsilon}\phi^{k+1} - \tfrac{q}{2}||\nabla u||^2 + (q+2)E(0) \geq 0. \tag{4.21}$$

Now, we choose $\epsilon = \epsilon_0 = \frac{q-k}{2} > 0$ so that

$$\alpha_0 = \tfrac{q+4}{2} - \tfrac{\epsilon_0}{2} = 2 + \tfrac{k}{2} > 1, 2\alpha_0 = 4 + k > k + 2, \tag{4.22}$$

where we have used the fact $q > k$. Then (4.21) becomes

$$\phi'' - \alpha_0 G + \tfrac{1}{\epsilon_0}\phi + \tfrac{(k+1)^2 C_0^{2(k+1)}}{\epsilon_0}\phi^{k+1} + (q+2)E(0) \geq \tfrac{q}{2}||\nabla u||^2 \geq 0. \tag{4.23}$$

From the Cauchy-Schwarz inequality, we have

$$(\phi')^2 \leq \phi G. \tag{4.24}$$

Then using (4.24) and (4.23) we arrive at the following second order differential inequality

$$\phi\phi'' - \alpha_0(\phi')^2 + \tfrac{1}{\epsilon_0}\phi^2 + \tfrac{(k+1)^2 C_0^{2(k+1)}}{\epsilon_0}\phi^{k+2} + (q+2)E(0)\phi \geq 0. \tag{4.25}$$

Comparing this differential inequality with inequality (4.1), we find that

$$\alpha = \alpha_0, \beta = \tfrac{1}{\epsilon_0}, \gamma = \tfrac{(k+1)^2 C_0^{2(k+1)}}{\epsilon_0}, \delta = (q+2)E(0), l = k. \tag{4.26}$$

By (4.26) and (4.22), we know that (4.2) and (4.3) are satisfied. By (2.8) and (2.9), we know that (4.4) and (4.5) are satisfied. Then by Lemma 4.1, we see that $\phi(t)$ blows up in finite time. This theorem is proved.

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# References


[1] S.A.Gabov. Mathematical fundamentals of linear theory of ionic sound waves in unmagnetized plasma. Matematicheskoe Modelirovanie, 1 (12) (1989), 133-148.

[2] M.O.Korpusov. Blow-up of ion acoustic waves in a plasma, Sbornik: Mathematics, 202 (1) (2011), 37-64.

[3] M.O.Korpusov, A.A.Panin. Blow-up of solutions of an abstract Cauchy problem for a formally hyperbolic equation with double non-linearity. Izvestiya: Mathematics, 78(5)(2014), 937-985.

[4] Y.D.Shang. Initial-boundary value problem for a class of fourth order nonlinear evolution equations. Mathematica Applicata, 13(1)(2000), 7-11.(in Chinese)

[5] Y.D.Shang. Initial boundary value problem of equation $u_{tt} - \Delta u - \Delta u_t - \Delta u_{tt} = f(u)$. Acta Mathematicae Applicatae Sinica, 23A(2000), 385-393. (in Chinese)

[6] H.W.Zhang, Q.Y.Hu. Existence of global weak solution and stability of a class nonlinear evolution equation. Acta Mathematiea Scientia, 24A(3) (2004), 329-336.(in Chinese)

[7] Y.Q.Xie, C.K.Zhong. The existence of global attractors for a class nonlinear evolution equation. Journal of Mathematical Analysis and Application, 336(2007), 54-69.

[8] R.Z.Xu, X.R.Zhao, J.H.Shen. Asymptotic behavior of solution for fourth order wave equation with dispersive and dissipative terms. Applied Mathematics and Mechanics, 29 (2008), 259-262.

[9] Y.Q.Xie, C.K.Zhong. Asymptotic behavior of a class of nonlinear evolution equations. Nonlinear Analysis, 71(11) (2009), 5095-5105.

[10] A.N.Carvalho, J.W.Cholewa. Local well posedness, asymptotic behavior and asymptotic bootstrapping for a class of semilinear evolution equations of the second order in time. Transactions of the American Mathematical Society, 361(5) (2009), 2567-2586.

[11] C.Y.Sun, L.Yang, J.Q.Duan. Asymptotic behavior for a semilinear second order evolution equation. Transactions of the American Mathematical Society, 363(11) (2011), 6085-6109

[12] C.S.Chen, H.Wang, S.L.Zhu. Global attractor and decay estimates of solutions to a class of nonlinear evolution equations. Mathematical Methods in the Applied Sciences, 34(5)(2011), 497-508.

[13] R.Z.Xu, Y.B.Yang. Finite time blow up for the nonlinear fourth-order dispersive-dissipative wave equation at high energy level. International Journal of Mathematics, 23(5) (2012), 1250060 (10 pages).

[14] H.F.Di, Y.D.Shang. Global existence and asymptotic behavior of solutions for the double dispersive-dissipative wave equation with nonlinear damping and source terms. Boundary Value Problems, 2015(2015) 29, DOI 10.1186/s13661-015-0288-6.

[15] V.Pata, M.Squassina. On the strongly damped wave equation. Communications in Mathematical Physics, 253(3) (2005), 511-533.

[16] A.N.Carvalho, J.W.Cholewa, T.Dlotko. Strongly damped wave problems: bootstrapping and regularity of solutions. Journal of Differential Equations, 244(2008), 2310-2333.

[17] V.Barbu, I.Lasiecka, M.A.Rammaha. On nonlinear wave equations with degenerate damping and source terms. Transactions of the American Mathematical Society, 357(7),(2005), 2571-2611.

[18] V.Barbu, I.Lasiecka, M.A.Rammaha. Blow-up of generalizedsolutions to wave equations with degenerate damping and source terms. Indiana University Mathematics Journal, 56(3),(2007), 995-1022.

[19] D.R.Pitts, M.A.Rammaha. Global existence and nonexistence theorems for nonlinear wave equations. Indiana University Mathematics Journal, 51(6)(2002), 1479-1509.

[20] M.A.Rammaha, T.A.Strei. Global existence and nonexistence for nonlinear wave equations with damping and source terms. Transactions of the American Mathematical Society, 354(9)(2002), 3621-3637.

[21] V.Barbu, I.Lasiecka, M.A.Rammaha. Existence and uniqueness of solutions to wave equations with degenerate damping and source terms. Control and Cybernetics, 34(3) (2005), 665-687.

[22] Q.Y. Hu, H.W. Zhang. Blowup and asymptotic stabiity of weak solutions to wave equations with nonlinear degenerate damping and source terms. Electronic Journal of Differential Equations. 2007(2007), No. 76, 1-10.

[23] M.A.Rammaha, S.Sakuntasathien. Global existence and blow up of solutions to systems of nonlinear wave equations with degenerate damping and source terms. Nonlinear Analysis Theory Methods and Applications, 2010, 72(5), 2658-2683.

[24] M.A. Rammaha, S.Sakuntasathien. Critically and degenerately damped systems of nonlinear wave equations with source terms. Applicable Analysis, 89(8)(2010), 1201-1227.

[25] X.S.Han, M.X.Wang. Global existence and blow-up of solutions for nonlinear viscoelastic wave equation with degenerate damping and source. Mathematische Nachrichten, 284(5-6)(2011), 703-716.

[26] S.T. Wu. General decay of solutions for a nonlinear system of viscoelastic wave equations with degenerate damping and source terms. Journal of Mathematical Analysis and Application, 406 (2013), 34-48.

[27] E.Piskin. Blow up of positive initial-energy solutions for coupled nonlinear wave equations with degenerate damping and source terms. Boundary Value Problems, 2015(2015),no.43, 1-11.

[28] L.E.Payne, D.Sattinger. Saddle points and instability of nonlinear hyperbolic equations. Israel Journal of Mathematics, 22(3-4) (1975), 273-303.

[29] M.Nakao. Asymptotic stability of the bounded or almost periodic solution of the wave equation. Journal of Mathematical Analysis and Application, 56 (1977), 336-343.

[30] H.A.Levine. Instability and nonexistence of global solutions to nonlinear wave equations of the form $Pu_{tt} = Au + F(u)$. Transactions of the American Mathematical Society, 192 (1974), 1-21.

[31] V.K.Kalantarov, O.A.Ladyzhenskaya. The occurrence of collapse for quasilinear equations of parabolic and hyperbolic types. Journal of Soviet Mathematics, 10(1)(1978), 77-102.

[32] E.Vitillaro. A potential well theory for the wave equation with nonlinear source and boundary damping terms. Glasgow Mathematical Journal, 44(2)(2002), 375-395.

[33] G. Todorova, Stable and unstable sets for the Cauchy problem for a nonlinear wave equation with nonlinear damping and source terms. Journal of Mathematical Analysis and Application, 239(1999), 213-226

[34] E.Vitillaro. Some new results on global nonexistence and blow-up for evolution problems with positive initial energy. Rendiconti Dellistituto Di Matematica Delluniversita Di Trieste, 31(2)(2000), 245-275.

[35] J.L.Lions. Quelques méthodes de résolution des problémes aux limites non linéaires, Dunod-Gauthier Villars, 1969.

# COMPACT AND MATRIX OPERATORS ON THE SPACE $|C,-1|_k$

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ABSTRACT. According to Hardy [5], Cesàro summability is usually considered for the range $\alpha \geq -1$. In a more recent paper [14], the space $|C_\alpha|_k$ is studied for $\alpha > -1$. In this paper we define $|C_{-1}|_k$ using the Cesàro mean $(C,-1)$ of Thorpe [26] , compute its $\alpha$-, $\beta$- and $\gamma$- duals, give some algebraic and topological properties, and characterize related matrix operators, and also obtain some identities or estimates for the their operator norms and the Hausdorff measure of noncompactness. Further, by applying the Hausdorff measure of noncompactness, we establish the necessary and sufficient conditions for such operators to be compact. So some results in [14] is also extended to the range $\alpha \geq -1$.

## 1. INTRODUCTION

Let $\omega$ be the set of all complex sequences, $c, \ell_\infty \subset w$ be the set of convergent and bounded sequences. For $c_s, b_s$ and $\ell_k$ $(k \geq 1,\ \ell_1 = \ell)$ , we write the sets of all convergent, bounded, $k$-absolutely convergent series, respectively. Let $A = (a_{nj})$ be an arbitrary infinite matrix of complex numbers. By $A(x) = (A_n(x))$ , we denote the $A$-transform of the sequence $x$, i.e., $A_n(x) = \sum_{j=0}^{\infty} a_{nj}x_j$, provided that the series converges for $n \geq 0$. We say that $A$ defines a matrix transformation from $U$ into $V$, and it denote by $A \in (U,V)$ or $A : U \to V$ if sequence $Ax = (A_n(x)) \in V$ for every sequence $x \in U$, where $U$ and $V$ are subspace of $w$ and also the sets $U^\alpha = \{\varepsilon \in w : (\varepsilon_v x_v) \in \ell \text{ for all } x \in U\}$, $U^\beta = \{\varepsilon \in w : (\varepsilon_v x_v) \in c_s \text{ for all } x \in U\}$, $U^\gamma = \{\varepsilon \in w : (\varepsilon_v x_v) \in b_s \text{ for all } x \in U\}$ and

$$U_A = \{x \in w : A(x) \in U\} \tag{1.1}$$

are said to be the $\alpha$-, $\beta$-, $\gamma$- duals of $U$ and the domain of the matrix $A$ in $U$, respectively. Further, $U$ is said to be an $BK$-space if it is a complete normed space with continuous coordinates $p_n : U \to \mathbb{C}$ defined by $p_n(x) = x_n$ for $n \geq 0$. The sequence $(e^n)$ is called a Schauder base (or briefly base) for a normed sequence space $U$ if for each $x \in U$ there exist unique sequence of

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coefficients $(x_v)$ such that $\|x-\sum_{v=0}^{m} x_v e^v\| \to 0$ $(m\to\infty)$, and in this case we write $x=\sum_{v=0}^{\infty} x_v e_v$. For example, the sequence $\left(e^{(n)}\right)$ is a base of $l_k$ with respect to the norm $\|x\|_{\ell_k}=\left(\sum_{v=0}^{\infty}|x_v|^k\right)^{1/k}$, $k\geq 1$, where $e^{(n)}$ is the sequence whose only non-zero term is 1 in the $nth$ place for each $n$.Throughout $k^*$ denotes the conjugate of $k>1$, i.e., $1/k+1/k^*=1$, and $1/k^*=0$ for $k=1$.

Let $\Sigma a_n$ be an infinite series with partial sum $s_n$. Let $(\sigma_n^{\alpha})$ be the nth Cesàro mean $(C,\alpha)$ of order $\alpha>-1$ of the sequence $(s_n)$, *e.i.*, $\sigma_n^{\alpha}=\frac{1}{A_n^{\alpha}}\sum_{v=0}^{n}A_{n-v}^{\alpha-1}s_v$.The summability $|C,\alpha|_k$ was defined by Flett [4] as follows. The series $\Sigma a_n$ is said to be summable $|C,\alpha|_k$ with index $k\geq 1$ if

$$\sum_{n=1}^{\infty} n^{k-1}\left|\sigma_n^{\alpha}-\sigma_{n-1}^{\alpha}\right|^k<\infty.$$

More recently the series space $|C_{\alpha}|_k$ has been studied by the second author for $\alpha>-1$, [14]. The Cesàro summability $(C,\alpha)$ is studied usually for range $\alpha\geq -1$ (see [5]). Since the above definition does not work for $\alpha=-1$, so it was separately defined by Thorpe [26] as follows. If the series to sequence transformation

$$T_n=\sum_{\nu=0}^{n-1} a_\nu+(n+1)\,a_n \tag{1.2}$$

tends to $s$ as $n$ tends to infinity, then the series $\Sigma a_n$ is summable by Cesàro summability $(C,-1)$ [26]. Now we define the space $|C_{-1}|_k$, $k\geq 1$, as the set of all series summable by the method $|C,-1|_k$. Then, it can be written that $|C_{-1}|_k=\left\{a=(a_v):\sum_{n=1}^{\infty} n^{k-1}\left|T_n-T_{n-1}\right|^k<\infty\right\}$, where $(T_n)$ is defined by (1.2), or

$$|C_{-1}|_k=\left\{a=(a_\nu):\sum_{n=1}^{\infty} n^{k-1}\left|(n+1)\,a_n-(n-1)\,a_{n-1}\right|^k<\infty\right\}.$$

The problems of absolute summability factors and comparision of these methods goes to old rather and uptill now were widely examined by many authors, (see, [1-3], [6], [10-12], [16-22] ,[24-25])) et al. There are a close relation between these problems and some special matrix transformations such as an identity matrix $I$ and a matrix $W=(w_{nv})$ defined by $w_{vv}=\varepsilon_v$ and $w_{nv}=0$ for $v\neq n$.

In this paper we derive a series space $|C_{-1}|_k$ using the Cesàro summability $(C,-1)$ of Thorpe [26] , compute its $\alpha$-, $\beta$- and $\gamma$- duals, give some algebraic and topological properties, and characterize certain matrix operators defined on that space, and also obtain some identities or estimates for the their operator norms and the Hausdorff measure of noncompactness. Moreover, by applying the Hausdorff measure of noncompactness, we establish the necessary and sufficient

conditions for such operators to be compact. So we also complete some open problems in the paper of Sarıgöl [14].

The following lemmas play important roles to prove our theorems.

**Lemma 1.1.** Let $1<k<\infty$. Then, $A\in(\ell_k,\ell)$ if and only if

$$\|A\|_{(\ell_k,\ell)}=\sup_N\left\{\sum_{v=0}^{\infty}\left|\sum_{n\in N}^{\infty}a_{nv}\right|^{k^*}\right\}^{1/k^*}<\infty,$$

where $N$ is any finite set of positive numbers [23].

The following lemma is more useful in many cases, which gives equivalent norm.

**Lemma 1.2.** Let $1<k<\infty$. Then, $A\in(\ell_k,\ell)$ if and only if

$$\|A\|'_{(\ell_k,\ell)}=\left\{\sum_{v=0}^{\infty}\left(\sum_{n=0}^{\infty}|a_{nv}|\right)^{k^*}\right\}^{1/k^*}<\infty,$$

and there exists $1\leq\xi\leq 4$ such that $\|A\|'_{(\ell_k,\ell)}=\xi\|A\|_{(\ell_k,\ell)}$ [15].

The second part of this lemma is easily seen by following the lines in [15] that $\|A\|_{(\ell_k,\ell)}\leq\|A\|'_{(\ell_k,\ell)}\leq 4\|A\|_{(\ell_k,\ell)}$.

**Lemma 1.3.** Let $1\leq k<\infty$. Then, $A\in(\ell,\ell_k)$ if and only if

$$\|A\|_{(\ell,\ell_k)}=\sup_v\left\{\sum_{n=0}^{\infty}|a_{nv}|^k\right\}^{1/k}<\infty,\ [7].$$

**Lemma 1.4.**

a-) $A\in(\ell,c)\Leftrightarrow(i)\ \lim_n a_{nv}$ exists, $v\geq 0$, $(ii)\ \sup_{n,v}|a_{nv}|<\infty$.

b-) $A\in(\ell,\ell_\infty)\Leftrightarrow(ii)$ holds.

c-) Let $1<k<\infty$.Then, $A\in(\ell_k,\ell_\infty)\Leftrightarrow(iii)\ \sup_n\sum_{v=0}^{\infty}|a_{nv}|^{k^*}<\infty$;

d-) $A\in(\ell_k,c)\Leftrightarrow(i)$ and $(iii)$ hold [23].

## 2. The Hausdorff measure of noncompactness

If $S$ and $H$ are subsets of a metric space $(X,d)$ and $\varepsilon>0$ then $S$ is called an $\varepsilon$-net of $H$ , if, for every $h\in H$, there exists an $s\in S$ such that $d(h,s)<\varepsilon$; if $S$ is finite, then the $\varepsilon$-net $S$ of $H$ is called a finite $\varepsilon$-*net* of $H$. Let $X$ and $Y$ be Banach spaces. A linear operator $L:X\to Y$ is called compact its domain is all of $X$ and, for every bounded sequence $(x_n)$ in $X$, the sequence $(L(x_n))$ has a convergent subsequence in $Y$. We denote the class of such operators by $\mathcal{C}(X,Y)$. If $Q$ is a bounded subset of the metric space $X$, then the Hausdorff measure of noncompactness of $Q$ is defined by $\chi(Q)=\{\varepsilon>0:Q$ has a finite $\varepsilon$-net in $X\}$ ,and $\chi$ is called the Hausdorff measure of noncompactness.

The following result is an important tool to compute the Hausdorff measure of noncompactness of a bounded subset of the $BK$ space $\ell_k,k\geq 1$.

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**Lemma 2.1.** Let $Q$ be a bounded subset of the normed space $X$ where $X = \ell_k$, for $1 \leq k < \infty$ or $X = c_0$.If $P_n : X \to X$ is the operator defined by $P_r(x) = (x_0, x_1, ..., x_r, 0, ...)$ for all $x \in X$, then $\chi(Q) = \lim_{r\to\infty} \left(\sup_{x\in Q} \|(I - P_r)(x)\|\right)$ ,where $I$ is the identity operator on $X$, [13].

If $X$ and $Y$ be Banach spaces and $\chi_1$ and $\chi_2$ be Hausdorff measures on $X$ and $Y$, then, the linear operator $L : X \to Y$ is said to be $(\chi_1, \chi_2)$-bounded if $L(Q)$ is bounded subset of $Y$ for every bounded subset $Q$ of $X$ and there exists a positive constant $M$ such that $\chi_2(L(Q)) \leq M\ \chi_1(Q)$ for every bounded $Q$ of $X$. If an operator $L$ is $(\chi_1, \chi_2)$-bounded then the number $\|L\|_{(\chi_1,\chi_2)} = \inf\{M > 0 : \chi_2(L(Q)) \leq M\chi_1(Q)$ for all bounded $Q \subset X\}$ is called the $(\chi_1, \chi_2)$-measure of noncompactness of $L$. In particular, we write $\|L\|_{(\chi,\chi)} = \|L\|_\chi$ for $\chi_1 = \chi_2 = \chi$.

**Lemma 2.2.** Let $X$ and $Y$ be Banach spaces, $L \in B(X, Y)$ and $S_X = \{x \in X : \|x\| \leq 1\}$ denote the unit sphere in $X$. Then, $\|L\|_\chi = \chi(L(S_X))$, and

$$L \in \mathcal{C}(X, Y) \text{ if and only if } \|L\|_\chi = 0, [8].$$

**Lemma 2.3.** Let $X$ be normed sequence space and $\chi_T$ and $\chi$ denote the Hausdorff measures of noncompactness on $\mathcal{M}_{X_T}$ and $\mathcal{M}_X$, the collections of all bounded sets in $X_T$ and $X$, respectively. Then, $\chi_T(Q) = \chi(T(Q)$ for all $Q \in \mathcal{M}_{X_T}$, where $T = (t_{nv})$ is a triangular infinite matrix, [9].

## 3. Continuous and compact matrix operators on the space $|C_{-1}|_k$

In this section by giving some topological properties, $\alpha$-, $\beta$-, $\gamma$- duals, base of $|C_{-1}|_k$ ,we characterize matrix operators on that space, determine their norms, and also establish the necessary and sufficient conditions for such operators to be compact by applying the Hausdorff measure of noncompactness, which also extends some results of Sarıgöl [14] to $\alpha \geq -1$. First of all, we define the matrix $T^{(k)} = (t^{(k)}_{nv})$ by $t^{(k)}_{00} = 1$,

$$t^{(k)}_{nv} = \begin{cases} -n^{1/k^*}(n-1), & v = n-1 \\ n^{1/k^*}(n+1), & v = n \\ 0, & \text{otherwise,} \end{cases} \tag{3.1}$$

for $1 \leq k < \infty$. Then it is clear that $|C_{-1}|_k = (\ell_k)_{T^{(k)}}$ according to (1.1). Further, since every triangular matrix has a unique inverse which is also triangle [27, p. 9]. there exists the inverse of $T^{(k)}$ and denote this inverse by $S^{(k)}$. Now it follows from (1.2) that $a_0 = y_0$ and $a_n = [n(n+1)]^{-1} \sum_{v=1}^{n} v^{1/k} y_\nu$, $n \geq 1$, and so $s^{(k)}_{00} = 1$,

$$s^{(k)}_{nv} = \begin{cases} \dfrac{\nu^{1/k}}{n(n+1)}, 1 \leq v \leq n \\ 0, \qquad v > n. \end{cases} \tag{3.2}$$

Finally, we define the following notations:

$$D_1=\left\{\varepsilon=(\varepsilon_v)\in w:\sum_{v=r}^{\infty}s_{vr}^{(1)}\varepsilon_v \text{ converges, } r\geq 1\right\},$$

$$D_2=\left\{\varepsilon=(\varepsilon_v)\in w:\sup_{m,r}\left|\sum_{v=r}^{m}s_{vr}^{(1)}\varepsilon_v\right|<\infty\right\},$$

$$D_3=\left\{\varepsilon=(\varepsilon_v)\in w:\sup_{m}\sum_{r=1}^{m}\left|\sum_{v=r}^{m}s_{vr}^{(k)}\varepsilon_v\right|^{k^*}<\infty\right\},$$

$$D_4=\left\{\varepsilon=(\varepsilon_v)\in w:\sup_{\nu}\sum_{n=\nu}^{\infty}\left|s_{nv}^{(1)}\varepsilon_n\right|<\infty\right\},$$

$$D_5=\left\{\varepsilon=(\varepsilon_v)\in w:\sum_{\nu=1}^{\infty}\left\{\sum_{n=\nu}^{\infty}\left|s_{nv}^{(k)}\varepsilon_n\right|\right\}^{k^*}<\infty\right\}.$$

**Theorem 3.1.** Let $1\leq k<\infty$ . Then,

**a-)** The space $|C_{-1}|_k$ is $BK$-space with respect to the norm $\|x\|_{|C_{-1}|_k}=\left\|T^{(k)}(x)\right\|_{l_k}$ and isomorphic to the space $\ell_k$, i.e., $|C_{-1}|_k\cong\ell_k$, where $T^{(k)}$ is defined by (3.1)

**b-)** $|C_{-1}|_k^{\beta}=D_1\cap D_3$ for $1<k<\infty$ and $|C_{-1}|^{\beta}=D_1\cap D_2$ for $k=1$.

**c-)** $|C_{-1}|_k^{\gamma}=D_3$ for $1<k<\infty$ and $|C_{-1}|_k^{\gamma}=D_2$ for $k=1$.

**d-)** $|C_{-1}|_k^{\alpha}=D_5$ for $1<k<\infty$ and $|C_{-1}|_k^{\alpha}=D_4$ for $k=1$.

**e-)** The sequence $b^{(v)}=(b_n^{(v)})$ is the Schauder base of the space $|C_{-1}|_k$ for $v,n\geq 0$, where $b_n^{(v)}=s_{nv}^{(k)}$.

**Proof a-)** Since $l_k$ is $BK$-spaces with its usual norm, $|C_{-1}|_k=(\ell_k)_{T^{(k)}}$ and $T^{(k)}$ is a triangle matrix, it follows from Theorem 4.3.2 of Wilansky [27, p. 61] that $|C_{-1}|_k$ is $BK$-spaces for $1\leq k<\infty$. To prove the second part, define $T^{(k)}:|C_{-1}|_k\to l_k$ by

$$T_n^{(k)}(x)=n^{1/k^*}\left[(n+1)\,x_n-(n-1)\,x_{n-1}\right],\ n\geq 1.$$

Then, it is clear that $T^{(k)}$ is linear operator and surjective, since, if $y\in\ell_k$, then $x_n=[n(n+1)]^{-1}\sum_{v=1}^{n}v^{1/k}y_\nu$, where $y=T^{(k)}(x)$, and also one to one. Further, it preserves the norm, since $\left\|T^{(k)}(x)\right\|_{l_k}=\|x\|_{|C_{-1}|_k}$ ,which completes the proof**.**

**b-)** Let $1<k<\infty$. Now, $\varepsilon\in|C_{-1}|_k^{\beta}$ if and only if $\Sigma\varepsilon_n x_n$ is convergent for every $x\in|C_{-1}|_k$. Let $y=T^{(k)}(x)$. Then, $y\in\ell_k$ if and only if $x\in|C_{-1}|_k$, where $x_n=(n(n+1))^{-1}\sum_{v=1}^{n}v^{1/k}y_\nu$ for $n\geq 1$, $x_0=y_0$, and also it can be written that

$$\sum_{v=0}^{m}\varepsilon_\nu x_\nu=\varepsilon_0y_0+\sum_{r=1}^{m}\left(\sum_{v=r}^{m}s_{vr}^{(k)}\varepsilon_v\right)y_r=\sum_{r=0}^{m}\mu_{mr}y_r$$

where, $\mu_{m0}=\varepsilon_0$,

$$\mu_{mr}=\begin{cases}\sum_{v=r}^{m} s_{vr}^{(k)}\varepsilon_v, 1\leq r\leq m\\ 0, \quad r>m.\end{cases}$$

So it follows from Lemma 1.4. that $\varepsilon\in|C_{-1}|_k^{\beta}$ iff $\mu\in(\ell_k,c)$, or equivalently, $\varepsilon\in D_1\cap D_3$, which completes the proof.

Since (c) and (d) can be proved easily as in (b), so we omit the detail.

**e-)** Since $|C_{-1}|_k=(\ell_k)_{T^{(k)}}$ and the sequence $\left(e^{(v)}\right)$ is a base of $l_k$, where $e^{(v)}=\left(e_n^{(v)}\right)$, it is clear that the sequence $\left(b^{(v)}\right)$ is the base for$|C_{-1}|_k$. In fact, if $x\in|C_{-1}|_k$, then there exists $y\in l_k$ such that $y=T^{(k)}(x)$, and so it follows

$$\left\|x-\sum_{v=0}^{m}x_v b^{(v)}\right\|_{|C_{-1}|_k}=\left\|y-\sum_{v=0}^{m}y_v e^{(v)}\right\|_{l_k}\to 0 \text{ as } m\to\infty,$$

where $x_v=\left(T^{(k)}\right)_v^{-1}(y)$, $v\geq 0$. Further, it has the unique representation by $x=\sum_{v=0}^{\infty}x_v b^{(v)}$, which is immediately seen from the triange inequality of norm.

Also, we state the following result which is immediate by Theorem 3.1.

**Theorem 3.2.** Let $1\leq k<\infty$ and $A=(a_{nv})$ is an infinite matrix with complex terms for all $n,v\geq 0$. Define the matrices $\overline{L}=\left(\overline{\ell}_{nr}\right)$, $D=(d_{nv})$ and $D^{(r)}=\left(\bar{d}_{n\nu}^{(r)}\right)$ by

$$\overline{\ell}_{nr}=r\sum_{v=r}^{\infty}\frac{a_{nv}}{\nu(\nu+1)},$$

$$d_{nv}=n^{1/k^*}\left[(n+1)\,\overline{\ell}_{n\nu}-(n-1)\,\overline{\ell}_{n-1,\nu}\right],\ n\geq 1, v\geq 1. \tag{3.3}$$

$$\bar{d}_{n\nu}^{(r)}=\begin{cases}0,\ 1\leq n\leq r\\ d_{nv},\quad n>r\end{cases}$$

Then each matrix in the class $A\in(|C_{-1}|,|C_{-1}|_k)$ defines a bounded linear operator $L_A:|C_{-1}|\to|C_{-1}|_k$ such that $L_A(x)=A(x)$ for all $x\in|C_{-1}|$, and $A\in(|C_{-1}|,|C_{-1}|_k)$ if and only if

$$\overline{L}\text{ is well defined } n,r\geq 1, \tag{3.4}$$

$$\sup_{m,r}\left|r\sum_{v=r}^{m}\frac{a_{nv}}{\nu(\nu+1)}\right|<\infty \text{ for each } n, \tag{3.5}$$

$$\sup_{v}\sum_{n=1}^{\infty}|d_{nv}|^k<\infty. \tag{3.6}$$

Morever, if $A\in(|C_{-1}|,|C_{-1}|_k)$, then

$$\|L_A\|_{(|C_{-1}|,|C_{-1}|_k)}=\|D\|_{(\ell,\ell_k)}\ \text{ and }\quad \|L_A\|_{\chi}=\lim_{r\to\infty}\left\|D^{(r)}\right\|_{(l,l_k)} \tag{3.7}$$

**Proof.**The first part is immediate by Theorem 4.2.8 of Wilansky [27, $p$.57], since $|C_{-1}|$ and $|C_{-1}|_k$ are a $BK$-spaces by Theorem 3.1. For second part, $A\in$

$(|C_{-1}|, |C_{-1}|_k)$ iff $(a_{nv})_{v=0}^{\infty} \in (|C_{-1}|)^{\beta}$ for each $n$, and $A(x) \in |C_{-1}|_k$ for every $x \in |C_{-1}|$. Now, by Theorem 3.1 $(b)$, $(a_{nv})_{v=0}^{\infty} \in (|C_{-1}|)^{\beta}$ iff $(a_{nv})_{\nu=0}^{\infty} \in D_1 \cap D_2$, or, equivalently, (3.4) and (3.5) hold for each $n$. Now, to obtain (3.6), consider operator $T^{(1)} : |C_{-1}| \to \ell_1$ by using (3.1) for $k = 1$ by

$$T_n^{(1)}(x) = (n+1)\, x_n - (n-1)\, x_{n-1} \tag{3.8}$$

As in the proof of Theorem 3.1, $T^{(1)}$ is bijection and the matrix corresponding to this operator is triangle. Further, let $x \in |C_{-1}|$ be given, then $y \in \ell$, where $y = T^{(1)}(x)$, for $n \geq 0$, $x_{-1} = 0$, and also, $x_n = (n(n+1))^{-1} \sum_{v=1}^{n} v y_\nu$ for $n \geq 1$ and $x_0 = y_0$. Now, we can write

$$\sum_{v=1}^{m} a_{nv} x_v = \sum_{r=1}^{m} \left( r \sum_{v=r}^{m} \frac{a_{n\nu}}{\nu(\nu+1)} \right) y_r = \sum_{r=1}^{m} \overline{\ell}_{mr}^{(n)} y_r,$$

where

$$\overline{\ell}_{mr}^{(n)} = \begin{cases} r \sum_{v=r}^{m} \dfrac{a_{n\nu}}{\nu(\nu+1)}, 1 \leq \nu \leq m \\ 0, \qquad \nu > m. \end{cases}$$

Moreover, if any matrix $H = (h_{nv}) \in (\ell, c)$, then, the series $H_n(x) = \Sigma_v h_{nv} x_v$ converges informly in n, since, by Lemma 1.4, the remaining term tends to zero informly in n, that is,

$$\left| \sum_{v=m}^{\infty} h_{nv} x_v \right| \leq \sup_{n,v} |h_{nv}| \sum_{v=m}^{\infty} |x_v| \to 0 \text{ as } m \to \infty,$$

and so

$$\lim_n H_n(x) = \sum_{v=0}^{\infty} \lim_n h_{nv} x_v \tag{3.9}$$

Hence, since (3.4) and (3.5) hold, $\overline{L}^{(n)} = \left( \overline{\ell}_{mr}^{(n)} \right) \in (\ell, c)$, then by (3.9), we get

$$A_n(x) = \sum_{r=0}^{\infty} \left( \lim_m \overline{\ell}_{mr}^{(n)} \right) y_r = \sum_{r=0}^{\infty} \overline{\ell}_{nr} y_r = \overline{L}_n(y),$$

This shows that $A(x) \in |C_{-1}|_k$ for every $x \in |C_{-1}|$ iff $\overline{L}(y) \in |C_{-1}|_k$ for every $y \in \ell$, or, equivalently, $D(y) \in \ell_k$, since $|C_{-1}|_k = (\ell_k)_{T^{(k)}}$, where $D = T^{(k)} \overline{L}$. So, it is clear that $A \in (|C_{-1}|, |C_{-1}|_k)$ iff (3.4), (3.5) hold, and $D \in (\ell, \ell_k)$. A few calculations show that

$$d_{nv} = \sum_{r=1}^{n} t_{nr}^{(k)} \overline{\ell}_{r\nu} = n^{1/k^*} \left[ (n+1)\, \overline{\ell}_{n\nu} - (n-1)\, \overline{\ell}_{n-1,\nu} \right], \quad v, n \geq 1,$$

Therefore, it can be obtained from Lemma 1.3 that $D \in (\ell, \ell_k)$ iff (3.6) is satisfied, which completes the proof of the second part.

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Now, to compute norm of $A$, $A \in (|C_{-1}|, |C_{-1}|_k)$ iff $D \in (\ell, \ell_k)$, where $D$ is defined by (3.3). On the other hand, consider the isomorphisms $T^{(1)} : |C_{-1}| \to \ell$ and $T^{(k)} : |C_{-1}|_k \to \ell_k$ defined in (3.8) and Theorem 3.1 (a), respectively. Then we get $A = \left(T^{(k)}\right)^{-1} oDoT^{(1)}$. So, by Theorem 3.1, $x \in |C_{-1}|$ iff $y = T^{(1)}(x) \in \ell$ and $\|A(x)\|_{|C_{-1}|_k} = \|D(y)\|_{\ell_k}$ for all $x \in |C_{-1}|$ and $y \in \ell$, which gives $\|A\|_{(|C_{-1}|,|C_{-1}|_k)} = \|D\|_{(l,l_k)}$.

Finally, let $S = \{x \in |C_{-1}| : \|x\| \leq 1\}$. Then, by Lemma 2.1-Lemma 2.3, and from the definition of $D^{(r)}$ we get

$$\begin{aligned} \|L_A\|_\chi &= \chi(L_A S) = \chi(DTS) = \lim_{r\to\infty} \sup_{y\in TS} \|(I - P_r) D(y)\|_{l_k} \\ &= \lim_{r\to\infty} \left\|D^{(r)}\right\|_{(l,l_k)} = \lim_{r\to\infty} \sup_v \left\{ \sum_{n=r+1}^{\infty} |d_{nv}|^k \right\}^{1/k} \end{aligned}$$

where $P_r : l_k \to l_k$ is defined by $P_r(y) = (y_0, y_1, ..., y_r, 0, ...)$ and the matrix $D^{(r)} = \left(\bar{d}_{n\nu}\right)$ is defined as: $\bar{d}_{n\nu} = d_{nv}$ for $n > r$, 0 otherwise. Thus, the proof is completed together with Lemma 1.3.

It is directly characterized from Theorem 3.2 the compact operators in the class $(|C_{-1}|, |C_{-1}|_k)$.

**Corollary 3.3.** Under hypotheses of Theorem 3.2, $L_A \in (|C_{-1}|, |C_{-1}|_k)$ is compact if and only if $\|L_A\|_\chi = \lim_{r\to\infty} \left\|D^{(r)}\right\|_{(l,l_k)} = 0$.

Now, If a matrix $A$ is chosen as a special matrix $W$ and $I$, then $A \in (|C_{-1}|, |C_{-1}|_k)$ means the form of summability factors that $\Sigma\varepsilon_v x_v$ is summable $|C_{-1}|_k$ when $\Sigma x_v$ is summable $|C_{-1}|$, and $I \in (|C_{-1}|, |C_{-1}|_k)$ means the comparisons of these methods, i.e., $|C_{-1}| \subset |C_{-1}|_k$. If we put $A = I$, then, since

$$\bar{\ell}_{nr} = \begin{cases} \dfrac{r}{n(n+1)}, & 1 \leq r \leq n \\ 0, & n < r \end{cases}$$

and

$$d_{nv} = \begin{cases} n^{1/k^*} \left[(n+1)\bar{\ell}_{nv} - (n-1)\bar{\ell}_{n-1,\nu}\right], & 1 \leq v \leq n \\ 0, & v > n, \end{cases}$$

it is also clear that condition (3.4), (3.5) and (3.6) are satisfied. So one can easily derive from Theorem 3.2 the following result.

**Corollary 3.4.** Let $1 \leq k < \infty$. Then, $|C_{-1}| \subset |C_{-1}|_k$.

**Theorem 3.5.** Let $1 < k < \infty$ and $A = (a_{nv})$ is an infinite matrix with complex terms for all $n, v \geq 0$. Let be define the matrices $\overline{L} = \left(\bar{\ell}_{n\nu}\right)$ as in Theorem 3.2 and $F = (f_{n\nu})$ and $F^{(r)} = \left(\tilde{f}_{n\nu}^{(n)}\right)$ by

$$f_{n\nu} = \nu^{-1/k^*} \left[(n+1)\bar{\ell}_{n\nu} - (n-1)\bar{\ell}_{n-1,\nu}\right], \ \nu \geq 1, n \geq 1, \tag{3.10}$$

and

$$\tilde{f}_{n\nu}^{(r)} = \begin{cases} 0, & 1 \leq n \leq r \\ f_{n\nu}, & n > r \end{cases}$$

Then each matrix in the class $A \in (|C_{-1}|_k, |C_{-1}|)$ defines a bounded linear operator $L_A : |C_{-1}|_k \to |C_{-1}|$ such that $L_A(x) = A(x)$ for all $x \in |C_{-1}|_k$, and $A \in (|C_{-1}|_k, |C_{-1}|)$ if and only if (3.4)

$$\sup_m \sum_{r=1}^{m} \left| r^{1/k} \sum_{v=r}^{m} \frac{a_{n\nu}}{\nu(\nu+1)} \right|^{k^*} < \infty, \tag{3.11}$$

$$\sum_{\nu=1}^{\infty} \left( \sum_{n=1}^{\infty} |f_{n\nu}| \right)^{k*} < \infty. \tag{3.12}$$

Morever, if $A \in (|C_{-1}|_k, |C_{-1}|)$, then, there exists $1 \leq \xi \leq 4$ such that

$$\|L_A\|_{(|C_{-1}|_k, |C_{-1}|)} = \frac{1}{\xi} \|F\|'_{(\ell_k, \ell)} \text{ and } \|L_A\|_\chi = \frac{1}{\xi} \lim_{r\to\infty} \left\| F^{(r)} \right\|'_{(l_k, l)}. \tag{3.13}$$

**Proof.** The first part is immediate by Theorem 4.2.8 of Wilansky [27, *p.*57], since $|C_{-1}|$ and $|C_{-1}|_k$ are a $BK$-spaces by Theorem 3.1. For the second part, let $A \in (|C_{-1}|_k, |C_{-1}|)$. Then, $(a_{n\nu})_{\nu=0}^{\infty} \in (|C_{-1}|_k)^\beta$ and $A(x) \in |C_{-1}|$ for every $x \in |C_{-1}|_k$. Now by Theorem 3.1 $(b)$, $(a_{n\nu})_{\nu=0}^{\infty} \in (|C_{-1}|_k)^\beta$ iff $(a_{n\nu})_{\nu=0}^{\infty} \in D_1 \cap D_3$ for each $n$. This also means that $(a_{n\nu})_{\nu=0}^{\infty} \in (|C_{-1}|_k)^\beta$ iff conditions (3.4) and (3.11) hold. Also to obtain (3.12), we consider the operators $T^{(k)} : |C_{-1}|_k \to \ell_k$ same as Theorem 3.1 (a). Then, since the space $|C_{-1}|_k$ is izomorf to $\ell_k$, it can be written that $x \in |C_{-1}|_k$ iff $y \in \ell_k$, where $T^{(k)}(x) = y$ , *i.e.*, $y_0 = x_0$ and $y_n = T_n^{(k)}(x) = n^{1/k^*}[(n+1)x_n - (n-1)x_{n-1}]$ for $n \geq 1$ , $x_{-1} = 0$. So by (3.2), as in the proof of Theorem 3.2 we get

$$\sum_{v=1}^{m} a_{nv} x_v = \sum_{r=1}^{m} r^{-1/k^*} \ell_{mr}^{(n)} y_r = \sum_{r=1}^{m} \overline{f}_{mr}^{(n)} y_r$$

where, $\overline{f}_{mr}^{(n)} = r^{-1/k^*} \ell_{mr}^{(n)}$ for $1 \leq r \leq m$, and $\overline{f}_{mr}^{(n)} = 0$ for $r > m$, and $\overline{L}^{(n)} = \left( \overline{\ell}_{mr}^{(n)} \right)$ is defined as in Theorem 3.2. Moreover, if any matrix $H = (h_{nv}) \in (\ell_k, c)$, then by Lemma1.4 and using Hölder's inequality, we get

$$\left| \sum_{v=m}^{\infty} h_{nv} x_v \right| \leq \sup_n \left( \sum_{v=0}^{\infty} |h_{nv}|^{k^*} \right)^{1/k^*} \left( \sum_{v=m}^{\infty} |x_v|^k \right)^{1/k}$$

Also, since $x \in \ell_k$, we obtain that the series $H_n(x) = \Sigma_v h_{nv} x_v$ converges informly in n, which implies

$$\lim_n H_n(x) = \sum_{v=0}^{\infty} \lim_n h_{nv} x_v. \tag{3.14}$$

Therefore, since (3.4) and (3.11) hold, $F^{(n)} = \left(\overline{f}_{mr}^{(n)}\right) \in (\ell_k, c)$, then by (3.14), we get

$$A_n(x) = \sum_{r=1}^{\infty} \left(\lim_m \overline{f}_{mr}^{(n)}\right) y_r = \sum_{r=1}^{\infty} r^{-1/k^*} \overline{\ell}_{nr} y_r = \sum_{r=1}^{\infty} \overline{f}_{nr} y_r = \overline{F}_n(y),\ n \geq 1$$

where, $\overline{f}_{nr} = \lim_m \overline{f}_{mr}^{(n)}$. This shows that $A(x) \in |C_{-1}|$ for every $x \in |C_{-1}|_k$ iff $\overline{F}(y) \in |C_{-1}|$ for every $y \in \ell_k$ or, equivalently $\left(T^{(1)}\overline{F}\right)(y) \in \ell$, since $|C_{-1}| = (\ell)_{T^{(1)}}$, so we obtain that $F \in (\ell_k, \ell)$, where $F = T^{(1)}\overline{F}$. Hence, it is clear that $A \in (|C_{-1}|_k, |C_{-1}|)$ iff (3.4), (3.11) hold, and $F \in (\ell_k, \ell)$.With a few calculations, it can be easily seen that $F$ is the same as (3.10), and so it follows from applying Lemma 1.2 to the matrix $F$ that $F \in (\ell_k, \ell)$ iff (3.12) is satisfied, and this proves the second part.

Also, considering that $T^{(k)} : |C_{-1}|_k \to \ell_k$ and $T^{(1)} : |C_{-1}| \to \ell$ are norm isomorphism, as in Theorem 3.2, it follows that $A = \left(T^{(1)}\right)^{-1} oFoT^{(k)}$ and so, by Lemma 1.2,

$$\|L_A\|_{\left(|C_{-1}|_k, |C_{-1}|\right)} = \|F\|_{(\ell_k, \ell)} = \frac{1}{\xi} \|F\|'_{(\ell_k, \ell)}$$

Finally, $S = \{x \in |C_{-1}|_k : \|x\| \leq 1\}$. Then, by considering Lemma 2.1-Lemma 2.3, and Lemma 1.2, there exists $1 \leq \xi \leq 4$ such that

$$\|L_A\|_\chi = \lim_{r\to\infty} \sup_{y \in T^{(k)}S} \|(I - P_r) F(y)\|_l = \lim_{r\to\infty} \left\|F^{(r)}\right\|_{(l_k, l)} = \frac{1}{\xi} \lim_{r\to\infty} \left\|F^{(r)}\right\|'_{(l_k, l)}$$

where $P_r : l \to l$ is defined by $P_r(y) = (y_0, y_1, ..., y_r, 0, ...)$ and the matrix $F^{(r)} = \left(\tilde{f}_{n\nu}^{(n)}\right)$ is defined as: $\tilde{f}_{n\nu}^{(n)} = 0$ for $1 \leq n \leq r$, and $f_{n\nu}$ for $n > r$, which proves the theorem together with Lemma 1.2.

The compact operators in this class are obtained from Theorem 3.5 as follows.

**Corollary 3.6.** Under hypotheses of Theorem 3.5,

$$L_A \in (|C_{-1}|_k, |C_{-1}|) \text{ is compact if and only if } \lim_{r\to\infty} \left\|F^{(r)}\right\|'_{(l_k, l)} = 0.$$

## References


[1] Bor, H. and Thorpe, B., On some absolute summability methods, *Analysis 7 (2) (1987), 145-152.*

[2] Borwein, D. and Cass, F.P., Strong Nörlund summability, *Math. Zeitschr.* 103 (1968), 94–111.

[3] Bosanquet L.S. and Das G., Absolute summability factors for Nörlund means, *Proc. London Math. Soc.* (3) 38 (1979), 1-52.

[4] Flett, T.M., On an extension of absolute summability and some theorems of Littlewood and Paley, *Proc. London Math. Soc. 7 (1957), 113-141.*

[5] Hardy, G. H., Divergent Series Oxford, 1949.

[6] Kuttner, B., Some remarks on summability factors, *Indian J. Pure Appl. Math. 16 (9) (1985), 1017-1027.*
[7] Maddox, I.J., Elements of functinal analysis, *Cambridge University Press, London,New York, (1970).*
[8] Malkowsky, E. and Rakočević, V., An introduction into the theory of sequence space and measures of noncompactness, Zb. Rad. (Beogr) 9, (17),(2000), 143-234.
[9] Malkowsky, E. and Rakočević, V., On matrix domain of triangles, Appl. Math. Comp. 189(2), (2007), 1146-1163.
[10] Mazhar, S.M., On the absolute summability factors of infinite series, *Tohoku Math. J. 23 (1971), 433-451.*
[11] Mohapatra, R.N. and Das, G., Summability factors of lower-semi matrix transformations, *Monatshefte für Mathematika, 79 (1975), 307-3015.*
[12] Orhan, C. and Sarıgöl, M.A., On absolute weighted mean summability, *Rocky Moun. J. Math. 23 (3) (1993), 1091-1097.*
[13] Rakočević, V., Measures of noncompactness and some applications, Filomat, 12, (1998), 87-120.
[14] Sarıgöl, M.A., Spaces of Series Summable by Absolute Cesàro and Matrix Operators, *Comm. Math Appl.* 7 (1) (2016*) 11-22.*
[15] _____ Extension of Mazhar's theorem on summability factors, *Kuwait J. Sci.* 42 (3) (2015), 28-35.
[16] _____ Matrix operators on $A_k$, *Math. Comp. Model.* 55 (2012), 1763-1769.
[17] _____ Matrix transformatins on fields of absolute weighted mean summability, *Studia Sci. Math. Hungar. 48 (3) (2011), 331-341.*
[18] _____ and Bor, H., Characterization of absolute summability factors, *J. Math. Anal. Appl. 195 (1995), 537-545.*
[19] _____ On two absolute Riesz summability factors of infinite series, *Proc. Amer. Math. Soc. 118 (1993), 485-488.*
[20] _____ A note on summability, *Studia Sci. Math. Hungar. 28 (1993), 395-400.*
[21] _____ On absolute weighted mean summability methods, *Proc. Amer. Math. Soc. 115 (1) (1992), 157-160.*
[22] _____ Necessary and sufficient conditions for the equivalence of the summability methods $\left|\overline{N},p_n\right|_k$ and $|C,1|_k$, *Indian J. Pure Appl. Math. 22(6) (1991), 483-489.*
[23] Stieglitz, M. and Tietz, H., Matrixtransformationen von folgenraumen eine ergebnisüberischt, *Math Z., 154 (1977), 1-16.*
[24] Sulaiman, W.T., On summability factors of infinite series, *Proc. Amer. Math. Soc. 115 (1992), 313-317.*
[25] Sunouchi, G., Notes on Fourier Analysis, 18, absolute summability of a series with constant terms, *Tohoku Math. J.*, 1, (1949), 57-65.
[26] Thorpe, B., Matrix Transformations of Cesàro Summable Series, Acta Math. Hung., 48(3-4), (1986), 255-265.
[27] Wilansky, A., Summability Through Functional Analysis, North-Holland Mathematical Studies, vol. 85, *Elsevier Science Publisher, 1984.*

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# On value distribution of meromorphic solutions of a certain second order difference equations *

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**Abstract.** In this paper, we consider difference equation $f(z+1)+f(z-1)=\frac{A(z)f(z)+C}{1-f^2(z)}$, where $C$ is a non-zero constant, $A(z)=\frac{m(z)}{n(z)}$, $m(z)$ and $n(z)$ are irreducible polynomials, we obtain the forms of rational solutions.

To the difference equation $f(z+1)+f(z-1)=\frac{A(z)f(z)+C(z)}{1-f^2(z)}$, where $A(z)$, $C(z)$ are non-constant small functions with respect to $f(z)$, the Borel exceptional value, the exponents of convergence of zeros, poles and fixed points of finite order transcendental meromorphic solution $f(z)$, and the exponents of convergence of poles of differences $\Delta f(z)=f(z+1)-f(z)$, $\Delta^2 f(z)=\Delta f(z+1)-\Delta f(z)$ and divided differences $\frac{\Delta f(z)}{f(z)}$, $\frac{\Delta^2 f(z)}{f(z)}$ are estimated.

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## 1 Introduction and Results

Halburd and Korhonen [6, 7] used value distribution theory and a reasoning related to the singularity confinement to single out the difference Painlevé I and II equations from difference equation

$$f(z+1)+f(z-1)=R(z,f), \tag{1.1}$$

where R is rational in both of its arguments. They obtained that if (1.1) has an admissible meromorphic solutions of finite order, then either $f$ satisfies a difference Riccati equation, or (1.1) may be transformed into some classical difference equations, which include difference Painlevé I equations

$$f(z+1)+f(z-1)=\frac{az+b}{f(z)}+c, \tag{1.2}$$

$$f(z+1)+f(z-1)=\frac{az+b}{f(z)}+\frac{c}{f^2(z)}, \tag{1.3}$$

$$f(z+1)+f(z)+f(z-1)=\frac{az+b}{f(z)}+c, \tag{1.4}$$

and difference Painlevé II equation

$$f(z+1)+f(z-1)=\frac{(az+b)f(z)+c}{1-f^2(z)}, \tag{1.5}$$

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where $a, b$ and $c$ are constants.

Recently, as the difference analogues of Nevanlinna's theory are investigated [2, 5], many results on the complex difference equations are rapidly obtained, such as [1, 3, 10−12]. However, there are a few papers concerning with the existence of rational solution of difference Painlevé equations. In this paper, we will consider the forms of rational solutions, and investigate the value distribution of meromorphic solution of finite order of a certain type of difference equation which originates from the difference Painlevé II equation.

We assume that the reader is familiar with the basic Nevanlinna's value distribution theory of meromorphic functions (see[4, 9]). In addition, we use the notation $\sigma(f)$ to denote the order of growth of the meromorphic function $f(z)$, $\lambda(f)$ and $\lambda(\frac{1}{f})$ to denote, respectively, the exponent of convergence of zeros and poles of $f(z)$. We also use the notation $\tau(f)$ to denote the exponent of convergence of fixed points of $f(z)$ which is defined as

$$\tau(f) = \overline{\lim_{r\to\infty}} \frac{\log N\left(r, \frac{1}{f(z)-z}\right)}{\log r}.$$

We denote by $S(r, f)$ any quantify satisfying $S(r, f) = o(T(r, f))$, as $r \to \infty$, possibly outside a set with finite measure. For every $n \in \mathbb{N}^*$, the forward difference $\Delta^n f(z)$ are defined in the standard way [14] by $\Delta f(z) = f(z + 1) - f(z)$, $\Delta^{n+1} f(z) = \Delta^n f(z + 1) - \Delta^n f(z)$.

Chen and Shon [3] considered the difference Painlevé II equation (1.5) and proved the following result.

**Theorem A.** (See [3]) *Let $a, b, c$ be constants, $ac \neq 0$. Suppose that a rational function*

$$f(z) = \frac{P(z)}{Q(z)} = \frac{pz^m + p_{m-1}z^{m-1} + \cdots + p_0}{qz^n + q_{n-1}z^{n-1} + \cdots + q_0}$$

*is a solution of* (1.5)*, where $P(z)$ and $Q(z)$ are relatively prime polynomials, $p, p_{m-1}, \ldots, p_0$ and $q, q_{n-1}, \ldots, q_0$ are constants. Then*

$$n = m + 1 \quad \text{and} \quad p = -\frac{c}{a}q.$$

In equation (1.5), if we replace $az + b$ with $A(z) = \frac{m(z)}{n(z)}$, where $m(z)$ and $n(z)$ are mutually prime polynomials, we still consider the rational solutions of equation (1.5), what will happen? Here, we obtain the following result.

**Theorem** 1.1. *Let $C$ be non-zero constant, and $A(z) = \frac{m(z)}{n(z)}$ be a rational function, where $m(z)$ and $n(z)$ are mutually prime polynomials with $\deg m(z) = m$, $\deg n(z) = n$. If difference equation*

$$f(z + 1) + f(z - 1) = \frac{A(z)f(z) + C}{1 - f^2(z)} \tag{1.6}$$

*has a rational solution $f(z) = \frac{P(z)}{Q(z)}$, where $P(z)$ and $Q(z)$ are relatively prime polynomials with $\deg P(z) = p$, $\deg Q(z) = q$, then*

*(i) Suppose that $m > n$, then $p - q = \frac{m-n}{2}$ and $m - n$ must be even or $q - p = m - n \geq 1$;*

*(ii) Suppose that $m = n$, then $p - q = 0$ and*

$$\lim_{z\to\infty} \frac{m(z)}{n(z)} = A \in \mathbb{C} \setminus \{0\}, \quad \lim_{z\to\infty} \frac{P(z)}{Q(z)} = B \in \mathbb{C} \setminus \{0\},$$

*$C = B[2(1 - B^2) - A]$ or $C = \pm A$;*

*(iii) Suppose that $m < n$, then $p - q = 0$ and*

$$\lim_{z\to\infty} \frac{P(z)}{Q(z)} = B \in \mathbb{C} \setminus \{0, \pm 1\}, \quad C = 2B(1 - B^2).$$

The following examples show that the difference equation (1.6) has rational solutions satisfying Theorem 1.1 (i), (ii) and (iii).

**Example** 1.1. *The difference equation*

$$f(z+1)+f(z-1)=\frac{-\frac{2z^6+2z^4+Cz^3+2z^2-Cz}{z^4-1}f(z)+C}{1-f^2(z)}$$

*has a rational solution* $f(z)=z+\frac{1}{z}$, *where* $C\neq 0,\ m=6,\ n=4,\ p=2,\ q=1$ *and* $p-q=\frac{m-n}{2}=1$.

**Example** 1.2. *The difference equation*

$$f(z+1)+f(z-1)=\frac{(-Cz+2)f(z)+C}{1-f^2(z)}$$

*has a rational solution* $f(z)=\frac{1}{z}$, *where* $C\neq 0,\ m=1,\ n=0,\ p=0,\ q=1$ *and* $q-p=m-n=1$.

**Example** 1.3. *The difference equation*

$$f(z+1)+f(z-1)=\frac{\frac{-2(z^3+5z+10)}{z^3+2z^2-z-2}f(z)+2}{1-f^2(z)}$$

*has a rational solution* $f(z)=\frac{z-1}{z+1}$, *where* $m=n=3,\ p=q=1$,

$$\lim_{z\to\infty}\frac{-2(z^3+5z+10)}{z^3+2z^2-z-2}=-2,\quad \lim_{z\to\infty}\frac{z-1}{z+1}=1,$$

*and* $C=2=B[2(1-B^2)-A]=1\cdot[2\cdot(1-1)-(-2)]$.

**Example** 1.4. *The difference equation*

$$f(z+1)+f(z-1)=\frac{\frac{-44z^3+36z^2+28z+12}{z^4-2z^3-z^2+2z}f(z)-12}{1-f^2(z)}$$

*has a rational solution* $f(z)=\frac{2(z+1)}{z-1}$, *where* $m=3,\ n=4,\ p=q=1$ *and*

$$\lim_{z\to\infty}\frac{2(z+1)}{z-1}=2,\quad C=-12=2B(1-B^2)=2\cdot 2\cdot(1-2^2).$$

In [3], Chen and Shon also investigated some properties of meromorphic solutions of finite order of difference Painlevé II equation (1.5) and obtained the following result.

**Theorem B.** (See [3]) *Let* $a,b,c$ *be constants with* $ac\neq 0$. *If* $f(z)$ *is a finite-order transcendental meromorphic solution of the difference Painlevé II equation* (1.5), *then:*
(*i*) $f(z)$ *has at most one non-zero finite Borel exceptional value;*
(*ii*) $\lambda\left(\frac{1}{f}\right)=\lambda(f)=\sigma(f)$;
(*iii*) $f(z)$ *has infinitely many fixed points and satisfies* $\tau(f)=\sigma(f)$.

In this paper, we investigate the properties of a transcendental meromorphic solution of the difference equation

$$f(z+1)+f(z-1)=\frac{A(z)f(z)+C(z)}{1-f^2(z)},\tag{1.7}$$

where $A(z),\ C(z)$ are nonconstant small functions with respect to $f(z)$. And we obtain the following result.

**Theorem** 1.2. *Suppose that the difference equation* (1.7) *admits a transcendental meromorphic solution of finite order, then*
(*i*) $\lambda\left(\frac{1}{f}\right)=\lambda(f)=\sigma(f)$;
(*ii*) $\lambda\left(\frac{1}{\Delta f(z)}\right)=\lambda\left(\frac{1}{\frac{\Delta f(z)}{f(z)}}\right)=\sigma(f),\quad \lambda\left(\frac{1}{\Delta^2 f(z)}\right)=\lambda\left(\frac{1}{\frac{\Delta^2 f(z)}{f(z)}}\right)=\sigma(f)$;
(*iii*) *If* $2z(z^2-1)+zA(z)+C(z)\not\equiv 0$, *then* $\tau(f)=\sigma(f)$;
(*iv*) *In particular, if* $A(z)\pm C(z)\equiv 0$, *then* $f(z)$ *has at most one non-zero finite Borel exceptional value.*

## 2 Lemmas for the Proof of Theorems

**Lemma 2.1.** (See [10, Theorem 2.4],[5]) *Let $f(z)$ be a transcendental meromorphic solution of finite order $\sigma$ of the difference equation*

$$P(z, f) = 0,$$

*where $P(z, f)$ is a difference polynomial in $f(z)$ and its shifts. If $P(z, a) \not\equiv 0$ for a slowly moving target meromorphic function $a$, that is, $T(r, a) = S(r, f)$, then*

$$m\left(r, \frac{1}{f-a}\right) = O(r^{\sigma-1+\varepsilon}) + S(r, f),$$

*outside of a possible exceptional set of finite logarithmic measure.*

**Lemma 2.2.** (See [10, *Theorem* 2.3]) *Let $f(z)$ be a transcendental meromorphic solution of finite order $\sigma$ of a difference equation of the form*

$$U(z, f)P(z, f) = Q(z, f),$$

*where $U(z, f), P(z, f)$ and $Q(z, f)$ are difference polynomials such that the total degree $\deg_f U(z, f) = n$ in $f(z)$ and its shifts, and $\deg_f Q(z, f) \leq n$. Moreover, we assume that $U(z, f)$ contains just one term of maximal total degree in $f(z)$ and its shifts. Then for each $\varepsilon > 0$,*

$$m(r, P(z, f)) = O(r^{\sigma-1+\varepsilon}) + S(r, f),$$

*possibly outside of an exceptional set of finite logarithmic measure.*

**Lemma 2.3.** (See [13]) *Let $f(z)$ be a meromorphic function. Then for all irreducible rational functions in $f(z)$,*

$$R(z, f(z)) = \frac{\sum_{i=0}^{m} a_i(z) f^i(z)}{\sum_{j=0}^{n} b_j(z) f^j(z)},$$

*with meromorphic coefficients $a_i(z), b_j(z)(a_m(z)b_n(z) \not\equiv 0)$ being small with respect to $f(z)$, the characteristic function of $R(z, f(z))$ satisfies*

$$T(r, R(z, f(z))) = \max\{m, n\}T(r, f) + S(r, f).$$

**Lemma 2.4.** (See [2, *Corollary* 2.5]) *Let $f(z)$ be a meromorphic function of finite order $\sigma$ and let $\eta$ be a non-zero complex number. Then for each $\varepsilon > 0$, we have*

$$m\left(r, \frac{f(z+\eta)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+\eta)}\right) = O(r^{\sigma-1+\varepsilon}).$$

**Lemma 2.5.** (See [2, *Theorem* 2.1]) *Let $f(z)$ be a meromorphic function with order $\sigma = \sigma(f), \sigma < +\infty$, and let $\eta$ be a fixed non-zero complex number, then for each $\varepsilon > 0$, we have*

$$T(r, f(z+\eta)) = T(r, f(z)) + O(r^{\sigma-1+\varepsilon}) + O(\log r).$$

**Lemma 2.6.** (See [2, *Theorem* 2.2]) *Let $f(z)$ be a meromorphic function with exponent of convergence of poles $\lambda\left(\frac{1}{f}\right) = \lambda < \infty, \eta \neq 0$ be fixed, then for each $\varepsilon > 0$,*

$$N(r, f(z+\eta)) = N(r, f(z)) + O(r^{\lambda-1+\varepsilon}) + O(\log r).$$

**Lemma 2.7.** (See [8, *Remark* 1]) *Let $f(z)$ be a transcendental meromorphic function. If $P(z, f)$ and $Q(z, f)$ are mutually prime polynomials in $f(z)$, there exist polynomials of $f(z)$, $U(z, f)$ and $V(z, f)$ such that*

$$U(z, f)P(z, f) + V(z, f)Q(z, f) = s(z),$$

*where $s(z)$ and coefficients of $U(z, f)$ and $V(z, f)$ are small functions with respect to $f(z)$.*

**Lemma 2.8.** *Let $f(z)$ be a transcendental meromorphic function and $A(z) \pm C(z) \not\equiv 0$, then $A(z)f(z)+C(z)$ and $f(z)(1-f^2(z))$ are mutually prime polynomials in $f(z)$, where $A(z)$, $C(z)$ are nonzero small functions with respect to $f(z)$.*

**Proof.** Since $A(z)$, $C(z)$ are non-zero small functions with respect to $f(z)$ and $A(z)\pm C(z) \not\equiv 0$, then $C(z)(C^2(z) - A^2(z)) \not\equiv 0$, $T(r, C(z)(C^2(z) - A^2(z))) = S(r, f)$. There exist polynomials of $f(z)$, $U(z, f) = A^2(z)f^2(z) - A(z)C(z)f(z) + C^2(z) - A^2(z)$ and $V(z, f) = A^3(z)$ such that

$$U(z, f)(A(z)f(z) + C(z)) + V(z, f)f(z)(1 - f^2(z)) = C(z)(C^2(z) - A^2(z)).$$

By Lemma 2.7, we see that $A(z)f(z) + C(z)$ and $f(z)(1 - f^2(z))$ are mutually prime.

**Lemma 2.9.** *Let $f(z)$ be a transcendental meromorphic function and $A(z) \pm C(z) \not\equiv 0$, then $2f^3(z) + (A(z) - 2)f(z) + C(z)$ and $1 - f^2(z)$ are mutually prime polynomials in $f(z)$, where $A(z)$, $C(z)$ are non-zero small functions with respect to $f(z)$.*

**Proof.** Since $A(z)$, $C(z)$ are non-zero small functions with respect to $f(z)$ and $A(z)\pm C(z) \not\equiv 0$, then $A^2(z) - C^2(z) \not\equiv 0$, $T(r, A^2(z) - C^2(z)) = S(r, f)$. There exist polynomials of $f(z)$, $U(z, f) = A(z)f(z) - C(z)$ and $V(z, f) = 2A(z)f^2(z) - 2C(z)f(z) + A^2(z)$ such that

$$U(z, f)(2f^3(z) + (A(z) - 2)f(z) + C(z)) + V(z, f)(1 - f^2(z)) = A^2(z) - C^2(z).$$

By Lemma 2.7, we see that $2f^3(z) + (A(z) - 2)f(z) + C(z)$ and $1 - f^2(z)$ are mutually prime.

**Lemma 2.10.** (See [15, *Theorem* 1.51]) *Suppose that $f_1(z), f_2(z), \ldots, f_n(z) (n \geq 2)$ are meromorphic functions and $g_1(z), g_2(z), \ldots, g_n(z)$ are entire functions satisfying the following conditions,*
*(1) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$;*
*(2) $g_j(z) - g_k(z)$ are not constants for $1 \leq j < k \leq n$;*
*(3) For $1 \leq j \leq n, 1 \leq h < k \leq n$, $T(r, f_j(z)) = o(T(r, e^{g_h(z)-g_k(z)}))(r \to \infty, r \notin E)$, where $E \subset [1, \infty)$ is finite linear measure or finite logarithmic measure.*
*Then $f_j(z) \equiv 0$ $(j = 1, \ldots, n)$.*

**Lemma 2.11.** *Suppose that $f(z)$ is a transcendental meromorphic solution of finite order of the difference equation*

$$f(z + 1) + f(z - 1) = \frac{\delta A(z)}{1 - \delta f(z)}, \tag{2.1}$$

*where $\delta = \pm 1$, $A(z)$ is a non-constant small function with respect to $f(z)$, then*
*(i) $\lambda\left(\frac{1}{f}\right) = \lambda(f) = \sigma(f)$;*
*(ii) $f(z)$ has at most one non-zero finite Borel exceptional value.*

**Proof.** (i) Since $f(z)$ is a finite order transcendental meromorphic solution of the equation (2.1), then we have

$$P(z, f) := \delta f(z)(f(z + 1) + f(z - 1)) - (f(z + 1) + f(z - 1)) + \delta A(z) \equiv 0. \tag{2.2}$$

By (2.2), we obtain

$$P(z, 0) = \delta A(z) \not\equiv 0. \tag{2.3}$$

It follows from (2.3) and Lemma 2.1 that

$$m\left(r, \frac{1}{f}\right) = S(r, f),$$

outside of a finite exceptional set of logarithmic measure. Then

$$N\left(r, \frac{1}{f}\right) = T(r, f) + S(r, f),$$

that is, $\lambda(f)=\sigma(f)$.

Next, we will prove $\lambda\left(\frac{1}{f}\right)=\sigma(f)$. It follows from (2.1) that

$$\delta f(z)(f(z+1)+f(z-1))=f(z+1)+f(z-1)-\delta A(z). \tag{2.4}$$

Set $\sigma(f)=\sigma<\infty$, by Lemma 2.2, we have

$$m(r,f(z+1)+f(z-1))=O(r^{\sigma-1+\varepsilon})+S(r,f), \tag{2.5}$$

possibly outside of an exceptional set of finite logarithmic measure. By (2.1) and Lemma 2.3, we obtain

$$T(r,f(z+1)+f(z-1))=T\left(r,\frac{\delta A(z)}{1-\delta f(z)}\right)=T(r,f)+S(r,f). \tag{2.6}$$

Since $\lambda\left(\frac{1}{f}\right)\leq\sigma(f)<\infty$, it follows from (2.5), (2.6) and Lemma 2.6 that

$$\begin{aligned} T(r,f)+O(r^{\sigma-1+\varepsilon})+S(r,f)&=N(r,f(z+1)+f(z-1))\\ &\leq 2N(r,f)+O(r^{\lambda\left(\frac{1}{f}\right)-1+\varepsilon})+O(\log r), \end{aligned} \tag{2.7}$$

then $\lambda\left(\frac{1}{f}\right)=\sigma(f)$.

(ii) By (i), we see that $0,\ \infty$ are not the Borel exceptional values of $f(z)$. Suppose that $f(z)$ has two distinct finite Borel exceptional values $a(\neq 0)$ and $b(\neq 0,a)$.

Set

$$g(z)=\frac{f(z)-a}{f(z)-b}. \tag{2.8}$$

Then

$$\sigma(g)=\sigma(f),\ \lambda(g)=\lambda(f-a)<\sigma(g),\ \lambda\left(\frac{1}{g}\right)=\lambda(f-b)<\sigma(g).$$

That is, 0 and $\infty$ are the Borel exceptional values of $g(z)$. By [15, Theorem 2.11], we see that $g(z)$ is of regular growth, then $g(z)$ can be rewritten as

$$g(z)=P(z)e^{dz^{\sigma}}, \tag{2.9}$$

where $d(\neq 0)$ is a constant, $\sigma(g)=\sigma(\geq 1)$ is a positive integer, $P(z)$ is a meromorphic function with $\sigma(P)<\sigma(g)=\sigma$. By (2.8) and (2.9), we have

$$f(z)=b+\frac{b-a}{g(z)-1}=b+\frac{b-a}{P(z)e^{dz^{\sigma}}-1} \tag{2.10}$$

and

$$\begin{aligned} f(z+1)&=b+\frac{b-a}{P(z+1)e^{d(z+1)^{\sigma}}-1}=b+\frac{b-a}{P(z+1)P_{+1}(z)e^{dz^{\sigma}}-1},\\ f(z-1)&=b+\frac{b-a}{P(z-1)e^{d(z-1)^{\sigma}}-1}=b+\frac{b-a}{P(z-1)P_{-1}(z)e^{dz^{\sigma}}-1}, \end{aligned} \tag{2.11}$$

where

$$P_{+1}(z)=\exp\left\{d\sum_{i=1}^{\sigma}\binom{\sigma}{i}z^{\sigma-i}\right\},\quad P_{-1}(z)=\exp\left\{d\sum_{i=1}^{\sigma}(-1)^i\binom{\sigma}{i}z^{\sigma-i}\right\}.$$

Substituting (2.10) and (2.11) into (2.1) yields

$$C_1(z)e^{3dz^{\sigma}}+C_2(z)e^{2dz^{\sigma}}+C_3(z)e^{dz^{\sigma}}+C_4(z)=0, \tag{2.12}$$

where

$$\begin{cases} C_1(z) &= [\delta A(z) - 2b(1-\delta b)]P(z)P(z+1)P_{+1}(z)P(z-1)P_{-1}(z); \\ C_2(z) &= [(a+b)(1-\delta b) - \delta A(z)]P(z)[P(z+1)P_{+1}(z) + P(z-1)P_{-1}(z)] \\ &\quad +[2b(1-\delta a) - \delta A(z)]P(z+1)P_{+1}(z)P(z-1)P_{-1}(z); \\ C_3(z) &= [\delta A(z) - (a+b)(1-\delta a)][P(z+1)P_{+1}(z) + P(z-1)P_{-1}(z)] \\ &\quad +[\delta A(z) - 2a(1-\delta b)]P(z); \\ C_4(z) &= 2a(1-\delta a) - \delta A(z). \end{cases} \tag{2.13}$$

It follows from (2.13) and Lemma 2.10 that

$$C_1(z) \equiv C_2(z) \equiv C_3(z) \equiv C_4(z) \equiv 0.$$

Since $P(z)P(z+1)P_{+1}(z)P(z-1)P_{-1}(z) \not\equiv 0$ and $C_1(z) \equiv C_4(z) \equiv 0$, we obtain

$$\delta A(z) \equiv 2b(1-\delta b) \quad and \quad \delta A(z) \equiv 2a(1-\delta a). \tag{2.14}$$

Note that $A(z)$ is a non-constant function, which shows that (2.14) is a contradiction.

# 3 Proof of Theorems

**Proof of Theorem** 1.1
Suppose that $f(z) = \frac{P(z)}{Q(z)}$ is a rational solution of (1.6). Then (1.6) can be rewritten as

$$\left[\frac{P(z+1)}{Q(z+1)} + \frac{P(z-1)}{Q(z-1)}\right]\left[1 - \frac{P^2(z)}{Q^2(z)}\right] = \frac{m(z)}{n(z)} \cdot \frac{P(z)}{Q(z)} + C \tag{3.1}$$

or

$$\begin{aligned} &n(z)[Q^2(z) - P^2(z)][P(z+1)Q(z-1) + P(z-1)Q(z+1)] \\ &= m(z)P(z)Q(z)Q(z+1)Q(z-1) + Cn(z)Q^2(z)Q(z+1)Q(z-1). \end{aligned} \tag{3.2}$$

(i) Suppose that $m > n$. If $p > q$, (3.2) yields

$$\begin{cases} \deg(n(z)[Q^2(z) - P^2(z)][P(z+1)Q(z-1) + P(z-1)Q(z+1)]) = n + 3p + q, \\ \deg(m(z)P(z)Q(z)Q(z+1)Q(z-1)) = m + 3q + p, \\ \deg(Cn(z)Q^2(z)Q(z+1)Q(z-1)) = n + 4q. \end{cases} \tag{3.3}$$

By $n + 3p + q > n + 4q$ and $m + 3q + p > n + 4q$, then we must have $n + 3p + q = m + 3q + p$, that is $p - q = \frac{m-n}{2}$, $m - n$ must be even.

If $p = q$, then $\frac{P(z)}{Q(z)} \to B$, $\frac{P(z+1)}{Q(z+1)} \to B$, $\frac{P(z-1)}{Q(z-1)} \to B$ as $z \to \infty$, where $B$ is a nonzero constant, while $\frac{m(z)}{n(z)} \to \infty$ as $z \to \infty$. If $B = 1$ or $B = -1$, then

$$\left[\frac{P(z+1)}{Q(z+1)} + \frac{P(z-1)}{Q(z-1)}\right]\left[1 - \frac{P^2(z)}{Q^2(z)}\right] \to 0, \quad \frac{m(z)}{n(z)} \cdot \frac{P(z)}{Q(z)} + C \to \infty \; as \; z \to \infty.$$

If $B \neq \pm 1$, then

$$\left[\frac{P(z+1)}{Q(z+1)} + \frac{P(z-1)}{Q(z-1)}\right]\left[1 - \frac{P^2(z)}{Q^2(z)}\right] \to 2B(1-B^2), \quad \frac{m(z)}{n(z)} \cdot \frac{P(z)}{Q(z)} + C \to \infty \; as \; z \to \infty.$$

These show that (3.1) is a contradiction.

If $p < q$, (3.2) yields

$$\begin{cases} \deg(n(z)[Q^2(z) - P^2(z)][P(z+1)Q(z-1) + P(z-1)Q(z+1)]) = n + 3q + p, \\ \deg(m(z)P(z)Q(z)Q(z+1)Q(z-1)) = m + 3q + p, \\ \deg(Cn(z)Q^2(z)Q(z+1)Q(z-1)) = n + 4q. \end{cases} \tag{3.4}$$

By $m+3q+p > n+3q+p$ and $n+4q > n+3q+p$, then we must have $m+3q+p = n+4q$, that is $q-p = m-n \geq 1$.

(ii) Suppose that $m = n$. If $p > q$, by (3.3), we see that $n+3p+q > m+3q+p > n+4q$. This shows that there is only one term in (3.2) which has the highest degree, a contradiction.

If $p = q$, then $\frac{P(z)}{Q(z)} \to B$, $\frac{P(z+1)}{Q(z+1)} \to B$, $\frac{P(z-1)}{Q(z-1)} \to B$ as $z \to \infty$, where $B$ is a nonzero constant, while $\frac{m(z)}{n(z)} \to A$ as $z \to \infty$, where $A$ is a nonzero constant. If $B = 1$ or $B = -1$, then

$$\left[\frac{P(z+1)}{Q(z+1)} + \frac{P(z-1)}{Q(z-1)}\right]\left[1 - \frac{P^2(z)}{Q^2(z)}\right] \to 0, \quad \frac{m(z)}{n(z)} \cdot \frac{P(z)}{Q(z)} + C \to AB + C \ as \ z \to \infty.$$

So we have $C \pm A = 0$. If $B \neq \pm 1$, then

$$\left[\frac{P(z+1)}{Q(z+1)} + \frac{P(z-1)}{Q(z-1)}\right]\left[1 - \frac{P^2(z)}{Q^2(z)}\right] \to 2B(1-B^2), \quad \frac{m(z)}{n(z)} \cdot \frac{P(z)}{Q(z)} + C \to AB + C \ as \ z \to \infty.$$

So we have $C = B[2(1-B^2) - A]$.

If $p < q$, by (3.4), we see that $n+4q > n+3q+p = m+3q+p$. This also shows that there is only one term in (3.2) which has the highest degree, a contradiction.

(iii) Suppose that $m < n$. If $p > q$, by (3.3), we see that $n+3p+q > m+3q+p$, $n+3p+q > n+4q$, that is, there is only one term in (3.2) which has the highest degree, a contradiction.

If $p = q$, then $\frac{P(z)}{Q(z)} \to B$, $\frac{P(z+1)}{Q(z+1)} \to B$, $\frac{P(z-1)}{Q(z-1)} \to B$ as $z \to \infty$, where $B$ is a nonzero constant, while $\frac{m(z)}{n(z)} \to 0$ as $z \to \infty$. If $B = 1$ or $B = -1$, then

$$\left[\frac{P(z+1)}{Q(z+1)} + \frac{P(z-1)}{Q(z-1)}\right]\left[1 - \frac{P^2(z)}{Q^2(z)}\right] \to 0, \quad \frac{m(z)}{n(z)} \cdot \frac{P(z)}{Q(z)} + C \to C \ as \ z \to \infty.$$

So we have $C = 0$, a contradiction. If $B \neq \pm 1$, then

$$\left[\frac{P(z+1)}{Q(z+1)} + \frac{P(z-1)}{Q(z-1)}\right]\left[1 - \frac{P^2(z)}{Q^2(z)}\right] \to 2B(1-B^2), \quad \frac{m(z)}{n(z)} \cdot \frac{P(z)}{Q(z)} + C \to C \ as \ z \to \infty.$$

So we have $C = 2B(1-B^2)$.

If $p < q$, by (3.4), we see that $n+4q > n+3q+p > m+3q+p$. That is, there is only one term in (3.2) which has the highest degree, a contradiction.

This completes the proof of Theorem 1.1.

**Proof of Theorem** 1.2

In what follows, we consider two cases: $A(z) \pm C(z) \not\equiv 0$ and $A(z) \pm C(z) \equiv 0$.

**Case** 1, $A(z) \pm C(z) \not\equiv 0$.

(i) Using the same method as in the proof of Lemma 2.11 (i), we can obtain that $\lambda\left(\frac{1}{f}\right) = \lambda(f) = \sigma(f)$.

(ii) First, we will prove $\lambda\left(\frac{1}{\frac{\Delta f(z)}{f(z)}}\right) = \sigma(f)$. By equation (1.7), Lemmas 2.3, 2.5 and 2.8, we have

$$\begin{aligned}
3T(r,f(z)) &= T\left(r, \frac{A(z)f(z)+C(z)}{f(z)(1-f^2(z))}\right) + S(r,f) \\
&= T\left(r, \frac{f(z+1)+f(z-1)}{f(z)}\right) + S(r,f) \\
&\leq T\left(r, \frac{f(z+1)}{f(z)}\right) + T\left(r, \frac{f(z)}{f(z-1)}\right) + S(r,f) \\
&= 2T\left(r, \frac{f(z+1)}{f(z)}\right) + S\left(r, \frac{f(z+1)}{f(z)}\right) + S(r,f) \\
&\leq 2T\left(r, \frac{f(z+1)}{f(z)}\right) + S(r,f) \\
&= 2T\left(r, \frac{\Delta f(z)}{f(z)}\right) + S(r,f),
\end{aligned}$$

that is,

$$\frac{3}{2}T(r,f(z)) \leq T\left(r,\frac{\Delta f(z)}{f(z)}\right)+S(r,w). \tag{3.5}$$

It follows from (3.5) and Lemma 2.4 that

$$\begin{aligned} N\left(r,\frac{\Delta f(z)}{f(z)}\right) &= T\left(r,\frac{\Delta f(z)}{f(z)}\right)-m\left(r,\frac{\Delta f(z)}{f(z)}\right) \\ &\geq \frac{3}{2}T(r,f(z))+S(r,f). \end{aligned}$$

Thus, $\lambda\left(\frac{1}{\frac{\Delta f(z)}{f(z)}}\right)=\sigma(f)$.

Next, we will prove $\lambda\left(\frac{1}{\Delta f(z)}\right)=\sigma(f)$. By equation (1.7),

$$\begin{aligned} \Delta f(z)-\Delta f(z-1) &= f(z+1)+f(z-1)-2f(z) \\ &= \frac{A(z)f(z)+C(z)}{1-f^2(z)}-2f(z) \\ &= \frac{2f^3(z)+(A(z)-2)f(z)+C(z)}{1-f^2(z)}. \end{aligned} \tag{3.6}$$

From (3.6), Lemmas 2.3, 2.5 and 2.9, we have

$$\begin{aligned} 3T(r,f(z)) &= T\left(r,\frac{2f^3(z)+(A(z)-2)f(z)+C(z)}{1-f^2(z)}\right)+S(r,f) \\ &= T(r,\Delta f(z)-\Delta f(z-1))+S(r,f) \\ &\leq 2T(r,\Delta f(z))+S(r,\Delta f(z))+S(r,f) \\ &\leq 2T(r,\Delta f(z))+S(r,f), \end{aligned}$$

that is,

$$\frac{3}{2}T(r,f(z)) \leq T(r,\Delta f(z))+S(r,f). \tag{3.7}$$

It follows from (3.7) and Lemma 2.4 that

$$\begin{aligned} N(r,\Delta f(z)) &= T(r,\Delta f(z))-m(r,\Delta f(z)) \\ &\geq T(r,\Delta f(z)-\left(m\left(r,\frac{\Delta f(z)}{f(z)}\right)+m(r,f(z))\right) \\ &\geq \frac{3}{2}T(r,f(z))-T(r,f(z))+S(r,f) \\ &= \frac{1}{2}T(r,f(z))+S(r,f). \end{aligned}$$

Hence, $\lambda\left(\frac{1}{\Delta f(z)}\right)=\sigma(f)$.

Furthermore, we will prove $\lambda\left(\frac{1}{\Delta^2 f(z)}\right)=\sigma(f)$. By (3.6), we have

$$\Delta^2 f(z-1)=\Delta f(z)-\Delta f(z-1)=\frac{2f^3(z)+(A(z)-2)f(z)+C(z)}{1-f^2(z)}. \tag{3.8}$$

From (3.8), Lemmas 2.3, 2.5 and 2.9, we have

$$\begin{aligned} 3T(r,f(z)) &= T\left(r,\frac{2f^3(z)+(A(z)-2)f(z)+C(z)}{1-f^2(z)}\right)+S(r,f) \\ &= T(r,\Delta^2 f(z-1))+S(r,f) \\ &= T(r,\Delta^2 f(z))+S(r,f). \end{aligned} \tag{3.9}$$

It follows from (3.9) and Lemma 2.4 that

$$\begin{aligned} N(r,\Delta^2 f(z)) &= T(r,\Delta^2 f(z)) - m(r,\Delta^2 f(z)) \\ &\geq 3T(r,f(z)) - \left(m\left(r,\frac{\Delta f(z)}{f(z)}\right) + m(r,f(z))\right) \\ &\geq 3T(r,f(z)) - T(r,f(z)) + S(r,f) \\ &= 2T(r,f(z)) + S(r,f). \end{aligned}$$

Thus, $\lambda\left(\frac{1}{\Delta^2 f(z)}\right) = \sigma(f)$.

Finally, we will prove $\lambda\left(\frac{1}{\frac{\Delta^2 f(z)}{f(z)}}\right) = \sigma(f)$. It follows from (3.9) and Lemma 2.4 that

$$\begin{aligned} N\left(r,\frac{\Delta^2 f}{f}\right) &= T\left(r,\frac{\Delta^2 f}{f}\right) - m\left(r,\frac{\Delta^2 f}{f}\right) \\ &\geq T(r,\Delta^2 f(z)) - T(r,f(z)) + S(r,f) \\ &= 3T(r,f(z)) - T(r,f(z)) + S(r,f) \\ &= 2T(r,f(z)) + S(r,f). \end{aligned}$$

Then, $\lambda\left(\frac{1}{\frac{\Delta^2 f(z)}{f(z)}}\right) = \sigma(f)$.

(iii) Suppose that $f(z)$ is a finite order transcendental meromorphic solution of equation (1.7). Set $g(z) = f(z) - z$. Then $g(z)$ is a transcendental meromorphic function with $\sigma(g) = \sigma(f) < \infty$ and $\tau(f) = \lambda(g)$. Substituting $f(z) = g(z) + z$ into (1.7) yields

$$P(z,g) := [g(z+1) + g(z-1) + 2z][(g(z)+z)^2 - 1] + A(z)g(z) + zA(z) + C(z) \equiv 0. \quad (3.10)$$

Since $P(z,0) = 2z(z^2-1) + zA(z) + C(z) \not\equiv 0$, it follows from (3.10) and Lemma 2.1 that

$$N\left(r,\frac{1}{g}\right) = T(r,g) + S(r,g) = T(r,f) + S(r,f).$$

Then, $\tau(f) = \lambda(g) = \sigma(g) = \sigma(f)$.

**Case** 2, $A(z) \pm C(z) \equiv 0$. Rewriting equation (1.7) as

$$f(z+1) + f(z-1) = \frac{\delta A(z)}{1 - \delta f(z)}, \quad (3.11)$$

where $\delta = \pm 1$.

By Lemma 2.11, we see that (i) and (iv) hold. Using the same method as in the proof of Case 1 (ii) and (iii), we can also obtain (ii) and (iii). If $2z(1-\delta z) - \delta A(z) \not\equiv 0$, then $\tau(f) = \sigma(f)$.

This completes the proof of Theorem 1.2.

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# References

[1] W. Bergweiler and J. K. Langley., Zeros of difference of meromorphic functions, *Math. Proc. Cambridge. Philos. Soc.* **142**(2007), 133-147.

[2] Y. M. Chiang and S. J. Feng., On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane, *Ramanujan J.* **16** (2008), 105-129.

[3] Z. X. Chen and K. H. Shon., Value distribution of meromorphic solutions of certain difference Painlevé equations, *J. Math. Anal. Appl.* **364** (2010), 556-566.

[4] W. K. Hayman., *Meromorphic Function*, Clarendon Press, Oxford, 1964.

[5] R. G. Halburd and R. Korhonen., Difference analogue of the lemma on the logarithmic derivative with applications to difference equations, *J. Math. Anal. Appl.* **314**(2006), 477-487.

[6] R. G. Halburd and R. Korhonen., Meromorphic solution of difference equation, integrability and discrete Painlevé equations, *J. Phys. A:Math. Theory.* **40**(2007), 1-38.

[7] R. G. Halburd and R. Korhonen., Finite-order meromorphic solutions and the discrete Painlevé equations, *Proc. Lond. Math. Soc.* **94**(2007), 443-474.

[8] K. Ishizaki., Admissible solutions of Schwarizan differential equation, *J. Austral. Math. Soc.* **50**(1991), 258-278.

[9] I. Laine, *Nevanlinna Theory and Complex Differential Equations*. W. de Gruyter, Berlin, 1993.

[10] I. Laine and C. C. Yang., Clunie theorems for difference and q-difference polynomials, *J. Lond. Math. Soc.* **76**(2007), 556-566.

[11] S. Li and Z. S. Gao., Finite order meromorphic solutions of linear difference equations, *Proc. Japan. Acad. Ser A. Math. Sci.* **86**(2010), 10-14.

[12] Q. Li and Z. B. Huang., Some results on a certain type of difference equation originated from difference Painlevé I equation, *Adv. Differ. Equ.* **2015**(2015):276, 1-11.

[13] A. Z. Mokhon'ko., On the Nevanlinna characteristics of some meromorphic functions, in "Theory of functions, functional analysis and their applicatins", Izd-vo Khar'kovsk. Un-ta, **14**(1971), 83-87.

[14] J. M. Whillaker., *Interpolatory Funtion Theory, Cambridge Tracts in Mathematics and Mathematical Physics. no. 33*. Cambridge University Press, Cambridge, 1935.

[15] C. C. Yang and H. X. Yi., *Uniqueness theory of meromorphic functions*, Science Press, Beijing, 1995. Kluwer Academic, Dordrecht, 2003.

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# Catalan Numbers, $k$-Gamma and $k$-Beta Functions, and Parametric Integrals

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**Abstract**

In the paper, the authors establish some new explicit formulas and integral representations of the Catalan numbers and a class of parametric integrals in terms of the $k$-gamma and $k$-beta functions.

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## 1 Introduction and main results

The Catalan numbers $C_n$ for $n \ge 0$ form a sequence of natural numbers that occur in various counting problems in combinatorial mathematics. The $n$th Catalan number can be expressed in terms of the central binomial coefficients $\binom{2n}{n}$ by

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!(n+1)!} = \frac{4^n\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(n+2)}, \tag{1}$$

where $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}\,t$ for $x > 0$ is the classical Euler gamma function, or say, the Euler integral of the second kind.

The Catalan numbers $C_n$ were described in the 18th century by Leonhard Euler and are named after the Belgian mathematician Eugéne Charles Catalan. In 1988, it came to light that the Catalan numbers $C_n$ had been used in China by the Mongolian mathematician Ming Antu by 1730, see [11, 12, 14, 16, 17, 18, 19, 20, 35]. In recent years, the Catalan numbers $C_n$ has been

analytically generalized and studied in [15, 21, 23, 25, 26, 27, 28, 30, 29, 31, 33, 36] and closely-related references therein. For more information on the Catalan numbers $C_n$, please refer to the monographs [2, 3, 8, 9, 10, 32, 34] and closely-related references therein.

It is common knowledge [1, p. 4] that the beta function, or say, the Euler integral of the first kind $B(x,y)$ can be defined by $B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}\,t$ and satisfies $B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ for $x,y>0$. The rising factorial, denoted by $(x)_n$, is defined [13, p. 13] by $(x)_n=x(x+1)\cdots(x+n-1)=\frac{\Gamma(x+n)}{\Gamma(x)}$ which is frequently called the Pochhammer symbol in mathematics.

Diaz and Pariguan [5, p. 180] introduced the Pochhammer $k$-symbol as

$$(x)_{n,k}=x(x+k)(x+2k)\cdots[x+(n-1)k].$$

It is clear that $(x)_{n,1}=(x)_n$.

Diaz et al. [4, 5, 6] introduced the $k$-gamma and $k$-beta functions and proved a number of their properties. They also studied the $k$-zeta function and the $k$-hypergeometric functions based on the Pochhammer $k$-symbols. The $k$-gamma function is defined in [5, p. 180] by

$$\Gamma_k(x)=\lim_{n\to\infty}\frac{n!k^n(kn)^{x/k-1}}{(x)_{n,\alpha}},\quad k>0.$$

It was showed [5, p. 180] that the Mellin transform of the exponential function $e^{-t^k/k}$ is the $k$-gamma function, that is,

$$\Gamma_k(x)=\int_0^\infty t^{x-1}e^{-t^k/k}\,\mathrm{d}\,t.$$

It is easy to see that

$$\Gamma(x)=\Gamma_1(x),\quad \Gamma_k(x)=k^{x/k-1}\Gamma\left(\frac{x}{k}\right),\quad \Gamma_k(x+k)=x\Gamma_k(x). \tag{2}$$

This gives rise to the $k$-beta function

$$B_k(x,y)=\frac{1}{k}\int_0^1 t^{x/k-1}(1-t)^{y/k-1}\,\mathrm{d}\,t \tag{3}$$

which satisfies

$$B_k(x,y)=\frac{1}{k}B\left(\frac{x}{k},\frac{y}{k}\right)\quad\text{and}\quad B_k(x,y)=\frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}. \tag{4}$$

The aim of this paper is to establish some new explicit formulas and integral representations of the Catalan numbers $C_n$ and parametric integrals

$$I_{\alpha,\beta;k}(a)=\int_0^a x^{(\alpha+1)k-1}\left(a^{2k}-x^{2k}\right)^\beta\,\mathrm{d}\,x$$

for $a,k>0$ and some $\alpha,\beta>-1$ in terms of the $k$-gamma function $\Gamma_k(x)$ and the $k$-beta function $B_k(x,y)$.

Our main results in this paper can be stated as the following theorems.

**Theorem 1.1.** *For $k>0$ and $n\in\mathbb{N}$, we have*

$$C_n=k^{3/2}\frac{4^n\Gamma_k\left(\frac{2n+1}{2}k\right)}{\sqrt{\pi}\,\Gamma_k((n+2)k)}=\frac{k2^{1+2n(1-k)}}{\pi(n+1)}\int_0^2\frac{x^{(2n+1)k-1}}{\sqrt{2^{2k}-x^{2k}}}\,\mathrm{d}\,x. \tag{5}$$

**Theorem 1.2.** *For $a, k > 0$ and $n \ge 0$, we have*

$$I_{n,1/2;k}(a) = \frac{k^{1/2}\sqrt{\pi}}{4} a^{(n+2)k} \frac{\Gamma_k\left(\frac{n+1}{2}k\right)}{\Gamma_k\left(\frac{n+4}{2}k\right)} = \frac{k^{1/2}}{2} a^{(n+2)k} B_k\left(\frac{n+1}{2}k, \frac{3}{2}k\right) \tag{6}$$

*and*

$$I_{n,-1/2;k}(a) = \sqrt{\frac{\pi}{k}}\, \frac{a^{nk}}{2} \frac{\Gamma_k\left(\frac{n+1}{2}k\right)}{\Gamma_k\left(\frac{n+2}{2}k\right)} = \frac{a^{nk}}{2} B\left(\frac{n+1}{2}k, \frac{1}{2}k\right). \tag{7}$$

*For $a, k > 0$ and $\alpha, \beta > -1$, we have*

$$I_{\alpha,\beta;k}(a) = \frac{a^{\alpha+2\beta k+1}}{2} B_k\left(\frac{\alpha+1}{2}, (\beta+1)k\right). \tag{8}$$

# 2 Proofs of main results

We are now in a position to prove our main results stated in Theorems 1.1 and 1.2.

From the second relation in (2), we have

$$\Gamma_k\left(\frac{2n+1}{2}k\right) = k^{(2n+1)k/2k-1}\Gamma\left(\frac{(2n+1)k}{2k}\right) = k^{(2n-1)/2}\Gamma\left(n+\frac{1}{2}\right) = k^{(2n-1)/2}\frac{(2n)!\sqrt{\pi}}{4^n n!}.$$

Accordingly, it follows that

$$\frac{\Gamma_k\left(\frac{2n+1}{2}k\right)}{\Gamma_k((n+2)k)} = \frac{k^{(2n-1)/2}\frac{(2n)!\sqrt{\pi}}{4^n n!}}{k^{n+1}(n+1)!} = k^{-3/2}\frac{(2n)!\sqrt{\pi}}{4^n n!(n+1)!} = k^{-3/2}\frac{\sqrt{\pi}\, C_n}{4^n}.$$

The explicit formulas in (5) thus follow.

By changing variables $x = at^{1/2k}$ for $t \in [0,1]$, we have

$$\begin{aligned} I_{n,1/2;k}(a) &= \int_0^a \left(at^{1/2k}\right)^{(n+1)k-1}\left[a^{2k} - \left(at^{1/2k}\right)^{2k}\right]^{1/2} \frac{a}{2k} t^{1/2k-1}\,\mathrm{d}\,t \\ &= \frac{a^{(n+2)k}}{2k}\int_0^1 t^{(n+1)k/2k-1}(1-t)^{1/2+1-1}\,\mathrm{d}\,t \\ &= \frac{a^{(n+2)k}}{2k}\int_0^1 t^{(n+1)k/2k-1}(1-t)^{3k/2k-1}\,\mathrm{d}\,t = \frac{a^{(n+2)k}}{2} B_k\left(\frac{n+1}{2}k, \frac{3}{2}k\right). \end{aligned}$$

Utilizing the second relation in (4) gives

$$I_{n,1/2;k}(a) = \frac{a^{(n+2)k}}{2}\frac{\Gamma_k\left(\frac{n+1}{2}k\right)\Gamma_k\left(\frac{3}{2}k\right)}{\Gamma_k\left(\frac{n+4}{2}k\right)}.$$

Further using

$$\Gamma_k\left(\frac{3}{2}k\right) = k^{1/2}\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{k\pi}}{2} \quad \text{and} \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi},$$

we obtain the formulas in (6).

By the Leibniz rule for differentiation

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\int_{g(t)}^{h(t)} F(x,t)\,\mathrm{d}\,x = h'(t)F(h(t),t) - g'(t)F(g(t),t) + \int_{g(t)}^{h(t)} \frac{\partial}{\partial t}F(x,t)\,\mathrm{d}\,x$$

in [7, p. 615], differentiating with respect to $a$ on both sides of (6) gives

$$I'_{n,1/2;k}(a) = ka^{2k-1}\int_0^a \frac{x^{(n+1)k-1}}{\left(a^{2k}-x^{2k}\right)^{1/2}}\,\mathrm{d}\,x = \frac{\sqrt{k\pi}}{4}(n+2)ka^{(n+2)k-1}\frac{\Gamma_k\left(\frac{n+1}{2}k\right)}{\Gamma_k\left(\frac{(n+4)k}{2}\right)}.$$

Therefore, it follows that

$$I_{n,-1/2;k}(a) = \frac{\sqrt{k\pi}}{4}(n+2)a^{nk}\frac{\Gamma_k\left(\frac{n+1}{2}k\right)}{\Gamma_k\left(\frac{n+4}{2}k\right)}. \tag{9}$$

The formulas in (7) are thus acquired.

Letting $a=2$ and replacing $n$ by $2n$ in (9) derive

$$\frac{k2^{1+2n(1-k)}}{\pi(n+1)}\int_0^2 \frac{x^{(2n+1)k-1}}{\sqrt{2^{2k}-x^{2k}}}\,\mathrm{d}\,x = k^{3/2}\frac{4^n\Gamma_k\left(\frac{2n+1}{2}k\right)}{\sqrt{\pi}\,\Gamma_k\left((n+2)k\right)} = C_n.$$

The integral representation in (5) is thus proved.

By changing variables $x = a\sin^{1/k}\theta$ for $\theta\in\left[0,\frac{\pi}{2}\right]$, we have

$$\begin{aligned} I_{n,k}(a) &= \int_0^{\pi/2}\left(a\sin^{1/k}\theta\right)^{(\alpha+1)k-1}\left(a^{2k}-a^{2k}\sin^2\theta\right)^{\beta}\frac{a}{k}\sin^{1/k-1}\theta\cos\theta\,\mathrm{d}\,\theta \\ &= \frac{a^{(\alpha+2\beta+1)k}}{k}\int_0^{\pi/2}\sin^{\alpha}\theta\cos^{2\beta+1}\theta\,\mathrm{d}\,\theta \\ &= \frac{a^{(\alpha+2\beta+1)k}}{k}\int_0^{\pi/2}\sin^{2k(\alpha+1)/2k-1}\theta\cos^{2k(\beta+1)/k-1}\theta\,\mathrm{d}\,\theta = \frac{a^{(\alpha+2\beta+1)k}}{2}B_k\left(\frac{\alpha+1}{2}k,(\beta+1)k\right), \end{aligned}$$

where we used in the last step the formula

$$\int_0^{\pi/2}\sin^{2x/k-1}\theta\cos^{2y/k-1}\theta\,\mathrm{d}\,\theta = \frac{k}{2}B_k(x,y), \quad \Re(x),\Re(y)>0,$$

which can be derived from using the change of the variable $t=\sin^2\theta$ in (3) by

$$\begin{aligned} B_k(x,y) &= \frac{1}{k}\int_0^{\pi/2}\left(\sin^2\theta\right)^{x/k-1}\left(1-\sin^2\theta\right)^{y/k-1}2\sin\theta\cos\theta\,\mathrm{d}\,\theta \\ &= \frac{2}{k}\int_0^{\pi/2}\sin^{2x/k-1}\theta\cos^{2y/k-1}\theta\,\mathrm{d}\,\theta. \end{aligned}$$

The proof of the formula (8) is thus complete.

# 3 Remarks

Finally we give some remarks about connections between our main results and some know conclusions.

*Remark* 3.1. When $k = 1$, the expression (5) becomes the last expression in (1).

Letting $k = 1$ in (6), we can deduce [22, Theorem 2.1].

Letting $k = 1$ in (7), we can obtain [22, Theorem 3.1].

Letting $k = 1$ in (8), we can recover [22, Theorem 5.1].

Letting $k = 1, \alpha = n$, and $\beta = \frac{1}{2}$ in (8), we recover [22, Remark 6.1].

Letting $k = 1, \alpha = 2n$, and $\beta = \frac{1}{2}$ in (8), we can obtain [22, Remark 6.2].

Let $k = 1, \alpha = n$, and $\beta = -\frac{1}{2}$ in (6), we can derive [22, Remark 6.3].

The Wallis ratio $W_n = \frac{(2n)!}{4^n (n!)^2}$ can also be expressed by

$$W_n = \sqrt{\frac{k}{\pi}} \frac{\Gamma_k\left(\frac{2n+1}{2}k\right)}{\Gamma_k((n+1)k)}$$

for $n \in \mathbb{N}$ and $k > 0$, which is a generalization of [22, Remark 6.5].

*Remark* 3.2. This paper is a slightly revised version of the preprint [24].

# References


[1] G. E. Andrews, R. Askey, and R. Roy, *Special Functions*, Encyclopedia of Mathematics and its Applications **71**, Cambridge University Press, Cambridge, 1999.

[2] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, Revised and Enlarged Edition, D. Reidel Publishing Co., Dordrecht and Boston, 1974.

[3] T. Dana-Picard, *Parametric integrals and Catalan numbers*, Internat. J. Math. Ed. Sci. Tech. **36** (2005), no. 4, 410–414; Available online at `http://dx.doi.org/10.1080/00207390412331321603`.

[4] R. Díaz, C. Ortiz, and E. Pariguan, *On the k-gamma q-distribution*, Cent. Eur. J. Math. **8** (2010), no. 3, 448–458; Available online at `http://dx.doi.org/10.2478/s11533-010-0029-0`

[5] R. Díaz and E. Pariguan, *On hypergeometric functions and Pochhammer k-symbol*, Divulg. Mat. **15** (2007), no. 2, 179–192.

[6] R. Díaz and C. Teruel, *q, k-generalized gamma and beta functions*, J. Nonlinear Math. Phys. **12** (2005), no. 1, 118–134; Available online at `http://dx.doi.org/10.2991/jnmp.2005.12.1.10`.

[7] H. Flanders, *Differentiation under the integral sign*, Amer. Math. Monthly **80** (1973), 615–627; correction, ibid. **81** (1974), 145; Available online at `http://dx.doi.org/10.2307/2319163`.

[8] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics—A Foundation for Computer Science*, 2nd ed., Addison-Wesley Publishing Company, Reading, MA, 1994.

[9] R. P. Grimaldi, *Fibonacci and Catalan Numbers: An Introduction*, John Wiley & Sons, Inc., Hoboken, NJ, 2012; Available online at http://dx.doi.org/10.1002/9781118159743.

[10] T. Koshy, *Catalan Numbers with Applications*, Oxford University Press, Oxford, 2009.

[11] P. Larcombe, *On the history of the Catalan numbers: a first record in China*, Math. Today (Southend-on-Sea) **35** (1999), no. 3, 89–89.

[12] P. J. Larcombe, *The 18th century Chinese discovery of the Catalan numbers*, Math. Spectrum **32** (1999/2000) no. 1, 5–7.

[13] N. N. Lebedev, *Special Functions and Their Applications*, Revised edition, translated from the Russian and edited by Richard A. Silverman. Unabridged and corrected republication. Dover Publications, Inc., New York, 1972.

[14] J. J. Liu, *Ming Antu and Catalan numbers*, J. Math. Res. Exposition **22** (2002), no. 4, 589–594. (Chinese)

[15] F.-F. Liu, X.-T. Shi, and F. Qi, *A logarithmically completely monotonic function involving the gamma function and originating from the Catalan numbers and function*, Glob. J. Math. Anal. **3** (2015), no. 4, 140–144; Available online at http://dx.doi.org/10.14419/gjma.v3i4.5187.

[16] J. J. Luo, *Ming Antu, the first discoverer of the Catalan numbers*, Neimenggu Daxue Xuebao **19** (1988), no. 2, 239–245. (Chinese)

[17] J. J. Luo, *Catalan numbers in the history of mathematics in China*, Combinatorics and Graph Theory (Hefei, 1992), 68–70, World Sci. Publ., River Edge, NJ, 1993.

[18] J. J. Luo, *Ming Antu and his power series expansions*, Seki, founder of modern mathematics in Japan, 299–310, Springer Proc. Math. Stat., Vol. 39, Springer, Tokyo, 2013; Available online at http://dx.doi.org/10.1007/978-4-431-54273-5_20.

[19] X. R. Ma, *Notes on a result due to Ming Antu*, J. Math. Res. Exposition **22** (2002), no. 4, 595–598. (Chinese)

[20] X. R. Ma, *The general solution of Ming Antu's problem*, Acta Math. Sin. (Engl. Ser.) **20** (2004), no. 1, 157–162; Available online at http://dx.doi.org/10.1007/s10114-003-0282-2.

[21] M. Mahmoud and F. Qi, *Three identities of the Catalan–Qi numbers*, Mathematics **4** (2016), no. 2, Article 35, 7 pages; Available online at http://dx.doi.org/10.3390/math4020035.

[22] F. Qi, *Parametric integrals, the Catalan numbers, and the beta function*, Elem. Math. **72** (2017), no. 2, in press; Available online at http://dx.doi.org/0013-6018/17/020001-?.

[23] F. Qi, *Some properties and generalizations of the Catalan, Fuss, and Fuss–Catalan numbers*, ResearchGate Research (2015), available online at http://dx.doi.org/10.13140/RG.2.1.1778.3128.

[24] F. Qi, A. Akkurt, and H. Yildirim, *Catalan numbers, k-gamma and k-beta functions, and parametric integrals*, ResearchGate Working Paper (2017), available online at http://dx.doi.org/10.13140/RG.2.2.19398.06721.

[25] F. Qi and B.-N. Guo, *Logarithmically complete monotonicity of a function related to the Catalan–Qi function*, Acta Univ. Sapientiae Math. **8** (2016), no. 1, 93–102; Available online at http://dx.doi.org/10.1515/ausm-2016-0006.

[26] F. Qi and B.-N. Guo, *Logarithmically complete monotonicity of Catalan–Qi function related to Catalan numbers*, Cogent Math. (2016), **3**:1179379, 6 pages; Available online at http://dx.doi.org/10.1080/23311835.2016.1179379.

[27] F. Qi, M. Mahmoud, X.-T. Shi, and F.-F. Liu, *Some properties of the Catalan–Qi function related to the Catalan numbers*, SpringerPlus (2016), **5**:1126, 20 pages; Available online at http://dx.doi.org/10.1186/s40064-016-2793-1.

[28] F. Qi, X.-T. Shi, and P. Cerone, *A unified generalization of the Catalan, Fuss, and Fuss–Catalan numbers and the Catalan–Qi function*, ResearchGate Working Paper (2015), available online at http://dx.doi.org/10.13140/RG.2.1.3198.6000.

[29] F. Qi, X.-T. Shi, M. Mahmoud, and F.-F. Liu, *Schur-convexity of the Catalan–Qi function related to the Catalan numbers*, Tbilisi Math. J. **9** (2016), no. 2, 141–150; Available online at http://dx.doi.org/10.1515/tmj-2016-0026.

[30] F. Qi, X.-T. Shi, F.-F. Liu, and D. V. Kruchinin, *Several formulas for special values of the Bell polynomials of the second kind and applications*, J. Appl. Anal. Comput. **7** (2017), no. 3, in press.

[31] F. Qi, X.-T. Shi, M. Mahmoud, and F.-F. Liu, *The Catalan numbers: a generalization, an exponential representation, and some properties*, J. Comput. Anal. Appl. **23** (2017), no. 5, 937–944.

[32] S. Roman, *An Introduction to Catalan Numbers*, with a foreword by Richard Stanley. Compact Textbook in Mathematics. Birkhäuser/Springer, Cham, 2015; Available online at http://dx.doi.org/10.1007/978-3-319-22144-1.

[33] X.-T. Shi, F.-F. Liu, and F. Qi, *An integral representation of the Catalan numbers*, Glob. J. Math. Anal. **3** (2015), no. 3, 130–133; Available online at http://dx.doi.org/10.14419/gjma.v3i3.5055.

[34] R. P. Stanley, *Catalan Numbers*, Cambridge University Press, New York, 2015; Available online at http://dx.doi.org/10.1017/CBO9781139871495.

[35] G. S. Te, *Catalan numbers in the Xiang shu yi yuan*, Collected research papers on the history of mathematics, Vol. 2, 105–112, Inner Mongolia Univ. Press, Hohhot, 1991. (Chinese)

[36] Q. Zou, *Analogues of several identities and supercongruences for the Catalan–Qi numbers*, J. Inequal. Spec. Funct. **7** (2016), no. 4, 235–241.

# Generalized von Neumann-Jordan and James Constants for Quasi-Banach Spaces

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**Abstract**

The generalized von Neumann-Jordan constant $C_{NJ}^{(p)}(\mathcal{B})$ and the James constant $J(\mathcal{B})$ for a quasi-Banach space were introduced in [7]. In this note, it is shown that $C_{NJ}^{(p)}(\mathcal{B}) \leq 2$ for any quasi-Banach space $\mathcal{B}$ and $C_{NJ}^{(p)}(\mathcal{B}) < 2$ if and only if $\mathcal{B}$ is uniformly non-square. Along with relationship between $J(\mathcal{B})$ and $C_{NJ}^{(p)}(\mathcal{B})$, the criterion for the uniformly smooth quasi-Banach space is also established.

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## 1 Introduction

Among various geometric constants of a Banach space $\mathcal{B}$, the von Neumann-Jordan constant $C_{NJ}(\mathcal{B})$ for a Banach space $\mathcal{B}$ introduced by Clarkson [2] as the smallest constant $C$, for which the estimates

$$\frac{1}{C} \leq \frac{\|x_1 + x_2\|^2 + \|x_1 - x_2\|^2}{2(\|x_1\|^2 + \|x_2\|^2)} \leq C$$

hold for any $x_1, x_2 \in \mathcal{B}$ with $(x_1, x_2) \neq (0, 0)$. Equivalently

$$C_{NJ}(\mathcal{B}) = \sup \left\{ \frac{\|x_1 + x_2\|^2 + \|x_1 - x_2\|^2}{2(\|x_1\|^2 + \|x_2\|^2)} : x_1, x_2 \in \mathcal{B} \text{ with } (x_1, x_2) \neq (0, 0) \right\}.$$

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This idea was further enhanced by many authors in [2, 4, 5, 8, 9].

The James constant $J(\mathcal{B})$ of a Banach space $\mathcal{B}$ is defined by

$$J(\mathcal{B}) = \sup\big\{\min(\|x_1+x_2\|, \|x_1-x_2\|) : x_1, x_2 \in S_{\mathcal{B}}\big\},$$

where $S_{\mathcal{B}}$ is unit sphere.

In [3], the authors introduced the generalized von Neumann-Jordan constant $C_{NJ}^{(p)}(\mathcal{B})$ which is defined as

$$C_{NJ}^{(p)}(\mathcal{B}) = \sup\left\{\frac{\|x_1+x_2\|^p + \|x_1-x_2\|^p}{2^{p-1}(\|x_1\|^p + \|x_2\|^p)} : x_1, x_2 \in \mathcal{B} \text{ with } (x_1, x_2) \neq (0,0)\right\}$$

and obtained the relationship between $C_{NJ}^{(p)}(\mathcal{B})$ and $J(\mathcal{B})$.

This has an analog in the quasi-Banach space, that was considered in [7]. In this note, It is shown that $C_{NJ}^{(p)}(\mathcal{B}) \leq 2$ for any quasi-Banach space $\mathcal{B}$ and $C_{NJ}^{(p)}(\mathcal{B}) < 2$ if and only if $\mathcal{B}$ is uniformly non-square. A relationship between $J(\mathcal{B})$ and $C_{NJ}^{(p)}(\mathcal{B})$ is established. We also give the criterion for the uniformly smooth quasi-Banach space.

## 2 Preliminaries

Recall [1], that a quasi-norm on $\|\cdot\|$ on vector space $\mathcal{B}$ over a field $K$ ($\mathbb{R}$ or $\mathbb{C}$) is a mapping $\mathcal{B} \to [0,\infty)$ with properties

- $\|x\| = 0 \Longleftrightarrow x = 0$.
- $\|\alpha x\| = |\alpha|\|x\|$ for all $\alpha \in K$ and $x \in \mathcal{B}$.
- There exists a constant $C \geq 1$ such that $\forall x_1, x_2 \in \mathcal{B}$ we have

$$\|x_1+x_2\| \leq C(\|x_1\| + \|x_2\|).$$

**Definition 2.1.** The generalized von Neumann-Jordan constant $C_{NJ}^{(p)}(\mathcal{B})$ for a quasi-Banach space is defined by

$$C_{NJ}^{(p)}(\mathcal{B}) = \sup\left\{\frac{\|x_1+x_2\|^p + \|x_1-x_2\|^p}{2^{p-1}C^p(\|x_1\|^p + \|x_2\|^p)} : x_1, x_2 \in \mathcal{B} \text{ with } (x_1, x_2) \neq (0,0)\right\},$$

where $1 \leq p < \infty$.

The parametrized formula for the constant $C_{NJ}^{(p)}(\mathcal{B})$ is given as

$$C_{NJ}^{(p)}(\mathcal{B}) = \sup\left\{\frac{\|x_1+tx_2\|^p + \|x_1-tx_2\|^p}{C^p 2^{p-1}(1+t^p)} : x_1, x_2 \in S_{\mathcal{B}},\ 0 \leq t \leq 1\right\},$$

By taking $t = 1$ and $x_1 = x_2$, we obtain the estimate

$$C_{NJ}^{(p)}(\mathcal{B}) \geq \frac{\|2x_1\|^p}{C2^p(1+1)} \geq \frac{2^p}{C2^{p-1}(1+1)} = \frac{1}{C}.$$

**Definition 2.2.** In a quasi-Banach space $\mathcal{B}$ the James constant is defined as

$$J(\mathcal{B}) = \sup\left\{\frac{1}{C}\min(\|x_1+x_2\|, \|x_1-x_2\|) : x_1, x_2 \in S_{\mathcal{B}}\right\}.$$

**Definition 2.3.** A quasi-Banach space $\mathcal{B}$ is said to be uniformly non-square if there exists a positive number $\delta < 2$ such that for any $x_1, x_2 \in S_{\mathcal{B}}$, we have

$$\min\left(\left\|\frac{x_1+x_2}{C}\right\|, \left\|\frac{x_1+x_2}{C}\right\|\right) \leq \delta.$$

**Remark 2.4.** As in classical case, the quasi-Banach space $\mathcal{B}$ is uniformly non-square if and only if $J(\mathcal{B}) < 2$

**Definition 2.5.** The modulus of uniform smoothness of a quasi-Banach space $\mathcal{B}$ is defined as

$$\rho_{\mathcal{B}}(t) = \sup\left\{\frac{\|x_1+tx_2\|+\|x_1-tx_2\|}{2C} - \frac{1}{C} : x_1, x_2 \in S_{\mathcal{B}},\, t \geq 0\right\}.$$

**Definition 2.6.** A quasi-Banach space $\mathcal{B}$ is said to be uniformly smooth if $(\rho_{\mathcal{B}})'_+(0) = \lim_{t\to 0^+}\frac{\rho_{\mathcal{B}}(t)}{t} = 0$.

**Definition 2.7.** For any quasi-Banach space $\mathcal{B}$ and a real number $p \in [0,\infty)$, $J_{\mathcal{B},p}(t)$ is defined by

$$J_{\mathcal{B},p}(t) = \sup\left\{\left(\frac{\|x_1+tx_2\|^p+\|x_1-tx_2\|^p}{2C^p}\right)^{\frac{1}{p}} : x_1, x_2 \in S_{\mathcal{B}}\right\}.$$

# 3 Main results

**Theorem 3.1.** *Let $\mathcal{B}$ be a non-trivial quasi-Banach space and $p \in [1,\infty)$. Then*

$$J_{\mathcal{B},p}(t) \geq \rho_{\mathcal{B}}(t) + \frac{1}{C}.$$

*Proof.* By using convexity of the function $f(u) = u^p$ on $(0,\infty)$, one can easily obtained

$$\left(\frac{\|x_1+tx_2\|+\|x_1-tx_2\|}{2}\right)^p \leq \frac{\|x_1+tx_2\|^p+\|x_1-tx_2\|^p}{2},$$

therefore

$$\left(\frac{\|x_1+tx_2\|+\|x_1-tx_2\|}{2C}\right)^p \leq \frac{\|x_1+tx_2\|^p+\|x_1-tx_2\|^p}{2C^p},$$

which implies that

$$\frac{1}{C} + \rho_{\mathcal{B}}(t) \leq J_{\mathcal{B},p}(t).$$

For $p = 1$, we have $J_{\mathcal{B},1}(t) = \frac{1}{C} + \rho_{\mathcal{B}}(t)$ and for $p = 2$, we have $2C^2 J^2_{\mathcal{B},2}(t) = E(t,\mathcal{B})$, where

$$E(t,\mathcal{B}) = \sup\left\{(\|x_1+tx_2\|^2 + \|x_1-tx_2\|^2) : x_1, x_2 \in S_{\mathcal{B}}\right\}.$$

This completes the proof. □

**Theorem 3.2.** *For any quasi-Banach space $\mathcal{B}$, we have*

(a) $J_{\mathcal{B},p}(t)$ *is a non-decreasing function.*
(b) $J_{\mathcal{B},p}(t)$ *is convex.*
(c) $J_{\mathcal{B},p}(t)$ *is continuous function.*
(d) $\frac{J_{\mathcal{B},p}(t)-1}{t}$ *is a non-decreasing function.*

*Proof.* We only prove $(a)$ and remaining are analogous to it. Let $g(t) = \|x_1 + tx_2\|^p + \|x_1 - tx_2\|^p$ be a convex and even function. Let $0 < t_1 \leq t_2$ and $x_1, x_2 \in S_{\mathcal{B}}$. Then we have

$$\begin{aligned}\|x_1 + t_1x_2\|^p + \|x_1 - t_1x_2\|^p = g(t_1) &= g\left(\frac{t_2 + t_1}{2t_2}t_2 + \frac{t_2 - t_1}{2t_2}(-t_2)\right)\\ &\leq \frac{t_2 + t_1}{2t_2}g(t_2) + \frac{t_2 - t_1}{2t_2}g(t_2)\\ &= g(t_2) = \|x_1 + t_2x_2\|^p + \|x_1 - t_2x_2\|^p\\ &\leq 2C^pJ^p_{\mathcal{B},p}(t_2),\end{aligned}$$

which implies that

$$\frac{\|x_1 + t_1x_2\|^p + \|x_1 - t_1x_2\|^p}{2C^p} \leq J^p_{\mathcal{B},p}(t_2).$$

Hence $J_{\mathcal{B},p}(t_1) \leq J_{\mathcal{B},p}(t_2)$. □

**Theorem 3.3.** *For any quasi-Banach space $\mathcal{B}$ and $1 \leq p < \infty$, the generalized von Neumann-Jordan constant $C^{(p)}_{NJ}(\mathcal{B})$ satisfy the inequality $C^{(p)}_{NJ}(\mathcal{B}) \leq 2$.*

*Proof.* As we have already defined

$$C^{(p)}_{NJ}(\mathcal{B}) = \sup\left\{\frac{\|x_1 + tx_2\|^p + \|x_1 - tx_2\|^p}{C^p2^{p-1}(1 + t^p)} : x_1, x_2 \in S_{\mathcal{B}} \text{ with } (x_1, x_2) \neq (0, 0)\right\},$$

where $0 \leq t \leq 1$.

By using definition of a quasi-Banach space, we have the following inequality

$$\begin{aligned}\|x_1 + tx_2\|^p + \|x_1 - tx_2\|^p &\leq C^p(\|x_1\| + t\|x_2\|)^p + C^p(\|x_1\| + t\|x_2\|)^p\\ &= 2C^p(\|x_1\| + t\|x_2\|)^p\\ &= 2C^p(1 + t)^p,\end{aligned}$$

therefore

$$\frac{\|x_1 + tx_2\|^p + \|x_1 - tx_2\|^p}{C^p2^{p-1}(1 + t^p)} \leq \frac{2(1 + t)^p}{2^{p-1}(1 + t^p)}. \tag{3.1}$$

The function $f(u) = u^p$ is convex, which leads

$$(1 + t)^p = \left(2 \cdot \frac{1 + t}{2}\right)^p \leq 2^p\left(\frac{1 + t^p}{2}\right) = 2^{p-1}(1 + t^p).$$

Using above inequality (3.1) become

$$\frac{\|x_1 + tx_2\|^p + \|x_1 - tx_2\|^p}{C^p2^{p-1}(1 + t^p)} \leq \frac{1}{2^{p-2}}2^{p-1} = 2.$$

Hence

$$C_{NJ}^{(p)}(\mathcal{B}) = \sup\left\{\frac{\|x_1+tx_2\|^p+\|x_1-tx_2\|^p}{C^p 2^{p-1}(1+t^p)} : x_1,x_2 \in S_{\mathcal{B}},\, 0\le t\le 1\right\}$$
$$\le 2.$$

This completes the proof. □

Next theorem presents the relationship between $C_{NJ}^{(p)}(\mathcal{B})$ and $J(\mathcal{B})$.

**Theorem 3.4.** *Let $\mathcal{B}$ be a non-trivial qausi-Banach space and $p \in (1,\infty)$. Then the following inequality hold:*

$$J(\mathcal{B}) \le 2^{\frac{p-1}{p}} \sqrt[p]{C_{NJ}^{(p)}(\mathcal{B})}.$$

*Proof.* For any $x_1, x_2 \in S_{\mathcal{B}}$, we have

$$\begin{aligned}
2(\min\{\|x_1+x_2\|,\|x_1-x_2\|\})^p &\le 2\left(\frac{\|x_1+x_2\|+\|x_1-x_2\|}{2}\right)^p \\
&\le 2\left(\frac{\|x_1+x_2\|^p+\|x_1-x_2\|^p}{2}\right) \\
&= \frac{\|x_1+x_2\|^p+\|x_1-x_2\|^p}{C^p 2^{p-1}(\|x_1\|^p+\|x_2\|^p)}.C^p 2^{p-1}(\|x_1\|^p+\|x_2\|^p) \\
&\le C^p C_{NJ}^{(p)}(\mathcal{B}) 2^{p-1}(\|x_1\|^p+\|x_2\|^p) \\
&= 2\cdot 2^{p-1} C^p C_{NJ}^{(p)}(\mathcal{B}),
\end{aligned}$$

$$(\{\min\{\|x_1+x_2\|,\|x_1-x_2\|\})^p \le 2^{p-1} C^p C_{NJ}^{(p)}(\mathcal{B}),$$

$$\frac{1}{C}\{\min\{\|x_1+x_2\|,\|x_1-x_2\|)\} \le 2^{\frac{p-1}{p}} \sqrt[p]{C_{NJ}^{(p)}(\mathcal{B})}.$$

Taking supremum both side,

$$\sup\left(\frac{1}{C}\min\{\|x_1+x_2\|,\|x_1-x_2\|\}\right) \le 2^{\frac{p-1}{p}} \sqrt[p]{C_{NJ}^{(p)}(\mathcal{B})}.$$

Therefore

$$J(\mathcal{B}) \le 2^{\frac{p-1}{p}} \sqrt[p]{C_{NJ}^{(p)}(\mathcal{B})}.$$

This completes the proof. □

**Theorem 3.5.** *For $p \in (1,\infty)$, a quasi-Banach space $\mathcal{B}$ is uniformly non-square if and only if there exists $\delta \in (0,1)$ such that for any $x_1, x_2 \in \mathcal{B}$, we have*

$$\left\|\frac{x_1+x_2}{2C}\right\|^p + \left\|\frac{x_1-x_2}{2C}\right\|^p \le (2-\delta)\frac{\|x_1\|^p+\|x_2\|^p}{2}. \tag{3.2}$$

*Proof.* Let $\mathcal{B}$ be an uniformly non-square quasi-Banach space and on contrary assume that (3.2) is not hold. Therefore for every positive integer $n$, there exists $x_n$ and $y_n$ in $\mathcal{B}$ such that

$$\left\|\frac{x_n+y_n}{2C}\right\|^p+\left\|\frac{x_n-y_n}{2C}\right\|^p>\left(2-\frac{1}{n}\right)\frac{\|x_n\|^p+\|y_n\|^p}{2}.$$

Let $x_n\in S_{\mathcal{B}}$ and $y_n\in B_{\mathcal{B}}=\{x\in\mathcal{B}:\|x\|\leq 1\}$ for all $n$. With out loss of generality we assume that $\{\|y_n\|\}$ converges to some $\gamma$, where $0\leq\gamma\leq 1$ (by Bolzano-Wiestrass) we have

$$\begin{aligned}\left(2-\frac{1}{n}\right)\frac{1+\|y_n\|^p}{2}&<\left\|\frac{x_n+y_n}{2C}\right\|^p+\left\|\frac{x_n-y_n}{2C}\right\|^p\\&\leq 2\left(\frac{C(\|x_n\|^p+\|y_n\|^p)}{2C}\right)^p\\&=2\left(\frac{1+\|y_n\|}{2}\right)^p\\&\leq 2\left(\frac{1+\|y_n\|^p}{2}\right),\end{aligned}$$

letting $n\to\infty$, we obtain

$$\frac{(1+\gamma)^p}{1+\gamma^p}=2^{p-1}\quad\Longrightarrow\quad\gamma=1,$$

therefore

$$\left\|\frac{x_n+y_n}{2C}\right\|^p+\left\|\frac{x_n-y_n}{2C}\right\|^p\longrightarrow 2,$$

which implies that

$$\left\|\frac{x_n+y_n}{2C}\right\|^p\longrightarrow 1\quad\text{and}\quad\left\|\frac{x_n-y_n}{2C}\right\|^p\longrightarrow 1.$$

This contradiction to the fact that $\mathcal{B}$ is uniformly non-square.

Conversely, suppose that

$$\left\|\frac{x_1+x_2}{2C}\right\|^p+\left\|\frac{x_1-x_2}{2C}\right\|^p\leq(2-\delta)\frac{\|x_1\|^p+\|x_2\|^p}{2}.$$

In particularly, we have

$$\left\|\frac{x_1+x_2}{2C}\right\|^p+\left\|\frac{x_1-x_2}{2C}\right\|^p\leq(2-\delta),$$

which implies that

$$\min\left\{\left\|\frac{x_1+x_2}{2C}\right\|^p,\left\|\frac{x_1-x_2}{2C}\right\|^p\right\}\leq\left(1-\frac{\delta}{2}\right)^{\frac{1}{p}}.$$

Hence $\mathcal{B}$ is uniformly non-square. This completes the proof. □

**Theorem 3.6.** *For $p\in(1,\infty)$, a quasi-Banach space $\mathcal{B}$ is uniformly non-square if and only if $C_{NJ}^{(p)}(\mathcal{B})<2$.*

*Proof.* From the Theorem 3.5, $\mathcal{B}$ is uniformly non-square if and only if there exists $0 < \delta < 1$ such that

$$\left\|\frac{x_1+x_2}{2C}\right\|^p + \left\|\frac{x_1-x_2}{2C}\right\|^p \leq (2-\delta)\frac{\|x_1\|^p+\|x_2\|^p}{2}.$$

Therefore

$$\frac{\|x_1+x_2\|^p+\|x_1-x_2\|^p}{C^p 2^{p-1}(\|x_1\|^p+\|x_2\|^p)} \leq 2, \quad \forall (x_1,x_2) \neq (0,0).$$

Hence $C_{NJ}^{(p)}(\mathcal{B}) < 2$. □

**Theorem 3.7.** *For any quasi-Banach space $\mathcal{B}$ and $p \in (1,\infty)$, the inequalities $C_{NJ}^{(p)}(\mathcal{B}) < 2$ and $J(\mathcal{B}) < 2$ are equivalent.*

*Proof.* From the Remark 2.4, $J(\mathcal{B}) < 2$ if and only if $\mathcal{B}$ is uniformly non-square. Therefore by using Theorem 3.6, we have $C_{NJ}^{(p)}(\mathcal{B}) < 2$.

Suppose that $C_{NJ}^{(p)}(\mathcal{B}) < 2$. Then by using the Theorem 3.4, we have

$$J(\mathcal{B}) < 2^{\frac{p-1}{p}} 2^{\frac{1}{p}} = 2.$$

This completes the proof. □

**Theorem 3.8.** *Let $\mathcal{B}$ be a quasi-Banach space, $p \in [1,\infty)$ and $t > 0$. Then the following conditions are equivalent:*

*(a)* $J_{\mathcal{B},p}(t) < 1+t$.

*(b)* $J(t,x_1) < 1+t$.

*Proof.* $(a) \Rightarrow (b)$ Suppose on contrary that $J(t,x_1) \geq 1+t$. Then it is enough to take $J(t,x_1) = 1+t$.

Since

$$J(t,\mathcal{B}) = \sup\left\{\frac{1}{C}\min(\|x_1+x_2\|,\|x_1-x_2\|) : x_1,x_2 \in S_{\mathcal{B}}\right\},$$

by using the definition of supremum, for any $\epsilon > 0$, there exist $x_1, x_2 \in S_{\mathcal{B}}$ such that

$$\begin{aligned}\frac{\|x_1+tx_2\|+\|x_1-tx_2\|}{2} &\geq \min\{\|x_1+tx_2\|,\|x_1-tx_2\|\} \\ &\geq c(1+t-\epsilon).\end{aligned} \tag{3.3}$$

Applying convexity of the function $f(u) = u^p$, we get

$$\left(\left(\frac{\|x_1+tx_2\|+\|x_1-tx_2\|}{2}\right)^p\right)^{\frac{1}{p}} \leq \left(\frac{\|x_1+tx_2\|^p+\|x_1-tx_2\|^p}{2}\right)^{\frac{1}{p}},$$

therefore from (3.3)

$$\begin{aligned}\left(\frac{\|x_1+tx_2\|^p+\|x_1-tx_2\|^p}{2}\right)^{\frac{1}{p}} &\geq \min\{\|x_1+tx_2\|,\|x_1-tx_2\|\} \\ &\geq c(1+t-\epsilon).\end{aligned}$$

Since $\epsilon$ is any arbitrary so

$$J_{\mathcal{B},p}(t) \geq 1 + t,$$

which leads a contradiction.

$(b) \Rightarrow (a)$ Suppose on contrary that $J_{\mathcal{B},p}(t) \geq 1 + t$. Then it is enough to take $J_{\mathcal{B},p}(t) = 1 + t$. Again using the definition of supremum, for any $\epsilon > 0$ there exists $x_1, x_2 \in S_{\mathcal{B}}$ such that

$$\|x_1 + tx_2\|^p + \|x_1 - tx_2\|^p \geq 2C^p(1 + t - \epsilon)^p,$$

also using

$$\|x_1 + tx_2\|^p + \|x_1 - tx_2\|^p \leq 2C^p(1 + t)^p,$$

$$2C^p(1 + t)^p \geq \|x_1 + tx_2\|^p + \|x_1 - tx_2\|^p \geq 2C^p(1 + t - \epsilon)^p$$

since $\epsilon$ is arbitrary, so

$$\|x_1 + tx_2\|^p + \|x_1 - tx_2\|^p = 2C^p(1 + t)^p,$$

which implies that

$$\|x_1 + tx_2\| = \|x_1 - tx_2\| = C(1 + t).$$

So using the definition of $J(t, x_1)$, we get $J(t, x_1) \geq 1+t$, which lead to a contradiction. □

**Corollary 3.9.** *Let $\mathcal{B}$ be a quasi-Banach space, $p \in [1, \infty)$ and $t > 0$. Then the following conditions are equivalent:*

*$(a)$ $\mathcal{B}$ is uniformly non-square.*
*$(b)$ $J_{\mathcal{B},p}(t) < 1 + t$.*
*$(c)$ $J(t, x_1) < 1 + t$.*

**Theorem 3.10.** *A quasi-Banach apace $\mathcal{B}$ is uniformly smooth if*

$$\lim_{t\to 0}\left(\frac{J_{\mathcal{B},p}(t) - \frac{1}{C}}{t}\right) = 0.$$

*Proof.* Suppose that

$$\lim_{t\to 0}\left(\frac{J_{\mathcal{B},p}(t) - \frac{1}{C}}{t}\right) = 0.$$

From Theorem 3.1 we know that

$$J_{\mathcal{B},p}(t) \geq \rho_{\mathcal{B}}(t) + \frac{1}{C},$$

which implies that

$$J_{\mathcal{B},p}(t) - \frac{1}{C} \geq \rho_{\mathcal{B}}(t),$$

dividing both side by $t$ and applying the $\lim_{t\to 0}$

$$\lim_{t\to 0}\frac{\rho_{\mathcal{B}}(t)}{t} \leq \lim_{t\to 0}\left(\frac{J_{\mathcal{B},p}(t) - \frac{1}{C}}{t}\right) = 0.$$

So by definition $\mathcal{B}$ is uniformly smooth. □

# 4 Quasi-Banach fixed point theorem (Contraction theorem)

**Definition 4.1.** ([6]) Let $\mathcal{B} = (\mathcal{B}, \tilde{d})$ be a quasi-metric space. A mapping $T : \mathcal{B} \to T$ is called a contraction on $\mathcal{B}$ if there exists a positive real number $\alpha < 1$ such that for all $x_1, x_2 \in \mathcal{B}$

$$\tilde{d}(Tx_1, Tx_2) \leq \alpha \tilde{d}(x_1, x_2).$$

**Theorem 4.2.** *Consider a quasi-metric space $\mathcal{B} = (\mathcal{B}, \tilde{d})$, where $\mathcal{B} \neq \emptyset$. Suppose $\mathcal{B}$ is complete and $T : \mathcal{B} \to T$ be a contraction on $\mathcal{B}$ and suppose that $C^n \beta^m \to 0$, where $\beta = \frac{\alpha}{C}$. Then $T$ has exactly one fixed point.*

*Proof.* Let any $x_0 \in \mathcal{B}$ and define the iterative sequence by

$$x_0, \quad x_1 = Tx_0, \quad x_2 = T^2 x_0, \quad x_3 = T^3 x_0, \quad \cdots, \quad x_n = T^n x_0, \quad \cdots.$$

First, we show that this iterative sequence $(x_n)$ is cauchy. For this we take

$$\begin{aligned}
\tilde{d}(x_{m+1}, x_m) &= \tilde{d}(Tx_m, Tx_{m-1}) \\
&\leq \alpha \tilde{d}(x_m, x_{m-1}) \\
&\leq \alpha \tilde{d}(Tx_{m-1}, Tx_{m-2}) \\
&\leq \alpha^2 \tilde{d}(x_{m-1}, x_{m-2}) \\
&\quad \cdots\cdots \\
&\leq \alpha^m \tilde{d}(x_1, x_0).
\end{aligned}$$

Suppose that $n > m$ and using the definition of quasi-metric

$$\begin{aligned}
\tilde{d}(x_m, x_n) &\leq C\big(\tilde{d}(x_m, x_{n-1}) + \tilde{d}(x_{n-1}, x_n)\big) \\
&\leq C^2\big(\tilde{d}(x_m, x_{n-2}) + \tilde{d}(x_{n-2}, x_{n-1})\big) + C\tilde{d}(x_{n-1}, x_n) \\
&\leq C^3\big(\tilde{d}(x_m, x_{n-3}) + \tilde{d}(x_{n-3}, x_{n-2})\big) + C^2\tilde{d}(x_{n-2}, x_{n-1}) + C\tilde{d}(x_{n-1}, x_n) \\
&\quad \cdots\cdots \\
&\leq C^{n-m}\tilde{d}(x_m, x_{m+1}) + \cdots + C\tilde{d}(x_{n-1}, x_n) \\
&\leq C^{n-m}\big(\tilde{d}(x_m, x_{m+1}) + \cdots + \tilde{d}(x_{n-1}, x_n)\big) \\
&\leq C^{n-m}\big[\alpha^m + \alpha^{m+1} + \cdots + \alpha^{n-1}\big]\tilde{d}(x_1, x_0) \\
&\leq C^{n-m}\frac{\alpha^m}{1-\alpha}\tilde{d}(x_1, x_0) \\
&= C^n\left(\frac{\alpha}{C}\right)^m \frac{1}{1-\alpha}\tilde{d}(x_1, x_0).
\end{aligned}$$

Since

$$\tilde{d}(x_m, x_n) \leq C^n\left(\frac{\alpha}{C}\right)^m \frac{1}{1-\alpha}\tilde{d}(x_1, x_0)$$

from our supposition $\tilde{d}(x_m, x_n) \to 0$ as $m \to \infty$. Since $\mathcal{B}$ is complete so $x_m \to x \in \mathcal{B}$.

Next we show that this limit $x$ is the fixed point of $T$. For this using the definition of quasi-metric we have

$$\begin{aligned}
\tilde{d}(x, Tx) &\leq C(\tilde{d}(x, x_m) + \tilde{d}(x_m, Tx)) \\
&\leq C(\tilde{d}(x, x_m) + \alpha\tilde{d}(x_{m-1}, x))
\end{aligned}$$

because $(x_m)$ converges to $x$, therefore we have $\tilde{d}(x, Tx) = 0$. This implies $Tx = x$. Now we prove that $x$ is the unique fixed point of $T$. For this let $Tx = x$ and $Ty = y$, therefore we have

$$\tilde{d}(x, y) = \tilde{d}(Tx, Ty) \leq \alpha \tilde{d}(x, y),$$

which implies $\tilde{d}(x, y) = 0$ because $\alpha < 1$. Hence $x = y$. This completes the proof. □

## 5 Conclusion

Generalized von Neumann-Jordan and James constants studied by many researcher for Banach space for example in [2, 4, 5, 8, 9] and the references therein. In this paper, we introduce the generalized von Neumann-Jordan constant and the James constant for a quasi-Banach space. Relationships between James constant and generalized von Neumann-Jordan constant are also presented.

## References

[1] C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, Inc., Boston, 1988.

[2] J. A. Clarkson, The von Neumann-Jordan constant for the Lebesgue spaces, *Ann. Math.*, **38** (1937), 114–115.

[3] Y. Cui, W. Huang, H. Hudzik and R. Kaczmarek, Generalized von Nuemann-Jordan constant and its relationship to the fixed point property, *Fixed Point Theory Appl.*, **2015** (2015), Article ID 40, 11 pages.

[4] M. Kato and L. Maligranda, On James and Jordan-von Neumann constants of Lorentz sequence spaces, *J. Math. Anal. Appl.*, **258** (2001), 457–465.

[5] M. Kato, L. Maligranda and Y. Takahashi, On James and Jordan-von Neumann constants and the normal structure coefficient of Banach spaces, *Stud. Math.*, **144** (2001), 275–295.

[6] E. Kreyszig, *Introductory Functional Analysis with Applications*, John Wiley & Sons, New York-London-Sydney, 1978.

[7] Y. C. Kwun, Q. Mehmood, W. Nazeer, A. U. Haq and S. M. Kang, Relations between generalized von Neumann-Jordan and James constants for quasi-Banach spaces, *J. Inequal. Appl.*, **2016** (2016), Article ID 171, 10 pages.

[8] Y. Takahashi and M. Kato, Von Neumann-Jordan constant and uniformly non-square Banach spaces, *Nihonkai Math. J.*, **9** (1998), 155–169.

[9] C. Yang, Jordan-von Neumann constant for Banaś-Frączek space, *Banach J. Math. Anal.*, **8** (2014), 185–192.

# Shared-values of meromorphic functions on annuli *

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**Abstract**

In this paper, we study the shared values and uniqueness of meromorphic functions on annulus, and obtain one theorem about meromorphic functions on annulus sharing some distinct values, and this result is an improvement of some theorems given by Cao, Yi [4, 5], Kondratyuk and Laine[9].
**Key words:** Meromorphic function, Nevanlinna theory, the annulus.
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## 1 Introduction and main resut

In 1929, R.Nevanlinna(see [14]) first investigated the uniqueness of meromorphic functions in the whole complex plane and obtained the well-known theorem—5 $IM$ theorem of two meromorphic functions sharing five distinct values.

After his theorem, there are vast references on the uniqueness of meromorphic functions sharing values and sets in the whole complex plane(see [2, 16, 18]). In recent, the uniqueness problem of meromorphic functions with shared values in some angular domain attracted many investigations (see [3, 11, 19, 20]). Thus, we always assumed that the reader is familiar with the notations of the Nevanlinna theory such as $T(r,f), m(r,f), N(r,f)$ and so on (see [6, 16, 17]).

We use $\mathbb{C}$ to denote the open complex plane, $\overline{\mathbb{C}}$ to denote the extended complex plane, and $\mathbb{X}$ to denote the subset of $\mathbb{C}$. Let $S$ be a set of distinct elements in $\widehat{\mathbb{C}}$ and $\mathbb{X} \subseteq \mathbb{C}$.

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Define

$$E_{\mathbb{X}}(S,f)=\bigcup_{a\in S}\{z\in\mathbb{X}|f_a(z)=0,\quad counting\quad multiplicities\},$$

$$\overline{E}_{\mathbb{X}}(S,f)=\bigcup_{a\in S}\{z\in\mathbb{X}|f_a(z)=0,\quad ignoring\quad multiplicities\},$$

where $f_a(z)=f(z)-a$ if $a\in\mathbb{C}$ and $f_\infty(z)=1/f(z)$.

For $a\in\overline{\mathbb{C}}$, we say that two meromorphic functions $f$ and $g$ share the value $a$ $CM$ $(IM)$ in $\mathbb{X}$ (or $\mathbb{C}$), if $f(z)-a$ and $g(z)-a$ have the same zeros with the same multiplicities(ignoring multiplicities) in $\mathbb{X}$ (or $\mathbb{C}$). In addition, we also use $f=a\rightleftharpoons g=a$ in $\mathbb{X}$ (or $\mathbb{C}$) to express that $f$ and $g$ share the value $a$ $CM$ in $\mathbb{X}$ (or $\mathbb{C}$), $f=a\Longleftrightarrow g=a$ in $\mathbb{X}$ (or $\mathbb{C}$) to express that $f$ and $g$ share the value $a$ $IM$ in $\mathbb{X}$ (or $\mathbb{C}$), and $f=a\Longrightarrow g=a$ in $\mathbb{X}$ (or $\mathbb{C}$) to express that $f=a$ implies $g=a$ in $\mathbb{X}$ (or $\mathbb{C}$).

As we know, the whole complex plane $\mathbb{C}$ and angular domain all can be regarded as simply-connected regions, many results about the uniqueness of shared values and sets in the complex plane and angular domain can also be regarded as the uniqueness of meromorphic functions in simply-connected regions. *Thus, it arises naturally an interesting subject on the uniqueness for the meromorphic functions in the multiply connected region?*

The main purpose of this paper is to study the uniqueness of meromorphic functions in doubly connected domains of complex plane $\mathbb{C}$. From the Doubly Connected Mapping Theorem [1], we can get that each doubly connected domain is conformally equivalent to the annulus $\{z:r<|z|<R\}$, $0\leq r<R\leq+\infty$. For two cases: $r=0$, $R=+\infty$ simultaneously and $0<r<R<+\infty$, the latter case the homothety $z\mapsto\frac{z}{\sqrt{rR}}$ reduces the given domain to the annulus $\{z:\frac{1}{R_0}<|z|<R_0\}$, where $R_0=\sqrt{\frac{R}{r}}$. Thus, every annulus is invariant with respect to the inversion $z\mapsto\frac{1}{z}$ in two cases. *The basic notions of the Nevanlinna theory on annuli will be showed in the next section.*

In recent, there have some results on the Nevannlina Theory of meromorphic functions on the annulus (see [7, 8, 10, 12, 13, 15]). In 2005, Khrystiyanyn and Kondratyuk [7, 8] proposed the Nevanlinna theory for meromorphic functions on annuli (see also [9]). Lund and Ye [12] in 2009 studied functions meromorphic on the annuli with the form $\{z:R_1<|z|<R_2\}$, where $R_1\geq 0$ and $R_2\leq\infty$. *However, there are few results about the uniqueness of meromorphic functions on the annulus.* In 2009 and 2011, Cao [4, 5] investigated the uniqueness of meromorphic functions on annuli sharing some values and some sets, and obtained an analog of Nevanlinna's famous five-value theorem as follows:

**Theorem 1.1** ([5, Thereom 3.2] or [4, Corollary 3.3]). Let $f_1$ and $f_2$ be two transcendental or admissible meromorphic functions on the annulus $\mathbb{A}=\{z:\frac{1}{R_0}<|z|<R_0\}$, where $1<R_0\leq+\infty$. Let $a_j$ $(j=1,2,3,4,5)$ be five distinct complex numbers in $\overline{\mathbb{C}}$. If $f_1,f_2$ share $a_j$ $IM$ for $j=1,2,3,4,5$, then $f_1(z)\equiv f_2(z)$.

**Remark 1.1** *For the case $R_0=+\infty$, the assertion was proved by Kondratyuk and Laine [9].*

In this paper, we will focus on the uniqueness problem of meromorphic functions in the field of complex analysis and obtain the main result below which improve Theorem 1.1.

**Theorem 1.2** *Let $f$ and $g$ be two transcendental or admissible meromorphic functions on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \leq +\infty$, $a_j \in \mathbb{C}(j = 1, 2, 3, 4)$ be four distinct values. We assume that $f$ and $g$ share four distinct values $a_j(j = 1, 2, 3, 4)$ IM on $\mathbb{A}$ and $\overline{E}_{\mathbb{A}}(S, f) \subset \overline{E}_{\mathbb{A}}(S, g)$, where $S = \{b_1, \ldots, b_m\}, m \geq 1$ and $b_1, \ldots, b_m \in \widehat{\mathbb{C}} \setminus \{a_1, a_2, a_3, a_4\}$. Then $f$ and $g$ share all values CM on $\mathbb{A}$, thus it follows that either $f \equiv g$ or $f$ is a Möbius transformation of $g$. Furthermore, if the number of the values in $S$ is odd, then $f \equiv g$.*

**Remark 1.2** *The special case $m = 1$ of Theorem 1.2 immediately yields Theorem 1.1. In fact, when $m = 1$, set $S = \{a_5\}$. If $f, g$ share $a_5$ IM on $\mathbb{A}$, which implies $\overline{E}_{\mathbb{A}}(S, f) \subset \overline{E}_{\mathbb{A}}(S, g)$, then by Theorem 1.2, we can get $f \equiv g$.*

## 2 Basic notions in the Nevanlinna theory on annuli

For a meromorphic function $f$ on whole plane $\mathbb{C}$, the classical notations of Nevanlinna theory are denoted as follows

$$N(R, f) = \int_0^R \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log R,$$
$$m(R, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(Re^{i\theta})| d\theta, \quad T(R, f) = N(R, f) + m(R, f),$$

where $\log^+ x = \max\{\log x, 0\}$, and $n(t, f)$ is the counting function of poles of the function $f$ in $\{z : |z| \leq t\}$.

Let $f$ be a meromorphic function on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R < R_0 \leq +\infty$, the notations of the Nevanlinna theory on annuli will be introduced as follows, let

$$N_1(R, f) = \int_{\frac{1}{R}}^1 \frac{n_1(t, f)}{t} dt, \quad N_2(R, f) = \int_1^R \frac{n_2(t, f)}{t} dt,$$
$$m_0(R, f) = m(R, f) + m(\frac{1}{R}, f), \quad N_0(R, f) = N_1(R, f) + N_2(R, f),$$

where $n_1(t, f)$ and $n_2(t, f)$ are the counting functions of poles of the function $f$ in $\{z : t < |z| \leq 1\}$ and $\{z : 1 < |z| \leq t\}$, respectively. Similarly, for $a \in \overline{\mathbb{C}}$, we have

$$\begin{aligned} \overline{N}_0(r, \frac{1}{f - a}) &= \overline{N}_1(R, \frac{1}{f - a}) + \overline{N}_2(R, \frac{1}{f - a}) \\ &= \int_{\frac{1}{R}}^1 \frac{\overline{n}_1(t, \frac{1}{f-a})}{t} dt + \int_1^R \frac{\overline{n}_2(t, \frac{1}{f-a})}{t} dt \end{aligned}$$

in which each zero of the function $f - a$ is counted only once. In addition, we use $\overline{n}_1^{k)}(t, \frac{1}{f-a})$ (or $\overline{n}_1^{(k}(t, \frac{1}{f-a})$) to denote the counting function of poles of the function $\frac{1}{f-a}$ with multiplicities $\leq k$ (or $> k$) in $\{z : t < |z| \leq 1\}$, each point counted only

once. Similarly, we have the notations $\overline{N}_1^{k)}(t,f)$, $\overline{N}_1^{(k}(t,f)$, $\overline{N}_2^{k)}(t,f)$, $\overline{N}_2^{(k}(t,f)$, $\overline{N}_0^{k)}(t,f)$, $\overline{N}_0^{(k}(t,f)$.

The Nevanlinna characteristic of $f$ on the annulus $\mathbb{A}$ is defined by

$$T_0(R,f) = m_0(R,f) - 2m(1,f) + N_0(R,f).$$

For a nonconstant meromorphic function $f$ on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R < R_0 \leq +\infty$, the following properties will be used in this paper (see [7])

$(i)$ $T_0(R,f) = T_0\left(R, \dfrac{1}{f}\right)$,

$(ii)$ $\max\{T_0(R, f_1 \cdot f_2), T_0(R, \dfrac{f_1}{f_2}), T_0(R, f_1 + f_2)\} \leq T_0(R,f_1) + T_0(R,f_2) + O(1)$,

$(iii)$ $T_0(R, \dfrac{1}{f-a}) = T_0(R,f) + O(1)$, $for$ $every$ $fixed$ $a \in \mathbb{C}$.

In 2005, the lemma on the logarithmic derivative on the the annulus $\mathbb{A}$ was obtained by Khrystiyanyn and Kondratyuk [8].

**Theorem 2.1** ([8]) (Lemma on the logarithmic derivative) Let $f$ be a nonconstant meromorphic function on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $R_0 \leq +\infty$, and let $\lambda > 0$. Then

(i) in the case $R_0 = +\infty$,

$$m_0\left(R, \frac{f'}{f}\right) = O(\log(RT_0(R,f)))$$

for $R \in (1, +\infty)$ except for the set $\triangle_R$ such that $\int_{\triangle_R} R^{\lambda-1}dR < +\infty$;
(ii) if $R_0 < +\infty$, then

$$m_0\left(R, \frac{f'}{f}\right) = O(\log(\frac{T_0(R,f)}{R_0 - R}))$$

for $R \in (1, R_0)$ except for the set $\triangle_R^{'}$ such that $\int_{\triangle_R^{'}} \frac{dR}{(R_0-R)^{\lambda-1}} < +\infty$.

In 2005, the second fundamental theorem on the the annulus $\mathbb{A}$ was first obtained by Khrystiyanyn and Kondratyuk [8]. Later, the other forms of the second fundamental theorem on annuli were given by Cao, Yi and Xu [5].

**Theorem 2.2** ([5, Theorem 2.3]) (The second fundamental theorem) Let $f$ be a nonconstant meromorphic function on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \leq +\infty$. Let $a_1, a_2, \ldots, a_q$ be $q$ distinct complex numbers in the extended complex plane $\overline{\mathbb{C}}$. Let $k_1, k_2, \ldots, k_q$ be $q$ positive integers, and let $\lambda \geq 0$. Then

$$(i) \quad (q-2)T_0(R,f) < \sum_{j=1}^{q} N_0(R, \frac{1}{f-a_j}) - N_0^{(1)}(R,f) + S(R,f),$$

$$(ii) \quad (q-2)T_0(R,f) < \sum_{j=1}^{q} \overline{N}_0(R, \frac{1}{f-a_j}) + S(R,f),$$

where

$$N_0^{(1)}(R,f) = N_0(R,\frac{1}{f'}) + 2N_0(R,f) - N_0(R,f'),$$

and (i) in the case $R_0 = +\infty$,

$$S(R,f) = O(\log(RT_0(R,f)))$$

for $R \in (1,+\infty)$ except for the set $\triangle_R$ such that $\int_{\triangle_R} R^{\lambda-1}dR < +\infty$;
(ii) if $R_0 < +\infty$, then

$$S(R,f) = O(\log(\frac{T_0(R,f)}{R_0 - R}))$$

for $R \in (1,R_0)$ except for the set $\triangle_R^{'}$ such that $\int_{\triangle_R^{'}} \frac{dR}{(R_0-R)^{\lambda-1}} < +\infty$.

**Definition 2.1** *Let $f(z)$ be a non-constant meromorphic function on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \leq +\infty$. The function $f$ is called a transcendental or admissible meromorphic function on the annulus $\mathbb{A}$ provided that*

$$\limsup_{R\to\infty} \frac{T_0(R,f)}{\log R} = \infty, \quad 1 < R < R_0 = +\infty$$

*or*

$$\limsup_{R\to R_0} \frac{T_0(R,f)}{-\log(R_0 - R)} = \infty, \quad 1 < R < R_0 < +\infty,$$

*respectively.*

Then for a transcendental or admissible meromorphic function on the annulus $\mathbb{A}$, $S(R,f) = o(T_0(R,f))$ holds for all $1 < R < R_0$ except for the set $\triangle_R$ or the set $\triangle_R^{'}$ mentioned in Theorem 2.1, respectively.

# 3 Some lemmas

To prove the above theorems, we need some Lemmas as follows.

From Theorem 2.1 and the definition of $m_0(R,f)$, $f$ is transcendental or admissible function on $\mathbb{A}$, we can get Lemma 3.1 by using the same argument as in Lemma 4.3 in [16]

**Lemma 3.1** *Suppose that $f$ is a transcendental or admissible meromorphic function on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \leq +\infty$. Let $P(f) = a_0 f^p + a_1 f^{p-1} + \cdots + a_p (a_0 \neq 0)$ be a polynomial of $f$ with degree $p$, where the coefficients $a_j (j = 0,1,\ldots,p)$ are constants, and let $b_j (j = 1,2,\ldots,q)$ be $q (q \geq p+1)$ distinct finite complex numbers. Then*

$$m_0\left(R, \frac{P(f)\cdot f'}{(f-b_1)(f-b_2)\cdots(f-b_q)}\right) = S(R,f).$$

**Lemma 3.2** *Let $f, g$ be two distinct transcendental or admissible meromorphic functions on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \leq +\infty$. Suppose that $f$ and $g$ share four distinct values $a_1, a_2, a_3, a_4$ IM on the annulus $\mathbb{A}$. Then*

*(i) $T_0(R, f) = T_0(R, g) + S(R, f)$ and $T_0(R, g) = T_0(R, f) + S(R, g)$;*

*(ii) $\sum_{j=1}^{4} \overline{N}_0\left(R, \frac{1}{f-a_j}\right) = 2T_0(R, f) + S(R, f)$;*

*(iii) $\overline{N}_0(R, \frac{1}{f-b}) = T_0(R, f) + S(R, f)$, $\overline{N}_0(r, \frac{1}{g-b}) = T_0(R, g) + S(R, g)$, where $b \neq a_j (j = 1, 2, 3, 4)$.*

*Proof:* From Theorem 2.2(ii), we have

$$\begin{aligned} 2T_0(R, f) &\leq \sum_{j=1}^{4} \overline{N}_0(R, \frac{1}{f - a_j}) + S(R, f) \leq N_0(R, \frac{1}{f - g}) + S(R, f) \\ &\leq T_0(R, f) + T_0(R, g) + S(R, f), \end{aligned}$$

then we can get $T_0(R, f) = T_0(R, g) + S(R, f)$ and $\sum_{j=1}^{4} \overline{N}_0\left(R, \frac{1}{f-a_j}\right) = 2T_0(R, f) + S(R, f)$. Similarly, we can get $T_0(R, g) = T_0(R, f) + S(R, g)$. Thus, we prove (i) and (ii).

Again by Theorem 2.2 and (ii), we get

$$\begin{aligned} 3T_0(R, f) &\leq \sum_{j=1}^{4} \overline{N}_0(R, \frac{1}{f - a_j}) + \overline{N}_0(R, \frac{1}{f - b}) + S(R, f) \\ &= 2T_0(R, f) + \overline{N}_0(R, \frac{1}{f - b}) + S(R, f), \end{aligned}$$

that is,

$$\overline{N}_0(R, \frac{1}{f - b}) = T_0(R, f) + S(R, f).$$

Similarly, we can get

$$\overline{N}_0(R, \frac{1}{g - b}) = T_0(R, g) + S(R, g).$$

Thus, we obtain (iii).

Therefore, we complete the proof of this lemma. □

**Lemma 3.3** *Let $f, g$ be two distinct transcendental or admissible meromorphic functions on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \leq +\infty$. Suppose that $f$ and $g$ share four distinct values $a_1, a_2, a_3, a_4$ CM on the annulus $\mathbb{A}$. Then $f$ is a Möbius transformation of $g$, two of the shared values, say $a_1$ and $a_2$, are Picard values on $\mathbb{A}$, and the cross ratio $(a_1, a_2, a_3, a_4) = -1$.*

*Proof:* Since $f, g$ share $a_1, a_2, a_3, a_4$ $CM$ on $\mathbb{A}$, from Lemma 3.2(i) and $f, g$ are transcendental or admissible, we have $S(R, f) = S(R, g)$. We assume that there exist three of $\overline{N}_0(R, \frac{1}{f-a_j})(j = 1, 2, 3, 4)$, say $\overline{N}_0(R, \frac{1}{f-a_j})(j = 1, 2, 3)$, such that $\overline{N}_0(R, \frac{1}{f-a_j}) = S(R, f)$, then from Theorem 2.2(ii) we have

$$T_0(R, f) \leq \sum_{j=1}^{3} \overline{N}_0(R, \frac{1}{f - a_j}) + S(R, f) = S(R, f).$$

Thus, we can get a contradiction with the condition of the lemma. Therefore, there are at least two of $\overline{N}_0(R, \frac{1}{f-a_j})(j=1,2,3,4)$, say $\overline{N}_0(R, \frac{1}{f-a_j})(j=1,2)$, such that

$$\overline{N}_0(R, \frac{1}{f-a_1}) = S(R,f), \quad \overline{N}_0(R, \frac{1}{f-a_2}) = S(R,f). \tag{1}$$

Set

$$L(z) = \frac{z-a_3}{z-a_4}\frac{a_2-a_4}{a_2-a_3}.$$

Then $L(a_3)=0, L(a_4)=\infty, L(a_2)=1$ and

$$L(a_1) = \frac{z-a_3}{z-a_4}\frac{a_2-a_4}{a_2-a_3} = (a_1,a_2,a_3,a_4),$$

which is the cross ratio of $a_1,a_2,a_3,a_4$. Let $\Phi(z)=L(f(z)), \Psi(z)=L(g(z))$. From $f(z)\not\equiv g(z)$, we get $\Phi(z)\not\equiv\Psi(z)$. From the assumptions of this lemma, we have $L(a_j)(j=1,2,3,4)$ are shared $CM$ by $\Phi(z)$ and $\Psi(z)$ on $\mathbb{A}$. Thus, we can get that $\Phi(z)$ and $\Psi(z)$ share $0,1,\infty,b$ $CM$ on $\mathbb{A}$, where $b=L(a_1)$. From Lemma 3.2(i) and (1), we have $S(R,\Phi)=S(R,\Psi)$ and

$$\overline{N}_0(R, \frac{1}{\Phi}) \neq S(R,f), \quad \overline{N}_0(R,\Phi) \neq S(R,f). \tag{2}$$

Set

$$H_1 = \frac{\Phi'}{\Phi(\Phi-1)(\Phi-b)} - \frac{\Psi'}{\Psi(\Psi-1)(\Psi-b)}. \tag{3}$$

Suppose that $H_1(z)\not\equiv 0$, from Lemma 3.1, we have $m_0(R,H_1)=S(R,\Phi)$. If $z_0\in\mathbb{A}$ is a point such that $\Phi(z_0)=\Psi(z_0)=L(a_j)$ for some $j=1,2,3,4$, then from (3) we can get that $H_1$ has no pole on $\mathbb{A}$. Thus, from we have $T_0(R,H_1)=m_0(R,H_1)+N_0(R,H_1)-2m(1,H_1)=S(R,\Phi)$. If $z_1\in\mathbb{A}$ is a pole of $\Phi$ with multiplicity $p$, then it must be a pole of $\Psi$ with multiplicity $p$. Hence from (3) we have that $z_1$ is a zero of $H$ with multiplicities as least $3p-(p+1)=2p-1$. Thus, we get

$$\overline{N}_0(R, \frac{1}{\Phi}) \leq N_0(R, \frac{1}{H_1}) \leq T_0(R,H_1) + 2m(1,H_1) + O(1) = S(R,f).$$

Therefore, we can get a contradiction with (2). Thus, we can get that $H_1(z)\equiv 0$.

Set

$$H_2 = \frac{\Phi\Phi'}{(\Phi-1)(\Phi-b)} - \frac{\Psi\Psi'}{(\Psi-1)(\Psi-b)}. \tag{4}$$

By using the same argument as in the above, we can get that $H_2(z)\equiv 0$. From $H_1(z)\equiv H_2(z)\equiv 0$ we have $\Phi^2(z)\equiv\Psi^2(z)$. Since $\Phi(z)\not\equiv\Psi(z)$, we have $\Phi(z)=-\Psi(z)$. Thus, both 1 and $-1$ are Picard values of $\Phi$ and $\Psi$ on $\mathbb{A}$. From Lemma 3.2(iii), we get that $b=-1$. Hence we have $L(a_1)=(a_1,a_2,a_3,a_4)=-1$. Therefore we get that $a_1$ and $a_2$ are Picard values of $f$ and $g$ on $\mathbb{A}$ and $L(f(z))=L(g(z))$. Thus, we get that $f$ is a Möbius transformation of $g$.

This completes the proof of this lemma. □

# 4 The proof of Theorem 1.2

Suppose that $f \not\equiv g$ and none of the $a_j (j = 1, 2, 3, 4)$ is $\infty$. From Lemma 3.2, we have $S(R, f) = S(R, g)$. Set $S(R) := S(R, f) = S(R, g)$. Let $\varphi$ be the function expressed as follows

$$\varphi = \frac{f'g'(f-g)^2}{(f-a_1)(f-a_2)(f-a_3)(f-a_4)(g-a_1)(g-a_2)(g-a_3)(g-a_4)}. \tag{5}$$

Then we can get $\varphi \not\equiv 0$. We will show that $T_0(R, \varphi) = S(R)$ as follows.

Suppose $z_0 \in \mathbb{A}$ and $f(z_0) = a_1$ (or $a_2, a_3, a_4$) with multiplicity $p$ and $g(z_0) = a_1$ (or $a_2, a_3, a_4$) with multiplicity $q$. From (5), we can get

$$\varphi(z) = O\left((z - z_0)^{2\min(p,q)-2}\right).$$

Hence $\varphi$ is an analytic function on $\mathbb{A}$. Then, we have

$$\begin{aligned}
T_0(R,\varphi) &= m_0(R,\varphi) \\
&\le m_0\left(R, \frac{f'}{(f-a_2)(f-a_3)(f-a_4)}\right) + m_0\left(R, \frac{f'}{(f-a_1)(f-a_2)(f-a_3)}\right) \\
&\quad + m_0\left(R, \frac{f'}{(f-a_1)(f-a_2)(f-a_3)(f-a_4)}\right) \\
&\quad + m_0\left(R, \frac{f'P_1(f)}{(f-a_1)(f-a_2)(f-a_3)(f-a_4)}\right) \\
&\quad + m_0\left(R, \frac{g'}{(g-a_2)(g-a_3)(g-a_4)}\right) + m_0\left(R, \frac{g'}{(g-a_1)(g-a_2)(g-a_3)}\right) \\
&\quad + m_0\left(R, \frac{g'}{(g-a_1)(g-a_2)(g-a_3)(g-a_4)}\right) \\
&\quad + m_0\left(R, \frac{g'P_2(g)}{(g-a_1)(g-a_2)(g-a_3)(g-a_4)}\right) + O(1) \\
&= S(R,f) + S(R,g) = S(R),
\end{aligned}$$

where $P_1(f)$ is a polynomial of degree no more than 2 in $f$ and $P_2(g)$ is a polynomial of degree no more than 2 in $g$. By Lemma 3.2 (iii), we have

$$m_0\left(R, \frac{1}{f-b_j}\right) = S(R,f), \quad m_0\left(R, \frac{1}{g-b_j}\right) = S(R,g), \tag{6}$$

for any $b_j \in S (j = 1, 2, \ldots, m)$.

Set

$$\varphi_1 := \frac{(g-b_1)\cdots(g-b_m)}{(f-b_1)\cdots(f-b_m)} \cdot \left(\frac{g'(f-g)}{(g-a_1)\cdots(g-a_4)}\right)^m$$

and

$$\varphi_2 := \frac{(f-b_1)\cdots(f-b_m)}{(g-b_1)\cdots(g-b_m)} \cdot \left(\frac{f'(f-g)}{(f-a_1)\cdots(f-a_4)}\right)^m.$$

By Lemma 3.1 and (6), we can get that

$$m_0\left(R, \frac{1}{f-b_j}\cdot\frac{g'(f-g)(g-b_j)}{(g-a_1)\cdots(g-a_4)}\right) = S(R)$$

and

$$m_0\left(R, \frac{1}{g-b_j}\cdot\frac{f'(f-g)(f-b_j)}{(f-a_1)\cdots(f-a_4)}\right) = S(R).$$

From the definitions of $\varphi_1$ and $\varphi_2$, we get $m_0(R,\varphi_j) = S(R), j = 1,2$. By Lemma 3.2(iii), we see that "almost all" of poles and $b_j$-points of $f$ and $g$ on the annulus $\mathbb{A}$ are simple. Since $f, g$ share the four distinct values $a_j, j = 1,2,3,4$ on the annulus $\mathbb{A}$ and $\overline{E}_{\mathbb{A}}(S,f) \subset \overline{E}_{\mathbb{A}}(S,g)$, we can easily get that $N_0(R,\varphi_1) = S(R)$. Therefore, we have

$$T_0(R,\varphi_1) = S(R). \tag{7}$$

Since $\varphi_1\varphi_2 \equiv \varphi^m$ and $T_0(R,\varphi) = S(R)$, we can have

$$T_0(R,\varphi_2) = S(R). \tag{8}$$

Let $\Gamma^{pq}_{\mathbb{A}}(a_j)$ be the set of those $a_j$-points of $f$ and $g$ on the annulus $\mathbb{A}$ such that the multiplicities of $f$ and $g$ at these points are $p$ and $q$, respectively. For any $z_0 \in \Gamma^{pq}_{\mathbb{A}}(a_1)$, by simple computation, we have

$$\varphi_1(z_0) = \left(q\cdot\frac{f'(z_0)-g'(z_0)}{(a_1-a_2)(a_1-a_3)(a_1-a_4)}\right)^m$$

and

$$\varphi_2(z_0) = \left(p\cdot\frac{f'(z_0)-g'(z_0)}{(a_1-a_2)(a_1-a_3)(a_1-a_4)}\right)^m.$$

Hence

$$\frac{1}{q^m}\varphi_1(z_0) - \frac{1}{p^m}\varphi_2(z_0) = 0. \tag{9}$$

Similarly, we can see that (9) holds for any $z_0 \in \Gamma^{pq}_{\mathbb{A}}(a_j), j = 2,3,4$.

Now we discuss two cases as follows.

**Case 1.** Suppose that $\varphi^{pq} := \frac{1}{q^m}\varphi_1 - \frac{1}{p^m}\varphi_2 \not\equiv 0$, for all positive integers $p, q$.

Next, for $j = 1,2,3,4$, we denote by

$$N^{pq}_0(R,\frac{1}{f-a_j}) = N^{pq}_1(R,\frac{1}{f-a_j}) + N^{pq}_2(R,\frac{1}{f-a_j}),$$

$$N^{pq}_1(R,\frac{1}{f-a_j}) = \int_{\frac{1}{R}}^1 \frac{n^{pq}_1(t,\frac{1}{f-a_j})}{t}dt, \quad N^{pq}_2(R,\frac{1}{f-a_j}) = \int_1^R \frac{n^{pq}_2(t,\frac{1}{f-a_j})}{t}dt,$$

where $n^{pq}_1(t,\frac{1}{f-a_j})$( $n^{pq}_2(t,\frac{1}{f-a_j})$) is the counting function of zeros of the function $f-a_j$ in $\{z : t < |z| \le 1\}$ (resp. $\{z : 1 < |z| \le t\}$) with respect to the set $\Gamma^{pq}_{\mathbb{A}}(a_j)$, similarly, we

have $\overline{N}_0^{pq}(R,\frac{1}{f-a_j}), \overline{N}_1^{pq}(R,\frac{1}{f-a_j})$ and $\overline{N}_2^{pq}(R,\frac{1}{f-a_j})$. Thus, we have

$$\begin{aligned} N_0\left(R,\frac{1}{f-a_j}\right) &= N_1\left(R,\frac{1}{f-a_j}\right)+N_2\left(R,\frac{1}{f-a_j}\right) \\ &= \sum_{p,q=1}^{\infty}\left(N_1^{pq}\left(R,\frac{1}{f-a_j}\right)+N_2^{pq}\left(R,\frac{1}{f-a_j}\right)\right) \\ &= \sum_{p,q=1}^{\infty} N_0^{pq}\left(R,\frac{1}{f-a_j}\right) \end{aligned}$$

and

$$\overline{N}_0\left(R,\frac{1}{f-a_j}\right)=\sum_{p,q=1}^{\infty}\overline{N}_0^{pq}\left(R,\frac{1}{f-a_j}\right).$$

Since $f,g$ are transcendental meromorphic on the annulus $\mathbb{A}$, and from the above two equations, (6),(7) and (8), we can see that $T_0(R,\varphi^{pq})=S(R,f)+S(R,g)$. And by (9) each zero of $f-a_j$ is a zero of $\varphi^{pq}$, so with the help of $\varphi^{pq}\not\equiv 0$, we can get

$$\begin{aligned} \overline{N}_0^{pq}\left(R,\frac{1}{f-a_j}\right) &\leq \overline{N}_0^{pq}\left(R,\frac{1}{\varphi^{pq}}\right)\leq T_0\left(R,\frac{1}{\varphi^{pq}}\right)+O(1) \\ &\leq T_0(R,\varphi^{pq})+O(1)=S(R,f)+S(R,g):=S(R), \end{aligned}$$

for some $p,q$. By Lemma 3.2 (ii), we have $T_0(R,f)+S(R,f)=T_0(R,g)+S(R,g)$. Thus, from the definition of $S(R)$, we can get $T_0(R,f)=T_0(R,g)+S(R)$. Therefore, for a positive integer $k(>4)$, we have

$$\begin{aligned} \overline{N}_0\left(R,\frac{1}{f-a_j}\right) &= \sum_{\max(p,q)\geq k}\overline{N}_0^{pq}\left(R,\frac{1}{f-a_j}\right)+S(R,f) \\ &\leq \frac{1}{k}\left(\sum_{\max(p,q)\geq k}N_0^{pq}\left(R,\frac{1}{f-a_j}\right)+\sum_{\max(p,q)\geq k}N_0^{pq}\left(R,\frac{1}{g-a_j}\right)\right) \\ &\quad +S(R,f) \\ &\leq \frac{1}{k}\left(N_0\left(R,\frac{1}{f-a_j}\right)+N_0\left(R,\frac{1}{g-a_j}\right)\right)+S(R,f) \\ &\leq \frac{2}{k}T(R,f)+S(R),\quad j=1,2,3,4. \end{aligned}$$

By the above inequality and Lemma 3.2(ii), we can get

$$T_0(R,f)\leq\frac{4}{k}T_0(R,f)+S(R). \tag{10}$$

Since $k(>4)$ is a positive integer, that is, $\frac{4}{k}<1$, from $f$ is transcendental or admissible on $\mathbb{A}$, thus, we can get a contradiction.

**Case 2.** Suppose that $\varphi^{pq} := \frac{1}{q^m}\varphi_1 - \frac{1}{p^m}\varphi_2 \equiv 0$, for some positive integers $p, q$. Thus, we have

$$\left(\frac{p}{q}\right)^m \cdot \frac{(g-b_1)^2\cdots(g-b_m)^2}{(f-b_1)^2\cdots(f-b_m)^2} \equiv \left(\frac{f'(g-a_1)\cdots(g-a_4)}{g'(f-a_1)\cdots(f-a_4)}\right)^m. \tag{11}$$

We will consider the two following subcases:

**Subcase 2.1.** $p \neq q$. Without loss of generality, we may assume that $p < q$. For some two positive integers $p_1$ and $q_1$, if $z_1 \in \Gamma_{\mathbb{A}}^{p_1 q_1}(a_j)$ for some $j \in \{1,2,3,4\}$, then (11) implies that $\frac{p}{q} = \frac{p_1}{q_1}$. Hence $q_1 > p_1 \geq 1$, and $q_1 \geq 2$ which means that any $a_j$-points $(j = 1,2,3,4)$ of $g$ on $\mathbb{A}$ are multiple. By Lemma 3.2 and $f, g$ are transcendental or admissible on $\mathbb{A}$, we can get

$$\begin{aligned} 2T_0(R,g) &= \sum_{j=1}^{4} \overline{N}_0\left(R, \frac{1}{g-a_j}\right) + S(R,g) \\ &\leq \frac{1}{2}\sum_{j=1}^{4} N_0\left(R, \frac{1}{g-a_j}\right) + S(R,g) \\ &\leq 2T_0(R,g) + S(R,g). \end{aligned}$$

Thus, we can get the following equalities easily

$$T_0(R,g) = N_0\left(R, \frac{1}{g-a_j}\right) + S(R), \quad j = 1,2,3,4 \tag{12}$$

and

$$N_0\left(R, \frac{1}{g-a_j}\right) = 2\overline{N}_0\left(R, \frac{1}{g-a_j}\right) + S(R), \quad j = 1,2,3,4. \tag{13}$$

From (12) and (13), we can see that "almost all" of $a_j$-points of $g$ have multiplicity 2, and "almost all" of $a_j$-points of $f$ are simple on the annulus $\mathbb{A}$. Without loss of generality, we may assume that $f$ and $g$ attain the values $a_3$ and $a_4$ on the annulus $\mathbb{A}$. Set

$$\phi_1 := \frac{2f'(f-a_4)}{(f-a_1)(f-a_2)(f-a_3)} - \frac{g'(g-a_4)}{(g-a_1)(g-a_2)(g-a_3)}$$

and

$$\phi_2 := \frac{2f'(f-a_3)}{(f-a_1)(f-a_2)(f-a_4)} - \frac{g'(g-a_3)}{(g-a_1)(g-a_2)(g-a_4)}.$$

Since $\phi_i (i = 1,2)$ is analytic at the poles of $f$ and of $g$ and also at those common $a_j$-points of $f$ and $g$ which have multiplicity 1 with respect to $f$ and multiplicity 2 with respect to $g$, by Lemma 3.1, we have $T_0(R,\phi_i) = S(R,f), i = 1,2$. If $\phi_i \not\equiv 0$, then $N_0\left(r, \frac{1}{f-a_4}\right) \leq N_0\left(R, \frac{1}{\phi_1}\right) = S(R,f)$, which contradicts to equation (13). Then $\phi_1 \equiv 0$. Similarly, we have $\phi_2 \equiv 0$. Therefore, from the definitions of $\phi_1$ and $\phi_2$, we have

$$\left(\frac{f-a_4}{f-a_3}\right)^2 \equiv \left(\frac{g-a_4}{g-a_3}\right)^2. \tag{14}$$

Since $f \not\equiv g$, and from (14), we have

$$\frac{f-a_4}{f-a_3} \equiv -\frac{g-a_4}{g-a_3},$$

which implies that $f$ and $g$ share $a_3, a_4$ $CM$ on the annulus $\mathbb{A}$. Since $f$ and $g$ assume the value $a_3$ there exist positive integers $p_1, q_1$ such that $\Gamma_{\mathbb{A}}^{p_1 q_1}(a_3) \neq \emptyset$. From the considerations above we get $q_1 > p_1$, contradicting the fact that $f$ and $g$ share $a_3$ $CM$.

**Subcase 2.2.** $p = q$.

In this subcase, (10) becomes

$$\frac{(g-b_1)^2 \cdots (g-b_m)^2}{(f-b_1)^2 \cdots (f-b_m)^2} \equiv \left(\frac{f'(g-a_1)\cdots(g-a_4)}{g'(f-a_1)\cdots(f-a_4)}\right)^m.$$

which implies that $f$ and $g$ share the four values $a_j (j = 1, 2, 3, 4)$ $CM$ on the annulus $\mathbb{A}$. From the conditions of this lemma and applying Lemma 3.3, $g$ is a Möbius transformation of $f$ on $\mathbb{A}$. Furthermore, two of the four values, say $a_1, a_2$ are Picard exceptional values of $f$ and $g$ on the annulus $\mathbb{A}$. Set

$$\Delta_1 := \frac{f'(f-a_4)}{(f-a_1)(f-a_2)(f-a_3)} - \frac{g'(g-a_4)}{(g-a_1)(g-a_2)(g-a_3)}$$

and

$$\Delta_2 := \frac{f'(f-a_3)}{(f-a_1)(f-a_2)(f-a_4)} - \frac{g'(g-a_3)}{(g-a_1)(g-a_2)(g-a_4)}.$$

Using the same argument as in **Subcase** 2.1 for $\Delta_1, \Delta_2$, we can get

$$\frac{f-a_3}{f-a_4} \equiv -\frac{g-a_3}{g-a_4}.$$

We take the Möbius transformations $T, M$ and $L$ satisfying

$$T(\omega) := \frac{w-a_3}{w-a_4}, \qquad M(w) := -w \qquad and \qquad L := T^{-1} \circ M \circ T.$$

Then we have

$$T \circ f = -T \circ g, \qquad hence \qquad g = L \circ f.$$

Thus, we can see that $a_3$ and $a_4$ are the fixed points of $L$. Therefore, there exist no fixed points of $L$ in the set $S$. If some $b \in S$ is given. Then from $b \neq a_1, a_2$, there exists a $z_0 \in \mathbb{C}$ such that $b = f(z_0)$, and from $\overline{E}_{\mathbb{A}}(S, f) \subseteq \overline{E}_{\mathbb{A}}(S, g)$ we obtain

$$L(b) = L(f(z_0)) = g(z_0) \in S.$$

So $S$ is invariant under $L$. Furthermore, we have $L \circ L = I$ where $I$ denotes the identical transformation. Thus, we can get that $S$ must contain an even number of values.

Thus, we complete the proof of Theorem 1.2.

# References


[1] S. Axler, *Harmomic functions from a complex analysis viewpoit*, Amer. Math. Monthly **93** (1986), 246-258.

[2] A. Banerjee, *Weighted sharing of a small function by a meromorphic function and its derivativer*, Comput. Math. Appl. **53** (2007), 1750-1761.

[3] T. B. Cao and H. X. Yi, *On the uniqueness of meromorphic functions that share four values in one angular domain*, J. Math. Anal. Appl. **358** (2009), 81-97.

[4] T. B. Cao and H. X. Yi, *Uniqueness theorems of meromorphic functions sharing sets $IM$ on annuli*, Acta Mathematica Sinica (Chinese Series), **54** (4) (2011), 623-632. (in Chinese).

[5] T. B. Cao, H. X. Yi and H. Y. Xu, *On the multiple values and uniqueness of meromorphic functions on annuli*, Comput. Math. Appl. **58** (2009), 1457-1465.

[6] W. Hayman, *Meromorphic functions*, Oxford: Clarendon Press, 1964.

[7] A. Ya. Khrystiyanyn and A. A. Kondratyuk, *On the Nevanlinna theory for meromorphic functions on annuli. I*, Mat. Stud. **23**(2005), No.1, 19-30.

[8] A. Ya. Khrystiyanyn and A. A. Kondratyuk, *On the Nevanlinna theory for meromorphic functions on annuli. II*, Mat. Stud. **24** (2005), No.2, 57-68.

[9] A. A. Kondratyuk and I. Laine, *Meromorphic functions in multiply connected domains*, Laine, Ilpo (ed.), fourier series methods in complex analysis. Proceedings of the workshop, Mekrijärvi, Finland, July 24-29, 2005. Joensuu: University of Joensuu, Department of Mathematics (ISBN 952-458-888-9/pbk). Report series. Department of mathematics, University of Joensuu 10, 9-111 (2006).

[10] R. Korhonen, *Nevanlinna theory in an annulus, value distribution theory and related topics*, Adv. Complex Anal. Appl. **3** (2004), 167-179.

[11] W. C. Lin, S. Mori and K. Tohge, *Uniqueness theorems in an angular domain*, Tohoku Math. J. 58(2006), 509-527.

[12] M. Lund and Z. Ye, *Logarithmic derivatives in annuli*, J. Math. Anal. Appl. **356** (2009), 441-452

[13] M. Lund and Z. Ye, *Nevanlinna theory of meromorphic functions on annuli*, Sci. China. Math. **53** (2010), 547-554.

[14] R. Nevanlinna, *Eindentig keitssätze in der theorie der meromorphen funktionen*, Acta. Math. **48** (1926), 367-391.

[15] H. Y. Xu and Z. X. Xuan, The uniqueness of analytic functions on annuli sharing some values, Abstract and Applied Analysis 2012 (2012), Art. 896596, 1-13.

[16] H. X. Yi and C. C. Yang, *Uniqueness theory of meromorphic functions*, Science Press 1995/Kluwer 2003.

[17] L. Yang, *Value distribution theory*, Berlin: Springer-Verlag/ Beijing: Science Press, 1993.

[18] X. Y. Zhang, J. F. Chen and W. C. Lin, *Entire or meromorphic functions sharing one value*, Comput. Math. Appl. **56** (2008), 1876-1883.

[19] J. H. Zheng, *On uniqueness of meromorphic functions with shared values in some angular domains*, Canad J. Math. **47** (2004), 152-160.

[20] J. H. Zheng, *On uniqueness of meromorphic functions with shared values in one angular domains*, Complex Var. Elliptic Equ. **48** (2003), 777-785.

# On the Henstock-Pettis Integral for Fuzzy Number Valued Functions

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**Abstract.** In this paper, we show that the fuzzy Henstock-Pettis integration of $n-$dimensional fuzzy-number-valued function could be translated into the sum of the fuzzy Henstock-Pettis integration of a $n-$dimensional fuzzy-number-valued function and the Henstock-Pettis integral of a vector-valued function which valued in the kernel sets. In addition, we give the Komlós-type convergence theorems for such integrals.

**Keywords.** Fuzzy number, fuzzy Henstock-Pettis integral, representation theorems, convergence theorems.

**AMS (MOS) subject classification:** Primary 26E50; Secondary 28B20.

# 1 Introduction

It is well-known that the Henstock integral includes the Riemann, improper Riemann, Lebesgue and Newton integrals [9, 12]. It is also equal to the Denjoy and Perron integrals [14]. In the theory of integrals, there are some integrals based on the Banach space-valued functions such as Pettis and Bochner integrals [3, 14, 21]. In particular, Ziat [28, 29] and Amri and Hess [1] presented a characterization of Pettis integral having as their values convex weakly compact subsets of a Banach space. Bochner and Pettis integrals are all defined by using the Lebesgue integrability of the support functions. The integrals of fuzzy-number-valued functions, as a natural generalization of set-valued functions, have been discussed by Puri and Ralescu [19], Kaleva [10], and other authors [7, 24, 25, 27]. Recently, Wu and Gong [6, 8] discussed the fuzzy Henstock integrals of fuzzy-number-valued functions which extended Kaleva [10] integration. However, for a fuzzy valued function in the $n-$dimensional fuzzy number space $E^n$, the integral and its characteristic theorems have not defined or discussed. In [2], the authors shown that a fuzzy-number valued function is fuzzy Henstock integrable if and only if it can be represented by a sum of a fuzzy McShane integrable fuzzy-number valued function and a fuzzy Henstock integrable fuzzy number valued function generated by a Henstock integrable function. In 2014, K. Musiał [18] established the following decomposition theorem for fuzzy mappings with values in a Banach space: a fuzzy mapping is fuzzy Henstock integrable if and only if it can be represented as a sum of a fuzzy McShane integrable fuzzy mapping and of a fuzzy Henstock integrable fuzzy mapping generated by a Henstock integrable function. As a continuation of our previous work [16, 17, 22], in this paper, we continue to develop the theory of Henstock-Pettis integrals in fuzzy number spaces. By means of replacing the Lebesgue integrability of support functions with their Henstock integrability, we give the definitions of Henstock-Pettis integral and Aumann-Henstock-Pettis integral for compact convex set-valued functions. In addition, the relationships among Henstock-Pettis integral, Aumann-Henstock integral and Pettis integral are investigated. Furthermore, we present the Henstock-Pettis integral, fuzzy Henstock-Pettis integral and Aumann-Henstock-Pettis integral of $n-$dimensional fuzzy-number-valued functions, and the relationships among them are studied. At the same time, the representation theorems and the calculations of fuzzy Henstock-Pettis integral are given. It shows that the fuzzy Henstock-Pettis integration of a $n-$dimensional fuzzy-number-valued function equals the sum of the Henstock-Pettis integration of an $n-$dimensional fuzzy-number-valued function and the Henstock-Pettis integration of a vector-valued function which valued in the kernel sets.

The rest of the paper is organized as follows. In section 2, the definitions of Pettis integral, Henstock-Pettis integral and Aumann-Henstock integral for compact convex set-valued functions are

given. In section 3, we disscus the characterization of Henstock-Pettis integral, fuzzy Henstock-Pettis integral and Aumann-Henstock integral for fuzzy-number-valued functions, and we give the representation theorems of fuzzy Henstock-Pettis integral. In section 4, we present a Komlós-type convergence theorem to the fuzzy Henstock-Pettis integral and give an existence theorem for a kind of fuzzy integral inclusion. And in section 5, we present some concluding remarks.

## 2 Preliminaries

Let $T$ be the closed interval on the real line $R$, i.e., $T = [a, b]$ $(a, b \in R)$. $|T|$ denotes the length of $T$.

Throughout this paper, we use $P_k(R^n)$ to denote the family of all nonempty compact convex subsets of $R^n$. For $A, B \in P_k(R^n), k \in R$, the addition and scalar multiplication are defined by the equations as follows respectively:

$$A + B = \big\{x + y \mid x \in A, y \in B\big\}, \qquad aA = \big\{ax \mid x \in A\big\}.$$

In addition, for $A, B \in P_k(R^n)$, the Hausdorff metric between them defined by:

$$d(A, B) = \max\big\{\sup_{a\in A}\inf_{b\in B}\|a - b\|, \sup_{b\in B}\inf_{a\in A}\|b - a\|\big\}.$$

A compact convex set-valued function $F : T \to P_k(R^n)$ is said to be measurable if $\{t \in T|F(t)\cap O \neq \phi\}$ is a measurable set for any open subset $O \subset R^n$. $F$ is said to be scalarly measurable if the map $\sigma(x, F(\cdot))$ is measurable for every $x \in S^{n-1}$. Certainly, a compact convex set-valued function $F : T \to P_k(R^n)$ is measurable if it is scalarly measurable.

A function $f : T \to R^n$ is called a selection of $F$ if $f(t) \in F(t)$ for any $t \in T$. A selection $f$ is said to be measurable if the function $f$ is strongly measurable, i.e., $f$ is a limit of an almost everywhere convergent sequence of measurable simple functions.

A compact convex set-valued function $F : T \to P_k(R^n)$ is said to be graph measurable if the set $\{(t, x) \in T \times R^n|x \in F(t)\}$ is a member of the product $\sigma-$algebra generated by $\mathcal{L}$ and the Borel subsets of $R^n$ in the norm topology. Here $\mathcal{L}$ denotes the family of all Lebesgue measurable subsets of $T$.

**Definition 2.1** ([27]). For $A \in P_k(R^n)$, $x \in S^{n-1}$, the support function of $A$ is defined by

$$\sigma(x, A) = \sup_{y\in A}\langle y, x\rangle,$$

where $S^{n-1}$ denotes the unit sphere of $R^n$, $\langle\cdot,\cdot\rangle$ is the inner product in $R^n$.

Next, we shall give the definitions of Pettis integral, Henstock-Pettis integral and Aumann-Henstock-Pettis integral for compact convex set-valued functions.

**Definition 2.2.** A set-valued function $F : T \to P_k(R^n)$ is said to be Henstock integrable to $I \in P_k(R^n)$ if for every $\varepsilon > 0$ there is a function $\delta(x) > 0$ such that for any $\delta-$fine division $\Pi = \{\xi_i, [x_{i-1}, x_i]\}$ of $T$, we have

$$d(I, \sum_i F(\xi_i)(x_i - x_{i-1})) < \varepsilon,$$

and write $(H)\int_T F(x)dx = I$.

**Lemma 2.3** ([27]). *If $A \in P_k(R^n), x \in S^{n-1}$, then $A = \big\{y \in R^n|\langle y, x\rangle \leq \sigma(x, A), x \in S^{n-1}\big\}$.*

**Lemma 2.4** ([26]). *If $A_r \in P_k(R^n), \{A_{r_m}\} \subset P_k(R^n)$, where $r_m$ is converging nondecreasingly to $r$ and $A_{r_m} \supset A_{r_{m+1}} \supset A_r (m = 1, 2, \cdots)$ for any $x \in S^{n-1}$, then $A_r = \bigcap_{m=1}^{\infty} A_{r_m}$ if $\sigma(x, A_{r_m})$ converge to $\sigma(x, A_r)$.*

**Theorem 2.5.** *A set-valued function $F : T \to P_k(R^n)$ is Henstock integrable on $T$ iff the real-valued function $\sigma\big(x, F(t)\big)$ is Henstock integrable uniformly on $T$ for any $x \in S^{n-1}$, and*

$$\sigma\big(x, (H)\int_T F(t)\mathrm{d}t\big) = (H)\int_T \sigma\big(x, F(t)\big)\mathrm{d}t. \tag{2.1}$$

**Definition 2.6.** Let $F : T \to P_k(R^n)$ be a measurable set-valued function. $F$ is said to be Pettis (Henstock-Pettis) integrable on $T$ if there is a nonempty set $A \in P_k(R^n)$ such that for any $x \in S^{n-1}$ we have

$$\sigma(x, A) = (L)\int_T \sigma(x, F(t))\mathrm{d}t$$

$$(\sigma(x, A) = (H)\int_T \sigma(x, F(t))\mathrm{d}t),$$

and write $A = (P)\int_T F(t)\mathrm{d}t$ $(A = (wH)\int_T F(t)\mathrm{d}t)$.

In particular, if the set-valued function above is defined by $F : T \to R^n$, then the set $A$ will become a vector in $R^n$, and for any $x \in S^{n-1}$ we have

$$< x, A >= (L)\int_T < x, F(t) > \mathrm{d}t$$

$$(< x, A >= (H)\int_T < x, F(t) > \mathrm{d}t).$$

In this case, $F$ is also said to be Pettis (Henstock-Pettis) integrable on $T$.

**Theorem 2.7.** *If $F : T \to P_k(R^n)$ is a measurable set-valued function, then the family of measurable selections of $F$ is not empty.*

*Proof.* Since $R^n$ is a separable space, we can prove the theorem easily. □

Now, we use $s_H(F)$ to denote the family of Henstock-Pettis integrable selections and $s_P(F)$ to denote the family of Pettis integrable selections of $F$.

**Definition 2.8.** The Aumann-Henstock integral of a measurable set-valued function $F : T \to P_k(R^n)$ defined by

$$(AH)\int_T F(t)\mathrm{d}t = \{(HP)\int_T f(t)\mathrm{d}t | f \in s_H(F)\}.$$

**Definition 2.9.** Pick a set-valued function $F : T \to P_k(R^n)$ and let $I \subset T$. The function $f : A \to P_k(R^n)$ is the weak derivative of $F$ on $T$ if the Banach valued function $(\sigma(x, F))'$ is differentiable almost everywhere on $I$ and $(\sigma(x, F))' = \sigma(x, f))$ almost everywhere on $T$.

*Example.* Let $F : T \to P_k(R^n)$ be weakly differentiable. Then its weak derivative $F'$ is Henstock-Pettis integrable and

$$(HP)\int_a^s F'(t)\mathrm{d}t = F(s) - F(a), s \in T.$$

Indeed, $F$ has the weak derivative at a point $t$ means that there is a point $F'(t) \in R^n$ such that for any $x \in S^{n-1}$, we have

$$\lim_{\triangle t \to 0} \frac{< x, F(t + \triangle t) > - < x, F(t) >}{\triangle t} =< x, F'(t) > .$$

Since $< x, F >$ is differentiable, so we have

$$< x, F(s) > - < x, F(a) >= (H)\int_a^s < x, F >' \mathrm{d}t, s \in T.$$

On the other hand, $< x, F >' = < x, F' >$, it implies that

$$< x, F(s) > - < x, F(a) >= (H)\int_a^s < x, F'(t) > \mathrm{d}t.$$

That is

$$< x, F(s) - F(a) >= (H)\int_a^s < x, F'(t) > \mathrm{d}t.$$

Hence

$$F(s) - F(a) = (HP)\int_a^s F'(t)\mathrm{d}t. \quad \square$$

**Theorem 2.10.** *If all measurable selections of $F : T \to P_k(R^n)$ are Henstock-Pettis integrable and $\sigma(x, F(t))$ is Henstock integrable, then $(AH)\int_T F(t)\mathrm{d}t$ is a compact convex set.*

*Proof.* Since $\sigma(x, F(t))$ is Henstock integrable, so $\sigma(x, F(t))$ is measurable. Now fix a measurable selection $f$ of $F$ and let $G(t) = F(t) - f(t)$. Since $f$ is Henstock-Pettis integrable, so $G$ is Aumann-Henstock integrable.

Let $I_T = (AH)\int_T G(t)\mathrm{d}t$ and $D$ be a countable dense subset of $S^{n-1}$. We prove the convexity of $I_T$ first. It can be proved that

$$(AH)\int_T G(t)\mathrm{d}t = \{(wH)\int_T g(t)\mathrm{d}t | g \in s_H(F - f)\}$$

is a convex set. In fact, for any $A, B \in I_T$, there exist $g_1(t), g_2(t) \in s_H(F - f)$ such that

$$A = (HP)\int_T g_1(t)\mathrm{d}t,\ B = (HP)\int_T g_2(t)\mathrm{d}t. \tag{2.2}$$

In addition, for any $\lambda \in [0, 1]$,

$$\begin{aligned} \lambda A + (1-\lambda)B &= \lambda(HP)\int_T g_1(t)\mathrm{d}t + (1-\lambda)\int_T g_2(t)\mathrm{d}t \\ &= (HP)\int_T (\lambda g_1(t) + (1-\lambda)g_2(t))\mathrm{d}t. \end{aligned}$$

That is, $\lambda A + (1 - \lambda)B \in I_T$.

In order to prove the compactness of $I_T$, we take a sequence of points $x_n \in I_T$, and then there exists $g_n \in s_H(G)$ with $x_n = (HP)\int_T g_n(t)\mathrm{d}t$. For each $n \in N, t \in T, x \in S^{n-1}$, we have the inequalities

$$-\sigma(-x, G(t)) \leq < x, g_n(t) > \leq \sigma(x, G(t)). \tag{2.3}$$

Since $f \in s_H(F)$ and the null function is included in $G(t)$, $\sigma(x, G(t))$ is nonnegative Henstock integrable. It implies that the support function $\sigma(x, G(t))$ is Lebesgue integrable. Thus, each $< x, g_n >$ is Lebesgue integrable and

$$(L)\int_T | < x, g_n(t) > |\mathrm{d}t \leq (L)\int_T \sigma(x, G(t))\mathrm{d}t + (L)\int_T \sigma(-x, G(t))\mathrm{d}t.$$

Furthermore, due to the countability of $D$ and $L_1-$boundeness of each $< x, g_n >$ we can find there exist $h_n \in conv\{g_n, g_{n+1}, \cdots\}$, such that for each $x \in D$ the sequence $< x, h_n >$ is almost everywhere convergent to a measurable function $h_x$.

As for each $t$ and $n$ we have $h_n(t) \in G(t)$ and $G(t)$ is compact, there is a cluster point $h(t) \in G(t)$. It follows that there is a set $N$ of Lebesgue measure zero such that for any $x \in D$ and $t \notin N$ we have

$$< x, h(t) >= \lim_{n\to\infty} < x, h_n(t) >= h_x(t).$$

Taking into formula Eq. (2.3) and the Lebesgue dominated convergence theorem, we have for any $x \in S^{n-1}$,

$$\begin{aligned} \lim_{n\to\infty} < x, (HP)\int_T h_n(t)\mathrm{d}t > &= \lim_{n\to\infty} (L)\int_T < x, h_n(t) > \mathrm{d}t \\ &= (L)\int_T < x, h(t) > \mathrm{d}t \\ &= < x, (HP)\int_T h(t)\mathrm{d}t > . \end{aligned} \tag{2.4}$$

We write $y_n = (HP)\int_T h_n(t)\mathrm{d}t$, then $y_n \in I_T, y_n \in conv\{x_n, x_{n+1}, \cdots\}$ and the sequence $\{y_n\}$ convergent to $y_0 = (HP)\int_T h(t)\mathrm{d}t$. Thus, given an arbitrary sequence $\{x_n\}$, $x_n \in I_T$, there is a convex combination of points $y_n \in conv\{x_n, x_{n+1}, \cdots\}$ and $y_0 \in I_T$ such that $y_n$ converge to $y_0$. Consequently, the set $I_T$ is compact, i.e., there exists $y_0 = (HP)\int_T h(t)\mathrm{d}t \in I_T$ such that $\lim_{n\to\infty} y_n = y_0$. □

# 3 The Henstock-Pettis integral for fuzzy number valued functions

In this section, we give the definition of Henstock-Pettis integral of fuzzy-number-valued functions and its representation theorems.

**Definition 3.1** ([4, 23]). Let $E^n = \{u | u : R^n \to [0,1]\}$. For any $u \in E^n$, $u$ is said to be a $n-$dimensional fuzzy number if the following conditions are satisfied:

(1) $u$ is a normal fuzzy set, i.e., there exists an $x_0 \in R^n$, such that $u(x_0) = 1$;

(2) $u$ is a convex fuzzy set, i.e., $u\big(tx + (1-t)y\big) \geq \min\big\{u(x), u(y)\big\}$ for any $x, y \in R^n, t \in [0,1]$;

(3) $u$ is upper semi-continuous;

(4) $\mathrm{supp}u = \overline{\{x \in R^n \mid u(x) > 0\}}$ is compact, here $\overline{A}$ denotes the closure of $A$.

For $r \in (0,1]$, denote $[u]^r = \{x \in R^n \mid u(x) \geq r\}$ and we call it the $r-$level set of $u$, and $[u]^0 = \overline{\bigcup_{r\in(0,1]} [u]^r}$.

$E^n$ denotes the $n-$dimensional fuzzy number space. If $u \in E^n$, then $[u]^r$ is a nonempty compact convex subset of $R^n$ for each $r \in [0,1]$.

**Theorem 3.2** ([4, 23]). *Define $D : E^n \times E^n \to [0,\infty)$ by the equation*

$$D(u,v) = \sup_{r\in[0,1]} d([u]^r, [v]^r), u, v \in E^n,$$

*then*

*(1) $(E^n, D)$ is a complete metric space;*

*(2) $D(\lambda u, \lambda v) = |\lambda| D(u,v), \lambda \in R$;*

*(3) $D(u+w, v+w) = D(u,v)$;*

*(4) $D(u+v, w+e) \leq D(u,w) + D(v,e)$;*

*(5) $D(u+v, \widetilde{0}) = D(u, \widetilde{0}) + D(v, \widetilde{0})$;*

*(6) $D(u+v, w) \leq D(u,w) + D(v, \widetilde{0})$.*

*where $u, v, w, e, \widetilde{0} \in E^n, \widetilde{0} = \chi_{(\{0\})}$.*

The metric space $(E^n, D)$ has a linear structure, it can be imbedded isomorphically as a convex cone with vertex $\theta$ into the Banach space of functions $u^* : I \times S^{n-1} \longrightarrow R$, where $S^{n-1}$ is the unit sphere in $R^n$, with an imbedding function $u^* = j(u)$ defined by

$$u^*(r,x) = \sup_{\alpha\in[u]^\alpha} < \alpha, x >$$

for all $< r, x > \in I \times S^{n-1}$.

**Theorem 3.3** ([23]). *There exists a real Banach space $X$ such that $E^n$ can be imbedding as a convex cone $C$ with vertex $\theta$ into $X$. Furthermore the following conditions hold true:*
*(1) the imbedding $j$ is isometric,*
*(2) addition in $X$ induces addition in $E^n$,*
*(3) multiplication by nonnegative real number in $X$ induces the corresponding operation in $E^n$,*
*(4) $C - C$ is dense in $X$,*
*(5) $C$ is closed.*

A fuzzy-number-valued function $\tilde{F} : [a, b] \to E^n$ is said to satisfy the condition $(H)$ on $[a, b]$, if for any $x_1 < x_2 \in [a, b]$ there exists $u \in E^n$ such that $f(x_2) = f(x_1) + u$. We call $u$ is the H-difference of $\tilde{F}(x_2)$ and $\tilde{F}(x_1)$, denoted $\tilde{F}(x_2) -_H \tilde{F}(x_1)$ ([10]).

For brevity, we always assume that the condition $(H)$ is satisfied when dealing with the operation of subtraction of fuzzy numbers throughout this paper.

**Definition 3.4** ([24, 25]). A fuzzy-number-valued function $\tilde{F} : T \to E^n$ is said to be fuzzy Henstock integrable on $T$ if there exists a fuzzy number $\tilde{A} \in E^n$ such that for every $\varepsilon > 0$ there is a function $\delta(x) > 0$ such that for any $\delta-$fine division $\Pi = \{\xi_i, [x_{i-1}, x_i]\}$ of $T$, we have

$$D(\tilde{A}, \sum_i \tilde{F}(\xi_i)(x_i - x_{i-1})) < \varepsilon.$$

We write $(FH) \int_T \tilde{F}(x)dx = \tilde{A}$.

**Definition 3.5** ([22]). A fuzzy-number-valued function $\tilde{F} : T \to E^n$ is said to be Pettis ( Henstock-Pettis) integrable on $T$ if $[F(t)]^r$ is Pettis ( Henstock-Pettis) integrable on $T$ for every $r \in [0, 1]$, and there exists a fuzzy number $\tilde{A} \in E^n$ such that for any $x \in S^{n-1}$ we have

$$\sigma(x, [A]^r) = (L) \int_T \sigma(x, [F(t)]^r)\mathrm{d}t$$

$$(\sigma(x, [A]^r) = (H) \int_T \sigma(x, [F(t)]^r)\mathrm{d}t).$$

We write $\tilde{A} = (FP) \int_T \tilde{F}(t)\mathrm{d}t$ $(\tilde{A} = (FHP) \int_T \tilde{F}(t)\mathrm{d}t)$.

*Remark* 3.6. In particular, if $\tilde{F}$ is degenerated into $F : T \to R^n$ and $\tilde{A}$ is degenerated into $A \in R^n$, then

$$\sigma(x, [A]^r) =< x, A > .$$

*Remark* 3.7. When $n = 1$, if the fuzzy-number-valued function $\tilde{F} : T \to E^1$ is Kaleva integrable on $T$ (refer to [25]), then $\tilde{F}$ is also Pettis integrable.

*Remark* 3.8. When $n = 1$, if the fuzzy-number-valued function $\tilde{F} : T \to E^1$ is fuzzy Henstock integrable on $T$ (refer to the Definition 3.2 of [24]), then $\tilde{F}$ is also fuzzy Henstock-Pettis integrable.

A fuzzy-number-valued function $\tilde{F} : T \to E^n$ is said to be measurable on $T$ iff the compact convex set-valued function $F^r : T \to P_k(R^n)$ is measurable for any $r \in [0, 1]$.

**Definition 3.9.** Let $\tilde{F} : T \to E^n$ be a measurable fuzzy-number-valued function, $\tilde{F}$ is said to be fuzzy Aumann-Henstock-Petiss integrable on $T$ if

$$(FAHP) \int_T [F(t)]^r \mathrm{d}t = \{(HP) \int_T f(t)\mathrm{d}t | f \in s_{HP}[F(t)]^r\}$$

determines a unique fuzzy number $\tilde{A} \in E^n$, where $s_H[F(t)]^r$ denotes the family of all fuzzy Henstock-Petiss integrable selections of $[F(t)]^r$. We write $(FAH) \int_T \tilde{F}(t)\mathrm{d}t = \tilde{A}$.

**Theorem 3.10.** *Let $\tilde{F}: T \to E^n$ be a fuzzy Aumann-Henstock-Pettis integrable function on $T$. If the set-valued function $[F(t)]^r$ is measurable and the measurable selections of $[F(t)]^r$ are Henstock-Pettis integrable for any $r \in [0,1]$, then for every $x \in S^{n-1}$ we have*

$$\sigma\big(x, (AHP)\int_T [F(t)]^r \mathrm{d}t\big) = (H)\int_T \sigma(x, [F(t)]^r)\mathrm{d}t.$$

*Proof.* Since the measurable selections of $[F(t)]^r$ are Henstock-Pettis integrable for any $r \in [0,1]$, we have

$$\sigma\big(x, (AHP)\int_T [F(t)]^r \mathrm{d}t\big) = (H)\int_T \sigma(x, [F(t)]^r)\mathrm{d}t$$

for any $x \in S^{n-1}, r \in [0,1]$. □

Furthermore, by Theorem 2.10, we can easily obtain the following Theorem 3.11.

**Theorem 3.11.** *Let $\tilde{F}: T \to E^n$ be a fuzzy-number-valued function. If all measurable selections of $[F(t)]^r$ are Henstock-Pettis integrable and $\sigma(x, [F(t)]^r)$ is Henstock integrable for any $r \in [0,1]$, then $(AHP)\int_T [F(t)]^r \mathrm{d}t$ is a compact convex set.*

**Theorem 3.12.** *Let $\tilde{F}: T \to E^n$ be a fuzzy-number-valued function on $T$. If $\tilde{F}$ is fuzzy Henstock-Pettis integrable on $T$, then each measurable selection of $[F(t)]^r$ is Henstock-Pettis integrable for any $r \in [0,1]$ and $t \in T$.*

*Proof.* Since $[F(t)]^r$ is Henstock-Pettis integrable on $T$, for any $r \in [0,1]$ and $t \in T$, by Lemma 3 of [5], the conclusion holds. □

**Theorem 3.13.** *If $\tilde{A}, \tilde{B} \in E^n$, then $\tilde{A} \subset \tilde{B}$ if and only if $\sigma(x, [A]^r) \le \sigma(x, [B]^r)$ for any $r \in [0,1]$ and $x \in S^{n-1}$.*

*Proof.* Necessity: If $\tilde{A} \subset \tilde{B}$, then for any $r \in [0,1]$ and $x \in S^{n-1}$ we have

$$\begin{aligned} \sigma(x, [A]^r) &= \sup\{< x, a > | a \in [A]^r\} \\ &\le \sup\{< x, b > | b \in [B]^r\} \\ &= \sigma(x, [B]^r). \end{aligned} \tag{3.1}$$

Sufficiency: If $\sigma(x, [A]^r) \le \sigma(x, [B]^r)$ for any $r \in [0,1]$ and $x \in S^{n-1}$, then for every $a \in A$, by the Lemma 2.3 we have $< x, a > \le \sigma(x, A) \le \sigma(x, B)$, thus $a \in [B]^r$, that is $[A]^r \subset [B]^r$. Hence, $\tilde{A} \subset \tilde{B}$. □

**Theorem 3.14.** *Let $\tilde{F}: T \to E^n$ be a fuzzy-number-valued function on $T$. If the integration $(FH)\int_T \tilde{F}(t)\mathrm{d}t$ exists, then the following statements are equivalent:*

*(1) $\tilde{F}$ is fuzzy Henstock-Pettis integrable on $T$;*

*(2) For every Henstock-Pettis integrable function $f \in s_{HP}([F(t)]^1)$, there exists a fuzzy-number-valued function $\tilde{G}: T \to E^n$ such that $\tilde{F}(t) = \tilde{G}(t) + \tilde{f}(t)$ and $\tilde{G}$ is fuzzy Pettis integrable on $T$;*

*(3) For every $f, h \in s_H([F(t)]^1)$, $h - f$ is Pettis integrable;*

*(4) $\tilde{F}(x)$ is fuzzy Aumann-Henstock-Pettis integrable on $T$ and for any $x \in S^{n-1}$, we have*

$$\sigma(x, (AHP)\int_T [F(t)]^r \mathrm{d}t) = (H)\int_T \sigma(x, [F(t)]^r)\mathrm{d}t \ (r \in [0,1]).$$

**Proof.** (1) ⇒ (2): For every $f(t) \in s_H([F(t)]^1)$, since $[F(t)]^1$ is Henstock-Pettis integrable on $T$, by Theorem 3.10, wa can infer that $f(t)$ is Henstock-Pettis integrable on $T$. Define $[G(t)]^r = [F(t)]^r - f(t)$, $[G]^r: T \to P_k(R^n)$, then for any $x \in S^{n-1}, t \in T$ we have $\sigma(x, [G(t)]^r) \ge 0$, and

$$\sigma(x, [F(t)]^r) = \sigma(x, [G(t)]^r) + < x, f(t) > .$$

By Theorem 3.12, $[G(t)]^r$ is Pettis integrable. In addition, we can prove that $\{[G(t)]^r, r \in [0,1]\}$ determines a fuzzy number. In fact, $\{[G(t)]^r\}$ satisfies the following conditions:

(i) $[G(t)]^r$ is a nonempty compact convex set;

(ii) if $0 \le r_1 \le r_2 \le 1$, then

$$\begin{aligned} \sigma(x,[G(t)]^{r_1}) &= \sigma(x,[F(t)]^{r_1}) - < x, f(t) > \\ &\ge \sigma(x,[F(t)]^{r_2}) - < x, f(t) > \\ &= \sigma(x,[G(t)]^{r_2}) \end{aligned} \tag{3.2}$$

That is $[G(t)]^{r_1} \supseteq [G(t)]^{r_2}$;

(iii) for any $\{r_m\}$ converging increasingly to $r \in (0,1]$, since for any $x \in S^{n-1}$ we have

$$\sigma(x,[F(t)]^{r_m}) \downarrow \sigma(x,[F(t)]^r),$$

so

$$\begin{aligned} \sigma(x,[G(t)]^{r_m}) &= \sigma(x,[F(t)]^{r_m}) - < x, f(t) > \\ &\downarrow \sigma(x,[F(t)]^{r}) - < x, f(t) > \\ &= \sigma(x,[G(t)]^{r}). \end{aligned} \tag{3.3}$$

(2) $\Rightarrow$ (3): Let $f \in s_{HP}([F(t)]^1)$, $[G(t)]^r = [F(t)]^r - f(t)$. If $h \in s_H([F(t)]^1)$, then $g = h - f$ is a measurable selection of $[G]^r$, and

$$-\sigma(-x,[G(t)]^r) \le < x, g(t) > \le \sigma(x,[G(t)]^r).$$

For every $E \in \mathcal{L}$, we denotes $w_E = (P)\int_E [G(t)]^r \mathrm{d}t \in P_k(R^n)$, then

$$\begin{aligned} -\sigma(-x,w_E) &= -(L)\int_E \sigma(-x,[G(t)]^r)\mathrm{d}t \\ &\le (L)\int_E < x, g(t) > \mathrm{d}t \\ &\le (L)\int_E \sigma(x,[G(t)]^r)\mathrm{d}t = \sigma(x,w_E). \end{aligned} \tag{3.4}$$

On the other hand, $w_E$ is compact and its support function $\sigma(x,w_E)$ is Lipschitz continuous uniformly with respect to $x$, therefore $x \to (L)\int_E < x, g(t) > \mathrm{d}t$ is continuous uniformly, it follows that $g = h - f$ is Pettis integrable.

(3) $\Rightarrow$ (2): For $f \in s_{HP}([F(t)]^1)$, define $[G(t)]^r = [F(t)]^r - f(t)$, then by assumption, each measurable selection $g$ of $[G(t)]^r$ is Pettis integrable, and by Theorem 3.12, $[G(t)]^r$ is also Pettis integrable on $T$. Furthermore, we can prove $\{[G(t)]^r, r \in [0,1]\}$ determines a fuzzy number similar to (1) $\Rightarrow$ (2). It shows that $\tilde{G}$ is Pettis integrable on $T$.

(2) $\Rightarrow$ (4) For $f \in s_H([F(t)]^1)$, the set-valued function $[G(t)]^r = [F(t)]^r - f(t)$ is Pettis integrable on $T$. By the Theorem 3.12, $[G(t)]^r$ is Aumann-Henstock-Pettis integrable on $T$, and

$$(P)\int_T [G(t)]^r \mathrm{d}t = \{(P)\int_T g(t)\mathrm{d}t | g \in s_P([G(t)]^r)\}.$$

Note that $(P)\int_T [G(t)]^r \mathrm{d}t$ is a compact convex set, then

$$(AHP)\int_T [F(t)]^r \mathrm{d}t = (P)\int_T [G(t)]^r \mathrm{d}t + (HP)\int_T f(t)\mathrm{d}t$$

is also a compact convex set.

We can prove that $\{(AHP)\int_T[F(t)]^r\mathrm{d}t \mid r\in[0,1]\}$ determines a unique fuzzy number. In fact, $\{(AHP)\int_T[F(t)]^r\mathrm{d}t \mid r\in[0,1]\}$ satisfies the following conditions:

(i) $(AHP)\int_T[F(t)]^r\mathrm{d}t$ is a compact convex set;

(ii) if $0\le r_1\le r_2\le 1$, then

$$\begin{aligned}\sigma(x,(AHP)\int_T[F(t)]^{r_1}\mathrm{d}t) &= \sigma(x,(P)\int_T[G(t)]^{r_1}\mathrm{d}t+(HP)\int_T f(t)\mathrm{d}t)\\ &= \sigma(x,(P)\int_T[G(t)]^{r_1}\mathrm{d}t)+\sigma(x,(HP)\int_T f(t)\mathrm{d}t)\\ &\ge \sigma(x,(P)\int_T[G(t)]^{r_2}\mathrm{d}t)+\sigma(x,(HP)\int_T f(t)\mathrm{d}t)\\ &= \sigma(x,(AHP)\int_T[F(t)]^{r_2}\mathrm{d}t). \qquad (3.5)\end{aligned}$$

(iii) for any $\{r_m\}$ converging increasingly to $r\in(0,1]$, since for every $x\in S^{n-1}$ we have

$$\sigma(x,[G(t)]^{r_m})\downarrow\sigma(x,[G(t)]^r).$$

Consequently,

$$\begin{aligned}&\sigma(x,(AHP)\int_T[F(t)]^{r_m}\mathrm{d}t)\\ =\ &\sigma(x,(P)\int_T[G(t)]^{r_m}\mathrm{d}t)+\sigma(x,(HP)\int_T f(t)\mathrm{d}t)\\ \downarrow\ &\sigma(x,(P)\int_T[G(t)]^{r}\mathrm{d}t)+\sigma(x,(HP)\int_T f(t)\mathrm{d}t)\\ =\ &\sigma(x,(AHP)\int_T[F(t)]^{r}\mathrm{d}t).\end{aligned}$$

Thus, $\{(AHP)\int_T[F(t)]^r\mathrm{d}t \mid r\in[0,1]\}$ determines a unique fuzzy number. That is, $\tilde{F}(x)$ is fuzzy Aumann-Henstock-Pettis integrable on $T$, and

$$\sigma(x,(AHP)\int_T[F(t)]^r\mathrm{d}t)=(H)\int_T\sigma(x,[F(t)]^r)\mathrm{d}t.$$

(4) $\Rightarrow$ (1): Since $\tilde{F}(t)$ is fuzzy Aumann-Henstock-Pettis integrable on $T$, so $[F(t)]^r$ is Aumann-Henstock-Pettis integrable on $T$ for any $r\in[0,1]$. By Theorem 3.12, $[F(t)]^r$ is Henstock-Pettis integrable on $T$. Similar to the proof of (2) $\Rightarrow$ (4), we can prove that $\{[F(t)]^r, r\in[0,1]\}$ determines a unique fuzzy number $(FAHP)\int_T\tilde{F}(t)\mathrm{d}t\in E^n$, i.e., $\tilde{F}(t)$ is fuzzy Henstock-Pettis integrable on $T$.

**Corollary 3.15.** *If $\tilde{F}:T\to E^n$, $\tilde{G}:T\to E^n$, $f\in s_{HP}([F(t)]^1$, then the fuzzy Henstock-Pettis integration of $\tilde{F}$ could be translated into the Henstock-Pettis integration of $\tilde{G}$, and*

$$(FHP)\int_T\tilde{F}(t)\mathrm{d}t=(FP)\int_T\tilde{G}(t)\mathrm{d}t+(HP)\int_T f(t)\mathrm{d}t.$$

**Theorem 3.16.** *Let $\tilde{F}:T\to E^n$ be a measurable fuzzy-number-valued function, $\sigma(x,[F(t)]^r)$ Henstock integrable on $T$. If $\tilde{F}$ is fuzzy Henstock-Pettis integrable on $T$, then $[G(t)]^r=[F(t)]^r-f(t)$ is Pettis integrable on $T$ for any measurable selection $f$ of $[F(t)]^1$, and*

$$(H)\int_T\sigma(x,[F(t)]^r)\mathrm{d}t=(L)\int_T\sigma(x,[G(t)]^r)\mathrm{d}t+(H)\int_T<x,f(t)>\mathrm{d}t.$$

*for every $t\in T, x\in S^{n-1}$.*

*Proof.* Suppose $f$ is a measurable selection of $[F(t)]^1$, $[G(t)]^r = [F(t)]^r - f(t)$. Since the support function $\sigma(x, [G(t)]^r)$ is a Henstock integrable set-valued function and $< x, f(t) >$ is Henstock integrable, we see that

$$[F(t)]^r = [G(t)]^r + f(t).$$

On the other hand, since $[G(t)]^r$ has at least one Bochner integrable selection (the null function), for every $E \in \mathcal{L}$ there is $w_E \in R^n$ with

$$\sigma(x, w_E) = (L) \int_E \sigma(x, [G(t)]^r) \mathrm{d}t$$

for any $x \in S^{n-1}$. Hence, we have for every $x \in S^{n-1}$

$$\sigma(x, w_T) + (H) \int_T < x, f(t) > \mathrm{d}t = (H) \int_T \sigma(x, [F(t)]^r) \mathrm{d}t \neq \pm\infty.$$

It follows that $\sigma(x, w_T) \neq \pm\infty$ for all $x \in S^{n-1}$. By Banach-Steinhaus Theorem, $w_T \in P_k(R^n)$. And we get that every $w_E$ is bounded, thus $[G(t)]^r$ is Pettis integrable on $T$. □

# 4 The Komlós-type convergence theorem for the Fuzzy Henstock-Pettis integrals and a fuzzy integral inclusion

The Komlós's classical theorem (see[11]) yields that from any $L^1-$bounded sequence of real functions one can extract a subsequence such that the arithmetic averages of all its subsequence converge point almost everywhere. In [20], the author extended these results by providing a Komlós-type theorem for set-valued functions under Henstock-Pettis integrability assumptions. In this section, we extend the Komlós theorem to the case of the fuzzy-number-valued Henstock-Pettis integrals. As an application, an existence theorem for a fuzzy integral inclusion involving the fuzzy Henstock-Pettis integral is obtained.

**Definition 4.1.** A sequence $(\tilde{F}_n)_n$ of fuzzy-number-valued functions is said to be Komlós convergent ($K-$convergent for short) to a fuzzy-number-valued function $\tilde{F}$ if for every subsequence $(\tilde{F}_{k_n})_n$ there exists a $\mu-$null set $N \subset T$, such that for all $t \in T \backslash N$,

$$\sigma(x, [F(t)]^r) = \lim_n \sigma(x, \frac{1}{n} \sum_{i=1}^{n} [F_{k_i}(t)]^r).$$

**Theorem 4.2.** *Let $\tilde{F}_n : T \to E^n$ be a sequence of $(FHP)-$integrable functions. Suppose*

*(i) there exists a real Henstock integrable function $\bar{f}$, such that*

$$\bar{f}(t) \leq \sigma(x, [F_n(t)]^r), \quad \forall t \in T, \forall n \in \mathbb{N};$$

*and*

$$\sup_{n \in \mathbb{N}} (H) \int_T \sigma(x, [F_n(t)]^r) \mathrm{d}t < +\infty;$$

*(ii) there exists a function $h : T \times R \to [0, +\infty)$ such that, for every $t \in T$, $h(t, \cdot)$ is convex and compact, and a countable measurable partition $(B_m)_m$ of $T$ satisfying:*

*(a) $\sup_n (H) \int_{B_m} |\sigma(x, [F_n(t)]^r)| \mathrm{d}t < +\infty$;*

*(b) $\sup_n (H) \int_{B_m} h(t, [F_n(t)]^r) \mathrm{d}t$.*

*Then there exist a $(FHP)-$integrable function $\tilde{F}$ and a subsequence of $(\tilde{F}_n)_n$ which $K-$converges to $\tilde{F}$. Moreover, $\int_{B_m} h(t, [F(t)]^r) \mathrm{d}t$ exist for each $m \in \mathbb{N}$.*

*Proof.* Consider the convex $Y \subset E^n$, let the function $g(t,C) = \sigma(x,C)$ be continuous on $Y$. By (ii) and defn 5.1, the fuzzy sequence $(\tilde{F}_n)_n$ $K-$converges to $\tilde{F}$ which is $(FHP)-$integrable and $\int_{B_m} h(t,[F(t)]^r)\mathrm{d}t$ exist for each $m \in \mathbb{N}$.

By (i), the function $-\bar{f}(t) + \sigma(x, \frac{1}{n}\sum_{i=1}^n [F_{k_i}(t)]^r)$ is Henstock integrable for every $n \in \mathbb{N}$. We are now able to apply Fatou's Lemma to the sequence $(-\bar{f}(t) + \sigma(x, \frac{1}{n}\sum_{i=1}^n [F_{k_i}(t)]^r))_n$, and have

$$\begin{aligned}
&\int_T (-\bar{f}(t) + \sigma(x,[F(t)]^r)\mathrm{d}t \\
\leq\ & \liminf_n (H)\int_T (-\bar{f}(t) + \sigma(x, \frac{1}{n}\sum_{i=1}^n [F_{k_i}(t)]^r))\mathrm{d}t \\
=\ & (H)\int_T (-\bar{f}(t)\mathrm{d}t + \liminf_n (H)\int_T \sigma(x, \frac{1}{n}\sum_{i=1}^n [F_{k_i}(t)]^r)\mathrm{d}t \\
\leq\ & (H)\int_T (-\bar{f}(t)\mathrm{d}t + \sup_{n\in\mathbb{N}}\int_T \sigma(x,[F_n(t)]^r)\mathrm{d}t \leq +\infty. \qquad (4.1)
\end{aligned}$$

Consequently, $-\bar{f}(t)+\sigma(x,[F(t)]^r$ is $(H)$-integrable and, since $\bar{f}(t)$ is $(H)$-integrable, the $H$-integrability of $\sigma(x,[F(t)]^r$ follows. Every measurable selection $\tilde{f}$ of $\tilde{F}$ is $HP-$integrable, so we have

$$-\sigma(-x,[F(t)]^r) \leq \langle x, f(t)\rangle \leq \sigma(x,F(t)]^r), \quad a.e.t \in T.$$

For every $[a,b] \subset T$, there exist $A$, such that $\langle x, A\rangle = (H)\int_a^b \langle x, f(t)\rangle$. Thus every measurable selection of $\tilde{F}$ is Hestock-Pettis integrable.

Finally, by implication (4) $\Rightarrow$ (1) in Theorem 3.14, we have

$$\lim_{n\to\infty} (FHP)\int_T \tilde{F}_n(t)\mathrm{d}t = (FHP)\int_T \tilde{F}(t)\mathrm{d}t.$$

□

**Corollary 4.3.** *Let $(\tilde{F}_n)_n$ be a sequence of $(FHP)-$integrable functions satisfying hypothesis (i) in Theorem 4.2 and for every $n \in \mathbb{N}$ there exist $\tilde{F}_n'(t)$, such that $\tilde{F}_n(t) \subset \tilde{F}_n'$ a.e. Then there exist a $FHP-$integrable function $\tilde{F}$ and a subsequence of $(\tilde{F}_n)_n$ which $K-$converges to $\tilde{F}$.*

*Proof.* Let $B_m = \{t \in T | m-1 \leq D(\tilde{F}_n'(t), \tilde{0}) < m, \quad \forall m \in \mathbb{N}\}$ satisfy hypothesis (ii) in Theorem 4.2. Then, for every $m \in \mathbb{N}$, we have

$$\begin{aligned}
\sup_{n\in\mathbb{N}}(H)\int_{B_m} |\sigma(x,[F_n(t)]^r)|\mathrm{d}t &\leq (H)\int_{B_m} |\sigma(x,[F_n'(t)]^r)|\mathrm{d}t \\
&\leq (H)\int_{B_m} D(\tilde{F}_n'(t),\tilde{0}) < +\infty.
\end{aligned}$$

By Theorem 4.2, the conclusion holds. □

**Theorem 4.4.** *Let $(\tilde{F}_n)_n$ be a sequence of $(FHP)-$integrable functions satisfying hypothesis (i) in Theorem 4.2 and*

*(ii') there exists a measurable countable partition $(B_m)_m$ of $T$ such that, for each $m \in \mathbb{N}$,*

$$\sup_{n\in\mathbb{N}}(H)\int_{B_m} D(\tilde{F}_n(t),\tilde{0})\mathrm{d}t < +\infty.$$

*Then there exist a $(FHP)-$integrable function $\tilde{F}$ and a subsequence of $(\tilde{F}_n)_n$ which $K-$converges to $\tilde{F}$ Moreover, $(H)\int_{B_m} D(\tilde{F}(t),\tilde{0})\mathrm{d}t < +\infty$ for every $m \in \mathbb{N}$.*

In the sequel, using above Komlós-type convergence theorem, we give an existence theorem of a fuzzy integral inclusion as the following:

$$x(t) \in \xi + (FHP)\int_0^t \tilde{F}(s, x(s))\mathrm{d}s, \quad t \in T$$

**Theorem 4.5.** *Let $U$ an open subset of fuzzy number space $(E^n, D)$, and $FHP$-integrable function $\tilde{F} : T \times U \to E^n$ and $\tilde{\Gamma} : T \to E^n$ satisfy*

*(1) $\tilde{F}(t, x) \subset \tilde{\Gamma}(t), \quad \forall t \in T, \forall x \in U$;*

*(2) $\tilde{F}(t, x)$ is upper semi-continuous for $t \in T$;*

*(3) $\sigma(x, [F(\cdot, x)]^r)$ is measurable for every $x \in U$.*

*Then, for every fixed $\xi \in U$, there exist $t_0 \in T$ such that $\xi + (FHP)\int_0^{t_0} \tilde{\Gamma}(s)\mathrm{d}s \subset U$ and*

$$x(t) \in \xi + (FHP)\int_0^t \tilde{F}(s, x(s))\mathrm{d}s$$

*has a solution in $C([0, t_0], E^n)$.*

*Proof.* By Theorem 3.14, for all $\tilde{f} \in s_{HP}([F(t)]^1)$, there exists $\tilde{G} : T \to E^n$ such that $\tilde{F}(t) = \tilde{G}(t) + \tilde{f}(t)$, and $\tilde{G}$ is fuzzy Pettis integrable on $T$, then $\tilde{f}$ is measurable.

Fixing $\xi \in U$, we consider the open subset $U_1$ and $U_2$ of $E^n$ such that $\xi \in U_1$ and $U_1 + U_2 \subset U$. Since $(FHP)\int_0^{(\cdot)} \tilde{f}(t)\mathrm{d}t$ is continuous, there exist $t_1 \in T$ such that $(FHP)\int_0^t \tilde{f}(t)\mathrm{d}t \in U_2$ for every $t \in [0, t_1]$. We define a fuzzy-number-valued function $\tilde{F}' : [0, t_1] \times U_1 \longrightarrow E^n$ as the following:

$$\tilde{F}'(t, x) = (-1) \cdot \tilde{f} + \tilde{F}(t, x + (FHP)\int_0^t \tilde{f}(\tau)\mathrm{d}\tau),$$

which satisfies the following conditions:

(1) $\tilde{F}'(t, x) \subset \tilde{G}(t), \quad \forall t \in T, \forall x \in U$;

(2) for evry $t \in T$, $\tilde{F}'(t, x)$ is upper semi-continuous;

(3) $\sigma(x, [F'(\cdot, x)]^r)$ is measurable for every $x \in U$.

Then we obtain that there exist $t_0 \in [0, t_1]$ such that $\xi + (FP)\int_0^{t_0} \tilde{G}(s)\mathrm{d}s \in U_1$, the integral inclusion

$$y(t) \in \xi + (FHP)\int_0^t \tilde{F}'(s, y(s))\mathrm{d}s \tag{4.2}$$

has a solution in $C([0, t_0], E^n)$ and the set of solution is compact in $C([0, t_0], E^n)$.

Therefore, we have

$$\xi + (FHP)\int_0^{t_0} \tilde{\Gamma}(s)\mathrm{d}s = \xi + (FHP)\int_0^{t_0} \tilde{f}(s)\mathrm{d}s + (FP)\int_0^{t_0} \tilde{G}(s)\mathrm{d}s \subset U$$

and we find $y(t) \in C([0, t_0], E^n)$ such that

$$y(t) \in \xi + (FP)\int_0^t (-1) \cdot \tilde{f}(s) + \tilde{F}(s, y(s) + (FHP)\int_0^s \tilde{f}(\tau)\mathrm{d}\tau)\mathrm{d}s,$$

That is

$$y(t) + (FHP)\int_0^t \tilde{f}(s)\mathrm{d}s \in \xi + (FHP)\int_0^t \tilde{F}(s, y(s) + (FHP)\int_0^s \tilde{f}(\tau)\mathrm{d}\tau)\mathrm{d}s.$$

Thus $x(\cdot) = y(\cdot) + (FHP)\int_0^{(\cdot)} \tilde{f}(\tau)\mathrm{d}\tau$ is a solution of the integral inclusion. □

# 5 Conclusions

In this paper, we study the Henstock-Pettis integral of compact convex set-valued functions and fuzzy-number-valued function and the $K-$convergence theorem of fuzzy Henstock-Pettis integrals. We emphasize that the outcomes of the second part in our paper are different from the results in L. Di Piazza's paper [5]. In the future research, we shall deals with a new derivative and Hestock-Pettis-$\Delta$-integral for fuzzy-number-valued functions on time scales. Also, we shall study and investigate fuzzy differential equations and fuzzy integral equations with $\Delta_H-$derivative and $FHP-\Delta-$integral on time scales.

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# References

[1] K. EI. Amri, C. Hess, On the Pettis integral of closed valued multifunctions. Set-Valued Anal. 8 (2000), 329-360.

[2] B. Bongiorno, L. Di Piazza, K. Musial, A decomposition theorem for the fuzzy Henstock integral. Fuzzy Sets and Syst. 200 (2012) 36-47.

[3] M. Cichoń, Convergence theorems for the Henstock-Kurzweil-Pettis integral. Aata. Math. Hungar. 92 (2001) 75-82.

[4] P. Diamond, P. Kloeden, Metric Space of Fuzzy Fets: Theory and Applications, World Scientific, Singapore, 1994.

[5] L. Dipiazza, K. Musiał, Set-valued Kurzweil-Henstock-Pettis integral. Set-valued Anal. 13 (2005) 167-169.

[6] Zengtai Gong, On the problem of characterizing derivatives for the fuzzy-valued functions(II). Fuzzy Sets and Syst. 145 (2004) 381-393.

[7] Zengtai Gong, Yabin Shao, The controlled convergence theorems for the strong Henstock integrals of fuzzy-number-valued functions. Fuzzy Sets and Syst. 160 (2009) 1528-1546.

[8] Zengtai Gong, Liangliang Wang, The Henstock-Stieltjes integral for fuzzy-number-valued functions. Info. Sci. 188 (2012) 276-297.

[9] R. Henstock, Theory of Integration, Butterworths, London, 1963.

[10] O. Kaleva, Fuzzy differential equations. Fuzzy Sets and Syst. 24 (1987) 301-317.

[11] J. Komlós, A generalization of a problem of Steinhaus. Acta Math. Acad. Sci. Hungar. 18 (1967), 217-229.

[12] J. Kurzweil, Generalized ordinary differential equations and continuous dependence on a parameter. Czechoslovak Math. J. 7 (1957) 418-446.

[13] H. Lebesgue, Integral, longueur, maire. Annali Math, Pura Applic. 7 (1902) 231-359.

[14] P. Y. Lee, Lanzhou Lectures on Henstock Integration, World Scientific, Singapore, 1989.

[15] Baolin Li, Haide Gou, On the existence of weak solutions of nonlinear integral equations in Banach spaces. Mediterr. J. Math. 13 (2016), 2633-2643.

[16] Qiang Ma, Yabin Shao, Zizhong Chen, A Kind of Generalized Fuzzy Integro-Differential Equations of Mixed Type and Strong Fuzzy Henstock Integrals, J. Comput. Anal. and Appl. 23(1) (2017), 92-107.

[17] Qiang Ma, Yabin Shao, Zizhong Chen, Existence of Generalized Solutions for Fuzzy Impulsive Retarded Differential Equations, J. Comput. Anal. and Appl. In press.

[18] K. Musiał, Adecomposition theorem for Banach space valued fuzzy Henstock. Fuzzy Sets and Syst. 259 (2015) 21-28.

[19] M. Puri, D. Ralescu, Fuzzy random variables. Math. Anal. Appl. 114 (1986) 409-422.

[20] B. Satco, A Komlós-type theorem for the set-valued Henstock-Kurzweil-Pettis integral and applications. Czechoslovak Math. J. 56 (2006) 1029-1047.

[21] S. Schwabik, Guoju Ye, Topics in Banach Space Integration, World Scientific, Singapore, 2005.

[22] Yabin Shao, Zengtai Gong, Zizhong Chen, On the existence of generalized weak solutions to discontinuous fuzzy differential equations, Submitted.

[23] Congxin Wu, Ming Ma, On embedding problem of fuzzy number spaces: Part 2. Fuzzy Sets and Syst. 45 (1992) 189-202.

[24] Congxin Wu, Zengtai Gong, On Henstock integral of fuzzy-number-valued functions(I). Fuzzy Sets and Syst. 120 (2001) 523-552.

[25] Congxin Wu, Zengtai Gong, On Henstock integral of interval-valued functions and fuzzy-number-valued functions. Fuzzy Sets and Syst. 115 (2000) 377-391.

[26] X. Xue, Y. Fu, Caratheodory solution of fuzzy differential equations. Fuzzy Sets and Syst. 125 (2002) 239-243.

[27] Bokan Zhang, Congxin Wu, On the representation of n-dimensional fuzzy numbers and their informational content. Fuzzy Sets and Syst. 128 (2002) 227-235.

[28] H. Ziat, Convergence theorems for Pettis integrable multifunctions. Bull. Polish Acad. Sci. Math. 45 (1997) 123-137.

[29] H. Ziat, On a characterization of Pettis integrable multifunctions. Bull. Polish Acad. Sci. Math. 48 (2000) 227-230.

# Some Inequalities for Riemann Diamond Integrals on Time Scales

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**Abstract**

In this paper, we investigate the Diamond integral on time scales. By using Darboux approach, we define the Riemann Diamond integral on time scales and prove the corresponding theorems. Our results extend and improve the corresponding results on inequality of [8].

*Keywords:* Diamond integral, generalized Hölder's inequality, generalized Jensen's inequality, time scales

*2010 MSC:* 26D15, 26E70

## 1. Introduction

The theory of time scales was born in 1988 with the Ph.D. thesis of Stefan Hilger, done under the supervision of Bernd Aulbach [9]. The aim of this theory was to unify various definitions and results from the theories of discrete and continuous dynamical systems, and to extend such theories to more general classes of dynamical systems. It has been extensively studied on various aspects by several authors [1,4,5,6,7,10,14,16].

Two versions of the calculus on time scales, the delta and nabla calculus, are now standard in the theory of time scales [5,6]. In 2006, the Diamond-alpha

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integral on time scales was introduced by Sheng, Fadag, Henderson, and Davis [16], as a linear combination of the delta and nabla integrals. The Diamond-alpha integral reduces to the standard delta integral for $\alpha = 1$ and to the standard nabla integral for $\alpha = 0$. We refer the reader to [2,3,11,12,13,15,16] for a complete account of the recent Diamond-alpha integral on time scales. In 2015, the Diamond integral on time scales, as a refined version of the diamond-alpha integral, was introduced by Artur M. C. Brito da Cruz et al., [8]. In this paper we define and study the Riemann Diamond integral on time scales. Basic properties of the theory are proved.

The paper is organized as follows. Section 2 contains basic concepts of time scales theory. In Section 3, definition of the Riemann diamond integral will be introduced. We will investigate basic properties of the Riemann diamond integral. In Section 4, we will establish generalized Hölder's inequality, Cauchy-Schwarz's inequality, Minkowski's inequality and Jensen's inequality on time scales.

## 2. Preliminaries

Let $\mathbb{T}$ be a time scale, i.e. a nonempty closed subset of $\mathbb{R}$. For $a, b \in \mathbb{T}$ we define the closed interval $[a, b]_{\mathbb{T}}$ by $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}$. The open and half-open intervals are defined in an similar way. For $t \in \mathbb{T}$ we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ where $\inf \emptyset = \sup \mathbb{T}$, while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$ where $\sup \emptyset = \inf \mathbb{T}$.

If $\sigma(t) > t$, we say that $t$ is right-scattered, while if $\rho(t) < t$, we say that $t$ is left-scattered. If $\sigma(t) = t$, we say that $t$ is right-dense, while if $\rho(t) = t$, we say that $t$ is left-dense. A point $t \in \mathbb{T}$ is dense if it is right and left dense; isolated if it is right and left scattered. The forward graininess function $\mu : \mathbb{T} \to [0, \infty)$ and the backward graininess function $\eta : \mathbb{T} \to [0, \infty)$ are defined by $\mu(t) = \sigma(t) - t$, $\eta(t) = t - \rho(t)$ for all $t \in \mathbb{T}$ respectively. If $\sup \mathbb{T}$ is finite and left-scattered, then we define $\mathbb{T}^k := \mathbb{T} \backslash \sup \mathbb{T}$, otherwise $\mathbb{T}^k := \mathbb{T}$; if $\inf \mathbb{T}$ is finite and right-scattered,

then $\mathbb{T}_k := \mathbb{T}\backslash \inf \mathbb{T}$, otherwise $\mathbb{T}_k := \mathbb{T}$. We set $\mathbb{T}_k^k := \mathbb{T}^k \bigcap \mathbb{T}_k$.

A function $f : \mathbb{T} \to \mathbb{R}$ is called *regulated* provided its right-sided limits exist (finite) at all right-dense point of $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense point of $\mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* provided it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$.

Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then we define $f^{\Delta}(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there exists a neighborhood $U$ of $t$ such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s|$$

for all $s \in U$. We call $f^{\Delta}(t)$ the delta derivative of $f$ at $t$ and we say that $f$ is delta differentiable on $\mathbb{T}^k$ provided $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^k$.

let $t \in \mathbb{T}_k$. We define $f^{\nabla}(t)$ to be the number with the property that given any $\varepsilon > 0$, there exists a neighborhood $U$ of $t$ such that

$$|f(\rho(t)) - f(s) - f^{\nabla}(t)(\rho(t) - s)| \leq \epsilon|\rho(t) - s|$$

for all $s \in U$. We call $f^{\nabla}(t)$ the nabla derivative of $f$ at $t$ and we say that $f$ is nabla differentiable on $\mathbb{T}_k$ provided $f^{\nabla}(t)$ exists for all $t \in \mathbb{T}_k$.

Let $t, s \in \mathbb{T}$ and define $\mu_{t,s} := \sigma(t) - s$ and $\eta_{t,s} := \rho(t) - s$. We define $f^{\Diamond_\alpha}(t)$ to be the number with the property that given any $\varepsilon > 0$, there exists a neighborhood $U$ of $t$ such that

$$|\alpha(f(\sigma(t)) - f(s))\eta_{t,s} + (1-\alpha)(f(\rho(t)) - f(s))\mu_{t,s} - f^{\Diamond_\alpha}(t)\mu_{t,s}\eta_{t,s}| \leq \epsilon|\mu_{t,s}\eta_{t,s}|$$

for all $s \in U$. We call $f^{\Diamond_\alpha}(t)$ the diamond-$\alpha$ derivative of $f$ at $t$ and we say that $f$ is diamond-$\alpha$ differentiable on $\mathbb{T}_k^k$ provided $f^{\Diamond_\alpha}(t)$ exists for all $t \in \mathbb{T}_k^k$.

The real function

$$\gamma(t) := \lim_{s \to t} \frac{\sigma(t) - s}{\sigma(t) + 2t - 2s - \rho(t)}.$$

## 3. The Riemann Diamond integral

A partition of $[a,b]_{\mathbb{T}}$ is any finite ordered subset

$$P=\{t_0,t_1,\ldots,t_n\}\subset[a,b]_{\mathbb{T}},\quad where\ \ a=t_0<t_1<\ldots<t_n=b.$$

Each partition $P=\{t_0,t_1,\ldots,t_n\}$ of $[a,b]_{\mathbb{T}}$ decomposes it into subintervals $[t_{i-1},t_i)_{\mathbb{T}},\ i=1,2,\ldots,n$, such that for $i\neq j$ one has $[t_{i-1},t_i)_{\mathbb{T}}\cap[t_{j-1},t_j)_{\mathbb{T}}=\emptyset$.

By $\mathcal{P}([a,b]_{\mathbb{T}})$ we denote the set of all partitions of $[a,b]_{\mathbb{T}}$. Let $P_n,P_m\in\mathcal{P}([a,b]_{\mathbb{T}})$. If $P_n\subset P_m$ we call $P_n$ a refinement of $P_m$. If $P_n,P_m$ are independently chosen, then the partition $P_n\cup P_m$ is a common refinement of $P_n$ and $P_m$.

Let $f:[a,b]_{\mathbb{T}}\to\mathbb{R}$ be a real-valued bounded function on $[a,b]_{\mathbb{T}}$. We denote

$$\overline{M}=\sup\{\gamma(t)f(t):t\in[a,b)_{\mathbb{T}}\},\quad \overline{m}=\inf\{\gamma(t)f(t):t\in[a,b)_{\mathbb{T}}\},$$

$$\underline{M}=\sup\{(1-\gamma(t))f(t):t\in(a,b]_{\mathbb{T}}\},\quad \underline{m}=\inf\{(1-\gamma(t))f(t):t\in(a,b]_{\mathbb{T}}\},$$

and for $1\leq i\leq n$,

$$\overline{M_i}=\sup\{\gamma(t)f(t):t\in[t_{i-1},t_i)_{\mathbb{T}}\},\quad \overline{m_i}=\inf\{\gamma(t)f(t):t\in[t_{i-1},t_i)_{\mathbb{T}}\},$$

$$\underline{M_i}=\sup\{(1-\gamma(t))f(t):t\in(t_{i-1},t_i]_{\mathbb{T}}\},\ \underline{m_i}=\inf\{(1-\gamma(t))f(t):t\in(t_{i-1},t_i]_{\mathbb{T}}\},$$

Let $\gamma(t)\in[0,1]$. The upper Darboux $\diamondsuit$-sum of $f$ with respect to the partition $P$, denoted by $U(f,P)$, is defined by

$$U(f,P)=\sum_{i=1}^{n}(\overline{M_i}+\underline{M_i})(t_i-t_{i-1}),$$

while the lower Darboux $\diamondsuit$-sum of $f$ with respect to the partition $P$, denoted by $L(f,P)$, is defined by

$$L(f,P)=\sum_{i=1}^{n}(\overline{m_i}+\underline{m_i})(t_i-t_{i-1}).$$

Note that

$$U(f,P)\leq\sum_{i=1}^{n}(\overline{M}+\underline{M})(t_i-t_{i-1})=(\overline{M}+\underline{M})(b-a)$$

and

$$L(f,P) \geq \sum_{i=1}^{n} (\overline{m} + \underline{m})(t_i - t_{i-1}) = (\overline{m} + \underline{m})(b-a).$$

Thus, we have $(\overline{m} + \underline{m})(b-a) \leq L(f,P) \leq U(f,P) \leq (\overline{M} + \underline{M})(b-a)$.

**Definition 3.1** Let $I = [a,b]_{\mathbb{T}}$, where $a, b \in \mathbb{T}$. The upper Darboux $\diamondsuit-$integral of $f$ from $a$ to $b$ is defined by

$$\overline{\int_a^b} f(t) \diamondsuit t = \inf_{P \in \mathcal{P}([a,b]_{\mathbb{T}})} U(f,P);$$

The lower Darboux $\diamondsuit-$integral of $f$ from $a$ to $b$ is defined by

$$\underline{\int_a^b} f(t) \diamondsuit t = \sup_{P \in \mathcal{P}([a,b]_{\mathbb{T}})} L(f,P).$$

If $\overline{\int_a^b} f(t) \diamondsuit t = \underline{\int_a^b} f(t) \diamondsuit t$, then we say that $f$ is Riemann $\diamondsuit-$integrable on $[a,b]_{\mathbb{T}}$, and the common value of the integrals, denoted by $\int_a^b f(t) \diamondsuit t$, is called the Riemann $\diamondsuit-$ integral.

**Definition 3.2** Let $I = [a,b]_{\mathbb{T}}$, where $a, b \in \mathbb{T}$. The upper Darboux $\Delta-$integral of $f$ from $a$ to $b$ is defined by

$$\overline{\int_a^b} f(t) \Delta t = \inf_{P \in \mathcal{P}([a,b]_{\mathbb{T}})} U(f,P)$$

where $U(f,P)$ denote the upper Darboux sum of $f$ with respect to the partition $P$ and

$$U(f,P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1}), M_i = \sup\{f(t) : t \in [t_{i-1}, t_i)_{\mathbb{T}}\}.$$

The lower Darboux $\Delta-$integral of $f$ from $a$ to $b$ is defined by

$$\underline{\int_a^b} f(t) \nabla t = \sup_{P \in \mathcal{P}([a,b]_{\mathbb{T}})} L(f,P).$$

where $L(f,P)$ denote the lower Darboux sum of $f$ with respect to the partition $P$ and

$$L(f,P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}), m_i = \inf\{f(t) : t \in [t_{i-1}, t_i)_{\mathbb{T}}\}.$$

If $\overline{\int_a^b} f(t)\Delta t = \underline{\int_a^b} f(t)\Delta t$, then we say that $f$ is $\Delta-$integrable on $[a,b]_{\mathbb{T}}$, and the common value of the integrals, denoted by $\int_a^b f(t)\Delta t$, is called the Riemann $\Delta-$ integral. Similarly, we can give the definition of the Riemann $\nabla-$ integral.

We can easily get the following two theorems.

**Theorem 3.1** If $\gamma f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is Riemann $\Delta-$integrable and $(1-\gamma)f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ Riemann $\nabla-$integrable on the interval $[a,b]_{\mathbb{T}}$, then $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is Riemann $\diamondsuit-$integral on $[a,b]_{\mathbb{T}}$ and

$$\int_a^b f(t)\diamondsuit t = \int_a^b \gamma(t)f(t)\Delta t + \int_a^b (1-\gamma(t))f(t)\nabla t.$$

**Theorem 3.2** Let $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is Riemann $\diamondsuit-$integrable on the interval $[a,b]_{\mathbb{T}}$.

(1) If $\gamma(t) \equiv 1$, then $f$ is Riemann $\Delta-$integrable on $[a,b]_{\mathbb{T}}$.

(2) If $\gamma(t) \equiv 0$, then $f$ is Riemann $\nabla-$integrable on $[a,b]_{\mathbb{T}}$.

(3) If $0 < \gamma(t) < 1$, then $f$ is Riemann $\Delta-$integrable and Riemann $\nabla-$integrable on $[a,b]_{\mathbb{T}}$.

The proofs of the following two Theorem are standard and similar to [6, Theorem 5.5 and Theorem 5.6].

**Theorem 3.3** Let $L(f,P) = U(f,P)$ for some $P \in \mathcal{P}([a,b]_{\mathbb{T}})$, then the function $f$ is Riemann $\diamondsuit-$integrable on the interval $[a,b]_{\mathbb{T}}$ and

$$\int_a^b f(t)\diamondsuit t = L(f,P) = U(f,P).$$

**Theorem 3.4** (Cauchy criterion) Let $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be a bounded function on the interval $[a,b]_{\mathbb{T}}$. Then the function $f$ is Riemann $\diamondsuit-$integrable on the interval $[a,b]_{\mathbb{T}}$ if and only if for every $\epsilon > 0$ there exists a partition $P \in \mathcal{P}([a,b]_{\mathbb{T}})$ such that $U(f,P) - L(f,P) < \epsilon$.

The following Lemma can be found in [7].

**Lemma 3.5** Let $I = [a,b]_{\mathbb{T}}$ be a closed (bounded) interval in $\mathbb{T}$. For every $\delta > 0$ there is a partition $P_\delta = \{t_0, t_1, \ldots, t_n\} \in \mathcal{P}([a,b]_{\mathbb{T}})$ such that for each $i$ one has:

$$t_i - t_{i-1} \leq \delta \quad or \quad t_i - t_{i-1} > \delta \wedge \rho(t_i) = t_{i-1}.$$

The next theorem gives another Cauchy criterion for integrability.

**Theorem 3.6** A bounded function $f$ on $[a,b]_{\mathbb{T}}$ is Riemann $\diamondsuit-$integrable if and only if for each $\epsilon > 0$ there exists $\delta > 0$ such that $P_\delta \in \mathcal{P}([a,b]_{\mathbb{T}})$ implies

$$U(f, P_\delta) - L(f, P_\delta) < \epsilon.$$

**Proof.** If for each $\epsilon > 0$ there exists $\delta > 0$ such that $P_\delta \in \mathcal{P}([a,b]_{\mathbb{T}})$ implies

$$U(f, P_\delta) - L(f, P_\delta) < \epsilon,$$

then we have that $f$ is integrable on $[a,b]_{\mathbb{T}}$ by Theorem 3.4.

Conversely, suppose that $f$ is Riemann $\diamondsuit-$integrable on $[a,b]_{\mathbb{T}}$. If $\gamma(t) \equiv 1$ or $\gamma(t) \equiv 0$, then $f$ is Riemann $\Delta-$integrable or $\nabla-$integrable on $[a,b]_{\mathbb{T}}$. Therefore condition holds from [6, Theorem 5.9]. Now, let $0 < \gamma(t) < 1$, $f$ is Riemann $\diamondsuit-$ integrable, then $\gamma f$ is Riemann $\Delta-$integrable and $(1-\gamma)f$ is Riemann $\nabla-$integrable. For each $\epsilon > 0$ there exists $\delta' > 0$ and $\delta'' > 0$ such that $P_{\delta'} \in \mathcal{P}([a,b]_{\mathbb{T}})$, $P_{\delta''} \in \mathcal{P}([a,b]_{\mathbb{T}})$ we have

$$\overline{U}(\gamma f, P_{\delta'}) - \overline{L}(\gamma f, P_{\delta'}) < \frac{\epsilon}{2}, \quad \underline{U}((1-\gamma)f, P_{\delta''}) - \underline{L}((1-\gamma)f, P_{\delta''}) < \frac{\epsilon}{2}.$$

If $P_\delta \in \mathcal{P}([a,b]_{\mathbb{T}})$ where $\delta = \min\{\delta', \delta''\}$, then we have

$$U(f,P_\delta) - L(f,P_\delta) = \overline{U}(\gamma f, P_\delta) - \overline{L}(\gamma f, P_\delta) + \underline{U}((1-\gamma)f, P_\delta) - \underline{L}((1-\gamma)f, P_\delta) < \epsilon.$$

The Riemann $\diamondsuit-$integral has the following properties. Here we will not dwell with the proofs.

**Theorem 3.7** Let functions $f,\ g : \mathbb{T} \to \mathbb{R}$ be Riemann $\diamondsuit-$integrable on the interval $[a,b]_{\mathbb{T}}$, $a < b < c$ and $\alpha, \beta$ be arbitrary real numbers. Then,

(1) $\alpha f \pm \beta g$ is Riemann $\diamondsuit-$integrable on the interval $[a,b]_{\mathbb{T}}$ and

$$\int_a^b (\alpha f(t) \pm \beta g(t)) \diamondsuit t = \alpha \int_a^b f(t) \diamondsuit t \pm \beta \int_a^b g(t) \diamondsuit t.$$

(2) $\int_a^c f(t) \diamondsuit t + \int_c^b f(t) \diamondsuit t = \int_a^b f(t) \diamondsuit t$.

(3) if $f \le g$ for $t \in [a,b]_{\mathbb{T}}$, then $\int_a^b f(t) \diamondsuit t \le \int_a^b g(t) \diamondsuit t$.

(4) $|f|$ is Riemann $\diamondsuit-$integrable on the interval $[a,b]_{\mathbb{T}}$ and $|\int_a^b f(t) \diamondsuit t| \le \int_a^b |f(t)| \diamondsuit t$.

(5) $fg$ is Riemann $\diamondsuit-$integrable on the interval $[a,b]_{\mathbb{T}}$.

The following theorem may be proved in much the same way as [6, Theorem 5.18, 5.19, 5.20, 5.21.].

**Theorem 3.8** Let $I=[a,b]_{\mathbb{T}}$, where $a,b\in\mathbb{T}$.

(¡) Every monotone function $f$ is Riemann $\diamondsuit-$integrable on $[a,b]_{\mathbb{T}}$.

(¡¡) Every continuous function $f$ is Riemann $\diamondsuit-$integrable on $[a,b]_{\mathbb{T}}$.

(¡¡¡) Every bounded function $f$ with only finitely many discontinuity points is Riemann $\diamondsuit-$integrable on $[a,b]_{\mathbb{T}}$.

(¡¡¡¡) Every regulated function $f$ is Riemann $\diamondsuit-$integrable on $[a,b]_{\mathbb{T}}$.

**Theorem 3.9** Let $f:\mathbb{T}\to\mathbb{R}$ and $t\in\mathbb{T}$. Then, $f$ is Riemann $\diamondsuit-$integrable on $[t,\sigma(t)]_{\mathbb{T}}$ and

$$\int_t^{\sigma(t)} f(s)\diamondsuit s=\mu(t)(f(t)+f^{\sigma}(t)).$$

**Proof** If $t=\sigma(t)$, then the equality is obvious. If $t<\sigma(t)$, then $\mathcal{P}([t,\sigma(t)]_{\mathbb{T}})$ contains only one element given by

$$t=s_0<s_1=\sigma(t).$$

Since $[s_0,s_1)_{\mathbb{T}}=\{t\}$ and $(s_0,s_1]_{\mathbb{T}}=\{\sigma(t)\}$, we have

$$U(f,P)=L(f,P)=\gamma(t)f(t)(\sigma(t)-t)+(1-\gamma(\sigma(t)))f^{\sigma}(t)(\sigma(t)-t)=\mu(t)(f(t)+f^{\sigma}(t)).$$

By Theorem 3.3, $f$ is Riemann $\diamondsuit-$integrable on $[t,\sigma(t)]_{\mathbb{T}}$ and

$$\int_t^{\sigma(t)} f(s)\diamondsuit s=\mu(t)(f(t)+f^{\sigma}(t)).$$

**Theorem 3.10** Let $f:\mathbb{T}\to\mathbb{R}$ and $t\in\mathbb{T}$. Then, $f$ is Riemann $\diamondsuit-$integrable on $[\rho(t),t]_{\mathbb{T}}$ and

$$\int_{\rho(t)}^{t} f(s)\diamondsuit s=\eta(t)(f^{\rho}(t)+f(t)).$$

**Proof** If $t=\rho(t)$, then the equality is obvious. If $t>\rho(t)$, then $[\rho(t),t]_{\mathbb{T}}$ contains only one element given by

$$\rho(t)=s_0<s_1=t.$$

Since $[s_0, s_1)_{\mathbb{T}} = \{\rho(t)\}$ and $(s_0, s_1]_{\mathbb{T}} = \{t\}$, we have

$$U(f,P) = L(f,P) = \gamma(\rho(t))f^{\rho}(t)(t-\rho(t))+(1-\gamma(t))f(t)(t-\rho(t)) = \eta(t)(f^{\rho}(t)+f(t)).$$

By Theorem 3.3, $f$ is Riemann $\diamondsuit-$integrable on $[\rho(t), t]_{\mathbb{T}}$ and

$$\int_{\rho(t)}^{t} f(s)\diamondsuit s = \eta(t)(f^{\rho}(t) + f(t)).$$

By the definition of the Riemann $\diamondsuit-$integral, we have the following Corollary:

**Corollary** Let $a, b \in \mathbb{T}$ and $a < b$. Then we have the following:

(1) If $\mathbb{T} = \mathbb{R}$, then a bounded function $f$ is Riemann $\diamondsuit-$integrable on the interval $[a, b]_{\mathbb{T}}$ if and only if $f$ is Riemann integrable on $[a, b]_{\mathbb{T}}$ in the classical sense, and in this case

$$\int_a^b f(t)\diamondsuit t = \int_a^b f(t)dt.$$

(2) If $\mathbb{T} = \mathbb{Z}$, then each function $f : \mathbb{Z} \to \mathbb{R}$ is Riemann $\diamondsuit-$integrable on the interval $[a, b]_{\mathbb{T}}$. Moreover

$$\int_a^b f(t)\diamondsuit t = \sum_{t=a+1}^{b-1} f(t) + f(a) + f(b).$$

(3) If $\mathbb{T} = h\mathbb{Z}$, then each function $f : h\mathbb{Z} \to \mathbb{R}$ is Riemann $\diamondsuit-$integrable on the interval $[a, b]_{\mathbb{T}}$. Moreover

$$\int_a^b f(t)\diamondsuit t = \sum_{k=\frac{a}{h}+1}^{\frac{b}{h}-1} f(kh) + f(a)h + f(b)h.$$

**Example** Let $f : \{1, 2, 3\} \to \mathbb{R}$ be defined by $f(t) = t$. Then,

$$\int_1^3 f(t)\diamondsuit t = \gamma(1)f(1)+(1-\gamma(2))f(2)+\gamma(2)f(2)+(1-\gamma(3))f(3) = 1+2+3 = 6.$$

## 4. Generalized Inequalities

In this section, we will establish generalized Hölder's inequality, Cauchy-Schwarz's inequality, Minkowski's inequality and Jensen's inequality on time scales.

**Theorem 4.1** (Generalized Hölder's inequality) Let $f, g, h \in C_{ld}([a,b]_{\mathbb{T}}, R)$ and $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$; then

$$\int_a^b |h(t)||f(t)g(t)|\diamondsuit t \leq \Big(\int_a^b |h(t)||f(t)|^p \diamondsuit t\Big)^{\frac{1}{p}} \Big(\int_a^b |h(t)||g(t)|^q \diamondsuit t\Big)^{\frac{1}{q}}.$$

**Proof** For nonnegative real numbers $\alpha, \beta$ and $p, q$ such that $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$, we have the well-known Young's inequality $\alpha^{\frac{1}{p}} \beta^{\frac{1}{q}} \leq \frac{\alpha}{p} + \frac{\beta}{q}$.

Without loss of generality, we suppose that

$$\Big(\int_a^b |h(t)||f(t)|^p \diamondsuit t\Big)\Big(\int_a^b |h(t)||g(t)|^q \diamondsuit t\Big) \neq 0.$$

Let

$$\alpha(t) = \frac{|h(t)||f(t)|^p}{\int_a^b |h(t)||f(t)|^p \diamondsuit t}, \quad \beta(t) = \frac{|h(t)||g(t)|^q}{\int_a^b |h(t)||g(t)|^q \diamondsuit t}.$$

Consequently we have that

$$\begin{aligned}
& \int_a^b \frac{|h(t)|^{\frac{1}{p}}|f(t)|}{\Big(\int_a^b |h(t)||f(t)|^p \diamondsuit t\Big)^{\frac{1}{p}}} \frac{|h(t)|^{\frac{1}{q}}|g(t)|}{\Big(\int_a^b |h(t)||g(t)|^q \diamondsuit t\Big)^{\frac{1}{q}}} \diamondsuit t \\
= & \int_a^b \alpha^{\frac{1}{p}}(t) \beta^{\frac{1}{q}}(t) \diamondsuit t \\
\leq & \int_a^b \Big(\frac{\alpha(t)}{p} + \frac{\beta(t)}{q}\Big) \diamondsuit t \\
= & \int_a^b \Big(\frac{1}{p} \frac{|h(t)||f(t)|^p}{\int_a^b |h(t)||f(t)|^p \diamondsuit t} + \frac{1}{q} \frac{|h(t)||g(t)|^q}{\int_a^b |h(t)||g(t)|^q \diamondsuit t}\Big) \diamondsuit t \\
= & \frac{1}{p} + \frac{1}{q} = 1,
\end{aligned}$$

which completes the proof. For the particular case $p = q = 2$ in Theorem 4.1, we obtain the Cauchy-Schwarz's inequality.

**Theorem 4.2** (Generalized Cauchy-Schwarz's Inequality) Let $f, g, h$ be $\diamondsuit-$integrable on the interval $[a,b]_{\mathbb{T}}$, then

$$\int_a^b |h(t)||f(t)g(t)|\diamondsuit t \leq \sqrt{\Big(\int_a^b |h(t)||f(t)|^2 \diamondsuit t\Big)\Big(\int_a^b |h(t)||g(t)|^2 \diamondsuit t\Big)}.$$

**Theorem 4.3** (Generalized Minkowski's inequality) Let $f, g, h$ be $\diamondsuit-$integrable on the interval $[a,b]_{\mathbb{T}}$ and $p > 1$, then

$$\Big(\int_a^b |h(t)||f(t)+g(t)|^p \diamondsuit t\Big)^{\frac{1}{p}} \leq \Big(\int_a^b |h(t)||f(t)|^p \diamondsuit t\Big)^{\frac{1}{p}} + \Big(\int_a^b |h(t)||g(t)|^p \diamondsuit t\Big)^{\frac{1}{p}}.$$

**Proof** It follows from Theorem 4.1 that

$$\begin{aligned}
&\int_a^b |h(t)||f(t)+g(t)|^p \diamondsuit t \\
= &\int_a^b |h(t)||f(t)+g(t)|^{p-1}(|f(t)+g(t)|)\diamondsuit t \\
\leq &\int_a^b |h(t)||f(t)+g(t)|^{p-1}(|f(t)|+|g(t)|)\diamondsuit t \\
= &\int_a^b |h(t)||f(t)+g(t)|^{p-1}|f(t)|\diamondsuit t + \int_a^b |h(t)||f(t)+g(t)|^{p-1}|g(t)|\diamondsuit t \\
\leq &\Big\{\int_a^b |h(t)|\Big(|f(t)+g(t)|^{p-1}\Big)^q \diamondsuit t\Big\}^{\frac{1}{q}}\Big(\int_a^b |h(t)||f(t)|^p\diamondsuit t\Big)^{\frac{1}{p}} \\
&+\Big\{\int_a^b |h(t)|\Big(|f(t)+g(t)|^{p-1}\Big)^q \diamondsuit t\Big\}^{\frac{1}{q}}\Big(\int_a^b |h(t)||g(t)|^p\diamondsuit t\Big)^{\frac{1}{p}} \\
= &\Big\{\int_a^b |h(t)||f(t)+g(t)|^p\diamondsuit t\Big\}^{\frac{1}{q}}\Big\{\Big(\int_a^b |h(t)||f(t)|^p\diamondsuit t\Big)^{\frac{1}{p}}+\Big(\int_a^b |h(t)||g(t)|^p\diamondsuit t\Big)^{\frac{1}{p}}.
\end{aligned}$$

Dividing both sides by

$$\Big\{\int_a^b |h(t)||f(t)+g(t)|^p\diamondsuit t\Big\}^{\frac{1}{q}},$$

we arrive to Minkowskis inequality:

$$\Big(\int_a^b |h(t)||f(t)+g(t)|^p\diamondsuit t\Big)^{\frac{1}{p}} \leq \Big(\int_a^b |h(t)||f(t)|^p\diamondsuit t\Big)^{\frac{1}{p}}+\Big(\int_a^b |h(t)||g(t)|^p\diamondsuit t\Big)^{\frac{1}{p}}.$$

**Theorem 4.4** (Jensen's inequality) Let $a, b \in \mathbb{T}$ and $c, d \in \mathbb{R}$. If $g : [a, b]_{\mathbb{T}} \to (c, d)$ is rd-continuous and $f : (c, d) \to \mathbb{R}$ is continuous and convex, then

$$f\Big(\frac{\int_a^b g(t)\diamondsuit t}{b-a}\Big) \leq \frac{\int_a^b f(g(t))\diamondsuit t}{b-a}.$$

**Proof** Let $x_0 \in (c, d)$. Then for each $x \in (c, d)$, there exists $\beta$ such that

$$f(x) - f(x_0) \geq \beta(x - x_0).$$

Let $x_0 = \frac{\int_a^b g(t)\diamondsuit t}{b-a}$, Thus,

$$\begin{aligned} & \int_a^b f(g(t))\diamondsuit t - (b-a)f\Big(\frac{\int_a^b g(t)\diamondsuit t}{b-a}\Big) \\ = & \int_a^b f(g(t))\diamondsuit t - (b-a)f(x_0) \\ = & \int_a^b (f(g(t)) - f(x_0))\diamondsuit t \\ \geq & \beta\int_a^b (g(t) - x_0)\diamondsuit t \\ = & \beta\int_a^b g(t)\diamondsuit t - (b-a)x_0 = 0, \end{aligned}$$

which completes our proof.

Similarly, we have the following Generalized Jensen's inequality.

**Theorem 4.5** (Generalized Jensen's inequality) Let $a, b \in \mathbb{T}$ and $c, d \in \mathbb{R}$. If $g : [a,b]_{\mathbb{T}} \to (c,d)$, $h : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is rd-continuous with $\int_a^b |h(t)|\diamondsuit t > 0$ and $f : (c,d) \to \mathbb{R}$ is continuous and convex, then

$$f\Big(\frac{\int_a^b |h(t)|g(t)\diamondsuit t}{\int_a^b |h(t)|\diamondsuit t}\Big) \leq \frac{\int_a^b |h(t)|f(g(t))\diamondsuit t}{\int_a^b |h(t)|\diamondsuit t}.$$

## References

[1] G. A. Anastassiou, Frontiers in Time Scales and Inequalities, Series on Concrete and Applicable Mathematics, Vol.17, World Scientific, 2015.

[2] M. R. Sidi Ammi, R. A. C. Ferreira, D. F. M. Torres, Diamond$-\alpha$ Jensen's inequality on time scales, J. Inequal. Appl., 2008, Art. ID 576876, 13 pp.

[3] M. R. Sidi Ammi, D. F. M. Torres, Hölder's and Hardy's two dimensional diamond-alpha inequalities on time scales, An. Univ. Craiova Ser. Mat. Inform., 2010, 37(1):1–11.

[4] M. Bohner, G. S. Guseinov, The convolution on time scales, Abstr. Appl. Anal., 2007, Art. ID 58373, 24 pp.

[5] M. Bohner, A. Peterson, Dynamic equations on time scales. An introduction with applications, Birkhauser, Boston, MA, 2001.

[6] M. Bohner, A. Peterson, Advances in Dynamic Equations on Time Scales, Birkhauser, Boston, 2004.

[7] A. M. C. Brito da Cruz, N. Martins, D. F. M. Torres, Symmetric differentiation on time scales, Appl. Math. Lett., 2013, 26(2):264–269.

[8] A. M. C. Brito da Cruz, N. Martins, D. F. M. Torres, The Diamond Integral on Time Scales, Bull. Malays. Math. Sci. Soc., 2015, 38(4):1453–1462.

[9] S. Hilger, Eιn Maßkettenkalkül mιt Anwendung auf Zentrumsmannιgfaltιgkeιten, Ph. D. Thesis, Universtät Würzburg, 1988.

[10] S. Hilger, Analysis on measure chains-A unified approach to continuous and discrete calculus, Results Math. 1990, (18):18–56.

[11] A. B. Malinowska, D. F. M. Torres, On the diamond-alpha Riemann integral and mean value theorems on time scales, Dynam. Systems Appl., 2009, 18(3-4):469–481.

[12] D. Mozyrska, D. F. M. Torres, Diamond$-\alpha$ polynomial series on time scales, Int. J. Math. Stat., 2009, 5:92–101.

[13] D. Mozyrska, D. F. M. Torres, A Study of Diamond-alpha dynamic equations on regular time scales, Afr. Diaspora J. Math., 2009, 8(1):35–47.

[14] A. Peterson, B. Thompson, Henstock-Kurzweil Delta and Nabla Integrals, J. Math. Anal. Appl. 2006, 323:162–178.

[15] J. W. Rogers. Jr., Q. Sheng, Notes on the diamond$-\alpha$ dynamic derivative on time scales, J. Math. Anal. Appl. 2007, 326(1):228–241.

[16] Q. Sheng, M. Fadag, J. Henderson, J. M. Davis, An exploration of combined dynamic derivatives on time scales and their applications, Nonlinear Anal. Real World Appl., 2006, 7(3):395–413.

# CONTINUITY AND CONTINUOUS HOMOGENEOUS SELECTIONS OF SET-VALUED METRIC GENERALIZED INVERSE IN BANACH SPACES

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Abstract. In this paper, upper semicontinuity and continuity for the set-valued metric generalized inverses $T^{\partial}$ in Banach spaces are investigated by metric projection operator. Moreover, criteria for the set-valued metric generalized inverses to have continuous homogeneous selections are given. Finally, the relation of continuity and continuous selection of the set-valued metric generalized inverse are given.

## 1. Introduction and preliminaries

Let $(X, \|\cdot\|)$ be a real Banach space. Let $S(X)$ and $B(X)$ denote the unit sphere and the unit ball of $X$, respectively. By $X^*$ we denote the dual space of $X$. Let $T$ denote a linear bounded operator from subspace of $X$ into Banach space $Y$. Let $D(T)$, $R(T)$ and $N(T)$ denote the domain, range and null space of $T$, respectively. Let $L$ be a subspace of $X$. The set-valued mapping $P_L : X \to L$

$$P_L(x) = \left\{ z \in L : \|x - z\| = \operatorname{dist}(x, L) := \inf_{y \in L} \|x - y\| \right\}$$

is said to be the metric projection operator from $X$ onto $L$. A subspace $L$ is said to be proximinal if $P_L(x) \neq \emptyset$ for all $x \in X$. Continuity of metric projection operator is an important content in geometry of Banach spaces. Moreover, metric projection operator plays an important role in the optimization, computational mathematics, theory of equation and control theory.

The concept of generalized inverses has been extensively studied in the last decades, which has its genetic in the context of the so-called "ill-posed" linear problems. If $N(T) \neq \{0\}$ or $R(T) \neq Y$, the operator equation $Tx = y$ is generally ill-posed, i.e., there exists $y_0 \in Y$ such that $\|Tx - y_0\| \neq 0$ for any $x \in D(T)$. In order to solve the best approximation problems for ill-posed linear operator equations in Banach spaces, it is necessary to study the set-valued metric generalized inverses of linear operators between Banach spaces. In 1974, Nashed and Votruba [9] introduced the concept of the set-valued metric generalized inverse of a linear operator between Banach spaces and they raised the following research

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suggestion "The problem of obtaining selections with nice properties for the metric generalized inverse merits study". Moreover, it is well known that set-valued metric generalized inverses is a set-valued mapping. Hence upper semicontinuity and continuity of set-valued metric generalized inverses merit study. In 2015, Shang and Cui [8] gave a criteria for upper semicontinuity of the set-valued metric generalized inverses in approximative compact spaces. In this paper, upper semicontinuity and continuity for the set-valued metric generalized inverses $T^{\partial}$ in Banach spaces are investigated by metric projection operator. Moreover, criteria for the set-valued metric generalized inverses to have continuous homogeneous selection are given. Finally, the relation of continuity and continuous selections of the set-valued metric generalized inverse are given. First let us recall some definitions that will be used in the further part of the paper.

**Definition 1.1.** (see [7]) A subspace $L \subset X$ is said to be $k$-Chebyshev subspace if $L$ is proximinal and for any $x \in X$, we have $\dim(\operatorname{span}\{x - P_L(x)\}) \leq k$.

I. Singer defined the $k$-strictly convex spaces in [15]. He proved that if $X$ is reflexive and $k$-strictly convex, then every closed subspace of $X$ is $k$-Chebyshev subspace. Moreover, it is easy to see that if $L$ is a Chebyshev subspace, then $L$ is $k$-Chebyshev.

**Definition 1.2.** (see [4]) Set-valued mapping $F : X \to Y$ is said to be upper semicontinuous at $x_0$, if for each norm open set $W$ with $F(x_0) \subset W$, there exists a norm neighborhood $U$ of $x_0$ such that $F(x) \subset W$ for all $x$ in $U$. $F$ is called lower semicontinuous at $x_0$, if for any $y \in F(x_0)$ and any $\{x_n\}_{n=1}^{\infty}$ in $X$ with $x_n \to x_0$, there exists $y_n \in F(x_n)$ such that $y_n \to y$ as $n \to \infty$. $F$ is called continuous at $x_0$, if $F$ is upper semicontinuous and is lower semicontinuous at $x_0$.

**Definition 1.3.** (see [14]) A closed subspace $N$ of $X$ is said to be a topologically complemented subspace of $X$, if there exists a closed subspace $M$ of $X$ such that $M \oplus N = X$.

**Definition 1.4.** A subspace $L \subset X$ is said to be maximal subspace of $X$ if there exists $x^* \in S(X^*)$ such that $L = \{x \in X : x^*(x) = 0\}$.

**Definition 1.5.** (see [6]) A Banach space $X$ is said to be nearly convex, if every closed convex set of $S(X)$ is compact.

**Definition 1.6.** (see [9]) A point $x_0 \in D(T)$ is said to be the best approximative solution to the operator equation $Tx = y$, if

$$\|Tx_0 - y\| = \inf\{\|Tx - y\| : x \in D(T)\}$$

and

$$\|x_0\| = \min\left\{\|v\| : v \in D(T),\ \|Tv - y\| = \inf_{x \in D(T)} \|Tx - y\|\right\}.$$

**Definition 1.7.** (see [9]) Let $X, Y$ be Banach spaces, $T$ be a linear bounded operator from subspace of $X$ to $Y$ and $D(T)$ be the domain of $T$. The set-valued mapping $T^{\partial} : D(T^{\partial}) \to X$ defined by

$$T^{\partial}(y) = \{x_0 \in D(T) : x_0 \text{ is a best approximative solution to } T(x) = y\}$$

for any $y \in D(T^{\partial})$, is said to be the (set-valued) metric generalized inverse of $T$, where

$$D(T^{\partial}) = \{y \in Y : T(x) = y \text{ has a best approximative solution in } X\}.$$

## 2. Continuity of the set-valued metric generalized inverse in Banach spaces

**Theorem 2.1.** *Let $X$ be a nearly convex space, $Y$ be a Banach space, $T$ be a linear bounded operator from subspace of $X$ into $Y$, $D(T)$ be a closed subspace of $X$ and $R(T)$ be a 2-Chebyshev maximal subspace of $Y$. Then*

*(1) $T^{\partial}$ is upper semicontinuous on $Y$ if and only if $P_{N(T)}$ is upper semicontinuous on $D(T)$;*

*(2) $T^{\partial}$ is continuous if and only if $TT^{\partial}$ is lower semicontinuous and $P_{N(T)}$ is continuous on $D(T)$.*

*Proof.* (1) "⇒" We first will prove that the metric projector operator $P_{R(T)}$ is continuous and $P_{R(T)}(y)$ is a line segment. In fact, by Lemma 1 of [8], we know that if closed subspace $H$ is a 2-Chebyshev subspace of $X$, then $P_H(x)$ is a line segment for any $x \in X$. Hence $P_{R(T)}(y)$ is a line segment.

Since $R(T)$ is an 2-Chebyshev maximal subspace of $Y$, there exists $f \in S(X^*)$ such that $R(T) = \{y \in Y : f(y) = 0\}$. Let $y \in Y$. Pick $z \in R(T)$ and $h \in S(Y)$. Then there exists $\alpha \in R$ such that $y - z = \alpha h$. It is easy to see that $\alpha = f(y)/f(h)$. Then $y - z = (f(y)/f(h))h$. Hence $\|y - z\| = |f(y)| / |f(h)| \geq |f(y)|$. Then it is easy to see that $z \in P_{R(T)}(y)$ if and only if $h \in A_f$. Hence $P_{R(T)}(y) = y - f(y)A_f$, where $A_f = \{y \in S(Y) : f(y) = 1\}$.

Suppose that metric projection operator $P_{R(T)}$ is not upper semicontinuous at $y_0$. Then there exist a sequence $\{y_n\}_{n=1}^{\infty} \subset Y$ and an open set $W \supset P_{R(T)}(y_0)$ such that $P_{R(T)}(y_n) \not\subset W$ and $y_n \to y_0$ as $n \to \infty$. Hence there exists $z_n \in P_{R(T)}(y_n)$ such that $z_n \notin W$. Then $z_n = y_n - f(y_n)h_n$, where $h_n \in A_f$. Since $P_{R(T)}(y)$ is a line segment and $P_{R(T)}(y) = y - f(y)A_f$, we obtain that $A_f$ is a line segment. Hence there exists a subsequence $\{h_{n_k}\}_{k=1}^{\infty}$ of $\{h_n\}_{n=1}^{\infty}$ such that $h_{n_k} \to h_0 \in A_f$ as $k \to \infty$. Let $z_0 = y_0 - f(y_0)h_0$. Then $z_0 \in P_{R(T)}(y_0)$ and

$$\lim_{k\to\infty} z_{n_k} = \lim_{k\to\infty} (y_{n_k} - f(y_{n_k})h_{n_k}) = y_0 - f(y_0)h_0 = z_0,$$

a contradiction. This implies that $P_{R(T)}$ is upper semicontinuous.

Let $y_n \to y_0$ as $n \to \infty$. Pick $z_0 \in P_{R(T)}(y_0)$. Then there exists $h_0 \in A_f$ such that $z_0 = y_0 - f(y_0)h_0$. Hence $z_n = y_n - f(y_n)h_0 \in P_{R(T)}(x_n)$ and

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} (y_n - f(y_n)h_0) = y_0 - f(y_0)h_0 = z_0$$

This implies that $P_{R(T)}$ is lower semicontinuous at $y_0$. Hence we obtain that $P_{R(T)}$ is continuous.

Pick $y_0 \in Y$. Suppose that $T^{\partial}$ is not upper semicontinuous at $y_0$. Then there exist a sequence $\{y_n\}_{n=1}^{\infty} \subset Y$, $y_n \to y_0 \in Y$ and norm open set $W$ with $T^{\partial}(y_0) \subset W$ such that $T^{\partial}(y_n) \not\subset W$ for all $n \in N$. Hence there exists $x_n \in T^{\partial}(y_n) \subset X$ such that $x_n \notin W$. Since $T$ is a bounded linear operator, we obtain that $N(T)$ is

a closed subspace of $D(T)$. Let

$$\overline{T} : D(T)/N(T) \to R(T), \qquad \overline{T}[x] = Tx,$$

where $[x] \in D(T)/N(T)$ and $x \in D(T)$. Then it is easy to see that $R(\overline{T}) = R(T)$. Moreover, $\overline{R(T)} = R(T)$. In fact, suppose that $\overline{R(T)} \neq R(T)$. Then there exists $y' \in \overline{R(T)}$ such that $y' \notin R(T)$. It is easy to see that $\{y \in R(T) : \|y' - y\| = \text{dist}(y', R(T))\} = \emptyset$. This implies that $R(T)$ is not a 2-Chebyshev subspace of $Y$, a contradiction. Since $\overline{R(T)} = R(T)$, we obtain that $R(T)$ is a Banach space. Moreover, it is easy to see that $\overline{T}$ is a bounded linear operator and $N(\overline{T}) = \{0\}$. This implies that the bounded linear operator $\overline{T}$ is both injective and surjective. Therefore, by the inverse operator theorem, we obtain that the operator $\overline{T}^{-1}$ is a bounded linear operator.

Let $P_{R(T)}(y_0) = [y(1,0), y(2,0)]$. Since $\overline{T}^{-1}$ is a bounded linear operator and $[y(1,0), y(2,0)]$ is a compact set, we obtain that the infimum

$$\inf\left\{\left\|\overline{T}^{-1}(z)\right\| : z \in [y(1,0), y(2,0)]\right\}$$

is attainable on $[y(1,0), y(2,0)]$. Let

$$A(0) = \left\{y \in [y(1,0), y(2,0)] : \left\|\overline{T}^{-1}(y)\right\| = \inf\{\left\|\overline{T}^{-1}(z)\right\| : z \in [y(1,0), y(2,0)]\}\right\}.$$

It is easy to see that $A(0)$ is a closed set. Moreover, if $z_1 \in A(0)$ and $z_2 \in A(0)$, then

$$\begin{aligned}\left\|\overline{T}^{-1}(\lambda z_1 + (1-\lambda)z_2)\right\| &\leq \lambda\left\|\overline{T}^{-1}(z_1)\right\| + (1-\lambda)\left\|\overline{T}^{-1}(z_2)\right\| \\ &= \inf\left\{\left\|\overline{T}^{-1}(z)\right\| : z \in [y(1,0), y(2,0)]\right\},\end{aligned}$$

where $\lambda \in [0,1]$. This implies that the set $A(0)$ is a closed convex set. Hence there exist $z(1,0) \in [y(1,0), y(2,0)]$ and $z(2,0) \in [y(1,0), y(2,0)]$ such that $A(0) = [z(1,0), z(2,0)]$. Let $P_{R(T)}(y_n) = [y(1,n), y(2,n)]$. Then there exist $z(1,n) \in [y(1,n), y(2,n)]$ and $z(2,n) \in [y(1,n), y(2,n)]$ such that

$$\begin{aligned}& A(n) \\ = \;& \left\{y \in [y(1,n), y(2,n)] : \left\|\overline{T}^{-1}(y)\right\| = \inf\left\{\left\|\overline{T}^{-1}(z)\right\| : z \in [y(1,n), y(2,n)]\right\}\right\} \\ = \;& [z(1,n), z(2,n)].\end{aligned}$$

Since $P_{R(T)}$ is continuous, we may assume without loss of generality that

$$\lim_{n\to\infty} y(1,n) = z_1 \in [y(1,0), y(2,0)] \quad \text{and} \quad \lim_{n\to\infty} y(2,n) = z_2 \in [y(1,0), y(2,0)]. \tag{2.1}$$

We claim that $[z_1, z_2] \in [z(1,0), z(2,0)]$. Otherwise, we may assume without loss of generality that $z_1 \notin [z(1,0), z(2,0)]$. Hence there exists $r > 0$ such that

$$\left\|\overline{T}^{-1}(z_1)\right\| > \inf\left\{\left\|\overline{T}^{-1}(z)\right\| : z \in [y(1,0), y(2,0)]\right\} + 4r.$$

Since $\overline{T}^{-1}$ is a bounded linear operator and $y(1,n) \to z_1$ as $n \to \infty$, we may assume without loss of generality that

$$\left\|\overline{T}^{-1}\left(z(1,n)\right)\right\| > \inf\left\{\left\|\overline{T}^{-1}(z)\right\| : z \in [y(1,0), y(2,0)]\right\} + 2r \tag{2.2}$$

for every $n \in N$. Since the metric projection operator $P_{R(T)}$ is continuous, there exists $z(n) \in [y(1,n), y(2,n)]$ such that $z(n) \to z(1,0)$ as $n \to \infty$. Since $\overline{T}^{-1}$ is a bounded linear operator and $z(n) \to z(1,0)$ as $n \to \infty$, we have

$$\lim_{n\to\infty}\left\|\overline{T}^{-1}(z(n))\right\| = \left\|\overline{T}^{-1}(z(1,0))\right\| = \inf\left\{\left\|\overline{T}^{-1}(z)\right\| : z \in [y(1,0), y(2,0)]\right\}.$$

Therefore, by formula (2.2), we may assume without loss of generality that

$$\left\|\overline{T}^{-1}\left(z(1,n)\right)\right\| > \left\|\overline{T}^{-1}(z(n))\right\| + r$$

for all $n \in N$, a contradiction.

Pick $x(1,0) \in \overline{T}^{-1}(z_1)$, $x(2,0) \in \overline{T}^{-1}(z_2)$, $x(1,n) \in \overline{T}^{-1}\left(z(1,n)\right)$ and $x(2,n) \in \overline{T}^{-1}\left(z(2,n)\right)$. Then we have $[x(1,n)] = \overline{T}^{-1}\left(z(1,n)\right)$ and $[x(2,n)] = \overline{T}^{-1}\left(z(2,n)\right)$. Since $\overline{T}^{-1}$ is a bounded linear operator, by formula (2.1), we have

$$\begin{aligned}\|[x(1,n) - x(1,0)]\| &= \|[x(1,n)] - [x(1,0)]\| \leq \left\|\overline{T}^{-1}\left(z(1,n)\right) - \overline{T}^{-1}z_1\right\| \\ &\leq \left\|\overline{T}^{-1}\right\| \|z(1,n) - z_1\| \to 0 \ \text{ as } \ n \to \infty.\end{aligned}$$

Hence we may assume without loss of generality that $x(1,n) \to x(1,0)$ as $n \to \infty$. Similarly, we may assume without loss of generality that $x(2,n) \to x(2,0)$ as $n \to \infty$. Moreover, by the definition of set-valued metric generalized inverse, there exists a sequence $\{\lambda_n\}_{n=1}^{\infty} \subset [0,1]$ such that

$$x_n = \lambda_n x(1,n) + (1-\lambda_n)x(2,n) - \pi_{N(T)}\left(\lambda_n x(1,n) + (1-\lambda_n)x(2,n)\right),$$

where

$$\pi_{N(T)}\left(\lambda_n x(1,n) + (1-\lambda_n)x(2,n)\right) \in P_{N(T)}\left(\lambda_n x(1,n) + (1-\lambda_n)x(2,n)\right).$$

We may assume without loss of generality that $\lambda_n \to \lambda$ as $n \to \infty$. Then

$$\lim_{n\to\infty} \lambda_n x(1,n) + (1-\lambda_n)x(2,n) = \lambda z_1 + (1-\lambda)z_2 \in [z_1, z_2]. \tag{2.3}$$

Since $P_{N(T)}$ is upper semicontinuous, by formula (2.3), we obtain that for any $\varepsilon > 0$, there exists $n_0 \in N$ such that

$$\pi_{N(T)}\left(\lambda_n x(1,n) + (1-\lambda_n)x(2,n)\right) \in \bigcup_{x \in P_{N(T)}(\lambda z_1 + (1-\lambda)z_2)} B(x,\varepsilon)$$

whenever $n > n_0$. This implies that

$$\operatorname{dist}\left(\{\lambda_n x(1,n) + (1-\lambda_n)x(2,n)\}_{n=1}^{\infty}, P_{N(T)}(\lambda z_1 + (1-\lambda)z_2)\right) = 0.$$

Hence, for any $k > 0$, there exists $h_{n_k} \in P_{N(T)}(\lambda z_1 + (1-\lambda)z_2)$ such that

$$\left\|\pi_{N(T)}(\lambda_{n_k} x(1,n_k) + (1-\lambda_{n_k})x(2,n_k)) - h_{n_k}\right\| < \frac{1}{k}. \tag{2.4}$$

Moreover, there exists $r > 0$ such that

$$\lambda z_1 + (1-\lambda)z_2 - P_{N(T)}(\lambda z_1 + (1-\lambda)z_2) \subset rS(X).$$

Since $X$ is a nearly convex space, we obtain that $\lambda z_1 + (1-\lambda)z_2 - P_{N(T)}(\lambda z_1 + (1-\lambda)z_2)$ is compact. Then the set $P_{N(T)}(\lambda z_1 + (1-\lambda)z_2)$ is compact. Hence we may assume without loss of generality that $h_{n_k} \to h \in P_{N(T)}(\lambda z_1 + (1-\lambda)z_2)$ as $n \to \infty$. Therefore, by formula (2.4), we have

$$\lim_{k\to\infty} \pi_{N(T)}\left(\lambda_{n_k} x(1,n_k) + (1-\lambda_{n_k})x(2,n_k)\right) = h \in P_{N(T)}(\lambda z_1 + (1-\lambda)z_2).$$

This implies that

$$\begin{aligned} x_k &= \lambda_{n_k} x(1,n_k) + (1-\lambda_{n_k})x(2,n_k) - \pi_{N(T)}\left(\lambda_{n_k} x(1,n_k) + (1-\lambda_{n_k})x(2,n_k)\right) \\ &\to \lambda z_1 + (1-\lambda)z_2 - h \\ &\in \lambda z_1 + (1-\lambda)z_2 - P_{N(T)}(\lambda z_1 + (1-\lambda)z_2). \end{aligned}$$

Moreover, by the definition of set-valued metric generalized inverse, we obtain that $\lambda z_1 + (1-\lambda)z_2 - P_{N(T)}(\lambda z_1 + (1-\lambda)z_2) \subset T^{\partial}(y) \subset W$. Since $W$ is a norm open set, we have $x_k \in W$ for $k$ large enough, a contradiction.

"$\Leftarrow$" Suppose that $P_{N(T)}$ is not upper semicontinuous on $D(T)$. Then there exist $\{x_n\}_{n=1}^{\infty} \subset D(T)$, $x_0 \in D(T)$ and a norm open set $W$ such that $x_n \to x_0$, $P_{N(T)}(x_0) \subset W$ and $P_{N(T)}(x_n) \not\subset W$. Hence there exists $\pi_{N(T)}(x_n) \in P_{N(T)}(x_n)$ such that $\pi_{N(T)}(x_n) \notin W$. We claim that there exists $\delta > 0$ such that

$$\bigcup_{z\in P_{N(T)}(x_0)} B(z, 2\delta) \subset W.$$

Otherwise, there exists $z_n \in P_{N(T)}(x_0)$ such that $B(z_n, 1/n) \not\subset W$. Since $P_{N(T)}(x_0)$ is compact, we may assume that $z_n \to z_0 \in P_{N(T)}(x_0)$ as $n \to \infty$. Hence there exists $\eta > 0$ such that $B(z_0, 4\eta) \subset W$. Moreover, there exists $n_0 \in N$ such that $1/n_0 < \eta$ and $\|z_{n_0} - z_0\| \le \eta$. Hence, for any $z \in B(z_{n_0}, 1/n_0)$, we have

$$\|z - z_0\| \le \|z - z_{n_0}\| + \|z_{n_0} - z_0\| \le \frac{1}{n_0} + \eta < \eta + \eta < 4\eta.$$

This implies that $z \in W$. Then $B(z_{n_0}, 1/n_0) \subset W$, a contradiction. Let $y_n = Tx_n$ and $y_0 = Tx_0$. Then

$$T^{\partial}(y_n) = x_n - P_{N(T)}(x_n), \quad T^{\partial}(y_0) = x_0 - P_{N(T)}(x_0) \ \text{ and } \ \lim_{n\to\infty} y_n = y_0.$$

Since $P_{N(T)}(x_0) \subset W$, we obtain that $T^{\partial}(y_0) = x_0 - P_{N(T)}(x_0) \subset x_0 - W$. We claim that

$$x_n - \pi_{N(T)}(x_n) \notin x_0 - \bigcup_{z\in P_{N(T)}(x_0)} B(z, \delta)$$

whenever $\|x_n - x_0\| < \delta$. In fact, suppose that $x_n - \pi_{N(T)}(x_n) \in x_0 - \bigcup_{z\in P_{N(T)}(x_0)} B(z,$ $\delta)$ whenever $\|x_n - x_0\| < \delta$. Then

$$\begin{aligned} \pi_{N(T)}(x_n) &= x_n - \big(x_n - \pi_{N(T)}(x_n)\big) \\ &\in x_n - \left(x_0 - \bigcup_{z\in P_{N(T)}(x_0)} B(z,\delta)\right) \\ &= \bigcup_{z\in P_{N(T)}(x_0)} B(z,\delta) + (x_n - x_0) \\ &\subset \bigcup_{z\in P_{N(T)}(x_0)} B(z, 2\delta) \subset W, \end{aligned}$$

a contradiction. Since $x_n-\pi_{N(T)}(x_n) \notin x_0-\bigcup_{z\in P_{N(T)}(x_0)} B(z,\delta)$ whenever $\|x_n-x_0\| < \delta$, we obtain that $T^\partial$ is not upper semicontinuous at $y_0$, a contradiction.

(2) "$\Rightarrow$" Let $y_0 \in Y$ and $y_n \to y_0$ as $n \to \infty$. Then, by the previous proof, there exist $z(1,n) \in P_{R(T)}(y_n)$ and $z(2,n) \in P_{R(T)}(y_n)$, $z(1,0) \in P_{R(T)}(y_0)$ and $z(2,0) \in P_{R(T)}(y_0)$ such that

$$[z(1,n),z(2,n)] = \left\{z : \left\|\overline{T}^{-1}(z)\right\| = \inf\{\left\|\overline{T}^{-1}(y)\right\| : y \in P_{R(T)}(y_n)\}\right\}$$

and

$$[z(1,0),z(2,0)] = \left\{z : \left\|\overline{T}^{-1}(z)\right\| = \inf\{\left\|\overline{T}^{-1}(y)\right\| : y \in P_{R(T)}(y_0)\}\right\}.$$

Moreover, by the previous proof, we may assume without loss of generality that

$$\lim_{n\to\infty} z(1,n) = z_1 \in [z(1,0),z(2,0)] \text{ and } \lim_{n\to\infty} z(2,n) = z_2 \in [z(1,0),z(2,0)]. \tag{2.5}$$

From the previous proof, there exist $[x(1,n),x(2,n)] \subset X$ and $[x(1,0),x(2,0)] \subset X$ such that

$$T\{x : x \in [x(1,n),x(2,n)]\} = [z(1,n),z(2,n)]$$

and

$$T\{x : x \in [x(1,0),x(2,0)]\} = [z(1,0),z(2,0)].$$

Moreover, by the definition of set-valued metric generalized inverse, we obtain that

$$TT^\partial(y_0) = [z(1,0),z(2,0)] \text{ and } TT^\partial(y_n) = [z(1,n),z(2,n)].$$

Since the set-valued mapping $TT^\partial$ is lower semicontinuous, by formula (2.5), we have

$$\lim_{n\to\infty} z(1,n) = z_1 = z(1,0) \text{ and } \lim_{n\to\infty} z(2,n) = z_2 = z(2,0).$$

Therefore, by the previous proof, we obtain that

$$\lim_{n\to\infty} x(1,n) = x(1,0) \text{ and } \lim_{n\to\infty} x(2,n) = x(2,0), \tag{2.6}$$

Moreover, by the definition of set-valued metric generalized inverse, we obtain that for any $x \in T^\partial(y_0)$, there exist $\lambda \in [0,1]$ and $h \in P_{N(T)}(\lambda x(1,0)+(1-\lambda)x(2,0))$ such that $x = \lambda x(1,0)+(1-\lambda)x(2,0)-h$. Since $P_{N(T)}$ is continuous, there exists $h_n \in P_{N(T)}(\lambda x(1,n)+(1-\lambda)x(2,n))$ such that $h_n \to h$ as $n \to \infty$. Therefore, by formula (2.6), we obtain that

$$\lim_{n\to\infty} \lambda x(1,n)+(1-\lambda)x(2,n) = \lambda x(1,0)+(1-\lambda)x(2,0).$$

This implies that

$$\lim_{n\to\infty} (\lambda x(1,n)+(1-\lambda)x(2,n)-h_n) = \lambda x(1,0)+(1-\lambda)x(2,0)-h = x. \tag{2.7}$$

Noticing that $\lambda x(1,n)+(1-\lambda)x(2,n)-h_n \in T^\partial(y_n)$ and formula (2.7), we obtain that $T^\partial$ is lower semicontinuous at $y_0$. Therefore, by (1), we obtain that $T^\partial$ is upper semicontinuous at $y_0$. Hence $T^\partial$ is continuous at $y_0$.

"$\Leftarrow$" Let $y_0 \in Y$ and $y_n \to y_0$ as $n \to \infty$. Then, by the previous proof, there exist $x_0 \in X$ and $\{x_n\}_{n=1}^\infty \subset X$ such that $P_{R(T)}(y_0) = Tx_0$, $P_{R(T)}(y_n) = Tx_n$ and

$x_n \to x_0$ as $n \to \infty$. Then $T^{\partial}(y_0) = x_0 - P_{N(T)}(x_0)$ and $T^{\partial}(y_n) = x_n - P_{N(T)}(x_n)$. Since $T^{\partial}$ is continuous, we obtain that for any $x_0 - z \in x_0 - P_{N(T)}(x_0)$, there exists $x_n - z_n \in x_n - P_{N(T)}(x_n)$ such that $x_n - z_n \to x_0 - z$ as $n \to \infty$, where $z_n \in P_{N(T)}(x_n)$. Hence, for any $z \in P_{N(T)}(x_0)$, there exists $z_n \in P_{N(T)}(x_n)$ such that $z_n \to z_0$ as $n \to \infty$. This implies that $P_{N(T)}$ is lower semicontinuous at $y_0$. Therefore, by (1), we obtain that $P_{N(T)}$ is continuous at $y_0$.

We next will prove that $TT^{\partial}$ is lower semicontinuous. Let $y_0 \in Y$ and $y_n \to y_0$ as $n \to \infty$. Pick $z \in TT^{\partial}(y_0)$. Then there exists $x_0 \in D(T)$ such that $Tx_0 = y_0$ and $x_0 \in T^{\partial}(y_0)$. Since $T^{\partial}$ is continuous, we obtain that $T^{\partial}$ is lower semicontinuous. Hence there exists $x_n \in T^{\partial}(y_n)$ such that $x_n \to x_0$ as $n \to \infty$. This implies that $Tx_n \to Tx_0 = y_0$ as $n \to \infty$. Since $x_n \in T^{\partial}(y_n)$, we obtain that $Tx_n \in TT^{\partial}(y_n)$. Hence $TT^{\partial}$ is lower semicontinuous at $y_0$, which completes the proof. □

## 3. Continuous selections of the set-valued metric generalized inverse in Banach spaces

**Theorem 3.1.** *Let $X$, $Y$ be Banach spaces, $T$ be a linear bounded operator from subspace of $X$ into $Y$, $D(T)$ be a closed subspace of $X$, $N(T)$ be a topologically complemented subspace of $D(T)$ and $R(T)$ be a proximinal subspace of $Y$. Then the following statements are equivalent:*

*(1) $T^{\partial}$ has a continuous homogeneous selection on $Y$;*

*(2) $P_{N(T)}$ has a continuous homogeneous selection on $D(T)$ and the set-valued mapping $TT^{\partial}$ has a continuous homogeneous selection on $Y$.*

*Proof.* (2) ⇒ (1). Since $N(T)$ is a topologically complemented subspace of $D(T)$, there exists a closed subspace $M(T)$ of $D(T)$ such that $M(T) \oplus N(T) = D(T)$. Moreover, by $M(T) \oplus N(T) = D(T)$, we obtain that $T^{-1}(y) \cap M(T)$ is singleton for any $y \in R(T)$. Define the mapping $G : R(T) \to M(T)$ such that

$$G(y) = T^{-1}(y) \cap M(T), \quad y \in R(T).$$

Since $T$ be a linear bounded operator from $D(T)$ into $R(T)$, by $M(T) \oplus N(T) = D(T)$, we obtain that $G$ is a linear bounded operator from $R(T)$ into $M(T)$. Let $TT^{\sigma}$ be a continuous homogeneous selection of $TT^{\partial}$ and $\pi_{N(T)}$ be a continuous homogeneous selection of $P_{N(T)}$. Therefore, by the definition of set-valued metric generalized inverse, we obtain that the mapping

$$G \circ TT^{\sigma} - \pi_{N(T)} \circ G \circ TT^{\sigma} : \quad Y \to D(T)$$

is a continuous homogeneous selection of $T^{\partial}$.

(1) ⇒ (2). Since $T^{\partial}$ has a continuous homogeneous selection on $Y$, by the definition of set-valued metric generalized inverse, we obtain that $N(T)$ is a proximinal subspace of $D(T)$. Let $T^{\sigma}$ be a continuous homogeneous selection of $T^{\partial}$. Pick $x \in D(T)$. Let

$$\pi_{N(T)}(x) = x - T^{\sigma}T(x).$$

We next will prove that $\pi_{N(T)}$ is a continuous homogeneous selection of $P_{N(T)}$. In fact, since

$$T\left(\pi_{N(T)}(x)\right) = T\left(x - T^{\sigma}T(x)\right) = Tx - TT^{\sigma}T(x) = Tx - Tx = 0, \qquad (3.1)$$

we obtain that $\pi_{N(T)}(x) \in N(T)$. Moreover, by the definition of set-valued metric generalized inverse, we obtain that $\|T^\sigma T(x)\| = \operatorname{dist}(x, N(T))$. Therefore, by

$$x - \pi_{N(T)}(x) = x - x + T^\sigma T(x) = T^\sigma T(x)$$

and formula (3.1), we obtain that $\pi_{N(T)}(x) \in P_{N(T)}(x)$. Moreover, since $T^\sigma$ is a continuous homogeneous selection of $T^\partial$ and $\pi_{N(T)}(x) = x - T^\sigma T(x)$, we obtain that $\pi_{N(T)}$ is a homogeneous selection. This implies that $P_{N(T)}$ has a continuous homogeneous selection on $D(T)$.

Since $T^\sigma$ is a continuous homogeneous selection of $T^\partial$, we obtain that $TT^\sigma$ is a continuous homogeneous selection of $TT^\partial$, which completes the proof. □

**Definition 3.2.** (see [5]) A nonempty subset $C$ of $X$ is said to be approximatively compact if for any $\{y_n\}_{n=1}^\infty \subset C$ and any $x \in X$ satisfying $\|x - y_n\| \to \inf_{y\in C} \|x - y\|$ as $n \to \infty$, there exists a subsequence of $\{y_n\}_{n=1}^\infty$ converging to an element in $C$. $X$ is called approximatively compact if every nonempty closed convex subset of $X$ is approximatively compact.

**Definition 3.3.** (see [5]) A Banach space $X$ is be said to be strictly convex if for any $x, y \in S(X)$ and $\|x + y\| = 2$ we have $x = y$.

**Theorem 3.4.** *Let $X$ be approximatively compact and strictly convex, $Y$ be a Banach spaces, $T$ be a linear bounded operator from subspace of $X$ into $Y$, $D(T)$ be a closed subspace of $X$, $N(T)$ be a topologically complemented subspace of $D(T)$ and $R(T)$ be a proximinal subspace of $Y$. Then the following statements are equivalent:*

*(1) $T^\partial$ has a continuous homogeneous selection on $Y$;*

*(2) $TT^\partial$ has a continuous homogeneous selection on $Y$.*

*Proof.* Since $X$ is approximatively compact, we obtain that $P_{N(T)}$ is upper semicontinuous. Since $X$ is a strictly convex space, we obtain that $P_{N(T)}$ is single value mapping. This implies that $P_{N(T)}$ is continuous. Therefore, by Theorem 3.1, it is easy to see that Theorem 3.4 is true, which completes the proof. □

**Theorem 3.5.** *Let $X$ be approximatively compact and strictly convex, $Y$ be a Banach space, $T$ be a linear bounded operator from subspace of $X$ into $Y$, $D(T)$ be a closed subspace of $X$ and $R(T)$ be a proximinal subspace of $Y$. Then the following statements are equivalent:*

*(1) $T^\partial$ has a continuous selection on $Y$;*

*(2) $TT^\partial$ has a continuous selection on $Y$.*

*Proof.* (2)⇒(1). Since $X$ is approximatively compact, we obtain that $P_{N(T)}$ is upper semicontinuous. Since $X$ is a strictly convex space, we obtain that $P_{N(T)}$ is a single value mapping. This implies that $P_{N(T)}$ is a continuous and single value mapping. Let $TT^\sigma$ be a continuous selection of $TT^\partial$ and $f^{-1}TT^\sigma$ be a selection of $T^{-1}TT^\sigma$. This implies that the mapping

$$f^{-1}TT^\sigma - P_{N(T)}f^{-1}TT^\sigma$$

is a selection of $T^\partial$. We next will prove that if $y_n \to y$ as $n \to \infty$, then

$$\lim_{n\to\infty} [f^{-1}TT^\sigma(y_n) - P_{N(T)}f^{-1}TT^\sigma(y_n)] = f^{-1}TT^\sigma(y) - P_{N(T)}f^{-1}TT^\sigma(y). \quad (3.2)$$

In fact, by the proof of Theorem 2.1, there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset D(T)$ such that

$$x_n \in f^{-1}TT^{\sigma}(y_n) - P_{N(T)}f^{-1}TT^{\sigma}(y_n) + N(T) \text{ and } \lim_{n\to\infty} x_n = f^{-1}TT^{\sigma}(y).$$

Since the mapping $P_{N(T)}$ and $f^{-1}TT^{\sigma}$ are single value mappings, we obtain that

$$Tx_n = T\left(f^{-1}TT^{\sigma}(y_n) - P_{N(T)}f^{-1}TT^{\sigma}(y_n)\right).$$

Hence there exists a sequence $\{z_n\}_{n=1}^{\infty} \subset N(T)$ such that

$$x_n - P_{N(T)}(x_n) - z_n = f^{-1}TT^{\sigma}(y_n) - P_{N(T)}f^{-1}TT^{\sigma}(y_n).$$

Moreover, by the definition of set-valued metric generalized inverse, we have

$$\left\|x_n - P_{N(T)}(x_n) - z_n\right\| = \left\|f^{-1}TT^{\sigma}(y_n) - P_{N(T)}f^{-1}TT^{\sigma}(y_n)\right\| = \operatorname{dist}\left(x_n, N(T)\right).$$

This implies that $P_{N(T)}(x_n)+z_n \subset P_{N(T)}(x_n)$. Since the mapping $P_{N(T)}$ is a single value mapping, we have $z_n = 0$. Hence

$$x_n - P_{N(T)}(x_n) = f^{-1}TT^{\sigma}(y_n) - P_{N(T)}f^{-1}TT^{\sigma}(y_n).$$

Since $P_{N(T)}$ is a continuous single value mapping, by $x_n \to f^{-1}TT^{\sigma}(y)$ as $n \to \infty$, we obtain that

$$\lim_{n\to\infty}[x_n - P_{N(T)}(x_n)] = f^{-1}TT^{\sigma}(y) - P_{N(T)}f^{-1}TT^{\sigma}(y).$$

Noticing that $x_n - P_{N(T)}(x_n) = f^{-1}TT^{\sigma}(y_n) - P_{N(T)}f^{-1}TT^{\sigma}(y_n)$, we obtain that (3.2) is true. Hence $f^{-1}TT^{\sigma} - P_{N(T)}f^{-1}TT^{\sigma}$ is a continuous selection of $T^{\partial}$.

(1)⇒(2). Let $T^{\sigma}$ be a continuous selection of $T^{\partial}$. Then $TT^{\sigma}$ is a continuous selection of $TT^{\partial}$, which completes the proof. □

**Theorem 3.6.** *Let $X$ be approximatively compact and strictly convex, $Y$ be a Banach spaces, $T$ be a linear bounded operator from subspace of $X$ into $Y$, $D(T)$ be a closed subspace of $X$, $N(T)$ be a topologically complemented subspace of $D(T)$ and $R(T)$ be a 2-Chebyshev maximal subspace of $Y$. Then the following statements are equivalent:*

*(1) For any $x \in T^{\partial}(y)$, there exists a selection $T^{\sigma}$ of $T^{\partial}$ such that $T^{\sigma}(y) = x$ and $T^{\sigma}$ is continuous at $y$;*

*(2) $TT^{\partial}$ is lower semicontinuous at $y$.*

*Proof.* (1)⇒(2). From the proof of Theorem 2.1, there exist $z(1) \in P_{R(T)}(y)$ and $z(2) \in P_{R(T)}(y)$ such that

$$TT^{\partial}(y) = [z(1,y), z(2,y)] \subset P_{R(T)}(y).$$

Since $N(T)$ is a topologically complemented subspace of $D(T)$, there exists a closed subspace $M(T)$ of $D(T)$ such that $M(T) \oplus N(T) = D(T)$. Pick $x(1,y) \in \overline{T}^{-1}(z(1,y))$ and $x(2,y) \in \overline{T}^{-1}(z(2,y))$. Then, by $M(T) \oplus N(T) = D(T)$, there exist $G(y(1)), G(y(2)) \in M(T)$ and $h(y(1)), h(y(2) \in N(T)$ such that

$$x(1,y) = G(y(1)) + h(y(1)) \text{ and } x(2,y) = G(y(2)) + h(y(2)).$$

Therefore, by $x(1,y) \in \overline{T}^{-1}(z(1,y))$ and $x(2,y) \in \overline{T}^{-1}(z(2,y))$, we obtain that

$$T(G(y(1))) = z(1,y) \text{ and } T(G(y(2))) = z(2,y).$$

Hence, for any $z \in TT^{\partial}(y)$, there exists $\lambda \in [0,1]$ such that

$$z = T\left(\lambda G(y(1)) + (1-\lambda)G(y(2))\right).$$

Pick $x \in T^{\sigma}(y)$. Then there exists a selection $T^{\sigma}$ of $T^{\partial}$ such that $T^{\sigma}(y) = x$ and $T^{\sigma}$ is continuous at $y$. This implies that

$$Tx = T\left(\lambda G(y(1)) + (1-\lambda)G(y(2))\right) = z.$$

Therefore, by the definition of set-valued metric generalized inverse, we have

$$T^{\sigma}(y) = \lambda G(y(1)) + (1-\lambda)G(y(2)) - P_{N(T)}\left(\lambda G(y(1)) + (1-\lambda)G(y(2))\right).$$

We next will prove that set-valued mapping $TT^{\partial}$ is lower semicontinuous at $y$. Let $y_n \to y$ as $n \to \infty$. Then, by $M(T) \oplus N(T) = D(T)$ and the previous proof, there exist $G(y_n(1)) \in M(T)$ and $G(y_n(2)) \in M(T)$ such that

$$T(G(y_n(1))) = z(1, y_n), \quad T(G(y_n(2))) = z(2, y_n)$$

and

$$TT^{\partial}(y_n) = [z(1, y_n), z(2, y_n)] \subset P_{R(T)}(y_n).$$

Hence there exists a sequence $\{\lambda_n\}_{n=1}^{\infty} \subset [0,1]$ such that

$$T^{\sigma}(y_n) = \lambda_n G(y_n(1)) + (1-\lambda_n)G(y_n(2)) - P_{N(T)}\left(\lambda_n G(y_n(1)) + (1-\lambda_n)G(y_n(2))\right).$$

Since

$$\lambda_n G(y_n(1)) + (1-\lambda_n)G(y_n(2)) \in M(T), \quad \lambda G(y(1)) + (1-\lambda)G(y(2)) \in M(T),$$

$$P_{N(T)}\left(\lambda_n G(y_n(1)) + (1-\lambda_n)G(y_n(2))\right) \in N(T)$$

and

$$P_{N(T)}\left(\lambda G(y(1)) + (1-\lambda)G(y(2))\right) \in N(T),$$

by $M(T) \oplus N(T) = D(T)$ and $T^{\sigma}(y_n) \to T^{\sigma}(y)$ as $n \to \infty$, we obtain that

$$\lim_{n\to\infty} \lambda_n G(y_n(1)) + (1-\lambda_n)G(y_n(2)) = \lambda G(y(1)) + (1-\lambda)G(y(2)).$$

This implies that

$$\lim_{n\to\infty} T\left(\lambda_n G(y_n(1)) + (1-\lambda_n)G(y_n(2))\right) = T\left(\lambda G(y(1)) + (1-\lambda)G(y(2))\right) = z.$$

where $T\left(\lambda_n G(y_n(1)) + (1-\lambda_n)G(y_n(2))\right) \in TT^{\sigma}(y_n)$. Hence the set-valued mapping $TT^{\partial}$ is lower semicontinuous at $y$.

(2)⇒(1). Let $x \in T^{\partial}(y)$. Then, by the previous proof, there exists $\lambda \in [0,1]$ such that

$$x = \lambda G(y(1)) + (1-\lambda)G(y(2)) - P_{N(T)}\left(\lambda G(y(1)) + (1-\lambda)G(y(2))\right).$$

Define the set-valued mapping $Y \to TT^{\partial}(Y)$ such that

$$F(z) = c, \quad \text{where} \quad \|c - Tx\| = \inf_{h \in TT^{\partial}(z)} \|h - Tx\| \quad \text{and} \quad c \in TT^{\partial}(z).$$

It is easy to see that $F(y) = Tx$. Let $f$ be a selection of $F$. Moreover, let $y_n \to y$ and $f(y_n) = c_n$. Since $TT^{\partial}$ is lower semicontinuous at $y$, we obtain that $c_n \to Tx$ as $n \to \infty$. This implies that $f$ is continuous at $y$. Since $N(T)$ is a topologically complemented subspace of $D(T)$, there exists a closed subspace $M(T)$ of $D(T)$

such that $M(T) \oplus N(T) = D(T)$. Define the mapping $G : R(T) \to M(T)$ such that

$$G(y) = T^{-1}(y) \cap M(T), \quad y \in R(T).$$

Then, by the previous proof, we obtain that $G$ is a linear bounded operator from $R(T)$ into $M(T)$. Since $M(T) \oplus N(T) = D(T)$, we have $x = x_1 - x_2$, where $x_1 \in M(T)$ and $x_2 \in N(T)$. Since $X$ is a strictly convex space, we obtain that $P_{N(T)}$ is a single value mapping. Since $x \in T^{\partial}(y)$, by the definition of set-valued metric generalized inverse, we have

$$\|x\| = \|x_1 - x_2\| = \inf\left\{\|z\| : z \in T^{-1}x = T^{-1}x_1\right\} = \inf\left\{\|x_1 - h\| : h \in N(T)\right\}.$$

Therefore, by $x_1 \in M(T)$ and $x_2 \in N(T)$, we obtain that $x_2 = P_{N(T)}(x_1)$. Since $f$ is a selection of $F$ and $F(y) = Tx$, we have $G \circ f(y) = G(Tx) = x_1$ Define the mapping

$$T^{\sigma} = G \circ f - P_{N(T)} \circ G \circ f.$$

Then $T^{\sigma}$ is a selection of $T^{\partial}$. Moreover, by $G \circ f(y) = G(Tx) = x_1$, we have

$$\begin{aligned} T^{\sigma}(y) &= G \circ f(y) - P_{N(T)} \circ G \circ f(y) \\ &= G(Tx) - P_{N(T)}(G(Tx)) \\ &= x_1 - P_{N(T)}(x_1) = x. \end{aligned}$$

Since $X$ is approximatively compact, we obtain that $P_{N(T)}$ is upper semicontinuous. Since $P_{N(T)}$ is a single value mapping, we obtain that $P_{N(T)}$ is continuous. Since $f$ and $G$ is continuous at $y$, we obtain that $T^{\sigma} = G \circ f - P_{N(T)} \circ G \circ f$ is continuous at $y$, which completes the proof. □

## 4. Relation of continuity and continuous selections of the set-valued metric generalized inverse in Banach spaces

**Theorem 4.1.** *Let $X$ be approximatively compact and strictly convex, $Y$ be a Banach space, $T$ be a linear bounded operator from subspace of $X$ into $Y$, $D(T)$ be a closed subspace of $X$, $N(T)$ be a topologically complemented subspace of $D(T)$ and $R(T)$ be a 2-Chebyshev maximal subspace of $Y$. Then the following statements are equivalent:*

*(1) For any $x \in \cup_{y \in Y} T^{\partial}(y)$, there exists a selection $T^{\sigma}$ of $T^{\partial}$ such that $T^{\sigma}(y) = x$ and $T^{\sigma}$ is continuous at $y$;*

*(2) $TT^{\partial}$ is lower semicontinuous;*

*(3) $T^{\partial}$ is continuous.*

*Proof.* Since $X$ is approximatively compact, we obtain that $P_{N(T)}$ is upper semicontinuous. Since $X$ is a strictly convex space, we obtain that $P_{N(T)}$ is single value mapping. This implies that $P_{N(T)}$ is continuous. Therefore, by Theorem 2.1, we obtain that (2)⇔ (3) is true. Moreover, by Theorem 3.5, we obtain that (1)⇔ (2) is true, which completes the proof. □

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## References

1. S. Shang and Y. Cui, *2-strict convexity and continuity of the set-valued metric generalized inverse in Banach spaces*, Abstr. Appl. Anal **2014** (2014) no. 1, 1–8.
2. L. Chen and L. Cheng, *Analytic characterizations of of Mazur intersection property via convex functions*, J. Funct. Anal. **262** (2012) no. 1, 4731–4745.
3. N.W. Jefimow and S.B. Stechkin, *Approximative compactness and Chebyshev sets*, Sov Math, **2** (1961) no. 1, 1226–1228.
4. J. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhauser, Boston, 1990.
5. S. Chen, H. Hudzik, W. Kowalewski, Y. Wang, M. Wisla, *Approximative compactness and continuity of metric projector in Banach spaces and applications*, Sci Sin Math, **50**(2007) no. 2, 75–84.
6. Sekowski, Stachura, *Noncompact smoothness and noncompact convexity*, Atti. Semin. Mat. Fis. Univ. Modena 36 (1988)no.1, 329–333.
7. I. Singer, *On the set of best approximation of an element in a normed linear space*, Rev. Roumaine Math. Pures Appl. 5(1960), no.1, 383–402.
8. S. Shang and Y. Cui, *Approximative compactness and continuity of the set-valued metric generalized inverse in Banach spaces*, J. Math. Anal. Appl. **422** (2015) no. 2, 1361–1375.
9. M.Z. Nashed , G.F. Votruba, *A unified apporoach to generalized inverses of linnear operator: II Extremal and proximinal properties*, Bull Amer Math Soc, **80** (1974) no. 1, 831–835.
10. H. Hudzik, Y.W. Wang and W. Zheng, *Criteria for the Metric Generalized Inverse and its Selections in Banach Spaces*, Set-Valued Anal. **16** (2008) no. 1, 51–65.
11. Y.W. Wang and J. Liu, *Metric generalized inverse of linear manifold and extremal solution of linear inclusion in Banach spaces*, J. Math. Anal. Appl. **302** (2005) no. 2, 360–371.
12. G. Chen and Y. Xue, *Perturbation analysis for the operator equation $Tx = b$ in Banach spaces*, J. Math. Anal. Appl. **212** (1997) no. 1, 107–125.
13. Y.W. Wang and H. Zhang, *Perturbation analysis for oblique projection generalized inverses of closed linear operators in Banach spaces*, Linear Algebra Appl. **426** (2007) no. 1, 1–11.
14. Y.W. Wang, *Generalized Inverse of Operator in Banach Spaces and Applications*, Science Press, Beijing, 2005.

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# Locally and globally small Riemann sums and Henstock-Stieltjes integral of fuzzy-number-valued functions

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**Abstract:** In this paper, we introduce and study locally and globally small Riemann sums with respect to $\alpha$ on $[a,b]$ for fuzzy-number-valued functions and obtain some of it's characterizations. Also, we shall prove two main theorems: (i) If a fuzzy-number-valued functions $\tilde{f}(x)$ is Henstock-Stieltjes integrable on $[a,b]$ then it has $(LSRS)$ and the converse is always true. (ii) If a fuzzy-number-valued functions $\tilde{f}(x)$ is Henstock-Stieltjes integrable on $[a,b]$ then it has $(GSRS)$ and the converse is always true. Finally, by Egorov's Theorem, we obtain the dominated convergence theorem for globally small Riemann sums $(GSRS)$ with respect to $\alpha$ on $[a,b]$ for fuzzy-number-valued functions.

**Keywords** : Fuzzy numbers; fuzzy integrals; Henstock-Stieltjes integral; locally small Riemann sums $(LSRS)$ with respect to $\alpha$ on $[a,b]$; globally small Riemann sums $(GSRS)$ with respect to $\alpha$ on $[a,b]$.

## 1 Introduction

Since the concept of fuzzy sets was firstly introduced by Zadeh in 1965 [21], it has been studied extensively from many different aspects of the theory and applications, such as fuzzy topology, fuzzy analysis, fuzzy decision making and fuzzy logic, information science and so on. It's well known that the concept of the Stieltjes integral for fuzzy-number-valued functions was originally introduced by Nanda [12] in 1989. Nonetheless, as Wu et al. [17] pointed out that the existence of supremum and infimum for a finite set of fuzzy numbers wasn't easy at first thought. That is, Nanda's concept of fuzzy Riemann-Stieltjes $(FRS)$ integral in [12] was incorrect. In 1998, Wu [18] introduced the notion of $(FRS)$ integral by means of the representation theorem of fuzzy-number-valued functions, whose membership function could be obtained by solving a nonlinear programming problem, but it's difficult to calculate and extend to the higher-dimensional space. In 2006, Ren et al. proposed the notion of two types of $(FRS)$ integral for fuzzy-number-valued functions [13, 14] and showed that a continuous fuzzy-number-valued function was $(FRS)$ integrable with respect to a real-valued increasing function. Gong et al. [2] defined and discussed the $(HS)$ integral for fuzzy-number-valued functions and proved two convergence theorems for sequences of the $(FHS)$ integrable functions in 2012. The locally and globally small Riemann sums have been introduced by many authors from different points of views. In 1986, Schurle characterized the Lebesgue integral in $(LSRS)$ (locally small Riemann sums) property [15]. The $(LSRS)$ property has been used to characterized the Perron $(P)$ integral on $[a,b]$ [16]. By considering the equivalency between the $(P)$ integral and the Henstock-Kurzweil $(HK)$ integral, the $(LSRS)$ property has been used to characterized the $(HK)$ integral on $[a,b]$ [10].

The $(LSRS)$ property brought a research to have global characterization on the Riemann sums of an $(HK)$ integrable function on $[a,b]$. This research has been done by considering the following fact: Every $(HK)$ integrable function on $[a,b]$ is measurable, however, there is no guarantee the boundedness of the function. A measurable function $f$ is $(HK)$ integrable on $[a,b]$ depends on it behaves on the set of $x$ in which $|f(x)|$ is large, i.e. $|f(x)| \geq N$

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for some $N$. This fact has been characterized in $(GSRS)$ (globally small Riemann sums) property [10]. The $(GSRS)$ property involves one characteristic of the primitive of an $(HK)$ integrable function. That is the primitive of the $(HK)$ integral on $[a,b]$ is $ACG^*$ (generalized strongly absolutely continuous) on $[a,b]$. This is not a simple concept. In 2015, Indrati [8] introduced a countably Lipschitz condition of a function which is simpler than the $ACG^*$, and proved that the $(HK)$ integrable function or it's primitive could be characterized in countably Lipschitz condition. Also, by considering the characterization of the $(HK)$ integral in the $(GSRS)$ property, it showed that the relationship between $(GSRS)$ property and countably Lipschitz condition of an $(HK)$ integrable function on $[a,b]$.

In 2018, Hamid et al. [6] investigated locally and globally small Riemann sums for fuzzy-number-valued functions and proved two main theorems: (1) A fuzzy-number-valued functions $\tilde{f}(x)$ is Henstock integrable on $[a,b]$ if and only if $\tilde{f}(x)$ has $(LSRS)$. (2) A fuzzy-number-valued functions $\tilde{f}(x)$ is Henstock integrable on $[a,b]$ if and only if $\tilde{f}(x)$ has $(GSRS)$.

In this paper, we introduce and study the locally and globally small Riemann sums with respect to $\alpha$ on $[a,b]$ for fuzzy-number-valued functions. We show that a fuzzy-number-valued functions is Henstock-Stieltjes integrable with respect to $\alpha$ on $[a,b]$ iff it has $(LSRS)$ with respect to $\alpha$ on $[a,b]$. Also it is shown that a fuzzy-number-valued functions is Henstock-Stieltjes integrable on $[a,b]$ iff it has $(GSRS)$ with respect to $\alpha$ on $[a,b]$. Finally, by Egorov's Theorem, we get the dominated convergence theorem for globally small Riemann sums $(GSRS)$ with respect to $\alpha$ on $[a,b]$ for fuzzy-number-valued functions.

The rest of this paper is organized as follows, in Section 2 we shall review the relevant concepts and properties of fuzzy sets and the definition of Henstock-Stieltjes integrals for fuzzy-number-valued functions. Section 3 is devoted to discussing the locally small Riemann sums $(LSRS)$ with respect to $\alpha$ on $[a,b]$ for fuzzy-number-valued functions. In Section 4 we shall investigate the globally small Riemann sums $(GSRS)$ with respect to $\alpha$ on $[a,b]$ for fuzzy-number-valued functions.

# 2 Preliminaries

**Definition 2.1** [7, 10] Let $\delta : [a,b] \to \mathbb{R}^+$ be a positive real-valued function. $P = \{[x_{i-1}, x_i]; \xi_i\}$ is said to be a $\delta$-fine division, if the following conditions are satisfied:

(1) $a = x_0 < x_1 < x_2 < ... < x_n = b$;

(2) $\xi_i \in [x_{i-1}, x_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))(i = 1, 2, \cdots, n)$.

For brevity, we write $P = \{[u,v]; \xi\}$, where $[u,v]$ denotes a typical interval in $P$ and $\xi$ is the associated point of $[u,v]$.

**Definition 2.2** [4] Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing function. A real function $f : [a,b] \to R$ is Henstock-Stieltjes $(HS)$ integrable to $A \in R$ with respect to $\alpha$ on $[a,b]$ if for every $\varepsilon > 0$, there is a function $\delta(x) > 0$ , such that for any $\delta$-fine division $P = \{[u_i, v_i]; \xi_i\}_{i=1}^n$ we have

$$\Big| \sum_{i=1}^{n} f(\xi_i)[\alpha(v_i) - \alpha(u_i)] - A \Big| < \varepsilon. \tag{2.1}$$

We write $(HS) \int_a^b f(x) \mathrm{d}\alpha = A$ , and $f \in HS_\alpha[a,b]$.

For the results about fuzzy number space $E^1$. we recall that $E^1 = \{u : \ R \to [0,1] : u$ satisfies (1)-(4) below$\}$:

(1) $u$ is normal, i.e., there exists a $x_0 \in R$ such that $u(x_0) = 1$;

(2) $u$ is a convex fuzzy set, i.e., $u(rx + (1-r)y) \geqslant \min(u(x), u(y)),\ x, y \in R,\ r \in [0,1]$;

(3) $u$ is upper semi-continuous;

(4) $cl\{x \in R : u(x) > 0\}$ is compact, where $clA$ denotes the closure of $A$.

For $0 < r \leqslant 1$, denote $[u]^r = \{x : u(x) \geqslant r\}$. Then from (1)-(4), it follows that the $r-$level set $[u]^r$ is a close interval for all $r \in [0,1]$ (refer to [1, 3, 5, 9, 11, 19, 20]). We write $u^r = [u]^r = [u_-^r, u_+^r]$ or $[u_-(r), u_+(r)]$.

For $u, v \in E^1$, the addition and scalar multiplication are defined by the equations:

$$[u+v]^r = [u]^r + [v]^r, \text{ i.e., } u_-^r + v_-^r = [u+v]_-^r \text{ and } u_+^r + v_+^r = [u+v]_+^r;$$

$$[ku]^r = k[u]^r, \text{ i.e., } [ku]^r_- = \min\{ku^r_-, ku^r_+\} \text{ and } [ku]^r_+ = \max\{ku^r_-, ku^r_+\},$$

respectively.

Define $D(u,v) = \sup\limits_{r\in[0,1]} d([u]^r,[v]^r) = \sup\limits_{r\in[0,1]} \max\{|u^r_- - v^r_-|, |u^r_+ - v^r_+|\}$, where $d$ is Hausdorff metric. Furthermore, we write

$$\|\tilde{u}\|_{E^1} = D(\tilde{u},\tilde{0}) = \sup_{\lambda\in[0,1]} \max\{|u^-_\lambda|, |u^+_\lambda|\}.$$

Notice that $\|\cdot\|_{E^1} = D(\cdot,\tilde{0})$ doesn't stands for the norm of $E^1$.

For $u,v \in E^1$, $u \leqslant v$ means $u^r_- \leqslant v^r_-, u^r_+ \leqslant v^r_+$ (see [1, 3, 5, 9, 11, 19, 20]).

Using the results of [1, 3, 5, 9, 11, 19, 20], we recall that:

(1) $(E^1, D)$ is a complete metric space,

(2) $D(u+w, v+w) = D(u,v)$,

(3) $D(u+v, w+e) \leqslant D(u,w) + D(v,e)$,

(4) $D(ku,kv) = |k|D(u,v), k \in R$,

(5) $D(u+v,\tilde{0}) \leqslant D(u,\tilde{0}) + D(v,\tilde{0})$,

(6) $D(u+v,w) \leqslant D(u,w) + D(v,\tilde{0})$, where $\tilde{0} = \chi_{\{0\}}$ and $u,v,w,e \in E^1$.

**Definition 2.3** [2] Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing function. A fuzzy-number-valued function $\tilde{f}(x)$ is said to be fuzzy Henstock-Stieltjes $(FHS)$ integrable with respect to $\alpha$ on $[a,b]$ if there exists a fuzzy number $\widetilde{H} \in E^1$ such that for every $\varepsilon > 0$, there is a function $\delta(x) > 0$ such that for any $\delta$-fine division $P = \{[u_i, v_i]; \xi_i\}_{i=1}^n$, we have

$$D\big(\sum_{i=1}^n \tilde{f}(\xi_i)[\alpha(v_i) - \alpha(u_i)], \widetilde{H}\big) < \varepsilon. \tag{2.2}$$

We write $(FHS)\int_a^b \tilde{f}(x)\mathrm{d}\alpha = \widetilde{H}$ , and $\tilde{f} \in FHS_\alpha[a,b]$.

**Definition 2.4** [6] A fuzzy-number-valued function $\tilde{f} : [a,b] \to E^1$ is said to be have locally small Riemann sums or $(LSRS)$ if for every $\varepsilon > 0$ there is a $\delta(\xi) > 0$ such that for every $t \in [a,b]$, we have

$$\big\|\sum \tilde{f}(\xi)(v-u)\big\|_{E^1} < \varepsilon, \tag{2.3}$$

whenever $P = \{[u,v];\xi\}$ is a $\delta$-fine division of an interval $[r,s] \subset (t-\delta(t), t+\delta(t))$, $t \in [r,s]$ and $\Sigma$ sums over $P$.

# 3 Locally small Riemann sums and Henstock-Stieltjes integral of fuzzy-number-valued functions

In this section, we shall define locally small Riemann sums with respect to $\alpha$ on $[a,b]$ for fuzzy-number-valued functions. Furthermore, we prove that a fuzzy-number-valued functions is Henstock-Stieltjes integrable with respect to $\alpha$ on $[a,b]$ if and only if it has $(LSRS)$ with respect to $\alpha$ on $[a,b]$. We begin with the following definition.

**Definition 3.1** Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing function. A fuzzy-number-valued function $\tilde{f} : [a,b] \to E^1$ is said to be have locally small Riemann sums (or $LSRS$) with respect to $\alpha$ on $[a,b]$, if for every $\varepsilon > 0$ there is a $\delta(\xi) > 0$ such that for every $t \in [a,b]$, we have

$$\Big\|\sum \tilde{f}(\xi)[\alpha(v) - \alpha(u)]\Big\|_{E^1} < \varepsilon, \tag{3.1}$$

whenever $P = \{[u,v];\xi\}$ is a $\delta$-fine division of an interval $[r,s] \subset (t-\delta(t), t+\delta(t))$, $t \in [r,s]$ and $\Sigma$ sums over $P$.

If there exists a $z \in E^1$ such that $x = y + z$, then we call $z$ the $H-$ difference of $x$ and $y$, denoted by $x - y$. According to the additivity of $FHS$, we have the following Lemma.

**Lemma 3.1** [2] Let $\tilde{f} \in FHS_\alpha[a,b]$ and $\tilde{F}$ be the primitive of $\tilde{f}(x)$ then $\tilde{F}$ satisfies the $H-$ difference.

**Lemma 3.2** (Henstock Lemma). Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing function. If a fuzzy-number-valued function $\tilde{f} : [a,b] \to E^1$ is Henstock-Stieltjes integrable with respect to $\alpha$ on $[a,b]$ with primitive $\tilde{F}$, i.e., for every $\varepsilon > 0$ there is a positive function $\delta(\xi) > 0$ such that for any $\delta$-fine division $P = \{[u,v];\xi\}$ of $[a,b]$, we have

$$D\Big(\sum \tilde{f}(\xi)[\alpha(v) - \alpha(u)], \sum \tilde{F}(u,v)\Big) < \varepsilon. \tag{3.2}$$

Then for any sum of parts $\sum\limits_{1}$ from $\sum$, we have

$$D\Big(\sum_{1} \tilde{f}(\xi)[\alpha(v) - \alpha(u)], \sum_{1} \tilde{F}(u,v)\Big) < \varepsilon. \tag{3.3}$$

The proof is similar to the Theorem 3.7 [10].

**Theorem 3.1** Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing function. If $\tilde{f}(x)$ is Henstock-Stieltjes integrable with respect to $\alpha$ on $[a,b]$ then $\tilde{f}(x)$ it has $LSRS$ with respect to $\alpha$ on $[a,b]$.

**Proof** Let $\tilde{F}$ be the primitive of $\tilde{f}(x)$. Given $\varepsilon > 0$ there is a $\delta(\xi) > 0$ such that for any $\delta$-fine division $P = \{[u,v];\xi\}$ of $[a,b]$, we have

$$D\Big(\sum \tilde{f}(\xi)[\alpha(v) - \alpha(u)], \sum \tilde{F}(u,v)\Big) < \varepsilon. \tag{3.4}$$

Where $\tilde{F}(u,v) = \tilde{F}(v) - \tilde{F}(u)$. By the continuity of $\tilde{F}$ at $\xi$,

$$D\big(\tilde{F}(u), \tilde{F}(v)\big) < \varepsilon \quad \text{whenever} \quad [u,v] \subset (\xi - \delta(\xi), \xi + \delta(\xi)).$$

Therefore for $t \in [a,b]$ and any $\delta$-fine division $P = \{[u,v];\xi\}$ of $[r,s] \subset (t - \delta(t), t + \delta(t))$, we have

$$\begin{aligned} \Big\| \sum \tilde{f}(\xi)[\alpha(v) - \alpha(u)] \Big\|_{E^1} &\leq D\Big(\sum \tilde{f}(\xi)[\alpha(v) - \alpha(u)], \sum \tilde{F}(u,v)\Big) + D\Big(\tilde{F}(r), \tilde{F}(s)\Big) \\ &< 2\varepsilon. \end{aligned}$$

That is $\tilde{f}(x)$ has $LSRS$ with respect to $\alpha$ on $[a,b]$.
This completes the proof. □

**Lemma 3.3** [2] (Cauchy criterion). Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing function. A fuzzy-number-valued function $\tilde{f} : [a,b] \to E^1$ is Henstock-Stieltjes integrable with respect to $\alpha$ on $[a,b]$ if and only if for every $\varepsilon > 0$ there is a positive function $\delta(\xi) > 0$ such that whenever $P_1 = \{[u_1,v_1];\xi_1\}$, $P_2 = \{[u_2,v_2];\xi_2\}$ are two $\delta$-fine divisions, we have

$$D\Big(\sum_{(P_1)} \tilde{f}(\xi_1)[\alpha(v_1) - \alpha(u_1)], \sum_{(P_2)} \tilde{f}(\xi_2)[\alpha(v_2) - \alpha(u_2)]\Big) < \varepsilon. \tag{3.5}$$

**Theorem 3.2** Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing function. If a fuzzy-number-valued function $\tilde{f} : [a,b] \to E^1$ has $LSRS$ with respect to $\alpha$ on $[a,b]$ then $\tilde{f}(x)$ is Henstock-Stieltjes integrable with respect to $\alpha$ on any closed sub-interval $C \subset (a,b)$. (Where $C = [r,s]$).

**Proof** A fuzzy-number-valued function $\tilde{f} : [a,b] \to E^1$ has $LSRS$ with respect to $\alpha$ on $[a,b]$ means that for every $\varepsilon > 0$ there is a $\delta(\xi) > 0$ such that for every $t \in [a,b]$, we have

$$\Big\| \sum \tilde{f}(\xi)[\alpha(v) - \alpha(u)] \Big\|_{E^1} < \varepsilon, \tag{3.6}$$

whenever $P = \{[u,v];\xi\}$ is a $\delta$-fine division of an interval $C \subset (t - \delta(t), t + \delta(t))$, $t \in C$ and $\Sigma$ sums over $P$.

(i) If there $t \in [a,b]$ with $C \subset (t - \delta(t), t + \delta(t))$ we have the following discussion:

(1) If $t \in C$ then for every $\varepsilon > 0$ there is a two $\delta$-fine divisions $P_1 = \{[u_1,v_1];\xi_1\}$, $P_2 = \{[u_2,v_2];\xi_2\}$ on $C$, such that

$$D\Big(\sum_{(P_1)} \tilde{f}(\xi_1)[\alpha(v_1) - \alpha(u_1)], \sum_{(P_2)} \tilde{f}(\xi_2)[\alpha(v_2) - \alpha(u_2)]\Big) < \varepsilon. \tag{3.7}$$

According to the Cauchy criterion, then $\tilde{f}(x)$ is Henstock-Stieltjes integrable on $C$.

(2) If $t \notin C$ then there is a closed interval $E \subset (t-\delta(t), t+\delta(t))$, with the result that $t \in E$ and $C \subset E$ (where $E = [g,h]$ ). As a result, for every $\varepsilon > 0$ there is a two $\delta$-fine divisions $P_1 = \{[u_1, v_1]; \xi_1\}$, $P_2 = \{[u_2, v_2]; \xi_2\}$ on $E$, such that

$$D\Big(\sum_{(P_1)} \tilde{f}(\xi_1)[\alpha(v_1) - \alpha(u_1)], \sum_{(P_2)} \tilde{f}(\xi_2)[\alpha(v_2) - \alpha(u_2)]\Big) < \varepsilon. \tag{3.8}$$

According to the Cauchy criterion, then $\tilde{f}(x)$ is Henstock-Stieltjes integrable on $E$. Because $C \subset E$ and $\tilde{f}(x)$ is Henstock-Stieltjes integrable on $E$ then $\tilde{f}(x)$ is Henstock-Stieltjes integrable on $C$.

(ii) If $C \nsubseteq (t-\delta(t), t+\delta(t))$ then there is a positive function $\delta$ on $[a,b]$ which resulted in the presence that $P = \{(C_i, t_i) : i = 1, 2, \cdots, k\}$ is a $\delta$-fine division of the interval $C$. It follows that $\tilde{f}(x)$ is Henstock-Stieltjes integrable on $C_i$ for $i = 1, 2, \cdots, k$.

Then $\tilde{f}(x)$ is Henstock-Stieltjes integrable on $C$.
This completes the proof. □

**Corollary 3.1** Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing function. If a fuzzy-number-valued function $\tilde{f} : [a,b] \to E^1$ has $LSRS$ with respect to $\alpha$ on $[a,b]$ then $\tilde{f}(x)$ is Henstock-Stieltjes integrable with respect to $\alpha$ on $C$ for any simple set $C \subset (a,b)$.

Notice that a simple set $C$ means that there exists finite closed sub-interval $C_i$ which belongs to $(a,b)$ such that $C = \bigcup_{i=1}^{k} C_i$.

**Theorem 3.3** Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing function. If a fuzzy-number-valued function $\tilde{f} : [a,b] \to E^1$ has $LSRS$ with respect to $\alpha$ on $[a,b]$ then $\tilde{f}(x)$ is Henstock-Stieltjes integrable with respect to $\alpha$ on $[a,b]$.

**Proof** A fuzzy-number-valued function $\tilde{f} : [a,b] \to E^1$ has $LSRS$ with respect to $\alpha$ on $[a,b]$, then for every $\varepsilon > 0$ there is $\delta^*(\xi) > 0$ such that for every $t \in [a,b]$, we have

$$\Big\| \sum \tilde{f}(\xi)[\alpha(v) - \alpha(u)] \Big\|_{E^1} < \varepsilon, \tag{3.9}$$

whenever $P = \{[u,v]; \xi\}$ is a $\delta^*$-fine division of an interval $C \subset (t-\delta(t), t+\delta(t))$, $t \in C$ and $\Sigma$ sums over $P$. According to the Corollary 3.1, $\tilde{f}(x)$ is Henstock-Stieltjes integrable on $C$ for any simple set $C \subset (a,b)$.

Rows set $\{E_i\}$, $E_i \bigcap E_j = \phi$, $\forall i \neq j$ with property $(a,b) = \bigcup E_i$, $E_i$ is a closed interval. Thus for a bove $\varepsilon > 0$, there is a positive numbers $n_0$ with property

$$\mu\{[a,b] - \bigcup_{i \leq n_0} E_i\} < \varepsilon, \tag{3.10}$$

where $\mu$ is Lebesgue measure.

For any $i$, there is a positive function $\delta_i$ such that for any $\delta_i$-fine division on $E_i$, we have

$$D\Big(\sum \tilde{f}(\xi)[\alpha(v) - \alpha(u)], (HS)\int_{E_i} \tilde{f}(x)\mathrm{d}\alpha\Big) < \varepsilon. \tag{3.11}$$

Define a positive function $\delta$ by the formula:

$$\delta(\xi) = \begin{cases} \min\{\delta^*(\xi), \frac{1}{2}d(\xi, \partial[a,b])\} & \text{if } \xi \in \bigcup_{i > n_0} E_i, \\ \min\{\delta^*(\xi), \delta_i(\xi)\}, & \text{if } \xi \in \bigcup_{i \leq n_0} E_i. \end{cases}$$

For each $\mathcal{C} = \{C\} = \{C_1, C_2, \cdots, C_k\}$ with $C_j = E_i \bigcap Q$ (where $Q = [u,v]$), for one $i \leq n_0$ and one $Q$ with $\{[u,v]; \xi\}$ is a $\delta$-fine division and $\xi \in (a,b)$, we have

(i) If $C_j = E_i$ for $i \leq n_0$. Because $\tilde{f}(x)$ is Henstock-Stieltjes integrable on $E_i$ and $\tilde{f}(x)$ is Henstock-Stieltjes integrable on $C_j$ consequently $\tilde{f}(x)$ is Henstock-Stieltjes integrable on $\bigcup_{j=1}^{k} C_j$. Selected a positive function $\delta_*$ with $\delta_*(\xi) = \min\{\delta_j(\xi) : j = 1, 2, \cdots, k\}$, then for each $\delta_*$-fine division $P = \{[u,v]; \xi\}$ on $\bigcup_{j=1}^{k} C_j$, we have

$$D\Big((HS)\int_{\bigcup_{j=1}^{k} C_j} \tilde{f}(x)\mathrm{d}\alpha, \sum \tilde{f}(\xi)[\alpha(v) - \alpha(u)]\Big) < \varepsilon. \tag{3.12}$$

Thus obtained:

$$\begin{aligned}\left\|\mathcal{C}\sum(HS)\int_C \tilde{f}(x)\mathrm{d}\alpha\right\|_{E^1} &\leq D\Big((HS)\int_{\bigcup_{j=1}^{k} C_j}\tilde{f}(x)\mathrm{d}\alpha, \sum\tilde{f}(\xi)[\alpha(v)-\alpha(u)]\Big)+\sum_{j=1}^{k}\Big\|\sum\tilde{f}(\xi)[\alpha(v)-\alpha(u)]\Big\|_{E^1}\\ &< \varepsilon+k\varepsilon.\end{aligned}$$

According to the properties of Cauchy, $\tilde{f}(x)$ is Henstock-Stieltjes integrable on $[a,b]$.

(ii) If $C_j = E_i\bigcap Q$, for $i\leq n_0$ and one $\delta$-fine $Q$ with $\{[u,v];\xi\}$ and $\xi\in(a,b)$ then $C_j\subset(\xi-\delta(\xi),\xi+\delta(\xi))$. According to the Theorem 3.2, then $\tilde{f}(x)$ is Henstock-Stieltjes integrable on $C_j$. As the result $\tilde{f}(x)$ is Henstock-Stieltjes integrable on $\bigcup_{j=1}^{k} C_j$. Selected a positive function $\delta_1$ with property $\delta_1(\xi)\leq\delta(\xi)$ then for each $\delta_*$-fine division $P=\{[u,v];\xi\}$ on $\bigcup_{j=1}^{k} C_j$, we have

$$D\Big((HS)\int_{\bigcup_{j=1}^{k} C_j}\tilde{f}(x)\mathrm{d}\alpha, \sum\tilde{f}(\xi)[\alpha(v)-\alpha(u)]\Big)<\varepsilon. \tag{3.13}$$

Thus obtained:

$$\begin{aligned}\left\|\mathcal{C}\sum(HS)\int_C \tilde{f}(x)\mathrm{d}\alpha\right\|_{E^1} &\leq D\Big((HS)\int_{\bigcup_{j=1}^{k} C_j}\tilde{f}(x)\mathrm{d}\alpha, \sum\tilde{f}(\xi)[\alpha(v)-\alpha(u)]\Big)+\sum_{j=1}^{k}\Big\|\sum\tilde{f}(\xi)[\alpha(v)-\alpha(u)]\Big\|_{E^1}\\ &< \varepsilon+k\varepsilon.\end{aligned}$$

According to the properties of Cauchy, $\tilde{f}(x)$ is Henstock-Stieltjes integrable on $[a,b]$.
This completes the proof. □

**Corollary 3.2** Let $\alpha:[a,b]\to\mathbb{R}$ be an increasing function. A fuzzy-number-valued function $\tilde{f}:[a,b]\to E^1$ has $LSRS$ with respect to $\alpha$ on $[a,b]$ if and only if $\tilde{f}(x)$ is Henstock-Stieltjes integrable with respect to $\alpha$ on $[a,b]$.

# 4 Globally small Riemann sums and Henstock-Stieltjes integral of fuzzy-number-valued functions

In this section, we shall define globally small Riemann sums with respect to $\alpha$ on $[a,b]$ for fuzzy-number-valued functions. Furthermore, we prove that a fuzzy-number-valued functions is Henstock-Stieltjes integrable with respect to $\alpha$ on $[a,b]$ if and only if it has $(GSRS)$ with respect to $\alpha$ on $[a,b]$. We begin with the following definition.

**Definition 4.1** Let $\alpha:[a,b]\to\mathbb{R}$ be an increasing function. A fuzzy-number-valued function $\tilde{f}:[a,b]\to E^1$ is said to be have globally small Riemann sums or $(GSRS)$ with respect to $\alpha$ on $[a,b]$ if for every $\varepsilon>0$ there exists a positive integer $N$ such that for every $n\geqslant N$ there is a $\delta_n(\xi)>0$ and for every $\delta_n$-fine division $P=\{[u,v];\xi\}$ of $[a,b]$, we have

$$\Big\|\sum_{\|\tilde{f}(\xi)\|_{E^1}>n}\tilde{f}(\xi)[\alpha(v)-\alpha(u)]\Big\|_{E^1}<\varepsilon, \tag{4.1}$$

where the $\sum$ is taken over $P$ and for which $\|\tilde{f}(\xi)\|_{E^1}>n$.

**Theorem 4.1** Let $\alpha:[a,b]\to\mathbb{R}$ be an increasing function and let $\tilde{f}(x)$ be Henstock-Stieltjes integrable to $\tilde{F}(a,b)$ with respect to $\alpha$ on $FHS_\alpha[a,b]$ and $\tilde{F}_n(a,b)$ the integral of $\tilde{f}_n(x)$ on $FHS_\alpha[a,b]$, where $\tilde{f}_n(x)=\tilde{f}(x)$ when $\left\|\tilde{f}(x)\right\|_{E^1}\leqslant n$ and $\tilde{0}$ otherwise. If $\tilde{F}_n(a,b)\to\tilde{F}(a,b)$ as $n\to\infty$ then $\tilde{f}(x)$ has $GSRS$ with respect to $\alpha$ on $[a,b]$.

**Proof** Given $\varepsilon>0$ there is a $\delta_n(\xi)>0$ such that for every $\delta_n$-fine division $P=\{[u,v];\xi\}$ of $[a,b]$, we have

$$D\big(\sum\tilde{f}_n(\xi)[\alpha(v)-\alpha(u)],\tilde{F}_n(a,b)\big)<\varepsilon, \tag{4.2}$$

where $(FHS)\int_a^b \tilde{f}_n(x)\mathrm{d}\alpha = \tilde{F}_n(a,b)$.

$$D\big(\sum \tilde{f}(\xi)[\alpha(v)-\alpha(u)], \tilde{F}(a,b)\big) < \varepsilon, \tag{4.3}$$

where $(FHS)\int_a^b \tilde{f}(x)\mathrm{d}\alpha = \tilde{F}(a,b)$.

Choose $N$ so that whenever $n \geqslant N$

$$D\big(\tilde{F}_n(a,b), \tilde{F}(a,b)\big) < \varepsilon. \tag{4.4}$$

Therefore for $n \geqslant N$ and $\delta_n$-fine division $P = \{[u,v];\xi\}$ of $[a,b]$, we have

$$\begin{aligned}
&\Big\| \sum_{\|\tilde{f}(\xi)\|_{E^1}>n} \tilde{f}(\xi)[\alpha(v)-\alpha(u)] \Big\|_{E^1} = D\Big(\sum \tilde{f}_n(\xi)[\alpha(v)-\alpha(u)], \sum \tilde{f}(\xi)[\alpha(v)-\alpha(u)]\Big) \\
\leqslant\ & D\Big(\sum \tilde{f}_n(\xi)[\alpha(v)-\alpha(u)], \tilde{F}_n(a,b)\Big) + D\Big(\tilde{F}_n(a,b), \tilde{F}(a,b)\Big) + D\Big(\tilde{F}(a,b), \sum \tilde{f}(\xi)[\alpha(v)-\alpha(u)]\Big) \\
<\ & 3\varepsilon.
\end{aligned}$$

Hence $\tilde{f}(x)$ has $GSRS$ with respect to $\alpha$ on $[a,b]$.
This completes the proof. □

**Theorem 4.2** Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing function. A fuzzy-number-valued function $\tilde{f}(x)$ has $GSRS$ with respect to $\alpha$ on $[a,b]$ if and only if $\tilde{f}(x)$ is Henstock-Stieltjes integrable with respect to $\alpha$ on $[a,b]$ and $\tilde{F}_n(a,b) \to \tilde{F}(a,b)$ as $n \to \infty$ where $\tilde{F}_n(a,b)$ and $\tilde{F}(a,b)$ are defined as in Theorem 4.1.

**Proof** Theorem 4.1 proves the sufficiency. We shall prove only the necessity. Suppose $\tilde{f}(x)$ has $GSRS$ with respect to $\alpha$ on $[a,b]$. Note that $\tilde{f}_n(x)$, as defined in Theorem 4.1, is fuzzy Henstock-Stieltjes integrable on $[a,b]$ for all $n$. Then for $n, m \geqslant N$ and a suitably chosen $\delta$-fine division $P = \{[u,v];\xi\}$, we have

$$\begin{aligned}
&D\Big(\tilde{F}_n(a,b), \tilde{F}_m(a,b)\Big) \\
\leqslant\ & D\Big(\tilde{F}_n(a,b), \sum_{\|\tilde{f}(\xi)\|_{E^1}\leqslant n} \tilde{f}(\xi)[\alpha(v)-\alpha(u)]\Big) + D\Big(\sum_{\|\tilde{f}(\xi)\|_{E^1}\leqslant m} \tilde{f}(\xi)[\alpha(v)-\alpha(u)], \tilde{F}_m(a,b)\Big) \\
+\ & \Big\| \sum_{\|\tilde{f}(\xi)\|_{E^1}>n} \tilde{f}(\xi)[\alpha(v)-\alpha(u)] \Big\|_{E^1} + \Big\| \sum_{\|\tilde{f}(\xi)\|_{E^1}>m} \tilde{f}(\xi)[\alpha(v)-\alpha(u)] \Big\|_{E^1} \\
<\ & 4\varepsilon.
\end{aligned}$$

That is, $\tilde{F}_n(a,b)$ converge to a fuzzy number, say $\tilde{F}(a,b)$, as $n \to \infty$. Again, for suitably chosen $N$ and $\delta(\xi)$ and for every $\delta$-fine division $P = \{[u,v];\xi\}$, we have

$$\begin{aligned}
&D\Big(\sum \tilde{f}(\xi)[\alpha(v)-\alpha(u)], \tilde{F}(a,b)\Big) \leqslant D\Big(\tilde{F}(a,b), \tilde{F}_N(a,b)\Big) \\
+\ & D\Big(\tilde{F}_N(a,b), \sum_{\|\tilde{f}(\xi)\|_{E^1}\leqslant N} \tilde{f}(\xi)[\alpha(v)-\alpha(u)]\Big) + \Big\| \sum_{\|\tilde{f}(\xi)\|_{E^1}>N} \tilde{f}(\xi)[\alpha(v)-\alpha(u)] \Big\|_{E^1} \\
<\ & 3\varepsilon.
\end{aligned}$$

That is, $\tilde{f}(x)$ is fuzzy Henstock-Stieljes integrable on $[a,b]$.
This completes the proof. □

**Theorem 4.3** Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing function and let $\tilde{f}_n(x) \in FHS_\alpha[a,b],\ n = 1,2,3\cdots$ and satisfy:
(1) $\lim\limits_{n\to\infty} \tilde{f}_n(x) = \tilde{f}(x)$ almost everywhere in $[a,b]$;
(2) there exists a Lebesgue-Stieljes integrable (Henstock-Stieljes integrable) function $h(x)$ on $[a,b]$ such that

$$D\big(\tilde{f}_n(x), \tilde{f}_m(x)\big) < h(x). \tag{4.5}$$

Then, $\tilde{f}_n(x)$ has $GSRS$ with respect to $\alpha$ on $[a,b]$ uniformly for any $n$. Naturally, $\tilde{f}$ is $(FHS)$ integrable with respect to $\alpha$ on $[a,b]$. Furthermore,

$$\lim_{n\to\infty} (FHS)\int_a^b \tilde{f}_n(x)\mathrm{d}\alpha = (FHS)\int_a^b \tilde{f}(x)\mathrm{d}\alpha. \tag{4.6}$$

**Proof** Let $\varepsilon > 0$. Since $H(x) = (LS)\int_a^x h(t)\mathrm{d}\alpha$ is absolutely continuous with respect to stieljes measurable on $[a,b]$, there exists a positive number $\eta > 0$ such that $\sum |H(b_i) - H(a_i)| < \varepsilon$ whenever $\{[a_i, b_i]\}$ is a finite collection of non-overlapping intervals in $[a,b]$ that satisfy $\sum \big(\alpha(b_i) - \alpha(a_i)\big) < \eta$. Since $\lim\limits_{n\to\infty} \tilde{f}_n(x) = \tilde{f}(x)$ almost everywhere in $[a,b]$, and

$$\begin{aligned} D(\tilde{f}_n, \tilde{f}) &= \sup_{r\in[0,1]} \max\{|(f_n(x))_-^r - (f(x))_-^r|, |(f_n(x))_+^r - (f(x))_+^r|\} \\ &= \sup_{r_k\in[0,1]} \max\{|(f_n(x))_-^{r_k} - (f(x))_-^{r_k}|, |(f_n(x))_+^{r_k} - (f(x))_+^{r_k}|\} \end{aligned}$$

is a sequence of Lebesgue-Stieljes measurable functions, where $r_k \in [0,1]$ is the set of rational numbers, by Egorov's Theorem, there exists an open set G with $LS(G) < \eta$ such that $\lim\limits_{n\to\infty} \tilde{f}_n(x) = \tilde{f}(x)$ uniformly for $x \in [a,b] \setminus G$. Then, there is an natural number $N$, such that for any $n, m > N$, and for any $x \in [a,b] \setminus G$, we have $D(\tilde{f}_n(x), \tilde{f}_m(x)) < \varepsilon$. Since $h(x)$ is Henstock-Stieljes integrable on $[a,b]$, there is a $\delta_h(\xi) > 0$ such that for any $\delta_h$-fine division $P = \{[u,v];\xi\}$ of $[a,b]$, we have

$$\left| \sum h(\xi)[\alpha(v) - \alpha(u)] - (LS)\int_a^b h(t)\mathrm{d}\alpha \right| < \varepsilon. \tag{4.7}$$

Define

$$\delta(\xi) = \begin{cases} \delta_h(\xi), & \text{if } \xi \in [a,b] \setminus G, \\ \delta(\xi), \text{ satisfying } (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i)) \subset G, & \text{if } \xi \in [a,b]. \end{cases}$$

Then, it follows that for a $\delta$-fine division $P_0 = \{[x_{i-1}, x_i];\xi_i\}$ of $[a,b]$,

$$\begin{aligned} & D\Big(\sum \tilde{f}_n(\xi_i)[\alpha(x_i) - \alpha(x_{i-1})], \sum \tilde{f}_m(\xi_i)[\alpha(x_i) - \alpha(x_{i-1})]\Big) \\ \le\ & D\Big(\sum_{\xi_i\in[a,b]\setminus G} \tilde{f}_n(\xi_i)[\alpha(x_i) - \alpha(x_{i-1})], \sum_{\xi_i\in[a,b]\setminus G} \tilde{f}_m(\xi_i)[\alpha(x_i) - \alpha(x_{i-1})]\Big) \\ +\ & D\Big(\sum_{\xi_i\in G} \tilde{f}_n(\xi_i)[\alpha(x_i) - \alpha(x_{i-1})], \sum_{\xi_i\in G} \tilde{f}_m(\xi_i)[\alpha(x_i) - \alpha(x_{i-1})]\Big) \\ \le\ & \sum_{\xi_i\in[a,b]\setminus G} D\big(\tilde{f}_n(\xi_i), \tilde{f}_m(\xi_i)\big)[\alpha(x_i) - \alpha(x_{i-1})] + \sum_{\xi_i\in G} D\big(\tilde{f}_n(\xi_i), \tilde{f}_m(\xi_i)\big)[\alpha(x_i) - \alpha(x_{i-1})] \\ <\ & \varepsilon(b-a) + \left| \sum_{\xi_i\in G} h(\xi_i)[\alpha(x_i) - \alpha(x_{i-1})] - \int_G h(t)\mathrm{d}\alpha \right| + \left| \int_G h(t)\mathrm{d}\alpha \right| \\ <\ & \varepsilon(b-a) + 3\varepsilon. \end{aligned}$$

Hence, there is an natural number $N$ such that for any $n, m > N$, we have

$$\begin{aligned} & D\Big(\tilde{F}_n[a,b], \tilde{F}_m[a,b]\Big) \\ \le\ & D\Big(\tilde{F}_n[a,b], \sum \tilde{f}_n(\xi_i)[\alpha(x_i) - \alpha(x_{i-1})]\Big) + D\Big(\tilde{F}_m[a,b], \sum \tilde{f}_m(\xi_i)[\alpha(x_i) - \alpha(x_{i-1})]\Big) \\ +\ & D\Big(\sum \tilde{f}_n(\xi_i)[\alpha(x_i) - \alpha(x_{i-1})], \sum \tilde{f}_m(\xi_i)[\alpha(x_i) - \alpha(x_{i-1})]\Big) \\ <\ & 3\varepsilon. \end{aligned}$$

Thus, $\tilde{F}_n[a,b]$ is a Cauchy sequence, and there is an natural number $N_1$ such that for any $n > N_1$, we have $D(\tilde{F}_n[a,b], \tilde{A}) < \varepsilon$. According to the $(FHS)$ integrability of $\tilde{f}_{N_1}(x)$, there is a $\delta_{N_1}(\xi) > 0$ such that for any $\delta_{N_1}$-fine division $P = \{[u,v];\xi\}$ of $[a,b]$, for any $n > N_{N_1}$, we have

$$\begin{aligned} & D\Big(\sum \tilde{f}_n(\xi)[\alpha(v) - \alpha(u)], \tilde{F}_n[a,b]\Big) \\ \le\ & D\Big(\tilde{F}_n[a,b], \tilde{F}_{N_1}[a,b]\Big) + D\Big(\sum \tilde{f}_{N_1}(\xi)[\alpha(v) - \alpha(u)], \tilde{F}_{N_1}[a,b]\Big) \\ +\ & D\Big(\sum \tilde{f}_n(\xi)[\alpha(v) - \alpha(u)], \sum \tilde{f}_{N_1}(\xi)[\alpha(v) - \alpha(u)]\Big) \\ <\ & 3\varepsilon. \end{aligned}$$

This completes the proof. □

# 5 conclusions

In this paper, we have investigated locally and globally small Riemann sums with respect to $\alpha$ on $[a,b]$ for fuzzy number-valued functions. Also we have stated and proved two main theorems: (1) If a fuzzy number-valued functions $\tilde{f}(x)$ is Henstock-Stieltjes integrable with respect to $\alpha$ on $[a,b]$ then $\tilde{f}(x)$ has $(LSRS)$ with respect to $\alpha$ on $[a,b]$ and the converse is always true. (2) If a fuzzy number-valued functions $\tilde{f}(x)$ is Henstock-Stieltjes integrable with respect to $\alpha$ on $[a,b]$ then $\tilde{f}(x)$ has $(GSRS)$ with respect to $\alpha$ on $[a,b]$ and the converse is always true. Finally, by Egorov's Theorem, we got the dominated convergence theorem for globally small Riemann sums $(GSRS)$ with respect to $\alpha$ on $[a,b]$ for fuzzy-number-valued functions.

## References

[1] P. Diamond, P.E. Kloeden, Metric spaces of fuzzy sets: theory and applications. World Scientific Publishing, Singapore, 1994.

[2] Z.T. Gong, L.L. Wang, The Henstock-Stieltjes integral for fuzzy-number-valued functions, Information Sciences, 188, 276-297(2012).

[3] Z.T. Gong, C.X. Wu, The Mcshane Integral of Fuzzy-Valued Functions, Southeast Asian Bulletin of Mathematics, 24(3), 365-373(2000).

[4] S. Jong, H. Ju, S. Gwang, On Henstock-Stieltjes integral, Kangweon-kyungki Math, 6, 87-96(1998).

[5] R. Goetschel Jr., W. Voxman, Elementary fuzzy calculus. Fuzzy Sets and Systems, 18, 31-43(1986).

[6] M.E. Hamid, L.S. Xu, Z.T. Gong, Locally and globally small Riemann sums and Henstock integral of fuzzy-number-valued functions, (accepted).

[7] R. Henstock, Theory of Integration, Butterworth, London, 1963.

[8] C.R. Indrati, Some Characteristics of the Henstock-Kurzweil in Countably Lipschitz Condition, The 7th SEAMS-UGM Conference 2015.

[9] O. Kaleva, Fuzzy differential equations, Fuzzy Sets and Systems, 24, 301-317(1987).

[10] P.Y. Lee, Lanzhou Lectures on Henstock Integration, World Scientific, Singapore, 1989.

[11] M. Ma, Embedding Problem of Fuzzy number Space, Part4 , Fuzzy Sets and Systems, 58, 185-193(1993).

[12] S. Nanda, On fuzzy integrals, Fuzzy Sets and Systems, 32, 95-101(1989).

[13] X.K. Ren, C.X. Wu, Z.G. Zhu, A new kind of fuzzy Riemann-Stieltjes integral, in: X.Z. Wang, D. Yeung, X.L. Wang (Eds.), Proceedings of 5th International Conference on Machine Learning and Cybernetics, ICMLC 2006, Dalian, China, pp. 1885-1888(2006).

[14] X.K. Ren, The Non-additive Measure and the Fuzzy Riemann-Stieltjes Integral, Ph.D dissertation, Harbin Institute of Technology, 2008.

[15] A.W. Schurle, A new property equivalent to Lebesgue integrability, Proceedings of the American Mathematical Society, 96(1), 103-106(1986) .

[16] A.W. Schurle, A function is Perron integrable if it has locally small Riemann sums, Journal of the Australian Mathematical Society (Series A), 41(2), 224-232(1986).

[17] C.X. Wu, C. Wu, The supremum and infimum of the set of fuzzy numbers and its application, Journal of Mathematical Analysis and Applications, 210, 499-511(1997).

[18] H.C. Wu, The fuzzy Riemann-Stieltjes integral, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, 6, 51-67(1998).

[19] C.X. Wu, M. Ma, Embedding Problem of Fuzzy number Space, Part(I), Fuzzy Sets and Systems, 44, 33-38(1991).

[20] C.X. Wu, M. Ma, Embedding Problem of Fuzzy number Space, Part(II), Fuzzy Sets and Systems, 45, 189-202(1992).

[21] L.A. Zadeh, Fuzzy sets, Information Control, 8, 338-353(1965).

# A note on special fuzzy differential subordinations using multiplier transformation and Ruschewehy derivative

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**Abstract**

In this paper we establish some fuzzy differential subordinations regarding the differential operator $RI^{\alpha}_{m,\lambda,l} : \mathcal{A}_n \to \mathcal{A}_n$, $RI^{\alpha}_{m,\lambda,l} f(z) = (1-\alpha)R^m f(z) + \alpha I(m,\lambda,l) f(z)$, where $R^m$ is the Ruscheweyh derivative,$I(m,\lambda,l)$ is the multiplier transformation and $\mathcal{A}_n = \{f \in \mathcal{H}(U),\ f(z) = z + a_{n+1}z^{n+1} + \ldots,\ z \in U\}$ is the class of normalized analytic functions. We introduce a fuzzy class $RI^{\delta}_{\mathcal{F}}(\alpha,m,\lambda,l)$ and by using the fuzzy differential subordinations we derive some properties of this class. Also, several fuzzy differential subordinations are established regarding the studied differential operator.

**Keywords:** fuzzy differential subordination, convex function, fuzzy best dominant, differential operator.
**2010 Mathematical Subject Classification:** 30C45, 30A20.

## 1 Introduction

The differential subordination method was introduced and developed by S.S. Miller and P.T. Mocanu. G.I. Oros and Gh.Oros in [6], [7] combine the notions from the complex functions domain with the fuzzy sets theory.

In this paper we obtain fuzzy differential subordinations regarding the differential operator studied in [4] using the methods from [2], [3].

Consider $U = \{z \in \mathbb{C} : |z| < 1\}$ the unit disc of the complex plane, $\mathcal{H}(U)$ the space of holomorphic functions in $U$, $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \ldots,\ z \in U\}$ and $\mathcal{H}[a,n] = \{f \in \mathcal{H}(U) : f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \ldots,\ z \in U\}$ for $a \in \mathbb{C}$ and $n \in \mathbb{N}$. The class of normalized convex functions in $U$ is denoted by $\mathcal{K} = \left\{f \in \mathcal{A}_n : \text{Re}\ \frac{zf''(z)}{f'(z)} + 1 > 0,\ z \in U\right\}$.

We need the following.

**Definition 1.1** *([6]) Let $D \subset \mathbb{C}$, $z_0 \in D$ be a fixed point and let the functions $f, g \in \mathcal{H}(D)$. The function $f$ is said to be fuzzy subordinate to $g$ and write $f \prec_{\mathcal{F}} g$ or $f(z) \prec_{\mathcal{F}} g(z)$, if are satisfied the conditions $f(z_0) = g(z_0)$ and $F_{f(D)} f(z) \leq F_{g(D)} g(z)$, $z \in D$.*

**Definition 1.2** *([7, Definition 2.2]) Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and $h$ univalent in $U$, with $\psi(a,0;0) = h(0) = a$. If $p$ is analytic in $U$, with $p(0) = a$ and satisfies the fuzzy differential subordination*

$$F_{\psi(\mathbb{C}^3 \times U)} \psi(p(z), zp'(z), z^2 p''(z); z) \leq F_{h(U)} h(z), \quad z \in U, \tag{1.1}$$

*then $p$ is called a fuzzy solution of the fuzzy differential subordination. The univalent function $q$ is called a fuzzy dominant of the fuzzy solutions of the fuzzy differential subordination, if $F_{p(U)} p(z) \leq F_{q(U)} q(z)$, $z \in U$, for all $p$ satisfying (1.1). A fuzzy dominant $\widetilde{q}$ that satisfies $F_{\widetilde{q}(U)} \widetilde{q}(z) \leq F_{q(U)} q(z)$, $z \in U$, for all fuzzy dominants $q$ of (1.1) is said to be the fuzzy best dominant of (1.1).*

**Lemma 1.1** *([5, Corollary 2.6g.2, p. 66]) Let $h \in \mathcal{A}_n$ and $L[f](z) = G(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt$, $z \in U$. If $Re\left(\frac{zh''(z)}{h'(z)} + 1\right) > -\frac{1}{2}$, $z \in U$, then $L(f) = G \in \mathcal{K}$.*

**Lemma 1.2** *([8]) Let $h$ be a convex function with $h(0) = a$, and let $\gamma \in \mathbb{C}^*$ be a complex number with $Re\ \gamma \geq 0$. If $p \in \mathcal{H}[a,n]$ with $p(0) = a$, $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$, $\psi(p(z), zp'(z); z) = p(z) + \frac{1}{\gamma} zp'(z)$ an analytic function in $U$ and*

$$F_{\psi(\mathbb{C}^2 \times U)} \left(p(z) + \frac{1}{\gamma} zp'(z)\right) \leq F_{h(U)} h(z), \quad z \in U, \tag{1.2}$$

*then $F_{p(U)} p(z) \leq F_{g(U)} g(z) \leq F_{h(U)} h(z)$, $z \in U$, where $g(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t) t^{\gamma/n-1} dt$, $z \in U$. The function $q$ is convex and is the fuzzy best dominant.*

**Lemma 1.3** *([8]) Let $g$ be a convex function in $U$ and let $h(z)=g(z)+n\alpha zg'(z)$, $z\in U$, where $\alpha>0$ and $n$ is a positive integer. If $p(z)=g(0)+p_nz^n+p_{n+1}z^{n+1}+\ldots$, $z\in U$, is holomorphic in $U$ and $F_{p(U)}\left(p(z)+\alpha zp'(z)\right)\leq F_{h(U)}h(z)$, $z\in U$, then $F_{p(U)}p(z)\leq F_{g(U)}g(z)$, $z\in U$, and this result is sharp.*

We need the following diferential operators.

**Definition 1.3** *(Ruscheweyh [9]) For $f\in\mathcal{A}_n$, $m,n\in\mathbb{N}$, the operator $R^m$ is defined by $R^m:\mathcal{A}_n\to\mathcal{A}_n$,*

$$\begin{aligned} R^0f(z) &= f(z),\ R^1f(z)=zf'(z),\ ... \\ (m+1)R^{m+1}f(z) &= z\left(R^mf(z)\right)'+mR^mf(z),\ \ z\in U. \end{aligned}$$

**Remark 1.1** *If $f\in\mathcal{A}_n$, $f(z)=z+\sum_{j=n+1}^{\infty}a_jz^j$, then $R^mf(z)=z+\sum_{j=n+1}^{\infty}\frac{(n+j-1)!}{n!(j-1)!}a_jz^j$, $z\in U$.*

**Definition 1.4** *([1]) For $f\in\mathcal{A}_n$, $m,n\in\mathbb{N}$, $\lambda,l\geq 0$, the operator $I(m,\lambda,l)f(z)$ is defined by the following infinite series $I(m,\lambda,l)f(z)=z+\sum_{j=n+1}^{\infty}\left(\frac{\lambda(j-1)+l+1}{l+1}\right)^m a_jz^j$.*

**Remark 1.2** *We have $I(0,\lambda,l)f(z)=f(z)$, $(l+1)I(m+1,\lambda,l)f(z)=(l+1-\lambda)I(m,\lambda,l)f(z)+\lambda z\left(I(m,\lambda,l)f(z)\right)'$, $z\in U$.*

**Definition 1.5** *([4]) Let $\alpha,\lambda,l\geq 0$, $m,n\in\mathbb{N}$. Denote by $RI^{\alpha}_{m,\lambda,l}$ the operator given by $RI^{\alpha}_{m,\lambda,l}:\mathcal{A}_n\to\mathcal{A}_n$, $RI^{\alpha}_{m,\lambda,l}f(z)=(1-\alpha)R^mf(z)+\alpha I(m,\lambda,l)f(z)$, $z\in U$.*

**Remark 1.3** *If $f\in\mathcal{A}_n$, $f(z)=z+\sum_{j=n+1}^{\infty}a_jz^j$, then*
*$RI^{\alpha}_{m,\lambda,l}f(z)=z+\sum_{j=n+1}^{\infty}\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_jz^j$, $z\in U$.*

# 2 Main results

Using the operator $RI^{\alpha}_{m,\lambda,l}$ we define the class $RI^{\delta}_{\mathcal{F}}(\alpha,m,\lambda,l)$ and we study fuzzy subordinations.

**Definition 2.1** *The class $RI^{\delta}_{\mathcal{F}}(\alpha,m,\lambda,l)$ contains all the functions $f\in\mathcal{A}_n$ which satisfy the inequality*

$$F_{\left(RI^{\alpha}_{m,\lambda,l}f\right)'(U)}\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'>\delta,\quad z\in U, \tag{2.1}$$

*where $\delta\in(0,1]$, $\alpha,\lambda,l\geq 0$ and $m,n\in\mathbb{N}$.*

**Theorem 2.1** *$RI^{\delta}_{\mathcal{F}}(\alpha,m,\lambda,l)$ is a convex set.*

**Proof.** Let $f_1$, $f_2\in RI^{\delta}_{\mathcal{F}}(\alpha,m,\lambda,l)$, $f_k(z)=z+\sum_{j=n+1}^{\infty}a_{jk}z^j$, $k=1,2$, $z\in U$. We show that the function $h(z)=\eta_1f_1(z)+\eta_2f_2(z)$ is in the class $RI^{\delta}_{\mathcal{F}}(\alpha,m,\lambda,l)$, where $\eta_1$ and $\eta_2$ are nonnegative such that $\eta_1+\eta_2=1$.

Differentiating, we obtain $h'(z)=(\mu_1f_1+\mu_2f_2)'(z)=\mu_1f_1'(z)+\mu_2f_2'(z)$, $z\in U$, and
$\left(RI^{\alpha}_{m,\lambda,l}h(z)\right)'=\left(RI^{\alpha}_{m,\lambda,l}(\mu_1f_1+\mu_2f_2)(z)\right)'=\mu_1\left(RI^{\alpha}_{m,\lambda,l}f_1(z)\right)'+\mu_2\left(RI^{\alpha}_{m,\lambda,l}f_2(z)\right)'$, so we have also
$F_{\left(RI^{\alpha}_{m,\lambda,l}h\right)'(U)}\left(RI^{\alpha}_{m,\lambda,l}h(z)\right)'=F_{\left(RI^{\alpha}_{m,\lambda,l}(\mu_1f_1+\mu_2f_2)\right)'(U)}\left(RI^{\alpha}_{m,\lambda,l}(\mu_1f_1+\mu_2f_2)(z)\right)'=$
$F_{\left(RI^{\alpha}_{m,\lambda,l}(\mu_1f_1+\mu_2f_2)\right)'(U)}\left(\mu_1\left(RI^{\alpha}_{m,\lambda,l}f_1(z)\right)'+\mu_2\left(RI^{\alpha}_{m,\lambda,l}f_2(z)\right)'\right)=$
$\frac{F_{\left(\mu_1RI^{\alpha}_{m,\lambda,l}f_1\right)'(U)}\left(\mu_1\left(RI^{\alpha}_{m,\lambda,l}f_1(z)\right)'\right)+F_{\left(\mu_2RI^{\alpha}_{m,\lambda,l}f_2\right)'(U)}\left(\mu_2\left(RI^{\alpha}_{m,\lambda,l}f_2(z)\right)'\right)}{2}=$
$\frac{F_{\left(RI^{\alpha}_{m,\lambda,l}f_1\right)'(U)}\left(RI^{\alpha}_{m,\lambda,l}f_1(z)\right)'+F_{\left(RI^{\alpha}_{m,\lambda,l}f_2\right)'(U)}\left(RI^{\alpha}_{m,\lambda,l}f_2(z)\right)'}{2}$.

Since $f_1,f_2\in RI^{\delta}_{\mathcal{F}}(\alpha,m,\lambda,l)$, we have $\delta<F_{\left(RI^{\alpha}_{m,\lambda,l}f_1\right)'(U)}\left(RI^{\alpha}_{m,\lambda,l}f_1(z)\right)'\leq 1$ and
$\delta<F_{\left(RI^{\alpha}_{m,\lambda,l}f_2\right)'(U)}\left(RI^{\alpha}_{m,\lambda,l}f_2(z)\right)'\leq 1$, $z\in U$.

Therefore $\delta<\frac{F_{\left(RI^{\alpha}_{m,\lambda,l}f_1\right)'(U)}\left(RI^{\alpha}_{m,\lambda,l}f_1(z)\right)'+F_{\left(RI^{\alpha}_{m,\lambda,l}f_2\right)'(U)}\left(RI^{\alpha}_{m,\lambda,l}f_2(z)\right)'}{2}\leq 1$ and we obtain that
$\delta<F_{\left(RI^{\alpha}_{m,\lambda,l}h\right)'(U)}\left(RI^{\alpha}_{m,\lambda,l}h(z)\right)'\leq 1$, which means that $h\in RI^{\delta}_{\mathcal{F}}(\alpha,m,\lambda,l)$ and $RI^{\delta}_{\mathcal{F}}(\alpha,m,\lambda,l)$ is a convex set. ■

**Theorem 2.2** *Consider $g$ a convex function in $U$ and $h(z)=g(z)+\frac{1}{c+2}zg'(z)$, with $z\in U$, $c>0$. When $f\in RI_{\mathcal{F}}^{\delta}(\alpha,m,\lambda,l)$ and $G(z)=\frac{c+2}{z^{c+1}}\int_0^z t^c f(t)\,dt$, $z\in U$, then*

$$F_{\left(RI_{m,\lambda,l}^{\alpha}f\right)'(U)}\left(RI_{m,\lambda,l}^{\alpha}f(z)\right)'\leq F_{h(U)}h(z),\quad z\in U, \tag{2.2}$$

*implies $F_{\left(RI_{m,\lambda,l}^{\alpha}G\right)'(U)}\left(RI_{m,\lambda,l}^{\alpha}G(z)\right)'\leq F_{g(U)}g(z)$, $z\in U$, and this result is sharp.*

**Proof.** Differentiating with respect to $z$ the following relation $z^{c+1}G(z)=(c+2)\int_0^z t^c f(t)\,dt$. We obtain $(c+1)G(z)+zG'(z)=(c+2)f(z)$ and $(c+1)RI_{m,\lambda,l}^{\alpha}G(z)+z\left(RI_{m,\lambda,l}^{\alpha}G(z)\right)'=(c+2)RI_{m,\lambda,l}^{\alpha}f(z)$, $z\in U$. Differentiating again we get $\left(RI_{m,\lambda,l}^{\alpha}G(z)\right)'+\frac{1}{c+2}z\left(RI_{m,\lambda,l}^{\alpha}G(z)\right)''=\left(RI_{m,\lambda,l}^{\alpha}f(z)\right)'$, $z\in U$, and the fuzzy differential subordination (2.2) becomes

$$F_{RI_{m,\lambda,l}^{\alpha}G(U)}\left(\left(RI_{m,\lambda,l}^{\alpha}G(z)\right)'+\frac{1}{c+2}z\left(RI_{m,\lambda,l}^{\alpha}G(z)\right)''\right)\leq F_{g(U)}\left(g(z)+\frac{1}{c+2}zg'(z)\right). \tag{2.3}$$

Consider

$$p(z)=\left(RI_{m,\lambda,l}^{\alpha}G(z)\right)',\quad z\in U, \tag{2.4}$$

evidently $p\in\mathcal{H}[1,n]$ and from relation (2.3) we get $F_{p(U)}\left(p(z)+\frac{1}{c+2}zp'(z)\right)\leq F_{g(U)}\left(g(z)+\frac{1}{c+2}zg'(z)\right)$, $z\in U$.

By using Lemma 1.3 we obtain $F_{\left(RI_{m,\lambda,l}^{\alpha}G\right)'(U)}\left(RI_{m,\lambda,l}^{\alpha}G(z)\right)'\leq F_{g(U)}g(z)$, $z\in U$, and $g$ is the fuzzy best dominant. ■

**Theorem 2.3** *Let $h(z)=\frac{1+(2\beta-1)z}{1+z}$, $\beta\in[0,1)$ and $c>0$. If $\lambda,l\geq 0$, $m,n\in\mathbb{N}$ and $I_c(f)(z)=\frac{c+2}{z^{c+1}}\int_0^z t^c f(t)\,dt$, $z\in U$, then*

$$I_c\left[RI_{\mathcal{F}}^{\beta}(\alpha,m,\lambda,l)\right]\subset RI_{\mathcal{F}}^{\beta^*}(\alpha,m,\lambda,l), \tag{2.5}$$

*where $\beta^*=2\beta-1+\frac{(c+2)(2-2\beta)}{n}\int_0^1\frac{t^{\frac{c+2}{n}-1}}{t+1}dt$.*

**Proof.** A similar proof with Theorem 2.2 for the convex function $h$ get us $F_{p(U)}\left(p(z)+\frac{1}{c+2}zp'(z)\right)\leq f_{h(U)}h(z)$, with $p(z)$ defined in (2.4).

Applying Lemma 1.2 we obtain $F_{\left(RI_{m,\lambda,l}^{\alpha}G\right)'(U)}\left(RI_{m,\lambda,l}^{\alpha}G(z)\right)'\leq F_{g(U)}g(z)\leq F_{h(U)}h(z)$, where $g(z)=\frac{c+2}{nz^{\frac{c+2}{n}}}\int_0^z t^{\frac{c+2}{n}-1}\frac{1+(2\beta-1)t}{1+t}dt=2\beta-1+\frac{(c+2)(2-2\beta)}{nz^{\frac{c+2}{n}}}\int_0^z\frac{t^{\frac{c+2}{n}-1}}{t+1}dt$. Since $g$ is convex and $g(U)$ is symmetric with respect to the real axis, we have

$$F_{I(m,\lambda,l)G(U)}\left(I(m,\lambda,l)G(z)\right)'\geq\min_{|z|=1}F_{g(U)}g(z)=F_{g(U)}g(1) \tag{2.6}$$

and $\beta^*=g(1)=2\beta-1+\frac{(c+2)(2-2\beta)}{n}\int_0^1\frac{t^{\frac{c+2}{n}-1}}{t+1}dt$. The inclusion (2.5) follows from (2.6). ■

**Theorem 2.4** *Consider $g$ a convex function with $g(0)=1$ and $h(z)=g(z)+zg'(z)$, $z\in U$. For $\lambda,l\geq 0$, $m,n\in\mathbb{N}$, $f\in\mathcal{A}_n$ and the fuzzy differential subordination*

$$F_{\left(RI_{m,\lambda,l}^{\alpha}f\right)'(U)}\left(RI_{m,\lambda,l}^{\alpha}f(z)\right)'\leq F_{h(U)}h(z),\quad z\in U, \tag{2.7}$$

*holds, we obtain $F_{RI_{m,\lambda,l}^{\alpha}f(U)}\frac{RI_{m,\lambda,l}^{\alpha}f(z)}{z}\leq F_{g(U)}g(z)$, $z\in U$, and this result is sharp.*

**Proof.** For $p(z)=\frac{RI_{m,\lambda,l}^{\alpha}f(z)}{z}$, $z\in U$, we have $p\in\mathcal{H}[1,n]$ and differentiating the relation $RI_{m,\lambda,l}^{\alpha}f(z)=zp(z)$, $z\in U$, we obtain $\left(RI_{m,\lambda,l}^{\alpha}f(z)\right)'=p(z)+zp'(z)$, $z\in U$.

The fuzzy differential subordination (2.7) becomes $F_{p(U)}(p(z)+zp'(z))\leq F_{h(U)}h(z)=F_{g(U)}(g(z)+zg'(z))$, $z\in U$, and applying Lemma 1.3, we deduce $F_{\left(RI_{m,\lambda,l}^{\alpha}f\right)'(U)}\frac{RI_{m,\lambda,l}^{\alpha}f(z)}{z}\leq F_{g(U)}g(z)$, $z\in U$. ■

**Theorem 2.5** *Let $h$ be a holomorphic function with* $\mathrm{Re}\left(1+\frac{zh''(z)}{h'(z)}\right)>-\frac{1}{2}$, $z\in U$, *and $h(0)=1$. When the fuzzy differential subordination*

$$F_{\left(RI_{m,\lambda,l}^{\alpha}f\right)'(U)}\left(RI_{m,\lambda,l}^{\alpha}f(z)\right)'\leq F_{h(U)}h(z),\quad z\in U, \tag{2.8}$$

*holds, where $\lambda,l\geq 0$, $m,n\in\mathbb{N}$, $f\in\mathcal{A}_n$, then $F_{RI_{m,\lambda,l}^{\alpha}f(U)}\frac{RI_{m,\lambda,l}^{\alpha}f(z)}{z}\leq F_{q(U)}q(z)$, $z\in U$. The function $q(z)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt$ is convex and it is the fuzzy best dominant.*

**Proof.** Since Re $\left(1+\frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}$, $z \in U$, from Lemma 1.1, we deduce that $q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt$ is a convex function and verifies the differential equation associated to the fuzzy differential subordination (2.8) $q(z) + zq'(z) = h(z)$, therefore it is the fuzzy best dominant.

Keeping $p(z) = \frac{RI^{\alpha}_{m,\lambda,l} f(z)}{z}$, $p \in \mathcal{H}[1,n]$ and differentiating it, we obtain $\left(RI^{\alpha}_{m,\lambda,l} f(z)\right)' = p(z) + zp'(z)$, $z \in U$ and (2.8) becomes $F_{p(U)}\left(p(z) + zp'(z)\right) \leq F_{h(U)} h(z)$, $z \in U$.

By using Lemma 1.3, we deduce $F_{RI^{\alpha}_{m,\lambda,l} f(U)} \frac{RI^{\alpha}_{m,\lambda,l} f(z)}{z} \leq F_{q(U)} q(z)$, $z \in U$. ■

**Example 2.1** *Let* $h(z) = \frac{1-z}{1+z}$ *with* $h(0) = 1$, $h'(z) = \frac{-2}{(1+z)^2}$ *and* $h''(z) = \frac{4}{(1+z)^3}$.

*Since* $Re\left(\frac{zh''(z)}{h'(z)} + 1\right) = Re\left(\frac{1-z}{1+z}\right) = Re\left(\frac{1-\rho\cos\theta - i\rho\sin\theta}{1+\rho\cos\theta+i\rho\sin\theta}\right) = \frac{1-\rho^2}{1+2\rho\cos\theta+\rho^2} > 0 > -\frac{1}{2}$, *the function* $h$ *is convex in* $U$.

*Let* $f(z) = z - z^2$, $z \in U$. *For* $n = 1$, $m = 1$, $l = 1$, $\lambda = 2$, $\alpha = \frac{2}{3}$, *we obtain* $RI^{\frac{2}{3}}_{1,2,1} f(z) = \frac{1}{3} R^1 f(z) + \frac{2}{3} I(1,2,1) f(z) = \frac{1}{3} z f'(z) + \frac{2}{3} z f'(z) = z - 2z^2$. *Then* $\left(RI^{\frac{2}{3}}_{1,2,1} f(z)\right)' = 1 - 4z$ *and* $\frac{RI^{\frac{2}{3}}_{1,2,1} f(z)}{z} = 1 - 2z$. *Function* $q(z) = \frac{1}{z}\int_0^z \frac{1-t}{1+t} dt = -1 + \frac{2\ln(1+z)}{z}$.

*From Theorem 2.5 we have* $F_U(1-4z) \leq F_U\left(\frac{1-z}{1+z}\right)$, $z \in U$, *imply* $F_U(1-2z) \leq F_U\left(-1 + \frac{2\ln(1+z)}{z}\right)$, $z \in U$.

**Theorem 2.6** *Let* $g$ *be a convex function with* $g(0) = 1$ *and* $h(z) = g(z) + zg'(z)$, $z \in U$. *When the fuzzy differential subordination*

$$F_{RI^{\alpha}_{m,\lambda,l} f(U)} \left(\frac{zRI^{\alpha}_{m+1,\lambda,l} f(z)}{RI^{\alpha}_{m,\lambda,l} f(z)}\right)' \leq F_{h(U)} h(z), \quad z \in U \tag{2.9}$$

*holds, for* $\lambda, l \geq 0$, $m, n \in \mathbb{N}$, $f \in \mathcal{A}_n$, *then* $F_{RI^{\alpha}_{m,\lambda,l} f(U)} \frac{RI^{\alpha}_{m+1,\lambda,l} f(z)}{RI^{\alpha}_{m,\lambda,l} f(z)} \leq F_{g(U)} g(z)$, $z \in U$, *and this result is sharp.*

**Proof.** Diferentiating $p(z) = \frac{RI^{\alpha}_{m+1,\lambda,l} f(z)}{RI^{\alpha}_{m,\lambda,l} f(z)}$ we get $p'(z) = \frac{\left(RI^{\alpha}_{m+1,\lambda,l} f(z)\right)'}{RI^{\alpha}_{m,\lambda,l} f(z)} - p(z) \cdot \frac{\left(RI^{\alpha}_{m+1,\lambda,l} f(z)\right)'}{RI^{\alpha}_{m,\lambda,l} f(z)}$ and $p(z) + z \cdot p'(z) = \left(\frac{zRI^{\alpha}_{m+1,\lambda,l} f(z)}{RI^{\alpha}_{m,\lambda,l} f(z)}\right)'$.

In this conditions, relation (2.9) becomes $F_{p(U)}\left(p(z) + zp'(z)\right) \leq F_{h(U)} h(z) = F_{g(U)}\left(g(z) + zg'(z)\right)$, $z \in U$, and aplying Lemma 1.3, we obtain $F_{RI^{\alpha}_{m,\lambda,l} f(U)} \frac{RI^{\alpha}_{m+1,\lambda,l} f(z)}{RI^{\alpha}_{m,\lambda,l} f(z)} \leq F_{g(U)} g(z)$, $z \in U$. ■

**Theorem 2.7** *For a convex function* $g$, *with* $g(0) = 1$, *consider* $h(z) = g(z) + \frac{z}{\delta} g'(z)$, $z \in U$. *The fuzzy differential subordination*

$$F_{RI^{\alpha}_{m,\lambda,l} f(U)} \left(\left(\frac{RI^{\alpha}_{m,\lambda,l} f(z)}{z}\right)^{\delta-1} \left(RI^{\alpha}_{m,\lambda,l} f(z)\right)'\right) \leq F_{h(U)} h(z), \quad z \in U, \tag{2.10}$$

*is satisfied, when* $\alpha, \lambda, l, \delta \geq 0$, $m, n \in \mathbb{N}$, $f \in \mathcal{A}_n$, *and implies the following fuzzy sharp differential subordination* $F_{RI^{\alpha}_{m,\lambda,l} f(U)} \left(\frac{RI^{\alpha}_{m,\lambda,l} f(z)}{z}\right)^{\delta} \leq F_{g(U)} g(z)$, $z \in U$.

**Proof.** Differentiating $p(z) = \left(\frac{RI^{\alpha}_{m,\lambda,l} f(z)}{z}\right)^{\delta}$, $z \in U$, we get $\left(\frac{RI^{\alpha}_{m,\lambda,l} f(z)}{z}\right)^{\delta-1} \left(RI^{\alpha}_{m,\lambda,l} f(z)\right)' = p(z) + \frac{1}{\delta} zp'(z)$, $z \in U$. Evidently $p \in \mathcal{H}[1,n]$, and (2.10) becomes $F_{p(U)}\left(p(z) + \frac{1}{\delta} zp'(z)\right) \leq F_{h(U)} h(z) = F_{g(U)}\left(g(z) + \frac{z}{\delta} g'(z)\right)$, $z \in U$.

By using Lemma 1.3, we have $F_{RI^{\alpha}_{m,\lambda,l} f(U)} \left(\frac{RI^{\alpha}_{m,\lambda,l} f(z)}{z}\right)^{\delta} \leq F_{g(U)} g(z)$, $z \in U$. ■

**Theorem 2.8** *Consider an holomorphic function* $h$ *which satisfies the inequality* Re $\left(1+\frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}$, $z \in U$, *and* $h(0) = 1$. *When the fuzzy differential subordination*

$$F_{RI^{\alpha}_{m,\lambda,l} f(U)} \left(\left(\frac{RI^{\alpha}_{m,\lambda,l} f(z)}{z}\right)^{\delta-1} \left(RI^{\alpha}_{m,\lambda,l} f(z)\right)'\right) \leq F_{h(U)} h(z), \quad z \in U, \tag{2.11}$$

*holds, with* $\alpha, \lambda, l, \delta \geq 0$, $m, n \in \mathbb{N}$, $f \in \mathcal{A}_n$, *then* $F_{RI^{\alpha}_{m,\lambda,l} f(U)} \left(\frac{RI^{\alpha}_{m,\lambda,l} f(z)}{z}\right)^{\delta} \leq F_{q(U)} q(z)$, $z \in U$, *and* $q(z) = \frac{\delta}{nz^{\frac{\delta}{n}}} \int_0^z h(t) t^{\frac{\delta}{n}-1} dt$.

**Proof.** Since Re $\left(1+\frac{zh''(z)}{h'(z)}\right)>-\frac{1}{2}$, $z\in U$, from Lemma 1.1, we deduce that $q(z)=\frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z h(t)t^{\frac{\delta}{n}-1}dt$ is a convex function and verifies the differential equation asscociated to the fuzzy differential subordination (2.11) $q(z)+\frac{1}{\delta}zq'(z)=h(z)$, therefore it is the fuzzy best dominant.

Consider $p(z)=\left(\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{z}\right)^{\delta}$, $z\in U$, then $p\in\mathcal{H}[1,n]$. Differentiating, we obtain $\left(\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{z}\right)^{\delta-1}\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'=p(z)+\frac{1}{\delta}zp'(z)$, $z\in U$, and the fuzzy differential subordination (2.11) becomes $F_{p(U)}\left(p(z)+\frac{1}{\delta}zp'(z)\right)\leq F_{h(U)}h(z)$, $z\in U$. Aplying Lemma 1.2, we get $F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{z}\right)^{\delta}\leq F_{q(U)}q(z)$ and $q(z)=\frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z h(t)t^{\frac{\delta}{n}-1}dt$, $z\in U$, is the best dominant. ■

**Theorem 2.9** *For a convex function $g$, with $g(0)=1$, define the function $h(z)=g(z)+\frac{z}{\delta}g'(z)$, $z\in U$. If $\alpha,\lambda,l,\delta\geq 0$, $m,n\in\mathbb{N}$, $f\in\mathcal{A}_n$ and the fuzzy differential subordination*

$$F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(\frac{z^2}{\delta}\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{\left(RI^{\alpha}_{m+1,\lambda,l}f(z)\right)^2}\left[\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m,\lambda,l}f(z)}-2\frac{\left(RI^{\alpha}_{m+1,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m+1,\lambda,l}f(z)}\right]\right.$$

$$\left.+z\frac{\delta+1}{\delta}\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{\left(RI^{\alpha}_{m+1,\lambda,l}f(z)\right)^2}\right)\leq F_{h(U)}h(z),\ z\in U, \tag{2.12}$$

*holds, then $F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(z\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{\left(RI^{\alpha}_{m+1,\lambda,l}f(z)\right)^2}\right)\leq F_{g(U)}g(z)$, $z\in U$, and this result is sharp.*

**Proof.** Differentiating $p(z)=z\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{\left(RI^{\alpha}_{m+1,\lambda,l}f(z)\right)^2}$, we get

$p(z)+\frac{z}{\delta}p'(z)=z\frac{\delta+1}{\delta}\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{\left(RI^{\alpha}_{m+1,\lambda,l}f(z)\right)^2}+\frac{z^2}{\delta}\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{\left(RI^{\alpha}_{m+1,\lambda,l}f(z)\right)^2}\left[\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m,\lambda,l}f(z)}-2\frac{\left(RI^{\alpha}_{m+1,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m+1,\lambda,l}f(z)}\right]$.

The fuzzy differential subordination (2.12) becomes

$F_{p(U)}\left(p(z)+\frac{z}{\delta}p'(z)\right)\leq F_{h(U)}h(z)F_{g(U)}\left(g(z)+\frac{z}{\delta}g'(z)\right)$, $z\in U$, and by using Lemma 1.3, we derive

$F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(z\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{\left(RI^{\alpha}_{m+1,\lambda,l}f(z)\right)^2}\right)\leq F_{g(U)}g(z)$, $z\in U$. ■

**Theorem 2.10** *Let $h$ be a holomorphic function with properties* Re $\left(1+\frac{zh''(z)}{h'(z)}\right)>-\frac{1}{2}$, $z\in U$, *and $h(0)=1$. The fuzzy differential subordination*

$$F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(\frac{z^2}{\delta}\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{\left(RI^{\alpha}_{m+1,\lambda,l}f(z)\right)^2}\left[\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m,\lambda,l}f(z)}-2\frac{\left(RI^{\alpha}_{m+1,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m+1,\lambda,l}f(z)}\right]\right.$$

$$\left.+z\frac{\delta+1}{\delta}\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{\left(RI^{\alpha}_{m+1,\lambda,l}f(z)\right)^2}\right)\leq F_{h(U)}h(z),\ \ z\in U, \tag{2.13}$$

*implies the following fuzzy differential subordination $F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(z\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{\left(RI^{\alpha}_{m+1,\lambda,l}f(z)\right)^2}\right)\leq F_{q(U)}q(z)$, $z\in U$, where $\alpha,\lambda,l,\delta\geq 0$, $m,n\in\mathbb{N}$, $f\in\mathcal{A}_n$, and $q(z)=\frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z h(t)t^{\frac{\delta}{n}-1}dt$.*

**Proof.** Since Re $\left(1+\frac{zh''(z)}{h'(z)}\right)>-\frac{1}{2}$, $z\in U$, from Lemma 1.1, we deduce that $q(z)=\frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z h(t)t^{\frac{\delta}{n}-1}dt$ is a convex function and verifies the differential equation associated to the fuzzy differential subordination (2.13) $q(z)+\frac{1}{\delta}zq'(z)=h(z)$, therefore it is the fuzzy best dominant.

Considering $p(z)=z\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{\left(RI^{\alpha}_{m+1,\lambda,l}f(z)\right)^2}$, $z\in U$, $p\in\mathcal{H}[1,n]$. Since

$p(z)+\frac{z}{\delta}p'(z)=z\frac{\delta+1}{\delta}\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{\left(RI^{\alpha}_{m+1,\lambda,l}f(z)\right)^2}+\frac{z^2}{\delta}\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{\left(RI^{\alpha}_{m+1,\lambda,l}f(z)\right)^2}\left[\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m,\lambda,l}f(z)}-2\frac{\left(RI^{\alpha}_{m+1,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m+1,\lambda,l}f(z)}\right]$, $z\in U$, from (2.13) we have $F_{p(U)}\left(p(z)+\frac{z}{\delta}p'(z)\right)\leq F_{h(U)}h(z)$, $z\in U$, and from Lemma 1.2, we obtain $F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(z\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{\left(RI^{\alpha}_{m+1,\lambda,l}f(z)\right)^2}\right)\leq F_{q(U)}q(z)$, $z\in U$, and the best dominant is $q(z)=\frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z h(t)t^{\frac{\delta}{n}-1}dt$. ■

**Theorem 2.11** *For a convex function $g$ with $g(0)=0$ define $h(z)=g(z)+\frac{z}{\delta}g'(z)$, $z\in U$. If $\alpha,\lambda,l,\delta\geq 0$, $m,n\in\mathbb{N}$, $f\in\mathcal{A}_n$ and the fuzzy differential subordination*

$$F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(z^2\frac{\delta+2}{\delta}\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m,\lambda,l}f(z)}+\right.$$

$$\left.\frac{z^3}{\delta}\left[\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)''}{RI^{\alpha}_{m,\lambda,l}f(z)}-\left(\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m,\lambda,l}f(z)}\right)^2\right]\right)\leq F_{h(U)}h(z),\ z\in U, \tag{2.14}$$

*holds, then the following result is sharp $F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(z^2\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m,\lambda,l}f(z)}\right)\leq F_{g(U)}g(z)$, $z\in U$.*

**Proof.** Considering $p(z)=z^2\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m,\lambda,l}f(z)}$, we deduce that $p\in\mathcal{H}[0,1]$ and differentiating it, we obtain

$p(z)+\frac{z}{\delta}p'(z)=z^2\frac{\delta+2}{\delta}\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m,\lambda,l}f(z)}+\frac{z^3}{\delta}\left[\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)''}{RI^{\alpha}_{m,\lambda,l}f(z)}-\left(\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m,\lambda,l}f(z)}\right)^2\right]$, $z\in U$.

The fuzzy differential subordination becomes $F_{p(U)}\left(p(z)+\frac{1}{\delta}zp'(z)\right)\leq F_{h(U)}h(z)=F_{g(U)}\left(g(z)+\frac{z}{\delta}g'(z)\right)$ and by using Lemma 1.3, we deduce $F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(z^2\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m,\lambda,l}f(z)}\right)\leq F_{g(U)}g(z)$, $z\in U$, and this result is sharp. ■

**Theorem 2.12** *Consider $h$ a holomorphic function with* Re $\left(1+\frac{zh''(z)}{h'(z)}\right)>-\frac{1}{2}$, $z\in U$, *and $h(0)=0$. If the fuzzy differential subordination*

$$F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(z^2\frac{\delta+2}{\delta}\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m,\lambda,l}f(z)}+\right.$$

$$\left.\frac{z^3}{\delta}\left[\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)''}{RI^{\alpha}_{m,\lambda,l}f(z)}-\left(\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m,\lambda,l}f(z)}\right)^2\right]\right)\leq F_{h(U)}h(z),\ z\in U, \tag{2.15}$$

*holds, for $\alpha,\lambda,l,\delta\geq 0$, $m,n\in\mathbb{N}$, $f\in\mathcal{A}_n$, then $F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(z^2\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m,\lambda,l}f(z)}\right)\leq F_{q(U)}q(z)$, $z\in U$, where $q(z)=\frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z h(t)t^{\frac{\delta}{n}-1}dt$.*

**Proof.** Since Re $\left(1+\frac{zh''(z)}{h'(z)}\right)>-\frac{1}{2}$, $z\in U$, from Lemma 1.1, we deduce that $q(z)=\frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z h(t)t^{\frac{\delta}{n}-1}dt$ is a convex function and verifies the differential equation asscociated to the fuzzy differential subordination (2.15) $q(z)+\frac{1}{\delta}zq'(z)=h(z)$, therefore it is the fuzzy best dominant.

Considering $p(z)=z^2\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m,\lambda,l}f(z)}$, $z\in U$, $p\in\mathcal{H}[0,n]$. Differentiating it, we obtain $p(z)+\frac{z}{\delta}p'(z)=z^2\frac{\delta+2}{\delta}\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m,\lambda,l}f(z)}+\frac{z^3}{\delta}\left[\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)''}{RI^{\alpha}_{m,\lambda,l}f(z)}-\left(\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m,\lambda,l}f(z)}\right)^2\right]$, $z\in U$, and (2.15) becomes $F_{p(U)}\left(p(z)+\frac{1}{\delta}zp'(z)\right)\leq F_{h(U)}h(z)$, $z\in U$.

Applying Lemma 1.2, we get $F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(z^2\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m,\lambda,l}f(z)}\right)\leq F_{q(U)}q(z)$, $z\in U$, and the best dominant is $q(z)=\frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z h(t)t^{\frac{\delta}{n}-1}dt$. ■

**Theorem 2.13** *Let $h(z)=g(z)+zg'(z)$, $z\in U$, where $g$ is a convex function such that $g(0)=1$. When the fuzzy differential subordination*

$$F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(1-\frac{RI^{\alpha}_{m,\lambda,l}f(z)\cdot\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)''}{\left[\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'\right]^2}\right)\leq F_{h(U)}h(z),\quad z\in U \tag{2.16}$$

*holds, for $\alpha,\lambda,l\geq 0$, $m,n\in\mathbb{N}$, $f\in\mathcal{A}_n$, then the followinf result is sharp $F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{z\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}\right)\leq F_{g(U)}g(z)$, $z\in U$.*

**Proof.** Let $p(z)=\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{z\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}$. We deduce that $p\in\mathcal{H}[1,n]$ and differentiating it, we get $1-\frac{RI^{\alpha}_{m,\lambda,l}f(z)\cdot\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)''}{\left[\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'\right]^2}=p(z)+zp'(z)$, $z\in U$.

The fuzzy differential subordination becomes $F_{p(U)}\left(p(z)+zp'(z)\right)\leq F_{h(U)}h(z)=F_{g(U)}\left(g(z)+zg'(z)\right)$ and we apply Lemma 1.3 to deduce the following sharp result $_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{z\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}\right)\leq F_{g(U)}g(z)$, $z\in U$. ■

**Theorem 2.14** *Let $h$ be a holomorphic function with $h(0)=1$ and* $\mathrm{Re}\left(1+\frac{zh''(z)}{h'(z)}\right)>-\frac{1}{2}$, $z\in U$. *The fuzzy differential subordination*

$$F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(1-\frac{RI^{\alpha}_{m,\lambda,l}f(z)\cdot\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)''}{\left[\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'\right]^2}\right)\leq F_{h(U)}h(z),\qquad z\in U,\tag{2.17}$$

*induce* $F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{z\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}\right)\leq F_{q(U)}q(z)$, $z\in U$, *where* $q(z)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt$, *for* $\alpha,\lambda,l\geq 0$, $m,n\in\mathbb{N}$, $f\in\mathcal{A}_n$.

**Proof.** Since $\mathrm{Re}\left(1+\frac{zh''(z)}{h'(z)}\right)>-\frac{1}{2}$, $z\in U$, from Lemma 1.1, we deduce that $\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt$ is a convex function and verifies the differential equation asscociated to the fuzzy differential subordination (2.17) $q(z)+zq'(z)=h(z)$, therefore it is the fuzzy best dominant.

Consider $p(z)=\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{z\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}$, $z\in U$, $p\in\mathcal{H}[1,n]$. Differentiating it, we obtain $1-\frac{RI^{\alpha}_{m,\lambda,l}f(z)\cdot\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)''}{\left[\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'\right]^2}=p(z)+zp'(z)$, $z\in U$, and the fuzzy differential subordination (2.17) is written $F_{p(U)}\left(p(z)+zp'(z)\right)\leq F_{h(U)}h(z)$, $z\in U$.

By using Lemma 1.2, we get $F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(\frac{RI^{\alpha}_{m,\lambda,l}f(z)}{z\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}\right)\leq F_{q(U)}q(z)$, $z\in U$, and the best dominant is $q(z)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt$. ■

**Example 2.2** *Let $h(z)=\frac{1-z}{1+z}$ a convex function in $U$ with $h(0)=1$ and Re $\left(\frac{zh''(z)}{h'(z)}+1\right)>-\frac{1}{2}$.*

*Let $f(z)=z-z^2$, $z\in U$. For $n=1$, $m=1$, $l=1,\lambda=2$, $\alpha=\frac{2}{3}$, we obtain $RI^{\frac{2}{3}}_{1,2,1}f(z)=z-2z^2$, $z\in U$.*

*Then $\left(RI^{\frac{2}{3}}_{1,2,1}f(z)\right)'=1-4z$ and $\left(RI^{\frac{2}{3}}_{1,2,1}f(z)\right)''=-4$, $\frac{RI^{\frac{2}{3}}_{1,2,1}f(z)}{z\left(RI^{\frac{2}{3}}_{1,2,1}f(z)\right)'}=\frac{z-2z^2}{z(1-4z)}=\frac{1-2z}{1-4z}$,*

$1-\frac{RI^{\frac{2}{3}}_{1,2,1}f(z)\cdot\left(RI^{\frac{2}{3}}_{1,2,1}f(z)\right)''}{\left[\left(RI^{\frac{2}{3}}_{1,2,1}f(z)\right)'\right]^2}=1-\frac{(z-2z^2)\cdot(-4)}{(1-4z)^2}=\frac{8z^2-4z+1}{(1-4z)^2}$. *We have $q(z)=\frac{1}{z}\int_0^z\frac{1-t}{1+t}dt=-1+\frac{2\ln(1+z)}{z}$.*

*Using Theorem 2.14 we obtain $F_U\left(\frac{8z^2-4z+1}{(1-4z)^2}\right)\leq F_U\left(\frac{1-z}{1+z}\right)$, $z\in U$, induce $F_U\left(\frac{1-2z}{1-4z}\right)\leq F_U\left(-1+\frac{2\ln(1+z)}{z}\right)$, $z\in U$.*

**Theorem 2.15** *Consider $h(z)=g(z)+zg'(z)$, $z\in U$, where $g$ is a convex function with $g(0)=1$. When the fuzzy differential subordination*

$$F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(\left[\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'\right]^2+RI^{\alpha}_{m,\lambda,l}f(z)\cdot\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)''\right)\leq F_{h(U)}h(z),\qquad z\in U\tag{2.18}$$

*holds, for $\alpha,\lambda,l\geq 0$, $m,n\in\mathbb{N}$, $f\in\mathcal{A}_n$, then the following result is sharp $F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(\frac{RI^{\alpha}_{m,\lambda,l}f(z)\cdot\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{z}\right)\leq F_{g(U)}g(z)$, $z\in U$.*

**Proof.** Let $p(z)=\frac{RI^{\alpha}_{m,\lambda,l}f(z)\cdot\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{z}$. We deduce that $p\in\mathcal{H}[1,n]$.

Differentiating it, we obtain $\left[\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'\right]^2+RI^{\alpha}_{m,\lambda,l}f(z)\cdot\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)''=p(z)+zp'(z)$, $z\in U$, and the fuzzy differential subordination becomes $F_{p(U)}\left(p(z)+zp'(z)\right)\leq F_{h(U)}h(z)=F_{g(U)}\left(g(z)+zg'(z)\right)$.

By using Lemma 1.3, we obtain the following result sharp $F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(\frac{RI^{\alpha}_{m,\lambda,l}f(z)\cdot\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{z}\right)\leq F_{g(U)}g(z)$, $z\in U$. ■

**Theorem 2.16** *For a holomorphic function $h$ which satisfies the inequality* $\operatorname{Re}\left(1+\frac{zh''(z)}{h'(z)}\right)>-\frac{1}{2}$, $z\in U$, *and* $h(0)=1$, *the fuzzy differential subordination*

$$F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(\left[\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'\right]^{2}+RI^{\alpha}_{m,\lambda,l}f(z)\cdot\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)''\right)\leq F_{h(U)}h(z),\ z\in U, \tag{2.19}$$

*induce* $F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(\frac{RI^{\alpha}_{m,\lambda,l}f(z)\cdot\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{z}\right)\leq F_{q(U)}q(z)$, $z\in U$, *where* $q(z)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt$, *and* $\alpha,\lambda,l\geq 0$, $m,n\in\mathbb{N}$, $f\in\mathcal{A}_n$

**Proof.** Since $\operatorname{Re}\left(1+\frac{zh''(z)}{h'(z)}\right)>-\frac{1}{2}$, $z\in U$, from Lemma 1.1, we deduce that $q(z)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt$ is a convex function and verifies the differential equation asscociated to the fuzzy differential subordination (2.19) $q(z)+zq'(z)=h(z)$, therefore it is the fuzzy best dominant.

Let $p(z)=\frac{RI^{\alpha}_{m,\lambda,l}f(z)\cdot\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{z}$, $z\in U$, evidently $p\in\mathcal{H}[1,n]$.

We have $p(z)+zp'(z)=\left[\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'\right]^{2}+RI^{\alpha}_{m,\lambda,l}f(z)\cdot\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)''$, $z\in U$, and (2.19) means $F_{p(U)}\left(p(z)+zp'(z)\right)\leq F_{h(U)}h(z)$, $z\in U$.

Applying Lemma 1.2, we get $F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(\frac{RI^{\alpha}_{m,\lambda,l}f(z)\cdot\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{z}\right)\leq F_{q(U)}q(z)$, $z\in U$, and the best dominant is $q(z)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt$. ■

**Example 2.3** *Let* $h(z)=\frac{1-z}{1+z}$ *a convex function in* $U$ *with* $h(0)=1$ *and Re* $\left(\frac{zh''(z)}{h'(z)}+1\right)>-\frac{1}{2}$.

*Consider* $f(z)=z-z^2$, $z\in U$. *For* $n=1$, $m=1$, $l=1$, $\lambda=2$, $\alpha=\frac{2}{3}$, *we obtain* $RI^{\frac{2}{3}}_{1,2,1}f(z)=z-2z^2$, $z\in U$.

*Then* $\left(RI^{\frac{2}{3}}_{1,2,1}f(z)\right)'=1-4z$, $\left(RI^{\frac{2}{3}}_{1,2,1}f(z)\right)''=-4$, $\frac{RI^{\frac{2}{3}}_{1,2,1}f(z)\cdot\left(RI^{\frac{2}{3}}_{1,2,1}f(z)\right)'}{z}=\frac{\left(z-2z^2\right)(1-4z)}{z}=8z^2-6z+1$,
$\left[\left(RI^{\frac{2}{3}}_{1,2,1}f(z)\right)'\right]^{2}+RI^{\frac{2}{3}}_{1,2,1}f(z)\cdot\left(RI^{\frac{2}{3}}_{1,2,1}f(z)\right)''=(1-4z)^2+\left(z-2z^2\right)\cdot(-4)=24z^2-12z+1$.

*We have* $q(z)=\frac{1}{z}\int_0^z\frac{1-t}{1+t}dt=-1+\frac{2\ln(1+z)}{z}$.

*From Theorem 2.16 we obtain* $F_U\left(24z^2-12z+1\right)\leq F_U\left(\frac{1-z}{1+z}\right)$, $z\in U$, *induce* $F_U\left(8z^2-6z+1\right)\leq F_U\left(-1+\frac{2\ln(1+z)}{z}\right)$, $z\in U$.

**Theorem 2.17** *Consider* $h(z)=g(z)+\frac{z}{1-\delta}g'(z)$, $z\in U$, *where* $g$ *is a convex function with* $g(0)=1$. *If the fuzzy differential subordination*

$$F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(\left(\frac{z}{RI^{\alpha}_{m,\lambda,l}f(z)}\right)^{\delta}\frac{RI^{\alpha}_{m+1,\lambda,l}f(z)}{1-\delta}\left(\frac{\left(RI^{\alpha}_{m+1,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m+1,\lambda,l}f(z)}-\delta\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m,\lambda,l}f(z)}\right)\right)\leq F_{h(U)}h(z), \tag{2.20}$$

$z\in U$, *holds, for* $\alpha,\lambda,l\geq 0$, $\delta\in(0,1)$, $m,n\in\mathbb{N}$, $f\in\mathcal{A}_n$, *then the following result is sharp*
$F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(\frac{RI^{\alpha}_{m+1,\lambda,l}f(z)}{z}\cdot\left(\frac{z}{RI^{\alpha}_{m,\lambda,l}f(z)}\right)^{\delta}\right)\leq F_{g(U)}g(z)$, $z\in U$.

**Proof.** Let $p(z)=\frac{RI^{\alpha}_{m+1,\lambda,l}f(z)}{z}\cdot\left(\frac{z}{RI^{\alpha}_{m,\lambda,l}f(z)}\right)^{\delta}$. We deduce that $p\in\mathcal{H}[1,n]$ and differentiating the function $p$, we obtain $\left(\frac{z}{RI^{\alpha}_{m,\lambda,l}f(z)}\right)^{\delta}\frac{RI^{\alpha}_{m+1,\lambda,l}f(z)}{1-\delta}\left(\frac{\left(RI^{\alpha}_{m+1,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m+1,\lambda,l}f(z)}-\delta\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m,\lambda,l}f(z)}\right)=p(z)+\frac{1}{1-\delta}zp'(z)$, $z\in U$.

The fuzzy differential subordination means $F_{p(U)}\left(p(z)+\frac{1}{1-\delta}zp'(z)\right)\leq F_{h(U)}h(z)=F_{g(U)}\left(g(z)+\frac{z}{1-\delta}g'(z)\right)$ and by using Lemma 1.3, we obtain the following sharp result $F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(\frac{RI^{\alpha}_{m+1,\lambda,l}f(z)}{z}\cdot\left(\frac{z}{RI^{\alpha}_{m,\lambda,l}f(z)}\right)^{\delta}\right)\leq F_{g(U)}g(z)$, $z\in U$. ■

**Theorem 2.18** *Let* $h$ *be an holomorphic function with* $h(0)=1$ *and* $\operatorname{Re}\left(1+\frac{zh''(z)}{h'(z)}\right)>-\frac{1}{2}$, $z\in U$. *The fuzzy differential subordination*

$$F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(\left(\frac{z}{RI^{\alpha}_{m,\lambda,l}f(z)}\right)^{\delta}\frac{RI^{\alpha}_{m+1,\lambda,l}f(z)}{1-\delta}\left(\frac{\left(RI^{\alpha}_{m+1,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m+1,\lambda,l}f(z)}-\delta\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m,\lambda,l}f(z)}\right)\right)\leq F_{h(U)}h(z), \tag{2.21}$$

$z \in U$, *induce* $F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(\frac{RI^{\alpha}_{m+1,\lambda,l}f(z)}{z}\cdot\left(\frac{z}{RI^{\alpha}_{m,\lambda,l}f(z)}\right)^{\delta}\right)\leq F_{q(U)}q(z)$, $z\in U$, *where* $q(z)=\frac{1-\delta}{nz^{\frac{1-\delta}{n}}}\int_0^z h(t)t^{\frac{1-\delta}{n}-1}dt$, *for* $\alpha,\lambda,l\geq 0$, $\delta\in(0,1)$, $m,n\in\mathbb{N}$, $f\in\mathcal{A}_n$.

**Proof.** Since Re $\left(1+\frac{zh''(z)}{h'(z)}\right)>-\frac{1}{2}$, $z\in U$, from Lemma 1.1, we deduce that $q(z)=\frac{1-\delta}{nz^{\frac{1-\delta}{n}}}\int_0^z h(t)t^{\frac{1-\delta}{n}-1}dtt$ is a convex function and verifies the differential equation asscociated to the fuzzy differential subordination (2.21) $q(z)+\frac{1}{1-\delta}zq'(z)=h(z)$, therefore it is the fuzzy best dominant.

Consider $p(z)=\frac{RI^{\alpha}_{m+1,\lambda,l}f(z)}{z}\cdot\left(\frac{z}{RI^{\alpha}_{m,\lambda,l}f(z)}\right)^{\delta}$, $z\in U$, $p\in\mathcal{H}[1,n]$.

Differentiating it, we get $\left(\frac{z}{RI^{\alpha}_{m,\lambda,l}f(z)}\right)^{\delta}\frac{RI^{\alpha}_{m+1,\lambda,l}f(z)}{1-\delta}\left(\frac{\left(RI^{\alpha}_{m+1,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m+1,\lambda,l}f(z)}-\delta\frac{\left(RI^{\alpha}_{m,\lambda,l}f(z)\right)'}{RI^{\alpha}_{m,\lambda,l}f(z)}\right)=p(z)+\frac{1}{1-\delta}zp'(z)$, $z\in U$, and (2.21) becomes $F_{p(U)}\left(p(z)+\frac{1}{1-\delta}zp'(z)\right)\leq F_{h(U)}h(z)$, $z\in U$.

From Lemma 1.2, we get $F_{RI^{\alpha}_{m,\lambda,l}f(U)}\left(\frac{RI^{\alpha}_{m+1,\lambda,l}f(z)}{z}\cdot\left(\frac{z}{RI^{\alpha}_{m,\lambda,l}f(z)}\right)^{\delta}\right)\leq F_{q(U)}q(z)$, $z\in U$, and the best dominant is $q(z)=\frac{1-\delta}{nz^{\frac{1-\delta}{n}}}\int_0^z h(t)t^{\frac{1-\delta}{n}-1}dt$. ■

# References


[1] A. Alb Lupaş, *A new comprehensive class of analytic functions defined by multiplier transformation*, Mathematical and Computer Modelling 54, 2011, 2355–2362.

[2] A. Alb Lupaş, Gh. Oros, *On special fuzzy differential subordinations using Sălăgean and Ruscheweyh operators*, Applied Mathematics and Computation, Volume 261, 2015, 119-127.

[3] A. Alb Lupaş, *A Note on Special Fuzzy Differential Subordinations Using Generalized Salagean Operator and Ruscheweyh Derivative*, Journal of Computational Analysis and Applications, Vol. 15, No. 8, 2013, 1476-1483.

[4] A. Alb Lupas, *Some differential subordinations using Ruscheweyh derivative and a multiplier transformation*, Computational Analysis: Contributions from AMAT 2015, Proceedings Volume, Springer - New York, Chapter 8, 97-117.

[5] S.S. Miller, P.T. Mocanu, *Differential Subordinations. Theory and Applications*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 225, Marcel Dekker Inc., New York, Basel, 2000.

[6] G.I. Oros, Gh. Oros, *The notion of subordination in fuzzy sets theory*, General Mathematics, vol. 19, No. 4 (2011), 97-103.

[7] G.I. Oros, Gh. Oros, *Fuzzy differential subordinations*, Acta Universitatis Apulensis, No. 30/2012, pp. 55-64.

[8] G.I. Oros, Gh. Oros, *Dominant and best dominant for fuzzy differential subordinations*, Stud. Univ. Babes-Bolyai Math. 57(2012), No. 2, 239-248.

[9] St. Ruscheweyh, *New criteria for univalent functions*, Proc. Amet. Math. Soc., 49(1975), 109-115.

# About some differential sandwich theorems involving a multiplier transformation and Ruscheweyh derivative

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**Abstract**

In this work we study a new operator $IR_{\lambda,l}^{m,n}$ defined as the Hadamard product of the multiplier transformation $I(m,\lambda,l)$ and Ruscheweyh derivative $R^n$, given by $IR_{\lambda,l}^{m,n}:\mathcal{A}\to\mathcal{A}$, $IR_{\lambda,l}^{m,n}f(z)=(I(m,\lambda,l)*R^n)f(z)$ and $\mathcal{A}_n=\{f\in\mathcal{H}(U):f(z)=z+a_{n+1}z^{n+1}+...,\ z\in U\}$ is the class of normalized analytic functions with $\mathcal{A}_1=\mathcal{A}$. The purpose of this paper is to derive certain subordination and superordination results involving the operator $IR_{\lambda.l}^{m,n}$ and we establish differential sandwich-type theorems.

**Keywords:** analytic functions, differential operator, differential subordination, differential superordination.
**2010 Mathematical Subject Classification:** 30C45.

## 1 Introduction

Let $\mathcal{H}(U)$ be the class of analytic function in the open unit disc of the complex plane $U=\{z\in\mathbb{C}:|z|<1\}$. Let $\mathcal{H}(a,n)$ be the subclass of $\mathcal{H}(U)$ consisting of functions of the form $f(z)=a+a_nz^n+a_{n+1}z^{n+1}+\dots$.

Let $\mathcal{A}_n=\{f\in\mathcal{H}(U):f(z)=z+a_{n+1}z^{n+1}+\dots,\ z\in U\}$ and $\mathcal{A}=\mathcal{A}_1$.

Let the functions $f$ and $g$ be analytic in $U$. We say that the function $f$ is subordinate to $g$, written $f\prec g$, if there exists a Schwarz function $w$, analytic in $U$, with $w(0)=0$ and $|w(z)|<1$, for all $z\in U$, such that $f(z)=g(w(z))$, for all $z\in U$. In particular, if the function $g$ is univalent in $U$, the above subordination is equivalent to $f(0)=g(0)$ and $f(U)\subset g(U)$.

Let $\psi:\mathbb{C}^3\times U\to\mathbb{C}$ and $h$ be an univalent function in $U$. If $p$ is analytic in $U$ and satisfies the second order differential subordination

$$\psi(p(z),zp'(z),z^2p''(z);z)\prec h(z),\quad \text{for } z\in U, \tag{1.1}$$

then $p$ is called a solution of the differential subordination. The univalent function $q$ is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p\prec q$ for all $p$ satisfying (1.1). A dominant $\widetilde{q}$ that satisfies $\widetilde{q}\prec q$ for all dominants $q$ of (1.1) is said to be the best dominant of (1.1). The best dominant is unique up to a rotation of $U$.

Let $\psi:\mathbb{C}^2\times U\to\mathbb{C}$ and $h$ analytic in $U$. If $p$ and $\psi\left(p(z),zp'(z),z^2p''(z);z\right)$ are univalent and if $p$ satisfies the second order differential superordination

$$h(z)\prec\psi(p(z),zp'(z),z^2p''(z);z),\qquad z\in U, \tag{1.2}$$

then $p$ is a solution of the differential superordination (1.2) (if $f$ is subordinate to $F$, then $F$ is called to be superordinate to $f$). An analytic function $q$ is called a subordinant if $q\prec p$ for all $p$ satisfying (1.2). An univalent subordinant $\widetilde{q}$ that satisfies $q\prec\widetilde{q}$ for all subordinants $q$ of (1.2) is said to be the best subordinant.

Miller and Mocanu [6] obtained conditions $h$, $q$ and $\psi$ for which the following implication holds $h(z)\prec\psi(p(z),zp'(z),z^2p''(z);z)\Rightarrow q(z)\prec p(z)$.

For two functions $f(z)=z+\sum_{j=2}^{\infty}a_jz^j$ and $g(z)=z+\sum_{j=2}^{\infty}b_jz^j$ analytic in the open unit disc $U$, the Hadamard product (or convolution) of $f(z)$ and $g(z)$, written as $(f*g)(z)$ is defined by $f(z)*g(z)=(f*g)(z)=z+\sum_{j=2}^{\infty}a_jb_jz^j$.

**Definition 1.1** *[5] For $f\in\mathcal{A}$, $m\in\mathbb{N}\cup\{0\}$, $\lambda,l\geq0$, the multiplier transformation $I(m,\lambda,l)f(z)$ is defined by the following infinite series $I(m,\lambda,l)f(z):=z+\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{1+l}\right)^m a_jz^j$.*

**Remark 1.1** *We have $(l+1)I(m+1,\lambda,l)f(z)=(l+1-\lambda)I(m,\lambda,l)f(z)+\lambda z(I(m,\lambda,l)f(z))'$, $z\in U$.*

**Remark 1.2** *For $l=0$, $\lambda\geq 0$, the operator $D_{\lambda}^{m}=I(m,\lambda,0)$ was introduced and studied by Al-Oboudi , which reduced to the Sălăgean differential operator $S^{m}=I(m,1,0)$ for $\lambda=1$.*

**Definition 1.2** *(Ruscheweyh [8]) For $f\in\mathcal{A}$ and $n\in\mathbb{N}$, the Ruscheweyh derivative $R^{n}$ is defined by $R^{n}:\mathcal{A}\rightarrow\mathcal{A}$,*

$$\begin{aligned} R^{0}f(z) &= f(z), \quad R^{1}f(z)=zf'(z), \quad ... \\ (n+1)R^{n+1}f(z) &= z\left(R^{n}f(z)\right)'+nR^{n}f(z), \quad z\in U. \end{aligned}$$

**Remark 1.3** *If $f\in\mathcal{A}$, $f(z)=z+\sum_{j=2}^{\infty}a_{j}z^{j}$, then $R^{n}f(z)=z+\sum_{j=2}^{\infty}\frac{(n+j-1)!}{n!(j-1)!}a_{j}z^{j}$ for $z\in U$.*

**Definition 1.3** *([2]) Let $\lambda,l\geq 0$ and $n,m\in\mathbb{N}$. Denote by $IR_{\lambda,l}^{m,n}:\mathcal{A}\rightarrow\mathcal{A}$ the operator given by the Hadamard product of the multiplier transformation $I(m,\lambda,l)$ and the Ruscheweyh derivative $R^{n}$, $IR_{\lambda,l}^{m,n}f(z)=\left(I(m,\lambda,l)*R^{n}\right)f(z)$, for any $z\in U$ and each nonnegative integers $m,n$.*

**Remark 1.4** *If $f\in\mathcal{A}$ and $f(z)=z+\sum_{j=2}^{\infty}a_{j}z^{j}$, then $IR_{\lambda,l}^{m,n}f(z)=z+\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m}\frac{(n+j-1)!}{n!(j-1)!}a_{j}^{2}z^{j}$, $z\in U$.*

Using simple computation one obtains the next result.

**Proposition 1.1** *[1]For $m,n\in\mathbb{N}$ and $\lambda,l\geq 0$ we have*

$$(n+1)IR_{\lambda,l}^{m,n+1}f(z)-nIR_{\lambda,l}^{m,n}f(z)=z\left(IR_{\lambda,l}^{m,n}f(z)\right)'. \tag{1.3}$$

The purpose of this paper is to derive the several subordination and superordination results involving a differential operator. Furthermore, we studied the results of Selvaraj and Karthikeyan [10], Shanmugam, Ramachandran, Darus and Sivasubramanian [11] and Srivastava and Lashin [12].

In order to prove our subordination and superordination results, we make use of the following known results.

**Definition 1.4** *[7] Denote by $Q$ the set of all functions $f$ that are analytic and injective on $\overline{U}\backslash E(f)$, where $E(f)=\{\zeta\in\partial U:\lim_{z\rightarrow\zeta}f(z)=\infty\}$, and are such that $f'(\zeta)\neq 0$ for $\zeta\in\partial U\backslash E(f)$.*

**Lemma 1.1** *[7] Let the function $q$ be univalent in the unit disc $U$ and $\theta$ and $\phi$ be analytic in a domain $D$ containing $q(U)$ with $\phi(w)\neq 0$ when $w\in q(U)$. Set $Q(z)=zq'(z)\phi(q(z))$ and $h(z)=\theta(q(z))+Q(z)$. Suppose that $Q$ is starlike univalent in $U$ and $Re\left(\frac{zh'(z)}{Q(z)}\right)>0$ for $z\in U$.*

*If $p$ is analytic with $p(0)=q(0)$, $p(U)\subseteq D$ and $\theta(p(z))+zp'(z)\phi(p(z))\prec\theta(q(z))+zq'(z)\phi(q(z))$, then $p(z)\prec q(z)$ and $q$ is the best dominant.*

**Lemma 1.2** *[4] Let the function $q$ be convex univalent in the open unit disc $U$ and $\nu$ and $\phi$ be analytic in a domain $D$ containing $q(U)$. Suppose that $Re\left(\frac{\nu'(q(z))}{\phi(q(z))}\right)>0$ for $z\in U$ and $\psi(z)=zq'(z)\phi(q(z))$ is starlike univalent in $U$.*

*If $p(z)\in\mathcal{H}[q(0),1]\cap Q$, with $p(U)\subseteq D$ and $\nu(p(z))+zp'(z)\phi(p(z))$ is univalent in $U$ and $\nu(q(z))+zq'(z)\phi(q(z))\prec\nu(p(z))+zp'(z)\phi(p(z))$, then $q(z)\prec p(z)$ and $q$ is the best subordinant.*

# 2 Main results

We begin with the following

**Theorem 2.1** *Let $\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}\in\mathcal{H}(U)$ and let the function $q(z)$ be analytic and univalent in $U$ such that $q(z)\neq 0$, for all $z\in U$. Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in $U$. Let*

$$Re\left(\frac{\xi}{\beta}q(z)+\frac{2\mu}{\beta}q^{2}(z)+1+z\frac{q''(z)}{q(z)}-z\frac{q'(z)}{q(z)}\right)>0, \tag{2.1}$$

*for $\alpha,\xi,\beta,\mu\in\mathbb{C}$, $\beta\neq 0$, $z\in U$ and*

$$\psi_{\lambda,l}^{m,n}(\alpha,\xi,\mu,\beta;z):=\left(\alpha-\xi n+\mu n^{2}\right)+(n+1)(\xi-2n\mu-\beta)\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)} \tag{2.2}$$

$$+\mu(n+1)^2\left(\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^2+\beta(n+1)\frac{(n+2)IR_{\lambda,l}^{m,n+2}f(z)-(n+1)IR_{\lambda,l}^{m,n+1}f(z)}{(n+1)IR_{\lambda,l}^{m,n+1}f(z)-nIR_{\lambda,l}^{m,n}f(z)}.$$

*If $q$ satisfies the following subordination*

$$\psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)\prec\alpha+\xi q(z)+\mu(q(z))^2+\beta\frac{zq'(z)}{q(z)}, \tag{2.3}$$

*for $\alpha,\xi,\beta,\mu\in\mathbb{C}$, $\beta\neq 0$, then*

$$\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}\prec q(z), \tag{2.4}$$

*and $q$ is the best dominant.*

**Proof.** Let the function $p$ be defined by $p(z):=\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}$, $z\in U$, $z\neq 0$, $f\in\mathcal{A}$. We have $p'(z)=(n+1)\frac{\left(IR_{\lambda,l}^{m,n+1}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}-(n+1)\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\cdot\frac{\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}$.

By using the identity (1.3), we obtain

$$\frac{zp'(z)}{p(z)}=(n+1)\frac{(n+2)IR_{\lambda,l}^{m,n+2}f(z)-(n+1)IR_{\lambda,l}^{m,n+1}f(z)}{(n+1)IR_{\lambda,l}^{m,n+1}f(z)-nIR_{\lambda,l}^{m,n}f(z)}-(n+1)\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}. \tag{2.5}$$

By setting $\theta(w):=\alpha+\xi w+\mu w^2$ and $\phi(w):=\frac{\beta}{w}$, it can be easily verified that $\theta$ is analytic in $\mathbb{C}$, $\phi$ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w)\neq 0$, $w\in\mathbb{C}\backslash\{0\}$.

Also, by letting $Q(z)=zq'(z)\phi(q(z))=\beta\frac{zq'(z)}{q(z)}$ and $h(z)=\theta(q(z))+Q(z)=\alpha+\xi q(z)+\mu(q(z))^2+\beta\frac{zq'(z)}{q(z)}$, we find that $Q(z)$ is starlike univalent in $U$.

We have $h'(z)=\xi q'(z)+2\mu q(z)q'(z)+\beta\frac{q'(z)}{q(z)}+\beta z\frac{q''(z)}{q(z)}-\beta z\left(\frac{q'(z)}{q(z)}\right)^2$ and $\frac{zh'(z)}{Q(z)}=\frac{\xi}{\beta}q(z)+\frac{2\mu}{\beta}q^2(z)+1+z\frac{q''(z)}{q(z)}-z\frac{q'(z)}{q(z)}$.

We deduce that $Re\left(\frac{zh'(z)}{Q(z)}\right)=Re\left(\frac{\xi}{\beta}q(z)+\frac{2\mu}{\beta}q^2(z)+1+z\frac{q''(z)}{q(z)}-z\frac{q'(z)}{q(z)}\right)>0$.

By using (2.5), we obtain

$\alpha+\xi p(z)+\mu(p(z))^2+\beta\frac{zp'(z)}{p(z)}=\left(\alpha-\xi n+\mu n^2\right)+(n+1)(\xi-2n\mu-\beta)\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}+$

$\mu(n+1)^2\left(\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^2+\beta(n+1)\frac{(n+2)IR_{\lambda,l}^{m,n+2}f(z)-(n+1)IR_{\lambda,l}^{m,n+1}f(z)}{(n+1)IR_{\lambda,l}^{m,n+1}f(z)-nIR_{\lambda,l}^{m,n}f(z)}$.

By using (2.3), we have $\alpha+\xi p(z)+\mu(p(z))^2+\beta\frac{zp'(z)}{p(z)}\prec\alpha+\xi q(z)+\mu(q(z))^2+\beta\frac{zq'(z)}{q(z)}$.

By an application of Lemma 1.1, we have $p(z)\prec q(z)$, $z\in U$, i.e. $\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}\prec q(z)$, $z\in U$ and $q$ is the best dominant. ■

**Corollary 2.2** *Let $m,n\in\mathbb{N}$, $\lambda,l\geq 0$. Assume that (2.1) holds. If $f\in\mathcal{A}$ and $\psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)\prec\alpha+\xi\frac{1+Az}{1+Bz}+\mu\left(\frac{1+Az}{1+Bz}\right)^2+\frac{\beta(A-B)z}{(1+Az)(1+Bz)}$, for $\alpha,\beta,\mu,\xi\in\mathbb{C}$, $\beta\neq 0$, $-1\leq B<A\leq 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.2), then $\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}\prec\frac{1+Az}{1+Bz}$, and $\frac{1+Az}{1+Bz}$ is the best dominant.*

**Proof.** For $q(z)=\frac{1+Az}{1+Bz}$, $-1\leq B<A\leq 1$ in Theorem 2.1 we get the corollary. ■

**Corollary 2.3** *Let $m,n\in\mathbb{N}$, $\lambda,l\geq 0$. Assume that (2.1) holds. If $f\in\mathcal{A}$ and $\psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)\prec\alpha+\xi\left(\frac{1+z}{1-z}\right)^{\gamma}+\mu\left(\frac{1+z}{1-z}\right)^{2\gamma}+\frac{2\beta\gamma z}{1-z^2}$, for $\alpha,\beta,\mu,\xi\in\mathbb{C}$, $0<\gamma\leq 1$, $\beta\neq 0$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.2), then $\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}\prec\left(\frac{1+z}{1-z}\right)^{\gamma}$, and $\left(\frac{1+z}{1-z}\right)^{\gamma}$ is the best dominant.*

**Proof.** Corollary follows by using Theorem 2.1 for $q(z)=\left(\frac{1+z}{1-z}\right)^{\gamma}$, $0<\gamma\leq 1$. ■

**Theorem 2.4** *Let $q$ be analytic and univalent in $U$ such that $q(z)\neq 0$ and $\frac{zq'(z)}{q(z)}$ be starlike univalent in $U$. Assume that*

$$Re\left(\frac{\xi}{\beta}q(z)q'(z)+\frac{2\mu}{\beta}q^2(z)q'(z)\right)>0,\ \text{for } \xi,\beta,\mu\in\mathbb{C},\ \beta\neq 0. \tag{2.6}$$

*If* $f \in \mathcal{A}$, $\frac{z\left(IR_{\lambda,l}^{m,n} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)} \in \mathcal{H}[q(0),1] \cap Q$ *and* $\psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)$ *is univalent in* $U$, *where* $\psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)$ *is as defined in (2.2), then*

$$\alpha + \xi q(z) + \mu (q(z))^2 + \frac{\beta z q'(z)}{q(z)} \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z) \tag{2.7}$$

*implies*

$$q(z) \prec \frac{z\left(IR_{\lambda,l}^{m,n} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)}, \quad z \in U, \tag{2.8}$$

*and* $q$ *is the best subordinant.*

**Proof.** Let the function $p$ be defined by $p(z) := \frac{z\left(IR_{\lambda,l}^{m,n} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)}$, $z \in U$, $z \neq 0$, $f \in \mathcal{A}$.

By setting $\nu(w) := \alpha + \xi w + \mu w^2$ and $\phi(w) := \frac{\beta}{w}$ it can be easily verified that $\nu$ is analytic in $\mathbb{C}$, $\phi$ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$.

Since $\frac{\nu'(q(z))}{\phi(q(z))} = \frac{q'(z)q(z)[\xi+2\mu q(z)]}{\beta}$, it follows that $Re\left(\frac{\nu'(q(z))}{\phi(q(z))}\right) = Re\left(\frac{\xi}{\beta} q(z) q'(z) + \frac{2\mu}{\beta} q^2(z) q'(z)\right) > 0$, for $\alpha,\beta,\mu \in \mathbb{C}$, $\mu \neq 0$.

By using (2.5) and (2.7) we obtain $\alpha + \xi q(z) + \mu (q(z))^2 + \frac{\beta z q'(z)}{q(z)} \prec \alpha + \xi p(z) + \mu (p(z))^2 + \frac{\beta z p'(z)}{p(z)}$.

Using Lemma 1.2, we have $q(z) \prec p(z) = \frac{z\left(IR_{\lambda,l}^{m,n} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)}$, $z \in U$, and $q$ is the best subordinant. ■

**Corollary 2.5** *Let* $m,n \in \mathbb{N}$, $\lambda,l \geq 0$. *Assume that (2.6) holds. If* $f \in \mathcal{A}$, $\frac{z\left(IR_{\lambda,l}^{m,n} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)} \in \mathcal{H}[q(0),1] \cap Q$ *and* $\alpha + \xi \frac{1+Az}{1+Bz} + \mu \left(\frac{1+Az}{1+Bz}\right)^2 + \frac{\beta(A-B)z}{(1+Az)(1+Bz)} \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)$, *for* $\alpha,\beta,\xi,\mu \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B < A \leq 1$, *where* $\psi_{\lambda,l}^{m,n}$ *is defined in (2.2), then* $\frac{1+Az}{1+Bz} \prec \frac{z\left(IR_{\lambda,l}^{m,n} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)}$, *and* $\frac{1+Az}{1+Bz}$ *is the best subordinant.*

**Proof.** For $q(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$ in Theorem 2.4 we get the corollary. ■

**Corollary 2.6** *Let* $m,n \in \mathbb{N}$, $\lambda,l \geq 0$. *Assume that (2.6) holds. If* $f \in \mathcal{A}$, $\frac{z\left(IR_{\lambda,l}^{m,n} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)} \in \mathcal{H}[q(0),1] \cap Q$ *and* $\alpha + \xi \left(\frac{1+z}{1-z}\right)^{\gamma} + \mu \left(\frac{1+z}{1-z}\right)^{2\gamma} + \frac{2\beta\gamma z}{1-z^2} \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)$, *for* $\alpha,\beta,\mu,\xi \in \mathbb{C}$, $\beta \neq 0$, $0 < \gamma \leq 1$, *where* $\psi_{\lambda,l}^{m,n}$ *is defined in (2.2), then* $\left(\frac{1+z}{1-z}\right)^{\gamma} \prec \frac{z\left(IR_{\lambda,l}^{m,n} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)}$, *and* $\left(\frac{1+z}{1-z}\right)^{\gamma}$ *is the best subordinant.*

**Proof.** For $q(z) = \left(\frac{1+z}{1-z}\right)^{\gamma}$, $0 < \gamma \leq 1$ in Theorem 2.4 we get the corollary. ■

Combining Theorem 2.1 and Theorem 2.4, we state the following sandwich theorem.

**Theorem 2.7** *Let* $q_1$ *and* $q_2$ *be analytic and univalent in* $U$ *such that* $q_1(z) \neq 0$ *and* $q_2(z) \neq 0$, *for all* $z \in U$, *with* $\frac{z q_1'(z)}{q_1(z)}$ *and* $\frac{z q_2'(z)}{q_2(z)}$ *being starlike univalent. Suppose that* $q_1$ *satisfies (2.1) and* $q_2$ *satisfies (2.6). If* $f \in \mathcal{A}$, $\frac{z\left(IR_{\lambda,l}^{m,n} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)} \in \mathcal{H}[q(0),1] \cap Q$ *and* $\psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)$ *is as defined in (2.2) univalent in* $U$, *then*

$$\alpha + \xi q_1(z) + \mu (q_1(z))^2 + \frac{\beta z q_1'(z)}{q_1(z)} \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z) \prec \alpha + \xi q_2(z) + \mu (q_2(z))^2 + \frac{\beta z q_2'(z)}{q_2(z)},$$

*for* $\alpha,\beta,\mu,\xi \in \mathbb{C}$, $\beta \neq 0$, *implies* $q_1(z) \prec \frac{z\left(IR_{\lambda,l}^{m,n} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)} \prec q_2(z)$, *and* $q_1$ *and* $q_2$ *are respectively the best subordinant and the best dominant.*

For $q_1(z) = \frac{1+A_1 z}{1+B_1 z}$, $q_2(z) = \frac{1+A_2 z}{1+B_2 z}$, where $-1 \leq B_2 < B_1 < A_1 < A_2 \leq 1$, we have the following corollary.

**Corollary 2.8** *Let* $m,n \in \mathbb{N}$, $\lambda,l \geq 0$. *Assume that (2.1) and (2.6) hold. If* $f \in \mathcal{A}$, $\frac{z\left(IR_{\lambda,l}^{m,n} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)} \in \mathcal{H}[q(0),1] \cap Q$ *and* $\alpha + \xi \frac{1+A_1 z}{1+B_1 z} + \mu \left(\frac{1+A_1 z}{1+B_1 z}\right)^2 + \frac{\beta(A_1-B_1)z}{(1+A_1 z)(1+B_1 z)} \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z) \prec \alpha + \xi \frac{1+A_2 z}{1+B_2 z} + \mu \left(\frac{1+A_2 z}{1+B_2 z}\right)^2 + \frac{\beta(A_2-B_2)z}{(1+A_2 z)(1+B_2 z)}$, *for* $\alpha,\beta,\mu,\xi \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$, *where* $\psi_{\lambda,l}^{m,n}$ *is defined in (2.2), then* $\frac{1+A_1 z}{1+B_1 z} \prec \frac{z\left(IR_{\lambda,l}^{m,n} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)} \prec \frac{1+A_2 z}{1+B_2 z}$, *hence* $\frac{1+A_1 z}{1+B_1 z}$ *and* $\frac{1+A_2 z}{1+B_2 z}$ *are the best subordinant and the best dominant, respectively.*

For $q_1(z) = \left(\frac{1+z}{1-z}\right)^{\gamma_1}$, $q_2(z) = \left(\frac{1+z}{1-z}\right)^{\gamma_2}$, where $0 < \gamma_1 < \gamma_2 \leq 1$, we have the following corollary.

**Corollary 2.9** *Let $m,n\in\mathbb{N}$, $\lambda,l\geq 0$. Assume that (2.1) and (2.6) hold. If $f\in\mathcal{A}$, $\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}\in\mathcal{H}\left[q\left(0\right),1\right]\cap Q$ and $\alpha+\xi\left(\frac{1+z}{1-z}\right)^{\gamma_1}+\mu\left(\frac{1+z}{1-z}\right)^{2\gamma_1}+\frac{2\beta\gamma_1 z}{1-z^2}\prec\psi_{\lambda,l}^{m,n}\left(\alpha,\beta,\mu;z\right)\prec\alpha+\xi\left(\frac{1+z}{1-z}\right)^{\gamma_2}+\mu\left(\frac{1+z}{1-z}\right)^{2\gamma_2}+\frac{2\beta\gamma_2 z}{1-z^2}$, for $\alpha,\beta,\mu,\xi\in\mathbb{C}$, $\beta\neq 0$, $0<\gamma_1<\gamma_2\leq 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.2), then $\left(\frac{1+z}{1-z}\right)^{\gamma_1}\prec\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}\prec\left(\frac{1+z}{1-z}\right)^{\gamma_2}$, hence $\left(\frac{1+z}{1-z}\right)^{\gamma_1}$ and $\left(\frac{1+z}{1-z}\right)^{\gamma_2}$ are the best subordinant and the best dominant, respectively.*

We have also

**Theorem 2.10** *Let $\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}\in\mathcal{H}\left(U\right)$, $f\in\mathcal{A}$, $z\in U$, $m,n\in\mathbb{N}$, $\lambda,l\geq 0$ and let the function $q\left(z\right)$ be convex and univalent in $U$ such that $q\left(0\right)=1$, $z\in U$. Assume that*

$$Re\left(\frac{\alpha+\beta}{\beta}+z\frac{q''\left(z\right)}{q'\left(z\right)}\right)>0, \tag{2.9}$$

*for $\alpha,\beta\in\mathbb{C}$, $\beta\neq 0$, $z\in U$, and*

$$\psi_{\lambda,l}^{m,n}\left(\alpha,\beta;z\right):=\beta\left(n+1\right)\left(n+2\right)\frac{IR_{\lambda,l}^{m,n+2}f\left(z\right)}{IR_{\lambda,l}^{m,n}f\left(z\right)}- \tag{2.10}$$

$$\beta\left(n+1\right)^2\left(\frac{IR_{\lambda,l}^{m,n+1}f\left(z\right)}{IR_{\lambda,l}^{m,n}f\left(z\right)}\right)^2+\left(\alpha-\beta\right)\left(n+1\right)\frac{IR_{\lambda,l}^{m,n+1}f\left(z\right)}{IR_{\lambda,l}^{m,n}f\left(z\right)}-\alpha n.$$

*If $q$ satisfies the following subordination*

$$\psi_{\lambda,l}^{m,n}\left(\alpha,\beta;z\right)\prec\alpha q\left(z\right)+\beta zq'\left(z\right), \tag{2.11}$$

*for $\alpha,\beta\in\mathbb{C}$, $\beta\neq 0$, $z\in U$, then*

$$\frac{z\left(IR_{\lambda,l}^{m,n}f\left(z\right)\right)'}{IR_{\lambda,l}^{m,n}f\left(z\right)}\prec q\left(z\right),\quad z\in U, \tag{2.12}$$

*and $q$ is the best dominant.*

**Proof.** Let the function $p$ be defined by $p\left(z\right):=\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}$, $z\in U$, $z\neq 0$, $f\in\mathcal{A}$. The function $p$ is analytic in $U$ and $p\left(0\right)=1$

We have $p'\left(z\right)=\left(n+1\right)\frac{\left(IR_{\lambda,l}^{m,n+1}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}-\left(n+1\right)\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\cdot\frac{\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}$.

By using the identity (1.3), we obtain

$$zp'\left(z\right)=\left(n+1\right)\left(n+2\right)\frac{IR_{\lambda,l}^{m,n+2}f\left(z\right)}{IR_{\lambda,l}^{m,n}f\left(z\right)}-\left(n+1\right)^2\left(\frac{IR_{\lambda,l}^{m,n+1}f\left(z\right)}{IR_{\lambda,l}^{m,n}f\left(z\right)}\right)^2-\left(n+1\right)\frac{IR_{\lambda,l}^{m,n+1}f\left(z\right)}{IR_{\lambda,l}^{m,n}f\left(z\right)}. \tag{2.13}$$

By setting $\theta\left(w\right):=\alpha w$ and $\phi\left(w\right):=\beta$, it can be easily verified that $\theta$ is analytic in $\mathbb{C}$, $\phi$ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi\left(w\right)\neq 0$, $w\in\mathbb{C}\backslash\{0\}$.

Also, by letting $Q\left(z\right)=zq'\left(z\right)\phi\left(q\left(z\right)\right)=\beta zq'\left(z\right)$, we find that $Q\left(z\right)$ is starlike univalent in $U$.

Let $h\left(z\right)=\theta\left(q\left(z\right)\right)+Q\left(z\right)=\alpha q\left(z\right)+\beta zq'\left(z\right)$.

We have $Re\left(\frac{zh'(z)}{Q(z)}\right)=Re\left(\frac{\alpha+\beta}{\beta}+z\frac{q''(z)}{q'(z)}\right)>0$.

By using (2.13), we obtain $\alpha p\left(z\right)+\beta zp'\left(z\right)=\beta\left(n+1\right)\left(n+2\right)\frac{IR_{\lambda,l}^{m,n+2}f(z)}{IR_{\lambda,l}^{m,n}f(z)}-$

$\beta\left(n+1\right)^2\left(\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^2+\left(\alpha-\beta\right)\left(n+1\right)\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}-\alpha n.$

By using (2.11), we have $\alpha p\left(z\right)+\beta zp'\left(z\right)\prec\alpha q\left(z\right)+\beta zq'\left(z\right)$.

From Lemma 1.1, we have $p\left(z\right)\prec q\left(z\right)$, $z\in U$, i.e. $\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}\prec q\left(z\right)$, $z\in U$, and $q$ is the best dominant. ■

**Corollary 2.11** *Let $q\left(z\right)=\frac{1+Az}{1+Bz}$, $z\in U$, $-1\leq B<A\leq 1$, $m,n\in\mathbb{N}$, $\lambda,l\geq 0$. Assume that (2.9) holds. If $f\in\mathcal{A}$ and $\psi_{\lambda,l}^{m,n}\left(\alpha,\beta;z\right)\prec\alpha\frac{1+Az}{1+Bz}+\frac{\beta(A-B)z}{(1+Bz)^2}$, for $\alpha,\beta\in\mathbb{C}$, $\beta\neq 0$, $-1\leq B<A\leq 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.10), then $\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}\prec\frac{1+Az}{1+Bz}$, and $\frac{1+Az}{1+Bz}$ is the best dominant.*

**Proof.** For $q(z)=\frac{1+Az}{1+Bz}$, $-1\leq B<A\leq 1$, in Theorem 2.10 we get the corollary. ■

**Corollary 2.12** *Let $q(z)=\left(\frac{1+z}{1-z}\right)^{\gamma}, m,n\in\mathbb{N}$, $\lambda,l\geq 0$. Assume that (2.9) holds. If $f\in\mathcal{A}$ and $\psi_{\lambda,l}^{m,n}(\alpha,\beta;z)\prec \alpha\left(\frac{1+z}{1-z}\right)^{\gamma}+\frac{2\beta\gamma z}{1-z^2}\left(\frac{1+z}{1-z}\right)^{\gamma}$, for $\alpha,\beta\in\mathbb{C}$, $0<\gamma\leq 1$, $\beta\neq 0$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.10), then $\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}\prec\left(\frac{1+z}{1-z}\right)^{\gamma}$, and $\left(\frac{1+z}{1-z}\right)^{\gamma}$ is the best dominant.*

**Proof.** Corollary follows by using Theorem 2.10 for $q(z)=\left(\frac{1+z}{1-z}\right)^{\gamma}$, $0<\gamma\leq 1$. ■

**Theorem 2.13** *Let $q$ be convex and univalent in $U$ such that $q(0)=1$. Assume that*

$$Re\left(\frac{\alpha}{\beta}q'(z)\right)>0,\ \text{for } \alpha,\beta\in\mathbb{C},\ \beta\neq 0. \tag{2.14}$$

*If $f\in\mathcal{A}$, $\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}\in\mathcal{H}[q(0),1]\cap Q$ and $\psi_{\lambda,l}^{m,n}(\alpha,\beta;z)$ is univalent in $U$, where $\psi_{\lambda,l}^{m,n}(\alpha,\beta;z)$ is as defined in (2.10), then*

$$\alpha q(z)+\beta z q'(z)\prec\psi_{\lambda,l}^{m,n}(\alpha,\beta;z) \tag{2.15}$$

*implies*

$$q(z)\prec\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)},\quad \delta\in\mathbb{C},\ \delta\neq 0,\ z\in U, \tag{2.16}$$

*and $q$ is the best subordinant.*

**Proof.** Let the function $p$ be defined by $p(z):=\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}$, $z\in U$, $z\neq 0$, $f\in\mathcal{A}$. The function $p$ is analytic in $U$ and $p(0)=1$.

By setting $\nu(w):=\alpha w$ and $\phi(w):=\beta$ it can be easily verified that $\nu$ is analytic in $\mathbb{C}$, $\phi$ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w)\neq 0$, $w\in\mathbb{C}\backslash\{0\}$.

Since $\frac{\nu'(q(z))}{\phi(q(z))}=\frac{\alpha}{\beta}q'(z)$, it follows that $Re\left(\frac{\nu'(q(z))}{\phi(q(z))}\right)=Re\left(\frac{\alpha}{\beta}q'(z)\right)>0$, for $\alpha,\beta\in\mathbb{C}$, $\beta\neq 0$.

Now, by using (2.15) we obtain $\alpha q(z)+\beta z q'(z)\prec\alpha p(z)+\beta z p'(z)$, $z\in U$. From Lemma 1.2, we have $q(z)\prec p(z)=\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}$, $z\in U$, and $q$ is the best subordinant. ■

**Corollary 2.14** *Let $q(z)=\frac{1+Az}{1+Bz}$, $-1\leq B<A\leq 1$, $z\in U$, $m,n\in\mathbb{N}$, $\lambda,l\geq 0$. Assume that (2.14) holds. If $f\in\mathcal{A}$, $\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}\in\mathcal{H}[q(0),1]\cap Q$, and $\alpha\frac{1+Az}{1+Bz}+\frac{\beta(A-B)z}{(1+Bz)^2}\prec\psi_{\lambda,l}^{m,n}(\alpha,\beta;z)$, for $\alpha,\beta\in\mathbb{C}$, $\beta\neq 0$, $-1\leq B<A\leq 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.10), then $\frac{1+Az}{1+Bz}\prec\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}$, and $\frac{1+Az}{1+Bz}$ is the best subordinant.*

**Proof.** For $q(z)=\frac{1+Az}{1+Bz}$, $-1\leq B<A\leq 1$, in Theorem 2.13 we get the corollary. ■

**Corollary 2.15** *Let $q(z)=\left(\frac{1+z}{1-z}\right)^{\gamma}, m,n\in\mathbb{N}$, $\lambda,l\geq 0$. Assume that (2.14) holds. If $f\in\mathcal{A}$, $\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}\in\mathcal{H}[q(0),1]\cap Q$ and $\alpha\left(\frac{1+z}{1-z}\right)^{\gamma}+\frac{2\beta\gamma z}{1-z^2}\left(\frac{1+z}{1-z}\right)^{\gamma}\prec\psi_{\lambda,l}^{m,n}(\alpha,\beta;z)$, for $\alpha,\beta\in\mathbb{C}$, $0<\gamma\leq 1$, $\beta\neq 0$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.10), then $\left(\frac{1+z}{1-z}\right)^{\gamma}\prec\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}$, and $\left(\frac{1+z}{1-z}\right)^{\gamma}$ is the best subordinant.*

**Proof.** Corollary follows by using Theorem 2.13 for $q(z)=\left(\frac{1+z}{1-z}\right)^{\gamma}$, $0<\gamma\leq 1$. ■

Combining Theorem 2.10 and Theorem 2.13, we state the following sandwich theorem.

**Theorem 2.16** *Let $q_1$ and $q_2$ be convex and univalent in $U$ such that $q_1(z)\neq 0$ and $q_2(z)\neq 0$, for all $z\in U$. Suppose that $q_1$ satisfies (2.9) and $q_2$ satisfies (2.14). If $f\in\mathcal{A}$, $\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}\in\mathcal{H}[q(0),1]\cap Q$, and $\psi_{\lambda,l}^{m,n}(\alpha,\beta;z)$ is as defined in (2.10) univalent in $U$, then $\alpha q_1(z)+\beta z q_1'(z)\prec\psi_{\lambda,l}^{m,n}(\alpha,\beta;z)\prec\alpha q_2(z)+\beta z q_2'(z)$, for $\alpha,\beta\in\mathbb{C}$, $\beta\neq 0$, implies $q_1(z)\prec\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}\prec q_2(z)$, $z\in U$, and $q_1$ and $q_2$ are respectively the best subordinant and the best dominant.*

For $q_1(z)=\frac{1+A_1z}{1+B_1z}$, $q_2(z)=\frac{1+A_2z}{1+B_2z}$, where $-1\leq B_2<B_1<A_1<A_2\leq 1$, we have the following corollary.

**Corollary 2.17** *Let $m,n\in\mathbb{N}$, $\lambda,l\geq 0$. Assume that (2.9) and (2.14) hold for $q_1(z)=\frac{1+A_1z}{1+B_1z}$ and $q_2(z)=\frac{1+A_2z}{1+B_2z}$, respectively. If $f\in\mathcal{A}$, $\frac{z\left(IR^{m,n}_{\lambda,l}f(z)\right)'}{IR^{m,n}_{\lambda,l}f(z)}\in\mathcal{H}[q(0),1]\cap Q$ and $\alpha\frac{1+A_1z}{1+B_1z}+\frac{\beta(A_1-B_1)z}{(1+B_1z)^2}\prec\psi^{m,n}_{\lambda,l}(\alpha,\beta;z)\prec\alpha\frac{1+A_2z}{1+B_2z}+\frac{\beta(A_2-B_2)z}{(1+B_2z)^2}$, $z\in U$, for $\alpha,\beta\in\mathbb{C}$, $\beta\neq 0$, $-1\leq B_2\leq B_1<A_1\leq A_2\leq 1$, where $\psi^{m,n}_{\lambda,l}$ is defined in (2.2), then $\frac{1+A_1z}{1+B_1z}\prec\frac{z\left(IR^{m,n}_{\lambda,l}f(z)\right)'}{IR^{m,n}_{\lambda,l}f(z)}\prec\frac{1+A_2z}{1+B_2z}$, $z\in U$, hence $\frac{1+A_1z}{1+B_1z}$ and $\frac{1+A_2z}{1+B_2z}$ are the best subordinant and the best dominant, respectively.*

For $q_1(z)=\left(\frac{1+z}{1-z}\right)^{\gamma_1}$, $q_2(z)=\left(\frac{1+z}{1-z}\right)^{\gamma_2}$, where $0<\gamma_1<\gamma_2\leq 1$, we have the following corollary.

**Corollary 2.18** *Let $m,n\in\mathbb{N}$, $\lambda,l\geq 0$. Assume that (2.9) and (2.14) hold for $q_1(z)=\left(\frac{1+z}{1-z}\right)^{\gamma_1}$ and $q_2(z)=\left(\frac{1+z}{1-z}\right)^{\gamma_2}$, respectively. If $f\in\mathcal{A}$, $\frac{z\left(IR^{m,n}_{\lambda,l}f(z)\right)'}{IR^{m,n}_{\lambda,l}f(z)}\in\mathcal{H}[q(0),1]\cap Q$ and $\alpha\left(\frac{1+z}{1-z}\right)^{\gamma_1}+\frac{2\beta\gamma_1z}{1-z^2}\left(\frac{1+z}{1-z}\right)^{\gamma_1}\prec\psi^{m,n}_{\lambda,l}(\alpha,\beta;z)\prec\alpha\left(\frac{1+z}{1-z}\right)^{\gamma_2}+\frac{2\beta\gamma_2z}{1-z^2}\left(\frac{1+z}{1-z}\right)^{\gamma_2}$, $z\in U$, for $\alpha,\beta\in\mathbb{C}$, $\beta\neq 0$, $0<\gamma_1<\gamma_2\leq 1$, where $\psi^{m,n}_{\lambda,l}$ is defined in (2.2), then $\left(\frac{1+z}{1-z}\right)^{\gamma_1}\prec\frac{z\left(IR^{m,n}_{\lambda,l}f(z)\right)'}{IR^{m,n}_{\lambda,l}f(z)}\prec\left(\frac{1+z}{1-z}\right)^{\gamma_2}$, $z\in U$, hence $\left(\frac{1+z}{1-z}\right)^{\gamma_1}$ and $\left(\frac{1+z}{1-z}\right)^{\gamma_2}$ are the best subordinant and the best dominant, respectively.*

# References


[1] A. Alb Lupas, *Some Differential Sandwich Theorems using a multiplier transformation and Ruscheweyh derivative*, Electronic Journal of Mathematics and Its Application, Vol. 1, No. 2 (2015), 76-86.

[2] A. Alb Lupas, *About some differential sandwich theorems using a multiplier transformation and Ruscheweyh derivative*, Journal of Computational Analysis and Applications, Vol. 21, No.7, 2016, 1218-1224.

[3] F.M. Al-Oboudi, *On univalent functions defined by a generalized Sălăgean operator*, Ind. J. Math. Math. Sci., 27 (2004), 1429-1436.

[4] T. Bulboacă, *Classes of first order differential superordinations*, Demonstratio Math., Vol. 35, No. 2, 287-292.

[5] A. Cătaş, *On certain class of p-valent functions defined by new multiplier transformations*, Proceedings Book of the International Symposium on Geometric Function Theory and Applications, August 20-24, 2007, TC Istanbul Kultur University, Turkey, 241-250.

[6] S.S. Miller, P.T. Mocanu, *Subordinants of Differential Superordinations*, Complex Variables, vol. 48, no. 10, 815-826, October, 2003.

[7] S.S. Miller, P.T. Mocanu, *Differential Subordinations: Theory and Applications*, Marcel Dekker Inc., New York, 2000.

[8] St. Ruscheweyh, *New criteria for univalent functions*, Proc. Amet. Math. Soc., 49(1975), 109-115.

[9] G. St. Sălăgean, *Subclasses of univalent functions*, Lecture Notes in Math., Springer Verlag, Berlin, 1013(1983), 362-372.

[10] C. Selvaraj, K.T. Karthikeyan, *Differential Subordination and Superordination for Analytic Functions Defined Using a Family of Generalized Differential Operators*, An. St. Univ. Ovidius Constanta, Vol. 17 (1) 2009, 201-210.

[11] T.N. Shanmugan, C. Ramachandran, M. Darus, S. Sivasubramanian, *Differential sandwich theorems for some subclasses of analytic functions involving a linear operator*, Acta Math. Univ. Comenianae, 16 (2007), no. 2, 287-294.

[12] H.M. Srivastava, A.Y. Lashin, *Some applications of the Briot-Bouquet differential subordination*, JIPAM. J. Inequal. Pure Appl. Math., 6 (2005), no. 2, Article 41, 7 pp. (electronic).

# Global stability of a quadratic anti-competitive system of rational difference equations in the plane with Allee effects.

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**Abstract.** We investigate global dynamics of the following systems of difference equations

$$\begin{cases} x_{n+1} &= \frac{y_n^2}{a+x_n^2} \\ \\ y_{n+1} &= \frac{x_n^2}{b+y_n^2} \end{cases}, \quad n=0,1,2,\ldots$$

where the parameters $a, b$ are positive numbers and initial conditions $x_0$ and $y_0$ are arbitrary nonnegative numbers. We find all possible dynamical scenario for this system. We show that this system has substantially different behavior than the corresponding linear fractional system.

**Keywords.** Competitive map, global stable manifold, monotonicity, period-two solution.

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# 1 Introduction and Preliminaries

We investigate global dynamics of the following systems of difference equations

$$\begin{cases} x_{n+1} &= \frac{y_n^2}{a+x_n^2} \\ \\ y_{n+1} &= \frac{x_n^2}{b+y_n^2} \end{cases}, \quad n=0,1,\ldots \tag{1}$$

where the parameters $a, b$ are positive numbers and initial conditions $x_0$ and $y_0$ are arbitrary nonnegative numbers. System (1) is related to an anti-competitive system considered in [21]

$$x_{n+1}=\frac{\gamma_1 y_n}{A_1+x_n}, \quad y_{n+1}=\frac{\beta_2 x_n}{A_2+y_n}, \quad n=0,1,..., \tag{2}$$

where the parameters $A_1, \gamma_1, A_2$ and $\beta_2$ are positive numbers and the initial conditions $(x_0, y_0)$ are arbitrary nonnegative numbers. In the classification of all linear fractional systems in [3], System (2) was mentioned as system $(16, 16)$.

The main result on the global behavior of System (2) is summarized in the following theorem, see [21].

**Theorem 1** *(a) If $\beta_2\,\gamma_1 - A_1\,A_2 < 0$, then $E_0(0,0)$ is a unique equilibrium and it is globally asymptotically stable.*

*(b) If $\beta_2\,\gamma_1 - A_1\,A_2 > 0$, then there exist two equilibrium points, namely a repeller $E_0$ and an interior saddle $E_+$. There exists a set $\mathcal{C} \subset \mathcal{R} = [0,\infty)\times[0,\infty)$ which is invariant subset of the basin of attraction of $E_+$. The set $\mathcal{C}$ is a graph of a strictly increasing continuous function of the first variable on an interval, and $E_0 \in \mathcal{C}$ and separates $\mathcal{R}$ into two connected and invariant components, namely*

$$\begin{aligned} \mathcal{W}_- &: = \{x \in \mathcal{R}\backslash\mathcal{C} : \exists y \in \mathcal{C} \text{ with } x \preceq_{se} y\}, \\ \mathcal{W}_+ &: = \{x \in \mathcal{R}\backslash\mathcal{C} : \exists y \in \mathcal{C} \text{ with } y \preceq_{se} x\}. \end{aligned}$$

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*which satisfy:*

*i) If* $(x_0, y_0) \in \mathcal{W}_+$, *then*

$$\lim_{n\to\infty} (x_{2n}, y_{2n}) = (\infty, 0) \quad \textit{and} \quad \lim_{n\to\infty} (x_{2n+1}, y_{2n+1}) = (0, \infty).$$

*ii) If* $(x_0, y_0) \in \mathcal{W}_-$,

$$\lim_{n\to\infty} (x_{2n}, y_{2n}) = (0, \infty) \quad \textit{and} \quad \lim_{n\to\infty} (x_{2n+1}, y_{2n+1}) = (\infty, 0).$$

*(c) If* $\beta_2\,\gamma_1 - A_1\,A_2 = 0$, *then,*

i. $E_0(0,0)$ *is the unique equilibrium, and every point of the positive semiaxes is a period-two point. Orbits of period-two solutions consist of the points* $(x, 0)$ *and* $(0, \frac{\beta_2}{A_2}\,x)$, *for some* $x > 0$.

ii. *All minimal period-two solutions and the equilibrium are stable but not asymptotically stable.*

iii. *There exists a family of strictly increasing curves* $\mathcal{C}_0$, $\mathcal{C}_x$ *and* $\mathcal{C}^x$ *for* $x > 0$, *that emanate from* $E_0$, $E_x := (x, 0)$ *and* $E^x := (0, \frac{\beta_2}{A_2}\,x)$ *respectively, such that the curves are pairwise disjoint, the union of all the curves equals* $\mathbb{R}^2_+$, *and solutions with initial point in* $\mathcal{C}_0$ *converge to* $E_0$, *solutions with initial point in* $\mathcal{C}_x$ *have even-indexed terms converging to* $E_x$ *and odd-indexed terms converging to* $E^x$, *and, solutions with initial point in* $\mathcal{C}^x$ *have even-indexed terms converging to* $E^x$ *and odd-indexed terms converging to* $E_x$.

As we will show in this paper System (1) has very different behavior than System (2), showing that introduction of quadratic terms can significantly change behavior of the system. As we will show in Section 4 there are three dynamic scenarios for System (1), each different than one of the three scenarios for System (2). For example System (1) always possesses the unique period-two solution which substantially effects the global behavior. Second System (1) exhibits the Allee's effect which is nonexistent in System (2). Third major difference between two systems lies in the techniques of the proof used in two results. While the results about the global stable and unstable manifolds in [18, 19, 20] were sufficient for the proofs of global dynamics of System (2), these results are not effective in the case of System (1) as the eigenvectors which correspond to the period-two solution of System (1) are parallel to the coordinate axes. Thus we used new techniques based on the properties of the basins of attraction of the period-two solution or the points at infinity $(0, \infty)$ and $(\infty, 0)$. Furthermore, we used the real algebraic geometry to prove some basic facts about the local stabilty of the equilibrium points and the period-two solutions. Our results show that the introduction of quadratic terms in the linear fractional systems of difference equations change substantially their behavior, see [2, 10] for similar results. In particular, introduction of quadratic terms creates the Allee's effect and introduces the periodic solutions.

The rest of this section contains some known results about competitive systems. Section 2 gives some basic facts about the global behavior of System (1). Section 3 presents local stability analysis of the equilibrium solutions and the period-two solution. Finally, Section 4 gives complete global dynamics of System (1).

A first order system of difference equations

$$\begin{cases} x_{n+1} = f(x_n, y_n) \\ y_{n+1} = g(x_n, y_n) \end{cases}, \quad n = 0, 1, ..., (x_0, y_0) \in \mathcal{R}, \tag{3}$$

where $\mathcal{R} \subset \mathbb{R}^2, (f, g) : \mathcal{R} \to \mathcal{R}, f, g$ are continuous functions is *competitive* if $f(x, y)$ is non-decreasing in $x$ and non-increasing in $y$, and $g(x, y)$ is non-increasing in $x$ and non-decreasing in $y$.

System (3) where the functions $f$ and $g$ have monotonic character opposite of the monotonic character in competitive system will be called *anti-competitive*. In other words (3) is anti-competitive if $f(x, y)$ is non-increasing in $x$ and non-decreasing in $y$, and $g(x, y)$ is non-decreasing in $x$ and non-increasing in $y$.

Consider a partial ordering $\preceq$ on $\mathbb{R}^2$. Two points $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are said to be *related* if $\mathbf{v} \preceq \mathbf{w}$ or $\mathbf{w} \preceq \mathbf{v}$. Also, a strict inequality between points may be defined as $\mathbf{v} \prec \mathbf{w}$ if $\mathbf{v} \preceq \mathbf{w}$ and $\mathbf{v} \neq \mathbf{w}$. A stronger inequality may be defined as $\mathbf{v} = (v_1, v_2) \ll \mathbf{w} = (w_1, w_2)$ if $\mathbf{v} \preceq \mathbf{w}$ with $v_1 \neq w_1$ and $v_2 \neq w_2$. For $\mathbf{u}$, $\mathbf{v}$ in $\mathbb{R}^2$, the *order interval* $[\![\mathbf{u}, \mathbf{v}]\!]$ is the set of all $\mathbf{x} \in \mathbb{R}^2$ such that $\mathbf{u} \preceq \mathbf{x} \preceq \mathbf{v}$. The interior of a set $A$ is deoned as $intA$.

A map $T$ on a nonempty set $\mathcal{S} \subset \mathbb{R}^2$ is a continuous function $T : \mathcal{S} \to \mathcal{S}$. The map $T$ is *monotone* if $\mathbf{v} \preceq \mathbf{w}$ implies $T(\mathbf{v}) \preceq T(\mathbf{w})$ for all $\mathbf{v}, \mathbf{w} \in \mathcal{S}$, and it is *strongly monotone* on $\mathcal{S}$ if $\mathbf{v} \prec \mathbf{w}$ implies that $T(\mathbf{v}) \ll T(\mathbf{w})$ for all $\mathbf{v}, \mathbf{w} \in \mathcal{S}$. The map is *strictly monotone* on $\mathcal{S}$ if $\mathbf{v} \prec \mathbf{w}$ implies that $T(\mathbf{v}) \prec T(\mathbf{w})$ for all $\mathbf{v}, \mathbf{w} \in \mathcal{S}$. Clearly, being related is invariant under iteration of a strongly monotone map.

Throughout this paper we shall use the *North-East ordering* (NE) for which the positive cone is the first quadrant, i.e. this partial ordering is defined by $(x_1, y_1) \preceq_{ne} (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \leq y_2$ and the South-East (SE) ordering defined as $(x_1, y_1) \preceq_{se} (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \geq y_2$.

A map $T$ on a nonempty set $\mathcal{S} \subset \mathbb{R}^2$ which is monotone with respect to the North-East ordering is called *cooperative* and a map monotone with respect to the South-East ordering is called *competitive*. A map $T$ on a nonempty set $\mathcal{S} \subset \mathbb{R}^2$ which second iterate $T^2$ is monotone with respect to the North-East ordering is called *anti-cooperative* and a map which second iterate $T^2$ is monotone with respect to the South-East ordering is called *anti-competitive*. A map $T$ that corresponds to System (3) is defined as $T = (f, g)$. An equilibrium $\mathbf{x}$ of anti-competitive system (3) is said to

be nonhyperbolic of stable (resp. unstable) type if one of the eigenvalues of the Jacobian matrix evaluated at $\mathbf{x}$ is by absolute value 1 and the second one is by absolute value less (resp. bigger) than 1.

Next, we give three results for competitive maps in the plane. There is an extensive literature on competitive systems in the plane, see [1, 2, 4, 5, 6, 8, 9, 18, 19, 20, 23] for different examples of planar competitive systems and their applications. The following definition is from [24].

**Definition 1** *Let $\mathcal{S}$ be a nonempty subset of $\mathbb{R}^2$. A competitive map $T : \mathcal{S} \to \mathcal{S}$ is said to satisfy condition (O+) if for every $x$, $y$ in $\mathcal{S}$, $T(x) \preceq_{ne} T(y)$ implies $x \preceq_{ne} y$, and $T$ is said to satisfy condition (O−) if for every $x$, $y$ in $\mathcal{S}$, $T(x) \preceq_{ne} T(y)$ implies $y \preceq_{ne} x$.*

The following theorem was proved by de Mottoni-Schiaffino [7] for the Poincaré map of a periodic competitive Lotka-Volterra system of differential equations. Smith generalized the proof to competitive and cooperative maps [23].

**Theorem 2** *Let $\mathcal{S}$ be a nonempty subset of $\mathbb{R}^2$. If $T$ is a competitive map for which (O+) holds then for all $x \in \mathcal{S}$, $\{T^n(x)\}$ is eventually componentwise monotone. If the orbit of $x$ has compact closure, then it converges to a fixed point of $T$. If instead (O−) holds, then for all $x \in \mathcal{S}$, $\{T^{2n}\}$ is eventually componentwise monotone. If the orbit of $x$ has compact closure in $\mathcal{S}$, then its omega limit set is either a period-two orbit or a fixed point.*

The following result is from [24], with the domain of the map specialized to be the cartesian product of intervals of real numbers. It gives a sufficient condition for conditions (O+) and (O−).

**Theorem 3** *Let $\mathcal{R} \subset \mathbb{R}^2$ be the cartesian product of two intervals in $\mathbb{R}$. Let $T : \mathcal{R} \to \mathcal{R}$ be a $C^1$ competitive map. If $T$ is injective and $\det J_T(x) > 0$ for all $x \in \mathcal{R}$ then $T$ satisfies (O+). If $T$ is injective and $\det J_T(x) < 0$ for all $x \in \mathcal{R}$ then $T$ satisfies (O−).*

The following result is a direct consequence of the Trichotomy Theorem of Dancer and Hess, see [11, 19], and is helpful for determining the basins of attraction of the equilibrium points.

**Corollary 1** *If the nonnegative cone of $\preceq$ is a generalized quadrant in $\mathbb{R}^n$, and if $T$ has no fixed points in $[\![u_1, u_2]\!]$ other than $u_1$ and $u_2$, then the interior of $[\![u_1, u_2]\!]$ is either a subset of the basin of attraction of $u_1$ or a subset of the basin of attraction of $u_2$.*

## 2 Some Basic Facts

Let

$$T_1(x, y) = \frac{y^2}{a + x^2}, \quad T_2(x, y) = \frac{x^2}{b + y^2}.$$

The map $T(x, y)$ associated to system (1) is given by

$$T(x, y) = (T_1(x, y), T_2(x, y)) = \left( \frac{y^2}{a + x^2}, \frac{x^2}{b + y^2} \right), \quad (x, y) \in [0, \infty) \times [0, \infty) \tag{4}$$

and the Jacobian matrix of the map $T$ at the point $(x, y)$ is given by:

$$J_T(x, y) = \begin{pmatrix} -\frac{2xy^2}{(x^2+a)^2} & \frac{2y}{x^2+a} \\ \frac{2x}{y^2+b} & -\frac{2x^2y}{(y^2+b)^2} \end{pmatrix}. \tag{5}$$

Determinant of the Jacobian matrix (5) is given by

$$\det J_T(x, y) = -\frac{4xy(bx^2 + a(y^2 + b))}{(x^2 + a)^2(y^2 + b)^2}, \tag{6}$$

and the trace of the Jacobian matrix (5) is given by

$$trJ_T(x, y) = -\left( \frac{2yx^2}{(y^2 + b)^2} + \frac{2y^2x}{(x^2 + a)^2} \right). \tag{7}$$

The map $T^2$ is given by $T^2(x, y) = (F(x, y), G(x, y))$, where

$$F(x, y) = \frac{x^4}{(b + y^2)^2 \left( \frac{y^4}{(a+x^2)^2} + a \right)}, \quad G(x, y) = \frac{y^4}{(a + x^2)^2 \left( \frac{x^4}{(b+y^2)^2} + b \right)}. \tag{8}$$

The Jacobian matrix of $T^2$ is given as

$$J_{T^2}(x,y)=\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{9}$$

where

$$A=\frac{4x^3\left(\frac{\left(2x^2+a\right)y^4}{\left(x^2+a\right)^3}+a\right)}{\left(y^2+b\right)^2\left(\frac{y^4}{\left(x^2+a\right)^2}+a\right)^2}, \quad B=-\frac{4x^4\left(x^2+a\right)^2y\left(2y^4+by^2+a\left(x^2+a\right)^2\right)}{\left(y^2+b\right)^3\left(y^4+a\left(x^2+a\right)^2\right)^2}, \tag{10}$$

$$C=-\frac{4xy^4\left(y^2+b\right)^2\left(\left(2x^2+a\right)x^2+b\left(y^2+b\right)^2\right)}{\left(x^2+a\right)^3\left(x^4+b\left(y^2+b\right)^2\right)^2}, \quad D=\frac{4y^3\left(\frac{\left(2y^2+b\right)x^4}{\left(y^2+b\right)^3}+b\right)}{\left(x^2+a\right)^2\left(\frac{x^4}{\left(y^2+b\right)^2}+b\right)^2}. \tag{11}$$

Determinant of the Jacobian matrix (9) is

$$\det J_{T^2}(x,y)=\frac{\mathcal{A}}{\left(a+x^2\right)\left(b+y^2\right)\left(a\left(a+x^2\right)^2+y^4\right)^2\left(b\left(b+y^2\right)^2+x^4\right)^2}, \tag{12}$$

where

$$\mathcal{A}=16x^3y^3\left(a\left(a+x^2\right)^2\left(b^3+x^4\right)\right.$$
$$\left.+by^4\left(a\left(a+x^2\right)^2+b^2\right)+2ab^2y^2\left(a+x^2\right)^2+2b^2y^6+by^8\right)\left(a\left(b+y^2\right)+bx^2\right).$$

The following lemma summarizes some basic facts about System (1).

**Lemma 1** *Let $(x_n,y_n):=T^n(x_0,y_0)$ be any solution of System (1). Then*

*(i) Assume that $x_0=0$ and $y_0>0$. Then the following holds:*

*(i.1) $x_{2n}=0$ and $y_{2n-1}=0$ for all $n\in\mathbb{N}$.*

*(i.2) If $0<y_0<\sqrt[3]{a^2b}$, then $0<x_1<\sqrt[3]{ab^2}$ and $y_{2n+2}\le y_{2n}$ for all $n\in\mathbb{N}$.*

*(i.3) If $y_0>\sqrt[3]{a^2b}$, then $x_1>\sqrt[3]{ab^2}$ and $y_{2n}\le y_{2n+2}$ for all $n\in\mathbb{N}$.*

*(ii) Assume that $x_0>0$ and $y_0=0$. Then the following holds*

*(ii.1) $x_{2n-1}=0$ and $y_{2n}=0$ for all $n\in\mathbb{N}$.*

*(ii.2) If $0<x_0<\sqrt[3]{ab^2}$, then $y_1<\sqrt[3]{a^2b}$ and $x_{2n+2}\le x_{2n}$ for all $n\in\mathbb{N}$.*

*(ii.3) If $x_0>\sqrt[3]{ab^2}$, then $y_1>\sqrt[3]{a^2b}$ and $x_{2n}\le x_{2n+2}$ for all $n\in\mathbb{N}$.*

*(iii) For all $n>0$ we have $x_ny_n<1$.*

**Proof.** We prove statement $(i.1)$. Take $x_0=0$ and $y_0>0$. The statement $(i)$ follows from

$$(0,y_0)-T^2(0,y_0)=(0,y_0)-\left(0,\frac{y_0^4}{a^2b}\right)=\left(0,\frac{a^2by_0-y_0^4}{a^2b}\right),$$

$$x_1-\sqrt[3]{ab^2}=\frac{y_0^2}{a}-\sqrt[3]{ab^2}=\frac{y_0^2-\sqrt[3]{a^4b^2}}{a}$$

and monotonicity of $T^2$. Similarly, we prove $(ii.1)$. Proofs of $(i.2),(ii.2),(i.3)$ and $(ii.3)$ are immediate.

Take $x_0,y_0\in[0,\infty)\times[0,\infty)$. Let $(x_n,y_n):=T^n(x_0,y_0)$. The proof of the statement (iii) follows from the fact

$$x_{n+1}y_{n+1}=\frac{y_n^2}{a+x_n^2}\frac{x_n^2}{b+y_n^2}=\frac{x_n^2}{a+x_n^2}\frac{y_n^2}{b+y_n^2}<1.$$

□

**Lemma 2** *The map $T$ is injective.*

**Proof.** We have to prove that

$$T(x_1, y_1) = T(x_2, y_2) \Rightarrow x_1 = x_2 \quad \text{and} \quad y_1 = y_2.$$

Let $x_1, y_1, x_2, y_2 > 0$. Then

$$T(x_1, y_1) - T(x_2, y_2) = \left( \frac{x_2^2 y_1^2 + a y_1^2 - x_1^2 y_2^2 - a y_2^2}{(x_1^2 + a)(x_2^2 + a)}, \frac{y_2^2 x_1^2 + b x_1^2 - b x_2^2 - x_2^2 y_1^2}{(y_1^2 + b)(y_2^2 + b)} \right).$$

From $T(x_1, y_1) - T(x_2, y_2) = 0$ we have

$$x_2^2 y_1^2 + a y_1^2 - x_1^2 y_2^2 - a y_2^2 = 0 \tag{13}$$

and

$$y_2^2 x_1^2 + b x_1^2 - b x_2^2 - x_2^2 y_1^2 = 0. \tag{14}$$

Equation (13) implies that

$$x_1^2 = \frac{x_2^2 y_1^2 + a y_1^2 - a y_2^2}{y_2^2}. \tag{15}$$

Substituting $x_1^2$ from (15) into equation (14) we have

$$\frac{(y_1 - y_2)(y_1 + y_2)\left(b x_2^2 + a y_2^2 + ab\right)}{y_2^2} = 0,$$

from which it follows that $y_1 = y_2$. Substituting in (15) we have $x_1 = x_2$. This proves Lemma. □

**Theorem 4** *Every bounded solution of System (1) converges to a period-two solution.*

**Proof.** In view of Lemma 2 the map $T$ associated to System (1) is injective which implies that the map $T^2$ is also injective. Relation (12) implies that determinant of the Jacobian matrix (9) is positive for all $x \in (0, \infty) \times (0, \infty)$. By using Theorem 3 we have that the condition $(O+)$ is satisfied for the map $T^2$ ($T^2$ is competitive). Theorem 2 implies that odd and even subsequences $\{x_{2n}\}_{n=0}^{\infty}, \{x_{2n+1}\}_{n=-1}^{\infty}, \{y_{2n}\}_{n=0}^{\infty}, \{y_{2n+1}\}_{n=-1}^{\infty}$ of any solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ are eventually monotonic, from which the proof follows. □

# 3 The local stability of the equilibrium solutions and the period-two solution

The equilibrium points $(\bar{x}, \bar{y})$ of System (1) satisfy equations

$$\frac{\bar{y}^2}{a + \bar{x}^2} = \bar{x}, \quad \frac{\bar{x}^2}{b + \bar{y}^2} = \bar{y}. \tag{16}$$

By eliminating $\bar{x}$ from (16) we get

$$\bar{y}^9 + 3b\bar{y}^7 + 2a\bar{y}^6 + 3b^2\bar{y}^5 + (4ab - 1)\bar{y}^4 + \left(b^3 + a^2\right)\bar{y}^3 + 2ab^2\bar{y}^2 + a^2 b\bar{y} = 0. \tag{17}$$

Similarly, we can eliminate variable $\bar{y}$ from system (16) to obtain

$$\bar{x}^9 + 3a\bar{x}^7 + 2b\bar{x}^6 + 3a^2\bar{x}^5 + (4ab - 1)\bar{x}^4 + \left(a^3 + b^2\right)\bar{x}^3 + 2a^2 b\bar{x}^2 + ab^2\bar{x} = 0. \tag{18}$$

In view of Descartes' Rule of Signs we obtain that Eq. (18) has zero equilibrium always and either zero, one or two positive equilibrium points if $4ab - 1 < 0$. By using (16) all its real roots are positive numbers. These equilibrium points will be denoted $E_0(0, 0)$, $E(\bar{x}, \bar{y})$, $E_{SW}(\bar{x}, \bar{y})$ and $E_{NE}(\bar{x}, \bar{y})$.

**Lemma 3** *Let*

$$\begin{aligned}\Delta_1 = {} & 186624b^2a^{15} + 55296b^5a^{13} + 2657664b^4a^{12} + 4096b^8a^{11} + 1619712b^3a^{11} \\ & + 754688b^7a^{10} - 12500b^2a^{10} + 10767632b^6a^9 + 55296b^{10}a^8 - 11550400b^5a^8 + 2657664b^9a^7 \\ & + 1980000b^4a^7 + 1619712b^8a^6 - 84375b^3a^6 + 186624b^{12}a^5 - 12500b^7a^5 \end{aligned} \tag{19}$$

*and*

$$\Delta_2 = 4ab - 1.$$

*Then the following holds:*

*a) If* $\Delta_2 \geq 0$, *then equation* (18) *has one real root and four pairs of conjugate imaginary roots. Consequently, System* (1) *has one equilibrium point* $E_0(0,0)$;

*b) If* $\Delta_2 < 0$, *and* $\Delta_1 < 0$, *then equation* (18) *has three distinct real roots and three pairs of conjugate imaginary roots. Consequently, System* (1) *has three equilibrium points* $E_0(0,0)$; $E_{SW}(\bar{x},\bar{y})$ *and* $E_{NE}(\bar{x},\bar{y})$;

*c) If* $\Delta_2 < 0$, *and* $\Delta_1 > 0$, *then equation* (18) *has four pairs of conjugate imaginary roots and one real root. Consequently, System* (1) *has one equilibrium point* $E_0(0,0)$;

*d) If* $\Delta_2 < 0$ *and* $\Delta_1 = 0$ *then equation* (18) *has three pairs of conjugate imaginary roots and one real root of multiplicity two and one root of multiplicity one. Consequently, System* (1) *has two equilibrium points* $E_0(0,0)$ *and* $E(\bar{x},\bar{y})$.

**Proof.** Let $\Delta$ be discriminant of

$$\tilde{f}(x) = x^9 + 3ax^7 + 2bx^6 + 3a^2x^5 + (4ab-1)x^4 + \left(a^3+b^2\right)x^3 + 2a^2bx^2 + ab^2x.$$

Then $\Delta = a^2b\Delta_1$. The rest of the proof follows from the fact that equation (18) has at most three real roots and Theorem 5.1 from [13]. □

Period-two solution $\{(\Phi,\Psi), T(\Phi,\Psi)\}$ satisfies the system

$$F(\Phi,\Psi) = \Phi,\ G(\Phi,\Psi) = \Psi,$$

which is equivalent to

$$\Phi\left(\Phi^3 - \left(b+\Psi^2\right)^2\left(\frac{\Psi^4}{\left(a+\Phi^2\right)^2}+a\right)\right) = 0, \quad \Psi\left(\Psi^3 - \left(a+\Phi^2\right)^2\left(\frac{\Phi^4}{\left(b+\Psi^2\right)^2}+b\right)\right) = 0. \tag{20}$$

For $\Phi = 0$ we have $\Psi = 0$ or $\Psi = \sqrt[3]{a^2b}$, and for $\Psi = 0$ we have $\Phi = 0$ or $\Phi = \sqrt[3]{ab^2}$. Hence, we have two minimal period-two points $P_1\left(0, \sqrt[3]{a^2b}\right)$ and $P_2\left(\sqrt[3]{ab^2}, 0\right)$.

**Lemma 4** *The period-two solution* $\{P_1, P_2\}$ *is a saddle point with corresponding eigenvectors which are coordinate axes.*

**Proof.** The proof follows from the fact that $J_{T^2}(P_1) = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}$. □

**Lemma 5** *Let* $\mathcal{C}_F := \{(x,y) : F(x,y) = x\}$ *and* $\mathcal{C}_G := \{(x,y) : G(x,y) = y\}$ *be the period-two curves, that is the curves which intersection is a period-two solution. Then for all* $y > \sqrt[3]{a^2b}$ *there exists exactly one* $x_G(y) > 0$ *such that* $G(x_G(y), y) = y$ *and for all* $x > \sqrt[3]{ab^2}$ *there exists exactly one* $y_F(x) > 0$ *such that* $F(x, y_F(x)) = x$. *Furthermore,* $x_G(y)$ *and* $y_F(x)$ *are continuous functions and* $x'_G(y) > 0$, $y'_F(x) > 0$.

**Proof.** Since $F(x,y) = x$ and $G(x,y) = y$ if and only if

$$-xy^4\left(a^3+2a^2x^2+ax^4+b^2\right) + x\left(a+x^2\right)^2\left(x^3-ab^2\right) - 2abxy^2\left(a+x^2\right)^2 - 2bxy^6 - xy^8 = 0,$$
$$-x^4\left(a^2+b\left(b+y^2\right)^2\right) - \left(b+y^2\right)^2\left(a^2b-y^3\right) - 2abx^2\left(b+y^2\right)^2 - 2ax^6 - x^8 = 0,$$

respectively, in view of Descartes' Rule of Signs we have that for all $y > \sqrt[3]{a^2b}$ there exists exactly one $x_G(y) > 0$ such that $G(x_G(y), y) = y$ and for all $x > \sqrt[3]{ab^2}$ there exists exactly one $y_F(x) > 0$ such that $F(x, y_F(x)) = x$. Taking derivatives of $F(x,y) = x$ with respect to $x$ we get

$$y'_F(x) = \frac{1 - F'_x(x,y)}{F'_y(x,y)}.$$

From $F(x,y) = x$ we have that $\left(b+y^2\right)^2 = \frac{x^3}{\frac{y^4}{\left(a+x^2\right)^2}+a}$, which implies

$$F'_x(x,y) = \frac{4y^4\left(a+2x^2\right) + 4a\left(a+x^2\right)^3}{y^4\left(a+x^2\right) + a\left(a+x^2\right)^3} > 1.$$

Since $F'_y(x,y) < 0$ we get $x'_F(y) > 0$. Taking derivatives of $G(x,y) = y$ with respect to $y$ we get

$$x'_G(y) = \frac{1 - G'_y(x,y)}{G'_x(x,y)}.$$

From $G(x,y)=y$ we have that $\left(b+y^2\right)^2=\frac{x^3}{\frac{y^4}{\left(a+x^2\right)^2+a}}$, which implies

$$G'_y(x,y)=\frac{4x^4y^2}{\left(b+y^2\right)\left(b\left(b+y^2\right)^2+x^4\right)}+4>4.$$

Since $G'_x(x,y)<0$ we get $x'_G(y)>0$. □

**Theorem 5** *If $T$ has a period-two solution $\{(\Phi,\Psi),T(\Phi,\Psi)\}$, then it is unstable. If $\mu_1$ and $\mu_2$, $(0<\mu_1<\mu_2)$ are the eigenvalues of $J_{T^2}(\Phi,\Psi)$ then $\mu_1>0$ and $\mu_2>1$. All period-two-solutions are ordered with respect to the North-East ordering.*

**Proof.** Since

$$F'_x(\Phi,\Psi)=\frac{4\Psi^4\left(a+2\Phi^2\right)+4a\left(a+\Phi^2\right)^3}{\Psi^4\left(a+\Phi^2\right)+a\left(a+\Phi^2\right)^3}>1$$

and

$$G'_y(\Phi,\Psi)=\frac{4\Phi^4\Psi^2}{\left(b+\Psi^2\right)\left(b\left(b+\Psi^2\right)^2+\Phi^4\right)}+4>4,$$

we obtain

$$trJ_{T^2}(\Phi,\Psi)=\mu_1+\mu_2>5.$$

The rest of the proof follows from the fact that $detJ_{T^2}(\Phi,\Psi)=\mu_1\mu_2>0$ and Lemma 5. □

**Theorem 6** *If map $T$ has a minimal period-two point $\{(\Phi_1,\Psi_1),T(\Phi_1,\Psi_1)\}$, which is non-hyperbolic, then $Dis(p)=0$, where $Dis(p)$ is the discriminant of polynomial*

$$p(x):=p_{16}x^{16}+p_{15}x^{15}+\cdots+p_1x+p_0,$$

*where the coefficients $p_i$, $i=0,...,16$ are given in appendix A. If $\{(\Phi_1,\Psi_1),T(\Phi_1,\Psi_1)\}$ and $\{(\Phi_2,\Psi_2),T(\Phi_2,\Psi_2)\}$ are two period-two points such that $T$ has no other period-two points in $[[(\Phi_1,\Psi_1),(\Phi_2,\Psi_2)]]=\{(x,y):(\Phi_1,\Psi_1)\preceq_{ne}(x,y)\preceq_{ne}(\Phi_2,\Psi_2)\}$, $Dis(\tilde{f})\neq0$ and $Dis(p)\neq0$, then one of them is a saddle point and the other one is repeller.*

**Proof.** Period-two solution curves $\mathcal{C}_F=\{(x,y)\in\mathcal{R}:\tilde{F}(x,y)=0\}$ and $\mathcal{C}_G=\{(x,y)\in\mathcal{R}:\tilde{G}(x,y)=0\}$, where

$$\begin{aligned}\tilde{F}(x,y)=&-a^3b^2x-2a^3bxy^2-a^3xy^4-2a^2b^2x^3-4a^2bx^3y^2+a^2x^4\\&-2a^2x^3y^4-ab^2x^5-2abx^5y^2+2ax^6-ax^5y^4-b^2xy^4-2bxy^6+x^8-xy^8,\\\tilde{G}(x,y)=&-a^2b^3y-2a^2b^2y^3-a^2by^5-a^2x^4y-2ab^3x^2y-4ab^2x^2y^3\\&-2abx^2y^5-2ax^6y-b^3x^4y-2b^2x^4y^3+b^2y^4-bx^4y^5+2by^6-x^8-y+y^8,\end{aligned}$$

are algebraic curves. By using software *Mathematica* one can see that the resultant of the polynomials $\tilde{F}(x,y)$ and $\tilde{G}(x,y)$ in variable $y$ is given by

$$\begin{aligned}R(\tilde{F},\tilde{G})=&x^{20}\left(a+x^2\right)^{16}\left(ab^2-x^3\right)\\&\left(a^3x^2+2a^2bx+3a^2x^4+ab^2+4abx^3+3ax^6+b^2x^2+2bx^5+x^8-x^3\right)p(y)\\=&x^{19}(a+x^2)^{16}(ab^2-x^3)\tilde{f}(x)p(x).\end{aligned}$$

The rest of the proof is the same as the proof of Theorem 15 in [10] so we skip it. □

It is easy to see that the following holds:

**Lemma 6** *The equilibrium point $E_0$ is locally asymptotically stable.*

Let $\mathcal{C}_1:=\{(x,y):T_1(x,y)=x\}$ and $\mathcal{C}_2:=\{(x,y):T_2(x,y)=y\}$ be the equilibrium curves, that is the curves which intersection is an equilibrium solution. Then for all $x\geq0$ there exist exactly one $y_1(x)>0$ such that $T_1(x,y_1(x))=x$ and exactly one $y_2(x)>0$ such that $T_2(x,y_2(x))=y$. Furthermore, it can be seen that $y_1(x)$ and $y_2(x)$ are continuous increasing functions.

**Theorem 7** *Assume that $\Delta_1 < 0$ and $\Delta_2 < 0$. Then there exist two positive equilibrium solutions $E_{SW}$ and $E_{NE}$ and the following holds true:*

*(i) The equilibrium solution $E_{SW}$ is repeller.*

*(ii) The equilibrium solution $E_{NE}$ is a saddle point.*

**Proof.** The existence of the equilibrium solution follows from Lemma 3.

(i) Since the map $T$ is anti-competitive then by results in [15], the eigenvalues $\lambda_1$ and $\lambda_2$ of the Jacobian matrix associated to the map $T$ at $E_{SW}(\bar{x},\bar{y}) \in int(\mathbb{R}^2_+)$ are real and distinct, and the following holds $|\lambda_2| < -\lambda_1$. By using (6), we see that $det_{J_T}(\bar{x},\bar{y}) = \lambda_1\lambda_2 < 0$, which implies $\lambda_2 > 0$. Let

$$J_T(\bar{x},\bar{y}) = \begin{pmatrix} e & f \\ g & h \end{pmatrix}. \tag{21}$$

Taking derivatives of $T_1(x,y) = x$ and $T_2(x,y) = y$ with respect to $x$ in the neighborhood of $\bar{x}$, we have

$$y_1'(\bar{x}) - y_2'(\bar{x}) = \frac{1-e}{f} - \frac{g}{1-h} = \frac{p(1)}{f(1-h)} = \frac{(\lambda_1-1)(\lambda_2-1)}{f(1-h)}, \tag{22}$$

where

$$p(\lambda) = \lambda^2 - (e+f)\lambda + (eh - fg)$$

is the characteristic equation of (21). One can see that $y_1'(\bar{x}) - y_2'(\bar{x}) < 0$. Since $\lambda_1 < 0$, and $f > 0$, $h < 0$, we obtain $\lambda_2 > 1$. In view of $|\lambda_2| < -\lambda_1$ we have $\lambda_1 < -1$ from which the proof follows.

(ii) Now, we consider the equilibrium point $E_{NE}(\bar{x},\bar{y})$. Same as in the previous case we have that $y_1'(\bar{x}) - y_2'(\bar{x}) > 0$ which implies that $0 < \lambda_2 < 1$. By Theorem 5 we have $\lambda_1^2 + \lambda_2^2 > 5$ from which it follows $\lambda_1^2 > 4$, i.e. $\lambda_1 < -2$. This completes the proof.

□

**Theorem 8** *Assume that $\Delta_2 < 0$ and $\Delta_1 = 0$. Then there exist one positive equilibrium point $E(\bar{x},\bar{y})$ which is a non-hyperbolic equilibrium point of unstable type. If $\lambda_1$ and $\lambda_2$ are the eigenvalues of the Jacobian matrix associated to the map $T$ at $E(\bar{x},\bar{y}) \in int(\mathbb{R}^2_+)$ then $\lambda_1 < -1$ and $\lambda_2 = 1$.*

**Proof.** In view of Lemmas 6 and 7 from [1], the curves $C_F$ and $C_G$ intersect tangentially at $E(\bar{x},\bar{y})$ (i.e. $y_1'(\bar{x}) - y_2'(\bar{x}) = 0$) if and only if $\bar{x}$ is zero of $\tilde{f}(x)$ of multiplicity greater then one. By Lemma 3, $\bar{x}$ is a root of $\tilde{f}(x)$ of multiplicity two. In view of

$$y_1'(\bar{x}) - y_2'(\bar{x}) = \frac{(\lambda_1-1)(\lambda_2-1)}{f(1-h)} = 0, \tag{23}$$

we obtain $\lambda_2 = 1$. Since $|\lambda_2| < -\lambda_1$ we have $\lambda_1 < -1$. □

# 4 The global behavior

Let $\mathcal{R} = [0,\infty)^2$, $\mathcal{C}_F = \{(x,y) \in \mathcal{R} : F(x,y) = x\}$, $\mathcal{C}_G = \{(x,y) \in \mathcal{R} : G(x,y) = y\}$ and

$$\begin{aligned}
\mathcal{R}_{T^2}(-,-) &= \{\, (x,y) \in \mathcal{R} : F(x,y) < x,\ G(x,y) < y \,\},\\
\mathcal{R}_{T^2}(+,-) &= \{\, (x,y) \in \mathcal{R} : F(x,y) > x,\ G(x,y) < y \,\},\\
\mathcal{R}_{T^2}(+,+) &= \{\, (x,y) \in \mathcal{R} : F(x,y) > x,\ G(x,y) > y \,\},\\
\mathcal{R}_{T^2}(-,+) &= \{\, (x,y) \in \mathcal{R} : F(x,y) < x,,\ G(x,y) > y \,\}.
\end{aligned}$$

By Lemma 5, $\mathcal{C}_F$ na $\mathcal{C}_G$ are the graphs of continuous strictly increasing functions $y_F$ and $y_G$, i.e. $\mathcal{C}_F = \{\, (x, y_F(x)) : x \geq \sqrt[3]{ab^2} \,\}$ and $\mathcal{C}_G = \{\, (x, y_G(x)) : x \geq 0 \,\}$.

In view of Lemma 4 [15] we have that $T(\mathcal{R}_{T^2}(+,-)) \subseteq \mathcal{R}_{T^2}(-,+)$ and $T(\mathcal{R}_{T^2}(-,+)) \subseteq \mathcal{R}_{T^2}(+,-)$ and $T^2(\mathcal{R}_{T^2}(-,+)) \subseteq \mathcal{R}_{T^2}(-,+)$ and $T^2(\mathcal{R}_{T^2}(+,-)) \subseteq \mathcal{R}_{T^2}(+,-)$. Since $T^2$ is competitive map, by using (iii) of Lemma 1, we obtain $T^2(x_0,y_0) \to (0,\infty)$ if $(x_0,y_0) \in \mathcal{R}_{T^2}(-,+)$ and $T^2(\tilde{x}_0,\tilde{y}_0) \to (\infty,0)$ if $(\tilde{x}_0,\tilde{y}_0) \in \mathcal{R}_{T^2}(+,-)$.

**Lemma 7** $int[[P_1,P_2]] \subset B(E_0)$.

**Proof.** If $P \in int[[P_1, P_2]]$ then there exist $x_0$ and $y_0$ such that

$$P_1 \prec_{se} (0, y_0) \prec_{se} P \prec_{se} (x_0, 0) \prec_{se} P_2.$$

Since $T^2$ is competitive map we have

$$P_1 \prec_{se} T^{2n}(0, y_0) \prec_{se} T^{2n}(P) \prec_{se} T^{2n}(x_0, 0) \prec_{se} P_2.$$

From Lemma 1 we obtain $T^{2n}(0, y_0) \to E_0$ and $T^{2n}(x_0, 0) \to E_0$ as $n \to \infty$ from which the proof follows. □

## 4.1 The case $\Delta_2 \geq 0$ or ($\Delta_2 < 0$ and $\Delta_1 > 0$)

In this case, by Lemma 3, there exists one equilibrium point, $E_0$ which is locally asymptotically stable and the minimal period two solution $\{P_1, P_2\}$ which is a saddle point. In this case we have that $y_F(x) < y_G(x)$ for $x \geq 0$ and

$$\begin{aligned}
\mathcal{R}_{T^2}(-,-) &= \{ (x,y) \in \mathcal{R} : y_F(x) < y < y_G(x) \}, \\
\mathcal{R}_{T^2}(+,-) &= \{ (x,y) \in \mathcal{R} : y_F(x) > y \} \subseteq \mathcal{B}(\infty, 0), \\
\mathcal{R}_{T^2}(-,+) &= \{ (x,y) \in \mathcal{R} : y > y_G(x) \} \subseteq \mathcal{B}(0, \infty).
\end{aligned}$$

Let $\mathcal{B}(0,\infty)$ denote the basin of attraction of $(0,\infty)$ and $\mathcal{B}(\infty,0)$ denote the basin of attraction of $(\infty,0)$ with respect to the map $T^2$.

In view of Theorem 4 and iii) of Lemma 1 it is clear that $\{T^n(x_0,y_0)\}$ is either asymptotic to $\{(0,\infty),(\infty,0)\}$ or converges to a period-two solution, for all $(x_0,y_0) \in \mathcal{R} = [0,\infty)^2$. Let $\mathcal{S}_1$ denote the boundary of $\mathcal{B}(0,\infty)$ and let $\mathcal{S}_2$ denote the boundary of $\mathcal{B}(\infty,0)$. It is easy to see that $P_1 \in \mathcal{S}_1$, $P_2 \in \mathcal{S}_2$, and $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathcal{R}_{T^2}(-,-)$. Similarly as in [10] it follows that $T^2(\mathcal{S}_1) \subseteq \mathcal{S}_1$, $T^2(\mathcal{S}_2) \subseteq \mathcal{S}_2$ and $T(\mathcal{S}_1) = \mathcal{S}_2$, $T(\mathcal{S}_2) = \mathcal{S}_1$. Further, $\mathcal{S}_1$ and $\mathcal{S}_2$ are the graphs of continuous strictly increasing functions. Since $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathcal{R}_{T^2}(-,-)$, we have, by the uniqueness of the global stable manifold of the map $T^2$, that $\mathcal{W}^s(P_1) = \mathcal{S}_1$ and $\mathcal{W}^s(P_2) = \mathcal{S}_2$.

**Theorem 9** *Assume that $\Delta_2 \geq 0$. Then System (1) possesses one equilibrium point $E_0(0,0)$ and one minimal period-two solution $\{P_1, P_2\}$. Equilibrium $E_0$ is locally asymptotically stable and $\{P_1, P_2\}$ is a saddle point. Global stable manifold $\mathcal{W}^s(\{P_1, P_2\})$, which is a union of two continuous increasing curves $\mathcal{S}_1$ and $\mathcal{S}_2$, divides the first quadrant such that the following holds:*

*i) Every initial point $(x_0, y_0) \in \mathcal{R}$ such that $(\tilde{x}_0, \tilde{y}_0) \ll_{se} (x_0, y_0) \ll_{se} (\bar{x}_0, \bar{y}_0)$ for some $(\tilde{x}_0, \tilde{y}_0) \in \mathcal{S}_1$ and $(\bar{x}_0, \bar{y}_0) \in \mathcal{S}_2$ is attracted to $E_0$.*

*ii) If $(x_0, y_0) \in \mathcal{R}$ such that $(x_0, y_0) \ll_{se} (\tilde{x}_0, \tilde{y}_0)$ for some $(\tilde{x}_0, \tilde{y}_0) \in \mathcal{S}_1$ then the subsequence of even-indexed terms $\{(x_{2n}, y_{2n})\}$ is asymptotic to $(0,\infty)$, and the subsequence of odd-indexed terms $\{(x_{2n+1}, y_{2n+1})\}$ is asymptotic to $(\infty, 0)$.*

*iii) If $(x_0, y_0) \in \mathcal{R}$ such that $(\bar{x}_0, \bar{y}_0) \ll_{se} (x_0, y_0)$ for some $(\bar{x}_0, \bar{y}_0) \in \mathcal{S}_2$ then the subsequence of even-indexed terms $\{(x_{2n}, y_{2n})\}$ is asymptotic to $(\infty, 0)$, and the subsequence of odd-indexed terms $\{(x_{2n+1}, y_{2n+1})\}$ is asymptotic to $(0,\infty)$.*

*See Figure 1 (a) for visual illustration.*

**Proof.** Since $S_1$ is invariant under $T^2$ and subset of $\mathcal{R}_{T^2}(-,-)$ we have that if $(x_0, y_0) \in S_1$ then $T^{2n+2}(x_0, y_0) \preceq_{ne} T^{2n}(x_0, y_0)$. This implies that subsequences $\{x_{2n}\}$ and $\{y_{2n}\}$ are decreasing and since they are bounded sequences, they are convergent. It must be that $T^{2n}(x_0, y_0) \to P_1$ as $n \to \infty$. By the uniqueness of the global stable manifold of $T^2$ we obtain $\mathcal{W}^s(P_1) = S_1$. Similarly we get $\mathcal{W}^s(P_2) = S_2$ from which it follows that $\mathcal{W}^s(\{P_1, P_2\}) = S_1 \cup S_2$. Take $(x_0, y_0) \in \mathcal{R}$, $(\tilde{x}_0, \tilde{y}_0) \in \mathcal{S}_1$ and $(\bar{x}_0, \bar{y}_0) \in \mathcal{S}_2$ such that $(\tilde{x}_0, \tilde{y}_0) \ll_{se} (x_0, y_0) \ll_{se} (\bar{x}_0, \bar{y}_0)$. By monotonicity of $T^2$ we have $T^{2n}(\tilde{x}_0, \tilde{y}_0) \ll_{se} T^{2n}(x_0, y_0) \ll_{se} T^{2n}(\bar{x}_0, \bar{y}_0)$. Since $T^{2n}(\tilde{x}_0, \tilde{y}_0) \to P_1$ and $T^{2n}(\bar{x}_0, \bar{y}_0) \to P_2$ as $n \to \infty$ and by the uniqueness of the global stable manifold we obtain that $T^{2n}(x_0, y_0)$ eventually enters $int[[P_1, P_2]]_{se}$. So it is enough to prove that $[[P_1, P_2]]_{se} \subseteq \mathcal{B}(E_0)$. Indeed, for $(x_0, y_0) \in int[[P_1, P_2]]_{se}$ there exist $\sqrt[3]{ab^2} > \bar{\bar{x}}_0 > 0$ and $\sqrt[3]{a^2b} > \bar{\bar{y}}_0 > 0$ such that $(0, \bar{\bar{y}}_0) \preceq_{se} (x_0, y_0) \preceq_{se} (\bar{\bar{x}}_0, 0)$. By Lemma 1 we have $T^{2n}(0, \bar{\bar{y}}_0) \to E_0$ and $T^{2n}(\bar{\bar{x}}_0, 0) \to E_0$ as $n \to \infty$. By monotonicity of $T^2$ we get $T^{2n}(x_0, y_0) \to E_0$ from which the proof follows. By construction of the sets $S_1$ and $S_2$ the statements ii) and iii) are valid. □

## 4.2 The case $\Delta_2 < 0$ and $\Delta_1 < 0$

In this case, by Lemma 3, there exist three equilibrium points $E_0$, $E_{SW}$ and $E_{NE}$. By Lemma 6 $E_0$ is locally asymptotically stable and by Theorem 7 $E_{SW}$ is repeller and $E_{NE}$ is a saddle point. If $Q_2(E_{SW}) = \{(x,y) : 0 \le x \le \bar{x}_{SW} \text{ and } y \ge \bar{x}_{SW}\}$ and $Q_4(E_{SW}) = \{(x,y) : x \ge \bar{x}_{SW} \text{ and } 0 \le y \le \bar{x}_{SW}\}$, then one can see that $Q_2(E_{SW}) \subseteq \mathcal{R}_{T^2}(-,+)$ and $Q_4(E_{SW}) \subseteq \mathcal{R}_{T^2}(+,-)$.

Let $\mathcal{B}(0,\infty)$ denote the basin of attraction of $(0,\infty)$ and $\mathcal{B}(\infty,0)$ denote the basin of attraction of $(\infty,0)$ with respect to the map $T^2$.

In view of Theorem 4 and iii) of Lemma 1 it is easy to see that $\{T^n(x_0,y_0)\}$ is either asymptotic to $(0,\infty)$ or $(\infty,0)$ or converges to a period-two solution, for all $(x_0,y_0) \in \mathcal{R} = [0,\infty)^2$. Let $\mathcal{S}_1$ denote the boundary of $\mathcal{B}(0,\infty)$ considered as a subset of $Q_3(E_{SW})$ and $\mathcal{S}_2$ denote the boundary of $\mathcal{B}(\infty,0)$ considered as a subset of $Q_3(E_{SW})$. It follows that $P_1, E_{SW} \in \mathcal{S}_1$ and $P_2, E_{SW} \in \mathcal{S}_2$. It is easy to see $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathcal{R}_{T^2}(-,-)$, from which, similarly as in [10], it follows that $T^2(\mathcal{S}_1) \subseteq \mathcal{S}_1$, $T^2(\mathcal{S}_2) \subseteq \mathcal{S}_2$ and $T(\mathcal{S}_1) = \mathcal{S}_2$, $T(\mathcal{S}_2) = \mathcal{S}_1$. Further, $\mathcal{S}_1$ and $\mathcal{S}_2$ are the graphs of continuous strictly increasing functions. Since $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathcal{R}_{T^2}(-,-)$, we have, by the uniqueness of the global stable manifold of $T^2$, that $\mathcal{W}^s(P_1) = \mathcal{S}_1$ and $\mathcal{W}^s(P_2) = \mathcal{S}_2$.

**Lemma 8** *$\mathcal{B}(E_0) = \{P \in [0,\infty)^2 : \widetilde{P} \prec_{se} P \prec_{se} \overline{P}$ for $\widetilde{P} \in \mathcal{S}_1$ and $\overline{P} \in \mathcal{S}_2\}$.*

**Proof.** Assume that $\widetilde{P} \prec_{se} P \prec_{se} \overline{P}$ for $\widetilde{P} \in \mathcal{S}_1$ and $\overline{P} \in \mathcal{S}_2$. By monotonicity of $T$ we get $T^{2n}(\widetilde{P}) \prec_{se} T^{2n}(P) \prec_{se} T^{2n}(\overline{P})$. Since $T^{2n}(\widetilde{P}) \to P_1$ and $T^{2n}(\overline{P}) \to P_2$ as $n \to \infty$. By the uniqueness of the global stable manifold we have that $T^{2n}$ eventually enters $int[[P_1,P_2]]$. The rest of the proof follows from Lemma 7. □

**Lemma 9** *Assume that $Dis(P) \neq 0$. Then System (1) does not have minimal period-two solution.*

**Proof.** For contradiction, assume that $P$ is a minimal period-two solution of System (1). It is clear from previous discussions that $P \in (Q_1(E_{SW}) \cap Q_3(E_{NE})) \cup Q_1(E_{NE})$. Furthermore, assume that $P \in Q_1(E_{SW}) \cap Q_3(E_{NE})$ and $T$ has no other minimal period-two solutions in $[[E_{SW}, P]]_{\preceq_{ne}}$. Since $E_{SW}$ is a repeller by Lemma 6 we obtain that $P$ is a saddle point. The map $T^2$ satisfy all conditions of Theorem 5 [20], which yields the existence of the global stable manifolds $\mathcal{W}^s(\{P, T(P)\})$, which is the union of two curves $\mathcal{W}^s(P)$ and $\mathcal{W}^s(T(P))$. Since $\mathcal{W}^s(T(P)) = T(\mathcal{W}^s(P))$ we have that these two curves have a common endpoint $E_{SW}$ and there exists minimal period-two solution $\{\tilde{P}, T(\tilde{P})\}$ such that $P \preceq_{ne} \tilde{P} \preceq_{ne} E_{NE}$ and the curve $\mathcal{W}^s(P)$ has the second endpoint at $\tilde{P}$ while the curve $\mathcal{W}^s(T(P))$ has the second endpoint at $T(\tilde{P})$. Furthermore, the minimal period-two solution $\{\tilde{P}, T(\tilde{P})\}$ is a repeller. Since all positive period-two solutions are ordered with respect to the North-East ordering it must be $\mathcal{W}^s(T(P)) \preceq_{ne} \mathcal{W}^s(P)$, i.e. $\mathcal{W}^s(T(P)) \subset Q_3(E_{SW})$ which is in contradiction to $\mathcal{W}^s(T(P)) \subset Q_1(E_{SW})$. Similarly, we have contradiction if $P \in Q_1(E_{NE})$. Hence, $T$ has no minimal period-two solutions. □

**Theorem 10** *Assume that $\Delta_2 < 0$ and $\Delta_1 < 0$. Then System (1) has three equilibrium solutions $E_0 \prec_{ne} E_{SW} \prec_{ne} E_{NE}$, where $E_0$ is locally asymptotically stable, $E_{SW}$ is a repeller and $E_{NE}$ is a saddle point and the minimal period-two solution $\{P_1, P_2\}$ which is a saddle point. In this case there exist three invariant continuous curves $\mathcal{W}^s(E_{NE})$, $\mathcal{W}^s(P_1)$, $\mathcal{W}^s(P_2)$, which have end point at $E_{SW}$ and they are graphs of increasing functions. Every solution $\{(x_n,y_n)\}$ which starts below $\mathcal{W}^s(E_{NE}) \cup \mathcal{W}^s(P_1)$ in South-East ordering is asymptotic to $(0,\infty)$ and every solution $\{(x_n,y_n)\}$ which starts above $\mathcal{W}^s(E_{NE}) \cup \mathcal{W}^s(P_2)$ in South-East ordering is asymptotic to $(\infty,0)$. Every solution $\{(x_n,y_n)\}$ which starts below $\mathcal{W}^s(P_2)$ and above $\mathcal{W}^s(P_1)$ in South-East ordering converges to $E_0$. The first quadrant of the initial conditions $Q_1 = \{(x_0,y_0) : x_0 \ge 0, y_0 \ge 0\}$ is the union of six disjoint basins of attraction, i.e.*

$$Q_1 = \mathcal{B}(0,\infty) \cup \mathcal{B}(\infty,0) \cup \mathcal{B}(E_0) \cup \mathcal{B}(\{P_1,P_2\}) \cup \mathcal{B}(E_{NE}) \cup \mathcal{B}(E_{SW}),$$

*where*

$$\begin{aligned}
\mathcal{B}(E_{SW}) &= \{E_{SW}\}, \quad \mathcal{B}(E_{NE}) = \mathcal{W}^s(E_{NE}), \quad \mathcal{B}(\{P_1,P_2\}) = \mathcal{W}^s(P_1) \cup \mathcal{W}^s(P_2),\\
\mathcal{B}(0,\infty) &= \{(x,y) | (x,y) \preceq_{se} (\tilde{x}_0,\tilde{y}_0) \quad \textit{for some} \quad (\tilde{x}_0,\tilde{y}_0) \in \mathcal{W}^s(E_{NE}) \cup \mathcal{W}^s(P_1)\},\\
\mathcal{B}(\infty,0) &= \{(x,y) | (\tilde{x}_1,\tilde{y}_1) \preceq_{se} (x,y) \quad \textit{for some} \quad (\tilde{x}_1,\tilde{y}_1) \in \mathcal{W}^s(E_{NE}) \cup \mathcal{W}^s(P_2)\},\\
\mathcal{B}(E_0) &= \{(x,y) | (\tilde{x}_1,\tilde{y}_1) \preceq_{se} (x,y) \preceq_{se} (\tilde{x}_2,\tilde{y}_2) \quad \textit{for some} \quad (\tilde{x}_1,\tilde{y}_1) \in \mathcal{W}^s(P_1), (\tilde{x}_2,\tilde{y}_2) \in \mathcal{W}^s(P_2)\}.
\end{aligned}$$

**Proof.** The proof which follows from previous discussions and Theorem 5 [20] will be ommited. See Figure 1 (c) for visual illustration. □

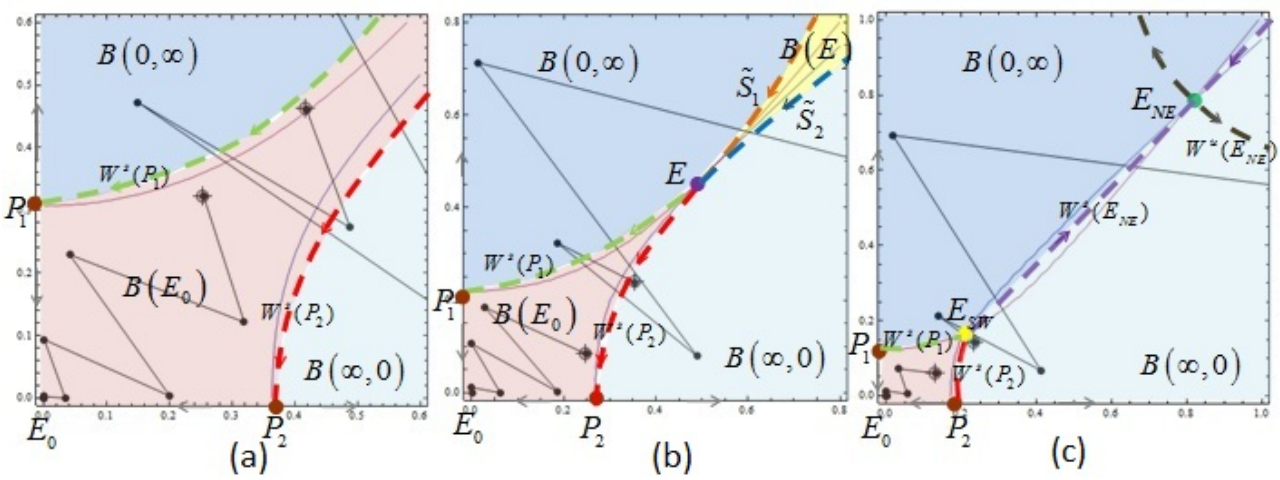



Figure 1: Visual illustration of (a) Theorem 9 (b) Theorem 11 and (c) Theorem 10.

### 4.3 The case $\Delta_2 < 0$ and $\Delta_1 = 0$

Let $S_1$ and $S_2$ be defined as in the previous case where $E_{SW} = E_{NE} = E$. In this case, by Lemma 3, there exist two equilibrium points $E_0$ and $E$. By Lemma 6 we have that $E_0$ is locally asymptotically stable and $E$ is a non-hyperbolic equilibrium point of unstable type. Let $\widetilde{\mathcal{S}}_1$ denote the boundary of $\mathcal{B}(0,\infty)$ considered as a subset of $Q_1(E)$ and $\widetilde{\mathcal{S}}_2$ denote the boundary of $\mathcal{B}(\infty,0)$ considered as a subset of $Q_1(E)$. It is clear that $E \in \widetilde{\mathcal{S}}_1 \cup \widetilde{\mathcal{S}}_2$. Furthermore, one can see $\widetilde{\mathcal{S}}_1, \widetilde{\mathcal{S}}_2 \subseteq \mathcal{R}_{T^2}(-,-)$. Similarly as in [10], we have that $T^2(\widetilde{\mathcal{S}}_1) \subseteq \widetilde{\mathcal{S}}_1$, $T^2(\widetilde{\mathcal{S}}_2) \subseteq \widetilde{\mathcal{S}}_2$ and $T(\widetilde{\mathcal{S}}_1) = \widetilde{\mathcal{S}}_2$, $T(\widetilde{\mathcal{S}}_2) = \widetilde{\mathcal{S}}_1$. Further, $\widetilde{\mathcal{S}}_1$ and $\widetilde{\mathcal{S}}_2$ are the graphs of continuous strictly increasing functions.

**Theorem 11** *Assume that $\Delta_2 < 0$ and $\Delta_1 = 0$. Then System (1) possesses two equilibrium solutions $E_0$ and $E$ and one minimal period-two solution $\{P_1, P_2\}$. Equilibrium $E_0$ is locally asymptotically stable, $E$ is non-hyperbolic of unstable type and $\{P_1, P_2\}$ is a saddle point. Global stable manifold $\mathcal{W}^s(\{P_1, P_2\})$ is the union of two continuous increasing curves $\mathcal{S}_1$ and $\mathcal{S}_2$ and the following holds*

- *i) Every initial point $(x_0, y_0) \in \mathcal{R}$ such that $(\tilde{x}_0, \tilde{y}_0) \ll_{se} (x_0, y_0) \ll_{se} (\bar{x}_0, \bar{y}_0)$ for some $(\tilde{x}_0, \tilde{y}_0) \in \mathcal{S}_1$ and $(\bar{x}_0, \bar{y}_0) \in \mathcal{S}_2$ is attracted to $E_0$.*
- *ii) Every initial point $(x_0, y_0) \in \mathcal{R}$ such that $(\tilde{x}_0, \tilde{y}_0) \ll_{se} (x_0, y_0) \ll_{se} (\bar{x}_0, \bar{y}_0)$ for some $(\tilde{x}_0, \tilde{y}_0) \in \widetilde{\mathcal{S}}_1$ and $(\bar{x}_0, \bar{y}_0) \in \widetilde{\mathcal{S}}_2$ is attracted to $E$.*
- *iii) If $(x_0, y_0) \in \mathcal{R}$ such that $(x_0, y_0) \ll_{se} (\tilde{x}_0, \tilde{y}_0)$ for some $(\tilde{x}_0, \tilde{y}_0) \in \mathcal{S}_1 \cup \widetilde{\mathcal{S}}_1$ then the subsequence of even-indexed terms $\{(x_{2n}, y_{2n})\}$ is asymptotic to $(0,\infty)$, and the subsequence of odd-indexed terms $\{(x_{2n+1}, y_{2n+1})\}$ is asymptotic to $(\infty, 0)$.*
- *iv) If $(x_0, y_0) \in \mathcal{R}$ such that $(\bar{x}_0, \bar{y}_0) \ll_{se} (x_0, y_0)$ for some $(\bar{x}_0, \bar{y}_0) \in \mathcal{S}_2 \cup \widetilde{\mathcal{S}}_2$ then the subsequence of even-indexed terms $\{(x_{2n}, y_{2n})\}$ is asymptotic to $(\infty, 0)$, and the subsequence of odd-indexed terms $\{(x_{2n+1}, y_{2n+1})\}$ is asymptotic to $(0,\infty)$.*

*See Figure 1 (b) for visual illustration.*

**Proof.** The proof of the statement i) is the same as the proof of the statement i) of Theorem 9. Since $\widetilde{S}_1$ is invariant under $T^2$ and subset of $\mathcal{R}_{T^2}(-,-)$ we have that if $(x_0, y_0) \in \widetilde{S}_1$ then $T^{2n+2}(x_0, y_0) \preceq_{ne} T^{2n}(x_0, y_0)$. This implies that subsequences $\{x_{2n}\}$ and $\{y_{2n}\}$ are decreasing and since they are bounded sequences, they are convergent. It must be that $T^{2n}(x_0, y_0) \to E$ as $n \to \infty$. Since $T$ is continuous map and $E$ is an equilibrium point we obtain $T^{2n+1}(x_0, y_0) \to E$ as $n \to \infty$ and $\widetilde{S}_2 = T(\widetilde{S}_1)$. Similarly we obtain that if $(x_0, y_0) \in \widetilde{S}_2$ then $T^{2n}(x_0, y_0) \in \widetilde{S}_2$, $T^{2n}(x_0, y_0) \to E$ as $n \to \infty$. Further, $T^{2n+1}(x_0, y_0) \in \widetilde{S}_1$, $T^{2n+1}(x_0, y_0) \to E$ as $n \to \infty$. Take $(x_0, y_0) \in \mathcal{R}$ and $(\tilde{x}_0, \tilde{y}_0) \in \widetilde{\mathcal{S}}_1$ and $(\bar{x}_0, \bar{y}_0) \in \widetilde{\mathcal{S}}_2$ such that $(\tilde{x}_0, \tilde{y}_0) \ll_{se} (x_0, y_0) \ll_{se} (\bar{x}_0, \bar{y}_0)$. By monotonicity of $T^2$ we have $T^{2n}(\tilde{x}_0, \tilde{y}_0) \ll_{se} T^{2n}(x_0, y_0) \ll_{se} T^{2n}(\bar{x}_0, \bar{y}_0)$. Since $T^{2n}(\tilde{x}_0, \tilde{y}_0) \to E$ and $T^{2n}(\bar{x}_0, \bar{y}_0) \to E$ as $n \to \infty$ we obtain that $T^{2n}(x_0, y_0) \to E$, which implies the statement ii). The statements iii) and iv) follow by construction of the sets $\widetilde{S}_1$ and $\widetilde{S}_2$. □

**Remark 1** The major results of this paper, Theorems 9 - 11, are actually the general results for general anti-competitive system (3). In fact, any anti-competitive system (3) with same configuration and local stability of the equilibrium and

period-two solutions will have the same global dynamics. So System (1) is actually an example of a global dynamics described in Theorems 9 - 11.

**Remark 2** In [22] we consider system

$$\begin{cases} x_{n+1} &= \frac{x_n^2}{a+y_n^2} \\ y_{n+1} &= \frac{y_n^2}{b+x_n^2} \end{cases}, \quad n=0,1,\ldots \tag{24}$$

where the parameters $a, b$ are positive numbers and initial conditions $x_0$ and $y_0$ are arbitrary nonnegative numbers, and obtain global dynamics similar to one described in Theorems 9 - 11, with the major difference that $P_1$ and $P_2$ are saddle point equilibrium solutions. Since the eigenvectors of the linearized system at $P_1$ and $P_2$ are parallel to the coordinate axes one can not apply at this time the results from [19, 20] to prove the existence of stable manifolds at these two points. However, existence of stable manifolds at these two points can be proved as in Theorems 9 - 11, where these two manifolds are obtained as Julia sets of the points $(\infty, 0)$ and $(0, \infty)$. Thus all results in [22] are correct with this adjustment of the proof.

# A Values of coefficients $p_i$ for $i = 0, ..., 16$.

$$\begin{aligned}
p_{16} =& a^4b^4 - 4a^3b^3 + 6a^2b^2 - 4ab + 1\\
p_{15} =& 4a^3b^5 - 12a^2b^4 + 12ab^3 - 4b^2\\
p_{14} =& 6a^2b^6 - 12ab^5 + 7a^5b^4 + 6b^4 - 20a^4b^3 + 18a^3b^2 - 4a^2b - a\\
p_{13} =& 4ab^7 - 4b^6 + 26a^4b^5 - 52a^3b^4 + 24a^2b^3 + 4ab^2 - 2b\\
p_{12} =& b^8 + 38a^3b^6 - 40a^2b^5 + 21a^6b^4 - 10ab^4 - 40a^5b^3 + 12b^3 + 10a^4b^2 + 8a^3b + a^7 + a^2\\
p_{11} =& 28a^2b^7 - 4ab^6 + 72a^5b^5 - 16b^5 - 87a^4b^4 - 30a^3b^3 + 26a^2b^2 + 4a^6b + 10ab + 7a^5 + 1\\
p_{10} =& 5a^8 + 35b^4a^7 - 40b^3a^6 - 12b^2a^5 + 101b^6a^4 + 32ba^4 - 44b^5a^3 + 13a^3 - 100b^4a^2\\
&+ 11b^8a + 16b^3a + 4b^7 + 11b^2\\
p_9 =& 2b^9 + 76a^3b^7 + 10a^2b^6 + 110a^6b^5 - 58ab^5 - 68a^5b^4 - 6b^4 - 102a^4b^3\\
&+ 34a^3b^2 + 18a^7b + 28a^2b + 21a^6 + 3a\\
p_8 =& 10a^9 + 35b^4a^8 - 20b^3a^7 + 2b^2a^6 + 145b^6a^5 + 58ba^5 - 12b^5a^4 + 15a^4 - 163b^4a^3 + 33b^8a^2\\
&- 20b^3a^2 + 8b^7a + 13b^2a + 8b^6 + 4b\\
p_7 =& 8ab^9 + b^8 + 104a^4b^7 + 16a^3b^6 + 100a^7b^5 - 66a^2b^5 - 22a^6b^4 - 33ab^4 - 66a^5b^3 + 29a^4b^2\\
&+ 32a^8b + 22a^3b + 22a^7 + 3a^2\\
p_6 =& 10a^{10} + 21b^4a^9 - 4b^3a^8 + 35b^2a^7 + 120b^6a^6 + 48ba^6 + 8b^5a^5 + 6a^5 - 90b^4a^4 + 43b^8a^3\\
&- 42b^3a^3 + 6b^7a^2 - b^2a^2 + 16b^6a + 4ba + b^{10} + 2b^5 + 1\\
p_5 =& 28ba^9 + 54b^5a^8 + 9a^8 + 10b^3a^6 + 76b^7a^5 + 16b^2a^5 + 6b^6a^4 + 10ba^4 - 22b^5a^3\\
&+ a^3 + 10b^9a^2 - 37b^4a^2 + 4b^8a - 10b^3a + b^2\\
p_4 =& 5a^{11} + 7b^4a^{10} + 33b^2a^8 + 56b^6a^7 + 14ba^7 + 4b^5a^6 + a^6 - 5b^4a^5 + 26b^8a^4 - 18b^3a^4\\
&+ 6b^7a^3 - b^2a^3 + 14b^6a^2 + 2ba^2 + b^{10}a - 2b^5a + 2b^9 + b^4\\
p_3 =& 12ba^{10} + 16b^5a^9 + a^9 + b^4a^8 + 20b^3a^7 + 28b^7a^6 + 3b^2a^6 + 4ba^5 + 2b^5a^4\\
&+ 4b^9a^3 - 9b^4a^3 + 8b^8a^2 - 6b^3a^2 - 4b^7a - 2b^2a + b^6\\
p_2 =& a^{12} + b^4a^{11} + 11b^2a^9 + 13b^6a^8 + 6b^4a^6 + 6b^8a^5 + 2b^3a^5\\
&+ 6b^7a^4 + 3b^2a^4 + 3b^6a^3 + 2b^5a^2 + 2b^9a - b^4a + b^8\\
p_1 =& 2ba^{11} + 2b^5a^{10} + 4b^3a^8 + 4b^7a^7 + 4b^4a^4 + 4b^8a^3 - 2b^3a^3 - 2b^7a^2\\
p_0 =& ab^8 + 2a^5b^7 + a^9b^6 + a^2b^4 + 2a^6b^3 + a^{10}b^2
\end{aligned}$$

# References

[1] S. Basu and O. Merino, On the Behavior of Solutions of a System of Difference Equations, *Comm. Appl. Nonlinear Anal.* 16 (2009), no. 1, 89–101.

[2] A. Brett and M. R. S. Kulenović, Two Species Competitive Model with the Allee Effect, *Adv. Difference Equ.*, Volume 2014 (2014):307, 28 p.

[3] E. Camouzis, M. R. S. Kulenović, G. Ladas, and O. Merino, Rational Systems in the Plane, *J. Difference Equ. Appl.* 15 (2009), 303-323.

[4] D. Clark and M. R. S. Kulenović, On a Coupled System of Rational Difference Equations, *Comput. Math. Appl.* 43(2002), 849-867.

[5] D. Clark, M. R. S. Kulenović, and J.F. Selgrade, Global Asymptotic Behavior of a Two Dimensional Difference Equation Modelling Competition, *Nonlinear Anal., TMA* 52(2003), 1765-1776.

[6] J. M. Cushing, S. Levarge, N. Chitnis and S. M. Henson, Some discrete competition models and the competitive exclusion principle , *J. Difference Equ. Appl.* 10(2004), 1139-1152.

[7] P. deMottoni and A. Schiaffino, Competition systems with periodic coefficients: A geometric approach, *J. Math. Biol.* 11 (1981), 319–335.

[8] J. E. Franke and A.-A. Yakubu, Mutual exclusion verses coexistence for discrete competitive systems, *J. Math. Biol.* 30 (1991), 161-168.

[9] J. E. Franke and A.-A. Yakubu, Geometry of exclusion principles in discrete systems, *J. Math. Anal. Appl.* 168 (1992), 385-400.

[10] V. Hadžiabdić, M. R. S. Kulenović and E. Pilav, Dynamics of a two-dimensional competitive system of rational difference equations with quadratic terms, *Adv. Difference Equ.* (2014), 2013:301, 32 p.

[11] P. Hess, *Periodic-parabolic boundary value problems and positivity.* Pitman Research Notes in Mathematics Series, 247. Longman Scientific & Technical, Harlow; 1991. viii+139 pp.

[12] M. Hirsch and H. Smith, Monotone dynamical systems. *Handbook of differential equations: ordinary differential equations.* Vol. II, 239-357, Elsevier B. V., Amsterdam, 2005.

[13] S. Janson, Resultant and discriminant of polynomials, http://www2.math.uu.se/ svante/ papers/, 2010.

[14] S. Kalabušić and M. R. S. Kulenović, Dynamics of Certain Anti-competitive Systems of Rational Difference Equations in the Plane, *J. Difference Equ. Appl.*, 17(2011), 1599–1615.

[15] S. Kalabušić, M. R. S. Kulenović and E. Pilav, Global Dynamics of Anti-Competitive Systems in the Plane, *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.*, 20(2013), 477-505

[16] M. R. S. Kulenović and G. Ladas, *Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures,* Chapman and Hall/CRC, Boca Raton, London, 2001.

[17] M. R. S. Kulenović and O. Merino, *Discrete Dynamical Systems and Difference Equations with Mathematica*, Chapman& Hall/CRC Press, Boca Raton, 2002.

[18] M. R. S. Kulenović and O. Merino, Competitive-Exclusion versus Competitive-Coexistence for Systems in the Plane, *Discrete Contin. Dyn. Syst. Ser. B* 6(2006), 1141-1156.

[19] M. R. S. Kulenović and O. Merino, Global Bifurcation for Competitive Systems in the Plane, *Discrete Contin. Dyn. Syst. B* 12(2009), 133-149.

[20] M. R. S. Kulenović and O. Merino, Invariant Manifolds for Competitive Discrete Systems in the Plane, *Int. J. of Bifurcations and Chaos*, 20(2010), 2471-2486.

[21] M. R. S. Kulenović and M. Nurkanović, Basins of Attraction of an Anti-competitive System of Difference Equations in the Plane, *Comm. Appl. Nonlinear Anal.*, 19(2012), 41–53.

[22] M. R. S. Kulenović and M. Pilling, Global Dynamics of a Certain Two-dimensional Competitive System of Rational Difference Equations with Quadratic Terms, *J. Comput. Anal. Appl.*, 19(2015), 156–166.

[23] H. L. Smith, Periodic competitive differential equations and the discrete dynamics of competitive maps, *J. Differential Equations* 64 (1986), 165-194.

[24] H. L. Smith, Planar Competitive and Cooperative Difference Equations,*J. Difference Equ. Appl.* 3(1998), 335-357.

# Solutions to Periodic Sylvester Matrix Equations Based on Matrices Splitting *

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**Abstract**

New iterative algorithms are introduced to solve periodic Sylvester matrix equations in this paper. The iterative algorithms are based on the principle of matrix splitting and gradient iteration method. Detailed iterative steps for solving equations are presented and their convergence property are strictly verified. A numerical test is employed to prove the correctness and effectiveness of the iterative algorithms.

**Keywords:** Periodic Sylvester matrix equations; iterative algorithm; matrix splitting.

## 1 Introduction

Analysis and design of time-varying systems are more charllenging than that of time-invariant dynamic systems since their coefficients are changing according to time. Take stability and stabilization for example, the stability concepts and criterion for (linear) time-varying systems are very difficult to characterize as they generally have no direct relationship with their coefficients (see [18] and [19] for detailed introductions). The periodic linear system as a special case of linear time-varying systems is thus important since it helps to understand that methods built for time-invariant systems can be generalized to time-varying setting. On the other, periodic linear systems also have important applications in engineering since they can be frequently used to describe cyclic temporal variation (seasonal or interannual) and to account for the operation of multiple processes. For example, Caswell analyzed in [1] the periodic models that must trace the effects of parameter changes and they applied the method to periodic system for periodic environments, and Verstraete introduced in [13] a picture to analyse the density matrix renormalization group (DMRG) numerical method from a quantum information prespective, which leads to a variational formulation of DMRG that allows for dramatic improvements in the case of problems with periodic boundary conditions. Therefore, in recent years, periodic linear systems have attracted significant attention in the literature.

Periodic Sylvester matrix equations play a major role in the analysis and design of discrete-time periodic linear systems. A general form of the periodic Sylvester matrix equation is as follows

$$A_t X_t + X_{t+1} B_t = C_t, \tag{1}$$

and

$$A_t X_{t+1} + X_t B_t = C_t, \tag{2}$$

where the coefficient matrices $A_t$, $B_t$, $C_t \in \mathbb{R}^{n\times n}$, $t = 0, 1, \cdots$, are given matrices and $X_t \in \mathbb{R}^{n\times n}$ are unknown matrices. These matrices are periodic with period $T$, i.e., $A_{t+T} = A_t$, $B_{t+T} = B_t$, $C_{t+T} = C_t$ and $X_{t+T} = X_t$. In [8], Korotyaev showns that it is related with the periodic matrix-valued Jacobi operators. We have shown recently that the aboveperiodic Sylvester matrix equation are helpful in the design of periodic

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Luenberger observers [10] and output regulator [11]. For more applications of this class of periodic Sylvester matrix equations, see [10], [11] and the refereces therein.

The (generalized periodic) Sylvester matrix equations were first put forward by Sylvester and applied in mathematical control theory. As the development of science and technology, they become more and more important. Now many scholars and experts have analyzed the existence and uniqueness of solutions to the Sylvester matrix equation. In [12], Sreedhar proposed an elegant and simple method for computing the periodic solution of Sylvester matrix equations. In [3], Chen used the matrix sign function to solve periodic Sylvester equations. In [16], Zhang offered a finite iterative algorithm for solving the complex generalized coupled Sylvester matrix equations. In [2], the author pays attention to solving the Lyapunov matrix equations and Sylvester matrix equations in control theory by numerical methods. [4] constructs an iterative algorithm to solve the generalized coupled Sylvester matrix equations over reflexive matrices. In [9], a comprehensive theory of the matrix linear equation $AX + XB = C$ is presented. In [6], Gu applied Jacobi iteration of solving linear equations to solve Sylvester matrix equations. In [5], M Dehghan propose two iterative algorithms for finding the Hermitian reflexive and skew-Hermitian solutions of the Sylvester matrix equation $AX + XB = C$.

Furthermore, it should be pointed that gradient iterative algorithm is attracting more and more researchers. Many experts apply it to solve the Sylvester matrix equations and a lot of cases show that it is a better way to solve matrix equations. In [17], zhang present a gradient iterative algorithm for solving coupled matrix equations based on the hierarchical identification principle. In [14], Li study solutions of general matrix equations by using the iterative method and present gradient iterative algorithms by applying the hierarchical identification principle. In [7], Hoskins discussed an iterative method for solving the matrix equation $XA + AY = F$ and compared it with existing techniques. In [15], an iterative algorithm is construct to solve the general coupled matrix equations over reflexive matrix solution. Of course, researchers have given numerical examples to demonstrate the correctness of the proposed algorithm.

However, to the best of our knlowedge, the iterative algorithms for the periodic Sylvester matrix equations have not been fully researched in the literature. Therefore, in this paper, we dedicate to give iterative algorithms for solving equations (1) and (2). The iterative algorithms are based on the principle of matrix splitting and gradient iteration method. Detailed iterative steps for solving equations are presented and their convergence property are strictly verified. A numerical test is employed to prove the correctness and effectiveness of the iterative algorithms.

The rest of this paper is arranged in the following ways. In section 2, new iterative algorithms are proposed to solve the Sylvester matrix equations and the convergences are validated. In section 3, a numerical example is provided to verify the correctness of the iterative algorithm. And in section 4, we draw some conclusions.

# 2 Main results

## 2.1 Iterative algorithm for equation (1)

Firstly, for $t = 0, 1, \cdots, T-1$, define $W_t^{'}$ and $W_t^{''}$ as

$$W_t^{'} = C_t - X_{t+1}B_t \tag{3}$$

$$W_t^{''} = C_t - A_tX_t \tag{4}$$

Rewrite matrices $A_t$ and $B_t$ as

$$A_t = \alpha_t I_{n\times n} + T_t^{'}, \tag{5}$$

$$B_t = \beta_t I_{n\times n} + T_t^{''}, \tag{6}$$

where $\alpha_t$ and $\beta_t$ are arbitrary constant numbers, $I_{n\times n}$ is the unit matrix and $T_t^{'}$, $T_t^{''}$ are the remaining matrices of $A_t$, $B_t$.

Based on the above division, it is easy to obtain that

$$W_t^{'} = (\alpha_t I_{n\times n} + T_t^{'})X_t \tag{7}$$

$$W_t^{''} = X_{t+1}(\beta_t I_{n\times n} + T_t^{''}) \tag{8}$$

Construct the following iteration:

$$X_t^{'}(k) = X_t^{'}(k-1) + \theta_t\alpha_t\left(C_t - X_{t+1}^{'}(k-1)B_t - A_tX_t^{'}(k-1)\right) \tag{9}$$

$$X_t^{''}(k) = X_t^{''}(k-1) + \theta_t\beta_t\left(C_t - X_{t+1}^{''}(k-1)B_t - A_tX_t^{''}(k-1)\right) \tag{10}$$

Where $k$ is iterative step and bigger than 1.

Further, let

$$\begin{aligned} X_t(k) &= \frac{X_t^{'}(k) + X_t^{''}(k)}{2} \\ &= X_t(k-1) + \frac{1}{2}\theta_t(\alpha_t+\beta_t)\left(C_t - X_{t+1}(k-1)B_t - A_tX_t(k-1)\right) \end{aligned} \tag{11}$$

In addition, denote

$$R_t(k) = \|X_t(k) - X_t(k-1)\| \tag{12}$$

**Algorithm 1** *(An iterative algorithm for equation (1))*

1. *Set error upper limit $\varepsilon$, freely select initial matrices $X_t^{'}(0)$ and $X_t^{''}(0)$, calculate*

$$X_t(0) = \frac{X_t^{'}(0) + X_t^{''}(0)}{2}$$

2. *Choose parameters of $\alpha_t$, $\beta_t$ and $\theta_t$ for $t = 0, 1, \cdots, T-1$, calculate*

$$\lambda_t = \sum_{t=0}^{T-1}\|(\mathrm{I}_t - \frac{1}{2}\theta_t(\alpha_t+\beta_t)A_t)\| + \sum_{t=0}^{T-1}\|\frac{1}{2}\theta_t(\alpha_t+\beta_t)\|\sum_{t=0}^{T-1}\|B_t\|) \tag{13}$$

$$k := 0;$$

3. *If $\lambda_t < 1$ for $t = 0, 1, \cdots, T-1$, go to next step; else, return to step 2.*
4. *Set $k$=$k+1$, according to (9),(10),(11), compute $X_t(k)$; Further more, compute $R_t(k)$ by (12).*
5. *If $R_t(k) \le \varepsilon$, stop; else, go to step 4.*

The convergence of the iterative algorithm will be proved by the following theorem.

**Theorem 1** *If equation (1) has solutions $X_t^*$ and $\lambda_t$ shown in (13) is less than 1, the iterative sequence of $X_t(k)$ generated by Algorithm 1 converges to the true solution $X_t^*$, which means, for any initial $X_t(0)$, there is*

$$\lim_{k\to\infty} X_t(k) = X_t^*$$

**Proof.** Define error matrix $\bar{X}_t(k)$=$X_t(k) - X_t^*$, where $X_t^*$ act as the real matrix, $X_t(k)$ is the iterative solution to k by the algorithm, then

$$\bar{X}_t^{'}(k) = X_t{'}(k) - X_t^* \tag{14}$$

$$\bar{X}_t^{''}(k) = X_t{''}(k) - X_t^* \tag{15}$$

We can easily get

$$\bar{X}_t^{'}(k) = \bar{X}_t^{'}(k-1) + \frac{1}{2}\theta_t(\alpha_t+\beta_t)(-\bar{X}_{t+1}^{'}(k-1)B_t - A_t\bar{X}_t^{'}(k-1)) \tag{16}$$

$$\bar{X}_t^{''}(k) = \bar{X}_t^{''}(k-1) + \frac{1}{2}\theta_t(\alpha_t+\beta_t)(-\bar{X}_{t+1}^{''}(k-1)B_t - A_t\bar{X}_t^{''}(k-1)) \tag{17}$$

Then, we get

$$\begin{aligned}
\bar{X}_t(k) &= \frac{\bar{X}_t^{'}(k) + \bar{X}_t^{''}(k)}{2} \\
&= \bar{X}_t(k-1) + \frac{1}{2}\theta_t(\alpha_t+\beta_t)(-\bar{X}_{t+1}(k-1)B_t - A_t\bar{X}_t(k-1)) \\
&= \bar{X}_t(k-1) - \frac{1}{2}\theta_t\alpha_t\bar{X}_{t+1}(k-1)B_t - \frac{1}{2}\theta_t\beta_t\bar{X}_{t+1}(k-1)B_t \\
&- \frac{1}{2}\theta_t\alpha_t A_t\bar{X}_t(k-1)) - \frac{1}{2}\theta_t\beta_t A_t\bar{X}_t(k-1)) \\
&= (I_t - \frac{1}{2}\theta_t\alpha_t A_t - \frac{1}{2}\theta_t\beta_t A_t)\bar{X}_t(k-1) - \frac{1}{2}\theta_t\alpha_t\bar{X}_{t+1}(k-1)B_t \\
&- \frac{1}{2}\theta_t\beta_t\bar{X}_{t+1}(k-1)B_t \\
&= (I_t - \frac{1}{2}\theta_t(\alpha_t+\beta_t)A_t)\bar{X}_t(k-1) - \frac{1}{2}\theta_t(\alpha_t+\beta_t)\bar{X}_{t+1}(k-1)B_t
\end{aligned}$$

Let

$$\begin{aligned}
\|\bar{X}_t(k)\| &= \|(I_t - \frac{1}{2}\theta_t(\alpha_t+\beta_t)A_t)\bar{X}_t(k-1) - \frac{1}{2}\theta_t(\alpha_t+\beta_t)\bar{X}_{t+1}(k-1)B_t\| \\
&\le \|(I_t - \frac{1}{2}\theta_t(\alpha_t+\beta_t)A_t)\bar{X}_t(k-1)\| + \|\frac{1}{2}\theta_t(\alpha_t+\beta_t)\bar{X}_{t+1}(k-1)B_t\| \\
&\le \|(I_t - \frac{1}{2}\theta_t(\alpha_t+\beta_t)A_t)\|\|\bar{X}_t(k-1)\| + \|\frac{1}{2}\theta_t(\alpha_t+\beta_t)\|\|\bar{X}_{t+1}(k-1)\|\|B_t\|
\end{aligned}$$

So we can obtain

$$\begin{aligned}
\sum_{t=0}^{T-1}\|\bar{X}_t(k)\| &\le \sum_{t=0}^{T-1}\|(I_t - \frac{1}{2}\theta_t(\alpha_t+\beta_t)A_t)\|\|\bar{X}_t(k-1)\| + \|\frac{1}{2}\theta_t(\alpha_t \\
&+ \beta_t)\|\|\bar{X}_{t+1}(k-1)\|\|B_t\| \\
&\le \sum_{t=0}^{T-1}\|(I_t - \frac{1}{2}\theta_t(\alpha_t+\beta_t)A_t)\|\sum_{t=0}^{T-1}\|\bar{X}_t(k-1)\| + \sum_{t=0}^{T-1}\|\frac{1}{2}\theta_t(\alpha_t \\
&+ \beta_t)\|\sum_{t=0}^{T-1}\|\bar{X}_{t+1}(k-1)\|\sum_{t=0}^{T-1}\|B_t\| \\
&\le (\sum_{t=0}^{T-1}\|(I_t - \frac{1}{2}\theta_t(\alpha_t+\beta_t)A_t)\| + \sum_{t=0}^{T-1}\|\frac{1}{2}\theta_t(\alpha_t \\
&+ \beta_t)\|\sum_{t=0}^{T-1}\|B_t\|)\sum_{t=0}^{T-1}\|\bar{X}_t(k-1)\|
\end{aligned}$$

According to assumption $\lambda_t < 1$, where $\lambda_t$ are shown in (14), we can obtain that

$$\sum_{t=0}^{T-1}\|\bar{X}_t(k)\| \le \lambda_t\sum_{t=0}^{T-1}\|\bar{X}_t(k-1)\| \le \lambda_t^2\sum_{t=0}^{T-1}\|\bar{X}_t(k-2)\| \le \ldots \le \lambda_t^k\sum_{t=0}^{T-1}\|\bar{X}_t(0)\| \tag{18}$$

By controlling parameters of $\alpha_t$, $\beta_t$, $\theta_t$ to make $\lambda_t < 1$.

When k is towards infinity,

$$\lim_{k\to\infty}\bar{X}_t(k) = 0$$

So

$$\lim_{k\to\infty} X_t(k) = X_t^*$$

■

## 2.2 Iterative algorithm for equation (2)

On periodic Sylvester matrix equation (2),we can also build an convergent algorithm which is similar to Algorithm 1.

Firstly construct the following iteration:

$$X_t^{'}(k) = X_t^{'}(k-1) + \theta_t\alpha_t\left(C_t - X_t^{'}(k-1)B_t - A_tX_{t+1}^{'}(k-1)\right) \tag{19}$$

$$X_t^{''}(k) = X_t^{''}(k-1) + \theta_t\beta_t\left(C_t - X_t^{''}(k-1)B_t - A_tX_{t+1}^{''}(k-1)\right) \tag{20}$$

Let

$$\begin{aligned} X_t(k) &= \frac{X_t^{'}(k) + X_t^{''}(k)}{2} \\ &= X_t(k-1) + \frac{1}{2}\theta_t(\alpha_t+\beta_t)\left(C_t - X_t(k-1)B_t - A_tX_{t+1}(k-1)\right) \end{aligned} \tag{21}$$

**Algorithm 2** *(An iterative algorithm for equation (2))*

1. *Set error upper limit $\varepsilon$, arbitrary select initial matrices $X_t^{'}(0)$ and $X_t^{''}(0)$, calculate $X_t(0)$ as*

$$X_t(0) = \frac{X_t^{'}(0) + X_t^{''}(0)}{2}$$

2. *Choose parameters of $\alpha_t$, $\beta_t$ and $\theta_t$ for $t = 0, 1, \cdots, T-1$, calculate $\lambda_t$ according to (13), and set*

$$k := 0;$$

3. *If $\lambda_t < 1$ for $t = 0, 1, \cdots, T-1$, go to next step; else, return to step 2.*
4. *Set k=k + 1, according to (19),(20),(21), compute $X_t(k)$.*
5. *Compute $R_t(k)$ according to (12). If $R_t(k) \le \varepsilon$, stop; else, go to step 4.*

We can make use of the following theorem to prove the convergence of the iterative algorithm.

**Theorem 2** *If equation (2) has solutions $X_t^*$ and $\lambda_t$ shown in (13) is less than 1, the iterative sequence of $X_t(k)$ generated by Algorithm 2 converges to the true solutions $X_t^*$, which means, for any initial $X_t(0)$, there is*

$$\lim_{k\to\infty} X_t(k) = X_t^*$$

**Proof.** According to Algorithm 2, we can acquire the following results

$$\bar{X}_t^{'}(k) = \bar{X}_t^{'}(k-1) + \frac{1}{2}\theta_t(\alpha_t+\beta_t)(-\bar{X}_t^{'}(k-1)B_t - A_t\bar{X}_{t+1}^{'}(k-1)) \tag{22}$$

$$\bar{X}_t^{''}(k) = \bar{X}_t^{''}(k-1) + \frac{1}{2}\theta_t(\alpha_t+\beta_t)(-\bar{X}_t^{''}(k-1)B_t - A_t\bar{X}_{t+1}^{''}(k-1)) \tag{23}$$

$$\bar{X}_t(k) = \bar{X}_t(k-1) + \frac{1}{2}\theta_t(\alpha_t+\beta_t)(-\bar{X}_t(k-1)B_t - A_t\bar{X}_{t+1}(k-1)) \tag{24}$$

Let

$$\begin{aligned}\|\bar{X}_t(k)\| &= \|(I_t - \frac{1}{2}\theta_t(\alpha_t+\beta_t)A_t)\bar{X}_{t+1}(k-1) - \frac{1}{2}\theta_t(\alpha_t+\beta_t)\bar{X}_t(k-1)B_t\| \\ &\le \|(I_t - \frac{1}{2}\theta_t(\alpha_t+\beta_t)A_t)\bar{X}_{t+1}(k-1)\| + \|\frac{1}{2}\theta_t(\alpha_t+\beta_t)\bar{X}_t(k-1)B_t\| \\ &\le \|(I_t - \frac{1}{2}\theta_t(\alpha_t+\beta_t)A_t)\|\|\bar{X}_{t+1}(k-1)\| + \|\frac{1}{2}\theta_t(\alpha_t+\beta_t)\|\|\bar{X}_t(k-1)\|\|B_t\|\end{aligned}$$

Further, let

$$\begin{aligned}\sum_{t=0}^{T-1}\|\bar{X}_t(k)\| &\le \sum_{t=0}^{T-1}\|(I_t - \frac{1}{2}\theta_t(\alpha_t+\beta_t)A_t)\|\|\bar{X}_{t+1}(k-1)\| + \|\frac{1}{2}\theta_t(\alpha_t \\ &+ \beta_t)\|\|\bar{X}_t(k-1)\|\|B_t\| \\ &\le \sum_{t=0}^{T-1}\|(I_t - \frac{1}{2}\theta_t(\alpha_t+\beta_t)A_t)\|\sum_{t=0}^{T-1}\|\bar{X}_{t+1}(k-1)\| + \sum_{t=0}^{T-1}\|\frac{1}{2}\theta_t(\alpha_t \\ &+ \beta_t)\|\sum_{t=0}^{T-1}\|\bar{X}_t(k-1)\|\sum_{t=0}^{T-1}\|B_t\| \\ &\le (\sum_{t=0}^{T-1}\|(I_t - \frac{1}{2}\theta_t(\alpha_t+\beta_t)A_t)\| + \sum_{t=0}^{T-1}\|\frac{1}{2}\theta_t(\alpha_t \\ &+ \beta_t)\|\sum_{t=0}^{T-1}\|B_t\|)\sum_{t=0}^{T-1}\|\bar{X}_t(k-1)\|\end{aligned}$$

According to assumption $\lambda_t < 1$, where $\lambda_t$ are shown in (13), we can obtain that

$$\sum_{t=0}^{T-1}\|\bar{X}_t(k)\| \le \lambda_t\sum_{t=0}^{T-1}\|\bar{X}_t(k-1)\| \le \lambda_t^2\sum_{t=0}^{T-1}\|\bar{X}_t(k-2)\| \le \ldots \le \lambda_t^k\sum_{t=0}^{T-1}\|\bar{X}_t(0)\| \tag{25}$$

When k is towards infinity and $\lambda < 1$, we can obtain

$$\lim_{k\to\infty}\bar{X}_t(k) = 0$$

So

$$\lim_{k\to\infty}X_t(k) = X_t^*$$

■

# 3 A numerical example

In this section, we will give an example to illustrate the correctness and effectiveness of the iterative algorithm.

**Example 1** *In this example, we consider the following periodic Sylvester matrix equation with $T = 3$:*

$$A_tX_t + X_{t+1}B_t = C_t$$

*For given matrices*

$$\begin{aligned}A_0 &= \begin{bmatrix} 2.1 & 0.8 \\ -1.0 & 1.3 \end{bmatrix}, A_1 = \begin{bmatrix} 3.2 & 1.3 \\ 0.9 & 3.1 \end{bmatrix}, A_2 = \begin{bmatrix} 5.2 & 2.8 \\ -3.1 & 5.3 \end{bmatrix} \\ B_0 &= \begin{bmatrix} 0.5 & -0.2 \\ 0.3 & 1.0 \end{bmatrix}, B_1 = \begin{bmatrix} 1.1 & -0.4 \\ 0.3 & 1.0 \end{bmatrix}, B_2 = \begin{bmatrix} 2.1 & -1.6 \\ 0.7 & 2.5 \end{bmatrix} \\ C_0 &= \begin{bmatrix} 12.2 & 10.6 \\ 0.6 & 7.4 \end{bmatrix}, C_1 = \begin{bmatrix} 25.6 & 21.4 \\ 1.2 & 15.1 \end{bmatrix}, C_2 = \begin{bmatrix} 37.4 & 30.2 \\ 1.6 & 24.4 \end{bmatrix}\end{aligned}$$

*Set the corresponding parameters as follows:*

$$\theta_0 = 0.22, \alpha_0 = 1, \beta_0 = 1$$

$$\theta_1 = 0.44, \alpha_1 = 1, \beta_1 = 0$$

$$\theta_2 = 0.44, \alpha_2 = 0, \beta_2 = 1$$

$$\varepsilon = 0.0000001$$

*By applying the iterative algorithm given in Algorithm 1 with* $X_0(0) = X_1(0) = X_2(0) = 10^{-6}\mathbf{1}(2)$*, we can compute the sequences* $X_0(k)$ *,* $X_1(k)$ *and* $X_2(k)$ *and finally obtain the convergent solution as*

$$X_0^* = \begin{bmatrix} 2.2792084 & 2.1443643 \\ -0.0053164232 & 2.8578095 \end{bmatrix}$$

$$X_1^* = \begin{bmatrix} 3.8959794 & 3.0173837 \\ 0.91457065 & 4.3687527 \end{bmatrix}$$

$$X_2^* = \begin{bmatrix} 3.7974423 & 2.1832934 \\ 1.8076325 & 3.3274102 \end{bmatrix}$$

*In order to demonstrate the convergent effectiveness,we define the relative iteration error as*

$$\delta(k) = \sqrt{\frac{\sum_{t=0}^{T-1} \|X_t(k) - X_t^*\|^2}{\sum_{t=0}^{T-1} \|X_t^*\|^2}}.$$

*The varying trajectory of relative iteration error with the time is shown in 1. It is cleared that* $\delta(k)$ *decreases quickly and converges to zero as* $k$ *increases.*

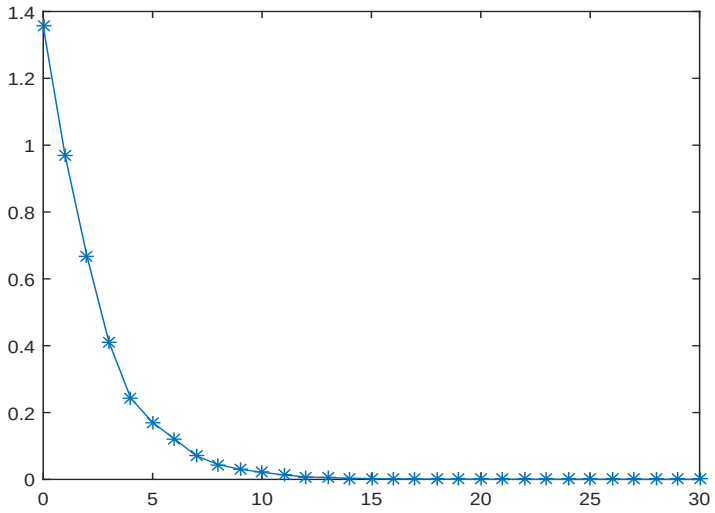



Figure 1: The changed trend of relative error

# 4 Conclusions

In this paper, we introduce a new iterative algorithm to solve a kind of periodic Sylvester matrix equation. The iterative algorithm is proven to converge the exact solutions in finite iteration steps without round-off errors. Finally, we give a numerical example to check the convergence and performance of the iterative algorithm.

# References


[1] Caswell H, Shyu E. Sensitivity analysis of periodic matrix population models.[J]. Theoretical Population Biology, 2012, 82(4):329.

[2] Chen T, Francis B A. Optimal Sampled-Data Control Systems[M]. Springer-Verlag New York, Inc. 1995.

[3] Chen X. Solving the (generalized) periodic Sylvester equation with the matrix sign function[J]. Mathematica Numerica Sinica, 2012, 34(2):153-162.

[4] Dehghan M, Hajarian M. An iterative algorithm for the reflexive solutions of the generalized coupled Sylvester matrix equations and its optimal approximation[J]. Applied Mathematics & Computation, 2008, 202(2):571-588.

[5] Dehghan M, Hajarian M. Two algorithms for finding the Hermitian reflexive and skew-Hermitian solutions of Sylvester matrix equations[J]. Applied Mathematics Letters, 2011, 24(4):444-449.

[6] Gu C, Jiang X. Improved gradient iteration algorithm for solving Sylvester matrix equations[J]. Communication on Applied Mathematics and Computation, 2014, 28(4): 432-439.

[7] Hoskins W D, Meek D S, Walton D J. The numerical solution of the matrix equation $XA + AY = F$[J]. Bit Numerical Mathematics, 1977, 17(2):184-190.

[8] Korotyaev E, Kutsenko A. Lyapunov functions for periodic matrix-valued Jacobi operators[J]. Mathematics, 2007.

[9] Kucera V. The matrix equation $AX + XB = C$[J]. Siam Journal on Applied Mathematics, 1974, 26(1):15-25.

[10] Lv L, Zhang L. On the periodic Sylvester equations and their applications in periodic Luenberger observers design[J]. Journal of the Franklin Institute, 2016, 353(5): 1005-1018.

[11] Lv L L, Zhang L. Parametric solutions to the discrete periodic regul ator equations[J]. Journal of the Franklin Institute, 2016, 353(5): 1089-1101.

[12] Sreedhar J, Van Dooren P. Periodic Schur forms and some matrix equations[C]// 1994.

[13] Verstraete F, Porras D, Cirac J I. Density matrix renormalization group and periodic boundary conditions: a quantum information perspective.[J]. Physical Review Letters, 2004, 93(22):227205.

[14] Xie L,Ding J,Ding F.Gradient based iterative solutions for general linear matrix equations. Computers Mathe-matics with Applications. 2009

[15] Yin F, Guo K, Huang G X. An iterative algorithm for the generalized reflexive solutions of the general coupled matrix equations[J]. Journal of Inequalities and Applications, 2013, 2013(1):952974-952974.

[16] Zhang H. A finite iterative algorithm for solving the complex generalized coupled Sylvester matrix equations by using the linear operators[J]. Journal of the Franklin Institute, 2016, http://dx.doi.org/10.1016/j.jfranklin.2016.12.011.

[17] Zhang L, Science F O. A Gradient Iterative Algorithm for Solving the Coupled Sylvester Matrix Equations[J]. Value Engineering, 2014.

[18] Zhou B. On asymptotic stability of linear time-varying systems[J]. Automatica, 2016, 68: 266-276.

[19] Zhou B, Egorov A V. Razumikhin and Krasovskii stability theorems for time-varying time-delay systems[J]. Automatica, 2016, 71: 281-291.

# FOURIER SERIES OF SUMS OF PRODUCTS OF EULER AND GENOCCHI FUNCTIONS AND THEIR APPLICATIONS

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Abstract. We study three types of sums of products of Euler and Genocchi functions and derive Fourier series expansions for them. Further, we will be able to express each of those functions in terms of Bernoulli functions.

## 1. Introduction

The *Genocchi polynomials* $G_m(x)$ are given by the generating function

$$\frac{2t}{e^t+1}e^{xt}=\sum_{m=0}^{\infty}G_m(x)\frac{t^m}{m!},\ (\text{see [1-5, 7, 11]}).$$

The first few Genocchi polynomials are as follows:

$$G_0(x)=0,\ G_1(x)=1,\ G_2(x)=2x-1,$$
$$G_3(x)=3x^2-3x,\ G_4(x)=4x^3-6x^2+1,$$
$$G_5(x)=5x^4-10x^3+5x,\ G_6(x)=6x^5-15x^4+15x^2-3,$$
$$G_7(x)=7x^6-21x^5+35x^3-21x.$$

The *Euler polynomials* $E_m(x)$ are defined by the generating function

$$\frac{2}{e^t+1}e^{xt}=\sum_{m=0}^{\infty}E_m(x)\frac{t^m}{m!},\ (\text{see [2-4, 6-8, 10]}).$$

When $x=0$, $E_m(0)=E_m$ are called the *Euler numbers*.

From the relation $G_m(x)=mE_{m-1}(x)$ $(m\geq 1)$, we have

$$\deg G_m(x)=m-1\ (m\geq 1),\ G_m=mE_{m-1}\ (m\geq 1),$$
$$G_0=0,\ G_1=1,\ G_{2m+1}=0\ (m\geq 1),\ \text{and}\ G_{2m}\neq 0\ (m\geq 1).$$

Moreover,

$$\frac{d}{dx}G_m(x)=mG_{m-1}(x),\ (m\geq 1),$$
$$G_m(x+1)+G_m(x)=2mx^{m-1},\ (m\geq 0).$$

From these, we have

$$G_m(1)+G_m(0)=2\delta_{m,1},\ (m\geq 0),$$

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* Corresponding author.

$$\begin{aligned}\int_0^1 G_m(x)dx =& \frac{1}{m+1}\left(G_{m+1}(1)-G_{m+1}(0)\right)\\ =& \frac{2}{m+1}\left(-G_{m+1}(0)+\delta_{m,0}\right)\\ =& \begin{cases} 0, & \text{for } m \text{ even},\\ -\dfrac{2}{m+1}G_{m+1}, & \text{for } m \text{ odd}.\end{cases}\end{aligned}$$

For any real number $x$, let

$$< x >= x - \lfloor x \rfloor \in [0,1)$$

denote the fractional part of $x$.

Let $B_m(x)$ denote the Bernoulli polynomials given by $\frac{t}{e^t-1}e^{tx} = \sum_{m=0}^{\infty} B_m(x)\frac{t^m}{m!}$. Then we recall the following about Bernoulli functions $B_m(< x >)$:

(a) for $m \geq 2$,

$$B_m(< x >) = -m! \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^{2\pi inx}}{(2\pi in)^m},$$

(b) for $m = 1$,

$$-\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^{2\pi inx}}{2\pi in} = \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z},\\ 0, & \text{for } x \in \mathbb{Z}.\end{cases}$$

Fourier series expansion of higher-order Bernoulli functions were treated in the recent paper [9]. Here we will study three types of sums of products of Euler and Genocchi functions and derive Fourier series expansions for them. Further, we will be able to express each of those functions in terms of Bernoulli functions.

## 2. Sums of products of Euler and Genocchi functions of the first type

Let

$$\alpha_m(x) = \sum_{k=0}^{m-1} E_k(x)G_{m-k}(x), \ (m \geq 2).$$

Note that $\deg \alpha_m(x) = m-1$. Then we will consider the function

$$\alpha_m(< x >) = \sum_{k=0}^{m-1} E_k(< x >)G_{m-k}(< x >), \ (m \geq 2)$$

defined on $(-\infty, \ \infty)$, which is periodic with period 1.

The Fourier series of $\alpha_m(< x >)$ is

$$\sum_{n=-\infty}^{\infty} A_n^{(m)} e^{2\pi inx},$$

where

$$\begin{aligned} A_n^{(m)} &= \int_0^1 \alpha_m(< x >) e^{-2\pi i n x} dx \\ &= \int_0^1 \alpha_m(x) e^{-2\pi i n x} dx. \end{aligned}$$

To proceed further, we need the following

$$\begin{aligned} \alpha'(x) &= \sum_{k=0}^{m-1} (kE_{k-1}(x)G_{m-k}(x) + (m-k)E_k(x)G_{m-k-1}(x)) \\ &= \sum_{k=1}^{m-1} kE_{k-1}(x)G_{m-k}(x) + \sum_{k=0}^{m-2} (m-k)E_k(x)G_{m-k-1}(x) \\ &= \sum_{k=0}^{m-2} (k+1)E_k(x)G_{m-1-k}(x) + \sum_{k=0}^{m-2} (m-k)E_k(x)G_{m-1-k}(x) \\ &= (m+1) \sum_{k=0}^{m-2} E_k(x)G_{m-1-k}(x) \\ &= (m+1)\alpha_{m-1}(x). \end{aligned}$$

So, $\alpha'_m(x) = (m+1)\alpha_{m-1}(x)$. From this,

$$\begin{aligned} \left(\frac{\alpha_{m+1}(x)}{m+2}\right)' &= \alpha_m(x), \\ \int_0^1 \alpha_m(x)dx &= \frac{1}{m+2}\left(\alpha_{m+1}(1) - \alpha_{m+1}(0)\right), \end{aligned}$$

and

$$\begin{aligned} \alpha_m(1) - \alpha_m(0) &= \sum_{k=0}^{m-1} (E_k(1)G_{m-k}(1) - E_k G_{m-k}) \\ &= \sum_{k=0}^{m-1} \left((-E_k + 2\delta_{0,k})\left(-G_{m-k} + 2\delta_{m-1,k}\right) - E_k G_{m-k}\right) \\ &= \sum_{k=0}^{m-1} (-2E_k\delta_{m-1,k} - 2\delta_{0,k}G_{m-k} + 4\delta_{k,0}\delta_{m-1,k}) \\ &= -2E_{m-1} - 2G_m + 4\delta_{m-1,0} \\ &= -2\left(E_{m-1} + G_m\right) \\ &= -2(m+1)E_{m-1}. \end{aligned}$$

Recall that

$$E_{2n} = 0 \ (n \geq 1), E_{2n-1} \neq 0 \ (n \geq 1), \text{ and } E_0 = 1.$$

So

$$\alpha_m(0) = \alpha(1) \ \Leftrightarrow m = 2n+1 \ (n \geq 1).$$

Also,

$$\begin{aligned}\int_0^1 \alpha_m(x)dx =& \frac{1}{m+2}\left(-2(m+2)E_m\right)\\ =& -2E_m.\end{aligned}$$

Now, we are going to determine the Fourier coefficients $A_n^{(m)}$.
**Case 1:** $n \neq 0$**.**

$$\begin{aligned}A_n^{(m)} =& \int_0^1 \alpha_m(x)e^{-2\pi inx}dx\\ =& -\frac{1}{2\pi in}\left[\alpha_m(x)e^{-2\pi inx}\right]_0^1 + \frac{1}{2\pi in}\int_0^1 \alpha_m'(x)e^{-2\pi inx}dx\\ =& -\frac{1}{2\pi in}(\alpha_m(1)-\alpha_m(0)) + \frac{m+1}{2\pi in}\int_0^1 \alpha_{m-1}(x)e^{-2\pi inx}dx\\ =& \frac{m+1}{2\pi in}A_n^{(m-1)} + \frac{2(m+1)}{2\pi in}E_{m-1}\\ =& \frac{m+1}{2\pi in}\left(\frac{m}{2\pi in}A_n^{(m-2)} + \frac{2m}{2\pi in}E_{m-2}\right) + \frac{2(m+1)}{2\pi in}E_{m-1}\\ =& \frac{(m+1)_2}{(2\pi in)^2}A_n^{(m-2)} + \sum_{k=1}^{2}\frac{2(m+1)_k}{(2\pi in)^k}E_{m-k}\\ =& \cdots\\ =& \frac{(m+1)_{m-1}}{(2\pi in)^{m-1}}A_n^{(1)} + \sum_{k=1}^{m-1}\frac{2(m+1)_k}{(2\pi in)^k}E_{m-k}\\ =& \sum_{k=1}^{m-1}\frac{2(m+1)_k}{(2\pi in)^k}E_{m-k},\end{aligned}$$

where $A_n^{(1)} = \int_0^1 e^{-2\pi inx}dx = 0$.

**Case 2:** $n = 0$**.**

$$A_0^{(m)} = \int_0^1 \alpha_m(x)dx = -2E_m.$$

$\alpha_m(< x >),\ (m \geq 2)$ is piecewise $C^\infty$. Moreover, $\alpha_m(< x >)$ is continuous for all odd integers $\geq 3$ and is discontinuous with jump discontinuities at integers for all even integers $\geq 2$.

Assume the first that $m$ is an odd integer $\geq 3$ . Then $\alpha_m(0) = \alpha_m(1)$. $\alpha_m(< x >)$ is piecewise $C^\infty$, and continuous. So the Fourier series of $\alpha_m(< x >)$ converges

uniformly to $\alpha_m(< x >)$, and

$$\begin{aligned}\alpha_m(< x >) =& -2E_m + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left( \sum_{k=1}^{m-1} \frac{2(m+1)_k}{(2\pi i n)^k} E_{m-k} \right) e^{2\pi i n x} \\ =& -2E_m - 2\sum_{k=1}^{m-1} \binom{m+1}{k} E_{m-k} \left( -k! \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^k} \right) \\ =& -2E_m - 2\sum_{k=2}^{m-1} \binom{m+1}{k} E_{m-k} B_k(< x >) - 2(m+1)E_{m-1} \\ & \times \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}\end{aligned}$$

Now, we can state our first theorem.

**Theorem 2.1.** *Let $m$ be an odd integer $\geq 3$. Then we have the following.*

(i) $\sum_{k=0}^{m-1} E_k(< x >) G_{m-k}(< x >)$ *has the Fourier series expansion*

$$\begin{aligned}&\sum_{k=0}^{m-1} E_k(< x >) G_{m-k}(< x >) \\ &= -2E_m + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left( \sum_{k=1}^{m-1} \frac{2(m+1)_k}{(2\pi i n)^k} E_{m-k} \right) e^{2\pi i n x},\end{aligned}$$

*for all $x \in (-\infty, \infty)$. Here the convergence is uniform.*

(ii)

$$\begin{aligned}&\sum_{k=0}^{m-1} E_k(< x >) G_{m-k}(< x >) \\ &= -2E_m - 2\sum_{k=2}^{m-1} \binom{m+1}{k} E_{m-k} B_k(< x >).\end{aligned}$$

*Here $B_k(< x >)$ is the Bernoulli function.*

Assume next that $m$ is an even integer $\geq 2$. Then $\alpha_m(0) \neq \alpha_m(1)$. Hence $\alpha_m(< x >)$ is piecewise $C^\infty$, and discontinuous with jump discontinuities at integers. The Fourier series of $\alpha_m(< x >)$ converges pointwise to $\alpha_m(< x >)$, for $x \notin \mathbb{Z}$, and converges to

$$\begin{aligned}&\frac{1}{2}\left(\alpha_m(0) + \alpha_m(1)\right) \\ =&\alpha_m(0) - (m+1)E_{m-1} \\ =&\sum_{k=0}^{m-1} E_k G_{m-k} - (m+1)E_{m-1},\end{aligned}$$

for $x \in \mathbb{Z}$. Hence we have the following theorem.

**Theorem 2.2.** *Let $m$ be an even integer $\geq 2$. Then we have the following.*

(i)

$$-2E_m+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(\sum_{k=1}^{m-1}\frac{2(m+1)_k}{(2\pi in)^k}E_{m-k}\right)e^{2\pi inx}$$
$$=\begin{cases}\sum_{k=0}^{m-1}E_k(<x>)G_{m-k}(<x>), & \text{for } x\notin\mathbb{Z},\\ \sum_{k=0}^{m-1}E_kG_{m-k}-(m+1)E_{m-1}, & \text{for } x\in\mathbb{Z}.\end{cases}$$

*Here the convergence is pointwise.*

(ii)

$$-2E_m-2\sum_{k=1}^{m-1}\binom{m+1}{k}E_{m-k}B_k(<x>)$$
$$=\sum_{k=0}^{m-1}E_k(<x>)G_{m-k}(<x>),\ \text{for } x\notin\mathbb{Z};$$

$$-2E_m-2\sum_{k=2}^{m-1}\binom{m+1}{k}E_{m-k}B_k(<x>)$$
$$=\sum_{k=0}^{m-1}E_kG_{m-k}-(m+1)E_{m-1},\ \text{for } x\in\mathbb{Z}.$$

*Here $B_k(<x>)$ is the Bernoulli function.*

## 3. Sums of products of Euler and Genocchi functions of the second type

Let

$$\beta_m(x)=\sum_{k=0}^{m-1}\frac{1}{k!(m-k)!}E_k(x)G_{m-k}(x),\ (m\geq 2).$$

Then we will consider the function

$$\beta_m(<x>)=\sum_{k=0}^{m-1}\frac{1}{k!(m-k)!}E_k(<x>)G_{m-k}(<x>),\ (m\geq 2)$$

defined on $(-\infty,\infty)$, which is periodic with period 1.

The Fourier series of $\beta_m(<x>)$ is

$$\sum_{n=-\infty}^{\infty}B_n^{(m)}e^{2\pi inx},$$

where

$$B_n^{(m)}=\int_0^1\beta_m(<x>)e^{-2\pi inx}dx$$
$$=\int_0^1\beta_m(x)e^{-2\pi inx}dx.$$

To proceed further, we need to observe the following.

$$\begin{aligned}
\beta'_m(x) =& \sum_{k=0}^{m-1} \left\{ \frac{k}{k!(m-k)!} E_{k-1}(x) G_{m-k}(x) + \frac{m-k}{k!(m-k)!} E_k(x) G_{m-k-1}(x) \right\} \\
=& \sum_{k=1}^{m-1} \frac{1}{(k-1)!(m-k)!} E_{k-1}(x) G_{m-k}(x) + \sum_{k=0}^{m-2} \frac{1}{k!(m-k-1)!} E_k(x) G_{m-k-1}(x) \\
=& \sum_{k=0}^{m-2} \frac{1}{k!(m-1-k)!} E_k(x) G_{m-1-k}(x) + \sum_{k=0}^{m-2} \frac{1}{k!(m-1-k)!} E_k(x) G_{m-1-k}(x) \\
=& 2 \sum_{k=0}^{m-2} \frac{1}{k!(m-1-k)!} E_k(x) G_{m-1-k}(x) \\
=& 2\beta_{m-1}(x).
\end{aligned}$$

So $\beta'_m(x) = 2\beta_{m-1}(x)$, and from this we obtain

$$\left( \frac{\beta_{m+1}(x)}{2} \right)' = \beta_m(x).$$

Thus

$$\int_0^1 \beta_m(x) dx = \frac{1}{2} \left( \beta_{m+1}(1) - \beta_{m+1}(0) \right),$$

and

$$\begin{aligned}
& \beta_m(1) - \beta_m(0) \\
=& \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} \left( E_k(1) G_{m-k}(1) - E_k G_{m-k} \right) \\
=& \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} \left( (-E_k + 2\delta_{k,0})(-G_{m-k} + 2\delta_{m-1,k}) - E_k G_{m-k} \right) \\
=& \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} \left( -2E_k \delta_{m-1,k} - 2\delta_{k,0} G_{m-k} + 4\delta_{k,0}\delta_{m-1,k} \right) \\
=& -\frac{2E_{m-1}}{(m-1)!} - \frac{2G_m}{m!} + \frac{4\delta_{m-1,0}}{m!} \\
=& -\frac{2E_{m-1}}{(m-1)!} - \frac{2mE_{m-1}}{m!} \\
=& -\frac{4}{(m-1)!} E_{m-1}.
\end{aligned}$$

So,

$$\begin{aligned}
\beta_m(0) = \beta_m(1) &\iff E_{m-1} = 0 \\
&\iff m = 2n+1 \ (n \geq 1).
\end{aligned}$$

Also,

$$\begin{aligned}
\int_0^1 \beta_m(x) dx =& \frac{1}{2} \left( \beta_{m+1}(1) - \beta_{m+1}(0) \right) \\
=& -\frac{2}{m!} E_m.
\end{aligned}$$

Now, we are ready to determine the Fourier coefficients $B_n^{(m)}$.

**Case** 1 : $n \neq 0$.

$$\begin{aligned}
B_n^{(m)} =& \int_0^1 \beta_m(x) e^{-2\pi inx} dx \\
=& -\frac{1}{2\pi in}\left[\beta_m(x)e^{-2\pi inx}\right]_0^1 + \frac{1}{2\pi in}\int_0^1 \beta_m'(x)e^{-2\pi inx}dx \\
=& -\frac{1}{2\pi in}\left(\beta_m(1)-\beta_m(0)\right) + \frac{2}{2\pi in}\int_0^1 \beta_{m-1}(x)e^{-2\pi inx}dx \\
=& \frac{2}{2\pi in}B_n^{(m-1)} + \frac{4}{2\pi in(m-1)!}E_{m-1} \\
=& \frac{2}{2\pi in}\left(\frac{2}{2\pi in}B_n^{(m-2)} + \frac{4}{2\pi in(m-2)!}E_{m-2}\right) + \frac{4}{2\pi in(m-1)!}E_{m-1} \\
=& \left(\frac{2}{2\pi in}\right)^2 B_n^{(m-2)} + \sum_{k=1}^{2}\frac{2^{k+1}}{(2\pi in)^k(m-k)!}E_{m-k} \\
=& \cdots \\
=& \left(\frac{2}{2\pi in}\right)^{m-1} B_n^{(1)} + \sum_{k=1}^{m-1}\frac{2^{k+1}}{(2\pi in)^k(m-k)!}E_{m-k} \\
=& \sum_{k=1}^{m-1}\frac{2^{k+1}}{(2\pi in)^k(m-k)!}E_{m-k},
\end{aligned}$$

where $B_n^{(1)} = \int_0^1 e^{-2\pi inx}dx = 0$.

**Case** 2 : $n = 0$.

$$B_0^{(m)} = \int_0^1 \beta_m(x)dx = -\frac{2}{m!}E_m.$$

$\beta_m(< x >)$, $(m \geq 2)$ is piecewise $C^\infty$. Moreover, $\beta_m(< x >)$ is continuous for all odd integers $\geq 3$ and discontinuous with jump discontinuities at integers for all even integers $\geq 2$.

Assume first that $m$ is an odd integer $\geq 3$. Then $\beta_m(0) = \beta_m(1)$. $\beta_m(< x >)$ is piecewise $C^\infty$, and continuous. So the Fourier series of $\beta_m(< x >)$ converges uniformly to $\beta_m(< x >)$, and

$$\begin{aligned}
\beta_m(< x >) =& -\frac{2}{m!}E_m + 2\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\left(\sum_{k=1}^{m-1}\frac{2^k}{(2\pi in)^k(m-k)!}E_{m-k}\right)e^{2\pi inx} \\
=& -\frac{2}{m!}E_m - \frac{2}{m!}\sum_{k=1}^{m-1}2^k\binom{m}{k}E_{m-k}\left(-k!\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^k}\right) \\
=& -\frac{2}{m!}E_m - \frac{2}{m!}\sum_{k=2}^{m-1}2^k\binom{m}{k}E_{m-k}B_k(< x >) \\
& -\frac{4}{(m-1)!}E_{m-1} \times \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}
\end{aligned}$$

We can now state our first theorem.

**Theorem 3.1.** *Let $m$ be an odd integer $\geq 3$. Then we have the following.*

(i)

$$\sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} E_k(< x >) G_{m-k}(< x >)$$

*has the Fourier series expansion*

$$\begin{aligned} &\sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} E_k(< x >) G_{m-k}(< x >) \\ &= -\frac{2}{m!} E_m + 2 \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left( \sum_{k=1}^{m-1} \frac{2^k}{(2\pi i n)^k (m-k)!} E_{m-k} \right) e^{2\pi i n x} \end{aligned}$$

*for all $x \in (-\infty, \infty)$. Here the convergence is uniform.*

(ii)

$$\begin{aligned} &\sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} E_k(< x >) G_{m-k}(< x >) \\ &= -\frac{2}{m!} E_m - \frac{2}{m!} \sum_{k=2}^{m-1} 2^k \binom{m}{k} E_{m-k} B_k(< x >) \end{aligned}$$

*Here $B_k(< x >)$ is the Bernoulli function.*

Assume next that $m$ is an even integer $\geq 2$. Then $\beta_m(0) \neq \beta_m(1)$. Hence $\beta_m(< x >)$ is piecewise $C^\infty$, and discontinuous with jump discontinuities at integers. The Fourier series of $\beta_m(< x >)$ converges pointwise to $\beta_m(< x >)$, for $x \notin \mathbb{Z}$, and converges to

$$\begin{aligned} &\frac{1}{2}\left(\beta_m(0) + \beta_m(1)\right) = \beta_m(0) - \frac{2}{(m-1)!} E_{m-1} \\ &= \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} E_k G_{m-k} - \frac{2}{(m-1)!} E_{m-1}, \end{aligned}$$

for $x \in \mathbb{Z}$.

Now, we can state our second theorem.

**Theorem 3.2.** *Let $m$ be an even integer $\geq 2$. Then we have the following.*

(i)

$$\begin{aligned} &-\frac{2}{m!} E_m + 2 \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left( \sum_{k=1}^{m-1} \frac{2^k}{(2\pi i n)^k (m-k)!} E_{m-k} \right) e^{2\pi i n x} \\ &= \begin{cases} \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} E_k(x) G_{m-k}(x), & \text{for } x \notin \mathbb{Z}, \\ \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} E_k G_{m-k} - \frac{2}{(m-1)!} E_{m-1}, & \text{for } x \in \mathbb{Z}. \end{cases} \end{aligned}$$

*Here the convergence is pointwise.*

(ii)

$$\begin{aligned} &-\frac{2}{m!}E_m - \frac{2}{m!}\sum_{k=1}^{m-1} 2^k \binom{m}{k} E_{m-k}B_k(< x >) \\ &= \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} E_k(< x >)G_{m-k}(< x >), \ \textit{for } x \notin \mathbb{Z}; \end{aligned}$$

$$\begin{aligned} &-\frac{2}{m!}E_m - \frac{2}{m!}\sum_{k=2}^{m-1} 2^k \binom{m}{k} E_{m-k}B_k(< x >) \\ &= \sum_{k=0}^{m-1} \frac{1}{k!(m-k)!} E_k G_{m-k} - \frac{2}{(m-1)!}E_{m-1}, \ \textit{for } x \in \mathbb{Z}. \end{aligned}$$

*Here $B_k(< x >)$ is the Bernoulli function.*

## 4. Sums of products of Euler and Genocchi functions of the third type

Let $\gamma_m(x) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} E_k(x)G_{m-k}(x)$, $(m \geq 3)$. Then we will consider the function

$$\gamma_m(< x >) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} E_k(< x >)G_{m-k}(< x >)$$

defined on $(-\infty, \infty)$, which is periodic with period 1.

The Fourier series of $\gamma_m(< x >)$ is

$$\sum_{n=-\infty}^{\infty} C_n^{(m)} e^{2\pi i n x},$$

where

$$\begin{aligned} C_n^{(m)} &= \int_0^1 \gamma_m(< x >)e^{-2\pi i n x} dx \\ &= \int_0^1 \gamma_m(x)e^{-2\pi i n x} dx. \end{aligned}$$

To proceed further, we need to note the following.

$$\begin{aligned}
\gamma_m'(x) &= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \{kE_{k-1}(x)G_{m-k}(x) + (m-k)E_k(x)G_{m-k-1}(x)\} \\
&= \sum_{k=1}^{m-1} \frac{1}{m-k} E_{k-1}(x)G_{m-k}(x) + \sum_{k=1}^{m-2} \frac{1}{k} E_k(x)G_{m-k-1}(x) \\
&= \sum_{k=0}^{m-2} \frac{1}{m-1-k} E_k(x)G_{m-1-k}(x) + \sum_{k=1}^{m-2} \frac{1}{k} E_k(x)G_{m-1-k}(x) \\
&= \frac{1}{m-1} G_{m-1}(x) + \sum_{k=1}^{m-2} \frac{1}{m-1-k} E_k(x)G_{m-1-k}(x) + \sum_{k=1}^{m-2} \frac{1}{k} E_k(x)G_{m-1-k}(x) \\
&= \frac{1}{m-1} G_{m-1}(x) + (m-1) \sum_{k=1}^{m-2} \frac{1}{k(m-1-k)} E_k(x)G_{m-1-k}(x) \\
&= \frac{1}{m-1} G_{m-1}(x) + (m-1)\gamma_{m-1}(x).
\end{aligned}$$

So $\gamma_m'(x) = \frac{1}{m-1} G_{m-1}(x) + (m-1)\gamma_{m-1}(x)$, and from this, we have

$$\left( \frac{1}{m} \left( \gamma_{m+1}(x) - \frac{1}{m(m+1)} G_{m+1}(x) \right) \right)' = \gamma_m(x).$$

Since $G_{m+1}(1) + G_{m+1}(0) = 2\delta_{m,0}$,

$$\begin{aligned}
\int_0^1 \gamma_m(x)dx &= \frac{1}{m} \left[ \gamma_{m+1}(x) - \frac{1}{m(m+1)} G_{m+1}(x) \right]_0^1 \\
&= \frac{1}{m} \left( \gamma_{m+1}(1) - \gamma_{m+1}(0) - \frac{1}{m(m+1)} \left( G_{m+1}(1) - G_{m+1}(0) \right) \right) \\
&= \frac{1}{m} \left( \gamma_{m+1}(1) - \gamma_{m+1}(0) + \frac{2}{m(m+1)} G_{m+1} \right),
\end{aligned}$$

and

$$\begin{aligned}
\gamma_m(1) - \gamma_m(0) &= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \left( E_k(1)G_{m-k}(1) - E_k G_{m-k} \right) \\
&= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \left( (-E_k + 2\delta_{k,0}) \left( -G_{m-k} + 2\delta_{m-1,k} \right) - E_k G_{m-k} \right) \\
&= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \left( -2E_k \delta_{m-1,k} \right) \\
&= -\frac{2E_{m-1}}{m-1}.
\end{aligned}$$

So

$$\begin{aligned}
\gamma_m(0) = \gamma_m(1) &\iff E_{m-1} = 0 \\
&\iff m = 2n+1, \ (n \geq 1).
\end{aligned}$$

Also,

$$\begin{aligned}\int_0^1 \gamma_m(x)dx =& \frac{1}{m}\left(-\frac{2E_m}{m}+\frac{2}{m(m+1)}G_{m+1}\right)\\ =& \frac{1}{m}\left(-\frac{2E_m}{m}+\frac{2E_m}{m}\right)\\ =& 0.\end{aligned}$$

Now, we are going to determine the Fourier coefficients $C_n^{(m)}$.
**Case 1** : $n \neq 0$.

$$\begin{aligned}C_n^{(m)} &= \int_0^1 \gamma_m(x)e^{-2\pi inx}dx\\ &= -\frac{1}{2\pi in}\left[\gamma_m(x)e^{-2\pi inx}\right]_0^1 + \frac{1}{2\pi in}\int_0^1 \gamma_m'(x)e^{-2\pi inx}dx\\ &= -\frac{1}{2\pi in}(\gamma_m(1)-\gamma_m(0)) + \frac{1}{2\pi in}\int_0^1 \left\{\frac{1}{m-1}G_{m-1}(x)+(m-1)\gamma_{m-1}(x)\right\}e^{-2\pi inx}dx\\ &= -\frac{1}{2\pi in}(\gamma_m(1)-\gamma_m(0)) + \frac{1}{2\pi in(m-1)}\int_0^1 G_{m-1}(x)e^{-2\pi inx}dx\\ &\quad + \frac{m-1}{2\pi in}\int_0^1 \gamma_{m-1}(x)e^{-2\pi inx}dx\\ &= \frac{2E_{m-1}}{2\pi in(m-1)} + \frac{2}{2\pi in(m-1)}\Phi_m + \frac{m-1}{2\pi in}C_n^{(m-1)},\end{aligned}$$

where $\Phi_m = \sum_{k=1}^{m-2}\frac{(m-1)_{k-1}G_{m-k}}{(2\pi in)^k}$, and, for $l \geq 2$,

$$\int_0^1 G_l(x)e^{-2\pi inx}dx = \begin{cases} 2\sum_{k=1}^{l-1}\frac{(l)_{k-1}G_{l-k+1}}{(2\pi in)^k}, & \text{for } n \neq 0,\\ -\frac{2G_{l+1}}{l+1}, & \text{for } n = 0.\end{cases}$$

Continuing our argument, we have

$$\begin{aligned}C_n^{(m)} &= \frac{m-1}{2\pi in}C_n^{(m-1)} + \frac{2E_{m-1}}{2\pi in(m-1)} + \frac{2}{2\pi in(m-1)}\Phi_m\\ &= \frac{m-1}{2\pi in}\left(\frac{m-2}{2\pi in}C_n^{(m-2)} + \frac{2E_{m-2}}{2\pi in(m-2)} + \frac{2}{2\pi in(m-2)}\Phi_{m-1}\right)\\ &\quad + \frac{2E_{m-1}}{2\pi in(m-1)} + \frac{2}{2\pi in(m-1)}\Phi_m\\ &= \frac{(m-1)_2}{(2\pi in)^2}C_n^{(m-2)} + \sum_{j=1}^{2}\frac{2(m-1)_{j-1}}{(2\pi in)^j(m-j)}E_{m-j} + \sum_{j=1}^{2}\frac{2(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Phi_{m-j+1}\\ &= \cdots\\ &= \frac{(m-1)!}{(2\pi in)^{m-2}}C_n^{(2)} + \sum_{j=1}^{m-2}\frac{2(m-1)_{j-1}}{(2\pi in)^j(m-j)}E_{m-j} + \sum_{j=1}^{m-2}\frac{2(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Phi_{m-j+1}\\ &= -\frac{(m-1)!}{(2\pi in)^{m-1}} + \sum_{j=1}^{m-2}\frac{2(m-1)_{j-1}}{(2\pi in)^j(m-j)}E_{m-j} + \sum_{j=1}^{m-2}\frac{2(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Phi_{m-j+1},\end{aligned}$$

where $C_n^{(2)} = \int_0^1 (x - \frac{1}{2}) e^{-2\pi i n x} dx = -\frac{1}{2\pi i n}$.
Here we note that

$$\begin{aligned}
&\sum_{j=1}^{m-2} \frac{2(m-1)_{j-1}}{(2\pi i n)^j (m-j)} \Phi_{m-j+1} \\
=&\sum_{j=1}^{m-2} \frac{2(m-1)_{j-1}}{(2\pi i n)^j (m-j)} \sum_{k=1}^{m-j-1} \frac{(m-j)_{k-1} G_{m-j-k+1}}{(2\pi i n)^k} \\
=&\sum_{j=1}^{m-2} \sum_{k=1}^{m-j-1} \frac{2(m-1)_{j+k-2}}{(2\pi i n)^{j+k} (m-j)} G_{m-j-k+1} \\
=&\frac{2}{m} \sum_{j=1}^{m-2} \sum_{s=j+1}^{m-1} \frac{(m)_{s-1}}{(2\pi i n)^s (m-j)} G_{m-s+1} \\
=&\frac{2}{m} \sum_{s=2}^{m-1} \frac{(m)_{s-1}}{(2\pi i n)^s} G_{m-s+1} \sum_{j=1}^{s-1} \frac{1}{m-j} \\
=&\frac{2}{m} \sum_{s=1}^{m-1} \frac{(m)_s}{(2\pi i n)^s} \frac{G_{m-s+1}}{m-s+1} (H_{m-1} - H_{m-s}).
\end{aligned}$$

Thus

$$\begin{aligned}
C_n^{(m)} =& -\frac{(m-1)!}{(2\pi i n)^{m-1}} + \frac{2}{m} \sum_{s=1}^{m-2} \frac{(m)_s}{(2\pi i n)^s (m-s)} E_{m-s} \\
&+ \frac{2}{m} \sum_{s=1}^{m-1} \frac{(m)_s}{(2\pi i n)^s} \frac{G_{m-s+1}}{m-s+1} (H_{m-1} - H_{m-s}) \\
=& -\frac{(m-1)!}{(2\pi i n)^{m-1}} + \frac{2}{m} \sum_{s=1}^{m-2} \frac{(m)_s}{(2\pi i n)^s (m-s)} E_{m-s} \\
&+ \frac{2}{m} \sum_{s=1}^{m-2} \frac{(m)_s}{(2\pi i n)^s} \frac{G_{m-s+1}}{m-s+1} (H_{m-1} - H_{m-s}) - \frac{(m-1)!}{(2\pi i n)^{m-1}} (H_{m-1} - 1) \\
=& -\frac{(m-1)!}{(2\pi i n)^{m-1}} H_{m-1} + \frac{2}{m} \sum_{s=1}^{m-2} \frac{(m)_s}{(2\pi i n)^s} \left\{ \frac{G_{m-s+1}}{m-s+1} (H_{m-1} - H_{m-s}) + \frac{E_{m-s}}{m-s} \right\}.
\end{aligned}$$

**Case** 2 : $n = 0$.

$$C_0^{(m)} = \int_0^1 \gamma_m(x) dx = 0.$$

$\gamma_m(< x >)$, $(m \geq 2)$ is piecewise $C^\infty$. Moreover, $\gamma_m(< x >)$ is continuous for all odd integers $\geq 3$, and discontinuous with jump discontinuities at integers for all even integers $\geq 2$.

Assume first that $m$ is an odd integer $\geq 3$. Then $\gamma_m(0) = \gamma_m(1)$. $\gamma_m(< x >)$ is piecewise $C^\infty$, and continuous. So the Fourier series of $\gamma_m(< x >)$ converges

uniformly to $\gamma_m(< x >)$, and

$$\begin{aligned}\gamma_m(< x >) = &\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left\{ -\frac{(m-1)!}{(2\pi in)^{m-1}} H_{m-1} \right. \\ &\left. + \frac{2}{m} \sum_{s=1}^{m-2} \frac{(m)_s}{(2\pi in)^s} \left( \frac{G_{m-s+1}}{m-s+1}(H_{m-1} - H_{m-s}) + \frac{E_{m-s}}{m-s} \right) \right\} e^{2\pi inx} \\ =&H_{m-1} \left( -(m-1)! \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^{2\pi inx}}{(2\pi in)^{m-1}} \right) \\ &- \frac{2}{m} \sum_{s=1}^{m-2} \binom{m}{s} \left( \frac{G_{m-s+1}}{m-s+1}(H_{m-1} - H_{m-s}) + \frac{E_{m-s}}{m-s} \right) \left( -s! \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^{2\pi inx}}{(2\pi in)^s} \right) \\ =&H_{m-1} B_{m-1}(< x >) - \frac{2}{m} \sum_{s=2}^{m-2} \binom{m}{s} \left( \frac{G_{m-s+1}}{m-s+1}(H_{m-1} - H_{m-s}) + \frac{E_{m-s}}{m-s} \right) \\ &\times B_s(< x >) - \frac{2E_{m-1}}{m-1} \times \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}\end{aligned}$$

Now, we can state our first theorem.

**Theorem 4.1.** *Let $m$ be an odd integer $\geq 3$. Then we have the following.*

(i)

$$\sum_{k=1}^{m-1} \frac{1}{k(m-k)} E_k(< x >) G_{m-k}(< x >)$$

*has the Fourier series expansion*

$$\begin{aligned}&\sum_{k=1}^{m-1} \frac{1}{k(m-k)} E_k(< x >) G_{m-k}(< x >) \\ =&\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left\{ -\frac{(m-1)!}{(2\pi in)^{m-1}} H_{m-1} \right. \\ &\left. + \frac{2}{m} \sum_{s=1}^{m-2} \frac{(m)_s}{(2\pi in)^s} \left( \frac{G_{m-s+1}}{m-s+1}(H_{m-1} - H_{m-s}) + \frac{E_{m-s}}{m-s} \right) \right\} e^{2\pi inx},\end{aligned}$$

*for all $x \in (-\infty, \infty)$. Here the convergence is uniform.*

(ii)

$$\begin{aligned}&\sum_{k=1}^{m-1} \frac{1}{k(m-k)} E_k(< x >) G_{m-k}(< x >) \\ =&H_{m-1} B_{m-1}(< x >) - \frac{2}{m} \sum_{s=2}^{m-2} \binom{m}{s} \left( \frac{G_{m-s+1}}{m-s+1}(H_{m-1} - H_{m-s}) + \frac{E_{m-s}}{m-s} \right) B_s(< x >).\end{aligned}$$

*Here $B_s(< x >)$ is the Bernoulli function.*

Assume next that $m$ is an even integer $\geq 4$. Then $\gamma_m(0) \neq \gamma_m(1)$. Hence $\gamma_m(< x >)$ is piecewise $C^\infty$ and discontinuous with jump discontinuities at integers. The Fourier series of $\gamma_m(< x >)$ converges pointwise to $\gamma_m(< x >)$, for $x \notin \mathbb{Z}$, and converges to

$$\begin{aligned}&\frac{1}{2}(\gamma_m(0)+\gamma_m(1)) = \gamma_m(0) - \frac{E_{m-1}}{m-1}\\ &= \sum_{k=1}^{m-2} \frac{1}{k(m-k)} E_k G_{m-k},\end{aligned}$$

for $x \in \mathbb{Z}$. Now, we can state our second theorem.

**Theorem 4.2.** *Let $m$ be an even integer $\geq 4$. Then we have the following.*

(i)

$$\begin{aligned}&\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \left\{ -\frac{(m-1)!}{(2\pi i n)^{m-1}} H_{m-1} \right.\\ &\left. + \frac{2}{m} \sum_{s=1}^{m-2} \frac{(m)_s}{(2\pi i n)^s} \left( \frac{G_{m-s+1}}{m-s+1}(H_{m-1} - H_{m-s}) + \frac{E_{m-s}}{m-s} \right) \right\} e^{2\pi i n x}\\ &= \begin{cases} \sum_{k=1}^{m-1} \frac{1}{k(m-k)} E_k(< x >) G_{m-k}(< x >), & \text{for } x \notin \mathbb{Z},\\ \sum_{k=1}^{m-2} \frac{1}{k(m-k)} E_k G_{m-k}, & \text{for } x \in \mathbb{Z}. \end{cases}\end{aligned}$$

*Here the convergence is pointwise.*

(ii)

$$\begin{aligned}&-\frac{2}{m} \sum_{s=1}^{m-1} \binom{m}{s} \left( \frac{G_{m-s+1}}{m-s+1}(H_{m-1} - H_{m-s}) + \frac{E_{m-s}}{m-s} \right) B_s(< x >)\\ &= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} E_k(< x >) G_{m-k}(< x >), \text{ for } x \notin \mathbb{Z};\\ &-\frac{2}{m} \sum_{s=2}^{m-1} \binom{m}{s} \left( \frac{G_{m-s+1}}{m-s+1}(H_{m-1} - H_{m-s}) + \frac{E_{m-s}}{m-s} \right) B_s(< x >)\\ &= \sum_{k=1}^{m-2} \frac{1}{k(m-k)} E_k G_{m-k}, \text{ for } x \in \mathbb{Z}.\end{aligned}$$

## References


[1] T. Agoh, *Convolution identities for Bernoulli and Genocchi polynomials*, Electron. J. Combin., **21** (2014), no. 1, Paper 1.65, 14 pp.

[2] A. Bayad, *Fourier expansions for Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials*, Math. Comp., **80** (2011), no. 276, 2219-2221.

[3] A. Bayad and T. Kim, *Identities for the Bernoulli, the Euler and the Genocchi numbers and polynomials*, Adv. Stud. Contemp. Math. (Kyungshang), **20** (2010), no. 2, 247-253.

[4] Y. He and T. Kim, *General convolution identities for Apostol-Bernoulli Euler and Genocchi polynomials*, J. Nonlinear Sci. Appl., **9** (2016), no. 6, 4780-4797.

[5] D. S. Kim and T. Kim, *Some identities involving Genocchi polynomials and numbers*, Ars Combin., **121** (2015), 403-412.

[6] T. Kim, *Euler numbers and polynomials associated with zeta functions*, Abstr. Appl. Anal. , **2008** (2008), 2008, Art. ID 581582, 11 pp.

[7] T. Kim, *On the multiple q-Genocchi and Euler numbers*, Russ. J. Math. Phys. **15** (2008), no. 4, 481486.
[8] T. Kim, *Note on the Euler numbers and polynomials*, Adv. Stud. Contemp. Math. **17** (2008), no. 2, 131-136.
[9] T. Kim, D. S. Kim, S.-H. Rim and D.-V. Dolgy, *Fourier series of higher-order Bernoulli functions and their applications*, J. Inequal. Appl. **2017** (2017), 2017:8.
[10] Q. M. Luo, *Some recursion formulae and relations for Bernoulli numbers and Euler numbers of higher order*, Adv. Stud. Contemp. Math. (Kyungsang), **10** (2005), no. 1, 63-70.
[11] H. Maïga, *Identities and congruences for Genocchi numbers*, Advances in non-Archimedean analysis, Contemp. Math., **551**, Amer. Math., Providence, RI, (2011), pp. 207-220.

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# MAJORIZATION PROPERTIES FOR CERTAIN FAMILIES OF ANALYTIC FUNCTIONS IN THE UNIT DISK

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ABSTRACT. The main object of this paper is to introduce the majorization properties for certain families of analytic functions associated with generalized Srivastava-Attiya operator in the unit disk. Also, some applications of our results are discussed which give a number of new results.

## 1. INTRODUCTION

Let $A(p)$ denote the class of functions $f(z)$ of the from

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p}, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Also, let $A = A(1)$.

**Definition 1.1.** *Let $f$ and $F$ be analytic functions in $\mathbb{U}$, $f$ is said to be majorized by $F$ in $\mathbb{U}$ (see* [15], [18]*), written $f \ll F$, $z \in \mathbb{U}$, if there exists a function $\varphi$, analytic in $\mathbb{U}$ such that*

$$|\varphi(z)| \leq 1 \quad \text{and} \quad f(z) = \varphi(z)F(z) \quad (z \in \mathbb{U}). \tag{1.2}$$

*Noting that the concept of majorization is closely related to the concept of quasi-subordination between analytic functions (see* [18]*).*

**Definition 1.2.** *Let $f$ and $F$ be analytic functions. The function $f$ is said to be subordinate to $F$, written $f \prec F$, if there exists a function $w$ analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$, and such that $f(z) = g(w(z))$, in particular, if $F$ is univalent, then $f \prec F$ if and only if $f(0) = F(0)$ and $f(\mathbb{U}) \subset F(\mathbb{U})$.*

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A general Hurwitz-Lerch Zeta function $\Phi(z, s, b)$ defined by (*cf., e.g.,* [20, P. 121 et seq.])

$$\Phi(z, s, b) = \sum_{k=0}^{\infty} \frac{z^k}{(k+b)^s}, \tag{1.3}$$

($b \in \mathbb{C} \setminus \mathbb{Z}_0^-$, $\mathbb{Z}_0^- = \mathbb{Z}^- \cup \{0\} = \{0, -1, -2, ...\}$, $s \in \mathbb{C}$ when $z \in \mathbb{U}$, $\mathrm{Re}(s) > 1$ when $|z| = 1$ ).

Many authors studied and invistagated various properties of $\Phi(z, s, b)$, see *e.g.* [2], [6], [5], [7], [8], [14], [10], [11], [19], [21], [22] and [17].

Now, let us define, the operator $J_{s,b}(f)$ which has been introduced by Srivastava and Attiya [19]

$$J_{s,b}(f)(z) = G_{s,b}(z) * f(z) \tag{1.4}$$

$$\left(z \in \mathbb{U};\ f \in A;\ b \in \mathbb{C}\setminus\mathbb{Z}_0^-;\ s \in \mathbb{C}\right)$$

where

$$G_{s,b}(z) = (1+b)^s \left[\Phi(z, s, b) - b^{-s}\right] \tag{1.5}$$

and $*$ denotes the Hadamard product .

Moreover, Attiya and Hakami [2] defined the function $G_{s,b,t}$ by

$$G_{s,b,t} = 1 + (t+b)^s\ z\Phi(z, s, 1+t+b) \tag{1.6}$$

$$\left(z \in \mathbb{U};\ b \in \mathbb{C}\setminus\mathbb{Z}_0^-;\ s \in \mathbb{C};\ t \in \mathbb{R}\right),$$

we denote by

$$\mathcal{J}_{s,b}^t(f):\ A(p) \longrightarrow\ A(p),$$

Attiya and Hakami [2] defined the operator $\mathcal{J}_{s,b}^t(f)(z)$ by:

$$\mathcal{J}_{s,b}^t(f)(z) = z^p G_{s,b,t} * f(z) \tag{1.7}$$

$$\left(z \in \mathbb{U};\ f \in A(p); b \in \mathbb{C}\setminus\mathbb{Z}_0^-;\ s \in \mathbb{C};\ t \in \mathbb{R}\ \right),$$

where $*$ denotes the Hadamard product

Noting that

$$\mathcal{J}_{s,b}^{t}(f)(z)=z^{p}+\sum_{k=1}^{\infty}\left(\frac{t+b}{k+t+b}\right)^{s}a_{k+p}\;z^{k+p}\;(z\in\mathbb{U}) \tag{1.8}$$

we note that

$$\mathcal{J}_{s,b}^{1}(f)=J_{s,b}^{p}(f)\;, \tag{1.9}$$

where $J_{s,b}^{p}(f)$ introduced by Liu [13] .

The operator $\mathcal{J}_{s,b}^{t}(f)$ generalizes many operators *e.g.*, Srivastava and Attiya operator [19], Liu operator [13], Alexander operator [1], Libera operator [12], Bernardi operator [4] and Jung-Kim-Srivastava integral operator [9].

Now, we begin by the following lemma due to Attiya and Hakami [2].

**Lemma 1.1.** *Let $f(z)\in A(p)$, then*

$$z\left(\mathcal{J}_{s+1,b}^{t}\;f(z)\right)'=(t+b)\mathcal{J}_{s,b}^{t}f(z)-(t+b-p)\mathcal{J}_{s+1,b}^{t}f(z), \tag{1.10}$$
$$\left(z\in\mathbb{U};\;b\in\mathbb{C}\backslash\mathbb{Z}_{0}^{-};\;s\in\mathbb{C};\;t\in\mathbb{R}\right)$$

**Definition 1.3.** *A function $f(z)\in A(p)$ is said to be in the class $S_{s,b,p}^{n,t}(A,B,\zeta)$ if it satisfies*

$$1+\frac{1}{\zeta}\left(\frac{z\left(\left(\mathcal{J}_{s+1,b}^{t}\right)^{(n+1)}(f)(z)\right)}{\left(\mathcal{J}_{s+1,b}^{t}\right)^{(n)}(f)(z)}-p+n\right)\prec\frac{1+Az}{1+Bz}, \tag{1.11}$$

*where $n\in\mathbb{N}_{0}=\{0,1,...\},-1\leq B<A\leq1,\zeta\in\mathbb{C}^{*}:=\mathbb{C}\setminus\{0\},s\in\mathbb{C},t\in\mathbb{R}$ and $b\in\mathbb{C}\setminus\mathbb{Z}_{0}^{-}$.*

We note that $S_{s-1,b,1}^{0,1}(A,B,1-\alpha)=H_{s,b,\alpha}(A,B)$ the class which introduced by Kutbi and Attiya [10], $S_{-1,b,1}^{0,1}(1,-1,1-\alpha)$ the well known class of starlike function of order $\alpha$. Also, using special cases of $n,b,p,t,A,B,\zeta$ we have many various classes associated with Alexander operator, Libera operator, Bernardi and Jung-Kim-Srivastava operator.

Also, we define the following classes:

(1) $S_{s,b,p}^{n,t}(-1,1,\zeta)=S_{s,b,p}^{n,t}(\zeta)$.
(2) $S_{s,b,p}^{n,t}(-1,1,1)=S_{s,b,p}^{n,t}$.
(3) $S_{0,0,p}^{n,t}(A,B,\zeta)=\mathcal{A}_{p}^{n,t}(A,B,\zeta)$.
(4) $S_{0,1,p}^{n,t}(A,B,\zeta)=\mathcal{L}_{p}^{n,t}(A,B,\zeta)$.
(5) $S_{0,\gamma,p}^{n,t}(A,B,\zeta)=\mathcal{L}_{\gamma,p}^{n,t}(A,B,\zeta)$ ( $\gamma$ real ; $\gamma>-1$ )

(6) $S^{n,t}_{\sigma,1,p}(A,B,\zeta) = \mathcal{I}^{n,t}_{\sigma,p}(A,B,\zeta)$ ( $\sigma$ real ; $\sigma > 0$ )

## 2. Main Results

To introduce our results, we need the following lemma which can be proved by using Lemma 1.1 and induction .

**Lemma 2.1.** *Let $f(z) \in A(p)$, then*

(2.1)

$$z\left(\mathcal{J}^t_{s+1,b}\right)^{(n+1)}(f)(z) = (t+b)\left(\mathcal{J}^t_{s,b}\right)^{(n)}(f)(z) - (t+n+b-p)\left(\mathcal{J}^t_{s+1,b}\right)^{(n)}(f)(z)$$
$$\left(n \in \mathbb{N}_0,\ z \in \mathbb{U};\ b \in \mathbb{C}\backslash\mathbb{Z}^-_0;\ s \in \mathbb{C};\ t \in \mathbb{R}\right)$$

We begin by proving the following main result.

**Theorem 2.1.** *Let the function $g(z) \in S^n_{s,b}(A,B,\zeta)$, if*

$$\mathcal{J}^{(n)}_{s+1,b}(f)(z) \ll \mathcal{J}^{(n)}_{s+1,b}(g)(z), \quad (z \in \mathbb{U}), \tag{2.2}$$

*then*

$$\left|\left(\mathcal{J}^t_{s+1,b}\right)^{(n)}(f)(z)\right| \le \left|\left(\mathcal{J}^t_{s+1,b}\right)^{(n)}(g)(z)\right| \ (|z| \le r_0), \tag{2.3}$$

*where $f(z) \in A(p)$ and $r_0 = r_0(\zeta,b,A,B)$ is the positive root and the smallest of the equation*

(2.4)

$$r^3|\zeta(A-B)+(t+b)B|-[|t+b|+2|B|]r^2-[|\zeta(A-B)+(t+b)B|+2]r+|t+b| = 0,$$
$$(-1 \le B < A \le 1, \zeta \in \mathbb{C}^*, s \in \mathbb{C}, b \in \mathbb{C} \setminus \mathbb{Z}^-_0, t \in \mathbb{R}),$$

*Proof.* Since $g(z) \in S^{n,t}_{s,b,p}(A,B,\zeta)$, then (1.11) gives

$$1 + \frac{1}{\zeta}\left(\frac{z\left(\left(\mathcal{J}^t_{s+1,b}\right)^{(n+1)}(g)(z)\right)}{\left(\mathcal{J}^t_{s+1,b}\right)^{(n)}(g)(z)} - p + n\right) = \frac{1+A\ \omega(z)}{1+B\ \omega(z)}, \tag{2.5}$$

where $\omega(z)$ is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \text{ and } |\omega(z)| < 1 \quad (z \in \mathbb{U}).$$

From (2.5), we get

(2.6)

$$\frac{z\left(\left(\mathcal{J}^t_{s+1,b}\right)^{(n+1)}(g)(z)\right)}{\left(\mathcal{J}^t_{s+1,b}\right)^{(n)}(g)(z)} = \frac{(p-n)+[(p-n)B+\zeta(A-B)]\omega(z)}{1+B\ \omega(z)}.$$

by using Lemma 2.1 and (2.6), we get

(2.7)

$$\left|\left(\mathcal{J}^t_{s+1,b}\right)^{(n)}(g)(z)\right| \le \frac{(t+b)[1+|B||z|]}{(t+b)-|\zeta(A-B)+(t+b)B||z|}\left|\left(\mathcal{J}^t_{s,b}\right)^{(n)}(g)(z)\right|.$$

Since $\left(\mathcal{J}_{s+1,b}^{t}\right)^{(n)}(f)(z))$ is majorized by $\left(\mathcal{J}_{s+1,b}^{t}\right)^{(n)}(g)(z)$, in $\mathbb{U}$, therefore, the equation (2.2), gives

$$\left(\mathcal{J}_{s+1,b}^{t}\right)^{(n)}(f)(z)) = \varphi(z)\left(\mathcal{J}_{s+1,b}^{t}\right)^{(n)}(g)(z), \tag{2.8}$$

where $|\varphi(z)| \leq 1$. Differentiating (2.8) with respect to $z$ , we have the following relation:

(2.9)

$$z(\left(\mathcal{J}_{s+1,b}^{t}\right)^{(n+1)}(f)(z)) = z\varphi'(z)\left(\mathcal{J}_{s+1,b}^{t}\right)^{(n)}(g)(z)+z\varphi(z)\left(\mathcal{J}_{s+1,b}^{t}\right)^{(n+1)}(g)(z).$$

Using (2.1) in the above equation, we get

(2.10)

$$\left(\mathcal{J}_{s,b}^{t}\right)^{(n)}(f)(z)) = \frac{z\varphi'(z)}{(t+b)}\left(\mathcal{J}_{s+1,b}^{t}\right)^{(n)}(g)(z) + \varphi(z)\left(\mathcal{J}_{s,b}^{t}\right)^{(n)}(g)(z).$$

since $\varphi \in \mathcal{P}$ satisfies the inequality (See, e.g., Nehari [16])

$$|\varphi'(z)| \leq \frac{1-|\varphi(z)|^2}{1-|z|^2}, \quad (z \in \mathbb{U}), \tag{2.11}$$

and making use of (2.7) and (2.11) in(2.10), it yields

(2.12)

$$\left|\left(\mathcal{J}_{s,b}^{t}\right)^{(n)}(f)(z)\right| \leq \left(|\varphi(z)| + \frac{1-|\varphi(z)|^2}{1-|z|^2}\frac{[1+|B||z|]|z|}{|t+b|-|\zeta(A-B)+(t+b)B||z|}\right)\left|\left(\mathcal{J}_{s,b}^{t}\right)^{(n)}(g)(z)\right|.$$

Setting

$$|z| = r \quad \textit{and} \quad |\varphi(z)| = \eta \quad (0 \leq \eta \leq 1)$$

this gives us the following inequality

(2.13)

$$\left|\left(\mathcal{J}_{s,b}^{t}\right)^{(n)}(f)(z)\right| \leq \frac{\Phi(\eta)}{(1-r^2)\left[|t+b|-|\zeta(A-B)+(t+b)B|r\right]}\left|\left(\mathcal{J}_{s,b}^{t}\right)^{(n)}(g)(z)\right|,$$

where

(2.14)

$$\Phi(\eta) = -r(1+|B|r)\eta^2+(1-r^2)\left[|t+b|-|\zeta(A-B)+(t+b)B|r\right]\eta+r(1+|B|r),$$

the function $\Phi(\eta)$ takes the maximum value at $\eta = 1$, with $r_0 = r_0(A, B, s, b, t)$ where $r_0$ is the smallest positive root of (2.4). Moreover, if $0 \leq \eta \leq r_0(A, B, s, b, t)$ then the function $\Psi(\eta)$ defined by

$$\Psi(\eta) = -\sigma(1+|B|\sigma)\eta^2+(1-\sigma^2)\left[|t+b|-|\zeta(A-B)+(t+b)B|\sigma\right]\eta+\sigma(1+|B|\sigma),$$

is an increasing function on the interval $[0, 1]$, therefore

$$\Psi(\eta) \leq \Psi(1) = (1-\sigma^2)[|t+b|-|\zeta(A-B)+(t+b)B|\sigma], \tag{2.15}$$

$$(0 \leq \eta \leq 1; 0 \leq \sigma \leq r_0(A, B, s, b)).$$

Hence putting $\eta = 1$, in (2.14), we conclude that (2.3) of Theorem 2.1 holds true for

$$|z| \leq r_0 = r_0(A, B, s, b),$$

where $r_0$ is the smallest positive root of equation (2.4). This completes the proof of Theorem 2.1. □

*Remark* 2.1. Putting $t = 1$ in Theorem 2.1, we have the result due to Attiya and Yassen [3].

Putting $A = 1$ and $B = -1$ in Theorem 2.1, we have the following result.

**Corollary 2.1.** *Let the function* $g(z) \in \mathcal{S}^{n,t}_{s,b,p}(\zeta)$, *if*

$$\left(\mathcal{J}^t_{s+1,b}\right)^{(n)}(f)(z) \ll \left(\mathcal{J}^t_{s+1,b}\right)^{(n)}(g)(z), \quad (z \in \mathbb{U}), \tag{2.16}$$

*then*

$$|\left(\mathcal{J}^t_{s+1,b}\right)^{(n)}(f)(z)| \leq |\left(\mathcal{J}^t_{s+1,b}\right)^{(n)}(g)(z)| \ (|z| \leq r_0), \tag{2.17}$$

*where* $f(z) \in A(p)$ *and* $r_0$ *given by*

$$r_0 = \begin{cases} \frac{m-\sqrt{m^2-4|b+t||2\zeta-b-t|}}{2|2\zeta-b-t|}, & \zeta \neq \frac{b+t}{2} \\ \frac{\sqrt{1+|b+t|(2+|b+t|)}-1}{2+|b+t|}, & \zeta = \frac{b+t}{2} \end{cases}, \tag{2.18}$$

$m = 2 + |b + t| + |2\zeta - b - t|$, $\zeta \in \mathbb{C}^*, s \in \mathbb{C}$ *and* $b \in \mathbb{C} \setminus \mathbb{Z}_0^-$).

Letting $A = 1$, $B = -1$ and $\zeta = 1$ in Theorem 2.1, we get the following property.

**Corollary 2.2.** *Let the function* $g(z) \in \mathcal{S}^{n,t}_{s,b,p}$, *if*

$$\left(\mathcal{J}^t_{s+1,b}\right)^{(n)}(f)(z) \ll \left(\mathcal{J}^t_{s+1,b}\right)^{(n)}(g)(z), \quad (z \in \mathbb{U}), \tag{2.19}$$

*then*

$$|\left(\mathcal{J}^t_{s+1,b}\right)^{(n)}(f)(z)| \leq |\left(\mathcal{J}^t_{s+1,b}\right)^{(n)}(g)(z)| \ (|z| \leq r_0), \tag{2.20}$$

*where* $f(z) \in A(p)$ *and* $r_0$ *given by*

$$r_0 = \begin{cases} \frac{m-\sqrt{m^2-4|b+t||2-b-t|}}{2|2-b-t|}, & b \neq 2-t \\ \frac{1}{2}, & b = 2-t \end{cases}, \tag{2.21}$$

$m = 2 + |b + t| + |2 - b - t|$, $s \in \mathbb{C}$ *and* $b \in \mathbb{C} \setminus \mathbb{Z}_0^-$).

Letting $s = b = 0$, in Theorem 2.1, we have the following corollary.

**Corollary 2.3.** *Let the function $g(z) \in \mathcal{A}_p^{n,t}(A,B,\zeta)$, if*

$$\left(\mathcal{A}_p^t\right)^{(n)}(f)(z) \ll \left(\mathcal{A}_p^t\right)^{(n)}(g)(z), \quad (z \in \mathbb{U}), \tag{2.22}$$

*then*

$$|\left(\mathcal{A}_p^t\right)^{(n)}(f)(z)| \leq |\left(\mathcal{A}_p^t\right)^{(n)}(g)(z)| \ (|z| \leq r_0), \tag{2.23}$$

*where $f(z) \in A(p)$ and $r_0 = r_0(\zeta, A, B)$ is the smallest positive root of the equation*

$$r^3|\zeta(A-B)+Bt|-[|t|+2|B|]r^2-[|\zeta(A-B)+Bt|+2]r+|t| = 0, \tag{2.24}$$
$$(-1 \leq B < A \leq 1, \zeta \in \mathbb{C}^*),$$

If we put $s = 0$ and $b = 1$,in Theorem 2.1, then we have the following result.

**Corollary 2.4.** *Let $g(z) \in \mathcal{L}_p^{n,t}(A,B,\zeta)$, if*

$$\left(\mathcal{L}_p^t\right)^{(n)}(f)(z) \ll \left(\mathcal{L}_p^t\right)^{(n)}(g)(z), \quad (z \in \mathbb{U}), \tag{2.25}$$

*then*

$$|\left(\mathcal{L}_p^t\right)^{(n)}(f)(z)| \leq |\left(\mathcal{L}_p^t\right)^{(n)}(g)(z)| \ (|z| \leq r_0), \tag{2.26}$$

*where $f(z) \in A(p)$ and $r_0 = r_0(\zeta, A, B)$ is the smallest positive root of the equation*

(2.27)
$$r^3|\zeta(A-B)+(t+1)B|-[|1+t|+2|B|]r^2-[|\zeta(A-B)+(1+t)B|+2]r+|1+t| = 0,$$
$$(-1 \leq B < A \leq 1, \zeta \in \mathbb{C}^*).$$

Putting $s = 0$ and $b = \gamma > -t$ in Theorem 2.1, we get the following corollary.

**Corollary 2.5.** *Let the function $g(z) \in \mathcal{L}_\gamma^n(A,B,\zeta)$, if*

$$\left(\mathcal{L}_{\gamma,p}^t\right)^{(n)}(f)(z) \ll \left(\mathcal{L}_{\gamma,p}^t\right)^{(n)}(g)(z), \quad (z \in \mathbb{U},\ \gamma > -1), \tag{2.28}$$

*then*

$$|\left(\mathcal{L}_{\gamma,p}^t\right)^{(n)}(f)(z)| \leq |\left(\mathcal{L}_{\gamma,p}^t\right)^{(n)}(g)(z)| \ (|z| \leq r_0), \tag{2.29}$$

*where $f(z) \in A(p)$ and $r_0 = r_0(\zeta, b, A, B)$ is the smallest positive root of the equation*

(2.30)
$$r^3|\zeta(A-B)+(t+\gamma)B|-[t+\gamma+2|B|]r^2-[|\zeta(A-B)+(t+\gamma)B|+2]r+(t+\gamma) = 0,$$
$$(-1 \leq B < A \leq 1,\ \gamma > -1,\ \zeta \in \mathbb{C}^*, s \in \mathbb{C}),$$

Putting $s = \sigma$ $(\sigma; \text{real},\ \sigma > 0)$ and $b = 1$ in Theorem 2.1, we get the following corollary.

**Corollary 2.6.** *Let the function $g(z) \in \mathcal{I}_{\sigma}^{n}(A, B, \zeta)$, if*

$$\left(\mathcal{I}_{\sigma,p}^{t}\right)^{(n)}(f)(z) \ll \left(\mathcal{I}_{\sigma,p}^{t}\right)^{(n)}(g)(z), \quad (z \in \mathbb{U}; \sigma > 0), \tag{2.31}$$

*then*

$$|\left(\mathcal{I}_{\sigma,p}^{t}\right)^{(n)}(f)(z)| \leq |\left(\mathcal{I}_{\sigma,p}^{t}\right)^{(n)}(g)(z)|\ (|z| \leq r_0), \tag{2.32}$$

*where $f(z) \in A(p)$ and $r_0 = r_0(\zeta, A, B)$ is the smallest positive root of the equation*

$$r^3|\zeta(A-B)+(t+1)B|-[|1+t|+2|B|]r^2-[|\zeta(A-B)+(1+t)B|+2]r+|1+t| = 0,$$
$$(-1 \leq B < A \leq 1, \zeta \in \mathbb{C}^*, s \in \mathbb{C}).$$

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## References

[1] J.W. Alexander, Functions which map the interior of the unit circle upon simple region, Annals of Math. 17(1915), 12-22.
[2] A.A. Attiya and A. Hakami, Some subordination results associated with generalized Srivastava-Attiya operator, Adv. Difference Equ. 2013, 2013:105, 14 pp.
[3] A.A. Attiya and M.F. Yassen, Majorization for some classes of analytic functions assocated with the Srivastava-Attiya operator, Filomat 30:7 (2016), 2067–2074.
[4] S.D. Bernardi, Convex and starlike univalent functions, Trans. Amer. Math. Soc. 135(1969), 429-449.
[5] N.E. Cho, I.H. Kim and H.M. Srivastava, Sandwich-type theorems for multivalent functions associated with the Srivastava-Attiya operator, Appl. Math. Comput. 217 (2010), no. 2, 918–928.
[6] J. Choi, D.S. Jang and H.M. Srivastava, A generalization of the Hurwitz-Lerch Zeta function, Integral Transforms Spec. Funct. 19(2008), no. 1-2, 65-79.
[7] C. Ferreira and J.L. López, Asymptotic expansions of the Hurwitz-Lerch zeta function, J. Math. Anal. Appl. 298(2004), 210-224.
[8] P.L. Gupta, R.C. Gupta, S. Ong and H.M. Srivastava, A class of Hurwitz-Lerch zeta distributions and their applications in reliability, Appl. Math. Comput. 196 (2008), no. 2, 521–531.
[9] J.B. Jung, Y.C. Kim and H.M. Srivastava, The Hardy space of analytic functions associated with certain one-parameter families of integral operator, J. Math. Anal. Appl. 176(1993), 138-147.

[10] M.A. Kutbi and A.A. Attiya, Differential subordination result with the Srivastava-Attiya integral operator, J. Inequal. Appl. 2010(2010), 1-10.
[11] M.A. Kutbi and A.A. Attiya, Differential subordination results for certain integrodifferential operator and it's applications, Abs. Appl. Anal., 2012(2012), 13 pp.
[12] R.J. Libera, Some classes of regular univalent functions, Proc. Amer. Math. Soc. 16(1965), 755–758.
[13] Jin-Lin Liu, Subordinations for certain multivalent analytic functions associated with the generalized Srivastava-Attiya operator, Integral Transforms Spec. Funct.19 (2008), no. 11-12, 893–901.
[14] Q.M. Luo and H.M. Srivastava, Some generalizations of the Apostol-Bernoulli and Apostol-Euler polynomials, J. Math. Anal. Appl. 308(2005), 290-302.
[15] T. H. MacGreogor, Majorization by univalent functions, Duke Math. J. 34(1967), 95–102.
[16] Z. Nehari, Conformal Mapping, MacGra-Hill Book Company, New York, Toronto and London (1955).
[17] S. Owa and A.A. Attiya, An application of differential subordinations to the class of certain analytic functions, Taiwanese J. Math., 13(2009), no. 2A, 369-375.
[18] G. Schober, Univalent Functions: Selected Topics, Springer-Verlag, Berlin, Heidelberg, New York, 1975.
[19] H.M. Srivastava and A.A. Attiya, An integral operator associated with the Hurwitz-Lerch zeta function and differential subordination, Integral Transforms Spec. Funct. 18 (2007), no. 3-4, 207-216.
[20] H.M. Srivastava and J. Choi, Series Associated with the Zeta and Related Functions, Kluwer Academic Publishers, Dordrecht, 2001.
[21] H.M. Srivastava, D. Răducanu, and G.S. Sălăgean, A new class of generalized close-to-starlike functions defined by the Srivastava-Attiya operator, Acta Math. Sin. (Engl. Ser.) 29 (2013), no. 5, 833-840.
[22] H.M. Srivastava, R.K. Saxena, T.K. Pogany and R. Saxena, Integral and computational representations of the extended Hurwitz–Lerch zeta function, Integral Transforms Spec. Funct. 22 (2011) 487–506.

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# On some applications of differential subordinations

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**Abstract**

The purpose of the presented paper is to derive several properties and characteristics of analytic functions, by applying the differential subordination techniques. Several consequences of the results stated here are also pointed out.

**Key words:** analytic functions; univalent functions; starlike functions; convex functions; differential subordination; best dominant.

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## 1 Introduction

Let $\mathcal{A}$ be the class of analytic functions which are analytic in the open unit disk $U = \{z : |z| < 1\}$, of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \quad z \in U. \tag{1.1}$$

By $S$ we denote the subclass of $\mathcal{A}$ consisting of univalent functions.
*Principle of Subordination* (see [6]): If $f$ and $g$ are two analytic functions in $U$, we say that $f$ is subordinate to $g$, written as $f \prec g$, if there exists a Schwarz function $w$ analytic in $U$, with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$, for all $z \in U$. In particular, if the function $g$ is univalent in $U$, the above subordination is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.
Important results in the theory of differential subordinations were elaborated by Miller and Mocanu in ([6]) and ([5]).

**Definition 1.1** *[6] Denote by $Q$ the set of all functions $q$ that are analytic and injective on $\overline{U}\backslash E(q)$, where $E(q) = \{\zeta \in \partial U : \lim_{z\to\zeta} q(z) = \infty\}$, and are such that $q'(\zeta) \neq 0$, for $\zeta \in \partial U\backslash E(q)$.*

If $q \in Q$, then $\Delta = q(U)$ is a simply connected domain.

In 1915, J. Alexander (see [1]) introduced the class of starlike functions defined as:

$$S^* = \left\{ f \in \mathcal{A} : \operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right] > 0,\ z \in U \right\}. \tag{1.2}$$

Let $S^*_\alpha$ denote the class of starlike univalent functions of order $\alpha$, $0 \leq \alpha < 1$, defined as:

$$S^*(\alpha) = \left\{ f \in \mathcal{A} : \operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right] > \alpha,\ z \in U,\ 0 \leq \alpha < 1 \right\}. \tag{1.3}$$

A function $f(z) \in \mathcal{A}$ is said to be close-to-convex if there exists a function $g(z) = z + \sum_{j=2}^{\infty} b_j z^j$ in $S^*$ such that

$$\operatorname{Re}\left[\frac{zf'(z)}{g(z)}\right] > 0, \text{ for all } z \in U. \tag{1.4}$$

This class of functions $f(z) \in \mathcal{A}$ with the condition (1.4) is denoted by $K$ and called the class of close-to-convex functions. The class $K$ was introduced by Kaplan [3], who showed that all close-to-convex functions are univalent. A function $f(z) \in \mathcal{A}$ is said to belongs to the class $K(\alpha)$ of close-to-convex functions of order $\alpha$ in $U$, if $g(z) \in S^*(\alpha)$ and satisfy

$$\operatorname{Re}\left[\frac{zf^{'}(z)}{g(z)}\right]>\alpha,\ \text{for all } z\in U,\ 0\leq\alpha<1. \tag{1.5}$$

In the present investigation, we will obtain different properties of $\frac{zf^{'}(z)}{g(z)}$, for $f\in\mathcal{A}$, given by (1.1) and $g(z)=z+\sum_{j=2}^{\infty}b_jz^j$ in $S^*$, by using differential subordination. Many studies of the classes involving $\frac{zf^{'}(z)}{g(z)}$ were appeared in the literature (see [2]). Several authors obtained several applications in the geometric functions theory by using differential subordination (see [4], [8], [7], [11], [9], [10], [12]). Also, sufficient conditions for close-to-convex and close-to-convex of order $\alpha$ of a function $f\in\mathcal{A}$ are obtained. Some consequences of the main results are mentioned.

The following lemmas will be required in order to prove our main results.

**Lemma 1.1** *[6] Let the function $q$ be univalent in the unit disk $U$ and $\theta$ and $\phi$ be analytic in a domain $D$ containing $q(U)$ with $\phi(w)\neq0$, when $w\in q(U)$. Set $Q(z)=zq'(z)\phi(q(z))$ and $h(z)=\theta(q(z))+Q(z)$. Suppose that*

*1. $Q$ is starlike univalent in the unit disk $U$*

*and*

*2. $\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right)=\operatorname{Re}\left[\frac{\theta'(q(z))}{\phi(q(z))}+\frac{zQ'(z)}{Q(z)}\right]>0$, for $z\in U$.*

*If $p$ is analytic in $U$, with $p(0)=q(0)$, $p(U)\subseteq D$ and*

$$\theta(p(z))+zp'(z)\phi(p(z))\prec\theta(q(z))+zq'(z)\phi(q(z))=h(z), \tag{1.6}$$

*then $p(z)\prec q(z)$ and $q$ is the best dominant of (1.6).*

**Lemma 1.2** *[6] Let the function $q\in Q$, with $q(0)=a$ and let $p(z)=a+a_nz^n+...$ be analytic in $U$ with $p(z)\neq a$ and $n\geq1$. If $p$ is not subordinate to $q$, then there exist points $z_0=r_0e^{i\theta_0}\in U$ and $\zeta_0\in\partial U\backslash E(q)$ and an $m\geq n\geq1$, for which $p(U_{r_0})\subset q(U)$,*

$$(1)\ p(z_0)=q(\zeta_0), \tag{1.7}$$

$$(2)\ z_0p'(z_0)=m\zeta_0q'(\zeta_0), \tag{1.8}$$

$$(3)\ \operatorname{Re}\left[\frac{z_0p''(z_0)}{p'(z_0)}+1\right]\geq m\operatorname{Re}\left[\frac{\zeta_0q''(\zeta_0)}{q'(\zeta_0)}+1\right]. \tag{1.9}$$

# 2 Main results

**Theorem 2.1** *Let $f(z)=z+a_2z^2+...$ and $g(z)=z+b_2z^2+...$ be analytic functions in $U$, with $f(0)=g(0)=0$, $g(z)\in S^*$, with $g(z)\neq zf^{'}(z)$ and*

$$\operatorname{Re}\left[\frac{2(f'(z))^2}{f'(z)g'(z)-f''(z)g(z)}-\frac{zf'(z)}{g(z)}\right]>0,\quad z\in U. \tag{2.1}$$

*Then $f\in K$.*

**Proof.** At the begining let us note that $f\in K$ is equivalent with the condition

$$\operatorname{Re}\left[\frac{zf^{'}(z)}{g(z)}\right]>0,\quad z\in U. \tag{2.2}$$

Let

$$p(z)=\frac{zf^{'}(z)}{g(z)}=1+p_1z+p_2z^2+...,\ z\in U, \tag{2.3}$$

with $p(z)\neq1$, and let

$$q(z)=\frac{1+z}{1-z}=1+q_1z+q_2z^2+\ldots,\ z\in U, \tag{2.4}$$

with $q(0)=1$.
So, $\Delta=q(U)=\{\omega:\operatorname{Re}\omega>0\}$ and $E(q)=\{1\}$, and $q\in Q$.
We have to prove that $\operatorname{Re}\{p(z)\}>0$, for all $z\in U$.

This condition is equivalent with

$$p(z) \prec q(z),\ z \in U. \tag{2.5}$$

From (2.3), perform calculations, we obtain:

$$\frac{2\left(f'(z)\right)^2}{f'(z)g'(z)-f''(z)\,g(z)} - \frac{zf'(z)}{g(z)} = \frac{p(z)\left[p(z)+zp'(z)\right]}{p(z)-zp'(z)}. \tag{2.6}$$

Let suppose that (2.5) does not hold, i.e.

$$p(z) \not\prec q(z) = \frac{1+z}{1-z},\ z \in U. \tag{2.7}$$

Hence from Lemma (1.2), there exists a point $z_0 \in U$ and a point $\zeta_0 \in \partial U \setminus \{1\}$, $|\zeta_0| = 1$, such that

$$\begin{aligned} p(z_0) &= q(\zeta_0);\ z_0 p'(z_0) = m\zeta_0 q'(\zeta_0),\ \text{for some } m \geq 1, \\ &\text{and} \\ \operatorname{Re}[p(z)] &\geq 0, \text{ for all } z \in U_{|z_0|}. \end{aligned}$$

This implies that

$$\operatorname{Re}[p(z_0)] = \operatorname{Re}\left[\frac{1+\zeta_0}{1-\zeta_0}\right] = 0, \tag{2.8}$$

so, we can take $p(z_0)$ on the form

$$p(z_0) := iy,\ y \in \mathbb{R}. \tag{2.9}$$

By (2.4) and (2.9), we have

$$\zeta_0 = q^{-1}(p(z_0)) = \frac{p(z_0)-1}{p(z_0)+1} = \frac{iy-1}{iy+1} = \frac{y^2+2iy-1}{1+y^2}. \tag{2.10}$$

From (2.4) and (2.10), we get

$$q'(\zeta_0) = \frac{2}{(1-\zeta_0)^2} = \frac{\left(1+y^2\right)^2}{4(1-iy)^2}. \tag{2.11}$$

We obtain

$$z_0 p'(z_0) = m\zeta_0 q'(\zeta_0) = -\frac{m}{2}\left(1+y^2\right), \text{ for some } m \geq 1.$$

Therefore, we have

$$\operatorname{Re}\left[\frac{2\left(f'(z_0)\right)^2}{f'(z_0)g'(z_0)-f''(z_0)\,g(z_0)} - \frac{zf'(z_0)}{g(z_0)}\right] = \operatorname{Re}\left[\frac{p(z_0)\left[p(z_0)+zp'(z_0)\right]}{p(z_0)-zp'(z_0)}\right] =$$

$$\operatorname{Re}\left[\frac{iy\left[iy-\frac{m}{2}\left(1+y^2\right)\right]}{iy+\frac{m}{2}\left(1+y^2\right)}\right] = -\frac{my^2\left(1+y^2\right)}{\frac{m^2}{4}\left(1+y^2\right)^2+y^2} < 0, \text{ for } y \in \mathbb{R} \text{ and } m \geq 1,$$

which contradicts the condition (2.1) from the hypothesis of the theorem, therefore $p \prec q$, i.e.

$$\operatorname{Re}\left[\frac{zf'(z)}{g(z)}\right] > 0, \quad z \in U, \tag{2.12}$$

and the proof is complete.

**Theorem 2.2** *Let $f(z) = z + a_2 z^2 + \ldots$ and $g(z) = z + b_2 z^2 + \ldots$ be analytic functions in $U$, with $f(0) = g(0) = 0$, $g(z) \in S^*$, with $g(z) \neq zf'(z)$ and let $0 < \mu \leq 1$. If*

$$\left|\frac{2\left(f'(z)\right)^2}{f'(z)g'(z)-f''(z)\,g(z)} - \frac{zf'(z)}{g(z)} - 1\right| < \mu,\ z \in U, \tag{2.13}$$

*then*

$$\left|\frac{zf'(z)}{g(z)} - 1\right| < \mu,\ z \in U. \tag{2.14}$$

**Proof.** Let

$$p(z)=\frac{zf^{'}(z)}{g(z)}=1+p_1z+p_2z^2+...,\ z\in U. \tag{2.15}$$

Taking account (2.15), the conditions (2.13) and (2.14) can be rewritten as

$$|\ \frac{p(z)\left[p(z)+zp^{'}(z)\right]}{p(z)-zp^{'}(z)}-1\ |<\mu\Rightarrow|\ p(z)-1\ |<\mu, \tag{2.16}$$

with $p(z)\neq 1$, and let

$$q(z)=1+\mu z,\ \text{with } q(0)=1. \tag{2.17}$$

We have $\Delta=q(U)=\{\omega:|\ \omega-1\ |<\mu\}$, $E(q)=\varnothing$ and $q\in Q$.
The relation $|\ p(z)-1\ |<\mu$ is equivalent to

$$p(z)\prec q(z),\ z\in U. \tag{2.18}$$

Let suppose that (2.18) does not hold, i.e.

$$p(z)\not\prec q(z)=1+\mu z,\ z\in U.$$

From Lemma (1.2), there exists a point $z_0\in U$ and a point $\zeta_0\in\partial U$, such that

$$\begin{aligned} p(z_0) &= q(\zeta_0);\ z_0p'(z_0)=m\zeta_0q'(\zeta_0),\ \text{for some } m\geq 1,\\ &\quad \text{and}\\ &|\quad p(z)-1\ |<\mu, \text{for all } z\in U_{|z_0|}\ . \end{aligned}$$

This implies that

$$|\ p(z_0)-1\ |=|\ q(\zeta_0)-1\ |=\mu,$$

so, we can take $p(z_0)$ on the form

$$p(z_0):=1+\mu e^{i\theta},\ \theta\in\mathbb{R}. \tag{2.19}$$

We have

$$\zeta_0=q^{-1}(p(z_0))=\frac{p(z_0)-1}{\mu}=e^{i\theta}. \tag{2.20}$$

So,

$$z_0p'(z_0)=m\zeta_0q'(\zeta_0)=m\mu e^{i\theta}, \text{for some } m\geq 1.$$

Therefore, we have

$$\begin{aligned} &|\quad \frac{p(z_0)\left[p(z_0)+zp^{'}(z_0)\right]}{p(z_0)-zp^{'}(z_0)}-1\ |=|\ \mu e^{i\theta}\ ||\ \frac{1+\mu e^{i\theta}+m\mu e^{i\theta}+2m}{1+\mu e^{i\theta}-m\mu e^{i\theta}}\ |=\\ &=\quad \mu\ |\ \frac{1+2m+(1+m)\,\mu e^{i\theta}}{1+(1-m)\,\mu e^{i\theta}}\ |\ , \text{for some } m\geq 1. \end{aligned}$$

We will prove that the last expression is greater or equal to $\mu$, that is equivalent with

$$|\ \frac{1+2m+(1+m)\,\mu e^{i\theta}}{1+(1-m)\,\mu e^{i\theta}}\ |\geq 1,\quad z\in U,$$

or

$$|\ 1+2m+(1+m)\,\mu e^{i\theta}\ |^2\geq|\ 1+(1-m)\,\mu e^{i\theta}\ |^2,\quad z\in U.$$

The last inequality is equivalent to

$$\mu^2+(m+2)\,\mu\cos\theta+m+1\geq\mu^2-(m+2)\,\mu+m+1\geq 0,\quad z\in U,$$

or

$$(\mu-1)\,[\mu-(m+1)]\geq 0,\quad z\in U.$$

The above inequality holds for every $m\geq 1$ and $0<\mu\leq 1$.
This contradicts with the condition (2.13) from the hypothesis of the theorem, therefore $p\prec q$.
So, $|\ p(z)-1\ |<\mu$ and the proof of the theorem is complete.

**Theorem 2.3** *Let the function $q(z)$ be analytic and univalent in $U$ such that $q(0)=1$ and $q(z)\neq 0$, for all $z\in U$ and satisfies*

$$\operatorname{Re}\left[1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right]>0,\quad z\in U. \tag{2.21}$$

*If $f,g\in\mathcal{A}$ with $f(0)=f'(0)-1=g(0)=g'(0)-1=0$, $g(z)\in S^*$ and*

$$1+\frac{zf''(z)}{f'(z)}-\frac{zg'(z)}{g(z)}\prec\frac{zq'(z)}{q(z)}=h(z), \tag{2.22}$$

*then*

$$\frac{zf'(z)}{g(z)}\prec q(z),\quad z\in U, \tag{2.23}$$

*and $q(z)$ is the best dominant of (2.22).*

**Proof.** Let the function $p$ be defined by

$$p(z):=\frac{zf'(z)}{g(z)},\quad z\in U, \tag{2.24}$$

where $f,g\in\mathcal{A}$ with $f(0)=f'(0)-1=g(0)=g'(0)-1=0$, $g(z)\in S^*$. The function $p$ is analytic in $U$ and $p(0)=1$.
Differentiating this function with respect to $g$, we get
$zp'(z)=\frac{zf'(z)}{g(z)}+\frac{z^2f''(z)}{g(z)}-\frac{z^2f'(z)g'(z)}{g^2(z)},\quad z\in U.$
By setting $\theta(w):=0$ and $\phi(w):=\frac{1}{w}$, it can be easily verified that $\phi$ is analytic in a domain $D=\mathbb{C}\backslash\{0\}$, which contains $q(U)$ and $\phi(w)\neq 0$, where $w\in q(U)$.
Also, by letting $Q(z)=zq'(z)\phi(q(z))=\frac{zq'(z)}{q(z)}$, we find that $Q(z)$ is starlike in $U$, since
$\operatorname{Re}\left[\frac{zQ'(z)}{Q(z)}\right]=\operatorname{Re}\left[1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right]>0.$
For the function $h(z)=\theta(q(z))+Q(z)=\frac{zq'(z)}{q(z)}$, we have $\operatorname{Re}\left[\frac{zh'(z)}{Q(z)}\right]=\operatorname{Re}\left[\frac{zQ'(z)}{Q(z)}\right]>0$.
Therefore, the conditions of Lemma 1.1 are met and we find that, if

$$\frac{zp'(z)}{p(z)}=\theta(p(z))+zp'(z)\phi(p(z))\prec\frac{zq'(z)}{q(z)},\quad z\in U, \tag{2.25}$$

then

$$\frac{zf'(z)}{g(z)}=p(z)\prec q(z),\quad z\in U, \tag{2.26}$$

and $q$ is the best dominant of (2.22). And the proof is complete.

**Corollary 2.4** *Let $f,g\in\mathcal{A}$ with $f(0)=f'(0)-1=g(0)=g'(0)-1=0$, $g(z)\in S^*$ and $0<\mu\leq 1$. If*

$$1+\frac{zf''(z)}{f'(z)}-\frac{zg'(z)}{g(z)}\prec\frac{\mu z}{1+\mu z}=h_1(z), \tag{2.27}$$

*then*

$$\frac{zf'(z)}{g(z)}\prec 1+\mu z,\quad z\in U, \tag{2.28}$$

*and $1+\mu z$ is the best dominant of (2.27). Furthermore,*

$$\left|\frac{zf'(z)}{g(z)}-1\right|<\mu,\quad z\in U, \tag{2.29}$$

*and this result is sharp.*

**Proof.** Let the function $q(z):=1+\mu z$, $z\in U$, $0<\mu\leq 1$, satisfies all the conditions of Theorem 2.3.
So,

$$h_1(z)=Q(z)=zq'(z)\phi(q(z))=\frac{zq'(z)}{q(z)}=\frac{\mu z}{1+\mu z}. \tag{2.30}$$

From (2.30), we find that the subordinations (2.22) and (2.30) are equivalent.
Hence, it follows directly from Theorem 2.3 that

$$\frac{zf'(z)}{g(z)} \prec 1+\mu z, \quad z \in U, \tag{2.31}$$

and $1+\mu z$ is the best dominant of (2.27).
Now, let us assume that the subordination (2.27) and inequality

$$|\frac{zf'(z)}{g(z)} - 1| < \mu_1, \quad z \in U, 0 < \mu_1 \leq 1, \tag{2.32}$$

hold, i.e.

$$\frac{zf'(z)}{g(z)} \prec 1+\mu_1 z, \quad z \in U, 0 < \mu_1 \leq 1. \tag{2.33}$$

But the function $1+\mu z$ is the best dominant of (2.27), meaning that $1+\mu z \prec 1+\mu_1 z$, i.e. $\mu \prec \mu_1$.
Therefore the conclusion is sharp, i.e. in the inequality (2.29) $\mu$ cannot be replaced by a smaller number such that the implication holds. This completes the proof of the corollary.

**Remark 2.1** *It is easy to verify that for* $0 < \mu \leq 1$, $h_1(z) = \frac{\mu z}{1+\mu z}$ *is an open disk with center* $c = \frac{h_1(1)+h_1(-1)}{2} = \frac{\frac{\mu}{1+\mu}-\frac{\mu}{1-\mu}}{2} = -\frac{\mu^2}{1-\mu^2}$, *and radius* $r = h_1(1) - c = \frac{\mu}{1+\mu} + \frac{\mu^2}{1-\mu^2} = \frac{\mu}{1-\mu^2}$. *For* $\mu = 1$, $h_1(U) = \{x+iy : x < 1, y \in \mathbb{R}\}$.

Therefore, the above corollary can be rewritten as:

**Corollary 2.5** *Let* $f, g \in \mathcal{A}$ *with* $f(0) = f'(0) - 1 = g(0) = g'(0) - 1 = 0$, $g(z) \in S^*$ *and* $0 < \mu \leq 1$.
*(i) If* $0 < \mu < 1$ *and*

$$|1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)} + \frac{\mu^2}{1-\mu^2}| < \frac{\mu}{1-\mu^2}, z \in U \tag{2.34}$$

*then*

$$|\frac{zf'(z)}{g(z)} - 1| < \mu, \quad z \in U. \tag{2.35}$$

*This result is sharp, i.e. the radius of the open disk from the conclusion is the smallest possible so that the corresponding implication holds.*
*(ii) If*

$$\mathrm{Re}\left[1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)}\right] < 1, \quad z \in U, \tag{2.36}$$

*then*

$$|\frac{zf'(z)}{g(z)} - 1| < 1, \tag{2.37}$$

*and these conclusions are sharp.*

From Corrolary 2.4 and Corrolary 2.5 we obtain the following result:

**Corollary 2.6** *Let* $f, g \in \mathcal{A}$ *with* $f(0) = f'(0) - 1 = g(0) = g'(0) - 1 = 0$, $g(z) \in S^*$ *and* $0 < \zeta \leq \frac{1}{2}$. *If*

$$|1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)}| < \zeta, \quad \textit{for all } z \in U, \tag{2.38}$$

*then*

$$|\frac{zf'(z)}{g(z)} - 1| < \frac{\zeta}{1-\zeta}, \quad z \in U. \tag{2.39}$$

**Proof.** Let $\frac{\zeta}{1-\zeta} = \mu$.
Then

$$|1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)}| < \zeta = \frac{\mu}{1+\mu}, \quad \text{for all } z \in U. \tag{2.40}$$

If $0<\zeta<\frac{1}{2}$, then $0<\mu<1$. We have

$$|\,1+\frac{zf''(z)}{f'(z)}-\frac{zg'(z)}{g(z)}+\frac{\mu^2}{1-\mu^2}-\frac{\mu^2}{1-\mu^2}\,|<\frac{\mu}{1+\mu},\quad z\in U, \tag{2.41}$$

i.e.

$$|\,1+\frac{zf''(z)}{f'(z)}-\frac{zg'(z)}{g(z)}+\frac{\mu^2}{1-\mu^2}\,|<\frac{\mu}{1+\mu}+\frac{\mu^2}{1-\mu^2}=\frac{\mu}{1-\mu^2},\quad z\in U. \tag{2.42}$$

From Corrolary 2.5, we get

$$|\,\frac{zf'(z)}{g(z)}-1\,|<\mu=\frac{\zeta}{1-\zeta},\quad z\in U. \tag{2.43}$$

If $\zeta=\frac{1}{2}$, then $\mu=1$. We have

$$|\,1+\frac{zf''(z)}{f'(z)}-\frac{zg'(z)}{g(z)}\,|<1,\text{ for all } z\in U, \tag{2.44}$$

i.e.

$$\operatorname{Re}\left[1+\frac{zf''(z)}{f'(z)}-\frac{zg'(z)}{g(z)}\right]<1,\quad z\in U. \tag{2.45}$$

From Corrolary 2.5, we get

$$|\,\frac{zf'(z)}{g(z)}-1\,|<1,\quad z\in U, \tag{2.46}$$

and the proof is complete.

**Corollary 2.7** *If $f,g\in\mathcal{A}$ with $f(0)=f'(0)-1=g(0)=g'(0)-1=0$, $g(z)\in S^*$. If*

$$1+\frac{zf''(z)}{f'(z)}-\frac{zg'(z)}{g(z)}\prec\frac{2z}{1-z^2}=h(z), \tag{2.47}$$

*then $f\in K$.*

**Proof.** Let $q(z)=\frac{1+z}{1-z}=1+q_1z+...$, with $q(0)=1$ satisfies the conditions of Theorem 2.3.
We have

$$h(z)=\theta(q(z))+Q(z)=\frac{zq'(z)}{q(z)},\quad z\in U. \tag{2.48}$$

So,

$$h(z)=\frac{2z}{1-z^2},\quad z\in U. \tag{2.49}$$

From (2.49), we find that the subordinations (2.22) and (2.47) are equivalent. Therefore, it follows directly from Theorem 2.3 that

$$\frac{zf'(z)}{g(z)}\prec\frac{1+z}{1-z},\quad z\in U, \tag{2.50}$$

and $\frac{1+z}{1-z}$ is the best dominant of (2.47).
Since the subordination (2.22) is equivalent with the condition

$$\operatorname{Re}\left[\frac{zf'(z)}{g(z)}\right]>0,\quad z\in U,$$

we get that $f\in K$, which completes the proof.

**Corollary 2.8** *Let $f,g\in\mathcal{A}$ with $f(0)=f'(0)-1=g(0)=g'(0)-1=0$, $g(z)\in S^*$ and $0\le\alpha<1$. If*

$$1+\frac{zf''(z)}{f'(z)}-\frac{zg'(z)}{g(z)}\prec\frac{2z(1-\alpha)}{(1-z)[1+(1-2\alpha)z]},\quad z\in U, \tag{2.51}$$

*then $f\in K(\alpha)$.*

**Proof.** For $0 \leq \alpha < 1$ and $z \in U$, setting $q(z) = \frac{1+(1-2\alpha)z}{1-z} = 1 + q_1 z + ...$, with $q(0) = 1$ in Theorem 2.3, we obtain

$$\operatorname{Re}\left[1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right] = \operatorname{Re}\left[\frac{1}{1-z} + \frac{1}{1+(1-2\alpha)z} - 1\right] > 0, \quad z \in U.$$

We have

$$h(z) = \frac{zq'(z)}{q(z)} = \frac{2z(1-\alpha)}{(1-z)[1+(1-2\alpha)z]}, \quad z \in U.$$

Hence, the hypotheses of Theorem 2.3 are satisfied. Therefore, it follows directly from Theorem 2.3 that

$$\frac{zf'(z)}{g(z)} \prec \frac{1+(1-2\alpha)z}{1-z}, \quad z \in U, 0 \leq \alpha < 1, \tag{2.52}$$

i.e.

$$\operatorname{Re}\left[\frac{zf'(z)}{g(z)}\right] > \alpha, \quad z \in U,$$

i.e. $f \in K(\alpha)$, and the proof is now complete.

# References


[1] Alexander, J.W., Functions which maps the interior of the unit circle upon simple regions, *Ann. of Math.*, 17, 12-22 (1915).

[2] R. Bucur, L. Andrei, D. Breaz, Geometric Properties of a New Integral Operator, *Abstract and Applied Analysis*, Article ID 430197 (2015).

[3] W. Kaplan, Close-to-convex schlicht functions, *Mich. Math. J.*, 169-185 (1952).

[4] S.Kanas, A.E. Tudor, Differential Subordinations and Harmonic Means, *Bull. Malays. Math. Sci. Soc.*, 38, 1243–1253 (2015).

[5] S.S. Miller, P.T. Mocanu, Subordinants of Differential Superordinations, *Complex Variables,* vol. 48, no. 10, 815-826, October (2003).

[6] S.S. Miller, P.T. Mocanu, *Differential Subordinations: Theory and Applications,* Marcel Dekker Inc., New York, 2000.

[7] M. H. Mohd, M. Darus, On a class of spiral-like functions with respect to a boundary point related to subordination, *Journal of Inequalities and Applications*, 274 (2013).

[8] M. Nunokawa, N. E. Cho, O.S. Kwon, J. Sokół, On differential subordinations in the complex plane, *Journal of Inequalities and Applications*, 7 (2015).

[9] H. Shiraishi, S. Owa, T. Hayami, K. Kuroki and H. M. Srivastava, Starlike problems for certain analytic functions concerned with subordinations, *Pan Amer. Math. J.,* 21, 63–77 (2011).

[10] H. M. Srivastava and S. S. Eker, Some applications of a subordination theorem for a class of analytic functions, *Appl. Math. Lett.,* 21, 394–399 (2008).

[11] N. Tuneski, M. Obradović, Some properties of certain expressions of analytic functions, *Computers and Mathematics with Applications,* 62, 3438–3445 (2011).

[12] Q.-H. Xu, C.-B. Lv, N.-C. Luo and H. M. Srivastava, Sharp coefficient estimates for a certain general class of spirallike functions by means of differential subordination, *Filomat,* 27, 1351–1356 (2013).

[1]

# DUALISTIC CONTRACTIONS OF RATIONAL TYPE AND FIXED POINT THEOREMS

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Abstract. In this paper, we introduce a dualistic contraction of rational type and use it to obtain some new fixed point results in dualistic partial metric spaces for dominating and dominated mappings. These results extend various comparable results existing in literature from set of positive real numbers to set of real numbers. Moreover, we give an example that shows the usefulness and effectiveness of these results among corresponding fixed point theorems established in partial metric spaces (nonnegative real numbers).

*Keywords:* fixed point; dualistic partial metric; dualistic contraction of rational type.
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## 1. Introduction and preliminaries

In [7], Matthews introduced the concept of partial metric space as a suitable mathematical tool for program verification and proved an analogue of Banach fixed point theorem in complete partial metric spaces. Fixed point theorems in complete partial metric spaces have been investigated in [3, 4, 11, 12, 14]. O'Neill [10] introduced the concept of dualistic partial metric, which is more general than partial metric and established a robust relationship between dualistic partial metric and quasi metric. In [13], Oltra and Valero presented a Banach fixed point theorem on complete dualistic partial metric spaces. Following Oltra and Valero, Nazam et al. [2, 8] established some fixed point results in dualistic partial metric spaces.

In this paper, we shall extend the following two fixed point theorems from nonnegative real numbers to real numbers to the theorems from partial metric spaces to dualistic partial metric spaces.

Harjani et al. [5] extended Banach's fixed point principle as follows:

**Theorem 1.** Let $M$ be a complete ordered metric space and $T : M \to M$ be a continuous and nondecreasing rational type contraction mapping. Then $T$ has a unique fixed point $m^* \in M$. Moreover, the Picard iterative sequence $\{T^n(j)\}_{n\in\mathbb{N}}$ converges to $m^*$ for every $j \in M$.

Isik and Tukroglu [6] established an ordered partial metric space version of Theorem 1, stated below.

**Theorem 2.** Let $M$ be a complete ordered partial metric space and $T : M \to M$ be a continuous and nondecreasing rational type contraction mapping. Then $T$ has a unique fixed point $m^* \in M$. Moreover, the Picard iterative sequence $\{T^n(j)\}_{n\in\mathbb{N}}$ converges to $m^*$ for every $j \in M$.

We need some mathematical concepts of dualistic partial metric space and results to make this paper self sufficient.

Throughout, in this paper, the letters $\mathbb{R}_0^+$, $\mathbb{R}$ and $\mathbb{N}$ will represent the set of nonnegative real numbers, set of real numbers and set of natural numbers, respectively.

**Definition 1.** [1]

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(1) Let $M$ be a nonempty set and $T : M \to M$ be a self mapping. A point $m^* \in M$ is called a fixed point of $T$ if $m^* = T(m^*)$.
(2) Let $(M, \preceq)$ be an ordered set and $T : M \to M$ be a self mapping defined on $M$ satisfying the property $j \preceq T(j)$ for all $j \in M$. Then $T$ is called a dominating mapping.
(3) Let $(M, \preceq)$ be an ordered set and $T : M \to M$ be a self mapping defined on $M$ satisfying the property $T(j) \preceq j$ for all $j \in M$. Then $T$ is called a dominated mapping.

According to O'Neill, a dualistic partial metric can be defined as follow:

**Definition 2.** [10] Let $M$ be a nonempty set. A function $D : M \times M \to \mathbb{R}$ is called a dualistic partial metric if for any $j, k, l \in M$, the following conditions hold:

$(D_1)$ $j = k \Leftrightarrow D(j, j) = D(k, k) = D(j, k)$.
$(D_2)$ $D(j, j) \leq D(j, k)$.
$(D_3)$ $D(j, k) = D(k, j)$.
$(D_4)$ $D(j, l) \leq D(j, k) + D(k, l) - D(k, k)$.

We observe that, as in the metric case, if $D$ is a dualistic partial metric then $D(j, k) = 0$ implies $j = k$. In case $D(j, k) \in \mathbb{R}_0^+$ for all $j, k \in M$, then $D$ is a partial metric on M. If $(M, D)$ is a dualistic partial metric space, then the function $d_D : M \times M \to \mathbb{R}_0^+$ defined by

$$d_D(j, k) = D(j, k) - D(j, j),$$

is a quasi metric on $M$ such that $\tau(D) = \tau(d_D)$. In this case, $d_D^s(j, k) = \max\{d_D(j, k), d_D(k, j)\}$ defines a metric on $M$.

**Remark 1.** It is obvious that every partial metric is a dualistic partial metric but the converse is not true. To support this comment, define $D_\vee : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$D_\vee(j, k) = j \vee k = \sup\{j, k\} \ \forall \ j, k \in \mathbb{R}.$$

It is easy to check that $D_\vee$ is a dualistic partial metric. Note that $D_\vee$ is not a partial metric, since $D_\vee(-1, -2) = -1 \notin \mathbb{R}_0^+$. Nevertheless, the restriction of $D_\vee$ to $\mathbb{R}_0^+$, $D_\vee|_{\mathbb{R}_0^+}$, is a partial metric.

**Definition 3.** [2] Let M be a nonempty set. Then $(M, \preceq, D)$ is said to be an ordered dualistic partial metric space if

(i) $(M, \preceq)$ is a partially ordered set.
(ii) $(M, D)$ is a dualistic partial metric space.

**Example 1.** If $(M, \preceq, d)$ is an ordered metric space and $c \in \mathbb{R}$ is an arbitrary constant, then $D_c : M \times M \to \mathbb{R}$ given by

$$D_c(j, k) = d(j, k) + c,$$

defines an ordered dualistic partial metric on $M$ and $(M, \preceq, D_c)$ is an ordered dualistic partial metric space.

Following [10], each dualistic partial metric $D$ on $M$ generates a $T_0$ topology $\tau(D)$ on $M$. The elements of the topology $\tau(D)$ are open balls of the form $\{B_D(j, \epsilon) : j \in M, \epsilon > 0\}$, where $B_D(j, \epsilon) = \{k \in M : D(j, k) < \epsilon + D(j, j)\}$. A sequence $\{j_n\}_{n\in\mathbb{N}}$ in $(M, D)$ converges to a point $j \in M$ if and only if $D(j, j) = \lim_{n\to\infty} D(j, j_n)$.

**Definition 4.** [8] Let $(M, D)$ be a dualistic partial metric space.

(1) A sequence $\{j_n\}_{n\in\mathbb{N}}$ in $(M, D)$ is called a Cauchy sequence if $\lim_{n,m\to\infty} D(j_n, j_m)$ exists and is finite.

(2) A dualistic partial metric space $(M, D)$ is said to be complete if every Cauchy sequence $\{j_n\}_{n\in\mathbb{N}}$ in $M$ converges, with respect to $\tau(D)$, to a point $j \in M$ such that $D(j,j) = \lim_{n,m\to\infty} D(j_n, j_m)$.

The following lemma will be helpful in the sequel.

**Lemma 1.** [8, 9]

(1) *A dualistic partial metric $(M, D)$ is complete if and only if the metric space $(M, d_D^s)$ is complete.*
(2) *A sequence $\{j_n\}_{n\in\mathbb{N}}$ in $M$ converges to a point $j \in M$, with respect to $\tau(d_D^s)$ if and only if $\lim_{n\to\infty} D(j, j_n) = D(j,j) = \lim_{n\to\infty} D(j_n, j_m)$.*
(3) *If $\lim_{n\to\infty} j_n = \upsilon$ such that $D(\upsilon,\upsilon) = 0$, then $\lim_{n\to\infty} D(j_n, k) = D(\upsilon, k)$ for every $k \in M$.*

Motivated by the results presented by Isik and Tukroglu [6] and Valero [9], we present a new fixed point theorem in an ordered dualistic partial metric space for both dominating and dominated mappings.

## 2. Main results

We begin with the following definition.

**Definition 5.** Let $(M, \preceq, D)$ be a complete ordered dualistic partial metric space. A mapping $T : M \to M$ is said to be a dualistic contraction of rational type if

$$|D(T(j), T(k))| \leq \alpha \left| \frac{D(j, T(j)) \cdot D(k, T(k))}{D(j,k)} \right| + \beta |D(j,k)| \tag{2.1}$$

holds for all comparable elements $j, k \in M$ and $0 < \alpha + \beta < 1$.

**Theorem 3.** *Let $(M, \preceq, D)$ be a complete ordered dualistic partial metric space. Suppose that $T : M \to M$ is a mapping such that*

(1) *$T$ is a dualistic contraction of rational type;*
(2) *$T$ is a dominating mapping;*
(3) *$T$ is a continuous mapping.*

*Then $T$ has a fixed point $m^*$. Moreover, $D(m^*, m^*) = 0$.*

*Proof.* Let $j_0$ be an initial point in $M$ and $j_n = T(j_{n-1})$ be an iterative sequence. If there exists a positive integer $r$ such that $j_{r+1} = j_r$, then $j_r$ is the fixed point of $T$ and it completes the proof. If $j_n \neq j_{n+1}$ for any $n \in \mathbb{N}$, then since $T$ is dominating, $j_n \preceq T(j_n)$ for all $n \in \mathbb{N}$. Therefore,

$$j_0 \preceq j_1 \preceq j_2 \preceq j_3 \preceq \cdots \preceq j_n \preceq j_{n+1} \cdots .$$

Now since $j_n \preceq j_{n+1}$, by (2.1), we have

$$\begin{aligned}
|D(j_n, j_{n+1})| &= |D(T(j_{n-1}), T(j_n))| \\
&\leq \alpha \left| \frac{D(j_{n-1}, j_n) \cdot D(j_n, j_{n+1})}{D(j_{n-1}, j_n)} \right| + \beta |D(j_{n-1}, j_n)| \\
&\leq \alpha |D(j_n, j_{n+1})| + \beta |D(j_{n-1}, j_n)|, \\
|D(j_n, j_{n+1})| - \alpha |D(j_n, j_{n+1})| &\leq \beta |D(j_{n-1}, j_n)|, \\
(1-\alpha)|D(j_n, j_{n+1})| &\leq \beta |D(j_{n-1}, j_n)|, \\
|D(j_n, j_{n+1})| &\leq (\frac{\beta}{1-\alpha}) |D(j_{n-1}, j_n)|.
\end{aligned}$$

If $\gamma = \dfrac{\beta}{1-\alpha}$, then $0 < \gamma < 1$ and we have

$$|D(j_n, j_{n+1})| \leq \gamma |D(j_{n-1}, j_n)|.$$

Thus

$$|D(j_n, j_{n+1})| \leq \gamma |D(j_{n-1}, j_n)| \leq \gamma^2 |D(j_{n-2}, j_{n-1})| \leq \cdots \leq \gamma^n |D(j_0, j_1)|.$$

Since $j_n \preceq j_n$, for each $n \in \mathbb{N}$, by (2.1), we have

$$\begin{aligned} |D(j_n, j_n)| = |D(T(j_{n-1}), T(j_{n-1}))| &\leq \frac{\alpha |D(j_{n-1}, j_n)|^2}{|D(j_{n-1}, j_{n-1})|} + \beta |D(j_{n-1}, j_{n-1})| \\ &\leq |D(j_{n-1}, j_{n-1})| \left\{ \frac{\alpha |D(j_{n-1}, j_n)|^2}{|D(j_{n-1}, j_{n-1})|^2} + \beta \right\} \\ &\leq (\alpha + \beta) |D(j_{n-1}, j_{n-1})|. \end{aligned}$$

Indeed, $\left| \dfrac{D(j_{n-1}, j_n)^2}{D(j_{n-1}, j_{n-1})^2} \right| = 1$. Thus we obtain that

$$|D(j_n, j_n)| \leq (\alpha + \beta)^n |D(j_0, j_0)|. \tag{2.2}$$

Now we show that $\{j_n\}$ is a Cauchy sequence in $(M, d_D^s)$. Note that $d_D(j_n, j_{n+1}) = D(j_n, j_{n+1}) - D(j_n, j_n)$, that is, $d_D(j_n, j_{n+1}) + D(j_n, j_n) = D(j_n, j_{n+1}) \leq |D(j_n, j_{n+1})|$. Thus we have

$$\begin{aligned} d_D(j_n, j_{n+1}) + D(j_n, j_n) &\leq \gamma^n |D(j_0, j_1)|, \\ d_D(j_n, j_{n+1}) &\leq \gamma^n |D(j_0, j_1)| + |D(j_n, j_n)| \\ &\leq \gamma^n |D(j_0, j_1)| + (\alpha + \beta)^n |D(j_0, j_0)|. \end{aligned}$$

Continuing this way, we obtain that

$$d_D(j_{n+k-1}, j_{n+k}) \leq \gamma^{n+k-1} |D(j_0, j_1)| + (\alpha + \beta)^{n+k-1} |D(j_0, j_0)|.$$

Now

$$\begin{aligned} d_D(j_n, j_{n+k}) &\leq d_D(j_n, j_{n+1}) + d_D(j_{n+1}, j_{n+2}) + \cdots + d_D(j_{n+k-1}, j_{n+k}) \\ &\leq \{\gamma^n + \gamma^{n+1} + \cdots + \gamma^{n+k-1}\} |D(j_0, j_1)| \\ &+ \{(\alpha + \beta)^n + (\alpha + \beta)^{n+1} + \cdots + (\alpha + \beta)^{n+k-1}\} |D(j_0, j_0)|. \end{aligned}$$

Thus, for $n + k = m > n$,

$$d_D(j_n, j_m) \leq \frac{\gamma^n}{1-\gamma} |D(j_0, j_1)| + \frac{(\alpha + \beta)^n}{1 - (\alpha + \beta)} |D(j_0, j_0)|.$$

Similarly, we have

$$d_D(j_m, j_n) \leq \frac{\gamma^n}{1-\gamma} |D(j_1, j_0)| + \frac{(\alpha + \beta)^n}{1 - (\alpha + \beta)} |D(j_0, j_0)|.$$

Taking limit as $n, m \to \infty$, we have

$$\lim_{n,m \to \infty} d_D(j_m, j_n) = 0 = \lim_{n,m \to \infty} d_D(j_n, j_m) \text{ and hence } \lim_{n,m \to \infty} d_D^s(j_m, j_n) = 0.$$

Thus $\{j_n\}$ is a Cauchy sequence in $(M, d_D^s)$. Since $(M, D)$ is a complete dualistic partial metric space, by Lemma 1, the metric space $(M, d_D^s)$ is also complete. Therefore, there exists $m^* \in M$ such that $\lim_{n \to \infty} d_D^s(j_n, m^*) = 0$. Again from Lemma 1, we have

$$\lim_{n \to \infty} d_D^s(j_n, m^*) = 0 \iff D(m^*, m^*) = \lim_{n \to \infty} D(j_n, m^*) = \lim_{n,m \to \infty} D(j_m, j_n).$$

Now $\lim_{n,m\to\infty} d_D(j_m, j_n) = 0$ implies that $\lim_{n,m\to\infty}[D(j_m, j_n) - D(j_n, j_n)] = 0$ and hence $\lim_{n,m\to\infty} D(j_n, j_m) = \lim_{n\to\infty} D(j_n, j_n)$. By (2.2), we have $\lim_{n\to\infty} D(j_n, j_n) = 0$. Consequently, $\lim_{n,m\to\infty} D(j_n, j_m) = 0$ and $\{j_n\}$ is a Cauchy sequence in $(M, D)$. Thus

$$D(m^*, m^*) = \lim_{n\to\infty} D(j_n, m^*) = 0. \tag{2.3}$$

Now $d_D(m^*, T(m^*)) = D(m^*, T(m^*)) - D(m^*, m^*) = D(m^*, T(m^*))$ implies that $D(m^*, T(m^*)) \geq 0$. Since $T$ is continuous, for a given $\epsilon > 0$, there exists a $\delta > 0$ such that $T(B_D(m^*, \delta)) \subseteq B_D(T(m^*), \epsilon)$. Since $\lim_{n\to\infty} D(j_{n+1}, m^*) = D(m^*, m^*) = 0$, there exists an $r \in \mathbb{N}$ such that $D(j_n, m^*) < D(m^*, m^*) + \delta$ for all $n \geq r$, and so $\{j_n\} \subset B_D(m^*, \delta)$ for all $n \geq r$. This implies that $T(j_n) \in T(B_D(m^*, \delta) \subseteq B_D(T(m^*), \epsilon)$ and so $D(T(j_n), T(m^*)) < D(T(m^*), T(m^*)) + \epsilon$ for all $n \geq r$. Now for any $\epsilon > 0$, we know that

$$-\epsilon + D(T(m^*), T(m^*)) < D(T(m^*), T(m^*)) \leq D(j_{n+1}, T(m^*)),$$

which implies that

$$|D(j_{n+1}, T(m^*)) - D(T(m^*), T(m^*))| < \epsilon.$$

That is, $D(T(m^*), T(m^*)) = \lim_{n\to\infty} D(j_{n+1}, T(m^*))$. The uniqueness of limit in $\mathbb{R}$ implies

$$\lim_{n\to\infty} D(j_{n+1}, T(m^*)) = D(T(m^*), T(m^*)) = D(m^*, T(m^*)). \tag{2.4}$$

Finally, we have $D(T(m^*), m^*) = \lim_{n\to\infty} D(T(j_n), j_n) = \lim_{n\to\infty} D(j_{n+1}, j_n) = 0$. This shows that $D(m^*, T(m^*)) = 0$. So from (2.3) and (2.4) we deduce that

$$D(m^*, T(m^*)) = D(T(m^*), T(m^*)) = D(m^*, m^*).$$

This leads us to conclude that $m^* = T(m^*)$ and hence $m^*$ is a fixed point of $T$. □

In order to prove the uniqueness of fixed point of a mapping $T$ in the above theorem, we give the following theorem.

**Theorem 4.** *Let $(M, \preceq, D)$ be a complete ordered dualistic partial metric space and $T : M \to M$ be a mapping which satisfies all the conditions of Theorem 3. Then $T$ has a unique fixed point provided that for each fixed point $m^*$, $n^*$ of $T$, there exists $\omega \in M$ which is comparable to both $m^*$ and $n^*$.*

*Proof.* From Theorem 3, it follows that the set of fixed points of $T$ is nonempty.

Assume that $n^*$ is another fixed point of $T$, that is, $n^* = T(n^*)$.

Case I: $m^*$ and $n^*$ are comparable.
In this case, we have

$$\begin{aligned} |D(m^*, n^*)| &= |D(T(m^*), T(n^*))| \\ &\leq \alpha \left| \frac{D(m^*, T(m^*)) \cdot D(n^*, T(n^*))}{D(m^*, n^*)} \right| + \beta |D(m^*, n^*)| \\ &\leq \alpha \left| \frac{D(m^*, m^*) \cdot D(n^*, n^*)}{D(m^*, n^*)} \right| + \beta |D(m^*, n^*)|. \end{aligned}$$

That is, $(1 - \beta)|D(m^*, n^*)| \leq 0$ which implies that $|D(m^*, n^*)| \leq 0$ and hence $D(m^*, n^*) = 0 = D(m^*, m^*) = D(n^*, n^*)$. So $m^*$ is a unique fixed point of $T$.

Case II: $m^*$ and $n^*$ are incomparable.
In this case, there exists $\omega$ which is comparable to both $m^*$, $n^*$. Without any loss of generality,

we assume that $m^* \preceq \omega$ and $n^* \preceq \omega$. Since $T$ is dominating, $m^* \preceq T(\omega)$ and $n^* \preceq T(\omega)$ imply that $m^* \preceq T^{n-1}(\omega)$ and $n^* \preceq T^{n-1}(\omega)$. Thus

$$\begin{aligned} |D(m^*, T^n(\omega))| &\leq \alpha \left| \frac{D(T^{n-1}(m^*), T^n(m^*)) \cdot D(T^{n-1}(\omega), T^n(\omega))}{D(T^{n-1}(m^*), T^{n-1}(\omega))} \right| \\ &+ \beta |D(T^{n-1}(m^*), T^{n-1}(\omega))|, \end{aligned}$$

which implies that $|D(m^*, T^n(\omega))| \leq \beta |D(m^*, T^{n-1}(\omega))|$. Thus $\lim_{n\to\infty} D(m^*, T^n(\omega)) = 0$. Similarly, $\lim_{n\to\infty} D(n^*, T^n(\omega)) = 0$. Moreover, by $D_4$,

$$\begin{aligned} D(n^*, m^*) &\leq D(n^*, T^n(\omega)) + D(T^n(\omega), m^*) - D(T^n(\omega), T^n(\omega)) \\ &\leq D(n^*, T^n(\omega)) + D(T^n(\omega), m^*) - D(T^n(\omega), m^*) - D(m^*, T^n(\omega)) + D(m^*, m^*). \end{aligned}$$

Letting $n \to \infty$, we obtain that $D(n^*, m^*) \leq 0$ but $d_D(m^*, m^*) = D(n^*, m^*) - D(n^*, n^*)$ implies that $D(n^*, m^*) \geq 0$. Hence $D(n^*, m^*) = 0$ which gives that $n^* = m^*$. □

We provide an example to explain the above result.

**Example 2.** Let $M = \mathbb{R}^2$. Define $D_\vee : M \times M \to \mathbb{R}$ by $D_\vee(j, k) = j_1 \vee k_1$, where $j = (j_1, j_2)$ and $k = (k_1, k_2)$. Note that $(M, D_\vee)$ is a complete dualistic partial metric space. Let $T : M \to M$ be given by

$$T(j) = e^{|D_\vee(j,j)|} \quad \text{for all } j \in M.$$

In $M$, we define the relation $\succeq$ in the following way:

$$j \succeq k \text{ if and only if } j_1 \geq k_1, \text{ where } j = (j_1, j_2) \text{ and } k = (k_1, k_2).$$

Clearly, $\succeq$ is a partial order on $M$ and $T$ is a continuous, dominating mapping with respect to $\succeq$ and satisfies the contractive condition (2.1).

**Remark 2.** Since every partial metric is a dualistic partial metric $D$ with $D(j, k) \in \mathbb{R}_0^+$ for all $j, k \in M$, Theorem 3 is an extension of Theorem 2.

There arises the following natural question:

Whether the contractive condition (2.1) in the statement of Theorem 3 can be replaced by the contractive condition in Theorem 2?

The following example provides a negative answer to the above question.

**Example 3.** *Define the mapping $T_0 : \mathbb{R} \to \mathbb{R}$ by*

$$T_0(j) = \begin{cases} 0 & \text{if } j > 1 \\ -\frac{1}{2} & \text{if } j = -5 \end{cases}.$$

*Clearly, for any $j, k \in \mathbb{R}$ and $\alpha = 0.1$ and $\beta = 0.09$, the following contractive condition is satisfied*

$$D_\vee(T_0(j), T_0(k)) \leq \frac{\alpha D_\vee(j, T_0(j)) \cdot D_\vee(k, T_0(k))}{D_\vee(j, k)} + \beta D_\vee(j, k),$$

*where $D_\vee$ is a complete dualistic partial metric on $\mathbb{R}$. Here $T$ has no fixed point. Thus a fixed point free mapping satisfies the contractive condition of Theorem 2. On the other hand, for $\alpha = 0.1$ and $\beta = 0.09$, we have*

$$0.5 = |D_\vee(-\frac{1}{2}, -\frac{1}{2})| = |D_\vee(T_0(-5), T_0(-5))|$$

$$> 0.455 = \left| \frac{\alpha D_\vee(-5, T_0(-5)) \cdot D(-5, T_0(-5))}{D_\vee(-5, -5)} \right| + \beta |D_\vee(-5, -5)|.$$

*Thus the contractive condition (2.1) of Theorem 3 does not hold.*

In next theorem, we show that the conclusion of Theorem 3 remains the same if the continuity of the mapping $T$ is replaced with the following condition:

(H): If $\{j_n\}$ is a nondecreasing sequence in $M$ such that $j_n \to \upsilon$, then $j_n \preceq \upsilon$ for all $n \in \mathbb{N}$.

For dominated mappings, the following condition will be needed:

(Q): If $\{j_n\}$ is a nonincreasing sequence in $M$ such that $j_n \to \upsilon$, then $j_n \succeq \upsilon$ for all $n \in \mathbb{N}$.

**Theorem 5.** *Let $(M, \preceq, D)$ be a complete ordered dualistic partial metric space. Suppose that $T : M \to M$ is a mapping such that*

(1) *$T$ is a dualistic contraction of rational type;*
(2) *$T$ is a dominating mapping;*
(3) *(H) holds.*

*Then $T$ has a fixed point $m^*$. Moreover, $D(m^*, m^*) = 0$.*

*Proof.* By the arguments similar to those in the proof of Theorem 3, we obtain that $\{j_n\}$ is a nondecreasing sequence in $M$ such that $j_n \to m^*$. By (H), we have $j_n \preceq m^*$. Since $T$ is a dominating mapping, we have $j_n \preceq T(m^*)$ and

$$m^* \preceq T(m^*). \tag{2.5}$$

From the proof of Theorem 3, we deduce that $\{T^n(m^*)\}$ is a nondecreasing sequence. Suppose that $\lim_{n\to+\infty} T^n(m^*) = \mu$ for some $\mu \in M$. Now $j_n \preceq m^*$ implies $j_n \preceq T^n(m^*)$ for all $n \geq 1$. Thus we have

$$j_n \preceq m^* \preceq T(m^*) \preceq T^n(m^*) \quad n \geq 1.$$

From (2.1), we have

$$\begin{aligned} |D(j_{n+1}, T^{n+1}(m^*))| &= |D(T(j_n), T(T^n(m^*)))| \\ &\leq \left| \frac{\alpha D(j_n, j_{n+1}) \cdot D(T^n(m^*), T^{n+1}(m^*))}{D(j_n, T^n(m^*))} \right| + \beta |D(j_n, T^n(m^*))|. \end{aligned}$$

Taking limit as $n \to +\infty$, we obtain that

$$|D(m^*, \mu)| \leq \beta |D(m^*, \mu)|,$$

which implies that $m^* = \mu$. Thus $\lim_{n\to+\infty} T^n(m^*) = \mu$ implies that $\lim_{n\to+\infty} T^n(m^*) = m^*$. Hence

$$T(m^*) \preceq m^*. \tag{2.6}$$

From (2.5) and (2.6), it follows that $m^* = T(m^*)$. □

Now we present some important consequences of our results.

**Corollary 1.** *Let $(M, \preceq, D)$ be a complete ordered dualistic partial metric space. Suppose that $T : M \to M$ is a mapping such that*

(1) *$T$ is a dualistic contraction of rational type with $\beta = 0$;*
(2) *$T$ is a dominating mapping;*
(3) *$T$ is a continuous mapping.*

*Then $T$ has a fixed point $m^*$. Moreover, $D(m^*, m^*) = 0$.*

**Corollary 2.** *Let $(M, \preceq, D)$ be a complete ordered dualistic partial metric space. Suppose that $T : M \to M$ is a mapping such that*

(1) *$T$ is a dualistic contraction of rational type with $\beta = 0$;*
(2) *$T$ is a dominating mapping;*
(3) *(H) holds.*

*Then $T$ has a fixed point $m^*$. Moreover, $D(m^*, m^*) = 0$.*

**Corollary 3.** *Let $(M, \preceq, D)$ be a complete ordered dualistic partial metric space. Suppose that $T : M \to M$ is a mapping such that*

(1) *$T$ is a dualistic contraction of rational type with $\alpha = 0$;*
(2) *$T$ is a dominating mapping;*
(3) *$T$ is a continuous mapping.*

*Then $T$ has a fixed point $m^*$. Moreover, $D(m^*, m^*) = 0$.*

**Corollary 4.** *Let $(M, \preceq, D)$ be a complete ordered dualistic partial metric space. Suppose that $T : M \to M$ is a mapping such that*

(1) *$T$ is a dualistic contraction of rational type with $\alpha = 0$;*
(2) *$T$ is a dominating mapping;*
(3) *(H) holds.*

*Then $T$ has a fixed point $m^*$. Moreover, $D(m^*, m^*) = 0$.*

For dominated mappings, we present the following results.

**Theorem 6.** *Let $(M, \preceq, D)$ be a complete ordered dualistic partial metric space. Suppose that $T : M \to M$ is a mapping such that*

(1) *$T$ is a dualistic contraction of rational type;*
(2) *$T$ is a dominated mapping;*
(3) *$T$ is a continuous mapping.*

*Then $T$ has a fixed point $m^*$. Moreover, $D(m^*, m^*) = 0$.*

*Proof.* Let $j_0 \in M$ be an initial element and $j_n = T(j_{n-1})$ for all $n \geq 1$. If there exists a positive integer $r$ such that $j_{r+1} = j_r$ then $j_r = T(j_r)$, and so we are done. Suppose that $j_n \neq j_{n+1}$ for all $n \in \mathbb{N}$. Since $T$ is a dominated mapping, $j_0 \succeq T(j_0) = j_1$, and $j_1 \succeq T(j_1)$ implies $j_1 \succeq j_2$, and $j_2 \succeq T(j_2)$ implies $j_2 \succeq j_3$. Continuing in the similar way, we get

$$j_0 \succeq j_1 \succeq j_2 \succeq j_3 \succeq \cdots \succeq j_n \succeq j_{n+1} \succeq j_{n+2} \succeq \cdots .$$

The remaining part of the proof follows from the proof of Theorem 3. □

The following example shall illustrate Theorem 6.

**Example 4.** Let $M = \mathbb{R}^2$. Define $D_\vee : M \times M \to \mathbb{R}$ by $D_\vee(j, k) = j_1 \vee k_1$, where $j = (j_1, j_2)$ and $k = (k_1, k_2)$. Note that $(M, D_\vee)$ is a complete dualistic partial metric space. Let $T : M \to M$ be given by

$$T(j) = \frac{j}{2} \quad \text{for all } j \in M.$$

In $M$, we define the relation $\succeq$ in the following way:

$$j \succeq k \text{ if and only if } j_1 \geq k_1, \text{ where } j = (j_1, j_2) \text{ and } k = (k_1, k_2).$$

Clearly, $\succeq$ is a partial order on $M$ and $T$ is a continuous, dominated mapping with respect to $\succeq$. Moreover, $T(-1, 0) \succeq (-1, 0)$. We shall show that for all $j, k \in M$, the contractive condition (2.1) is satisfied. For this, note that

$$\begin{aligned}
|D_\vee(T(j), T(k))| = \left| D_\vee\left(\frac{j}{2}, \frac{k}{2}\right) \right| &= \left| \frac{j_1}{2} \right| \text{ for all } j_1 \geq k_1, \\
|D_\vee(j, T(j))| = \left| D_\vee\left(j, \frac{j}{2}\right) \right| &= \begin{cases} \left|\frac{j_1}{2}\right| & \text{if } j_1 \leq 0 \\ |j_1| & \text{if } j_1 \geq 0, \end{cases} \\
|D_\vee(k, T(k))| = \left| D_\vee\left(k, \frac{k}{2}\right) \right| &= \begin{cases} \left|\frac{k_1}{2}\right| & \text{if } k_1 \leq 0 \\ |k_1| & \text{if } k_1 \geq 0, \end{cases} \\
\text{and } |D_\vee(j, k)| &= |j_1| \text{ for all } j_1 \geq k_1.
\end{aligned}$$

Now for $\alpha = \frac{1}{3}$, $\beta = \frac{1}{2}$. If $j_1 \leq 0$, $k_1 \leq 0$, then

$$|D_\vee(T(j), T(k))| \leq \alpha \left| \frac{D_\vee(j, T(j)) \cdot D_\vee(k, T(k))}{D_\vee(j, k)} \right| + \beta |D_\vee(j, k)|$$

holds for all $j \succeq k$ if and only if $6|j_1| \leq |k_1| + 6|j_1|$.

If $j_1 \geq 0$, $k_1 \geq 0$, then the contractive condition

$$|D_\vee(T(j), T(k))| \leq \alpha \left| \frac{D_\vee(j, T(j)) \cdot D_\vee(k, T(k))}{D_\vee(j, k)} \right| + \beta |D_\vee(j, k)|$$

holds for all $j \succeq k$ if and only if $j_1 \leq \frac{2}{3}k_1 + j_1$.

Finally, if $j_1 \geq 0$, $k_1 \leq 0$, then

$$|D_\vee(T(j), T(k))| \leq \alpha \left| \frac{D_\vee(j, T(j)) \cdot D_\vee(k, T(k))}{D_\vee(j, k)} \right| + \beta |D_\vee(j, k)|$$

holds for all $j \succeq k$ if and only if $3j_1 \leq |k_1| + 3j_1$. Thus all the conditions of Theorem 5 are satisfied. Moreover, $(0, 0)$ is a fixed point of $T$.

**Theorem 7.** *Let $(M, \preceq, D)$ be a complete ordered dualistic partial metric space. Suppose that $T : M \to M$ is a mapping such that*

(1) *$T$ is a dualistic contraction of rational type;*
(2) *$T$ is a dominated mapping;*
(3) *(Q) holds.*

*Then $T$ has a fixed point $m^*$. Moreover, $D(m^*, m^*) = 0$.*

The proof can be obtained by the proofs of Theorems 5 and 6. Some consequences of Theorems 6 and 7 are given below.

**Corollary 5.** *Let $(M, \preceq, D)$ be a complete ordered dualistic partial metric space. Suppose that $T : M \to M$ is a mapping such that*

(1) *$T$ is a dualistic contraction of rational type with $\beta = 0$;*
(2) *$T$ is a dominated mapping;*
(3) *$T$ is a continuous mapping.*

*Then $T$ has a fixed point $m^*$. Moreover, $D(m^*, m^*) = 0$.*

**Corollary 6.** *Let $(M, \preceq, D)$ be a complete ordered dualistic partial metric space. Suppose that $T : M \to M$ is a mapping such that*

(1) *$T$ is a dualistic contraction of rational type with $\beta = 0$;*
(2) *$T$ is a dominated mapping;*
(3) *(Q) holds.*

*Then $T$ has a fixed point $m^*$. Moreover, $D(m^*, m^*) = 0$.*

**Corollary 7.** *Let $(M, \preceq, D)$ be a complete ordered dualistic partial metric space. Suppose that $T : M \to M$ is a mapping such that*

(1) *$T$ is a dualistic contraction of rational type with $\alpha = 0$;*
(2) *$T$ is a dominated mapping;*
(3) *$T$ is a continuous mapping.*

*Then $T$ has a fixed point $m^*$. Moreover, $D(m^*, m^*) = 0$.*

**Corollary 8.** *Let $(M, \preceq, D)$ be a complete ordered dualistic partial metric space. Suppose that $T : M \to M$ is a mapping such that*

(1) *$T$ is a dualistic contraction of rational type with $\alpha = 0$;*
(2) *$T$ is a dominated mapping;*
(3) *(Q) holds.*

*Then $T$ has a fixed point $m^*$. Moreover, $D(m^*, m^*) = 0$.*

## References


[1] M. Arshad, A. Azam, M. Abbas, A. Shoaib, *Fixed point results of dominated mappings on a closed ball in ordered partial metric spaces without continuity,* U.P.B. Sci. Bull. Series A **76** (2014), 123–134.

[2] M. Arshad, M. Nazam, I. Beg, *Fixed point theorems in ordered dualistic partial metric spaces*, Korean J. Math. **24** (2016), 169–179.

[3] X. Fan, Z. Wang, *Some fixed point theorems in generalized quasi-partial metric spaces*, J. Nonlinear Sci. Appl. **9** (2016), 1658–1674.

[4] A. Farajzadeh, P. Chuadchawna, A. Kaewcharoen, *Fixed point theorems for $(\alpha, \eta, \psi, \xi)$-contractive multivalued mappings on $\alpha$-$\eta$-complete partial metric spaces*, J. Nonlinear Sci. Appl. **9** (2016), 1977–1990.

[5] J. Harjani, B. Lopez, K. Sadarangani, *A fixed point theorem for mappings satisfying a contractive condition of rational type on a partially ordered metric space*, Abstr. Appl. Anal. **2010**, Art. ID 190701 (2010).

[6] H. Isik, D. Tukroglu *Some fixed point theorems in ordered partial metric spaces,* J. Inequal. Spec. Funct. **4** (2013), No. 2, 13–18.

[7] S.G. Matthews, *Partial metric topology*, in Proceedings of the $11^th$ Summer Conference on General Topology and Applications **728**, pp. 183–197, New York Acad. Sci., New York, 1994.

[8] M. Nazam, M. Arshad, *On a fixed point theorem with application to integral equations*, Int. J. Anal. **2016**, Art. ID 9843207 (2016).

[9] S. Oltra, O. Valero, *Banach's fixed point theorem for partial metric spaces*, Rend. Ist. Mat. Univ. Trieste **36** (2004),17–26.

[10] S.J. O'Neill, *Partial metric, valuations and domain theory*, Ann. New York Acad. Sci. **806** (1996), 304–315.

[11] A. Shoaib, *$\alpha$-$\eta$ dominated mappings and related common fixed point results in closed ball*, J. Concrete Appl. Math. **13** (2015), 152–170.

[12] A. Shoaib, M. Arshad, M. A. Kutbi, *Common fixed points of a pair of Hardy Rogers type mappings on a closed ball in ordered partial metric spaces*, J. Comput. Anal. Appl. **17** (2014), 255–264.

[13] O. Valero, *On Banach fixed point theorems for partial metric spaces*, Appl. Gen. Topol. **6** (2005), 229–240.

[14] H-H. Zheng, F. Gu, *Some results of common fixed point for four self-maps satisfying a new $\Psi$-contractive condition in partial metric spaces*, J. Nonlinear Sci. Appl. **9** (2016), 2258–2272.



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# HOMOMORPHISMS AND DERIVATIONS IN PROPER LIE $CQ^*$-ALGEBRAS

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Abstract. In this paper, we investigate homomorphisms in proper Lie $CQ^*$-algebras and derivations on proper Lie $CQ^*$-algebras associated with the following Pexiderized functional equation

$$f(x+y) = f_0(x) + f_1(y).$$

Moreover, we prove the Hyers-Ulam stability of homomorphisms in proper Lie $CQ^*$-algebras and of derivations on proper Lie $CQ^*$-algebras.

## 1. Introduction and preliminaries

Ulam [24] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

*Let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality*

$$d(h(x * y), h(x) \diamond h(y)) < \delta$$

*for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with*

$$d(h(x), H(x)) < \epsilon$$

*for all $x \in G_1$?*

Hyers [10] considered the case of approximately additive mappings $f : E \to E'$, where $E$ and $E'$ are Banach spaces and $f$ satisfies *Hyers inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon$$

for all $x, y \in E$. It was shown that the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and that $L : E \to E'$ is the unique additive mapping satisfying

$$\|f(x) - L(x)\| \leq \epsilon.$$

Rassias [16] provided a generalization of Hyers' Theorem which allows the *Cauchy difference to be unbounded.* See [18, 19, 20, 21, 22] for more information on functional equations and their stability.

Let $A$ be a linear space and $A_0$ is a $*$-algebra contained in $A$ as a subspace.We say that $A$ is a quasi $*$-algebra over $A_0$ if

$(i)$ the right and left multiplications of an element of $A$ and an element of $A_0$ are defined and linear;

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$(ii)$ $x_1(x_2a) = (x_1x_2)a, (ax_1)x_2 = a(x_1x_2)$ and $x_1(ax_2) = (x_1a)x_2$ for all $x_1, x_2 \in A_0$ and all $a \in A$;

$(iii)$ an involution $*$, which extends the involution of $A_0$, is defined in $A$ with the property $(ab)^* = b^*a^*$ whenever the multiplication is defined.

A quasi $*$-algebra $(A, A_0)$ is said to be a locally convex quasi $*$-algebra if in $A$ a locally convex topology $\tau$ is defined such that

$(i)$ the involution is continuous and the multiplications are separately continuous;

$(ii)$ $A_0$ is dense in $A[\tau]$.

Throughout this paper, we suppose that a locally convex quasi $*$-algebra $(A, A_0)$ is complete. For an overview on partial $*$-algebra and related topics we refer to [1].

In a series of papers [2, 4, 5, 6] many authors have considered a special class of quasi $*$-algebras, called proper $CQ^*$-algebras, which arise as completions of $C^*$-algebras. They can be introduced in the following way:

Let $A$ be a Banach module over the $C^*$-algebra $A_0$ with involution $*$ and $C^*$-norm $\| \, . \, \|_0$ such that $A_0 \subset A$. We say that $(A, A_0)$ is a proper $CQ^*$-algebra if

$(i)$ $A_0$ is dense in $A$ with respect to its norm $\| \, . \, \|$;

$(ii)$ $(ab)^* = b^*a^*$ whenever the multiplication is defined;

$(iii)$ $\| \, y \, \|_0 = sup_{a \in A, \|a\| \leq 1} \| \, ay \, \|$ for all $y \in A_0$.

**Definition 1.1.** A proper $CQ^*$-algebra $(A, A_0)$, endowed with the Lie product

$$[z, x] = \frac{zx - xz}{2}$$

for all $x \in A$ and all $z \in A_0$, is called a proper Lie $CQ^*$-algebra.

**Definition 1.2.** Let $(A, A_0)$ and $(B, B_0)$ be proper Lie $CQ^*$-algebras.

$(i)$ A $\mathbb{C}$-linear mapping $H : A \longrightarrow B$ is called a proper Lie $CQ^*$-algebra homomorphism if $H(z) \in B_0$ and $H([z, x]) = [H(z), H(x)]$ for all $z \in A_0$ and all $x \in A$.

$(ii)$ A $\mathbb{C}$-linear mapping $\delta : A_0 \to A$ is called a Lie derivation if

$$\delta([z, x]) = [\delta(z), x] + [z, \delta(x)]$$

for all $x, z \in A_0$ (see [12]).

Park and Rassias [14] investigated homomorphisms in proper $JCQ^*$-triples and derivations on proper $JCQ^*$-triples. Park [15] investigated homomorphisms in proper $CQ^*$-ternary algebras and derivations on proper $CQ^*$-ternary algebras. Najati and Park [11] investigated homomorphisms in quasi-Banach algebras associated to the Pexiderized Cauchy function equation.

In this paper, we investigate homomorphisms in proper Lie $CQ^*$-algebras and derivations on proper Lie $CQ^*$-algebras associated with the following Pexiderized functional equation

$$f(x + y) = f_0(x) + f_1(y).$$

This is applied to investigate homomorphisms in proper Lie $CQ^*$-algebras.

Throughout this paper, assume that $k$ is a fixed positive integer.

This paper is organized as follows: In Sections 2 and 3, we investigate homomorphisms in proper Lie $CQ^*$-algebras and derivations in proper Lie $CQ^*$-algebras.

In Sections 4 and 5, we prove the Hyers-Ulam stability of homomorphisms in proper Lie $CQ^*$-algebras and stability of derivations on proper Lie $CQ^*$-algebras.

## 2. Homomorphism in proper Lie $CQ^*$-algebra

Throughout this section, assume that $(A, A_0)$ is a proper Lie $CQ^*$-algebra with $C^*$-norm $\| \, . \, \|_{A_0}$ and norm $\| \, . \, \|_A$, and that $(B, B_0)$ is a proper Lie $CQ^*$-algebras with $C^*$-norm $\| \, . \, \|_{B_0}$ and norm $\| \, . \, \|_B$.

**Theorem 2.1.** *Let $\varphi : A_0 \times A \longrightarrow [0, +\infty)$ be a mapping such that*

$$\lim_{n\to\infty} \frac{1}{2^n}\varphi(2^n w, x) = 0 \tag{2.1}$$

*for all $w \in A_0$ and all $x \in A$, and let $f, f_0, f_1 : A \longrightarrow B$ be mappings with $f(0) = 0$ and $f(w), f_0(0) \in B_0$ for all $w \in A_0$ and*

$$\|\mu f(x) - f_0(y) - f_1(z)\|_B \leq \| \, kf(\frac{\mu x + y + z}{k}) \, \|_B, \tag{2.2}$$

$$\|f([w, x]) - [f_0(w), f_1(x)]\|_B \leq \varphi(w, x) \tag{2.3}$$

*for all $\mu \in \mathbb{T}^1 := \{\mu \in \mathbb{C} : |\mu| = 1\}$, all $w \in A_0$ and all $x, y, z \in A$. Then the mapping $f : A \longrightarrow B$ is a proper Lie $CQ^*$-algebra homomorphism. Moreover,*

$$f(x) = f_0(0) - f_0(x) = f_1(0) - f_1(x)$$

*for all $x \in A$.*

*Proof.* Letting $\mu = 1$, $y = -x$ and $z = 0$ in (2.2), we get

$$f(x) = f_0(-x) - f_0(0)$$

for all $x \in A$. So $f_0(w) \in B_0$ for all $w \in A_0$. Similarly, we have

$$f(x) = f_1(-x) - f_1(0)$$

for all $x \in A$. By (2.2), we have

$$\|f(x + y) - f(x) - f(y)\|_B \leq \|f(x + y) - f_0(-x) - f_1(-y)\|_B = 0$$

for all $x, y \in A$. So the mapping $f : A \longrightarrow B$ is additive. Letting $y = -\mu x$, and $z = 0$ in (2.2), we get

$$\mu f(x) = f_0(-\mu x) + f_1(0) = f(\mu x)$$

for all $x \in A$. By the same reasoning as in the proof of [13, Theorem 2.1], the mapping $f : A \longrightarrow B$ is $\mathbb{C}$-linear. By (2.1) and (2.3), we have

$$\begin{aligned}\|f([w, x]) - [f(w), f(x)]\|_B &= \lim_{n\to\infty} \frac{1}{2^n}\|f(2^n[w, x]) - [f_0(2^n w), f_1(x)]\|_B \\ &\leq \lim_{n\to\infty} \frac{\varphi(2^n w, x)}{2^n} = 0\end{aligned}$$

for all $w \in A_0$ and all $x \in A$. So

$$f([w, x]) = [f(w), f(x)]$$

for all $w \in A_0$ and all $x \in A$.

Therefore, the mapping $f : A \longrightarrow B$ is a proper Lie $CQ^*$-algebra homomorphism, as desired. □

**Remark 2.2.** We can formulate a similar theorem if we replace the condition (2.1) by

$$\lim_{n\to\infty} \frac{1}{2^n}\varphi(w, 2^n x) = 0$$

for all $w \in A_0$ and all $x \in A$.

**Corollary 2.3.** *Let $\theta, r_0, r_1$ be non-negative real numbers such that $r_j \in [0,1)$ for some $0 \leq j \leq 1$ and let $f, f_0, f_1 : A \longrightarrow B$ be mappings with $f(0) = 0$ and $f(w), f_0(0) \in B_0$ for all $w \in A_0$ and satisfying (2.2) and*

$$\|f([w,x]) - [f_0(w), f_1(x)]\|_B \leq \theta \|w\|_A^{r_0} \|x\|_A^{r_1}$$

*for all $w \in A_0$ and all $x \in A$ (by putting $\|.\|_A^0 = 1$). Then the mapping $f : A \longrightarrow B$ is a proper Lie $CQ^*$-algebra homomorphism. Moreover,*

$$f(x) = f_0(0) - f_0(x) = f_1(0) - f_1(x)$$

*for all $x \in A$.*

*Proof.* The proof follows from Theorem 2.1. □

**Corollary 2.4.** *Let $\theta, r_0, r_1$ be non-negative real numbers such that $r_j \in [0,1)$ for some $0 \leq j \leq 1$ and let $f, f_0, f_1 : A \longrightarrow B$ be mappings with $f(0) = 0$ and $f(w), f_0(0) \in B_0$ for all $w \in A_0$ and satisfying (2.2) and*

$$\|f([w,x]) - [f_0(w), f_1(x)]\|_B \leq \theta(\|w\|_{A_0}^{r_0} + \|x\|_A^{r_1})$$

*for all $w \in A_0$ and all $x \in A$ (by putting $\|.\|_A^0 = 1$). Then the mapping $f : A \longrightarrow B$ is a proper Lie $CQ^*$-algebra homomorphism. Moreover,*

$$f(x) = f_0(0) - f_0(x) = f_1(0) - f_1(x)$$

*for all $x \in A$.*

*Proof.* The proof follows from Theorem 2.1. □

## 3. Derivations on proper Lie $CQ^*$-algebras

Throughout this section, assume that $(A, A_0)$ is a proper Lie $CQ^*$-algebra with $C^*$-norm $\| \, . \, \|_{A_0}$ and norm $\| \, . \, \|_A$.

**Theorem 3.1.** *Let $\varphi : A_0 \times A_0 \to [0, +\infty)$ be a mapping such that*

$$\lim_{n\to\infty} \frac{1}{2^n} \varphi(2^n w_0, w_1) = 0 \tag{3.1}$$

*for all $w_0, w_1 \in A_0$ and let $f, f_0, f_1 : A_0 \longrightarrow A$ be mappings with $f(0) = 0$ and $f(w), f_0(0) \in A_0$ for all $w \in A_0$ and*

$$\|\mu f(w) - f_0(w_0) - f_1(w_1)\|_A \leq \| \, kf(\frac{\mu w + w_0 + w_1}{k}) \, \|_A,$$

$$\|f([w_0, w_1]) + [f_0(w_0), w_1] + [w_0, f_1(w_1)] \, \|_A \leq \varphi(w_0, w_1) \tag{3.2}$$

*for all $\mu \in \mathbb{T}^1$ and all $w, w_0, w_1 \in A_0$. Then the mapping $f : A_0 \longrightarrow A$ is a Lie derivation. Moreover,*

$$f(x) = f_0(0) - f_0(x) = f_1(0) - f_1(x)$$

*for all $x \in A_0$.*

*Proof.* By the same reasoning as in the proof of Theorem 2.1, the mapping $f : A_0 \longrightarrow A$ is $\mathbb{C}$-linear and $f(x) = f_0(0) - f_0(x) = f_1(0) - f_1(x)$ for all $x \in A_0$. By (3.1) and (3.2), we have

$$\begin{aligned}
&\|f([w_0, w_1]) - [f(w_0), w_1] - [w_0, f(w_1)]\|_A \\
&= \lim_{n\to\infty} \frac{1}{2^n}\|f(2^n[w_0, w_1]) + [f_0(2^n w_0), w_1] + [2^n w_0, f_1(w_1)]\|_A \\
&\leq \lim_{n\to\infty} \frac{1}{2^n}\varphi(2^n w_0, w_1) = 0
\end{aligned}$$

for all $w_0, w_1 \in A_0$. So

$$f([w_0, w_1]) = [f(w_0), w_1] + [w_0, f(w_1)]$$

for all $w_0, w_1 \in A_0$. Therefore, the mapping $f : A_0 \longrightarrow A$ is a Lie derivation, as desired. □

**Remark 3.2.** We can formulate a similar theorem if we replace the condition (3.1) by

$$\lim_{n\to\infty} \frac{1}{2^n}\varphi(w_0, 2^n w_1) = 0$$

for all $w_0, w_1 \in A_0$.

**Corollary 3.3.** *Let $\theta, r_0, r_1$ be non-negative real numbers such that $r_j \in [0, 1)$ for some $0 \leq j \leq 1$ and let $f, f_0, f_1 : A \longrightarrow B$ be mappings with $f(0) = 0$,$f(w), f_0(0) \in A_0$ for all $w \in A_0$ and satisfying (3.2) such that*

$$\|f([w_0, w_1]) + [f_0(w_0), w_1] + [w_0, f_1(w_1)]\|_A \leq \theta(\|w_0\|_{A_0}^{r_0} + \|w_1\|_{A_0}^{r_1})$$

*for all $w_0, w_1 \in A_0$ (by putting $\|.\|_A^0 = 1$). Then the mapping $f : A_0 \longrightarrow A$ is a Lie derivation. Moreover,*

$$f(x) = f_0(0) - f_0(x) = f_1(0) - f_1(x)$$

*for all $x \in A_0$.*

*Proof.* The proof follows from Theorem 3.1. □

**Corollary 3.4.** *Let $\theta, r_0, r_1$ be non-negative real numbers such that $r_j \in [0, 1)$ for some $0 \leq j \leq 1$ and let $f, f_0, f_1 : A \longrightarrow B$ be mappings with $f(0) = 0$ and satisfying (3.2) such that*

$$\|f([w_0, w_1]) + [f_0(w_0), w_1] + [w_0, f_1(w_1)]\|_A \leq \theta\|w_0\|_{A_0}^{r_0}\|w_1\|_{A_0}^{r_1}$$

*for all $w_0, w_1 \in A_0$ (by putting $\|.\|_A^0 = 1$). Then the mapping $f : A_0 \longrightarrow A$ is a Lie derivation. Moreover,*

$$f(x) = f_0(0) - f_0(x) = f_1(0) - f_1(x)$$

*for all $x \in A_0$.*

*Proof.* The proof follows from Theorem 3.1. □

## 4. Stability of homomorphism in proper Lie $CQ^*$-algebras

Using an idea of Găvruta [7], we prove the Hyers-Ulam stability of homomorphisms in proper Lie $CQ^*$-algebras.

**Theorem 4.1.** *Let $\varphi : A \times A \to [0, +\infty)$ be a mapping such that $\varphi(0, 0) = 0$ and*

$$\lim_{n\to\infty} \frac{1}{2^n}\varphi(2^n x, 2^n y) = 0, \tag{4.1}$$

$$\widetilde{\varphi}(x) := \sum_{i=0}^{\infty} \frac{1}{2^i}\Big[\varphi(2^i x, 2^i x) + \varphi(2^i x, 0) + \varphi(0, 2^i x)\Big] < \infty \tag{4.2}$$

*for all $x, y \in A$ and let $f, f_0, f_1 : A \to B$ be mappings with $f(0) = 0$ and $f(w), f_0(w), f_1(w) \in B_0$ for all $w \in A_0$ such that*

$$\|f(\mu x + \mu y) - \mu f_0(x) - \mu f_1(y)\|_B \leq \varphi(x, y), \tag{4.3}$$

$$\|f(w_0 + w_1) - f_0(w_0) - f_1(w_1)\|_{B_0} \leq \varphi(w_0, w_1), \tag{4.4}$$

$$\|f([w_0, x]) - [f_0(w_0), f_1(x)]\|_B \leq \varphi(w_0, x) \tag{4.5}$$

*for all $\mu \in \mathbb{T}^1$, all $w_0, w_1 \in A_0$ and all $x, y \in A$. Then there exists a unique proper Lie $CQ^*$-algebra homomorphism $H : A \longrightarrow B$ such that*

$$\begin{aligned} \|f(x) - H(x)\|_B &\leq \frac{1}{2}\widetilde{\varphi}(x), \\ \|f_0(x) - f_0(0) - H(x)\|_B &\leq \frac{1}{2}\widetilde{\varphi}(x) + \varphi(x, 0), \\ \|f_1(x) - f_1(0) - H(x)\|_B &\leq \frac{1}{2}\widetilde{\varphi}(x) + \varphi(0, x) \end{aligned} \tag{4.6}$$

*for all $x \in A$.*

*Proof.* Letting $y = 0$ and $\mu = 1$ in (4.3), we get

$$\|f(x) - f_0(x) - f_1(0)\|_B \leq \varphi(x, 0) \tag{4.7}$$

Similarly, we get

$$\|f(y) - f_1(y) - f_0(0)\|_B \leq \varphi(0, y) \tag{4.8}$$

for all $x, y \in A$. Using (4.7) and (4.8), we get

$$\|f(\mu x + \mu y) - \mu f(x) - \mu f(y)\|_B \leq \psi(x, y) \tag{4.9}$$

where

$$\psi(x, y) := \varphi(x, y) + \varphi(x, 0) + \varphi(0, y)$$

for all $\mu \in \mathbb{T}^1$ and all $x, y \in A$. Letting $x = y$ and $\mu = 1$ in (4.9), we get

$$\|f(2x) - 2f(x)\|_B \leq \psi(x, x) \tag{4.10}$$

Replacing $x$ by $2^n x$ in (4.10) and dividing both sides of (4.10) by $2^{n+1}$, we get

$$\Big\|\frac{f(2^{n+1}x)}{2^{n+1}} - \frac{f(2^n x)}{2^n}\Big\|_B \leq \frac{\psi(2^n x, 2^n x)}{2^{n+1}} \tag{4.11}$$

for all $x \in A$ and all non-negative integers $n$. By (4.11), we have

$$\Big\|\frac{f(2^{n+1}x)}{2^{n+1}} - \frac{f(2^m x)}{2^m}\Big\|_B \leq \frac{1}{2}\sum_{i=m}^{n} \frac{\psi(2^i x, 2^i x)}{2^i} \tag{4.12}$$

for all $x \in A$ and all non-negative integers $n$ and $m$ with $n \geq m$. Thus we conclude from (4.2) and (4.12) that the sequence $\{\frac{1}{2^n} f(2^n x)\}$ is Cauchy in $B$ for all $x \in A$. Since $B$ is complete, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ converges in $B$ for all $x \in A$. So one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x) = \lim_{n\to\infty} \frac{1}{2^n} f_i(2^n x) \quad (i = 0, 1) \tag{4.13}$$

for all $x \in A$. Letting $m = 0$ and passing the limit when $n \to \infty$ in (4.12), we get (4.6). It follows from (4.1), (4.3) and (4.13) that

$$\begin{aligned}&\Big\|H(\mu x+\mu y)-\mu H(x)-\mu H(y)\Big\|_B\\&=\lim_{n\to\infty}\frac{1}{2^n}\Big\|f(2^n\mu x+2^n\mu y)-\mu f_0(2^n x)-\mu f_1(2^n y)\Big\|_B\\&\leq\lim_{n\to\infty}\frac{\varphi(2^n x,2^n y)}{2^n}=0\end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y \in A$. Hence

$$H(\mu x+\mu y)=\mu H(x)+\mu H(y)$$

for all $\mu \in \mathbb{T}^1$ and all $x, y \in A$. By the same reasoning as in the proof of [13, Theorem 2.1], the mapping $H : A \to B$ is $\mathbb{C}$-linear. It follows from (4.4) and (4.13) that $H(w) \in B_0$ for all $w \in A_0$. It follows from (4.1), (4.5) and (4.13) that

$$\begin{aligned}\big\|H([w,x])-[H(w),H(x)]\big\|_B&\leq\lim_{n\to\infty}\frac{1}{4^n}\Big\|f(4^n[w,x])-[f_0(2^n w),f_1(2^n x)]\Big\|_B\\&\leq\lim_{n\to\infty}\frac{\varphi(2^n w,2^n x)}{4^n}=0\end{aligned}$$

for all $w \in A_0$ and all $x \in A$. Hence

$$H([w,x])=[H(w),H(x)]$$

for all $w \in A_0$ and all $x \in A$. So $H : A \longrightarrow B$ is a proper Lie $CQ^*$-algebra homomorphism. Now, we show that $H : A \longrightarrow B$ is unique. Let $T : A \to B$ be another proper Lie $CQ^*$-algebra homomorphism satisfying (4.6). It follows from (4.2), (4.6) and (4.13) that

$$\begin{aligned}\|H(x)-T(x)\|_B&=\lim_{n\to\infty}\frac{1}{2^n}\|f(2^n x)-T(2^n x)\|_B\\&\leq\frac{1}{2}\lim_{n\to\infty}\frac{1}{2^n}\widetilde{\varphi}(2^n x)=0\end{aligned}$$

for all $x \in A$. So $H = T$. □

**Corollary 4.2.** *Let $\theta, r_0, r_1$ be non-negative real numbers such that $0 \leq r_0, r_1 < 1$ and let $f, f_0, f_1 : A \longrightarrow B$ be mappings with $f(0) = 0$ and $f(w), f_0(w), f_1(w) \in B_0$ for all $w \in A_0$ and*

$$\|f(\mu x+\mu y)-\mu f_0(x)-\mu f_1(y)\|_B\leq\theta(\|x\|_A^{r_0}+\|y\|_A^{r_1}),$$

$$\|f(w_0+w_1)-f_0(w_0)-f_1(w_1)\|_{B_0}\leq\theta(\|w_0\|_A^{r_0}+\|w_1\|_A^{r_1}),$$

$$\|f([w_0,x])-[f_0(w_0),f_1(x)]\|_B\leq\theta(\|w_0\|_A^{r_0}+\|x\|_A^{r_1})$$

*for all $\mu \in \mathbb{T}^1$, all $w_0, w_1 \in A_0$ and all $x, y \in A$ (by putting $\|.\|_A^0 = 1$). Then there exists a unique proper Lie $CQ^*$-algebra homomorphism $H : A \longrightarrow B$ such that*

$$\|f(x)-H(x)\|_B\leq 2\Big(\frac{\|x\|_A^{r_0}}{2-2^{r_0}}+\frac{\|x\|_A^{r_1}}{2-2^{r_1}}\Big)\theta$$

$$\|f_i(x)-f_i(0)-H(x)\|_B\leq 2\Big(\frac{\|x\|_A^{r_0}}{2-2^{r_0}}+\frac{\|x\|_A^{r_1}}{2-2^{r_1}}\Big)\theta+\theta\|x\|_A^{r_i}\quad(i=0,1)$$

*for all $x \in A$.*

*Proof.* The proof follows from Theorem 4.1. □

**Corollary 4.3.** *Let $\theta, r_0, r_1$ be non-negative real numbers such that $\lambda := r_0 + r_1 < 1$ and let $f, f_0, f_1 : A \longrightarrow B$ be mappings with $f(0) = 0$ and $f(w), f_0(w), f_1(w) \in B_0$ for all $w \in A_0$ and*

$$\|f(\mu x + \mu y) - \mu f_0(x) - \mu f_1(y)\|_B \leq \theta \|x\|_A^{r_0} \|y\|_A^{r_1},$$

$$\|f(w_0 + w_1) - f_0(w_0) - f_1(w_1)\|_{B_0} \leq \theta \|w_0\|_A^{r_0} \|w_1\|_A^{r_1},$$

$$\|f([w_0, x]) - [f_0(w_0), f_1(x)]\|_B \leq \theta \|w_0\|_A^{r_0} \|x\|_A^{r_1}$$

*for all $\mu \in \mathbb{T}^1$, all $w_0, w_1 \in A_0$ and all $x, y \in A$ (by putting $\|.\|_A^0 = 1$). Then there exists a unique proper Lie $CQ^*$-algebra homomorphism $H : A \longrightarrow B$ such that*

$$\|f(x) - H(x)\|_B \leq \frac{\theta}{2 - 2^\lambda} \|x\|_A^\lambda,$$
$$\|f_i(x) - f_i(0) - H(x)\|_B \leq \frac{\theta}{2 - 2^\lambda} \|x\|_A^\lambda \quad (i = 0, 1)$$

*for all $x \in A$.*

*Proof.* The proof follows from Theorem 4.1. □

For $r_0 = r_1 = 0$, we have the following theorem:

**Theorem 4.4.** *Let $\theta$ be non-negative real number and let $f, f_0, f_1 : A \longrightarrow B$ be mappings such that $f(w), f_0(w), f_1(w) \in B_0$ for all $w \in A_0$ and*

$$\|f(\mu x + \mu y) - \mu f_0(x) - \mu f_1(y)\|_B \leq \theta,$$

$$\|f(w_0 + w_1) - f_0(w_0) - f_1(w_1)\|_{B_0} \leq \theta,$$

$$\|f([w_0, x]) - [f_0(w_0), f_1(x)]\|_B \leq \theta$$

*for all $\mu \in \mathbb{T}^1$, all $w_0, w_1 \in A_0$ and all $x, y \in A$. Then there exists a unique proper Lie $CQ^*$-algebra homomorphism $H : A \longrightarrow B$ such that*

$$\|f(x) - H(x)\|_B \leq 3\theta + M,$$

$$\|f_0(x) + f_1(0) - H(x)\|_B \leq 4\theta + M,$$

$$\|f_1(x) + f_0(0) - H(x)\|_B \leq 4\theta + M$$

*for all $x \in A$, where $M = \|f_0(0) + f_1(0)\|_B$.*

*Proof.* The proof is similar to the proof of Theorem 4.1. □

## 5. Stability of derivation on proper Lie $CQ^*$-algebras

We prove the Hyers-Ulam stability of derivations on proper Lie $CQ^*$-algebras.

**Theorem 5.1.** *Let $\varphi : A_0 \times A_0 \longrightarrow [0, +\infty)$ be a mapping satisfying (4.1) and (4.2) for all $x, y \in A_0$ such that $\varphi(0,0) = 0$ and let $f, f_0, f_1 : A_0 \longrightarrow A$ be mappings with $f(0) = 0$ and $f(w), f_0(w), f_1(w) \in A_0$ for all $w \in A_0$ such that*

$$\|f(\mu w_0 + \mu w_1) - \mu f_0(w_0) - \mu f_1(w_1)\|_{A_0} \leq \varphi(w_0, w_1),$$

$$\big\|f([w_0, w_1]) - [f_0(w_0), w_1] - [w_0, f_1(w_1)]\big\|_A \leq \varphi(w_0, w_1) \tag{5.1}$$

*for all $\mu \in \mathbb{T}^1$ and all $w_0, w_1 \in A_0$. Then there exists a unique Lie derivation $\delta : A_0 \longrightarrow A$ such that*

$$\begin{aligned} \|f(x) - \delta(x)\|_A &\leq \frac{1}{2}\widetilde{\varphi}(x), \\ \|f_0(x) - f_0(0) - \delta(x)\|_A &\leq \frac{1}{2}\widetilde{\varphi}(x) + \varphi(x, 0), \\ \|f_1(x) - f_1(0) - \delta(x)\|_A &\leq \frac{1}{2}\widetilde{\varphi}(x) + \varphi(0, x) \end{aligned} \tag{5.2}$$

*for all $x \in A_0$.*

*Proof.* By the same reasoning as in the proof of Theorem 4.1, there exists a unique $\mathbb{C}$-linear mapping $\delta : A_0 \longrightarrow A$ satisfying (5.2). The mapping $\delta : A_0 \longrightarrow A$ is defined by

$$\delta(w) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n w) = \lim_{n\to\infty} \frac{1}{2^n} f_i(2^n w) \quad (i = 0, 1) \tag{5.3}$$

for all $w \in A_0$ and $\delta(w) \in A_0$ for all $w \in A_0$. It follows from (4.1), (5.1) and (5.3) that

$$\begin{aligned} &\big\|\delta([w_0, w_1]) - [\delta(w_0), w_1] - [w_0, \delta(w_1)]\big\|_A \\ &= \lim_{n\to\infty} \frac{1}{4^n}\big\|f(4^n[w_0, w_1]) - [f_0(2^n w_0), 2^n w_1] - [2^n w_0, f_1(2^n w_1)]\big\|_A \\ &\leq \lim_{n\to\infty} \frac{\varphi(2^n w_0, 2^n w_1)}{4^n} = 0 \end{aligned}$$

for all $w_0, w_1 \in A_0$. So

$$\delta([w_0, w_1]) = [\delta(w_0), w_1] + [w_0, \delta(w_1)]$$

for all $w_0, w_1 \in A_0$. Hence the mapping $\delta : A_0 \to A$ is a unique Lie derivation satisfying (5.2). □

**Corollary 5.2.** *Let $\theta, r_0, r_1$ be non-negative real numbers such that $0 \leq r_0, r_1 < 1$ and let $f, f_0, f_1 : A_0 \to A$ be mappings with $f(0) = 0$ and $f(w), f_0(w), f_1(w) \in A_0$ for all $w \in A_0$ such that*

$$\|f(\mu w_0 + \mu w_1) - \mu f_0(w_0) - \mu f_1(w_1)\|_{A_0} \leq \theta(\|w_0\|_{A_0}^{r_0} + \|w_1\|_{A_0}^{r_1}),$$

$$\big\|f([w_0, w_1]) - [f_0(w_0), w_1] - [w_0, f_1(w_1)]\big\|_A \leq \theta(\|w_0\|_{A_0}^{r_0} + \|w_1\|_{A_0}^{r_1})$$

*for all $\mu \in \mathbb{T}^1$ and all $w_0, w_1 \in A_0$ (by putting $\|.\|_A^0 = 1$). Then there exists a unique Lie derivation $\delta : A_0 \to A$ such that*

$$\begin{aligned} \|f(x) - \delta(x)\|_A &\leq 2\Big(\frac{\|x\|_{A_0}^{r_0}}{2 - 2^{r_0}} + \frac{\|x\|_{A_0}^{r_1}}{2 - 2^{r_1}}\Big)\theta, \\ \|f_i(x) - f_i(0) - \delta(x)\|_A &\leq 2\Big(\frac{\|x\|_{A_0}^{r_0}}{2 - 2^{r_0}} + \frac{\|x\|_{A_0}^{r_1}}{2 - 2^{r_1}}\Big)\theta + \theta\|x\|_{A_0}^{r_i} \qquad (i = 0, 1) \end{aligned}$$

*for all* $x \in A_0$.

*Proof.* The proof follows from Theorem 5.1. □

**Corollary 5.3.** *Let* $\theta, r_0, r_1$ *be non-negative real numbers such that* $0 \leq \lambda < 1$ *and let* $f, f_0, f_1 : A_0 \longrightarrow A$ *be mappings with* $f(0) = 0$ *and* $f(w), f_0(w), f_1(w) \in A_0$ *for all* $w \in A_0$ *such that*

$$\|f(\mu w_0 + \mu w_1) - \mu f_0(w_0) - \mu f_1(w_1)\|_{A_0} \leq \theta \|w_0\|_{A_0}^{r_0} \|w_1\|_{A_0}^{r_1},$$

$$\|f([w_0, w_1]) - [f_0(w_0), w_1] - [w_0, f_1(w_1)]\|_A \leq \theta \|w_0\|_{A_0}^{r_0} \|w_1\|_{A_0}^{r_1}$$

*for all* $\mu \in \mathbb{T}^1$ *and all* $w_0, w_1 \in A_0$ *(by putting* $\|.\|_A^0 = 1$*). Then there exists a unique Lie derivation* $\delta : A_0 \longrightarrow A$ *such that*

$$\|f(x) - \delta(x)\|_A \leq \frac{\theta}{2 - 2^\lambda} \|x\|_{A_0}^\lambda,$$

$$\|f_i(x) - f_i(0) - \delta(x)\|_A \leq \frac{\theta}{2 - 2^\lambda} \|x\|_{A_0}^\lambda \qquad (i = 0, 1)$$

*for all* $x \in A_0$.

*Proof.* The proof follows from Theorem 5.1. □

For $r_0 = r_1 = 0$, we have the following theorem:

**Theorem 5.4.** *Let* $\theta$ *be non-negative real number and let* $f, f_0, f_1 : A_0 \longrightarrow A$ *be mappings and* $f(w), f_0(w), f_1(w) \in A_0$ *for all* $w \in A_0$ *and*

$$\|f(\mu w_0 + \mu w_1) - \mu f_0(w_0) - \mu f_1(w_1)\|_{A_0} \leq \theta,$$

$$\|f([w_0, w_1]) - [f_0(w_0), f_1(w_1)]\|_A \leq \theta$$

*for all* $\mu \in \mathbb{T}^1$ *and all* $w_0, w_1 \in A_0$*. Then there exists a unique Lie derivation* $\delta : A \longrightarrow B$ *such that*

$$\|f(x) - \delta(x)\|_A \leq 3\theta + M,$$

$$\|f_0(x) + f_1(0) - \delta(x)\|_A \leq 4\theta + M,$$

$$\|f_1(x) + f_0(0) - \delta(x)\|_A \leq 4\theta + M$$

*for all* $x \in A_0$*, where* $M = \|f_0(0) + f_1(0)\|_A$.

*Proof.* The proof is similar to Theorem 5.1. □

## References

[1] J.P. Antoine, A. Inoue, C. Trapani, Partial $*$-Algebras and Their Operator Realizations, Kluwer, Dordrecht, 2002.

[2] F. Bagarello, A. Inoue, C. Trapani, Some classes of topological quasi $*$-algebras, Proc. Amer. Math. Soc. **129** (2001), 2973–2980.

[3] F. Bagarello, G. Morchio, Dynamics of mean-field spin models from basic results in abstract differential equations, J. Stat. Phys. **66** (1992), 849–866.

[4] F. Bagarello C.Trapani, States and representations of $CQ^*$-algebras, Ann. Inst. H. Poincaré **61** (1994), 103–133.

[5] F. Bagarello, C. Trapani, $CQ^*$-algebras: Structure properties, Publ. Res. Inst. Math. Sci. **32** (1996), 85–116.

[6] F. Bagarello, C. Trapani, Morphisms of certain Banach $C^*$-modules, Publ. Res. Inst. Math. Sci. **36** (2000), 681–705.

[7] P. Găvruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, J. Math. Anal. Appl. **184** (1994), 431–436.
[8] G.O.S. Ekhaguere, Partial $W^*$-dynamical systems, in: Current Topics in Operator Algebras, Proceedings of the Satellite Conference of ICM- 90, World Scientific, Singapore, 1991, pp. 202–217.
[9] R. Haag, D. Kastler, An algebraic approach to quantum field theory, J. Math. Phys. **5** (1964), 848–861.
[10] D.H. Hyers, On the stability of the linear functional equation, Proc. Nat. Acad. Sci. USA **27** (1941), 222–224.
[11] A. Najati, C. Park, Hyers-Ulam-Rassias stability of homomrphisims in quasi-Banach algebras associated to the Pexiderized Cauchy functional equation, J. Math. Anal. Appl. **335** (2007), 763–778.
[12] C. Park, Homomorphisms between Lie $JC^*$-algebras and Cauchy-Rassias stability of Lie $JC^*$-algebra derivations, J. Lie Theory **15** (2005), 393–414.
[13] C. Park, Homomorphisms between Poisson $JC^*$-algebras, Bull. Braz. Math. Soc. **36** (2005), 79–97.
[14] C. Park, Th.M. Rassias, Homomorphisms and derivations in proper $JCQ^*$-triples, J. Math. Anal. Appl. **337** (2008), 1404–1414.
[15] C. Park, Proper $CQ^*$-ternary algebras, J. Nonlinear Sci. Appl. **6** (2013), 152–161.
[16] Th.M. Rassias, On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. **72** (1978), 297–300.
[17] G.L. Sewell, Quantum Mechanics and its Emergent Macrophysics, Princeton Univ. Press, Princeton, NJ, 2002.
[18] S. Schin, D. Ki, J. Chang, M. Kim, *Random stability of quadratic functional equations: a fixed point approach*, J. Nonlinear Sci. Appl. **4** (2011), 37–49.
[19] S. Shagholi, M. Bavand Savadkouhi, M. Eshaghi Gordji, *Nearly ternary cubic homomorphism in ternary Fréchet algebras,* J. Comput. Anal. Appl. **13** (2011), 1106–1114.
[20] S. Shagholi, M. Eshaghi Gordji, M. Bavand Savadkouhi, *Stability of ternary quadratic derivation on ternary Banach algebras,* J. Comput. Anal. Appl. **13** (2011), 1097–1105.
[21] D. Shin, C. Park, Sh. Farhadabadi, *On the superstability of ternary Jordan $C^*$-homomorphisms,* J. Comput. Anal. Appl. **16** (2014), 964–973.
[22] D. Shin, C. Park, Sh. Farhadabadi, *Stability and superstability of $J^*$-homomorphisms and $J^*$-derivations for a generalized Cauchy-Jensen equation,* J. Comput. Anal. Appl. **17** (2014), 125–134.
[23] C. Trapani, Quasi-$*$-algebras of operators and their applications, Rev. Math. Phys. **7** (1995), 1303–1332.
[24] S.M. Ulam, A Collection of the Mathematical Problems, Interscience Publ., New York, 1960.

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# WOVEN FRAMES IN HILBERT $C^*$-MODULES

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ABSTRACT. In this paper we introduce woven frames in Hilbert $C^*$-modules. We then investigate some properties of woven frames and prove some results. For a given frame $\{\varphi_i\}_{i\in I}$ in a Hilbert $C^*$-module $U$, we obtain some conditions on a perturbed family $\{\psi_i\}_{i\in I}$ of $U$ which imply that $\{\varphi_i\}_{i\in I}$ and $\{\psi_i\}_{i\in I}$ are woven.

## 1. INTRODUCTION

1.1. **Hilbert $C^*$-modules.** Let $\mathcal{A}$ be a $C^*$-algebra. A pre-Hilbert $\mathcal{A}$-module is a complex vector space $U$ which is also a left $\mathcal{A}$-module equipped with an $\mathcal{A}$-valued inner product $\langle .,. \rangle : U \times U \to \mathcal{A}$ such that

(1) $\langle x, x\rangle \geqslant 0, \quad x \in U$,
(2) $\langle x, x\rangle = 0 \Leftrightarrow x = 0, \quad x \in U$,
(3) $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle, \quad \alpha, \beta \in \mathbb{C}$ and $x, y, z \in U$,
(4) $\langle ax, y\rangle = a\langle x, y\rangle, \quad x, y \in U, a \in \mathcal{A}$,
(5) $\langle x, y\rangle = \langle y, x\rangle^*, \quad x, y \in U$.

It is easy to see that scalar multiplication and the left $\mathcal{A}$-module structure of a pre-Hilbert $\mathcal{A}$-module $U$ are compatible in the sense that

$$(\lambda a)x = a(\lambda x) = \lambda(ax), \quad \lambda \in \mathbb{C},\ a \in A,\ x \in U.$$

For $x \in U$, we set $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$. It is well known that $U$ is a normed space with $\|.\|$ (see [10]). If $U$ is complete with norm $\|.\|$, it is called a Hilbert $\mathcal{A}$-module or a Hilbert $C^*$-module over $\mathcal{A}$. We are especially interested in finitely and countably generated $C^*$-modules over unital $C^*$-algebras $\mathcal{A}$. A Hilbert $\mathcal{A}$-module $U$ is called *countably generated* if there exists a countable set $\{x_i : i \in I\} \subseteq U$ such that $U$ equals the closed linear span of the set $\{a_i x_i : i \in I, a_i \in \mathcal{A}\}$. A Hilbert $\mathcal{A}$-module $U$ is *finitely generated* if there exists a finite set $\{x_1, x_2, \cdots, x_m\} \subseteq U$ such that each $x \in U$ can be expressed as an $\mathcal{A}$-linear combination of $\{x_1, x_2, \cdots, x_m\}$.

For a unital $C^*$-algebra $\mathcal{A}$, let $\ell^2(\mathcal{A}) = \{\{a_i\}_{i=1}^{\infty} \subseteq \mathcal{A} :\ \sum_{i=1}^{\infty} a_i a_i^*$ converges in norm in $\mathcal{A}\}$. The pointwise operations

$$\{a_i\}_{i=1}^{\infty} + \{b_i\}_{i=1}^{\infty} := \{a_i + b_i\}_{i=1}^{\infty}, \quad a\{a_i\}_{i=1}^{\infty} := \{aa_i\}_{i=1}^{\infty}$$

and the inner product

$$\langle \{a_i\}_{i=1}^{\infty}, \{b_i\}_{i=1}^{\infty}\rangle := \sum_{i=1}^{\infty} a_i b_i^*$$

turn $\ell^2(\mathcal{A})$ into a Hilbert $\mathcal{A}$-module. The sequence $\{e_i\}_{i=1}^{\infty}$, where $e_i$ takes value $1_{\mathcal{A}}$ at $i$ and $0_{\mathcal{A}}$ everywhere else, is an orthonormal basis for $\ell^2(\mathcal{A})$ and it is called the *standard orthonormal basis*.

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For Hilbert $\mathcal{A}$-modules $U$ and $V$, a map $T : U \to V$ is called *bounded $\mathcal{A}$-linear* if

$$T(\lambda x + y) = \lambda T(x) + T(y), \quad T(ax) = aT(x), \quad \lambda \in \mathbb{C},\ x, y \in U,\ a \in \mathcal{A},$$

and if

$$\|T\| := \sup\Big\{\|T(x)\| :\ x \in U,\ \|x\| \leqslant 1\Big\} < \infty.$$

For Hilbert $C^*$-modules $U$ and $V$, a map $T : U \to V$ is called *adjointable* if there is a map $T^* : V \to U$ such that

$$\langle Tx, y\rangle = \langle x, T^*y\rangle, \quad x \in U, y \in V.$$

We denote by $\mathbf{L}(U, V)$ the set of all adjointable maps from $U$ to $V$. It is easy to show that every $T \in \mathbf{L}(U, V)$ is $\mathcal{A}$-linear and bounded. In fact $\mathbf{L}(U, V)$ is a Banach space with respect to the operator norm. Moreover, $\mathbf{L}(U, U)$ is a $C^*$-algebra and we will denote it by $\mathbf{L}(U)$ [10]. In spite of the fact that every (linear) bounded operator in Hilbert space is adjointable, not all bounded linear operators on Hilbert $C^*$-modules are adjointable, since Riesz representation theorem for continuous functionals in Hilbert spaces does not hold for general Hilbert $C^*$-modules (see [7]).

**Proposition 1.1.** [10] *Let $U$ be a Hilbert $\mathcal{A}$-module and let $T$ be a bounded $\mathcal{A}$-linear operator on $U$. Then the following conditions are equivalent:*

(1) *$T$ is a positive element of $\mathbf{L}(U)$.*
(2) *$\langle Tx, x\rangle \geqslant 0$ for all $x \in U$.*

**Proposition 1.2.** *Let $\{e_i\}_{i=1}^{\infty}$ be the standard orthonormal basis of $\ell^2(\mathcal{A})$. For $\sigma \subset \mathbb{N}$, we define $P_\sigma : \ell^2(\mathcal{A}) \to \ell^2(\mathcal{A})$ by $P_\sigma(\sum_{i=1}^{\infty} a_i e_i) = \sum_{i\in\sigma} a_i e_i$. Then $P_\sigma$ is an adjointable projection onto $\overline{span}\{e_i\}_{i\in\sigma}$.*

*Proof.* We have $P_\sigma^2 = P_\sigma$, and so $P_\sigma$ is a projection. We show that $P_\sigma$ is adjointable and $P_\sigma^* = P_\sigma$. For this, let $a = \sum_{i=1}^{\infty} a_i e_i, b = \sum_{i=1}^{\infty} b_i e_i \in \ell^2(\mathcal{A})$. Then

$$\begin{aligned}\langle P_\sigma a, b\rangle &= \langle \sum_{i\in\sigma} a_i e_i, \sum_{j=1}^{\infty} b_j e_j\rangle = \sum_{i\in\sigma} \langle a_i e_i, \sum_{j=1}^{\infty} b_j e_j\rangle \\ &= \sum_{i\in\sigma} \langle a_i e_i, b_i e_i\rangle = \sum_{i\in\sigma} \langle \sum_{j=1}^{\infty} a_j e_j, b_i e_i\rangle \\ &= \langle \sum_{j=1}^{\infty} a_j e_j, \sum_{i\in\sigma} b_i e_i\rangle = \langle a, P_\sigma b\rangle.\end{aligned}$$

This completes the proof. □

1.2. **Frames in Hilbert $C^*$-modules.** The concept of frame for a separable Hilbert space $H$ was introduced by Duffin and Schaeffer [4], and it was defined as a finite or countable sequence $\{\varphi_i\}_{i\in I}$ in $H$ such that there exist constants $A, B > 0$ satisfying

$$A\|x\|^2 \leqslant \sum_{i\in I} \|\langle x, \varphi_i\rangle\|^2 \leqslant B\|x\|^2, \quad x \in H.$$

Frames for Hilbert spaces have natural analogues for Hilbert $C^*$-modules. Frames for Hilbert $C^*$-modules were introduced by Frank and Larson [7]. Hilbert $C^*$-module frames are generalization of Hilbert space frames. A *frame* for a countably (or finitely) generated Hilbert $C^*$-module $U$ is a sequence $\{\varphi_i\}_{i\in I}$ ($I$ is a finite or countable subset of $\mathbb{N}$) in $U$ for which there are constants $A, B > 0$ such that

$$A\langle x, x\rangle \leqslant \sum_{i\in I} \langle x, \varphi_i\rangle\langle \varphi_i, x\rangle \leqslant B\langle x, x\rangle, \quad x \in U. \tag{1.1}$$

We consider *standard frames* for which $\sum_{i\in I}\langle x,\varphi_i\rangle\langle\varphi_i,x\rangle$ converges in norm for every $x\in U$. The constants $A, B$ are called the lower and upper frame bounds, respectively. If only the second inequality in (1.1) is satisfied, we say that $\{\varphi_i\}_{i\in I}$ is a *Bessel sequence* with a *Bessel bound* $B$. A frame $\{\varphi_i\}_{i\in I}$ is called a *tight frame* if we can choose $A = B$ and a *Parseval frame* (or *normalized tight frame*) if $A = B = 1$.

Frank and Larson [7] showed that standard frames always exist in a countably (or finitely) generated Hilbert $C^*$-module over a unital $C^*$-algebra. Li [11] showed that every infinite-dimensional commutative unital $C^*$-algebra has a Hilbert $C^*$-module admitting no frames. Many related concepts of frames in Hilbert spaces such as $g$-frame, perturbation of frames and Riesz bases and stability of $g$-frames were introduced and investigated in Hilbert $C^*$-module spaces [5, 6, 8, 9, 14, 15].

In this paper we introduce the concept of woven frames and investigate whose relation with perturbation of frames in Hilbert $C^*$-modules.

Arambašić proved the following result which states that the condition (1.1) from the definition of standard frames can be replaced with a weaker one.

**Theorem 1.3.** [1] *Let $\mathcal{A}$ be a $C^*$-algebra, $U$ a countably generated Hilbert $\mathcal{A}$-module, and $\{\varphi_i\}_{i\in I}$ a sequence in $U$ such that $\sum_{i\in I}\langle x,\varphi_i\rangle\langle\varphi_i,x\rangle$ converges in norm for every $x\in U$. Then $\{\varphi_i\}_{i\in I}$ is a standard frame for $U$ if and only if there are constants $C, D > 0$ such that*

$$C\|x\|^2 \leqslant \Big\|\sum_{i\in I}\langle x,\varphi_i\rangle\langle\varphi_i,x\rangle\Big\| \leqslant D\|x\|^2, \quad x\in U.$$

Let $\{\varphi_i\}_{i\in I}$ be a standard frame for a Hilbert $C^*$-module $U$. We define the *synthesis operator* $T : \ell^2(\mathcal{A}) \to U$ by $T(\{a_i\}_{i\in I}) = \sum_{i\in I} a_i\varphi_i$. It is well known that $T$ is adjointable and its adjoint $T^* : U \to \ell^2(\mathcal{A})$ which is called the *analysis operator* fulfills $T^*x = \{\langle x,\varphi_i\rangle\}_{i\in I}$. By composing $T$ and $T^*$, we obtain the frame operator $S : U \to U$ which is given by

$$Sx = TT^*x = \sum_{i\in I}\langle x,\varphi_i\rangle\varphi_i, \quad x\in U.$$

The frame operator is positive, invertible, and is the unique operator in $\mathbf{L}(U)$ such that the reconstruction formula

$$x = \sum_{i\in I}\langle x, S^{-1}\varphi_i\rangle\varphi_i = \sum_{i\in I}\langle x,\varphi_i\rangle S^{-1}\varphi_i,$$

holds for all $x\in U$. It is easy to see that the sequences $\{S^{-1}\varphi_i\}_{i\in I}$ and $\{S^{-\frac{1}{2}}\varphi_i\}_{i\in I}$ are frames for $U$. The frame $\{S^{-1}\varphi_i\}_{i\in I}$ is said to be the *canonical dual frame* of $\{\varphi_i\}_{i\in I}$ and the frame $\{S^{-\frac{1}{2}}\varphi_i\}_{i\in I}$ is said to be the *canonical Parseval frame* of $\{\varphi_i\}_{i\in I}$ [7].

**Theorem 1.4.** [1] *Let $\mathcal{A}$ be a $C^*$-algebra, $U$ a countably generated Hilbert $\mathcal{A}$-module, $\{\varphi_i\}_{i\in I}$ a sequence in $U$, and $\theta x = \{\langle x,\varphi_i\rangle\}_{i\in I}$ for $x\in U$. The following statements are mutually equivalent:*

1. *$\{\varphi_i\}_{i\in I}$ is a standard frame for $U$.*
2. *$\theta\in\mathbf{L}(U,\ell^2(A))$ and $\theta$ is bounded below with respect to norm, i.e., there is $m > 0$ such that $\|\theta x\| \geqslant m\|x\|$ for all $x\in U$.*
3. *$\theta\in\mathbf{L}(U,\ell^2(A))$ and $\theta$ is bounded below with respect to the inner product, i.e., there is $m' > 0$ such that $\langle\theta x,\theta x\rangle \geqslant m'\langle x,x\rangle$ for all $x\in U$.*
4. *$\theta\in\mathbf{L}(U,\ell^2(A))$ and $\theta^*$ is surjective.*

**Theorem 1.5.** [1] *Let $\mathcal{A}$ be a $C^*$-algebra, $V$ and $W$ countably generated Hilbert $\mathcal{A}$-modules, and $T\in\mathbf{L}(V,W)$ surjective. If $\{\varphi_i : i\in I\}$ is a standard frame for $V$ with frame bounds $C$ and $D$, then $\{T\varphi_i : i\in I\}$ is a standard frame for $W$ with frame bounds $\frac{C}{\|(TT^*)^{-1}\|}$ and $D\|T\|^2$.*

For the converse, we have the following.

**Proposition 1.6.** *Let $\{\varphi_i\}_{i\in I}$ be a frame for $U$ and $P$ be an adjointable operator on $U$ such that $\{P\varphi_i\}_{i\in I}$ is a frame for $U$. Then $P$ is a surjective operator.*

*Proof.* Let $S$ be the frame operator of $\{\varphi_i\}_{i\in I}$. Then

$$\sum_{i\in I}\langle x, P\varphi_i\rangle P\varphi_i = P\left(\sum_{i\in I}\langle P^*x, \varphi_i\rangle\varphi_i\right) = PSP^*(x), \quad x\in U.$$

So $PSP^*$ is the frame operator of $\{P\varphi_i\}_{i\in I}$. Since $PSP^*$ is invertible, we infer that $P$ is surjective. □

**Definition 1.7.** A frame $\{\varphi_i\}_{i\in I}$ for a Hilbert $\mathcal{A}$-module $U$ is said to be a *(standard) Riesz basis* for $U$ if it satisfies

(*i*) $\varphi_i \neq 0$ for all $i\in I$;
(*ii*) if $\sum_{j\in J} a_j\varphi_j = 0$ for $\{a_j : j\in J\}\subset\mathcal{A}$ and $J\subset I$, then every summand $a_jx_j$ is equal to zero.

We need the following lemmas to obtain optimal frame bounds.

**Lemma 1.8.** [12] *Let $a, b$ be positive elements of a unital $C^*$-algebra $\mathcal{A}$. If $a\leqslant b$, then $0\leqslant b^{-1}\leqslant a^{-1}$.*

**Lemma 1.9.** [13] *Let $T$ be a bounded and $\mathcal{A}$-linear map on Hilbert $\mathcal{A}$-module $U$. Then*

$$\|T\| = \inf\{K^{\frac{1}{2}} : \langle Tx, Tx\rangle \leqslant K\langle x, x\rangle, \quad x\in U\}.$$

In the rest of paper we suppose that $\mathcal{A}$ is a unital $C^*$-algebra and $U$ is a countably or finitely generated Hilbert $\mathcal{A}$-module.

**Theorem 1.10.** *Let $\{\varphi_i\}_{i\in I}$ be a frame for $U$ with lower and upper frame bounds $A$ and $B$, respectively. Let $S$ be the frame operator of $\{\varphi_i\}_{i\in I}$. Then*

(1) *$\{S^{-1}\varphi_i\}_{i\in I}$ is a frame with lower and upper frame bounds $B^{-1}$ and $A^{-1}$, respectively.*
(2) *The optimal frame bounds $A, B$ for $\{\varphi_i\}_{i\in I}$ are given by*

$$A = \|S^{-1}\|^{-1}, \quad B = \|S\|.$$

*Proof.* (1) Since $S^{-1}$ is an adjointable and invertible operator, by Theorem 1.5, $\{S^{-1}\varphi_i\}_{i\in I}$ is a frame for $U$. We show that $B^{-1}$ and $A^{-1}$ are frame bounds of $\{S^{-1}\varphi_i\}_{i\in I}$. Since $AId\leqslant S\leqslant BId$, we have $B^{-1}Id\leqslant S^{-1}\leqslant A^{-1}Id$ by Lemma 1.8. This means $B^{-1}\langle x, x\rangle\leqslant\langle S^{-1}x, x\rangle\leqslant A^{-1}\langle x, x\rangle$ for all $x\in U$.

On the other hand, for every $x\in U$ we have

$$\sum_{i\in I}\langle x, S^{-1}\varphi_i\rangle\langle S^{-1}\varphi_i, x\rangle = \sum_{i\in I}\langle S^{-1}x, \varphi_i\rangle\langle\varphi_i, S^{-1}x\rangle = \langle SS^{-1}x, S^{-1}x\rangle = \langle S^{-1}x, x\rangle.$$

Therefore,

$$B^{-1}\langle x, x\rangle\leqslant\sum_{i\in I}\langle x, S^{-1}\varphi_i\rangle\langle S^{-1}\varphi_i, x\rangle\leqslant A^{-1}\langle x, x\rangle, \quad x\in U.$$

(2) It is clear that $\langle S^{\frac{1}{2}}x, S^{\frac{1}{2}}x\rangle\leqslant B\langle x, x\rangle$ for all $x\in U$. By Lemma 1.9, we have

$$\|S^{\frac{1}{2}}\| = \inf\{K^{\frac{1}{2}} : \langle S^{\frac{1}{2}}x, S^{\frac{1}{2}}\rangle\leqslant K\langle x, x\rangle \quad , x\in U\}.$$

So $\|S\| = \|S^{\frac{1}{2}}S^{\frac{1}{2}}\| = \|S^{\frac{1}{2}}\|^2$ is the optimal upper frame bound for $\{\varphi_i\}_{i\in I}$

$$\|S\| = \|S^{\frac{1}{2}}S^{\frac{1}{2}}\| = \|S^{\frac{1}{2}}\|^2 = B.$$

For the optimal lower frame bound, since the frame operator of $\{S^{-1}\varphi_i\}_{i\in I}$ is $S^{-1}$, we get $\|S^{-1}\|$ is the optimal upper frame bound for $\{S^{-1}\varphi_i\}_{i\in I}$. Thus $\|S^{-1}\|^{-1}$ is the optimal lower frame bound for $\{\varphi_i\}_{i\in I}$.

This completes the proof. □

**Proposition 1.11.** *Let $\{\varphi_i\}_{i\in I}$ be a Bessel sequence in $U$ and $S : U \to U$ be given by*

$$Sx = \sum_{i\in I}\langle x, \varphi_i\rangle\varphi_i, \quad x \in U.$$

*If there exists a constant $A > 0$ such that*

$$\|Sx\| \geqslant A\|x\|, \quad x \in U, \tag{1.2}$$

*then $\{\varphi_i\}_{i\in I}$ is a frame for $U$ with lower frame bound $A$.*

*Proof.* We define the operator $T : \ell^2(\mathcal{A}) \to U$ by $T(\{a_i\}_{i\in I}) = \sum_{i\in I} a_i\varphi_i$. Since $\{\varphi_i\}_{i\in I}$ is a Bessel sequence in $U$, $T$ is well defined, adjointable, and $S = TT^*$. By (1.2), we have

$$\|T^*x\| \geqslant \frac{A}{\|T\|}\|x\|, \quad x \in U.$$

Hence by Theorem 1.4, $\{\varphi_i\}_{i\in I}$ is a frame for $U$. Therefore $S$ is invertible and $\|S^{-1}\| \leqslant A^{-1}$. Since the optimal lower frame bound for $\{\varphi_i\}_{i\in I}$ is $\|S^{-1}\|^{-1}$, we get $A$ is a lower frame bound for $\{\varphi_i\}_{i\in I}$. □

If $\{\varphi_i\}_{i\in I}$ is a frame for $U$ with lower bound $A$ and frame operator $S$, then

$$\|Sx\| = \sup_{\|y\|=1}\|\langle Sx, y\rangle\| \geqslant \frac{1}{\|x\|}\|\langle Sx, x\rangle\| \geqslant A\|x\|, \quad x \in U \setminus \{0\}.$$

Therefore, $\|Sx\| \geqslant A\|x\|$ for all $x \in U$.

## 2. Woven frames

The concept of woven frames for Hilbert spaces was introduced in [2].

In this section we introduce woven frames for finitely or countably generated Hilbert $C^*$-modules. We use the notation $[m] = \{1, ..., m\}$ in the rest of the paper for $m \in \mathbb{N}$.

**Definition 2.1.** A family $\left\{\{\varphi_{ij}\}_{i\in I}\right\}_{j\in[M]}$ of frames for $U$ is called *woven* if there exist universal constants $0 < A < B < \infty$ such that for every partition $\{\sigma_j\}_{j\in[M]}$ of $I$, the family $\{\varphi_{ij}\}_{i\in\sigma_j, j\in[M]}$ is a frame for $U$ with lower and upper frame bounds $A$ and $B$, respectively. Each family $\{\varphi_{ij}\}_{i\in\sigma_j, j\in[M]}$ is called a *weaving*.

To verify that a family of frames is woven, it is enough to check that there exists a universal lower frame bound, since every weaving automatically has a universal upper frame bound.

**Proposition 2.2.** *If $\left\{\{\varphi_{ij}\}_{i\in I}\right\}_{j\in[M]}$ is a family of Bessel sequences with bounds $B_j$ for $j \in [M]$, then every weaving is a Bessel sequence with Bessel bound $\sum_{j=1}^M B_j$.*

*Proof.* For every partition $\{\sigma_j\}_{j\in[M]}$ of $I$ and every $x \in U$, we have

$$\sum_{j=1}^M\sum_{i\in\sigma_j}\langle x, \varphi_{ij}\rangle\langle\varphi_{ij}, x\rangle \leqslant \sum_{j=1}^M\sum_{i\in I}\langle x, \varphi_{ij}\rangle\langle\varphi_{ij}, x\rangle \leqslant \sum_{j=1}^M B_j\langle x, x\rangle.$$

So $\sum_{j=1}^M B_j$ is a Bessel bound for every weaving. □

By Theorem 1.5 and Proposition 1.6, we get the following.

**Proposition 2.3.** *Let $\left\{\{\varphi_{ij}\}_{i\in I}\right\}_{j\in[M]}$ be a woven family of frames for $U$ and $P : U \to U$ be an adjointable operator. Then $\left\{\{P\varphi_{ij}\}_{i\in I}\right\}_{j\in[M]}$ are woven frames if and only if $P$ is surjective.*

**Corollary 2.4.** *Let $\{\varphi_i\}_{i\in I}$ and $\{\psi_i\}_{i\in I}$ be two woven frames. We can always assume that one of this two woven frames is Parseval.*

*Proof.* Let $S$ be the frame operator of $\{\varphi_i\}_{i\in I}$. Then by Proposition 2.3, $\{S^{-\frac{1}{2}}\varphi_i\}_{i\in I}$ and $\{S^{-\frac{1}{2}}\psi_i\}_{i\in I}$ are woven frames, where $\{S^{-\frac{1}{2}}\varphi_i\}_{i\in I}$ is a Parseval frame. □

In the following proposition, we assume that $\mathcal{Z}(\mathcal{A}) = \{a \in \mathcal{A} : ab = ba, \forall b \in \mathcal{A}\}$.

**Proposition 2.5.** *Let $\{\varphi_i\}_{i\in I}$ and $\{\psi_i\}_{i\in I}$ be woven frames for Hilbert $\mathcal{A}$-module $U$ with universal lower and upper bounds $A, B$, respectively. Assume that $\{a_i\}_{i\in I}$ and $\{b_i\}_{i\in I}$ are sequences in $\mathcal{Z}(\mathcal{A})$ such that $0 < C1_{\mathcal{A}} \leqslant a_i^* a_i \leqslant D1_{\mathcal{A}}$ and $0 < C1_{\mathcal{A}} \leqslant b_i^* b_i \leqslant D1_{\mathcal{A}}$ for all $i \in I$. Then $\{a_i\varphi_i\}_{i\in I}$ and $\{b_i\psi_i\}_{i\in I}$ are woven frames for $U$ with universal lower and upper bounds $AC$ and $BD$, respectively.*

*Proof.* Let $\sigma \subset I$ and $x \in U$. Then

$$\begin{aligned}
&\sum_{i\in\sigma}\langle x, a_i\varphi_i\rangle\langle a_i\varphi_i, x\rangle + \sum_{i\in\sigma^c}\langle x, b_i\psi_i\rangle\langle b_i\psi_i, x\rangle \\
&= \sum_{i\in\sigma} a_i^*\langle x, \varphi_i\rangle a_i\langle\varphi_i, x\rangle + \sum_{i\in\sigma^c} b_i^*\langle x, \psi_i\rangle b_i\langle\psi_i, x\rangle \\
&= \sum_{i\in\sigma} a_i^* a_i\langle x, \varphi_i\rangle\langle\varphi_i, x\rangle + \sum_{i\in\sigma^c} b_i^* b_i\langle x, \psi_i\rangle\langle\psi_i, x\rangle.
\end{aligned}$$

The last equality implies that

$$AC\langle x, x\rangle \leqslant \sum_{i\in\sigma}\langle x, a_i\varphi_i\rangle\langle a_i\varphi_i, x\rangle + \sum_{i\in\sigma^c}\langle x, b_i\psi_i\rangle\langle b_i\psi_i, x\rangle \leqslant BD\langle x, x\rangle.$$

Thus two frames $\{a_i\varphi_i\}_{i\in I}$ and $\{b_i\psi_i\}_{i\in I}$ are woven. □

In the following result the commutativity of coefficients is not necessary.

**Corollary 2.6.** *Let $\{\varphi_i\}_{i\in I}$ and $\{\psi_i\}_{i\in I}$ be woven frames for $U$ with universal lower and upper bounds $A, B$, respectively. Assume that $\{\lambda_i\}_{i\in I}$ and $\{\mu_i\}_{i\in I}$ are sequences in $\mathbb{C}$ such that $0 < C \leqslant |\lambda_i|^2 \leqslant D < \infty$ and $0 < C \leqslant |\mu_i|^2 \leqslant D < \infty$ for all $i \in I$. Then $\{\lambda_i\varphi_i\}_{i\in I}$ and $\{\mu_i\psi_i\}_{i\in I}$ are woven frames for $U$ with universal lower and upper bounds $AC$ and $BD$, respectively.*

**Proposition 2.7.** *Assume that $\{\varphi_i\}_{i\in J}$ and $\{\psi_i\}_{i\in J}$ are woven frames. If $J \subset I$ such that $\{\varphi_i\}_{i\in I}$ and $\{\psi_i\}_{i\in I}$ are Bessel sequences, then $\{\varphi_i\}_{i\in I}$ and $\{\psi_i\}_{i\in I}$ are woven.*

*Proof.* By Proposition 2.2, it is enough to show that there is a universal lower bound. Let $\sigma \subset I$ and $x \in U$. Then

$$A\langle x, x\rangle \leqslant \sum_{i\in\sigma\cap J}\langle x, \varphi_i\rangle\langle\varphi_i, x\rangle + \sum_{i\in\sigma^c\cap J}\langle x, \psi_i\rangle\langle\psi_i, x\rangle \leqslant \sum_{i\in\sigma}\langle x, \varphi_i\rangle\langle\varphi_i, x\rangle + \sum_{i\in\sigma^c}\langle x, \psi_i\rangle\langle\psi_i, x\rangle.$$

This completes the proof. □

The next result shows that we can remove some vectors from woven frames.

**Proposition 2.8.** *Let $\{\varphi_i\}_{i\in I}$ and $\{\psi_i\}_{i\in I}$ be woven frames for $U$ with universal lower and upper frame bounds $A$ and $B$, respectively. If $J \subset I$ and there exists $0 < D < A$ such that*

$$\sum_{i\in J}\langle x, \varphi_i\rangle\langle\varphi_i, x\rangle \leqslant D\langle x, x\rangle, \quad x \in U,$$

*then $\{\varphi_i\}_{i\in(I\setminus J)}$ and $\{\psi_i\}_{i\in(I\setminus J)}$ are woven frames for $U$ with universal lower and upper frame bounds $A - D$ and $B$, respectively.*

*Proof.* Suppose that $\sigma \subset (I \setminus J)$. Then

$$\sum_{i\in\sigma}\langle x,\varphi_i\rangle\langle\varphi_i,x\rangle + \sum_{i\in(I\setminus J)\setminus\sigma}\langle x,\psi_i\rangle\langle\psi_i,x\rangle \leqslant \sum_{i\in\sigma\cup J}\langle x,\varphi_i\rangle\langle\varphi_i,x\rangle + \sum_{i\in(I\setminus J)\setminus\sigma}\langle x,\psi_i\rangle\langle\psi_i,x\rangle \leqslant B\langle x,x\rangle,$$

and

$$\begin{aligned}&\sum_{i\in\sigma}\langle x,\varphi_i\rangle\langle\varphi_i,x\rangle + \sum_{i\in(I\setminus J)\setminus\sigma}\langle x,\psi_i\rangle\langle\psi_i,x\rangle\\ &= \left(\sum_{i\in\sigma\cup J}\langle x,\varphi_i\rangle\langle\varphi_i,x\rangle - \sum_{i\in J}\langle x,\varphi_i\rangle\langle\varphi_i,x\rangle\right) + \sum_{i\in(I\setminus J)\setminus\sigma}\langle x,\psi_i\rangle\langle\psi_i,x\rangle\\ &\geqslant (A-D)\langle x,x\rangle\end{aligned}$$

for all $x \in U$. So $A - D$ and $B$ are lower and upper weaving bounds as desired. If we set $\sigma = J^c$ (respectively $\sigma = \emptyset$), then $\{\varphi_i\}_{i\in(I\setminus J)}$ (respectively $\{\psi_i\}_{i\in(I\setminus J)}$) is a frame for $U$. □

Since every frame is woven with itself, we have the following.

**Corollary 2.9.** *Let $\{\varphi_i\}_{i\in I}$ be a frame with lower frame bound $A$ for $U$. If $J \subset I$ and there exists $0 < D < A$ such that*

$$\sum_{i\in J}\langle x,\varphi_i\rangle\langle\varphi_i,x\rangle \leqslant D\langle x,x\rangle, \quad x \in U,$$

*then $\{\varphi_i\}_{i\in I\setminus J}$ is a frame with lower frame bound $A - D$.*

In the next example, we show that if a frame is woven with a Riesz basis, it may be not a Riesz basis. However in Hilbert spaces a Riesz basis only is woven with a Riesz basis and it cannot be woven with a frame which is not a Riesz basis [2].

*Example* 2.10. Let $l^\infty$ be the set of all bounded complex-valued sequences. If the multiplication is defined pointwise and the involution is defined by conjugate, then $\mathcal{A} = l^\infty$ will be a $C^*$-algebra with supremum norm. Let $U = c_0$ be the set of all sequences converging to zero. For any $u = \{u_i\}_{i=1}^\infty, v = \{v_i\}_{i=1}^\infty \in U$ and $\{c_i\}_{i=1}^\infty \in \mathcal{A}$, we define

$$\langle u,v\rangle = uv^* = \{u_i\overline{v_i}\}_{i=1}^\infty, \quad \{c_i\}_{i=1}^\infty\{u_i\}_{i=1}^\infty = \{c_iu_i\}_{i=1}^\infty.$$

Then $U$ is a Hilbert $\mathcal{A}$-module. Obviously, $\{e_i\}_{i\in\mathbb{N}}$, where $e_i$ takes value 1 at $i$ and 0 everywhere else, is an orthonormal basis of $U$. Now let $x_i = e_i$ for each $i$, and

$$y_i = \begin{cases} e_1 + e_2 & \text{if } i = 1,2\\ e_i & \text{if } i \neq 1,2.\end{cases}$$

Then $\{x_i\}_{i\in\mathbb{N}}$ is a Riesz basis that is woven with $\{y_i\}_{i\in\mathbb{N}}$, but $\{y_i\}_{i\in\mathbb{N}}$ is not a Riesz basis.

## 3. Perturbation and woven frames

In this section we show that under some conditions frames that are perturbation of each other are woven.

**Theorem 3.1.** [8] *Let $U$ be a finitely or countably generated Hilbert $\mathcal{A}$-module over a unital $C^*$-algebra $\mathcal{A}$, and $\{\varphi_i\}_{i\in I}$ be a frame for $U$ with frame bounds $A$ and $B$. Suppose that $\{\psi_i\}_{i\in I}$ is a*

*sequence of $U$ and that there exist $\lambda_1, \lambda_2, \mu > 0$ such that $\max\{\lambda_1 + \frac{\mu}{\sqrt{A}}, \lambda_2\} < 1$. Then $\{\psi_i\}_{i\in I}$ is also a frame for $U$ with frame bounds*

$$\left(\frac{(1-\lambda_1)\sqrt{A}-\mu}{1+\lambda_2}\right)^2 \quad \text{and} \quad \left(\frac{(1+\lambda_1)\sqrt{B}+\mu}{1-\lambda_2}\right)^2,$$

*if one of the following conditions is fulfilled for any finite sequence $\{a_i\}_{i=1}^n \subseteq \mathcal{A}$ and all $x \in U$:*

$$\left\|\sum_{i=1}^n \langle x, \varphi_i - \psi_i\rangle\langle \varphi_i - \psi_i, x\rangle\right\|^{\frac{1}{2}} \leqslant \lambda_1 \left\|\sum_{i=1}^n \langle x, \varphi_i\rangle\langle \varphi_i, x\rangle\right\|^{\frac{1}{2}} + \lambda_2 \left\|\sum_{i=1}^n \langle x, \psi_i\rangle\langle \psi_i, x\rangle\right\|^{\frac{1}{2}} + \mu\|x\|;$$

*or*

$$\left\|\sum_{i=1}^n a_i(\varphi_i - \psi_i)\right\| \leqslant \lambda_1 \left\|\sum_{i=1}^n a_i\varphi_i\right\| + \lambda_2 \left\|\sum_{i=1}^n a_i\psi_i\right\| + \mu \left\|\sum_{i=1}^n a_i a_i^*\right\|^{\frac{1}{2}}.$$

We show that if $\{\varphi_i\}_{i\in I}$ and $\{\psi_i\}_{i\in I}$ have the above perturbations, then they may be woven frames.

**Theorem 3.2.** *Let $\{\varphi_i\}_{i\in I}$ be a frame for $U$ with frame bounds $A, B$ and $\{\psi_i\}_{i\in I}$ be a sequence in $U$. If for all sequences $\{a_i\}_{i\in I} \in \ell^2(\mathcal{A})$*

$$\left\|\sum_{i\in I} a_i(\varphi_i - \psi_i)\right\| \leqslant \lambda_1 \left\|\sum_{i\in I} a_i\varphi_i\right\| + \lambda_2 \left\|\sum_{i\in I} a_i\psi_i\right\| + \mu \left\|\sum_{i\in I} a_i a_i^*\right\|^{\frac{1}{2}}, \tag{3.1}$$

*or*

$$\left\|\sum_{i\in I} \langle x, \varphi_i - \psi_i\rangle\langle \varphi_i - \psi_i, x\rangle\right\|^{\frac{1}{2}} \leqslant \lambda_1 \left\|\sum_{i\in I} \langle x, \varphi_i\rangle\langle \varphi_i, x\rangle\right\|^{\frac{1}{2}} + \lambda_2 \left\|\sum_{i\in I} \langle x, \psi_i\rangle\langle \psi_i, x\rangle\right\|^{\frac{1}{2}} + \mu\|x\| \tag{3.2}$$

*for some $\lambda_1, \lambda_2, \mu > 0$ such that $\max\{\lambda_1 + \frac{\mu}{\sqrt{A}}, \lambda_2\} < 1$ and*

$$\left(\lambda_1\sqrt{B} + \lambda_2\frac{(1+\lambda_1)\sqrt{B}+\mu}{1-\lambda_2} + \mu\right)\left(\sqrt{B} + \frac{(1+\lambda_1)\sqrt{B}+\mu}{1-\lambda_2}\right) \leqslant \alpha < A,$$

*then $\{\psi_i\}_{i\in I}$ is a frame for $U$, and two frames $\{\psi_i\}_{i\in I}$ and $\{\varphi_i\}_{i\in I}$ are woven.*

*Proof.* By Theorem 3.1, $\{\psi_i\}_{i\in I}$ is a frame for $U$ with bounds

$$\left(\frac{(1-\lambda_1)\sqrt{A}-\mu}{1+\lambda_2}\right)^2 \quad \text{and} \quad \left(\frac{(1+\lambda_1)\sqrt{B}+\mu}{1-\lambda_2}\right)^2.$$

Suppose that $\sigma \subset I$ and $P_\sigma : \ell^2(\mathcal{A}) \to \ell^2(\mathcal{A})$ denotes the projection described in Proposition 1.2. Let $T, R$ be the synthesis operators for the frames $\{\varphi_i\}_{i\in I}$ and $\{\psi_i\}_{i\in I}$, respectively. We define $T_\sigma, R_\sigma : \ell^2(\mathcal{A}) \to U$ by $T_\sigma = TP_\sigma$ and $R_\sigma = RP_\sigma$. It is clear that

$$T_\sigma(\{a_i\}_{i\in I}) = TP_\sigma(\{a_i\}_{i\in I}) = \sum_{i\in\sigma} a_i\varphi_i,$$

and

$$R_\sigma(\{a_i\}_{i\in I}) = RP_\sigma(\{a_i\}_{i\in I}) = \sum_{i\in\sigma} a_i\psi_i.$$

Since $\|P_\sigma\| \leqslant 1$, we have $\|T_\sigma - R_\sigma\| \leqslant \|T - R\|, \|T_\sigma\| \leqslant \|T\|$ and $\|R_\sigma\| \leqslant \|R\|$. Also the inequalities in (3.1) and (3.2) become

$$\|T - R\| \leqslant \lambda_1\sqrt{B} + \lambda_2\frac{(1+\lambda_1)\sqrt{B}+\mu}{1-\lambda_2} + \mu,$$

and

$$\|T^* - R^*\| \leqslant \lambda_1\sqrt{B} + \lambda_2\frac{(1+\lambda_1)\sqrt{B}+\mu}{1-\lambda_2} + \mu.$$

For every $x \in U$, we have

$$\begin{aligned}
&\left\|\sum_{i\in\sigma}\langle x,\varphi_i\rangle\varphi_i - \sum_{i\in\sigma}\langle x,\psi_i\rangle\psi_i\right\| \\
&= \|T_\sigma T_\sigma^* x - R_\sigma R_\sigma^* x\| \\
&\leqslant \|(T_\sigma T_\sigma^* - T_\sigma R_\sigma^*)x\| + \|(T_\sigma R_\sigma^* - R_\sigma R_\sigma^*)x\| \\
&\leqslant \|T_\sigma\|\|T_\sigma^* - R_\sigma^*\|\|x\| + \|T_\sigma - R_\sigma\|\|R_\sigma^*\|\|x\| \\
&\leqslant \|T\|\|T - R\|\|x\| + \|T - R\|\|R\|\|x\| \\
&= \|T - R\|(\|T\| + \|R\|)\|x\| \\
&\leqslant \left(\lambda_1\sqrt{B} + \lambda_2\frac{(1+\lambda_1)\sqrt{B}+\mu}{1-\lambda_2} + \mu\right)(\|T\| + \|R\|)\|x\| \\
&\leqslant \left(\lambda_1\sqrt{B} + \lambda_2\frac{(1+\lambda_1)\sqrt{B}+\mu}{1-\lambda_2} + \mu\right)\left(\sqrt{B} + \frac{(1+\lambda_1)\sqrt{B}+\mu}{1-\lambda_2}\right)\|x\| \\
&\leqslant \alpha\|x\|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left\|\sum_{i\in\sigma}\langle x,\psi_i\rangle\psi_i + \sum_{i\in\sigma^c}\langle x,\varphi_i\rangle\varphi_i\right\| &= \left\|\sum_{i\in I}\langle x,\varphi_i\rangle\varphi_i + \left(\sum_{i\in\sigma}\langle x,\psi_i\rangle\psi_i - \sum_{i\in\sigma}\langle x,\varphi_i\rangle\varphi_i\right)\right\| \\
&\geqslant \left\|\sum_{i\in I}\langle x,\varphi_i\rangle\varphi_i\right\| - \left\|\sum_{i\in\sigma}\langle x,\psi_i\rangle\psi_i - \sum_{i\in_\sigma}\langle x,\varphi_i\rangle\varphi_i\right\| \\
&\geqslant A\|x\| - \left\|\sum_{i\in\sigma}\langle x,\psi_i\rangle\psi_i - \sum_{i\in\sigma}\langle x,\varphi_i\rangle\varphi_i\right\| \\
&\geqslant A\|x\| - \alpha\|x\| = (A-\alpha)\|x\|
\end{aligned}$$

for all $x \in U$. So by Proposition 1.11, $\{\varphi_i\}_{i\in I}$ and $\{\psi_i\}_{i\in I}$ are woven under the perturbations (3.1) and (3.2). □

**Theorem 3.3.** *Let $\{\varphi_i\}_{i\in I}$ and $\{\psi_{ij}\}_{i\in I}$ be frames with bounds $A, B$ and $A_j, B_j$ for $j = 1, ..., n-1$, respectively. Suppose that $\lambda \in (0,1)$ such that $\lambda(\sqrt{B} + \sqrt{B_j}) \leqslant \alpha$ for all $j = 1, \ldots, n-1$, where $(n-1)\alpha < A$ . If*

$$\left\|\sum_{i\in I} a_i(\varphi_i - \psi_{ij})\right\| \leqslant \lambda\|\{a_i\}_{i\in I}\|, \quad j = 1, \ldots, n-1 \tag{3.3}$$

*for all* $\{a_i\}_{i\in I} \in \ell^2(\mathcal{A})$*, then* $\{\varphi_i\}_{i\in I}$ *and* $\{\psi_{ij}\}_{i\in I}$ *for* $j = 1, ..., n-1$ *are woven frames with bounds* $A - (n-1)\alpha$ *and* $B + \sum_{j=1}^{n-1} B_j$.

*Proof.* Suppose that $P_\sigma : \ell^2(\mathcal{A}) \to \ell^2(\mathcal{A})$ denotes the projection (with respect to $\sigma \subset I$) described in Proposition 1.2. Let $T$ and $T_j$ be the synthesis operators for the frames $\{\varphi_i\}_{i\in I}$ and $\{\psi_{ij}\}_{i\in I}$, respectively. Let $\sigma \cup \{\sigma_j\}_{j=1}^{n-1}$ be an arbitrary partition of $I$. We define

$$T_\sigma = TP_\sigma, \quad T_{j_{\sigma_j}} = T_j P_{\sigma_j}, \quad j = 1, \ldots, n-1.$$

It follows from (3.3) that

$$\|T - T_j\| \leqslant \lambda.$$

For every $x \in U$, we have

$$\begin{aligned}\left\|\sum_{j=1}^{n-1}\left(\sum_{i\in\sigma_j}\langle x,\psi_{ij}\rangle\psi_{ij} - \sum_{i\in\sigma_j}\langle x,\varphi_i\rangle\varphi_i\right)\right\| &= \left\|\sum_{j=1}^{n-1}\left(T_{j_{\sigma_j}}T^*_{j_{\sigma_j}}(x) - T_{\sigma_j}T^*_{\sigma_j}(x)\right)\right\| \\ &\leqslant \sum_{j=1}^{n-1}\|T_{j_{\sigma_j}}T^*_{j_{\sigma_j}}(x) - T_{\sigma_j}T^*_{\sigma_j}(x)\|,\end{aligned}$$

and

$$\begin{aligned}&\sum_{j=1}^{n-1}\|T_{j_{\sigma_j}}T^*_{j_{\sigma_j}}(x) - T_{\sigma_j}T^*_{\sigma_j}(x)\| \\ &\leqslant \sum_{j=1}^{n-1}\|T_{j_{\sigma_j}}T^*_{j_{\sigma_j}}(x) - T_{j_{\sigma_j}}T^*_{\sigma_j}(x)\| + \sum_{j=1}^{n-1}\|T_{j_{\sigma_j}}T^*_{\sigma_j}(x) - T_{\sigma_j}T^*_{\sigma_j}(x)\| \\ &\leqslant \sum_{j=1}^{n-1}\|T_j\|\|T - T_j\|\|x\| + \sum_{j=1}^{n-1}\|T\|\|T - T_j\|\|x\| \\ &\leqslant \sum_{j=1}^{n-1}\lambda(\sqrt{B} + \sqrt{B_j})\|x\| \\ &\leqslant (n-1)\alpha\|x\|.\end{aligned}$$

Hence

$$\left\|\sum_{j=1}^{n-1}\left(\sum_{i\in\sigma_j}\langle x,\psi_{ij}\rangle\psi_{ij} - \sum_{i\in\sigma_j}\langle x,\varphi_i\rangle\varphi_i\right)\right\| \leqslant (n-1)\alpha\|x\|.$$

Therefore,

$$\begin{aligned}&\left\|\sum_{j=1}^{n-1}\sum_{i\in\sigma_j}\langle x,\psi_{ij}\rangle\psi_{ij} + \sum_{i\in\sigma}\langle x,\varphi_i\rangle\varphi_i\right\| \\ &= \left\|\sum_{i\in I}\langle x,\varphi_i\rangle\varphi_i + \sum_{j=1}^{n-1}\left(\sum_{i\in\sigma_j}\langle x,\psi_{ij}\rangle\psi_{ij} - \sum_{i\in\sigma_j}\langle x,\varphi_i\rangle\varphi_i\right)\right\| \\ &\geqslant \left\|\sum_{i\in I}\langle x,\varphi_i\rangle\varphi_i\right\| - \left\|\sum_{j=1}^{n-1}\left(\sum_{i\in\sigma_j}\langle x,\psi_{ij}\rangle\psi_{ij} - \sum_{i\in\sigma_j}\langle x,\varphi_i\rangle\varphi_i\right)\right\| \\ &\geqslant A\|x\| - (n-1)\alpha\|x\| = [A - (n-1)\alpha]\|x\|\end{aligned}$$

for all $x \in U$. So by Proposition 1.11, we get the result. □

**Proposition 3.4.** *Let $\{\varphi_i\}_{i\in I}$ be a frame for $U$ with frame bounds $A, B$ and $\{\psi_i\}_{i\in I}$ be a sequence in $U$. Let $\lambda_1, \lambda_2, \mu > 0$ such that $\max\{\lambda_1 + \frac{\mu}{\sqrt{A}}, \lambda_2\} < 1$ and*

$$\lambda_1\sqrt{B} + \lambda_2\left(\frac{(1+\lambda_1)\sqrt{B}+\mu}{1-\lambda_2}\right) + \mu < \sqrt{A}.$$

*If*

$$\begin{aligned}\left\|\sum_{i\in I}\langle x, \varphi_i - \psi_i\rangle\langle\varphi_i - \psi_i, x\rangle\right\|^{\frac{1}{2}} \leqslant \lambda_1 &\left\|\sum_{i\in I}\langle x, \varphi_i\rangle\langle\varphi_i, x\rangle\right\|^{\frac{1}{2}} \\ &+ \lambda_2\left\|\sum_{i\in I}\langle x, \psi_i\rangle\langle\psi_i, x\rangle\right\|^{\frac{1}{2}} + \mu\|x\|,\end{aligned} \tag{3.4}$$

*or*

$$\left\|\sum_{i\in I} a_i(\varphi_i - \psi_i)\right\| \leqslant \lambda_1\left\|\sum_{i\in I} a_i\varphi_i\right\| + \lambda_2\left\|\sum_{i\in I} a_i\psi_i\right\| + \mu\left\|\sum_{i\in I} a_i a_i^*\right\|^{\frac{1}{2}} \tag{3.5}$$

*for all $x \in U$ and all $\{a_i\}_i \in \ell^2(\mathcal{A})$, then $\{\psi_i\}_{i\in I}$ is a frame and it is woven with $\{\varphi_i\}_{i\in I}$.*

*Proof.* It follows from Theorem 3.2 that $\{\psi_i\}_{i\in I}$ is a frame for $U$. We only show that $\{\psi_i\}_{i\in I}$ is woven with $\{\varphi_i\}_{i\in I}$. Suppose that $\sigma \subset I$. If the inequality (3.4) holds, then we have

$$\begin{aligned}&\left\|\sum_{i\in\sigma}\langle x, \varphi_i\rangle\langle\varphi_i, x\rangle + \sum_{i\in\sigma^c}\langle x, \psi_i\rangle\langle\psi_i, x\rangle\right\|^{\frac{1}{2}} \\ &= \|\{\langle x, \varphi_i\rangle\}_{i\in\sigma} \cup \{\langle x, \psi_i\rangle\}_{i\in\sigma^c}\| \\ &= \|\{\langle x, \varphi_i\rangle\}_{i\in\sigma} \cup \{\langle x, \varphi_i\rangle - \langle x, \varphi_i - \psi_i\rangle\}_{i\in\sigma^c}\| \\ &= \|\{\langle x, \varphi_i\rangle\}_{i\in I} - \{\langle x, \varphi_i - \psi_i\rangle\}_{i\in\sigma^c}\| \\ &\geqslant \|\{\langle x, \varphi_i\rangle\}_{i\in I}\| - \|\{\langle x, \varphi_i - \psi_i\rangle\}_{i\in\sigma^c}\| \\ &\geqslant \sqrt{A}\|x\| - \left(\lambda_1\sqrt{B} + \lambda_2\frac{(1+\lambda_1)\sqrt{B}+\mu}{1-\lambda_2} + \mu\right)\|x\| \\ &= \left[\sqrt{A} - \left(\lambda_1\sqrt{B} + \lambda_2\frac{(1+\lambda_1)\sqrt{B}+\mu}{1-\lambda_2} + \mu\right)\right]\|x\|\end{aligned}$$

for all $x \in U$.

If the inequality (3.5) holds, then $T : \ell^2(\mathcal{A}) \to U$ given by $T(\{a_i\}_i) = \sum_{i\in I} a_i(\varphi_i - \psi_i)$ is well defined and bounded with $\|T\| \leqslant \lambda_1\sqrt{B} + \lambda_2\left(\frac{(1+\lambda_1)\sqrt{B}+\mu}{1-\lambda_2}\right) + \mu$. It is easy to see $T^*(x) = \{\langle x, \varphi_i - \psi_i\rangle\}_{i\in I}$

for all $x \in U$. Therefore,

$$\begin{aligned}
&\left\| \sum_{i\in\sigma} \langle x, \varphi_i\rangle\langle \varphi_i, x\rangle + \sum_{i\in\sigma^c} \langle x, \psi_i\rangle\langle \psi_i, x\rangle \right\|^{\frac{1}{2}} \\
&= \|\{\langle x, \varphi_i\rangle\}_{i\in\sigma} \cup \{\langle x, \psi_i\rangle\}_{i\in\sigma^c}\| \\
&= \|\{\langle x, \varphi_i\rangle\}_{i\in\sigma} \cup \{\langle x, \varphi_i\rangle - \langle x, \varphi_i - \psi_i\rangle\}_{i\in\sigma^c}\| \\
&= \|\{\langle x, \varphi_i\rangle\}_{i\in I} - \{\langle x, \varphi_i - \psi_i\rangle\}_{i\in\sigma^c}\| \\
&\geqslant \|\{\langle x, \varphi_i\rangle\}_{i\in I}\| - \|\{\langle x, \varphi_i - \psi_i\rangle\}_{i\in\sigma^c}\| \\
&\geqslant \sqrt{A}\|x\| - \|T^*\|\|x\| \\
&= \left[\sqrt{A} - \left(\lambda_1\sqrt{B} + \lambda_2\frac{(1+\lambda_1)\sqrt{B} + \mu}{1-\lambda_2} + \mu\right)\right]\|x\|
\end{aligned}$$

for all $x \in U$. Hence $\{\varphi_i\}_{i\in I}$ and $\{\psi_i\}_{i\in I}$ are woven by Proposition 1.11. □

**Corollary 3.5.** *Let $\{\varphi_i\}_{i\in I}$ be a frame for $U$ with bounds $A, B$, and let $\{\psi_i\}_{i\in I}$ be a sequence in $U$. If there exists a constant $R < A$ such that*

$$\left\| \sum_{i\in I} \langle x, \varphi_i - \psi_i\rangle\langle \varphi_i - \psi_i, x\rangle \right\| \leqslant R\|x\|^2, \quad x \in U,$$

*then $\{\psi_i\}_{i\in I}$ is a frame for $U$ and $\{\psi_i\}_{i\in I}$ is woven with $\{\varphi_i\}_{i\in I}$.*

**Lemma 3.6.** [3] *Let $X$ be a Banach space and let $T : X \to X$ be a linear operator. Assume that there exist constants $\lambda_1, \lambda_2 \in [0, 1)$ such that*

$$\|Tx - x\| \leqslant \lambda_1\|x\| + \lambda_2\|Tx\|, \quad x \in X.$$

*Then $T$ is a bounded and invertible operator on $X$, and*

$$\frac{1-\lambda_1}{1+\lambda_2}\|x\| \leqslant \|Tx\| \leqslant \frac{1+\lambda_1}{1-\lambda_2}\|x\|, \quad \frac{1-\lambda_2}{1+\lambda_1}\|x\| \leqslant \|T^{-1}x\| \leqslant \frac{1+\lambda_2}{1-\lambda_1}\|x\|, \quad x \in X.$$

**Theorem 3.7.** *Let $\{\varphi_i\}_{i\in I}$ be a frame for $U$ with frame bounds $A, B$ and let $T : U \to U$ be an adjointable operator. Assume that $\lambda_1, \lambda_2 \in [0, 1)$ such that $\sqrt{B}(\lambda_1 + \lambda_2\|T\|) < \sqrt{A}$. If $\|T - Id\| < \lambda_1 + \lambda_2\|T\|$, then $\{\varphi_i\}_{i\in I}$ and $\{T\varphi_i\}_{i\in I}$ are woven.*

*Proof.* By Lemma 3.6, $T$ is invertible and we get that $\{T\varphi_i\}_{i\in I}$ is a frame for $U$. For every $\sigma \subset I$ and for every $x \in U$, we have

$$\begin{aligned}
\left\| \sum_{i\in\sigma} \langle x, \varphi_i\rangle\langle \varphi_i, x\rangle + \sum_{i\in\sigma^c} \langle x, T\varphi_i\rangle\langle T\varphi_i, x\rangle \right\|^{\frac{1}{2}} &= \|\{\langle x, \varphi_i\rangle\}_{i\in\sigma} \cup \{\langle x, T\varphi_i\rangle\}_{i\in\sigma^c}\| \\
&= \|\{\langle x, \varphi_i\rangle\}_{i\in\sigma} \cup \{\langle x, \varphi_i\rangle - \langle (Id - T^*)x, \varphi_i\rangle\}_{i\in\sigma^c}\| \\
&= \|\{\langle x, \varphi_i\rangle\}_{i\in I} - \{\langle (Id - T^*)x, \varphi_i\rangle\}_{i\in\sigma^c}\| \\
&\geqslant \|\{\langle x, \varphi_i\rangle\}_{i\in I}\| - \|\{\langle (Id - T^*)x, \varphi_i\rangle\}_{i\in\sigma^c}\| \\
&\geqslant \sqrt{A}\|x\| - \sqrt{B}\|(Id - T^*)x\| \\
&\geqslant (\sqrt{A} - \sqrt{B}\|Id - T^*\|)\|x\| \\
&\geqslant (\sqrt{A} - \sqrt{B}(\lambda_1 + \lambda_2\|T\|))\|x\|.
\end{aligned}$$

So by Theorem 1.3, we get the result. □

**Corollary 3.8.** *Let $\{\varphi_i\}_{i\in I}$ be a frame for $U$ with frame bounds $A, B$ and let $T : U \to U$ be an adjointable operator. If $\|Id - T\|^2 \leq \alpha$ with $\alpha < \frac{A}{B}$, then $\{\varphi_i\}_{i\in I}$ and $\{T\varphi_i\}_{i\in I}$ are woven.*

*Remark* 3.9. Corollary 3.8 can be generalized to a finite number of invertible operators, and in this case we assume that the sum over all $j$ of $\|Id - T_j\|$ is less than $\sqrt{\frac{A}{B}}$.

**Corollary 3.10.** *Let $\{\varphi_i\}_{i\in I}$ be a frame with frame bounds $A, B$ and frame operator $S$. If $\frac{B}{A} < 2$, then $\{\varphi_i\}_{i\in I}$ is woven with $\{\frac{2AB}{A+B}S^{-1}\varphi_i\}_{i\in I}$ and if $\frac{B}{A} \leqslant (\sqrt{2}+1)^2$, then $\{\varphi_i\}_{i\in I}$ is woven with $\{\frac{2\sqrt{AB}}{\sqrt{A}+\sqrt{B}}S^{-\frac{1}{2}}\varphi_i\}_{i\in I}$.*

## References

[1] L. Arambašić, *On frames for countably generated Hilbert $C^*$-modules,* Proc. Am. Math. Soc. **135** (2007), 469–478.
[2] T. Bemrose, P. G. Casazza, K. Gröchenig, M. C. Lammers, R. G. Lynch, *Weaving frames,* Oper. Matrices **10** (2016), 1093–1116.
[3] P. G. Casazza, O. Christensen, *Perturbation of operators and applications to frame theory,* J. Fourier Anal. Appl. **3** (1997), 543–557.
[4] R. J. Duffin, A. C. Schaeffer, *A class of nonharmonic Fourier series,* Trans. Am. Math. Soc. **72** (1952), 341–366.
[5] Y. Feng, Y. Li, Y. Shen, S. Yang, *Construction of nonseparable symmetric tight wavelet frames of $L_2(\mathbb{R}^n)$,* J. Comput. Anal. Appl. **15** (2013), 300–307.
[6] Y. Feng, D. Yuan, S. Yang, *Construction of orthogonal shearlet t tight frames with symmetry,* J. Comput. Anal. Appl. **15** (2013), 300–307.
[7] M. Frank, D. R. Larson, *Frames in Hilbert $C^*$-modules and $C^*$-algebras,* J. Operator Theory **48** (2002), 273–314.
[8] D. Han, W. Jing, R. Mohapatra, *Perturbation of frames and Riesz bases in Hilbert $C^*$-modules,* Linear Algebra Appl. **431** (2009), 746–759.
[9] A. Khosravi, B. Khosravi, *g-Frames and modular Riesz bases in Hilbert $C^*$-modules,* Int. J. Wavelets Multiresolut. Inf. Process. **10** (2012), Art. ID 1250013, 12 pp.
[10] E. C. Lance, *Hilbert $C^*$-modules - a toolkit for operator algebraists,* London Mathematical Society Lecture Note Series, vol. 210, Cambridge University Press, England, 1995.
[11] H. Li, *A Hilbert $C^*$-module admitting no frames,* Bull. London Math. Soc. **42** (2010), 388–394.
[12] G. J. Murphy, *$C^*$-algebras and operator theory,* Academic Press, San Diego, 1990.
[13] W. Paschke, *Inner product modules over $B^*$-algebras,* Trans. Am. Math. Soc. **182** (1973), 443–468.
[14] M. Rashidi-Kouchi, A. Nazari, M. Amini, *On stability of g-frames and g-Riesz bases in Hilbert $C^*$-modules,* Int. J. Wavelets Multiresolut. Inf. Process. **12** (2014), Art. ID 1450036, 16 pp.
[15] Z.-Q. Xiang, *A note on the stability of g-frames in Hilbert $C^*$-modules,* Int. J. Wavelets Multiresolut. Inf. Process. **14** (2016), Art. ID 1650031, 9 pp.

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# THE GLOBAL ATTRACTIVITY OF SOME RATIONAL DIFFERENCE EQUATIONS

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**Abstract**

In this paper we investigate the asymptotic behavior of the solutions and the global attractivity of the equilibrium point of the following rational difference equation which was conjectured in ([**6**], *Conjecture 5.170.1*),

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{C x_{n-1} + D x_{n-2}}, \ n = 0, 1, \cdots, \tag{1}$$

with positive parametres $\alpha$, $\beta$, $\gamma$, $C$, $D$ and with arbitrary positive initial conditions $x_{-2}$, $x_{-1}$, $x_0$.

**Key Words :** Difference equation, equilibrium point, locally asymptotically stable, global attractor.

**Mathematics Subject Classification :** 39A10

# 1 Introduction and Preliminaries

Recently it is very interesting to investigate the asymptotic behavior of solutions of a rational difference equations and there has been a lot of interest in studying the global attractivity of their equilibrium points. One of the reasons is that difference equations have been applied in several mathematical models in biology, economics, genetics, physiology, ecology, physics etc. See, for example, **[1], [2], [3], [4] , [5], [8].**

We begin by introducing some basic definitions and some theorems needed in the sequel. For details, see **[7], [9].** Firstly, we investigate the local asymptotic stability of the equilibrium of the normalized form of Eq.(1) and then we study the global attractor of the equilibrium point of this rational difference equation. Finally, some numerical examples are presented to verify our theoretical results and graphed by Mathematica.

Let $I$ be some interval of real numbers and let $f : I^{k+1} \to I$ be a continuously differentiable function. A difference equation of order $(k + 1)$ is an equation of the form

$$x_{n+1} = f(x_n, x_{n-1}, \cdots, x_{n-k}), \quad n = 0, 1, \cdots. \tag{2}$$

A solution of Eq.(2) is a sequence $\{x_n\}_{n=-k}^{\infty}$ that satisfies Eq.(2) for all $n \geq -k$. As a special case of Eq.(2), for every set of initial conditions $x_0, x_{-1}, x_{-2} \in I$, the third order difference equation

$$x_{n+1} = f(x_n, x_{n-1}, x_{n-2}), \quad n = 0, 1, \cdots, \tag{3}$$

has a unique solution $\{x_n\}_{n=-2}^{\infty}$.

**Definition 1** *A solution of Eq.(2) that is constant for all $n \geq -k$ is called an equilibrium solution of Eq.(2). If*

$$x_n = \overline{x}, \text{ for all } n \geq -k$$

*is an equilibrium solution of Eq.(2), then $\overline{x}$ is called an equilibrium point, or simply an equilibrium of Eq.(2).*

So a point $\overline{x} \in I$ is called an equilibrium point of Eq.(2) if

$$\overline{x} = f(\overline{x}, \overline{x}, \cdots, \overline{x}),$$

that is,

$$x_n = \overline{x} \quad \text{for} \quad n \geq -k$$

is a solution of Eq.(2).

**Definition 2 (Stability)** *Let $\overline{x}$ an equilibrium point of Eq.(2).*

**(a)** *An equilibrium point $\overline{x}$ of Eq.(2) is called locally stable if, for every $\varepsilon > 0$; there exists $\delta > 0$ such that if $\{x_n\}_{n=-k}^{\infty}$ is a solution of Eq.(2) with*

$$|x_{-k} - \overline{x}| + |x_{1-k} - \overline{x}| + \cdots + |x_0 - \overline{x}| < \delta,$$

*then*

$$|x_n - \overline{x}| < \varepsilon, \quad \text{for all } n \geq -k.$$

**(b)** *An equilibrium point $\overline{x}$ of Eq.(2) is called locally asymptotically stable if, it is locally stable, and if in addition there exists $\gamma > 0$ such that if $\{x_n\}_{n=-k}^{\infty}$ is a solution of Eq.(2) with*

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \cdots + |x_0 - \overline{x}| < \gamma,$$

*then we have*

$$\lim_{n\to\infty} x_n = \overline{x}.$$

**(c)** *An equilibrium point $\overline{x}$ of Eq.(2) is called a global attractor if, for every solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(2), we have*

$$\lim_{n\to\infty} x_n = \overline{x}.$$

**(d)** *An equilibrium point $\overline{x}$ of Eq.(2) is called globally asymptotically stable if it is locally stable, and a global attractor.*

**(e)** *An equilibrium point $\overline{x}$ of Eq.(2) is called unstable if it is not locally stable.*

Now we present some important results and definitions which will be useful for our investigation.

Suppose that the function $f$ is continuously differentiable in some open neighborhood of an equilibrium point $\overline{x}$. Let

$$q_i = \frac{\partial f}{\partial u_i}(\overline{x}, \overline{x}, \cdots, \overline{x}), \quad \text{for } i = 0, 1, \cdots, k$$

denote the partial derivative of $f(u_0, u_1, \cdots, u_k)$ with respect to $u_i$ evaluated at the equilibrium point $\overline{x}$ of Eq.(2)

**Definition 3** *The equation*

$$z_{n+1} = q_0 z_n + q_1 z_{n-1} + \cdots + q_k z_{n-k}, \ n = 0, 1, \cdots, \tag{4}$$

*is called the linearized equation of Eq.(2) about the equilibrium point $\overline{x}$, and the equation*

$$\lambda^{k+1} - q_0 \lambda^k - \cdots - q_{k-1}\lambda - q_k = 0 \tag{5}$$

*is called the characteristic equation of Eq.(4) about $\overline{x}$.*

**Theorem 4 (Clark Theorem)** ***([6],*** *p.6) Assume that $q_0, q_1, ..., q_k$ are real numbers such that*

$$|q_0| + |q_1| + \cdots + |q_k| < 1.$$

*Then all roots of Eq.(5) lie inside the unit disk.*

We give the following theorems without proofs. The *Linearized Stability Theorem*, is very useful in determining the local stability character of the equilibrium point $\overline{x}$ of Eq.(2)

**Theorem 5 (The Linearized Stability Theorem)** ***([6],*** *p.5) Assume that the function $f$ is a continuously differentiable function defined on some open neighborhood of an equilibrium point $\overline{x}$. Then the following statements are true:*

**(a)** *When all the roots of Eq.(5) have absolute value less than one, then the equilibrium point $\overline{x}$ of Eq.(2) is locally asymptotically stable.*

**(b)** *If at least one root of Eq.(5) has absolute value greater than one, then the equilibrium point $\overline{x}$ of Eq.(2) is unstable.*

# 2 Local Stability Analysis

In this section we investigate the local asymptotic stability of the positive equilibrium $\overline{x}$ of the normalized form of Eq.(1).

**Lemma 6**

**(a)** *Eq.(1) can be written in the normalized form*

$$x_{n+1} = \frac{\alpha + \beta x_n + x_{n-1}}{x_{n-1} + D x_{n-2}}, \ n = 0, 1, \cdots, \tag{6}$$

*with positive parameters* $\alpha, \beta, D$ *and with arbitrary positive initial conditions* $x_{-2}, x_{-1}, x_0$.

**(b)** *Positive equilibrium point of Eq.(6) is*

$$\overline{x} = \frac{(\beta+1) + \sqrt{(\beta+1)^2 + 4\alpha(1+D)}}{2(1+D)}.$$

**(c)** *The linearized equation of Eq.(6) about its positive equilibrium* $\overline{x}$ *is*

$$z_{n+1} - \frac{\beta}{(1+D)\overline{x}} z_n + \frac{\overline{x}-1}{(1+D)\overline{x}} z_{n-1} + \frac{D}{1+D} z_{n-2} = 0.$$

**Proof.**

**(a)** The Eq.(1) which by the change of variables

$$x_n = \frac{\gamma}{C} y_n$$

reduces to the difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + x_{n-1}}{x_{n-1} + D x_{n-2}}, \qquad n = 0, 1, \cdots,$$

where

$$\alpha := \frac{\alpha C}{\gamma^2}, \quad \beta := \frac{\beta}{\gamma}, \quad D := \frac{D}{C}.$$

**(b)** The positive equilibrium points of Eq.(6) are the non-negative solutions of the equation

$$\overline{x} = \frac{\alpha + \beta\overline{x} + \overline{x}}{\overline{x} + D\overline{x}}$$

or equivalently

$$(1+D)\overline{x}^2 - (1+\beta)\overline{x} - \alpha = 0. \tag{7}$$

Hence, one can easily obtain the solutions of Eq.(7) are

$$\overline{x} = \frac{1 + \beta + \sqrt{(1+\beta)^2 + 4\alpha(1+D)}}{2(1+D)} \tag{8}$$

and

$$\overline{x} = \frac{1 + \beta - \sqrt{(1+\beta)^2 + 4\alpha(1+D)}}{2(1+D)}$$

So, the positive equilibrium point of Eq.(6) is unique and is given by (8).

**(c)** Now, let $I$ be some interval of real numbers and let

$$f : I \times I \times I \to I$$

be a continuously differentiable function such that $f$ is defined by

$$f(x_n, x_{n-1}, x_{n-2}) = \frac{\alpha + \beta x_n + x_{n-1}}{x_{n-1} + Dx_{n-2}}.$$

Thus, we obtain that

$$\begin{aligned} q_0 &= \frac{\partial f}{\partial x_n}(\overline{x}, \overline{x}, \overline{x}) = \left[\frac{\beta.(x_{n-1} + Dx_{n-2})}{(x_{n-1} + Dx_{n-2})^2}\right](\overline{x}, \overline{x}, \overline{x}) \\ &= \frac{\beta.(\overline{x} + D\overline{x})}{\overline{x}^2(1+D)^2} = \frac{\beta\overline{x}(1+D)}{\overline{x}^2(1+D)^2} = \frac{\beta}{\overline{x}(1+D)} \end{aligned}$$

$$\begin{aligned} q_1 &= \frac{\partial f}{\partial x_{n-1}}(\overline{x}, \overline{x}, \overline{x}) = \left[\frac{(x_{n-1} + Dx_{n-2}) - (\alpha + \beta x_n + x_{n-1})}{(x_{n-1} + Dx_{n-2})^2}\right](\overline{x}, \overline{x}, \overline{x}) \\ &= \frac{\overline{x} + D\overline{x} - \overline{x}.(\overline{x} + D\overline{x})}{\overline{x}^2(1+D)^2} = \frac{(\overline{x} + D\overline{x})(1-\overline{x})}{\overline{x}^2(1+D)^2} = \frac{1-\overline{x}}{\overline{x}(1+D)}. \end{aligned}$$

and

$$\begin{aligned} q_2 &= \frac{\partial f}{\partial x_{n-2}}(\overline{x}, \overline{x}, \overline{x}) = \left[\frac{-(\alpha + \beta x_n + x_{n-1}).D}{(x_{n-1} + Dx_{n-2})^2}\right](\overline{x}, \overline{x}, \overline{x}) \\ &= \frac{-(\alpha + \beta\overline{x} + \overline{x}).D}{(\overline{x} + D\overline{x})^2} = \frac{-\overline{x}.(\overline{x} + D\overline{x}).D}{(\overline{x} + D\overline{x})^2} = \frac{-D}{1+D} \end{aligned}$$

If $\overline{x}$ denotes an equilibrium point of Eq.(6), then the linearized equation associated with Eq.(6) about the equilibrium point $\overline{x}$ is

$$z_{n+1} = q_0 z_n + q_1 z_{n-1+} q_2 z_{n-2}$$

or

$$z_{n+1} - \frac{\beta}{(1+D)\overline{x}} z_n + \frac{\overline{x} - 1}{(1+D)\overline{x}} z_{n-1} + \frac{D}{1+D} z_{n-2} = 0. \tag{9}$$

**Lemma 7** *The positive equilibirium $\overline{x}$ of Eq.(6) is locally asymptotically stable when*

$$\frac{(1+\beta)^2 (D-1)}{4} < \alpha \quad \textit{and} \quad \beta < 1.$$

**Proof.** From Theorem 4 it follows that all roots of the characteristic equation of Eq.(9) lie in an open disc $|\lambda| < 1$, if

$$|q_0| + |q_1| + |q_2| < 1.$$

This implies that

$$\left|\frac{\beta}{(1+D)\overline{x}}\right|+\left|\frac{1-\overline{x}}{(1+D)\overline{x}}\right|+\left|\frac{-D}{1+D}\right|<1.$$

Hence

$$\begin{aligned}\frac{\beta}{(1+D)\overline{x}}+\frac{|1-\overline{x}|}{(1+D)\overline{x}}+\frac{D\overline{x}}{(1+D)\,\overline{x}} &< 1 &(10)\\ \beta+D\overline{x}+|1-\overline{x}| &< (1+D)\overline{x} &(11)\\ |1-\overline{x}| &< \overline{x}+D\overline{x}-\beta-D\overline{x} &(12)\\ |1-\overline{x}| &< \overline{x}-\beta &(13)\end{aligned}$$

and so we have two following two cases to consider.

**Case I :** Since

$$1-\overline{x}<\overline{x}-\beta$$

we have

$$\frac{1+\beta}{2}<\overline{x}.$$

$$\frac{1+\beta}{2}<\frac{(\beta+1)+\sqrt{(\beta+1)^2+4\alpha(1+D)}}{2(1+D)}$$

$$\frac{(1+\beta)^2\,(D-1)}{4}<\alpha.$$

**Case II :** Since

$$-\overline{x}+\beta<1-\overline{x}$$

it follows that

$$\beta<1.$$

# 3 Main Result

In this section we are concerned with the global attractor of Eq.(6). The following two theorems will be needed in the main result of this paper.

**Theorem 8** *([10], p.205) Let $[a,b]$ be an interval of real numbers and assume that*

$$f:[a,b]\times[a,b]\times[a,b]\longrightarrow[a,b]$$

*is a continuous function satisfying the following properties:*

**(a)** *$f(x,y,z)$ is non-decreasing in $x$ for each $y$ and $z\in[a,b]$ and is non-increasing in $y$ and $z$ for each $x\in[a,b]$ of its arguments;*

**(b)** *If* $(m, M) \in [a, b] \times [a, b]$ *is a solution of the system*

$$M = f(M, m, m) \quad \text{and} \quad m = f(m, M, M),$$

*then* $m = M$.

*Then Eq.(3) has a unique equilibrium* $\overline{x} \in [a, b]$ *and every solution of Eq.(3) converges to* $\overline{x}$.

**Theorem 9** ***([10],*** *p.202) Let* $[a, b]$ *be an interval of real numbers and assume that*

$$f : [a, b] \times [a, b] \times [a, b] \longrightarrow [a, b]$$

*is a continuous function satisfying the following properties:*

**(a)** $f(x, y, z)$ *is non-decreasing in* $x$ *and* $y \in [a, b]$ *for each* $z \in [a, b]$, *and is non-increasing in* $z \in [a, b]$ *for each* $x$ *and* $y \in [a, b]$

**(b)** *If* $(m, M) \in [a, b] \times [a, b]$ *is a solution of the system*

$$m = f(m, m, M) \quad \text{and} \quad M = f(M, M, m),$$

*then* $m = M$.

*Then Eq.(3) has a unique equilibrium* $\overline{x} \in [a, b]$ *and every solution of Eq.(3) converges to* $\overline{x}$.

We are now in a position to give the main result of this work.

**Lemma 10** *The equilibrium point* $\overline{x}$ *is a global attractor of Eq.(6) if one of the following statements holds:*

$$(i)\ Dw < \alpha + \beta u \text{ and } \beta \neq 1, \tag{14}$$

$$(ii)\ Dw > \alpha + \beta u \text{ and } D > 1,\ \frac{(\beta + 1)^2 (D - 1)}{4} < \alpha. \tag{15}$$

***Proof.*** *Let* $\alpha$ *and* $\beta$ *be real numbers and assume that* $g : [\alpha, \beta]^3 \to [\alpha, \beta]$ *is a function defined by*

$$g(u, v, w) = \frac{\alpha + \beta u + v}{v + Dw}.$$

*Then it follows that*

$$\begin{aligned}
\frac{\partial g(u, v, w)}{\partial u} &= \frac{\beta}{v + Dw}, \\
\frac{\partial g(u, v, w)}{\partial v} &= \frac{Dw - \alpha - \beta u}{(v + Dw)^2}, \\
\frac{\partial g(u, v, w)}{\partial w} &= \frac{-(\alpha + \beta u + v)D}{(v + Dw)^2}.
\end{aligned}$$

*We consider two cases :*

***Case I :*** *If $Dw < \alpha+\beta u$ then we can easily see that the function $g(u,v,w)$ is increasing in $u$ and decreasing in $v,w$.*

*Suppose that $(m,M)$ is a solution of the system $M = g(M,m,m)$ and $m = g(m,M,M)$ then from (6),we see that*

$$M = \frac{\alpha+\beta M+m}{m+Dm}, \qquad m = \frac{\alpha+\beta m+M}{M+DM}.$$

*Since*

$$Mm + DMm - \beta M - m - \alpha = 0$$

$$Mm + DMm - \beta m - M - \alpha = 0$$

*we have*

$$(m-M)(\beta-1) = 0$$

*When $\beta \neq 1$, we have*

$$M = m$$

*which the result follows.*

*It follows from Theorem 8 that $\overline{x}$ is global attractor of Eq.(6) and then the proof is complete.*

***Case II :*** *If $Dw > \alpha+\beta u$, then we can easily see that the function $g(u,v,w)$ is increasing in $u,v$ and decreasing in $w$.*

*Suppose that $(m,M)$ is a solution of the system $M = g(M,M,m)$ and $m = g(m,m,M)$. Then from (6), we see that*

$$M = \frac{\alpha+\beta M+M}{M+Dm}, \qquad m = \frac{\alpha+\beta m+m}{m+DM}.$$

*Since*

$$M^2 + DMm - M(\beta+1) - \alpha = 0$$

$$m^2 + DMm - m(\beta+1) - \alpha = 0$$

*we have*

$$(m-M)((m+M)-(\beta+1)) = 0$$

*with simple calculations. Now if $m+M \neq \beta+1$, then $M = m$. On the other hand if $m+M = \beta+1$ then $m$ and $M$ satisfy the equation*

$$m^2 + Dm(\beta+1-m) = \alpha+\beta m+m$$

*and so*

$$m^2(1-D) + (\beta+1)(D-1)m - \alpha = 0. \tag{16}$$

*The discriminant of the Eq.(16)*

$$\begin{aligned}\Delta &= [(\beta+1)(D-1)]^2 + 4(1-D)\alpha \\ &= (D-1)\left[(\beta+1)^2(D-1) - 4\alpha\right]\end{aligned}$$

*is negative when*

$$D > 1 \text{ and } (\beta+1)^2 (D-1) < 4\alpha$$

*then we have*

$$M = m$$

*which the result follows.*

*It follows from Theorem 9 that* $\overline{x}$ *is global attractor of Eq.(6) and then the proof is complete.* ■

# 4 Numerical Examples

In this section we give some numerical examples to support our theoretical discussion which was mentioned in the previous section.

**Example 11** *Consider the equation* $x_{n+1} = \dfrac{1+3x_n+2x_{n-1}}{2x_{n-1}+x_{n-2}}$ *with initial conditions* $x_{-2} = 0.1$ , $x_{-1} = 0.2$, $x_0 = 0.4$ *to verify our theoretical results. (See Fig. 1)*

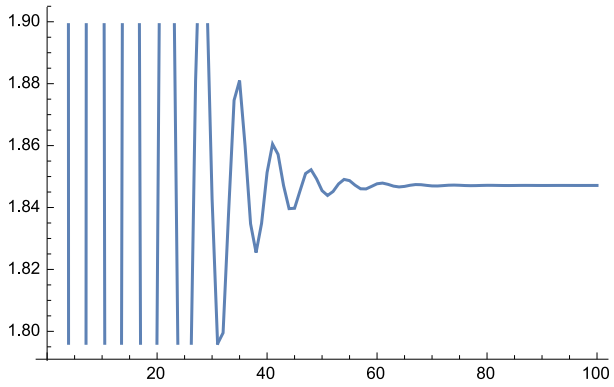



Figure 1: Plot of the difference equation $x_{n+1} = \frac{1+3x_n+2x_{n-1}}{2x_{n-1}+x_{n-2}}$.

**Example 12** *Consider the equation* $x_{n+1} = \dfrac{0.5+2x_n+x_{n-1}}{x_{n-1}+(0.1)\,x_{n-2}}$ *with initial conditions* $x_{-2} = 20$ , $x_{-1} = 1/3$, $x_0 = 0.1$ *to verify our theoretical results. (See Fig. 2)*

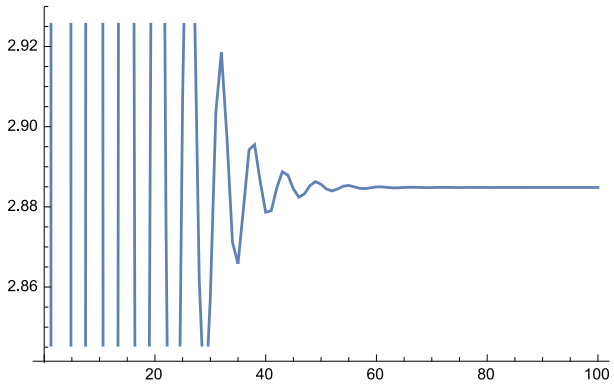



Figure 2: Plot of the difference equation $x_{n+1} = \frac{0.5+2x_n+x_{n-1}}{x_{n-1}+(0.1)x_{n-2}}$

# References

[1] R. P. Agarwal. *Difference Equations and Inequalities*, 1st edition, Marcel Dekker, New York, 1992, 2nd edition, 2000.

[2] A. M. Amleh, V. Kirk, and G. Ladas. On the dynamics of $x_{n+1} = \frac{a+bx_{n-1}}{A+Bx_{n-2}}$, *Math. Sci. Res. Hot-Line*, **5** (2001), 1-15.

[3] A. Breet, E. J. Janowski and M. R. S. Kulenovic. Global asymptotic stability for linear fractional difference equation. *Journal of Difference Equations,* vol.2014, Article ID 275312, 11 pages, 2014.

[4] C. H. Gibbons, M. R. S. Kulenovic, and G. Ladas, On the dynamics of $x_{n+1} = \frac{\alpha+\beta x_n+\gamma x_{n-1}}{A+Bx_n}$, *Proceedings of the Fifth International Conference on Difference Equations and Applications*, Temuco, Chile, Taylor & Francis, London 2002, 141-158.

[5] E. Camouzis and G. Ladas, "When does local asymptoticstability imply global attractivity in rational equations?", *Journal of Difference Equations and Applications*, vol. 12, no. 8, 2006, pp. 863–885.

[6] E. Camouzis and G. Ladas**.** *Dynamics of third order rational difference equations with open problems and conjectures, volume 5 of Advances in Discrete Mathematics and Applications.* Chapman & Hall/CRC, Boca Raton, FL, 2008.

[7] S. Elaydi. *An Introduction to Difference Equations*, Springer-Verlag, New York, Inc, 1996.

[8] G. Ladas, On the recursive sequence $x_{n+1} = \dfrac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}}$, *J. Difference Equ. Appl.*, **1** (1995), 317-321.

[9] V. L. Kocić and G. Ladas. *Global behavior of nonlinear difference equations of higher order with applications*, Kluwer Academic Publishers, Dordrecht, (1993).

[10] M. R. S. Kulenovic and G. Ladas. *Dynamics of second order rational difference equations.* Chapman & Hall/CRC, Boca Raton, FL, *With Open Problems and Conjectures*, 2002.

# Hybrid pair via $\alpha$-admissible Geraghty $F$-contraction with applications

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**Abstract:** In this paper, we introduce a notion of hybrid pair $(f, T)$, which is $\alpha_*$-admissible with respect to $\eta$ for generalized multivalued Geraghty $F$-contraction mappings and obtain coincidence and common fixed point results for such mappings. We provide some examples to support our results and give applications to dynamic programming and integral equations. Our results provide extension as well as substantial generalizations and improvements of several well known results in the existing comparable literature.

## 1 Introduction and preliminaries

Let $(X, d)$ be a metric space. Let $CB(X)$ $(CL(X))$ be the family of all nonempty closed and bounded (nonempty closed) subsets of $X$. For $A, B \in CL(X)$, define

$$E_{A,B} = \{\varepsilon > 0 : A \subseteq N_\varepsilon(B), B \subseteq N_\varepsilon(A)\}.$$

The Hausdorff metric $H$ on $CL(X)$ induced by metric $d$ is given as:

$$H(A, B) = \begin{cases} \inf E_{A,B} & \text{if } E_{A,B} \neq \emptyset \\ \infty & \text{if } E_{A,B} = \emptyset \end{cases}.$$

Let $f : X \to X$ and $T : X \to CL(X)$. A hybrid pair $\{f, T\}$ is said to satisfy a *range inclusion condition* if $f(X) \subseteq T(X)$.

A point $x$ in $X$ is called a fixed point of $T$ if $x \in Tx$. The set of all fixed points of $T$ is denoted by $F(T)$. Furthermore, a point $x$ in $X$ is called a coincidence point of $f$ and $T$ if $fx \in Tx$. The set of all such points is denoted by $C(f, T)$. If for some point $x$ in $X$, we have $x = fx \in Tx$, then a point $x$ is called a common fixed point of $f$ and $T$. We denote the set of all common fixed points of $f$ and $T$ by $F(f, T)$. A mapping $T : X \to CL(X)$ is said to be continuous at $p \in X$ if for any sequence $\{x_n\}$ in $X$ with $\lim_{n\to\infty} d(x_n, p) = 0$, we have $\lim_{n\to\infty} H(Tx_n, Tp) = 0$.

**Definition 1.1** [28] *Let $\alpha : X \times X \to [0, \infty)$. A self mapping $T$ on $X$ is said to be $\alpha$-admissible if for any $x, y \in X$, $\alpha(x, y) \geq 1$ implies that $\alpha(Tx, Ty) \geq 1$.*

Hussain et al. [17] introduced the notion of $\alpha_*$-admissible mappings as follows:

**Definition 1.2** [17] *Let $\alpha, \eta : X \times X \to [0, \infty)$ be two functions where $\eta$ is bounded. A mapping $T : X \to 2^X$ is called $\alpha_*$-admissible with respect to $\eta$ if $\alpha(x, y) \geq \eta(x, y)$ implies $\alpha_*(Tx, Ty) \geq \eta_*(Tx, Ty)$, $x, y \in X$, where $\alpha_*(A, B) = \inf\{\alpha(x, y) : x \in A,\ y \in B\}$ and $\eta_*(A, B) = \sup\{\eta(x, y) : x \in A,\ y \in B\}$.*

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If $\eta(x,y)=1$ for all $x,y\in X$, then the above definition reduces to [17, Definition 4.1].

In Definition 1.2, if $\alpha(x,y)=1$ for all $x,y\in X$, then $T$ is called an $\eta_*$-subadmissible mapping.

We extend Definition 1.2 to hybrid pair of mappings as follows:

**Definition 1.3** *Let $f:X\to X$, $T:X\to CL(X)$, and $\alpha,\eta:X\times X\to[0,\infty)$ be two functions where $\eta$ is bounded. We say that the hybrid pair $(f,T)$ is $\alpha_*$-admissible with respect to $\eta$ if $\alpha(fx,fy)\geq\eta(fx,fy)$ implies $\alpha_*(Tx,Ty)\geq\eta_*(Tx,Ty)$, $x,y\in X$, where*

$$\alpha_*(A,B)=\inf\{\alpha(fx,fy):fx\in A,\ fy\in B\}\ \text{ and }\ \eta_*(A,B)=\sup\{\eta(fx,fy):fx\in A,\ fy\in B\}.$$

**Definition 1.4** *Let $f,T:X\to X$ and $\alpha,\eta:X\times X\to[0,+\infty)$. We say that the pair $(f,T)$ is an $\alpha$-admissible mapping with respect to $\eta$ if $\alpha(fx,fy)\geq(fx,fy)$ implies $\alpha(Tx,Ty)\geq\eta(Tx,Ty)$, $x,y\in X$.*

In 1973, Geraghty [14] studied most interesting generalization of Banach contraction principle.

**Theorem 1.5** [14] *Let $(X,d)$ be a metric space. Let $T:X\to X$ be a self mapping. Suppose that there exists $\beta\in\Omega$ such that for all $x,y\in X$,*

$$d(Tx,Ty)\leq\beta\left(d(x,y)\right)d(x,y),$$

*then $S$ has a fixed unique point $p\in X$ and $\{T^nx\}$ converges to $p$ for each $x\in X$.*

We denote by $\Omega$ the family of all functions $\beta:[0,+\infty)\to[0,1)$ such that, for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n)\to1$ implies $t_n\to0$.

For more discussion on Geraghty contraction mappings, we refer to [22, 23] and references therein.

Berinde and Berinde [12] extended the notion of weak contraction mappings as follows:

**Definition 1.6** [12, 13] *A mapping $T:X\to CL(X)$ is called a multivalued weak contraction if there exist two constants $\theta\in(0,1)$ and $L\geq0$ such that*

$$H(Tx,Ty)\leq\theta d(x,y)+Ld(y,Tx)$$

*holds for all $x,y$ in $X$.*

The following definition of a generalized multivalued $(\theta,L)$-strict almost contraction mapping is due to Berinde and Păcurar [13].

**Definition 1.7** [13] *A mapping $T:X\to CL(X)$ is called a generalized multivalued $(\theta,L)$-strict almost contraction mapping if there exist two constants $\theta\in(0,1)$ and $L\geq0$ such that*

$$H(Tx,Ty)\leq\theta d(x,y)+L\min\{d(y,Tx),d(x,Ty),d(x,Tx),d(y,Ty)\}$$

*holds for all $x,y$ in $X$.*

We have the following fixed point theorem given in [13].

**Theorem 1.8** *Let $(X,d)$ be a complete metric space and $T:X\to CL(X)$ a generalized multivalued $(\theta,L)$-strict almost contraction mapping. Then $F(T)\neq\emptyset$. Moreover, for any $p\in F(T)$, $T$ is continuous at $p$.*

Kamran [21] extended the notion of a multivalued weak contraction mapping to a hybrid pair $\{f,T\}$ of single valued mapping $f$ and multivalued mapping $T$. For more discussion on multivalued mappings, we refer to [4, 15] and references therein.

**Definition 1.9** *Let $(X,d)$ be a metric space and $f$ a self map on $X$. A multivalued mapping $T:X\to CL(X)$ is called a generalized multivalued $(f,\theta,L)$-weak contraction mapping if there exist two constants $\theta\in(0,1)$ and $L\geq0$ such that*

$$H(Tx,Ty)\leq\theta d(fx,fy)+Ld(fy,Tx)$$

*holds for all $x,y$ in $X$.*

Abbas [1] extended the above definition as follows.

**Definition 1.10** [1] *Let $(X,d)$ be a metric space and $f$ a self mapping on $X$. A multivalued mapping $T: X \to CL(X)$ is called a generalized multivalued $(f,\theta,L)$-almost contraction mapping if there exist two constants $\theta \in (0,1)$ and $L \geq 0$ such that*

$$H(Tx,Ty) \leq \theta M(x,y) + LN(x,y)$$

*holds for all $x, y$ in $X$, where*

$$\begin{aligned} M(x,y) &= \max\{d(fx,fy), d(fx,Tx), d(fy,Ty), \frac{d(fx,Ty)+d(fy,Tx)}{2}\}, \\ N(x,y) &= \min\{d(fx,Tx), d(fy,Ty), d(fx,Ty), d(fy,Tx)\}. \end{aligned}$$

Let $\digamma$ be the collection of all mappings $F: \mathbb{R}_+ \to \mathbb{R}$ which satisfy the following conditions:
$(C1)$ $F$ is strictly increasing, that is, for all $\alpha, \beta \in \mathbb{R}_+$ such that $\alpha < \beta \Rightarrow F(\alpha) < F(\beta)$;
$(C2)$ For every sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive numbers $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;
$(C3)$ There exists $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.
Wardowski [32] introduced the following concept of $F$-contraction mappings.

**Definition 1.11** [32] *Let $(X,d)$ be a metric space. A self mapping $f$ on $X$ is said to be an $F$-contraction on $X$ if there exists $\tau > 0$ such that*

$$d(fx,fy) > 0 \Rightarrow \tau + F(d(fx,fy)) \leq F(d(x,y))$$

*for all $x, y \in X$, where $F \in \digamma$.*

**Remark 1.12** [32] *Every $F$-contraction mapping is continuous.*

Abbas et al. [2] extended the concept of $F$-contraction mapping and obtained common fixed point results. They employed their results to obtain fixed points of a generalized nonexpansive mappings on star shaped subsets of normed linear spaces. Recently, Minak [24] proved some fixed point results for Ćirić type generalized $F$-contractions on complete metric spaces.

Sgroi and Vetro [29] proved the following result to obtain fixed point of multivalued mappings as a generalization of Nadler's Theorem [25].

**Theorem 1.13** [29] *Let $(X,d)$ be a complete metric space and $T: X \to CL(X)$ a multivalued mapping. Assume that there exist $F \in \digamma$ and $\tau \in \mathbb{R}_+$ such that*

$$2\tau + F(H(Tx,Ty)) \leq F(\alpha d(x,y) + \beta d(x,Tx) + \gamma d(y,Ty) + \delta d(x,Ty) + L d(y,Tx))$$

*for all $x, y \in X$, with $Tx \neq Ty$, where $\alpha, \beta, \gamma, \delta, L \geq 0$, $\alpha + \beta + \gamma + 2\delta = 1$ and $\gamma \neq 1$. Then $T$ has a fixed point.*

Acar et al. [3] proved the following result.

**Theorem 1.14** [3] *Let $(X,d)$ be a complete metric space and $T: X \to K(X)$ (i.e., compact subsets of $X$). Assume that there exist an $F \in \digamma$ and $\tau \in \mathbb{R}_+$ such that for any $x, y \in X$, we have*

$$H(Tx,Ty) > 0 \Longrightarrow \tau + F(H(Tx,Ty)) \leq F(M(x,y)),$$

*where*

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}.$$

*Then $T$ has a fixed point if $T$ or $F$ is continuous,*

Recently, Altun et al. [6] proved the following result.

**Theorem 1.15** [6] *Let $(X,d)$ be a complete metric space and $T: X \to CB(X)$. Assume that there exist an $F \in \digamma$ and $\tau, \lambda \in \mathbb{R}_+$ such that for any $x, y \in X$, we have*

$$H(Tx,Ty) > 0 \text{ implies that } \tau + F(H(Tx,Ty)) \leq F(d(x,y) + \lambda d(y,Tx)).$$

*Then the mapping $T$ is a multivalued weakly Picard operator.*

For the definition of multivalued weakly Picard operator and the related results, we refer to [12].

**Definition 1.16** *Let $f$ be a self mapping on a metric space $X$ and $T: X \to CL(X)$ a multivalued mapping. Then $T$ is called a generalized multivalued $(f,L)$-almost $F$-contraction mapping if there exist $F \in \digamma$ and $\tau \in \mathbb{R}_+$ and $L \geq 0$ such that*

$$2\tau + F(H(Tx,Ty)) \leq F(M(x,y) + LN(x,y)) \tag{1}$$

*for all $x, y$ in $X$ with $Tx \neq Ty$ and*

$$\begin{aligned} M(x,y) &= \max\{d(fx,fy), d(fx,Tx), d(fy,Ty), \frac{d(fx,Ty)+d(fy,Tx)}{2}\}, \\ N(x,y) &= \min\{d(fx,Tx), d(fy,Ty), d(fx,Ty), d(fy,Tx)\}). \end{aligned}$$

**Remark 1.17** *Take $F(x) = \ln x$ in Definition* 1.16. *Then* (1) *becomes*

$$2\tau + \ln(H(Tx,Ty)) \leq \ln(M(x,y) + LN(x,y),$$

*that is,*

$$\begin{aligned} H(Tx,Ty)) &\leq e^{-2\tau}M(x,y) + e^{-2\tau}LN(x,y) \\ &= \theta_1 M(x,y) + L_1 N(x,y), \end{aligned}$$

*where $\theta_1 = e^{-2\tau} \in (0,1)$ and $L_1 = e^{-2\tau}L \geq 0$. Thus we obtain the generalized multivalued $(f, \theta_1, L_1)$-almost contraction mapping* [1].

**Remark 1.18** *Take $\alpha = \beta = \gamma = \frac{1}{4}$, $\delta = \frac{1}{8} = L$. Note that $\alpha + \beta + \gamma + 2\delta = 1$. Then the contraction condition in Theorem* 1.13 *becomes*

$$\begin{aligned} 2\tau + F(H(Tx,Ty)) &\leq F\left(\frac{1}{4}\left(d(x,y) + (d(x,Tx) + d(y,Ty)) + \frac{d(x,Ty)+d(y,Tx)}{2}\right)\right) \\ &\leq F\left(\frac{1}{4}\left(4M(x,y)\right)\right) = F\left((M(x,y) + 0N(x,y))\right) \end{aligned}$$

*for all $x, y \in X$ with $Tx \neq Ty$. Thus, for $L = 0$ and $f = I_X$,*

$$\begin{aligned} M(x,y) &= \max\{d(fx,fy), d(fx,Tx), d(fy,Ty), \frac{d(fx,Ty)+d(fy,Tx)}{2}\}, \\ N(x,y) &= \min\{d(fx,fy), d(fx,Tx), d(fy,Ty)\}, \end{aligned}$$

*and the contraction condition in Theorem* 1.14 *is an $(f,0)$-almost $F$-contraction, a special case of generalized multivalued $(f,L)$-almost $F$-contraction (for $L = 0$ and $\tau = 2\tau_1$).*

Now, we give the following definition.

**Definition 1.19** *Let $f$ and $T$ be a pair of self mappings on a metric space $X$. Suppose that $\alpha, \eta : X \times X \to [0,\infty)$ and $\beta : [0,+\infty) \to [0,1)$ be three functions. Then $T$ is called a Geraghty $F$-contraction with respect to $\eta$ if for any $x, y \in X$ with $\alpha(Tx,Ty) \geq \eta(Tx,Ty)$, $Tx \neq Ty$, we have*

$$2\tau + F(d(Tx,Ty)) \leq F(\beta\left(M(x,y)\right) M(x,y)),$$

*where*

$$M(x,y) = \max\{d(fx,fy), d(fx,Tx), d(fy,Ty), \frac{d(fx,Ty)+d(fy,Tx)}{2}\},$$

*for $F \in \digamma, \beta \in \Omega$ and $\tau \in \mathbb{R}_+$.*

**Definition 1.20** *Let $f$ be a self mapping on a metric space $X$ and $T : X \to CL(X)$ a multivalued mapping. Suppose that $\alpha, \eta : X \times X \to [0, \infty)$ and $\beta : [0, +\infty) \to [0, 1)$ be three functions. Then $T$ is called a generalized multivalued Geraghty $F$-contraction with respect to $\eta$ if for any $x, y \in X$ with $\eta_*(Tx, Ty) \leq \alpha_*(Tx, Ty)$ and $Tx \neq Ty$, we have*

$$2\tau + F(H(Tx, Ty)) \leq F(\beta\,(M(x, y))\, M(x, y)),$$

*where*

$$M(x, y) = \max\{d(fx, fy), d(fx, Tx), d(fy, Ty), \frac{d(fx, Ty) + d(fy, Tx)}{2}\}$$

*for $F \in \digamma, \beta \in \Omega$ and $\tau \in \mathbb{R}_+$.*

**Definition 1.21** [20] *Let $f : X \to X$ and $T : X \to CL(X)$ a multivalued mapping. The pair $(f, T)$ is called* (a) *commuting if $Tfx = fTx$ for all $x \in X$* (b) *weakly compatible if they commute at their coincidence points, that is, $fTx = Tfx$ whenever $x \in C(f, T)$.*

The mapping $f$ is called *$T$-weakly commuting* at $x \in X$ if $f^2x \in Tfx$ If a hybrid pair $(f, T)$ is weakly compatible at $x \in C(f, T)$, then $f$ is $T$-weakly commuting at $x$ and hence $f^n(x) \in C(f, T)$. However, the converse is not true in general. For detailed discussion on the above mentioned notions and their implications, we refer to [5, 16, 18, 19, 20, 30, 31] and references therein.

# 2 Main results

Throughout this section, we assume that the mapping $F$ is right continuous.

Now we state our main result.

**Theorem 2.1** *Let $f$ be a self mapping on a metric space $(X, d)$ and $T : X \to CL(X)$ a multivalued mapping with $\overline{T(X)} \subseteq f(X)$ satisfying the following assertions:*

(i) *the pair $(f, T)$ is an $\alpha_*$-admissible mapping with respect to $\eta$;*

(ii) *$T$ is a generalized multivalued Geraghty $F$-contraction with respect to $\eta$;*

(iii) *there exists $x_0, x_1 \in X$ such that $fx_1 \in Tx_0$ and $\alpha(fx_0, fx_1) \geq \eta(fx_0, fx_1)$;*

(iv) *if $\{fx_n\}$ is a sequence in $f(X)$ such that $\alpha(fx_n, fx_{n+1}) \geq \eta(fx_n, fx_{n+1})$ for all $n \in \mathbb{N}$ and $fx_n \to fu^*$, then $\alpha(fx_n, fu^*) \geq \eta(fx_n, fu^*)$ for all $n \in \mathbb{N}$.*

*If $\overline{T(X)}$ is complete, then $C(f, T) \neq \phi$ provided that $F$ is continuous. Moreover, $F(f, T) \neq \emptyset$ if one of the following conditions holds:*

(a) *for some $x \in C(f, T)$ with $f$ is $T$-weakly commuting at $x$, $f^2x = fx$;*

(b) *$f(C(f, T))$ is a singleton subset of $C(f, T)$.*

**Proof.** We first note that, by Definition 1.16, $H(Tx, Ty) < \infty$ for all $x, y \in X$. Now we shall show that $C(f, T) \neq \emptyset$. Let $x_0$ and $x_1$ be given points in $X$ such that $fx_1 \in Tx_0$ and $\alpha(fx_0, fx_1) \geq \eta(fx_0, fx_1)$. If $H(Tx_0, Tx_1) = 0$, then $Tx_0 = Tx_1$ and so $x_1 \in C(f, T)$. Assume that $H(Tx_0, Tx_1) > 0$. Since $F$ is right continuous, there exists $h > 1$ such that

$$F(hH(Tx_0, Tx_1)) < F(H(Tx_0, Tx_1)) + \tau.$$

Then $d(fx_1, Tx_1) \leq H(Tx_0, Tx_1)$, and hence there exists $y_1 \in Tx_1$ such that

$$d(fx_1, y_1) < hH(Tx_0, Tx_1).$$

Pick an element $x_2$ in $X$ such that $fx_2 = y_1$. Then the above inequality becomes

$$d(fx_1, fx_2) < hH(Tx_0, Tx_1).$$

If $fx_1 = fx_2$, then $fx_1 \in Tx_1$. In this case $x_1$ becomes a coincidence point of $f$ and $T$ and the proof is finished.

Assume that $fx_1 \neq fx_2$, that is, $d(fx_1, fx_2) > 0$. Since $F$ is strictly increasing, we obtain

$$F(d(fx_1, fx_2)) < F(hH(Tx_0, Tx_1)) < F(H(Tx_0, Tx_1)) + \tau.$$

Since $(f, T)$ is an $\alpha_*$-admissible mapping with respect to $\eta$, $\alpha(fx_0, fx_1) \geq \eta(fx_0, fx_1)$ implies that $\alpha_*(Tx_0, Tx_1) \geq \eta_*(Tx_0, Tx_1)$. Since $T$ is a generalized multivalued Geraghty $F-$contraction with respect to $\eta$, we have

$$\begin{aligned}
F(d(fx_1, fx_2)) &< F(H(Tx_0, Tx_1)) + \tau \\
&\leq F(\beta\,(M(x_0, x_1))\, M(x_0, x_1)) - 2\tau + \tau \\
&= F(\beta\left(\max\{d(fx_0, fx_1), d(fx_0, Tx_0), d(fx_1, Tx_1), \frac{d(fx_0, Tx_1) + d(fx_1, Tx_0)}{2}\}\right) \\
&\quad \times \max\{d(fx_0, fx_1), d(fx_0, Tx_0), d(fx_1, Tx_1), \frac{d(fx_0, Tx_1) + d(fx_1, Tx_0)}{2}\}) - \tau \\
&\leq F(\beta\left(\max\{d(fx_0, fx_1), d(fx_0, fx_1), d(fx_1, fx_2), \frac{d(fx_0, fx_2) + d(fx_1, fx_1)}{2}\}\right) \\
&\quad \times \max\{d(fx_0, fx_1), d(fx_0, fx_1), d(fx_1, fx_2), \frac{d(fx_0, fx_2) + d(fx_1, fx_1)}{2}\}) - \tau \\
&\leq F(\beta\left(\max\{d(fx_0, fx_1), d(fx_1, fx_2), \frac{d(fx_0, fx_1) + d(fx_1, fx_2)}{2}\}\right) \\
&\quad \times \max\{d(fx_0, fx_1), d(fx_1, fx_2), \frac{d(fx_0, fx_1) + d(fx_1, fx_2)}{2}\}) - \tau \\
&= F(\beta\,(\max\{d(fx_0, fx_1), d(fx_1, fx_2)\})\max\{d(fx_0, fx_1), d(fx_1, fx_2)\}) - \tau.
\end{aligned}$$

Suppose that $d(fx_1, fx_2) \not< d(fx_0, fx_1)$. Then we obtain

$$F(d(fx_2, fx_1)) < F(\beta\,(d(fx_1, fx_2))\, d(fx_1, fx_2) - \tau.$$

Since $\beta \in \Omega$, we have

$$F(d(fx_2, fx_1)) < F(d(fx_1, fx_2)) - \tau,$$

which implies $\tau \leq 0$, a contradiction. Hence $d(fx_1, fx_2) < d(fx_0, fx_1)$ and so

$$\tau + F(d(fx_2, fx_1)) \leq F(\beta\,(d(fx_0, fx_1))\, d(fx_0, fx_1)\}.$$

Note that $\alpha(fx_1, fx_2) \geq \alpha_*(Tx_0, Tx_1) \geq \eta_*(Tx_0, Tx_1) \geq \eta(fx_1, fx_2)$. That is, $\alpha(fx_1, fx_2) \geq \eta(fx_1, fx_2)$ which further implies that $\alpha_*(Tx_1, Tx_2) \geq \eta_*(Tx_1, Tx_2)$. If $H(Tx_1, Tx_2) = 0$, then $Tx_1 = Tx_2$ and so $x_2 \in C(f, T)$. Assume that $H(Tx_1, Tx_2) > 0$. Since $F$ is right continuous, there exists $h > 1$ such that

$$F(hH(Tx_1, Tx_2)) < F(H(Tx_1, Tx_2)) + \tau.$$

Then $d(fx_2, Tx_2) \leq H(Tx_1, Tx_2)$, and hence there exists $y_2 \in Tx_2$ such that

$$d(fx_2, y_2) < hH(Tx_1, Tx_2).$$

Pick an element $x_3$ in $X$ such that $fx_3 = y_2$. Then the above inequality becomes

$$d(fx_2, fx_3) < hH(Tx_1, Tx_2).$$

If $fx_2 = fx_3$, then $fx_2 \in Tx_2$. In this case $x_2$ becomes a coincidence point of $f$ and $T$ and the proof is finished.

Assume that $fx_2 \neq fx_3$, that is, $d(fx_2, fx_3) > 0$. Since $F$ is strictly increasing, we obtain

$$F(d(fx_2, fx_3)) < F(hH(Tx_1, Tx_2)) < F(H(Tx_1, Tx_2)) + \tau.$$

Now $\alpha_*(Tx_1, Tx_2) \geq \eta_*(Tx_1, Tx_2)$ implies that

$$\begin{aligned}
F(d(fx_2, fx_3)) &< F(H(Tx_1, Tx_2)) + \tau. \\
&\leq F(\beta\,(M(x_1, x_2))\, M(x_1, x_2)) - 2\tau + \tau \\
&= F(\beta\left(\max\{d(fx_1, fx_2), d(fx_1, Tx_1), d(fx_2, Tx_2), \frac{d(fx_1, Tx_2) + d(fx_2, Tx_1)}{2}\}\right) \\
&\quad \times \max\{d(fx_1, fx_2), d(fx_1, Tx_1), d(fx_2, Tx_2), \frac{d(fx_1, Tx_2) + d(fx_2, Tx_1)}{2}\}) - \tau \\
&\leq F(\beta\left(\max\{d(fx_1, fx_2), d(fx_1, fx_2), d(fx_2, fx_3), \frac{d(fx_1, fx_3) + d(fx_2, fx_2)}{2}\}\right) \\
&\quad \times \max\{d(fx_1, fx_2), d(fx_1, fx_2), d(fx_2, fx_3), \frac{d(fx_1, fx_3) + d(fx_2, fx_2)}{2}\}) - \tau \\
&\leq F(\beta\left(\max\{d(fx_1, fx_2), d(fx_2, fx_3), \frac{d(fx_1, fx_2) + d(fx_2, fx_3)}{2}\}\right) \\
&\quad \times \max\{d(fx_1, fx_2), d(fx_2, fx_3), \frac{d(fx_1, fx_2) + d(fx_2, fx_3)}{2}\}) - \tau \\
&= F(\beta\,(\max\{d(fx_1, fx_2), d(fx_2, fx_3)\})\max\{d(fx_1, fx_2), d(fx_2, fx_3)) - \tau.
\end{aligned}$$

Thus we have

$$\tau + F(d(fx_3, fx_2)) < F(d(fx_1, fx_2)).$$

Note that $\alpha(fx_2, fx_3) \geq \alpha_*(Tx_1, Tx_2) \geq \eta_*(Tx_1, Tx_2) \geq \eta(fx_2, fx_3)$. That is, $\alpha(fx_2, fx_3) \geq \eta(fx_2, fx_3)$ which further implies that $\alpha_*(Tx_2, Tx_3) \geq \eta_*(Tx_2, Tx_3)$. By continuing this process, we obtain a sequence $\{fx_n\} \subset f(X)$ such that $fx_n \in Tx_{n-1}$,

$$\alpha(fx_{n-1}, fx_n) \geq \eta(fx_{n-1}, fx_n) \text{ implies that } \alpha_*(Tx_{n-1}, Tx_n) \geq \eta_*(Tx_{n-1}, Tx_n)$$

and we have

$$\begin{aligned}
&F(d(fx_n, fx_{n+1})) < F(H(Tx_{n-1}, Tx_n)) + \tau \\
&\leq F(\beta\left(\max\{d(fx_{n-1}, fx_n), d(fx_{n-1}, Tx_{n-1}), d(fx_n, Tx_n), \frac{d(fx_{n-1}, Tx_n) + d(fx_n, Tx_{n-1})}{2}\}\right) \\
&\times \max\{d(fx_{n-1}, fx_n), d(fx_{n-1}, Tx_{n-1}), d(fx_n, Tx_n), \frac{d(fx_{n-1}, Tx_n) + d(fx_n, Tx_{n-1})}{2}\}) - \tau \\
&\leq F(\beta\left(\max\{d(fx_{n-1}, fx_n), d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1}), \frac{d(fx_{n-1}, fx_{n+1}) + d(fx_n, fx_n)}{2}\}\right) \\
&\times \max\{d(fx_{n-1}, fx_n), d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1}), \frac{d(fx_{n-1}, fx_{n+1}) + d(fx_n, fx_n)}{2}\}) - \tau \\
&\leq F(\beta\left(\max\{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1}), \frac{d(fx_{n-1}, fx_n) + d(fx_n, fx_{n+1})}{2}\}\right) \\
&\times \max\{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1}), \frac{d(fx_{n-1}, fx_n) + d(fx_n, fx_{n+1})}{2}\}) - \tau \\
&= F(\beta\,(\max\{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1})\})\max\{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1})\}) - \tau
\end{aligned}$$

and

$$\tau + F(d(fx_n, fx_{n+1})) \leq F(\beta\,(\max\{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1})\})\max\{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1})\})$$

for all $n \in \mathbb{N}$. Since $F$ is strictly increasing, we have

$$d(fx_n, fx_{n+1}) < \beta\,(\max\{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1})\})\max\{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1})\}.$$

If

$$\max\{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1}) = d(fx_n, fx_{n+1})$$

for some $n$, then

$$d(fx_n, fx_{n+1}) < d(fx_n, fx_{n+1}),$$

which is a contradiction since $\beta \in \Omega$ and hence we have

$$d(fx_n, fx_{n+1}) < d(fx_{n-1}, fx_n).$$

Consequently,

$$\tau + F(d(fx_n, fx_{n+1})) \leq F(d(fx_{n-1}, fx_n))$$

for all $n \in \mathbb{N}$. Thus we obtain that

$$\begin{aligned} F(d(fx_n, fx_{n+1})) &\leq F(d(fx_{n-1}, fx_n)) - \tau \\ &\leq F(d(fx_{n-2}, fx_{n-1})) - 2\tau \\ &\vdots \\ &\leq F(d(fx_0, fx_1)) - n\tau. \end{aligned}$$

Taking limit as $n \to \infty$, we have $\lim_{n\to\infty} F(d(fx_n, fx_{n+1})) = -\infty$. By (C2), $\lim_{n\to\infty} d(fx_n, fx_{n+1}) = 0$. By (C3), there exists an $r \in (0,1)$ such that

$$\lim_{n\to\infty} \{d(fx_n, fx_{n+1})\}^r F(d(fx_n, fx_{n+1})) = -\infty.$$

Hence it follows that

$$\begin{aligned} & \{d(fx_n, fx_{n+1})\}^r F(d(fx_n, fx_{n+1})) - \{d(fx_n, fx_{n+1})\}^r F(d(fx_0, fx_1)) \\ \leq\ & d(fx_n, fx_{n+1})^r [F(d(fx_0, fx_1) - n\tau)] - d(x_n, x_{n+1})^r F(d(fx_0, fx_1)) \\ =\ & -n\tau [d(fx_n, fx_{n+1})]^r \leq 0. \end{aligned}$$

Taking limit as $n$ tends to $\infty$, we obtain that $\lim_{n\to\infty} n\{d(fx_n, fx_{n+1})\}^r = 0$, i.e., $\lim_{n\to\infty} n^{1/r} d(fx_n, fx_{n+1}) = 0$. This implies that $\sum_{n=1}^{\infty} d(fx_n, fx_{n+1})$ is convergent and hence the sequence $\{fx_n\}_{n\geq 1}$ is a Cauchy sequence in $T(X) \subseteq \overline{T(X)}$. Since $\overline{T(X)}$ is complete, there exists $p \in \overline{T(X)}$ such that $\lim_{n\to\infty} fx_n = p$. Now $\overline{T(X)} \subseteq f(X)$ implies that there exists $u^*$ in $X$ such that $fu^* = p$. Since $\{fx_n\}$ is a sequence in $f(X)$ such that $\alpha(fx_n, fx_{n+1}) \geq \eta(fx_n, fx_{n+1})$ for all $n \in \mathbb{N}$ and $fx_n \to fu^*$, $\alpha(fx_n, fu^*) \geq \eta(fx_n, fu^*)$ for all $n \in \mathbb{N}$. Since the hybrid pair $(f, T)$ is $\alpha_*$-admissible with respect to $\eta$, we have $\alpha_*(Tx_n, Tu^*) \geq \eta_*(Tx_n, Tu^*)$, which implies that

$$\begin{aligned} 2\tau + F(d(fx_{n+1}, Tu^*)) &\leq 2\tau + F(H(Tx_n, Tu^*)) \\ &\leq F(\beta(M(x_n, u^*)) M(x_n, u^*) \end{aligned}$$

for all $n \in \mathbb{N}$ by the contractive condition.

Next suppose that $F$ is continuous. Then

$$\lim_{n\to\infty} d(fx_n, Tu^*) = d(fu^*, Tu^*). \tag{2}$$

Note that

$$\begin{aligned} d(fu^*, Tu^*) &\leq M(x_n, u^*) = \max\left\{d(fx_n, fu^*), d(fx_n, Tx_n), d(fu^*, Tu^*), \frac{d(fx_n, Tu^*) + d(fu^*, Tx_n)}{2}\right\} \\ &\leq \max\left\{d(fx_n, fu^*), d(fx_n, fx_{n+1}), d(fu^*, Tu^*), \frac{d(fx_n, Tu^*) + d(fu^*, fx_{n+1})}{2}\right\}. \end{aligned}$$

Taking limit as $n \to \infty$, we obtain that

$$\lim_{n\to\infty} M(x_n, u^*) = d(fu^*, Tu^*). \tag{3}$$

Since $\beta \in \Omega$, it follows from (2), (3) and the continuity of $F$ that

$$2\tau + F(d(fu^*, Tu^*)) \leq F(d(fu^*, Tu^*)).$$

that is, $d(fu^*, Tu^*) = 0$ and thus $fu^* \in Tu^*$.

Now let (a) hold, that is, for $x \in C(f,T)$, $f$ is $T$-weakly commuting at $x$. So we get $f^2x \in Tfx$. By the given hypothesis, $fx = f^2x$ and hence $fx = f^2x \in Tfx$. Consequently, $fx \in F(f,T)$.

Let (b) hold. Since $f(C(f,T)) = \{x\}$ and $x \in C(f,T)$, $x = fx \in Tx$. Thus $F(f,T) \neq \emptyset$. ■

**Example 2.2** *Let $X = [1, \infty)$ be the usual metric space. Define $f : X \to X$, $\alpha, \eta : X \times X \to [0, \infty)$, $\beta : [0, +\infty) \to [0, 1)$ and $T : X \to CL(X)$ by $fx = x^2$, $Tx = [x + 2, \infty)$ for all $x \in X$, $\alpha(x,y) = \begin{cases} e^{xy} & \text{if } x, y \geq 0 \\ 0 & \text{otherwise.} \end{cases}$, $\eta(x,y) = \begin{cases} e^{x-y} & \text{if } x \geq y \\ 0 & \text{otherwise.} \end{cases}$, $\beta(t) = \frac{1}{t+1}$, $\forall t > 0$, where $\tau = \ln \sqrt{2}$, and $F(t) = \ln(t)$ for all $t > 0$. Note that $\overline{T(X)} = T(X) = [3, \infty)$ and $\alpha(fx, fy) \geq \eta(fx, fy)$ imply that $\alpha_*(Tx, Ty) \geq \eta_*(Tx, Ty)$ for all $x, y \in X$ with $Tx \neq Ty$ (equivalently, $x \neq y$) and we have*

$$2\tau + F(H(Tx, Ty)) \leq F\left(\beta\left(M(x,y)\right) M(x,y)\right).$$

*Thus all the conditions of Theorem* 2.1 *are satisfied.*

If we take $\eta(x,y) = 1$ in Theorem 2.1, then we have the following result.

**Theorem 2.3** *Let $f$ be a self mapping on a metric space $(X,d)$ and $T : X \to CL(X)$ a multivalued mapping with $\overline{T(X)} \subseteq f(X)$ satisfying the following assertions:*

(i) *the pair $(f,T)$ is an $\alpha_*$-admissible mapping;*

(ii) *$T$ is a generalized multivalued Geraghty $F$-contraction;*

(iii) *there exists $x_0, x_1 \in X$ such that $fx_1 \in Tx_0$ and $\alpha(fx_0, fx_1) \geq 1$;*

(iv) *if $\{fx_n\}$ is a sequence in $f(X)$ such that $\alpha(fx_n, fx_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $fx_n \to fu^*$, then $\alpha(fx_n, fu^*) \geq 1$ for all $n \in \mathbb{N}$.*

*If $\overline{T(X)}$ is complete, then $C(f,T) \neq \phi$ provided that $F$ is continuous. Moreover, $F(f,T) \neq \emptyset$ if one of the following conditions holds:*

(a) *for some $x \in C(f,T)$ with $f$ is $T$-weakly commuting at $x$, $f^2x = fx$;*

(b) *$f(C(f,T))$ is a singleton subset of $C(f,T)$.*

**Corollary 2.4** *Let $f$ and $T$ be a self mapping on a metric space $(X,d)$ such that $T(X) = f(X)$. Suppose that*

(i) *the pair $(f,T)$ is an $\alpha$-admissible mapping with respect to $\eta$;*

(ii) *$T$ is a Geraghty $F$-contraction with respect to $\eta$;*

(iii) *there exists $x_0, x_1 \in X$ such that $fx_1 = Tx_0$ and $\alpha(fx_0, fx_1) \geq \eta(fx_0, fx_1)$;*

(iv) *if $\{fx_n\}$ is a sequence in $f(X)$ such that $\alpha(fx_n, fx_{n+1}) \geq \eta(fx_n, fx_{n+1})$ for all $n \in \mathbb{N}$ and $fx_n \to fu^*$, then $\alpha(fx_n, fu^*) \geq \eta(fx_n, fu^*)$ for all $n \in \mathbb{N}$.*

*If $T(X)$ is complete, then $C(f,T) \neq \phi$ provided that $F$ is continuous. Moreover, $F(f,T) \neq \emptyset$ if $f$ and $T$ commute at their coincidence point.*

**Proof.** If we take $X = CL(X)$ in Theorem 2.1, then we obtain the required result. ■

# 3 Applications

In this section, we obtain the existence and uniqueness of common solution of: $(I)$ system of functional equations arising in dynamical programing problems and $(II)$ system of integral equations.

### $(I)$ Application in dynamic programming

Decision space and a state space are two basic components of dynamic programming problems. State space is a set of states including initial states, action states and transitional states. So a state space is set of parameters representing different states. A decision space is the set of possible actions that can be taken to solve the problem. These general settings allow us to formulate many problems in mathematical optimization and computer programming. In particular, the problem of dynamic programming related to multistage process reduces to the problem of solving functional equations

$$p(x) = \sup_{y\in D}\{g(x,y)+G_1(x,y,p(\xi(x,y)))\},\ \text{for } x\in W, \tag{4}$$

$$q(x) = \sup_{y\in D}\{g'(x,y)+G_2(x,y,q(\xi(x,y)))\},\ \text{for } x\in W, \tag{5}$$

where $U$ and $V$ are Banach spaces, $W\subseteq U$ and $D\subseteq V$ and

$$\begin{aligned} \xi &: W\times D\longrightarrow W\\ g,g' &: W\times D\longrightarrow \mathbb{R}\\ G_1,G_2 &: W\times D\times\mathbb{R}\longrightarrow\mathbb{R}\\ \beta &: \mathbb{R}^+\to[0,1)\,. \end{aligned}$$

For more discussions and results on dynamic programming problems, we refer to [8, 9, 10, 11, 27] and references mentioned therein. Suppose that $W$ and $D$ are the state and decision spaces, respectively. We aim to give the existence and uniqueness of common and bounded solution of functional equations given in (4) and (5). Let $B(W)$ denote the set of all bounded real valued functions on $W$. For an arbitrary $h\in B(W)$, define $\|h\|=\sup_{x\in W}|h(x)|$. Then $(B(W),\|\cdot\|)$ is a Banach space endowed with the metric $d$ given by

$$d(h,k)=\sup_{x\in W}|h(x)-k(x)|\,.$$

Suppose that the following conditions hold:

$(C1)$ $G_1,G_2,g$ and $g'$ are bounded.

$(C2)$ For $x\in W$, $h\in B(W)$ and $b>0$, define

$$Kh(x)=\sup_{y\in D}\{g(x,y)+G_1(x,y,h(\xi(x,y)))\}, \tag{6}$$

$$Jh(x)=\sup_{y\in D}\{g'(x,y)+G_2(x,y,h(\xi(x,y)))\}. \tag{7}$$

Moreover, assume that there exist $\tau>0$ and $L\geq 0$ such that for every $(x,y)\in W\times D$, $h,k\in B(W)$ and $t\in W$ imply

$$|G_1(x,y,h(t))-G_1(x,y,k(t))|\leq e^{-2\tau}[\beta\,(M(h,k))\,M(h,k)], \tag{8}$$

where

$$M((h,k))=\max\{d(Jh,Jk),d(Jk,Kk),d(Jh,Kh),\frac{d(Jh,Kk)+d(Jk,Kh)}{2}\}.$$

$(C3)$ For any $h\in B(W)$, there exists $k\in B(W)$ such that $Kh(x)=Jk(x)$ for $x\in W$.

$(C4)$ There exists $h\in B(W)$ such that $Kh(x)=Jh(x)$ implies that $JKh(x)=KJh(x)$.

$(C5)$ There exist $\alpha,\eta: B(W)\times B(W)\to\mathbb{R}^+$ such that $\alpha(Jh_1,Jh_2)\geq\eta(Jh_1,Jh_2)$.

**Theorem 3.1** *Assume that the conditions* $(C1)-(C5)$ *are satisfied. If* $J(B(W))$ *is a closed convex subspace of* $B(W)$*, then the functional equations* (4) *and* (5) *have a unique, common and bounded solution.*

**Proof.** Note that $(B(W),d)$ is a complete metric space. By $(C1)$, $J,K$ are self-mappings of $B(W)$ and each element is mapped into a singleton set. The condition $(C3)$ implies that $K(B(W))\subseteq J(B(W))$ is satisfied. It follows from $(C4)$ that $J$ and $K$ commute at their coincidence points. Let $\lambda$ be an arbitrary positive number and $h_1,h_2\in B(W)$. Choose $x\in W$ and $y_1,y_2\in D$ such that

$$Kh_j<g(x,y_j)+G_1(x,y_j,h_j(x_j)+\lambda, \tag{9}$$

where $x_j = \xi(x, y_j)$, $j = 1, 2$. Further, from (6) and (7), we have

$$\begin{aligned} Kh_1 &\geq g(x, y_2) + G_1(x, y_2, h_1(x_2)), \\ Kh_2 &\geq g(x, y_1) + G_1(x, y_1, h_2(x_1)). \end{aligned} \tag{10}$$

Then (9) and (10) together with (8) imply that

$$\begin{aligned} Kh_1(x) - Kh_2(x) &< G_1(x, y_1, h_1(x_1)) - G_1(x, y_1, h_2(x_2)) + \lambda \\ &\leq |G_1(x, y_1, h_1(x_1)) - G_1(x, y_1, h_2(x_2))| + \lambda \\ &\leq e^{-2\tau}(\beta\,(M(h,k))\,M(h,k)) + \lambda. \end{aligned} \tag{11}$$

Then (8) and (9) together with (7) imply that

$$\begin{aligned} Kh_2(x) - Kh_1(x) &\leq G_1(x, y_1, h_2(x_2)) - G_1(x, y_1, h_1(x_1)) \\ &\leq |G_1(x, y_1, h_1(x_1)) - G_1(x, y_1, h_2(x_2))| \\ &\leq e^{-2\tau}(\beta\,(M(h,k))\,M(h,k)). \end{aligned} \tag{12}$$

From (11) and (12), we have

$$|Kh_1(x) - Kh_2(x)| \leq e^{-2\tau}(\beta\,(M(h,k))\,M(h,k)). \tag{13}$$

The inequality (13) implies

$$d(Kh_1, Kh_2) \leq e^{-2\tau}[(\beta\,(M(h,k))\,M(h,k))].$$

Since $\alpha(Jh_1(x), Jh_2(x)) \geq \eta(Jh_1(x), Jh_2(x))$ implies that $\alpha(Kh_1(x), Kh_2(x)) \geq \eta(Kh_1(x), Kh_2(x))$ and so we have

$$2\tau + \ln[d(Kh_1(x) - Kh_2(x))] \leq \ln[(\beta\,(M(h(t),k(t)))\,M(h(t),k(t)))].$$

Therefore, by Corollary 2.4, the pair $(K, J)$ has a common fixed point $h^*$, that is, $h^*(x)$ is a unique, bounded and common solution of (4) and (5). ■

### $(II)$ Application of integral equations

Now we discuss an application of fixed point theorem, proved in the previous section, to solve the system of Volterra type integral equations. Such system is given by the following equations:

$$u(t) = \int_0^t K_1(t, s, u(s))ds + g(t), \tag{14}$$

$$w(t) = \int_0^t K_2(t, s, w(s))ds + f(t). \tag{15}$$

for $t \in [0, a]$, where $a > 0$. We find the solution of the system (14) and (15). Let $C([0, a], \mathbb{R})$ be the space of all continuous functions defined on $[0, a]$. For $u \in C([0, a], \mathbb{R})$, define supremum norm as: $\|u\|_\tau = \sup_{t\in[0,a]} \{u(t)e^{-\tau(t)t}\}$, where $\tau > 0$. Let $C([0, a], \mathbb{R})$ be endowed with the metric

$$d_\tau(u, v) = \sup_{t\in[0,a]} \| \, |u(t) - v(t)| \, e^{-\tau(t)t} \|_\tau$$

for all $u, v \in C([0, a], \mathbb{R})$. With these setting $C([0, a], \mathbb{R}, \|\cdot\|_\tau)$ becomes a Banach space.

Now we prove the following theorem to ensure the existence of solution of system of integral equations. For more details on such applications, we refer the reader to [7, 26].

**Theorem 3.2** *Assume the following conditions are satisfied:*
(i) $K_1, K_2 : [0,a] \times [0,a] \times \mathbb{R} \to \mathbb{R}$ *and* $f, g : [0,a] \to \mathbb{R}$ *are continuous;*
(ii) *Define*

$$\begin{aligned} Tu(t) &= \int_0^t K_1(t,s,u(s))ds + g(t), \\ Su(t) &= \int_0^t K_2(t,s,u(s))ds + f(t). \end{aligned}$$

*Suppose* $\alpha, \eta : [0,a] \times [0,a] \to \mathbb{R}^+, \beta : [0,a] \to [0,1)$. *Then there exist* $\tau \geq 1$ *and* $L \geq 0$ *such that*

$$|K_1(t,s,u) - K_2(t,s,v)| \leq \tau e^{-2\tau}[\beta\left(M(u,v)\right) M(u,v)]$$

*for all* $t, s \in [0,a]$ *and* $u, v \in C([0,a], \mathbb{R})$, *where*

$$M(u,v) = \max\{|Su(t) - Sv(t)|, |Sv(t) - Tv(t)|, |Su(t) - Tu(t)|, \frac{|Su(t) - Tv(t)| + |Sv(t) - Tu(t)|}{2}\};$$

(iii) *There exists* $u \in C([0,a], \mathbb{R})$ *such that* $Tu(t) = Su(t)$ *implies* $TSu(t) = STu(t)$.
*Then the system of the integral equations* (14) *and* (15) *has a solution.*

**Proof.** Define the mapping $\alpha, \eta : [0,a] \times [0,a] \to \mathbb{R}^+$ by

$$\alpha(u,v) = \begin{cases} e^{uv} \text{ if } u, v \in [0,a] \\ 0, \quad \text{otherwise} \end{cases} \quad \text{and } \eta(u,v) = \begin{cases} e^{u-v} \text{ if } u, v \in [0,a] \\ 0, \quad \text{otherwise.} \end{cases}$$

Then $\alpha(u,v) \geq \eta(u,v)$ and so $\alpha(Su, Sv) \geq \eta(Su, Sv)$. By the assumption (iii),

$$\begin{aligned} |Tu(t) - Tv(t)| &= \int_0^t |K_1(t,s,u(s) - K_2(t,s,v(s)))| \, ds \\ &\leq \int_0^t \tau e^{-2\tau}([\beta\left(M(u,v)\right) M(u,v)]e^{-\tau s})e^{\tau s} ds \\ &\leq \int_0^t \tau e^{-2\tau} \|\beta\left(M(u,v)\right) M(u,v)\|_\tau e^{\tau s} ds \\ &\leq \tau e^{-2\tau} \|\beta\left(M(u,v)\right) M(u,v)\|_\tau \int_0^t e^{\tau s} ds \\ &\leq \tau e^{-2\tau} \|\beta\left(M(u,v)\right) M(u,v)\|_\tau \frac{1}{\tau} e^{\tau t} \\ &\leq e^{-2\tau} \|\beta\left(M(u,v)\right) M(u,v)\|_\tau e^{\tau t}, \end{aligned}$$

which implies

$$|Tu(t) - Tv(t)| \, e^{-\tau t} \leq e^{-2\tau} \|\beta\left(M(u,v)\right) M(u,v)\|_\tau.$$

That is,

$$\|Tu(t) - Tv(t)\|_\tau \leq e^{-2\tau} \|\beta\left(M(u,v)\right) M(u,v)\|_\tau$$

and

$$\alpha(Su, Sv) \geq \eta(Su, Sv) \text{ implies } \alpha(Tu, Tv) \geq \eta(Tu, Tv)$$

and so we have

$$2\tau + \ln \|Tu(t) - Tv(t)\|_\tau \leq \ln \|\beta\left(M(u,v)\right) M(u,v)\|_\tau.$$

So all the conditions of Corollary 2.4 are satisfied. Hence the system of integral equations (14) and (15) has a unique common solution. ■

# References


[1] M. Abbas, Coincidence points of multivalued $f$-almost nonexpansive mappings, Fixed Point Theory **13** (2012), 3–10.

[2] M. Abbas, B. Ali, S. Romaguera, Fixed and periodic points of generalized contractions in metric spaces, Fixed Point Theory Appl. **2013**, 2013:243.

[3] O. Acar, G. Durmaz, G. Minak, Generalized multivalued $F$-contraction on complete metric spaces, Bull. Iranian Math. Soc. **40** (2014), 1469–1478.

[4] J. Ahmad, N. Hussain, A. R. Khan, A. Azam, Fixed point results for generalized multi-valued contractions, J. Nonlinear Sci. Appl. **8** (2015), 909–918.

[5] M. A. Al-Thagafi, N. Shahzad, Coincidence points, generalized $I$-nonexpansive multimaps and applications, Nonlinear Anal. **67** (2007), 2180–2188.

[6] I. Altun, G. Mınak, H. Dag, Multivalued $F$-contraction on complete metric spaces, J. Nonlinear Convex Anal. (in press).

[7] A. Augustynowicz, Existence and uniqueness of solutions for partial differential-functional equations of the first order with deviating argument of the derivative of unknown function, Serdica Math. J. **23** (1997), 203–210.

[8] R. Baskaran, P.V. Subrahmanyam, A note on the solution of a class of functional equations. Appl. Anal. **22** (1986), 235–241.

[9] R. Bellman, Methods of Nonlinear Analysis. Vol. $II$. Mathematics in Science and Engineering, vol. 61. Academic Press, New York 1973.

[10] R. Bellman, E.S. Lee, Functional equations in dynamic programming. Aequationes. Math. **17** (1978), 1–18.

[11] P.C. Bhakta, S. Mitra, Some existence theorems for functional equations arising in dynamic programming. J. Math. Anal. Appl. **98** (1984), 348–362 .

[12] M. Berinde, V. Berinde, On general class of multivalued weakly Picard mappings, J. Math. Anal. Appl. **326** (2007), 772–782.

[13] V. Berinde, M. Păcurar, Fixed points and continuity of almost contractions, Fixed Point Theory **9** (2008), 23–34.

[14] M. Geraghty, On contractive mappings, Proc. Am. Math. Soc. **40** (1973), 604–608.

[15] N. Hussain, J. Ahmad, L. Ćirić, A. Azam, Coincidence point theorems for generalized contractions with application to integral equations, Fixed Point Theory Appl. **2015**, 2015:78.

[16] N. Hussain, G. Jungck, Common fixed point and invariant approximation results for noncommuting generalized $(f, g)$-nonexpansive maps, J. Math. Anal. Appl. **321** (2006), 851–861.

[17] N. Hussain, P Salimi, A. Latif, Fixed point results for single and set-valued $\alpha$-$\eta$-$\psi$-contractive mappings, Fixed Point Theory Appl. **2013**, 2013:212.

[18] G. Jungck, Commuting mappings and fixed points, Am. Math. Monthly **83** (1976), 261–263.

[19] G. Jungck, Common fixed points for commuting and compatible maps on compacta, Proc. Am. Math. Soc. **103** (1988), 977–983.

[20] G. Jungck, B. E. Rhoades, Fixed points for set valued functions without continuity, Indian J. Pure Appl. Math. **29** (1998), 227–238.

[21] T. Kamran, Multivalued $f$-weakly Picard mappings, Nonlinear Anal. **67** (2007), 2289–2296.

[22] E. Karapinar, $\alpha$-$\psi$-Geraghty contraction type mappings and some related fixed point results, Filomat **28** (2014), 37–48.

[23] E. Karapinar, A discussion on "$\alpha$-$\psi$-Geraghty contraction type mappings, Filomat **28** (2014), 761–766.

[24] G. Minak, A. Halvaci, I. Altun, Ćirić type generalized $F$-contractions on complete metric spaces and fixed point results, Filomat **28** (2014), 1143–1151.

[25] S. B. Nadler, Multivalued contraction mappings, Pacific J. Math. **30** (1969), 475–488.

[26] D. O'Regan, A. Petruşel, Fixed point theorems for generalized contractions in ordered metric spaces, J. Math. Anal. Appl. **341** (2008), 1241–1252.

[27] H. K. Pathak, Y. Cho, S. Kang, B. Lee, Fixed point theorems for compatible mappings of type $P$ and applications to dynamic programming. Matematiche **50** (1995), 15–33.

[28] B. Samet, C. Vetro, P. Vetro, Fixed point theorems for $\alpha$-$\psi$-contractive type mappings, Nonlinear Anal. **75** (2012), 2154–2165.

[29] M. Sgroi, C. Vetro, Multi-valued $F$-contractions and the solution of certain functional and integral equations, Filomat **27** (2013), 1259–1268.

[30] A. Shoaib, $\alpha$-$\eta$ dominated mappings and related common fixed point results in closed ball, J. Concrete Appl. Math. **13** (2015), 152–170.

[31] A. Shoaib, M. Arshad, M. A. Kutbi, Common fixed points of a pair of Hardy Rogers type mappings on a closed ball in ordered partial metric spaces, J. Comput. Anal. Appl. **17** (2014), 255–264.

[32] D. Wardowski, Fixed points of new type of contractive mappings in complete metric spaces, Fixed Point Theory Appl. **2012**, 2012:94.

# $\alpha$-$\psi$-GERAGHTY CONTRACTIONS IN GENERALIZED METRIC SPACES VIA NEW FUNCTIONS

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ABSTRACT. In this paper, we introduce the class of $\alpha$-$h$-$F$-$\psi$-Greaghty contractions where the pair $(F, h)$ is up class of type $I$ and establish several fixed point theorems for this newly introduced class. Our results extend some recent results of Asadi et al. [1] as well as other corresponding results.

## 1. INTRODUCTION

Banach contraction principle is one of the most pivotal results of fixed point theory. Since this principle plays a very crucial role in solving various kinds of nonlinear equations, it has been extended in several possible ways. Geraghty [3] generalized the Banach contraction mapping principle by introducing the class of auxiliary function in the following way:

Let $F^{\#}$ denote all functions $\beta : [0, \infty) \to [0, 1)$ which satisfy the condition:

$$\lim_{n\to\infty} \beta(t_n) = 1 \text{ implies } \lim_{n\to\infty} t_n = 0.$$

By using the function $\beta \in F^{\#}$, Geraghty [3] proved the following remarkable theorem.

**Theorem 1.1.** [3] *Let $(X, d)$ be a complete metric space and $T : X \to X$ be an operator. Suppose that there exists $\beta : [0, \infty) \to [0, 1)$ satisfying the condition*

$$\beta(t_n) \to 1 \text{ implies } t_n \to 0$$

*If $T$ satisfies the following inequality*

$$d(Tx, Ty) \leq \beta(d(x, y))d(x, y) \tag{1.1}$$

*for all $x, y \in X$, then $T$ has a unique fixed point*

Recently, Samet et al. [8] introduced the class of $\alpha$-$\psi$ contractive mappings and established several fixed point theorems for such mappings in the set up of complete metric space which generalized and extend the Banach contraction principle as well as many other well known fixed point theorems existing in the literature. Branciari [2] replaces the triangle inequality in the metric space with the quadrilateral inequality and introduced a new space called generalized metric space or rectangular metric space. He extended the Banach contraction principle to this newly defined space. Very recently, Asadi et al. [1] utilized the concept of Geraghty [3] and Samet et al. [8] and introduced the notion of $\alpha$-$\psi$-Greaghty contractions in the context of generalized metric space and extended several well known contractions existing in the literature. For more information, see [9, 10].

In the present work, we extend the notion $\alpha$-$\psi$-Geraghty contractions announced by Asadi et al. [1] by introducing the class of $\alpha$-$h$-$F$-$\psi$-Geraghty contractions and investigate the existence and uniqueness of fixed points for this newly introduced class in the setting of generalized metric space.

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Before going to the main work, we recall some useful definitions and auxiliary results that will be needed in the sequel. Throughout this paper, $\mathbb{N}$ and $\mathbb{R}$ denote the set of natural numbers and the set of real numbers, respectively.

**Definition 1.2.** [2] Let $X$ be a nonempty set and let $d : X \times X \to [0, \infty]$ satisfy the following conditions for all $x, y \in X$ and all distinct $u, v \in X$ each of which is different from $x$ and $y$:

$$\begin{array}{ll} (1) & d(x, y) = 0 \text{ if and only if } x = y \\ (2) & d(x, y) = d(y, x) \\ (3) & d(x, y) \leq d(x, u) + d(u, v) + d(v, y) \end{array} \tag{1.2}$$

Then the pair $(X, d)$ is called a generalized metric space and abbreviated as GMS. Every metric space is GMS but the converse need not be true (see [1, Example 39]).

Given a generalized metric $d$ on $X$ and $\epsilon > 0$, we call $B_d(x, \epsilon) = \{y \in X | d(x, y) < \epsilon\}$ as $\epsilon$-ball centered at $x$. It is to be noted that a GMS becomes a topological space, when a subset $U$ of $X$ is said to be open if to each $a \in U$, there exists a positive number $\epsilon_a$ such that $B_d(a, \epsilon_a) \subseteq U$. The concepts of convergence, Cauchy sequence, completeness and continuity on a GMS are defined below.

**Definition 1.3.** Let $(X, d)$ be a generalized metric space.

(1) A sequence $\{x_n\}$ in $(X, d)$ is GMS convergent to a limit $x$ if and only if $d(x_n, x) \to 0$ as $n \to \infty$.
(2) A sequence $\{x_n\}$ in $(X, d)$ is GMS Cauchy if and only if for every $\epsilon > 0$ there exists a positive integer $N(\epsilon)$ such that $d(x_n, x_m) < \epsilon$ for all $n > m > N(\epsilon)$.
(3) A space $(X, d)$ is called complete if every GMS Cauchy sequence in $X$ is GMS convergent.
(4) A mapping $T : (X, d) \to (X, d)$ is continuous if for any sequence $\{x_n\}$ in $X$ such that $d(x_n, x) \to 0$ as $n \to \infty$, we have $d(Tx_n, Tx) \to 0$ as $n \to \infty$.

It is to be noted that any generalized metric space need not be complete, neither the respective topology needs to be Hausdorff and a convergent sequence in GMS need not be Cauchy.

**Lemma 1.4.** [4] *Let $(X, d)$ be a generalized metric space and let $\{x_n\}$ be a Cauchy sequence in $X$ such that $x_m \neq x_n$ whenever $m \neq n$. Then $\{x_n\}$ can converge to at most one point.*

**Lemma 1.5.** [4] *Let $(X, d)$ be a generalized metric space and let $\{x_n\}$ be a sequence in $X$ with distinct elements ($x_n \neq x_m$ for $n \neq m$). Suppose that $d(x_n, x_{n+1})$ and $d(x_n, x_{n+2})$ tend to 0 as $n \to \infty$ and that $\{x_n\}$ is not a Cauchy sequence. Then there exist $\epsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $n_k > m_k > k$ and the following four sequences*

$$d(x_{m_k}, x_{n_k}), \quad d(x_{m_k}, x_{n_{k+1}}), \quad d(x_{m_{k-1}}, x_{n_k}), \quad d(x_{m_{k-1}}, x_{n_{k+1}}) \tag{1.3}$$

*tend to $\epsilon$ as $k \to \infty$.*

**Proposition 1.6.** [7] *Suppose that $\{x_n\}$ is a Cauchy sequence in a GMS $(X, d)$ with $\lim_{n\to\infty} d(x_n, u) = 0$, where $u \in X$. Then $\lim_{n\to\infty} d(x_n, z) = d(u, z)$ for all $z \in X$.*

Samet et al. [8] introduced the notion of $\alpha$-admissible mappings as follows.

**Definition 1.7.** Let $X$ be a nonempty set, and let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be mappings. Then $T$ is called $\alpha$-admissible if for all $x, y \in X$, we have

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1. \tag{1.4}$$

Recently, Karapinar et al. [5] defined the notion of triangular $\alpha$-admissible mappings as follows.

**Definition 1.8.** Let $X$ be a nonempty set, and let $T : X \to X$ and $\alpha : X \times X \to \mathbb{R}$ be mappings. Then $T$ is called triangular $\alpha$-admissible if

(1) $x, y \in X$, $\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1$;
(2) $x, y, z \in X$, $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1 \Rightarrow \alpha(x, y) \geq 1$.

**Lemma 1.9.** [5] *Let $T : X \to X$ be a triangular $\alpha$-admissible map. Assume that there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$. Then we have $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N}$ with $n < m$.*

Now we recall the following class of auxiliary functions (or altering distance functions) (see [6]) which will be used densely in the sequel: Let $\Psi$ denote the class of the functions $\psi : [0, \infty) \to [0, \infty)$ which satisfy the following conditions:

($a$) $\psi$ is nondecreasing;
($b$) $\psi$ is continuous;
($c$) $\psi(t) = 0 \Leftrightarrow t = 0$.

## 2. Main results

**Definition 2.1.** A function $h : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is of subclass of type $I$ if it is continuous and satisfies the following:
for $y, z \in \mathbb{R}^+, y \geq 1 \Longrightarrow h(1, z) \leq h(y, z)$.

**Example 2.2.** The following functions $h : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ are of subclass of type $I$. For $y, z \in \mathbb{R}^+$,
(1) $h(y, z) = (z + l)^y, l > 1$;
(2) $h(y, z) = (y + l)^z, l > 1$;
(3) $h(y, z) = yz$;
(4) $h(y, z) = (\frac{1+y}{2})z$;
(5) $h(y, z) = y^k z$;
(6) $h(y, z) = z$;
(7) $h(y, z) = \frac{1+2y}{3}z$;
(8) $h(y, z) = (\frac{\sum_{i=0}^{n} y^i}{n+1})z$;
(9) $h(y, z) = (\frac{\sum_{i=0}^{n} y^i}{n+1} + l)^z, l > 1$.

**Definition 2.3.** Let $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be a function. We say that the pair $(F, h)$ is a upclass of type $I$ if $F$ is continuous, $h$ is a function of subclass of type $I$ and the following hold:
(1) for $0 \leq s \leq 1$, $t \in \mathbb{R}^+ \Longrightarrow F(s, t) \leq F(1, t)$;
(2) for $z, s, t \in \mathbb{R}^+$ if $h(1, z) \leq F(s, t) \Longrightarrow z \leq st$.

**Example 2.4.** The following functions $h : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ are of upclass of type $I$. For $x, y, z, s, t \in \mathbb{R}^+$,
(1) $h(y, z) = (z + l)^y, l > 1, F(s, t) = st + l$;
(2) $h(y, z) = (xy + l)^z, l > 1, F(s, t) = (1 + l)^{st}$;
(3) $h(y, z) = yz, F(s, t) = st$;
(4) $h(y, z) = (\frac{1+y}{2})z, F(s, t) = st$;
(5) $h(y, z) = y^k z, F(s, t) = t$;
(6) $h(y, z) = \frac{1+2y}{3}z, F(s, t)) = st$;
(7) $h(y, z) = (\frac{1+y}{2})z, F(s, t) = st$;
(8) $h(y, z) = (\frac{\sum_{i=0}^{n} y^i}{n+1})z, F(s, t) = st$;
(9) $h(y, z) = (\frac{\sum_{i=0}^{n} y^i}{n+1} + l)^z, l > 1, F(s, t) = (1 + l)^{st}$.

We start to this section with the following definition.

**Definition 2.5.** Let $(X,d)$ be a generalized metric space and let $\alpha : X \times X \to \mathbb{R}$ be a function. A map $T : X \to X$ is called an $\alpha$-$h$-$F$-$\psi$-Geraghty contraction type map if there exists $\beta \in F^{\#}$ such that for all $x, y \in X$,

$$h(\alpha(x,y), \psi(d(Tx,Ty))) \leq F(\beta(\psi d(x,y)), \psi(d(x,y))), \tag{2.1}$$

where $\psi \in \Psi$ and the pair $(F,h)$ is a upclass of type $I$.

**Theorem 2.6.** *Let $(X,d)$ be a complete generalized metric space, $\alpha : X \times X \to \mathbb{R}$ a function, and let $T : X \to X$ be a map. Suppose that the following conditions are satisfied:*

(1) *$T$ is an $\alpha$-$h$-$F$-$\psi$-Geraghty contraction type map;*
(2) *$T$ is triangular $\alpha$-admissible;*
(3) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$;*
(4) *$T$ is continuous.*

*Then $T$ has a fixed point $x^* \in X$ and $\{T^n x_0\}$ converges to $x^*$.*

*Proof.* By (3), from $x_0 \in X$, construct the sequence $\{x_n\}$ as $x_{n+1} = Tx_n$, $n \in \mathbb{N}$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N} \cup \{0\}$, then $x^* = x_n$ is a fixed point of $T$. Assume further that $x_n \neq x_{n+1}$ for each $n \in \mathbb{N} \cup \{0\}$. Since $T$ is triangular $\alpha$-admissible, it follows from (3) that

$$\alpha(x_1, x_2) = \alpha(x_1, Tx_1) \geq 1 \quad \text{and} \quad \alpha(x_1, x_3) = \alpha(x_1, T^2x_1) \geq 1.$$

By Lemma 1.9, we have

$$\alpha(x_n, x_{n+1}) \geq 1 \quad \text{and} \quad \alpha(x_n, x_{n+2}) \geq 1. \tag{2.2}$$

for all $n \in \mathbb{N}$. Notice that we also find $\alpha(x_n, x_{n+m}) \geq 1$ for each $m, n \in \mathbb{N}$.

Now we shall prove that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. By taking $x = x_{n-1}$ and $y = x_n$ in (2.1) and regarding (2.2), we get that

$$\begin{aligned} h(1, \psi(d(x_n, x_{n+1}))) &\leq h(\alpha(x_{n-1}, x_n), \psi(d(Tx_{n-1}, Tx_n))) \\ &\leq F(\beta(\psi(d(x_{n-1}, x_n))), \psi(d(x_{n-1}, x_n))) \Longrightarrow \\ \psi(d(x_n, x_{n+1})) &\leq \beta(\psi(d(x_{n-1}, x_n))), \psi(d(x_{n-1}, x_n)) < \psi(d(x_{n-1}, x_n)) \end{aligned} \tag{2.3}$$

for each $n \in \mathbb{N}$.

Since $\psi$ is nondecreasing, we conclude from (2.3) that

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$$

for each $n \in \mathbb{N}$. Thus we conclude that the sequence $\{d(x_n, x_{n+1})\}$ is nonnegative and nonincreasing. As a result, there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$. We claim that $r = 0$. Suppose, on the contrary, that $r > 0$. Then, on account of (2.3), we get that

$$\frac{\psi(d(x_n, x_{n+1}))}{\psi(d(x_{n-1}, x_n))} \leq \beta(\psi(d(x_{n-1}, x_n))) < 1,$$

which yields that $\lim_{n\to\infty} \beta(\psi(d(x_n, x_{n+1}))) = 1$. We obtain

$$\lim_{n\to\infty} \psi(d(x_n, x_{n+1})) = 0, \tag{2.4}$$

due to the fact that $\beta \in F^{\#}$. On the other hand, the continuity of $\psi$ together with (2.4) yield that

$$r = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.5}$$

Analogously, we shall prove that $\lim_{n\to\infty} d(x_n, x_{n+2}) = 0$. By substituting $x = x_{n-1}$ and $y = x_{n+1}$ in (2.1) and taking (2.2) into account, we find that

$$\begin{array}{rcl} h(1, \psi(d(x_n, x_{n+2}))) & \leq & h(\alpha(x_{n-1}, x_{n+1}), \psi(d(Tx_{n-1}, Tx_{n+1})) \\ & \leq & F(\beta(\psi(d(x_{n-1}, x_{n+1}))), \psi(d(x_{n-1}, x_{n+1}))) \Longrightarrow \\ \psi(d(x_n, x_{n+2})) & \leq & \beta(\psi(d(x_{n-1}, x_{n+1})))\psi(d(x_{n-1}, x_{n+1})) < \psi(d(x_{n-1}, x_{n+1})) \end{array} \tag{2.6}$$

for each $n \in \mathbb{N}$. Since $\psi$ is nondecreasing, we conclude from (2.6) that

$$d(x_n, x_{n+2}) < d(x_{n-1}, x_{n+1})$$

for each $n \in \mathbb{N}$. Thus we observe that the sequence $\{d(x_{n-1}, x_{n+1})\}$ is nonnegative and nonincreasing. Consequently, there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_{n-1}, x_{n+1}) = r$. We assert that $r = 0$. Suppose, on the contrary, that $r > 0$. Then, by regarding (2.6), we get that

$$\frac{\psi(d(x_n, x_{n+2}))}{\psi(d(x_{n-1}, x_{n+1}))} \leq \beta(\psi(d(x_{n-1}, x_{n+1}))) < 1,$$

which implies that $\lim_{n\to\infty} \beta(\psi(d(x_{n-1}, x_{n+1}))) = 1$. We obtain

$$\lim_{n\to\infty} \psi(d(x_{n-1}, x_{n+1})) = 0, \tag{2.7}$$

due to the fact that $\beta \in F^{\#}$. On the other hand, the continuity of $\psi$ together with (2.7) yield that

$$r = \lim_{n\to\infty} d(x_{n-1}, x_{n+1}) = 0 = \lim_{n\to\infty} d(x_n, x_{n+2}). \tag{2.8}$$

Suppose that $x_n = x_m$ for some $m, n \in \mathbb{N}$, $m < n$. Then

$$\begin{array}{rcl} \psi(d(x_m, x_{m+1})) & = & \psi(d(x_n, x_{n+1})) \\ & \leq & \beta(\psi(d(x_{n-1}, x_n)))\psi(d(x_{n-1}, x_n)) \\ & < & \psi(d(x_{n-1}, x_n)) \\ & \leq & \psi^{n-m}(d(x_m, x_{m+1})) \\ & < & \psi(d(x_m, x_{m+1})), \end{array}$$

a contradiction. Hence all elements of the sequence $\{x_n\}$ are distinct.

We are ready to prove that $\{x_n\}$ is a Cauchy sequence in $(X, d)$. Suppose, on the contrary, that we have

$$\varepsilon = \limsup_{m,n\to\infty} d(x_n, x_m) > 0. \tag{2.9}$$

Regarding the quadrilateral inequality, we need to examine two possible cases:

**Case 1.** Suppose $k = n - m$ is odd, where $k \geq 1$. Then we have

$$\begin{array}{rl} d(x_n, x_m) & \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{m+1}) + d(x_{m+1}, x_m) \\ & = d(x_n, x_{n+1}) + d(Tx_n, Tx_m) + d(x_{m+1}, x_m), \end{array} \tag{2.10}$$

which is equivalent to

$$d(x_n, x_m) - d(x_n, x_{n+1}) - d(x_{m+1}, x_m) \leq d(Tx_n, Tx_m) \tag{2.11}$$

Since $T$ is triangular $\alpha$-admissible, by applying $\psi$, we get that

$$\psi(d(x_n, x_m) - d(x_n, x_{n+1}) - d(x_{m+1}, x_m)) \leq \psi(d(Tx_n, Tx_m))$$

and

$$\begin{array}{rl} h(1, \psi(d(Tx_n, Tx_m))) & \leq h(\alpha(x_n, x_m), \psi(d(Tx_n, Tx_m))) \\ & \leq F(\beta(\psi\left(d(x_n, x_m)\right)), \psi\left(d(x_n, x_m)\right)) \Longrightarrow \end{array}$$

$$\psi(d(Tx_n, Tx_m))) \leq \beta(\psi\left(d(x_n, x_m)\right))\,, \psi\left(d(x_n, x_m)\right). \tag{2.12}$$

Letting $m, n \to \infty$, we deduce that

$$\lim_{m,n\to\infty} \psi(d(x_n, x_m) - d(x_n, x_{n+1}) - d(x_{m+1}, x_m)) \leq \lim_{m,n\to\infty} \beta(\psi\left(d(x_n, x_m)\right)) \lim_{m,n\to\infty} \psi\left(d(x_m, x_n)\right).$$

So, by using (2.5), (2.9) and the continuity of $\psi$, we get

$$1 \leq \lim_{m,n\to\infty} \beta(\psi\left(d(x_n, x_m)\right)),$$

which implies that $\lim_{m,n\to\infty} \beta(\psi\left(d(x_n, x_m)\right)) = 1$. Consequently, we get $\lim_{m,n\to\infty} d(x_n, x_m) = 0$, which is a contradiction.

**Case 2.** Suppose $k = n - m$ is even, where $k \geq 1$. So we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+2}) + d(x_{n+2}, x_{m+2}) + d(x_{m+2}, x_m) \\ &= d(x_n, x_{n+2}) + d(Tx_{n+1}, Tx_{m+1}) + d(x_{m+2}, x_m), \end{aligned} \tag{2.13}$$

that can be written as

$$d(x_n, x_m) - d(x_n, x_{n+2}) - d(x_{m+2}, x_m) \leq d(Tx_{n+1}, Tx_{m+1}). \tag{2.14}$$

Due to the fact that $T$ is triangular $\alpha$-admissible, by applying $\psi$, we obtain that

$$\psi(d(x_n, x_m) - d(x_n, x_{n+2}) - d(x_{m+2}, x_m)) \leq \psi(d(Tx_{n+1}, Tx_{m+1}))$$

and

$$\begin{aligned} h(1, \psi(d(Tx_{n+1}, Tx_{m+1}))) &\leq h(\alpha(x_{n+1}, x_{m+1}), \psi(d(Tx_{n+1}, Tx_{m+1}))) \\ &\leq F(\beta(\psi\left(d(x_{n+1}, x_{m+1})\right))\,, \psi\left(d(x_{n+1}, x_{m+1})\right)) \Longrightarrow \\ \psi(d(Tx_{n+1}, Tx_{m+1})) &\leq \beta(\psi\left(d(x_{n+1}, x_{m+1})\right))\, \psi\left(d(x_{n+1}, x_{m+1})\right). \end{aligned} \tag{2.15}$$

Letting $m, n \to \infty$, we find that

$$\lim_{m,n\to\infty} \psi(d(x_n, x_m) - d(x_n, x_{n+2}) - d(x_{m+2}, x_m)) \leq \lim_{m,n\to\infty} \beta(\psi\left(d(x_{n+1}, x_{m+1})\right)) \lim_{m,n\to\infty} \psi\left(d(x_{n+1}, x_{m+1})\right)$$

So, by using (2.8), (2.9) and the continuity of $\psi$, we observe

$$1 \leq \lim_{m,n\to\infty} \beta(\psi\left(d(x_{n+1}, x_{m+1})\right)),$$

which implies that $\lim_{m,n\to\infty} \beta(\psi\left(d(x_{n+1}, x_{m+1})\right)) = 1$. Thus we conclude that $\lim_{m,n\to\infty} d(x_{n+1}, x_{m+1}) = 0$, which is a contradiction.

From Case 1 and Case 2, we conclude that $\{x_n\}$ is a Cauchy sequence. Since $(X, d)$ is a complete generalized metric space, there exists $x^* \in X$ such that $\lim_{n\to\infty} d(x_n, x^*) = 0$. Since $T$ is continuous, we have

$$\lim_{n\to\infty} d(Tx_n, x^*) = \lim_{n\to\infty} d(x_{n+1}, Tx^*) = 0.$$

By Lemma 1.4, we have that $Tx^* = x^*$. □

It is also possible to remove the continuity of the mapping $T$ by replacing it with a weaker condition:

**Definition 2.7.** Let $(X, d)$ be a complete generalized metric space, $\alpha : X \times X \to \mathbb{R}$ a function and let $T : X \to X$ be a map. We say that the sequence $\{x_n\}$ is $\alpha$-regular if the following condition is satisfied:
If $\{x_n\}$ is a sequence in $X$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n$ and $x_n \to x \in X$ as $n \to +\infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all $k$.

**Theorem 2.8.** *Let $(X,d)$ be a complete generalized metric space, $\alpha : X \times X \to \mathbb{R}$ a function and let $T : X \to X$ be a map. Suppose that the following conditions are satisfied:*

(1) *$T$ is an $\alpha$-$h$-$F$-$\psi$-Geraghty contraction type map;*
(2) *$T$ is triangular $\alpha$-admissible;*
(3) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$;*
(4) *either $T$ is continuous or $\{x_n\}$ is $\alpha$-regular.*

*Then $T$ has a fixed point $x^* \in X$ and $\{T^n x_0\}$ converges to $x^*$.*

*Proof.* Following the proof of Theorem 2.6, we know that the sequence $\{x_n\}$, defined by $x_{n+1} = Tx_n$ for all $n \geq 0$, converges to some $x^* \in X$. From (2.2) and the assumption (4) of the theorem, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x^*) \geq 1$. Applying (2.1), for all $k$, we get that

$$\begin{aligned} h(1, \psi(d(x_{n(k)+1}, Tx^*))) &\leq h(\alpha(x_{n(k)}, x^*), \psi(d(Tx_{n(k)}, Tx^*)) \\ &\leq F(\beta(\psi(d(x_{n(k)}, x^*))), \psi(d(x_{n(k)}, x^*))) \\ &\Longrightarrow \end{aligned}$$

$$\psi(d(x_{n(k)+1}, Tx^*)) \leq \beta(\psi(d(x_{n(k)}, x^*)))\psi(d(x_{n(k)}, x^*)) < \psi(d(x_{n(k)}, x^*)). \tag{2.16}$$

Letting $k \to \infty$ in (2.16), we have

$$\lim_{k\to\infty} \psi(d(x_{n(k)+1}, Tx^*)) \leq 0.$$

Therefore, in view of Proposition 1.6, we obtain $x^* = Tx^*$. □

Now we introduce the notion of generalized $\alpha$-$h$-$F$-$\psi$-Geraghty contraction.

**Definition 2.9.** Let $(X,d)$ be a generalized metric space and let $\alpha : X \times X \to \mathbb{R}$ be a function. A map $T : X \to X$ is called an $\alpha$-$h$-$F$-$\psi$-Geraghty contraction type map if there exists $\beta \in F^{\#}$ such that for all $x, y \in X$,

$$h(\alpha(x,y), \psi(d(Tx,Ty))) \leq F(\beta(\psi(M(x,y)), \psi(M(x,y))), \tag{2.17}$$

where $M(x,y) = max\{d(x,y), d(x,Tx), d(y,Ty)\}$, $\psi \in \Psi$ and the pair $(F,h)$ is a upclass of type $I$.

**Theorem 2.10.** *Let $(X,d)$ be a complete generalized metric space, $\alpha : X \times X \to \mathbb{R}$ a function and let $T : X \to X$ be a map. Suppose that the following conditions are satisfied:*

(1) *$T$ is a generalized $\alpha$-$h$-$F$-$\psi$-Geraghty contraction type map;*
(2) *$T$ is triangular $\alpha$-admissible;*
(3) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$;*
(4) *$T$ is continuous.*

*Then $T$ has a fixed point $x^* \in X$ and $\{T^n x_0\}$ converges to $x^*$.*

*Proof.* By (3), from $x_0 \in X$, construct the sequence $\{x_n\}$ as $x_{n+1} = Tx_n$, $n \in \mathbb{N}$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N} \cup \{0\}$, then $x^* = x_n$ is a fixed point of $T$. Assume further that $x_n \neq x_{n+1}$ for each $n \in \mathbb{N} \cup \{0\}$. Since $T$ is triangular $\alpha$-admissible, it follows from (3) that

$$\alpha(x_1, x_2) = \alpha(x_1, Tx_1) \geq 1 \quad \text{and} \quad \alpha(x_1, x_3) = \alpha(x_1, T^2x_1) \geq 1$$

and so by induction, we get

$$\alpha(x_n, x_{n+1}) \geq 1 \quad \text{and} \quad \alpha(x_n, x_{n+2}) \geq 1 \tag{2.18}$$

for $n \in \mathbb{N}$. And we also find $\alpha(x_n, x_{n+m}) \geq 1$ for each $m, n \in \mathbb{N}$.
Therefore, by (2.17)

$$\begin{aligned} h(1, \psi(d(x_n, x_{n+1}))) &\leq h(\alpha(x_{n-1}, x_n), \psi(d(Tx_{n-1}, Tx_n))) \\ &\leq F(\beta(\psi(M(x_{n-1}, x_n))), \psi(M(x_{n-1}, x_n))) \Longrightarrow \\ \psi(d(x_n, x_{n+1})) &\leq \beta(\psi(M(x_{n-1}, x_n)))\psi(M(x_{n-1}, x_n)) < \psi(M(x_{n-1}, x_n)) \end{aligned} \tag{2.19}$$

for each $n \geq 1$, where

$$M(x_{n-1}, x_n) = \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}. \tag{2.20}$$

If $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n+1})$, then by (2.19), we get

$$\psi(d(x_n, x_{n+1})) < \psi(d(x_n, x_{n+1})),$$

which is a contradiction. Hence $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_{n-1}, x_n)$ and (2.19) gives

$$\psi(d(x_n, x_{n+1})) < \psi(d(x_{n-1}, x_n)), \text{ for all } n \in \mathbb{N}. \tag{2.21}$$

This yields that, for each $n \in \mathbb{N}$,

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n). \tag{2.22}$$

Thus we conclude that the sequence $\{d(x_n, x_{n+1})\}$ is nonnegative and nonincreasing. As a result, there exists $t \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = t$. We claim that $t = 0$. Suppose, on the contrary, that $t > 0$. Then, on account of (2.19), we get that

$$\frac{\psi(d(x_n, x_{n+1}))}{\psi(M(x_{n-1}, x_n))} \leq \beta(\psi(M(x_{n-1}, x_n))) < 1,$$

which yields that $\lim_{n\to\infty} \beta(\psi(d(x_n, x_{n+1}))) = 1$. We obtain

$$\lim_{n\to\infty} \psi(d(x_n, x_{n+1})) = 0, \tag{2.23}$$

due to the fact that $\beta \in F^{\#}$. On the other hand, the continuity of $\psi$ together with (2.23) yield that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.24}$$

Now we shall show

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0. \tag{2.25}$$

Regarding (2.17) and (2.18), we find that

$$\begin{aligned} h(1, \psi(d(x_n, x_{n+2}))) &\leq h(\alpha(x_{n-1}, x_{n+1})\psi(d(Tx_{n-1}, Tx_{n+1})) \\ &\leq F(\beta(\psi(M(x_{n-1}, x_{n+1}))), \psi(M(x_{n-1}, x_{n+1}))) \Longrightarrow \end{aligned}$$

$$\psi(d(x_n, x_{n+2}))) \leq \beta(\psi(M(x_{n-1}, x_{n+1})))\psi(M(x_{n-1}, x_{n+1}))) < \psi(M(x_{n-1}, x_{n+1})) \tag{2.26}$$

for all $n \in \mathbb{N}$, where

$$\begin{aligned} M(x_{n-1}, x_{n+1}) &= \max\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, Tx_{n-1}), d(x_{n+1}, Tx_{n+1})\} \\ &= \max\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n), d(x_{n+1}, x_{n+2})\}. \end{aligned} \tag{2.27}$$

In view of (2.22), we obtain

$$M(x_{n-1}, x_{n+1}) = \max\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n)\}.$$

Define $a_n = d(x_n, x_{n+2})$ and $b_n = d(x_n, x_{n+1})$. Then, taking (2.26) into account, we get

$$\psi(a_n) < \psi(\max\{a_{n-1}, b_{n-1}\}).$$

This yields that, for each $n \in \mathbb{N}$,

$$a_n < \max\{a_{n-1}, b_{n-1}\}. \tag{2.28}$$

By (2.22), we have

$$b_n < \max\{a_{n-1}, b_{n-1}\}. \tag{2.29}$$

Therefore,

$$\max\{a_n, b_n\} < \max\{a_{n-1}, b_{n-1}\}$$

for all $n \in \mathbb{N}$. Thus the sequence $\max\{a_n, b_n\}$ is nonegative and nonincreasing and so it converges to some $r \geq 0$. Clearly, by (2.24),

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \max\{a_n, b_n\} = r.$$

Now we will show that $r = 0$. If $r > 0$, then in view of (2.26), we have

$$\frac{\psi(d(x_n, x_{n+2}))}{\psi(M(x_{n-1}, x_{n+1}))} \leq \beta(\psi(M(x_{n-1}, x_{n+1}))) < 1,$$

which yields that $\lim_{n\to\infty} \beta(\psi(M(x_{n-1}, x_{n+1}))) = 1$. We obtain

$$\lim_{n\to\infty} \psi(M(x_{n-1}, x_{n+1})) = 0, \tag{2.30}$$

due to the fact that $\beta \in F^{\#}$. On the other hand, the continuity of $\psi$ together with (2.30) yield that

$$\psi(r) = \psi(\lim_{n\to\infty} \max\{a_{n-1}, b_{n-1}\}) = \lim_{n\to\infty} \psi(\max\{a_{n-1}, b_{n-1}\}) = 0,$$

which is a contradiction and hence $r = 0$.

Suppose that $x_n = x_m$ for some $m, n \in \mathbb{N}$, $m < n$. Then

$$\begin{aligned} \psi(d(x_m, x_{m+1})) &= \psi(d(x_n, x_{n+1})) \\ &\leq \beta(\psi(M(x_{n-1}, x_n)))\psi(M(x_{n-1}, x_n)) \\ &< \psi(d(x_{n-1}, x_n)) \\ &\leq \psi^{n-m}(d(x_m, x_{m+1})) \\ &< \psi(d(x_m, x_{m+1})), \end{aligned}$$

a contradiction. Hence all elements of the sequence $\{x_n\}$ are distinct.

In order to prove that $\{x_n\}$ is a Cauchy sequence in $(X, d)$, suppose that it is not. Then by Lemma 1.5, using (2.24) and (2.25), we assert that there exist $\epsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $n_k > m_k > k$ and the sequences (1.3) tend to $\epsilon$ as $k \to \infty$. By substituting $x = x_{m_k}$ and $y = x_{n_{k+1}}$ in (2.17) and taking (2.18) into account, we obtain

$$\begin{aligned} h(1, \psi(d(x_{m_k}, x_{n_{k+1}}))) &\leq h(\alpha(x_{m_{k-1}}, x_{n_k}), \psi(d(Tx_{m_{k-1}}, Tx_{n_k}))) \\ &\leq F(\beta(\psi(M(x_{m_{k-1}}, x_{n_k}))), \psi(M(x_{m_{k-1}}, x_{n_k}))). \end{aligned} \tag{2.31}$$

On the other hand, we have

$$\begin{aligned} M(x_{m_{k-1}}, x_{n_k}) &= \max\{d(x_{m_{k-1}}, x_{n_k}), d(x_{m_{k-1}}, Tx_{m_{k-1}}), d(x_{n_k}, Tx_{n_k})\} \\ &= \max\{d(x_{m_{k-1}}, x_{n_k}), d(x_{m_{k-1}}, x_{m_k}), d(x_{n_k}, x_{n_{k+1}})\} \end{aligned} \tag{2.32}$$

and hence

$$\lim_{k\to\infty} \psi(M(x_{m_{k-1}}, x_{n_k})) = \psi(\epsilon). \tag{2.33}$$

From (2.31), we have

$$\frac{\psi(d(x_{m_k}, x_{n_{k+1}}))}{\psi(M(x_{m_{k-1}}, x_{n_k}))} \leq \beta(\psi(M(x_{m_{k-1}}, x_{n_k}))) < 1.$$

Letting $k \to \infty$, we get

$$\lim_{k\to\infty} \beta(\psi(M(x_{m_{k-1}}, x_{n_k}))) = 1.$$

Thus $\lim_{k\to\infty} \psi(M(x_{m_{k-1}}, x_{n_k})) = 0$ and so (2.33) gives $\psi(\epsilon) = 0$, which is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence. Since $(X, d)$ is a complete generalized metric space, there exists $x^* \in X$ such that $\lim_{n\to\infty} d(x_n, x^*) = 0$. Since $T$ is continuous, we have

$$\lim_{n\to\infty} d(Tx_n, x^*) = \lim_{n\to\infty} d(x_{n+1}, Tx^*) = 0.$$

By Lemma 1.4, we get that $Tx^* = x^*$. □

**Theorem 2.11.** *Let $(X, d)$ be a complete generalized metric space, $\alpha : X \times X \to \mathbb{R}$ a function and let $T : X \to X$ be a map. Suppose that the following conditions are satisfied:*

(1) *$T$ is a generalized $\alpha$-h-F-$\psi$-Geraghty contraction type map;*
(2) *$T$ is triangular $\alpha$-admissible;*
(3) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$;*
(4) *$\{x_n\}$ is $\alpha$-regular.*

*Then $T$ has a fixed point $x^* \in X$ and $\{T^n x_0\}$ converges to $x^*$.*

*Proof.* Following the proof of Theorem 2.10, we know that the sequence $\{x_n\}$, defined by $x_{n+1} = Tx_n$ for all $n \geq 0$, converges to some $x^* \in X$. Now we shall show that $Tx^* = x^*$. Suppose, on the contrary, that $Tx^* \neq x^*$, i.e,, $d(x^*, Tx^*) > 0$. Since $x_n$ is $\alpha$-regular, from (2.18), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x^*) \geq 1$. Applying (2.17), for all $k$, we get that

$$\begin{aligned} h(1, \psi(d(x_{n(k)+1}, Tx^*))) &\leq h(\alpha(x_{n(k)}, x^*), \psi(d(Tx_{n(k)}, Tx^*))) \\ &\leq F(\beta(\psi(M(x_{n(k)}, x^*))), \psi(M(x_{n(k)}, x^*))), \end{aligned} \tag{2.34}$$

where $M(x_{n(k)}, x^*) = \max\{d(x_{n(k)}, x^*), d(x_{n(k)}, Tx_{n(k)}), d(x^*, Tx^*)\}$.

Letting $k \to \infty$ in (2.34), we have

$$\lim_{n\to\infty} \psi(d(x_{n(k)+1}, Tx^*)) < \psi(d(x^*, Tx^*)).$$

In view of Proposition 1.6, we get a contradiction and hence $x^* = Tx^*$. □

For the uniqueness of a fixed point of $\alpha$-$\psi$-Geraghty contractive mapping, we will consider the following condition.

Condition $(U)$: For all $x, y \in F(T)$, we have $\alpha(x, y) \geq 1$, where $F(T)$ denotes the set of fixed points of $T$.

**Theorem 2.12.** *If the condition $(U)$ is added to the hypothesis of Theorem 2.6 (respectively, Theorem 2.8), then we obtain that $u$ is the unique fixed point of $T$.*

*Proof.* We will show that $u$ is a unique fixed point of $T$. Let $v$ be another fixed point of $T$ with $v \neq u$. By hypothesis $(U)$,

$$1 \leq \alpha(u, v) = \alpha(Tu, Tv).$$

Now, using (2.5), we have

$$\begin{aligned} h(1, \psi(d(u, v))) &\leq h(\alpha(u, v), \psi(d(Tu, Tv))) \\ &\leq F(\beta(\psi(d(u, v))), \psi(d(u, v))) \Longrightarrow \end{aligned}$$

$$\psi(d(u, v)) \leq \beta(\psi(d(u, v)))\psi(d(u, v)) < \psi(d(u, v)),$$

which is a contradiction. Hence $u = v$. □

**Theorem 2.13.** *If the condition $(U)$ is added to the hypothesis of Theorem 2.10 (respectively, Theorem 2.11), then we obtain that $u$ is the unique fixed point of $T$.*

*Proof.* Let $v$ be another fixed point of $T$ with $v \neq u$. Then by the assumption $(U)$,

$$1 \leq \alpha(u,v) = \alpha(Tu,Tv).$$

Now, using (2.17), we have

$$\begin{aligned} h(1,\psi(d(u,v))) &\leq h(\alpha(u,v),\psi(d(Tu,Tv))) \\ &\leq F(\beta(\psi(M(u,v)))\psi(M(u,v))), \end{aligned}$$

$$\psi(d(u,v))) \leq \beta(\psi(M(u,v)))\psi(M(u,v)) < \psi(M(u,v)),$$

where

$$M(u,v) = \max\{d(u,v), d(u,Tu), d(v,Tv)\}.$$

Therefore,

$$\psi(d(u,v)) < \psi(d(u,v)),$$

which is a contradiction. Hence $u = v$. □

Now we give a useful example.

**Example 2.14.** Let $X = A \cup B$, where $A = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\}$ and $B = [\frac{3}{4}, 1]$. Define the function $d : X \times X \to \mathbb{R}$ as follows:

$$\begin{aligned} d\left(\frac{1}{2},\frac{1}{3}\right) &= \frac{1}{2}, \\ d\left(\frac{1}{2},\frac{1}{4}\right) &= d\left(\frac{1}{3},\frac{1}{4}\right) = \frac{1}{5}, \\ d\left(\frac{1}{4},\frac{1}{5}\right) &= d\left(\frac{1}{2},\frac{1}{5}\right) = d\left(\frac{1}{3},\frac{1}{5}\right) = \frac{1}{6}, \\ d(x,x) &= 0 \text{ for all } x \in A, \\ d(x,y) &= d(y,x) \text{ for all } x,y \in A \end{aligned}$$

and $d(x,y) = |x-y|$, whenever $(x \in A, y \in B)$ or $(x \in B, y \in A)$ or $(x, y \in B)$. It is easy to check that $(X,d)$ is a generalized metric space. Let $T : X \to X$ be a mapping defined by

$$T(x) = \begin{cases} \frac{2x+1}{x+2} & \text{if } x \in B \\ \frac{1}{5} & \text{otherwise} \end{cases}$$

and the function $\alpha : X \times X \to [0,\infty)$ defined by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x,y \in B \\ 0 & \text{otherwise.} \end{cases}$$

Define $\beta(t) = \frac{1}{1+t}$. Using routine calculation it is easy to check that $T$ is an $\alpha$-$h$-$F$-$\psi$-Geraghty contraction for $h(y,z) = yz$, $F(s,t) = st$ and $\psi(t) = t$. For $x_0 = \frac{3}{4}$, we have $\alpha(x_0, Tx_0) \geq 1$ and $\alpha\left(x_0, T^2x_0\right) \geq 1$. Using the definition of the maps $\alpha$ and $T$, we observe that $\alpha(x,y) \geq 1 \implies \alpha(Tx,Ty) \geq 1$ for all $x,y \in X$ and also $\alpha(x,z) \geq 1$ and $\alpha(y,z) \geq 1 \implies \alpha(x,y) \geq 1$ and so $T$ is triangular $\alpha$-admissible. Moreover, if $\{x_n\}$ is a sequence in $X$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n$ with $x_n \to x$ as $n \to \infty$, then the definition of $\alpha$ gives $x_n \in B$ for each $n$. Since $B$ is closed, it follows that $x \in B$. Thus $\alpha(x_n, x) = 1$ for each $n$ and hence the sequence $\{x_n\}$ is $\alpha$-regular. So all

the hypotheses of Theorem 2.8 are satisfied and therefore $T$ has a fixed point. Here $x = 1$ is such a fixed point.

*Remark* 2.15. If the self map $T$ is an $\alpha$-$\psi$-Geraghty contraction type map (see [1]), then $T$ is an $\alpha$-$h$-$F$-$\psi$-Geraghty contraction type map for $h(y, z) = yz$ and $F(s, t) = st$. Therefore the results of Asadi et al. [1] can be obtained as a particular case of our results.

## References

[1] M. Asadi, E. Karapinar, A. Kumar, *$\alpha$-$\psi$-Geraghty contractions on generalized metric spaces*, J. Inequal. Appl. **2014**, 2014:423.
[2] A. Branciari, Â fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces, Publ. Math. Debrecen **57** (2000), 31–37.
[3] M, Geraghty, *On contractive mappings*, Proc. Amer. Math. Soc. **40** (1973), 604–608.
[4] Z. Kadelburg, S. Radenović, *Fixed point results in generalized metric spaces without Hausdorff property*, Math. Sci. **8** (2014), Art. ID 125.
[5] E. Karapinar, P. Kumam, P. Salimi, *On $\alpha$-$\psi$-Meir-Keeler contractive mappings*, Fixed Point Theory Appl. **2013**, 2013:94.
[6] M.S. Khan, M. Swaleh, S. Sessa, *Fixed point theorems by altering distances between the points*, Bull. Aust. Math. Soc. **30** (1984), 1–9.
[7] W.A. Kirk, N. Shahzad, Generalized metrics and Caristi's theorem, Fixed Point Theory Appl. **2013**, 2013:129.
[8] B. Samet, C. Vetro, P. Vetro, Fixed point theorems for $\alpha$-$\psi$-contractive type mappings, Nonlinear Anal. **75** (2012), 2154–2165.
[9] A. Shoaib, *$\alpha$-$\eta$ dominated mappings and related common fixed point results in closed ball*, J. Concrete Appl. Math. **13** (2015), 152–170.
[10] A. Shoaib, M. Arshad, M. A. Kutbi, *Common fixed points of a pair of Hardy Rogers type mappings on a closed ball in ordered partial metric spaces*, J. Comput. Anal. Appl. **17** (2014), 255–264.

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# *Oscillation of nth-order nonlinear dynamic equations on time scales*

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**Absract** In this paper, we investigate the oscillation of the following $n$th-order nonlinear dynamic equation

$$(r(t)\Phi_\gamma(a_{n-1}(t)(a_{n-2}(t)(\cdots(a_1(t)x^\Delta(t))^\Delta\cdots)^\Delta)^\Delta)^\Delta+\sum_{i=0}^{k}q_i(t)\Phi_{\alpha_i}(x(\delta_i(t)))=0$$

on a time scale $\mathbb{T}$ with $n\geq 2$. We obtain some new oscillation criteria of the above equation.

**Keywords:** Oscillation; Dynamic equation; Time scale

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## 1. Introduction

In this paper, we study the following $n$th-order nonlinear dynamic equation

$$(r(t)\Phi_\gamma(a_{n-1}(t)(a_{n-2}(t)(\cdots(a_1(t)x^\Delta(t))^\Delta\cdots)^\Delta)^\Delta)^\Delta+\sum_{i=0}^{k}q_i(t)\Phi_{\alpha_i}(x(\delta_i(t)))=0, \quad (1.1)$$

on a time scale $\mathbb{T}$ satisfying $\inf\mathbb{T}=t_0$ and $\sup\mathbb{T}=\infty$, where $n,k\in\mathbb{N}\equiv\{1,2,\cdots\}$ with $n\geq 2$ and $\gamma,\alpha_i>0$, $i=0,1,2,\cdots,k$, are ratios of odd numbers, $q_i(t)\in C_{rd}(\mathbb{T},[0,\infty))$ and $q_i(t)\not\equiv 0$, $i=0,1,2,\cdots,k$. And we also assume the following conditions are satisfied:

(H1) $\Phi_p(u)=|u|^{p-1}u$ for any $p>0$.

(H2) There is an integer $m \in [1, k)$ such that

$$\alpha_1 > \alpha_2 > \cdots > \alpha_m > \alpha_0 = \gamma > \alpha_{m+1} > \cdots > \alpha_k. \tag{1.2}$$

(H3) $a_j(t) \in C_{rd}(\mathbb{T}, (0, \infty)),\ 1 \le j \le n-1,\ r(t) \in C_{rd}(\mathbb{T}, (0, \infty))$ satisfying

$$\int_{t_0}^{\infty} \frac{1}{r^{1/\gamma}(t) a_{n-1}(t)} \Delta t = \infty.$$

(H4) $\delta_i(t) \in C_{rd}(\mathbb{T}, \mathbb{T})$ such that $\delta_i^{\Delta}(t) > 0$ and $\lim\limits_{t\to\infty} \delta_i(t) = \infty$.

Write

$$S_l(t) = \begin{cases} x(t), & l = 0, \\ a_l(t) S_{l-1}^{\Delta}(t), & 1 \le l \le n-1. \end{cases}$$

Then (1.1) reduces to the equation

$$(r(t)\Phi_\gamma(S_{n-1}(t)))^{\Delta} + \sum_{i=0}^{k} q_i(t)\Phi_{\alpha_i}(x(\delta_i(t))) = 0. \tag{1.3}$$

Few decades, much attention has been paid to various dynamic equations, see [1-5]. In recent years, there has been more interest in obtaining conditions for oscillation of solutions of dynamic equations on time scales, see [6-10]. Erbe et al.[1] researched the oscillation of the equation

$$x^{\Delta^n}(t) + p(t)x^{\alpha}(\sigma(t)) = 0, t \in \mathbb{T},$$

where $\alpha > 1$ is a ratio of odd numbers; and Jia et al.[2] investigated oscillation for the linear equation

$$x^{\Delta^n}(t) + p(t)x(t) = 0, t \in \mathbb{T}.$$

On a general time scale $\mathbb{T}$. Recently, Zhang et al.[3] established the oscillation criteria for the equation

$$(r(t)\Phi_\gamma(x^{\Delta^{n-1}}(t))^{\Delta} + \sum_{i=0}^{k} q_i(t)\Phi_{\alpha_i}(x(\delta_i(t))) = 0, \tag{1.4}$$

where $\Phi_p(u), \gamma, \alpha_i, \delta_i(t)$ is the same as (1.1).

The purpose of this paper is to generalize results in [3] to more general equation and obtain some new oscillation criteria of (1.1).

## 2. Preliminary results

**Lemma 2.1**[3] Assume (1.2) holds. Then there exist $\eta_i \in (0, 1),\ i = 1, \cdots, k$, such that

$$\sum_{i=1}^{k} \alpha_i \eta_i = \gamma \quad \text{and} \quad \sum_{i=1}^{k} \eta_i = 1. \qquad \square$$

**Lemma 2.2**[4] Assume that

$$\int_{t_0}^{\infty} \frac{\triangle s}{a_i(s)} = \infty, \quad 1 \le i \le n-2, \tag{2.1}$$

and $1 \le m \le n-2$. Then,

(1) $\liminf_{t\to\infty} S_m(t) > 0$ implies $\lim_{t\to\infty} S_i(t) = +\infty$, $0 \le i \le m-1$;

(2) $\limsup_{t\to\infty} S_m(t) < 0$ implies $\lim_{t\to\infty} S_i(t) = -\infty$, $0 \le i \le m-1$. □

**Lemma 2.3** Assume (2.1) holds, and suppose that $x(t) > 0$ and $(r(t)\Phi_\gamma(S_{n-1}(t)))^\Delta < 0$ for $t \ge t_0$. Then there exist an integer $m \in [0, n-1]$ with $m+n-1$ even and a sufficiently large $T \in \mathbb{T}$ such that, for any $t \ge T$,

(1) $(-1)^{m+i} S_i(t) > 0$, $m \le i \le n-1$;

(2) $S_i(t) > 0$, $1 \le i \le m-1$, when $m > 1$.

**Proof** First we claim that $S_{n-1}(t) > 0$ for $t > t_0$. Otherwise, there exists some $t_1 > t_0$ such that $S_{n-1}(t_1) < 0$. By (1.3) and (H1), $r(t)S_{n-1}^\gamma(t)$ is strictly decreasing on $[t_0, \infty)_{\mathbb{T}}$. It follows that $r^{\frac{1}{\gamma}}(t)(-S_{n-1}(t))$ is positive and strictly increasing on $[t_1, \infty)_{\mathbb{T}}$. Thus,

$$\begin{aligned} S_{n-2}(t) &= S_{n-2}(t_1) - \int_{t_1}^{t} \frac{r^{\frac{1}{\gamma}}(s)(-S_{n-1}(s))}{r^{\frac{1}{\gamma}}(s)a_{n-1}(s)}\Delta s \\ &\le S_{n-2}(t_1) - r^{\frac{1}{\gamma}}(t_1)(-S_{n-1}(t_1)) \int_{t_1}^{t} \frac{1}{r^{\frac{1}{\gamma}}(s)a_{n-1}(s)}\Delta s. \end{aligned}$$

By (H3), we have $\lim_{t\to\infty} S_{n-2}(t) = -\infty$. From Lemma 2.2, we get $\lim_{t\to\infty} S_0(t) = -\infty$, i.e., $\lim_{t\to\infty} x(t) = -\infty$, which contradicts the fact that $x(t) > 0$ for $t \ge t_0$. Consequently, $S_{n-1}(t) > 0$ for $t > t_0$. Then we get two cases as follows:

(i) $S_i(t) > 0$, $0 \le i \le n-1$;

(ii) there exists an integer $j \in [1,\ n-2]$ such that $S_j(t) < 0$.

For case (ii), let $m$ be the smallest integer $m \in [0, n-1]$ with $m+n-1$ even such that $(-1)^{m+i} S_i(t) > 0$ for $t \ge t_0$ and $m \le i \le n-1$. Note that $S_{m-1}^\Delta(t) = \frac{S_m(t)}{a_m(t)} > 0$ for $t \ge t_0$. So, when $m > 1$, we have that either $S_{m-1}(t) < 0$ for $t \ge t_0$ or there is $t_2 \in \mathbb{T}$ such that $S_{m-1}(t) \ge S_{m-1}(t_2) > 0$ for $t \in \ge t_2$.

If $S_{m-1}(t) < 0$ for $t \ge t_0$, then using the above arguments similar to the case of $S_{n-1}(t) < 0$, we have $S_{m-2}(t) > 0$ for $t \ge t_0$, which is a contradiction to the definition of $m$.

If $S_{m-1}(t) \ge S_{m-1}(t_2) > 0$ for $t \ge t_2$, then from (1) of Lemma 2.2 we have $\lim_{t\to\infty} S_i(t) = \infty$ for $0 \le i \le m-1$. This completes the proof. □

**Lemma 2.4** Assume that (2.1) holds and one of the following two conditions is satisfied:

$$\int_{t_0}^{\infty} \sum_{i=0}^{k} q_i(s)\Delta s = \infty \tag{2.2}$$

or

$$\int_{t_0}^{\infty}\left[\Big(\int_{u_2}^{\infty}\frac{(\int_s^{\infty}\sum_{i=0}^{k}q_i(u_1)\Delta u_1)^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}(s)a_{n-1}(s)}\Delta s\Big)\Big/a_{n-2}(u_2)\right]\Delta u_2=\infty. \tag{2.3}$$

Let $x(t)$ be an eventually positive solution of $Eq.(1.3)$. Then there exists $T\in\mathbb{T}$ such that, for any $t\geq T$, $(r(t)\Phi_\gamma(S_{n-1}(t)))^{\Delta}<0$, and the following statements hold:

(1) when $n$ is even,

$$S_j(t)>0,\quad j=1,2,\cdots,n-1. \tag{2.4}$$

(2) when $n$ is odd, either (2.4) holds or $\lim_{t\to\infty}x(t)=0$.

**Proof** Let $x(t)$ be an eventually positive solution of $Eq.(1.3)$, then by (H4), we may assume that $x(t)>0$ and $x(\delta_i(t))>0$ $(i=0,1,\cdots,k)$ for $t\geq t_0$. By $Eq.(1.3)$ and (H1), we have

$$(r(t)\Phi_\gamma(S_{n-1}(t)))^{\Delta}=(r(t)(S_{n-1}(t))^{\gamma})^{\Delta}<0,\quad t\geq t_0.$$

From Lemma 2.3, there exists $t_1\in\mathbb{T}$ such that

$$S_j(t)>0\ \text{ for }\ t\geq t_1\text{ and }j\in[1,m-1], \tag{2.5}$$

$$(-1)^{m+j}S_j(t)>0\ \text{ for }\ t\geq t_1\text{ and }j\in[m,n-1]. \tag{2.6}$$

When $n$ is even, by Lemma 2.3, $m$ must be an odd number. By (2.5), $x^{\Delta}(t)=\frac{S_1(t)}{a_1(t)}>0$. Hence, $\lim_{t\to\infty}x(t)$ exists, and it's positive or $\lim_{t\to\infty}x(t)=\infty$. In this case, we will show that $m=n-1$. Otherwise, the odd integer $m\leq n-3$. Now (2.6) implies that, for any $t\geq t_1$,

$$S_{n-2}(t)<0\ \text{ and }\ S_{n-3}(t)>0.$$

Note that there exist $T\geq t_1$ and $a>0$ such that $x(t)\geq a$ and $x(\delta_i(t))\geq a$ $(i=0,1,\cdots,k)$ for $t\geq T$. Taking $b:=\min_{0\leq i\leq k}\{a^{\alpha_i}\}$, we have

$$(r(t)(S_{n-1}(t))^{\gamma})^{\Delta}\leq-\sum_{i=0}^{k}q_i(t)a^{\alpha_i}\leq-b\sum_{i=0}^{k}q_i(t). \tag{2.7}$$

If (2.2) holds, integrating (2.7) from $T$ to $t$ with $t\geq T$, we obtain that

$$r(t)(S_{n-1}(t))^{\gamma}\leq r(T)(S_{n-1}(T))^{\gamma}-b\int_T^t\sum_{i=0}^{k}q_i(s)\Delta s\to-\infty,\quad\text{as }t\to\infty,$$

which contradicts the fact that $S_{n-1}(t)>0$ for $t\in[t_1,\infty)_{\mathbb{T}}$. Hence, (2.4) holds. If (2.3) holds, integrating (2.7) from $t$ to $u_1$ with $T\leq t\leq u_1$, we obtain that

$$r(t)(S_{n-1}(t))^{\gamma}\geq r(u_1)(S_{n-1}(u_1))^{\gamma}+b\int_t^{u_1}\sum_{i=0}^{k}q_i(s)\Delta s\geq b\int_t^{u_1}\sum_{i=0}^{k}q_i(s)\Delta s.$$

Taking $u_1 \to \infty$, we have

$$S_{n-1}(t) \geq b^{\frac{1}{\gamma}} \frac{(\int_t^\infty \sum_{i=0}^k q_i(s)\Delta s)^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}(t)} = b^{\frac{1}{\gamma}} \left( r^{-1}(t) \int_t^\infty \sum_{i=0}^k q_i(s)\Delta s \right)^{\frac{1}{\gamma}}. \tag{2.8}$$

Since $S_{n-2}(t) < 0$, integrating (2.8) from $t$ to $u_2$ with $T \leq t \leq u_2$, we have

$$-S_{n-2}(t) \geq S_{n-2}(u_2) - S_{n-2}(t) \geq b^{\frac{1}{\gamma}} \int_t^{u_2} \frac{(\int_t^\infty \sum_{i=0}^k q_i(u_1)\Delta u_1)^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}(s)a_{n-1}(s)}\Delta s.$$

Taking $u_2 \to \infty$, we have

$$-S_{n-2}(t) \geq b^{\frac{1}{\gamma}} \int_t^\infty \frac{(\int_t^\infty \sum_{i=0}^k q_i(u_1)\Delta u_1)^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}(s)a_{n-1}(s)}\Delta s. \tag{2.9}$$

Since $S_{n-3}(t) > 0$, integrating (2.9) from $T$ to $t$ with $t \geq T$, we get

$$S_{n-3}(T) \geq -S_{n-3}(t) + S_{n-3}(T) \geq b^{\frac{1}{\gamma}} \int_T^t \left[ \left( \int_t^\infty \frac{(\int_t^\infty \sum_{i=0}^k q_i(u_1)\Delta u_1)^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}(s)a_{n-1}(s)}\Delta s \right) \Big/ a_{n-2}(u_2) \right] \Delta u_2.$$

Let $t \to \infty$, we obtain

$$\int_T^\infty \left[ \left( \int_t^\infty \frac{(\int_t^\infty \sum_{i=0}^k q_i(u_1)\Delta u)^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}(s)a_{n-1}(s)}\Delta s \right) \Big/ a_{n-2}(u_2) \right] \Delta u_2 \leq b^{-\frac{1}{\gamma}} S_{n-3}(T) < \infty,$$

which contradicts (2.3). Hence, $m = n-1$ and (2.4) holds.

When $n$ is odd, by Lemma 2.3, $m$ must be even. From (2.5) and (2.6), either $x^\Delta(t) > 0$ or $x^\Delta(t) < 0$ hold, which implies $\lim_{t\to\infty} x(t) = c \geq 0$. If $c > 0$, we claim that $m = n-1$. Otherwise, $m \leq n-3$. Similar as above, we can arrive a contradiction. This completes the proof. □

In the sequel, for any $n \in \mathbb{N}$ and $t, T \in \mathbb{T}$ with $t \geq T$, we define $\beta_k(t,T)$ as follows:

$$\beta_i(t,T) = \int_T^t \frac{\beta_{i-1}(s,T)}{a_{n-i}(s)}\Delta s, \quad i = 1, 2, \cdots, k,$$

where $\beta_0(t,T) = \frac{1}{r^\gamma(t)}$.

**Lemma 2.5** Suppose that (2.1) and either (2.2) or (2.3) hold. Let $x(t)$ be an eventually positive solution of *Eq.*(1.3) satisfied (2.4). Then there exists $T \in \mathbb{T}$ such that, for $t \in [T, \infty)_{\mathbb{T}}$,

$$S_1(t) \geq r^{\frac{1}{\gamma}}(t) S_{n-1}(t) \beta_{n-2}(t,T) \quad \text{and} \quad x(t) \geq r^{\frac{1}{\gamma}}(t) S_{n-1}(t) \beta_{n-1}(t,T).$$

**Proof** By the hypothesis and (H4), there exists $T \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) > 0,\ x(\delta_i(t)) > 0$ $(i = 0, 1, \cdots, k)$ and (2.4) hold for $t \geq T$. It's easy to see, from Lemma 2.4, that $r^{\frac{1}{\gamma}}(t)S_{n-1}(t)$ is decreasing on $[T, \infty)_{\mathbb{T}}$. From (2.4), we have

$$\begin{aligned} S_{n-2}(t) &= S_{n-2}(T) + \int_T^t \frac{S_{n-1}(s)}{a_{n-1}(s)}\Delta s \\ &= S_{n-2}(T) + \int_T^t \frac{r^{\frac{1}{\gamma}}(s)S_{n-1}(s)}{r^{\frac{1}{\gamma}}(s)a_{n-1}(s)}\Delta s \\ &\geq r^{\frac{1}{\gamma}}(t)S_{n-1}(t)\int_T^t \frac{1}{r^{\frac{1}{\gamma}}(s)a_{n-1}(s)}\Delta s \\ &= r^{\frac{1}{\gamma}}(t)S_{n-1}(t)\beta_1(t,T). \end{aligned}$$

Integrating above inequality from $T$ to $t$ for $t \geq T$, we have

$$\begin{aligned} S_{n-3}(t) &= S_{n-3}(T) + \int_T^t \frac{S_{n-2}(s)}{a_{n-2}(s)}\Delta s \\ &\geq r^{\frac{1}{\gamma}}(t)S_{n-1}(t)\int_T^t \frac{\beta_1(s,T)}{a_{n-2}(s)}\Delta s \\ &= r^{\frac{1}{\gamma}}(t)S_{n-1}(t)\beta_2(t,T). \end{aligned}$$

By induction, we can show that

$$S_1(t) \geq r^{\frac{1}{\gamma}}(t)S_{n-1}(t)\beta_{n-2}(t,T), x(t) \geq r^{\frac{1}{\gamma}}(t)S_{n-1}(t)\beta_{n-1}(t,T).$$

This completes the proof. □

**Lemma 2.6**[3] Let $g(y) = By - Ay^{\frac{\gamma+1}{\gamma}}$, where $A, B$ and $y$ are positive numbers. Then $g(y)$ attains its maximum value on $[0, \infty)$ at $y^* = (\frac{B\gamma}{A(\gamma+1)})^\gamma$, and

$$\max_{y\in[0,\infty)} g(y) = g(y^*) = \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}}\frac{B^{\gamma+1}}{A^\gamma}.$$ □

## 3. Main Results

For the convenience of presentation, we give some notations. Write $\mathbb{D} = \{(t, s) \in \mathbb{T}^2 :\ t \geq s \geq 0\}$, for any $z \in C^1_{rd}(\mathbb{T}, (0, \infty))$, we denote

$$\mathcal{H} = \{H(t, s) \in C^1_{rd}(\mathbb{D}, [0, \infty)) : H^\Delta_s(t, s) \leq 0 \text{ and } H(t, s) = 0 \text{ iff } t = s\},$$

$$C(t, s) = H^\Delta_s(t, s)z^\sigma(s) + H(t, s)z^\Delta(s),$$

and

$$C_+(t,s) = \max\{H_s^{\Delta}(t,s)z^{\sigma}(s) + H(t,s)z_+^{\Delta}(s), 0\},$$

where $H(t,s) \in \mathcal{H}$ and $z_+^{\Delta}(s) = \max\{z^{\Delta}(s), 0\}$.

**Theorem 3.1** Suppose that (2.1) and either (2.2) or (2.3) hold, and $\eta_i$ $(i = 1, \cdots, k)$ are defined as in Lemma 2.1. Assume that for sufficiently large $T \in \mathbb{T}$, one of the following conditions is satisfied:

(C1) either

$$\int_T^{\infty} Q(s)\Delta s = \infty,$$

or

$$\int_T^{\infty} Q(s)\Delta s < \infty \quad \text{and} \quad \limsup_{t\to\infty} \beta_{n-1}^{\gamma}(\delta(t), T)\int_t^{\infty} Q(s)\Delta s > 1,$$

(C2) there is a function $z \in C_{rd}^1(\mathbb{T}, (0,\infty))$ such that

$$\limsup_{t\to\infty}\int_T^t \left[Q(s)z(s) - \frac{z_+^{\Delta}(s)}{\beta_{n-1}^{\gamma}(\delta^{\sigma}(s), T)}\right]\Delta s = \infty,$$

(C3) there is a function $z \in C_{rd}^1(\mathbb{T}, (0,\infty))$ such that

$$\limsup_{t\to\infty}\int_T^t \left[Q(s)z(s) - \frac{1}{(\gamma+1)^{\gamma+1}}\frac{a_1^{\gamma}(s)(z_+^{\Delta}(s))^{\gamma+1}}{z^{\gamma}(s)\beta_{n-2}^{\gamma}(\delta(s),T)(\delta^{\Delta}(s))^{\gamma}}\right]\Delta s = \infty,$$

(C4) there are a function $z \in C_{rd}^1(\mathbb{T}, (0,\infty))$ and $H \in \mathcal{H}$ such that

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t \Bigg[H(t,s)Q(s)z(s) - \frac{a_1^{\gamma}(s)C_+^{\gamma+1}(t,s)}{H^{\gamma}(t,s)(\gamma+1)^{\gamma+1}z^{\gamma}(s)\beta_{n-2}^{\gamma}(\delta(s),T)(\delta^{\Delta}(s))^{\gamma}}\Bigg]\Delta s = \infty$$

where $Q(t) = q_0(t) + \prod_{i=1}^k \left(\eta_i^{-1}q_i(t)\right)^{\eta_i}$ and $\delta(t) = \min\{t, \delta_i(t),\ i = 0, 1, 2, \cdots, k\}$. Then

(i) every solution of $Eq.(1.3)$ either oscillates or tends to zero as $t \to \infty$ when $n$ is odd,

(ii) every solution of $Eq.(1.3)$ is oscillatory when $n$ is even.

**Proof** Assume that $Eq.(1.3)$ has a nonoscillatory solution $x(t)$. Without loss of generality, we may suppose that $x(t)$ is eventually positive. Then, Lemma 2.4 and Lemma 2.5 hold, and by (H4), there exists a sufficiently large $T \in \mathbb{T}$ such that $x(t) > 0$ and $x(\delta_i(t)) > 0$ $(i = 0, 1, \cdots, k)$ for $t \geq T$.

When $n$ is odd, from Lemma 2.4, we have that (2.4) holds or $\lim_{t\to\infty} x(t) = 0$. If (2.4) holds, $Eq.(1.3)$ reduces to

$$(r(t)(S_{n-1}(t))^{\gamma})^{\Delta} + \sum_{i=0}^{k} q_i(t)x^{\alpha_i}(\delta_i(t)) = 0. \tag{3.1}$$

We consider four parts corresponding conditions (C1)-(C4).

Part 1: Suppose that (C1) holds.

Write $\psi(t) = r(t)(S_{n-1}(t))^{\gamma}$, then $\psi(t) > 0$ and $\psi^{\Delta}(t) + \sum_{i=0}^{k} q_i(t)x^{\alpha_i}(\delta(t)) \leq 0$ for $t \geq T$. So we have $\psi^{\Delta}(t) < 0$ and $\lim_{t\to\infty} \psi(t) = b \geq 0$. Put $a_i = \eta_i^{-1} q_i(t)x^{\alpha_i}(\delta(t))$, by the arithmetic-geometric mean inequality [11], we get

$$\sum_{i=1}^{k} \eta_i a_i \geq \prod_{i=1}^{k} a_i^{\eta_i} \text{ and } a_i \geq 0.$$

So we have

$$\sum_{i=0}^{k} q_i(t)x^{\alpha_i}(\delta(t)) \geq q_0(t)x^{\gamma}(\delta(t)) + (\eta_i^{-1}q_i(t))^{\eta_i}x^{\gamma}(\delta(t)) = Q(t)x^{\gamma}(\delta(t)). \tag{3.2}$$

Consequently,

$$\psi^{\Delta}(t) + Q(t)x^{\gamma}(\delta(t)) \leq 0. \tag{3.3}$$

Integrating (3.3) from $t$ to $\infty$, we have

$$b - \psi(t) + \int_t^{\infty} Q(s)x^{\gamma}(\delta(s))\Delta s \leq 0.$$

If $\int_t^{\infty} Q(s)\Delta s = \infty$, using (2.4), we can arrive a contradiction. If $\int_t^{\infty} Q(s)\Delta s < \infty$, then

$$\psi(\delta(t)) \geq \psi(t) \geq \int_t^{\infty} Q(s)x^{\gamma}(\delta(s))\Delta s \geq x^{\gamma}(\delta(t)) \int_t^{\infty} Q(s)\Delta s.$$

From Lemma 2.5, we have

$$\beta_{n-1}^{\gamma}(\delta(t), T) \int_t^{\infty} Q(s)\Delta s \leq 1,$$

which contradicts to (C1). Therefore, every solution of $Eq.(1.3)$ either oscillates or tends to zero as $t \to \infty$.

Part 2: Suppose that (C2) holds.

Define

$$w(t) = \frac{z(t)r(t)(S_{n-1}(t))^{\gamma}}{x^{\gamma}(\delta(t))}, \quad t \geq T. \tag{3.4}$$

We get $w(t) > 0$ immediately. From Lemma 2.5, we have

$$\begin{aligned} w^{\Delta}(t) &= \Big(r(t)(S_{n-1}(t))^{\gamma}\Big)^{\Delta}\left(\frac{z(t)}{x^{\gamma}(\delta(t))}\right) + \Big(r(t)(S_{n-1}(t))^{\gamma}\Big)^{\sigma}\left(\frac{z(t)}{x^{\gamma}(\delta(t))}\right)^{\Delta} \\ &= \Big(r(t)(S_{n-1}(t))^{\gamma}\Big)^{\sigma}\left[\frac{z^{\Delta}(t)x^{\gamma}(\delta(t)) - z(t)(x^{\gamma}(\delta(t)))^{\Delta}}{x^{\gamma}(\delta(t))x^{\gamma}(\delta^{\sigma}(t))}\right] \end{aligned}$$

$$+ z(t)\frac{-\sum_{i=0}^{k} q_i(t)x^{\alpha_i}(\delta_i(t))}{x^{\gamma}(\delta(t))}$$

$$\leq \frac{z_+^{\Delta}(t)(r(t)(S_{n-1}(t))^{\gamma})^{\sigma}}{x^{\gamma}(\delta^{\sigma}(t))} - z(t)\frac{\sum_{i=0}^{k} q_i(t)x^{\alpha_i}(\delta(t))}{x^{\gamma}(\delta(t))}$$

$$- \frac{(r(t)(S_{n-1}(t))^{\gamma})^{\sigma} z(t)(x^{\gamma}(\delta(t)))^{\Delta}}{x^{\gamma}(\delta(t)x^{\gamma}(\delta^{\sigma}(t)))}. \tag{3.5}$$

Noting that $x^{\Delta}(t) = \frac{S_1(t)}{a_1(t)} > 0$. When $\gamma \geq 1$, by Keller's chain rule, we have

$$\begin{aligned}(x^{\gamma}(t))^{\Delta} &= \gamma\Big[\int_0^1 (x(t) + h\mu(t)x^{\Delta}(t))^{\gamma-1}dh\Big]x^{\Delta}(t)\\ &\geq \gamma x^{\Delta}(t)\int_0^1 ((1-h)x(t) + hx(t))^{\gamma-1}dh = \gamma x^{\gamma-1}(t)x^{\Delta}(t).\end{aligned}$$

So $(x^{\gamma}(\delta(t)))^{\Delta} \geq \gamma x^{\gamma-1}(\delta(t))(x(\delta(t)))^{\Delta}$. It's easy to see, from (H4), that $\delta^{\Delta}(t) > 0$ for $t \in \mathbb{T}$. Hence, from [12, Theorem 1.93], we have

$$(x^{\gamma}(\delta(t)))^{\Delta} \geq \gamma x^{\gamma-1}(\delta(t))x^{\Delta}(\delta(t))\delta^{\Delta}(t) \geq 0. \tag{3.6}$$

When $0 < \gamma < 1$,

$$(x^{\gamma}(t))^{\Delta} \geq \gamma x^{\Delta}(t)\int_0^1 ((1-h)x^{\sigma}(t) + hx^{\sigma}(t))^{\gamma-1}dh = \gamma(x^{\sigma}(t))^{\gamma-1}x^{\Delta}(t).$$

So $(x^{\gamma}(\delta(t)))^{\Delta} \geq \gamma x^{\gamma-1}(\delta^{\sigma}(t))(x(\delta(t)))^{\Delta}$. From [12, Theorem 1.93], we have

$$(x^{\gamma}(\delta(t)))^{\Delta} \geq \gamma x^{\gamma-1}(\delta^{\sigma}(t))x^{\Delta}(\delta(t))\delta^{\Delta}(t) \geq 0. \tag{3.7}$$

Noting that $r(t) > 0$. From (3.6) and (3.7), we get

$$\frac{(r(t)S_{n-1}^{\gamma}(t))^{\sigma} z(t)(x^{\gamma}(\delta(t)))^{\Delta}}{x^{\gamma}(\delta(t))x^{\gamma}(\delta^{\sigma}(t))} \geq 0.$$

Since $(r(t)S_{n-1}^{\gamma}(t))^{\Delta} < 0,\ \delta(t) \leq t \leq \sigma(t)$ and $\delta^{\sigma}(t) \leq \sigma(t)$, we have

$$r(\sigma(t))S_{n-1}^{\gamma}(\sigma(t)) \leq r(t)S_{n-1}^{\gamma}(t) \leq r(\delta(t))S_{n-1}^{\gamma}(\delta(t)). \tag{3.8}$$

and

$$r(\sigma(t))S_{n-1}^{\gamma}(\sigma(t)) \leq r(\delta^{\sigma}(t))S_{n-1}^{\gamma}(\delta^{\sigma}(t)). \tag{3.9}$$

Hence, from (3.2), (3.9), Lemma 2.5 and $x^{\Delta}(t) > 0$, we have

$$w^{\Delta}(t) \leq -z(t)Q(t) + \frac{z_+^{\Delta}(t)}{\beta_{n-1}^{\gamma}(\delta^{\sigma}(t), T)}.$$

Integrating the above inequality from $T$ to $t$ for $t \geq T$, we get

$$\int_T^t \left[ z(s)Q(s) - \frac{z_+^{\Delta}(s)}{\beta_{n-1}^{\gamma}(\delta^{\sigma}(s), T)} \right] \Delta s \leq w(T) - w(t) < w(T).$$

Taking the lim sup on both sides as $t \to \infty$, we obtain a contradiction to (C2). Therefore, every solution of $Eq.(1.3)$ either oscillates or tends to zero as $t \to \infty$.

Part 3: Suppose that (C3) holds.

When $\gamma \geq 1$, from (2.4), (3.5) and (3.6), we have

$$w^{\Delta}(t) \leq -z(t)Q(t) + \frac{z_+^{\Delta}(t)}{z^{\sigma}(t)} w^{\sigma}(t) - (r(t)S_{n-1}^{\gamma}(t))^{\sigma} \frac{z(t)\gamma x^{\Delta}(\delta(t))\delta^{\Delta}(t)}{x^{\gamma+1}(\delta^{\sigma}(t))}. \tag{3.10}$$

Since $x^{\Delta}(t) = \frac{S_1(t)}{a_1(t)} \geq \frac{r^{\frac{1}{\gamma}(t)} S_{n-1}(t)\beta_{n-2}(t,T))}{a_1(t)}$, from (3.8) we get

$$\begin{aligned} -(r(t)S_{n-1}^{\gamma}(t))^{\sigma} \frac{z(t)\gamma x^{\Delta}(\delta(t))\delta^{\Delta}(t)}{x^{\gamma+1}(\delta^{\sigma}(t))} &= \frac{-(r^{\sigma}(t))^{\frac{\gamma+1}{\gamma}} S_{n-1}^{\gamma+1}(t)}{x^{\gamma+1}(\delta^{\sigma}(t))} \frac{z(t)\gamma x^{\Delta}(\delta(t))\delta^{\Delta}(t)}{r^{\frac{1}{\gamma}}(\sigma(t)) S_{n-1}(\sigma(t))} \\ &\leq \frac{-(r^{\sigma}(t))^{\frac{\gamma+1}{\gamma}} S_{n-1}^{\gamma+1}(t)}{x^{\gamma+1}(\delta^{\sigma}(t))} \frac{z(t)\gamma x^{\Delta}(\delta(t))\delta^{\Delta}(t)}{r^{\frac{1}{\gamma}}(\delta(t)) S_{n-1}(\delta(t))} \\ &\leq -\frac{z(t)\gamma\beta_{n-2}(t,T)\delta^{\Delta}(t)}{a_1(t) z^{\frac{\gamma+1}{\gamma}}(\sigma(t))} w^{\frac{\gamma+1}{\gamma}}(\sigma(t)). \end{aligned}$$

Then

$$w^{\Delta}(t) \leq -z(t)Q(t) + \frac{z_+^{\Delta}(t)}{z^{\sigma}(t)} w^{\sigma}(t) - \frac{z(t)\gamma\beta_{n-2}(t,T)\delta^{\Delta}(t)}{a_1(t) z^{\frac{\gamma+1}{\gamma}}(\sigma(t))} w^{\frac{\gamma+1}{\gamma}}(\sigma(t)). \tag{3.11}$$

When $0 < \gamma < 1$, by (3.2), (3.5) and (3.7), we have

$$w^{\Delta}(t) \leq -z(t)Q(t) + \frac{z_+^{\Delta}(t)}{z^{\sigma}(t)} w^{\sigma}(t) - (r(t)S_{n-1}^{\gamma}(t))^{\sigma} \frac{z(t)\gamma (x(\delta^{\sigma}(t)))^{\gamma-1} x^{\Delta}(\delta(t))\delta^{\Delta}(t)}{x^{\gamma}(\delta(t))(x(\delta^{\sigma}(t)))^{\gamma}}.$$

Then, by (3.8) and Lemma 2.5, we have

$$\begin{aligned} & -(r(t)S_{n-1}^{\gamma}(t))^{\sigma} \frac{z(t)\gamma (x(\delta^{\sigma}(t)))^{\gamma-1} x^{\Delta}(\delta(t))\delta^{\Delta}(t)}{x^{\gamma}(\delta(t))(x(\delta^{\sigma}(t)))^{\gamma}} \\ = & \frac{-(r^{\sigma}(t))^{\frac{\gamma+1}{\gamma}} S_{n-1}^{\gamma+1}(t)}{x^{\gamma}(\delta(t)) x(\delta^{\sigma}(t))} \frac{z(t)\gamma x^{\Delta}(\delta(t))\delta^{\Delta}(t)}{r^{\frac{1}{\gamma}}(\sigma(t)) S_{n-1}(\sigma(t))} \\ \leq & \frac{-(r^{\sigma}(t))^{\frac{\gamma+1}{\gamma}} S_{n-1}^{\gamma+1}(t)}{x^{\gamma}(\delta(t)) x(\delta^{\sigma}(t))} \frac{z(t)\gamma x^{\Delta}(\delta(t))\delta^{\Delta}(t)}{r^{\frac{1}{\gamma}}(\delta(t)) S_{n-1}(\delta(t))} \\ \leq & -\frac{z(t)\gamma\beta_{n-2}(t,T)\delta^{\Delta}(t)}{a_1(t) z^{\frac{\gamma+1}{\gamma}}(\sigma(t))} w^{\frac{\gamma+1}{\gamma}}(\sigma(t)). \end{aligned}$$

It follows that

$$w^{\Delta}(t) \leq -z(t)Q(t) + \frac{z_{+}^{\Delta}(t)}{z^{\sigma}(t)}w^{\sigma}(t) - \frac{z(t)\gamma\beta_{n-2}(t,T)\delta^{\Delta}(t)}{a_1(t)z^{\frac{\gamma+1}{\gamma}}(\sigma(t))}w^{\frac{\gamma+1}{\gamma}}(\sigma(t)). \tag{3.12}$$

Let

$$B = \frac{z_{+}^{\Delta}(t)}{z^{\sigma}(t)}, \quad A = \frac{z(t)\gamma\beta_{n-2}(t,T)\delta^{\Delta}(t)}{a_1(t)z^{\frac{\gamma+1}{\gamma}}(\sigma(t))}, \quad y = w^{\sigma}(t).$$

Then, by Lemma 2.6 and (3.12), for all $t \geq T$,

$$w^{\Delta}(t) \leq -z(t)Q(t) + \frac{1}{(\gamma+1)^{\gamma+1}}\frac{a_1^{\gamma}(t)(z_{+}^{\Delta}(t))^{\gamma+1}}{z^{\gamma}(t)\beta_{n-2}^{\gamma}(t,T)(\delta^{\Delta}(t))^{\gamma}}.$$

Integrating the above inequality from $T$ to $t$ for $t \geq T$, we get

$$\int_T^t \left[z(s)Q(s) - \frac{1}{(\gamma+1)^{\gamma+1}}\frac{a_1^{\gamma}(s)(z_{+}^{\Delta}(s))^{\gamma+1}}{z^{\gamma}(s)\beta_{n-2}^{\gamma}(s,T)(\delta^{\Delta}(s))^{\gamma}}\right]\Delta s \leq w(T) - w(t) < w(T).$$

By taking the lim sup on both sides as $t \to \infty$, we obtain a contradiction to (C3). Therefore, every solution of $Eq.(1.3)$ either oscillates or tends to zero as $t \to \infty$.

Part 4: Suppose that (C4) holds.

From (3.11) and (3.12), we have that for $H \in \mathcal{H}$ and $t \geq T$

$$\begin{aligned}
\int_T^t H(t,s)z(s)Q(s)\Delta s &\leq -\int_T^t H(t,s)w^{\Delta}(s)\Delta s + \int_T^t H(t,s)w^{\sigma}(s)\frac{z_{+}^{\Delta}(s)}{z^{\sigma}(s)}\Delta s \\
&\quad - \int_T^t H(t,s)\frac{z(s)\gamma\beta_{n-2}(\delta(s),T)\delta^{\Delta}(s)}{a_1(s)z^{\frac{\gamma+1}{\gamma}}(\sigma(s))}w^{\frac{\gamma+1}{\gamma}}(\sigma(s))\Delta s \\
&\leq H(t,T)w(T) + \int_T^t \left[H_s^{\Delta}(t,s) + H(t,s)\frac{z_{+}^{\Delta}(s)}{z^{\sigma}(s)}\right]w^{\sigma}(s)\Delta s \\
&\quad - \int_T^t H(t,s)\frac{z(s)\gamma\beta_{n-2}(\delta(s),T)\delta^{\Delta}(s)}{a_1(s)z^{\frac{\gamma+1}{\gamma}}(\sigma(s))}w^{\frac{\gamma+1}{\gamma}}(\sigma(s))\Delta s \\
&\leq H(t,T)w(T) + \int_T^t \left[C_{+}(t,s)\right]w^{\sigma}(s)\Delta s \\
&\quad - \int_T^t H(t,s)\frac{z(s)\gamma\beta_{n-2}(\delta(s),T)\delta^{\Delta}(s)}{a_1(s)z^{\frac{\gamma+1}{\gamma}}(\sigma(s))}w^{\frac{\gamma+1}{\gamma}}(\sigma(s))\Delta s.
\end{aligned}$$

Let

$$B = C_{+}(t,s), \quad A = H(t,s)\frac{z(s)\gamma\beta_{n-2}(\delta(s),T)\delta^{\Delta}(s)}{a_1(s)z^{\frac{\gamma+1}{\gamma}}(\sigma(s))}, \quad y = w^{\sigma}(s).$$

By Lemma 2.6, for all $t \geq T$,

$$\begin{aligned}\int_T^t H(t,s)z(s)Q(s)\Delta s &\leq H(t,T)w(T) \\ &\quad + \int_T^t \frac{[C_+(t,s)]^{\gamma+1}(z^\sigma(s))^{\gamma+1}a_1^\gamma(s)}{\beta_{n-2}^\gamma(\delta(s),T)(\delta^\Delta(s))^\gamma H^\gamma(t,s)(\gamma+1)^{\gamma+1}z^\gamma(s)}\Delta s,\end{aligned}$$

i.e.,

$$\begin{aligned}w(T) &\geq \frac{1}{H(t,T)}\int_T^t \Bigg[H(t,s)z(s)Q(s) \\ &\quad - \frac{[C_+(t,s)]^{\gamma+1}a_1^\gamma(s)}{\beta_{n-2}^\gamma(\delta(s),T)(\delta^\Delta(s))^\gamma H^\gamma(t,s)(\gamma+1)^{\gamma+1}z^\gamma(s)}\Bigg]\Delta s.\end{aligned}$$

Taking the lim sup on both sides as $t \to \infty$, we obtain a contradiction to (C4). Therefore, every solution of $Eq.(1.3)$ either oscillates or tends to zero as $t \to \infty$.

When $n$ is even. From Lemma 2.4, (2.4) holds. Similar to above four parts, we can show that every solution of $Eq.(1.3)$ is oscillatory. This completes the proof. □

**Theorem 3.2** Let $\gamma \geq 1$. Suppose that (2.1) and either (2.2) or (2.3) hold. For sufficiently large $T \in \mathbb{T}$ and $z \in C_{rd}^1(\mathbb{T},(0,\infty))$, one of the following two conditions is satisfied:

$$\text{(I)}\ \limsup_{t\to\infty}\int_T^t\left[z(s)Q(s) - \frac{a_1(s)(z^\Delta(s))^2}{4\gamma\beta^*(\delta(s),T)z(s)\delta^\Delta(s)}\right]\Delta s = \infty,$$

(II) there exists $H \in \mathcal{H}$ such that

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[H(t,s)z(s)Q(s) - \frac{a_1(s)C^2(t,s)}{4\gamma z(s)\delta^\Delta(s)\beta^*(\delta(s),T)H(t,s)}\right]\Delta s = \infty,$$

where $\beta^*(t,T) = \beta_{n-1}^{\gamma-1}(t,T)\beta_{n-2}(t,T)$. Then

(i) every solution of $Eq.(1.3)$ either oscillates or tends to zero as $t \to \infty$ when $n$ is odd,

(ii) every solution of $Eq.(1.3)$ is oscillatory when $n$ is even.

**Proof** Assume that $Eq.(1.3)$ has a non-oscillatory solution $x(t)$. Without loss of generality, we may suppose that $x(t)$ is eventually positive. Then, Lemma 2.4 and Lemma 2.5 hold, and by (H4) there exists a sufficiently large $T \in [t_0,\infty)$ such that $x(t) > 0$ and $x(\delta_i(t)) > 0$ $(i = 0,1,\cdots,k)$ for $t \geq T$.

When $n$ is odd, from Lemma 2.4 we see that (2.4) holds or $\lim_{t\to\infty} x(t) = 0$. If (2.4) holds, we separate the rest of the proof into two parts.

Part one: Suppose that (I) holds.

Define $w(t)$ as in (5.4). Because $x(t) > 0,\ \sigma(t) \geq t$, by (3.2), (3.5) and (3.6), we get that

$$w^\Delta(t) \leq -z(t)Q(t) + \frac{z^\Delta(t)}{z^\sigma(t)}w^\sigma(t) - (r(t)S_{n-1}^\gamma(t))^\sigma\frac{z(t)\gamma x^{\gamma-1}(\delta(t))x^\Delta(\delta(t))\delta^\Delta(t)}{x^\gamma(\delta^\sigma(t))x^\gamma(\delta(t))}$$

$$\leq \quad -z(t)Q(t) + \frac{z^{\Delta}(t)}{z^{\sigma}(t)}w^{\sigma}(t) - \frac{z(t)\gamma x^{\gamma-1}(\delta(t))x^{\Delta}(\delta(t))\delta^{\Delta}(t)}{(r(t)S_{n-1}^{\gamma}(t))^{\sigma}(z^{\sigma}(t))^2}(w^{\sigma}(t))^2.$$

From (3.8) and Lemma 2.4, we get

$$\begin{aligned} w^{\Delta}(t) &\leq -z(t)Q(t) + \frac{z^{\Delta}(t)}{z^{\sigma}(t)}w^{\sigma}(t) - \frac{z(t)\delta^{\Delta}(t)}{(z^{\sigma}(t))^2 r(\sigma(t))}\frac{x^{\gamma-1}(\delta(t))}{S_{n-1}^{\gamma-1}(\delta(t))}\frac{x^{\Delta}(\delta(t))}{S_{n-1}(\delta(t))}(w^{\sigma}(t))^2 \\ &\leq -z(t)Q(t) + \frac{z^{\Delta}(t)}{z^{\sigma}(t)}w^{\sigma}(t) - \frac{z(t)\delta^{\Delta}(t)}{a_1(t)(z^{\sigma}(t))^2}\beta^{*}(\delta(t),T)(w^{\sigma}(t))^2. \end{aligned}$$

By completing the square for $w^{\sigma}(t)$ on the right-hand side, we have

$$w^{\Delta}(t) \leq -z(t)Q(t) + \frac{a_1(t)(z^{\Delta}(t))^2}{4\gamma\beta^{*}(\delta(t),T)z(t)\delta^{\Delta}(t)}.$$

Integrating the above inequality from $T$ to $t$ for $t \geq T$, we get

$$\int_T^t \left[z(s)Q(s) - \frac{a_1(s)(z^{\Delta}(s))^2}{4\gamma\beta^{*}(\delta(s),T)z(s)\delta^{\Delta}(s)}\right]\Delta s \leq w(T) - w(t) < w(T).$$

Taking the lim sup on both sides as as $t \to \infty$, we obtain a contradiction to (I). Therefore, every solution of $Eq.(1.3)$ either oscillates or tends to zero as $t \to \infty$.

Part two: Suppose that (II) holds.

The proof is similar to Part 4 of Theorem 3.1 and Part one of Theorem 3.2.

When $n$ is even. From Lemma 2.4, (2.4) holds. Similarly, we can show that every solution of $Eq.(1.3)$ is oscillatory. The proof is completed. □

## 4. Examples

**Example 4.1** Consider the equation

$$\left(t^{-1}\Phi_{\frac{1}{2}}(S_n(t))\right)^{\Delta} + t^{-\frac{1}{2}}\Phi_{\frac{1}{2}}(x(t+1)) + t^{-2}\Phi_{\frac{19}{3}}(x(t-1)) + t^{-3}\Phi_{\frac{1}{3}}(x(t+2)) = 0, \quad t \in \mathbb{T}$$

where $S_n(t)$ satisfies $Eq.(1.3)$, $a_i(t) = t^{-1}$ $(1 \leq i \leq n)$, $\mathbb{T} = [2,\infty)_{\mathbb{R}}$. Here, we have

(1) $n \geq 2,\ \gamma = \alpha_0 = \frac{1}{2},\ \alpha_1 = \frac{19}{3},\ \alpha_2 = \frac{1}{3}$;

(2) $r(t) = t^{-1},\ q_0(t) = t^{-\frac{1}{2}},\ q_1(t) = t^{-2},\ q_2(t) = t^{-3}$;

(3) $\delta_0(t) = t+1,\ \delta_1(t) = t-1$ and $\delta(t) = t-1$.

Clearly,

$$\int_{t_0}^{\infty} \frac{1}{r^{1/\gamma}(t)a_n(t)}\Delta t = \int_2^{\infty} t^3\Delta t = \infty,$$

$$\int_{t_0}^{\infty} \frac{1}{a_i(t)}\Delta t = \int_2^{\infty} t\Delta t = \infty, \quad 1 \leq i \leq n,$$

hence (H1)-(H4) hold. Note that

$$\int_2^\infty \sum_{i=0}^{2} q_0(s)\Delta s = \int_2^\infty \left(s^{-\frac{1}{2}} + s^{-2} + s^{-3}\right)\Delta s \geq \int_2^\infty s^{-\frac{1}{2}}\Delta s = \infty,$$

i.e., (2.2) holds. Let $\eta_1 = \frac{1}{36},\ \eta_2 = \frac{35}{36}$. Then $\eta_1, \eta_2$ satisfies Lemma 2.1. With $z(t) = 1$ we see that for sufficiently large $T \in \mathbb{T}$,

$$\begin{aligned} & \limsup_{t\to\infty} \int_T^t \left[Q(s)z(s) - \frac{1}{(\gamma+1)^{\gamma+1}} \frac{a_1^\gamma(s)(z_+^\Delta(s))^{\gamma+1}}{z^\gamma(s)\beta_{n-2}^\gamma(\delta(s),T)(\delta^\Delta(s))^\gamma}\right]\Delta s \\ = & \limsup_{t\to\infty} \int_T^t \left[s^{-\frac{1}{2}} + \left(36s^{-2}\right)^{\frac{1}{36}}\left(\frac{36}{35}s^{-3}\right)^{\frac{35}{36}}\right]\Delta s \\ \geq & \limsup_{t\to\infty} \int_T^t s^{-\frac{1}{2}}\Delta s = \infty. \end{aligned}$$

Hence, the condition (C3) of Theorem 3.1 is satisfied.

By Theorem 3.1, every solution of $Eq.(1.3)$ either oscillates or tends to zero as $t \to \infty$ when $n$ is odd and oscillatory when $n$ is even. □

**Example 4.2** Consider the equation

$$\left(t\Phi_2(S_n(t))\right)^\Delta + \Phi_2(x(t)) + 2\Phi_3(x(t-1)) + 3t^{-3}\Phi_{\frac{3}{2}}(x(t+2)) = 0, \quad t \in \mathbb{T}$$

where $S_n(t)$ satisfies $Eq.(1.3)$, $\mathbb{T} = [1,\infty)_{\mathbb{R}},\ a_i(t) = t^{-4}(1 \leq i \leq n)$. Here, we have

(1) $n \geq 2,\ k = 2,\ \gamma = \alpha_0 = 3,\ \alpha_1 = 3,\ \alpha_2 = \frac{3}{2}$;

(2) $r(t) = t,\ q_0(t) = 1,\ q_1(t) = 2,\ q_2(t) = 3t^{-3}$;

(3) $\delta_0(t) = t,\ \delta_1(t) = t-1,\ \delta_2(t) = t+2$ and $\delta(t) = t-1$.

Clearly,

$$\int_{t_0}^\infty \frac{1}{r^{1/\gamma}(t)a_n(t)}\Delta t = \int_1^\infty t^{\frac{7}{2}}\Delta t = \infty,$$
$$\int_{t_0}^\infty \frac{1}{a_i(t)}\Delta t = \int_1^\infty t^4\Delta t = \infty, (1 \leq i \leq n)$$

hence (H1)-(H4) hold. Note that

$$\begin{aligned} & \int_1^\infty \left[\left(\int_v^\infty \frac{\left(\int_s^\infty \sum_{i=0}^k q_i(u)\Delta u\right)^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}(s)a_{n-1}(s)}\Delta s\right) \Big/ a_{n-2}(v)\right]\Delta v \\ = & \int_1^\infty \left[\left(\int_v^\infty \frac{\left(\int_s^\infty (1+2+3u^{-3})\Delta u\right)^{\frac{1}{2}}}{s^{\frac{1}{2}}s^{-4}}\Delta s\right) \Big/ v^{-4}\right]\Delta v \\ \geq & \int_1^\infty \left(\int_v^\infty \frac{\left(\int_s^\infty (3u^{-3})\Delta u\right)^{\frac{1}{2}}}{s^{\frac{1}{2}}s^{-4}}\Delta s\right) v^4\Delta v \end{aligned}$$

$$\geq \int_1^\infty \left( \int_v^\infty s^{\frac{5}{2}} \Delta s \right) v^4 \Delta v = \infty,$$

i.e., (2.3) holds. Let $\eta_1 = \frac{1}{3},\ \eta_2 = \frac{2}{3}$. Then $\eta_1, \eta_2$ satisfies Lemma 2.1. With $z(t) = 1$ we see that for sufficiently large $T \in \mathbb{T}$,

$$\limsup_{t\to\infty} \int_T^t Q(s)\Delta s = \limsup_{t\to\infty} \int_T^t \left[ 1 + (3\times 2)^{\frac{1}{3}} \left( \frac{3}{2} \times 3s^{-3} \right)^{\frac{2}{3}} \right] \Delta s = \infty.$$

Hence, the condition (I) of Theorem 3.2 is satisfied. By Theorem 3.2, every solution of $Eq.(1.3)$ either oscillates or tends to zero as $t \to \infty$ when $n$ is odd and is oscillatory when $n$ is even. □

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# REFERENCES

[1] L.Erbe, B.Jia, A.Peterson, Oscillation of $n$th-order superlinear dynamic equations on time scales, Rocky Mt. J. Math., 41(2011), 471-491.

[2] B.Jia, L.Erbe, A.Peterson, Asymptotic behavior of $n$th-order sublinear dynamic equations on time scales, Common. Appl. Anal., 15(2011), 183-194.

[3] S.Zhang, Q.Wang, Q.Kong, Asymptotic and oscillation of $n$th-order nonlinear dynamic equations on time scales, Applied Mathematics and computation, 275(2016), 324-334.

[4] T.Sun, Q.He, H.Xi, W.Y, Oscillation for higher order dynamic equations on time scales, Abstract and Applied Analysis, Volume 2013, Article ID 267281, 8 pages.

[5] R.P.Agarwal, M.Bohner, Basic calculus on time scales and some of its applications, Results Math., 35(1999), 3-22.

[6] T.Sun, W.Y, H.Xi, Oscillation behavior and comparison for higher order nonlinear dynamic equations on time scales, J. Appl. Math. Informatics, 30(2012), 289-304.

[7] S.R.Grace, R.P.Agarwal, A.Zafer, Oscillation of higher order nonlinear dynamic equations on time scales, Adv. Differ.Eq., 1(2012), 1-18.

[8] Q.R.Wang, Oscillation criteria for even order nonlinear damped differential equations, Acta Math. Hung., 95(3)(2002), 169-178.

[9] S.Zhang, Q.Wang, Oscillation of second-order nonlinear neutral dynamic equations on time scales, Appl. Math. Comput., 216(2010), 2837-2848.

[10] S.R.Grace, On the oscillation of $n$th order dynamic equations on time scales, Mediterr. J. Math., 10(1)(2013), 147-156.

[11] E.F.Beckenbach, R.Bellman, Inequalities, Springer, Berlin, 1961.

[12] M.Bohner, A.Peterson, Dynamic Equations on Time Scales: An Introduction with Applications, Birkhuser, Boston, 2001.

# Normality of meromorphic functions concerning sharing values

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Abstract: Let $n$, $k$ be positive integers satisfying $n \geq 2k+4$, $a \neq 0$ be a complex number, and let $\mathcal{F}$ be a family of functions meromorphic in a domain $D$, if for each $f \in \mathcal{F}$, $f^n + af^{(k)}$ and $g^n + ag^{(k)}$ share $b$, and all zeros have multiplicity at least $k+1$, then $\mathcal{F}$ is normal in $D$.

*Keywords*: meromorphic function; normal family; shared value.

## 1. Introduction and results

In this paper, we denote by $\mathbb{C}$ the whole complex plane. Let $f$ be a meromorphic function in a domain $D \subset \mathbb{C}$. For $a \in \mathbb{C}$, set $\overline{E}_f(a) = \{z \in D : f(z) = a\}$. We say that two meromorphic functions $f$ and $g$ share the value $a$ provided that $\overline{E}_f(a) = \overline{E}_g(a)$ in $D$. When $a = \infty$ the zeros of $f - a$ means the poles of $f$.

Let $\mathcal{F}$ a family of meromorphic functions defined on $D \subset \mathbb{C}$. $\mathcal{F}$ is said to be normal on $D$, in the sense of Montel, if for any sequence $f_j \in \mathcal{F}$ there exists a subsequence $f_{n_j}$ converges spherically locally uniformly on $D$, to a meromorphic function or $\infty$ (see [1], [2], [3]).

According to Bloch's principle, every condition which reduces a meromorphic function in the plane $\mathbb{C}$ to a constant, makes a family of meromorphic functions in a domain $D$ normal. It is also more interesting to find normality criteria from the point of view of shared values. In this area, Schwick[4] first proved an interesting result that a family

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of meromorphic functions in a domain is normal if in which every function shares three distinct finite complex numbers with its first derivatives. And later, Sun[5] proved that a family of meromorphic functions in a domain is normal if in which each pair of functions share three fixed distinct values, which is an improvement of the famous Montel's Normal Criterion [6] by the idea of shared values. More results about normality criteria concerning shared values can be found, for instance, in [7-9] and so on.

In 2008, Zhang[10] proved

**Theorem A.** (see [10]). let $\mathcal{F}$ be a family of functions meromorphic in a domain $D$, $n$ be a positive integer and $a$, $b$ be two constants such that $a \neq 0, \infty$ and $b \neq \infty$. If $n \geq 4$ and for each $f$ and $g$ in $\mathcal{F}$, $f' - af^n$ and $g' - ag^n$ share the value $b$, then $\mathcal{F}$ is normal in $D$.

In this paper,we replace $f'$ by $f^{(k)}$ in Theorem A and obtain the following theorem.

**Theorem 1.** Let $n$, $k$ be a positive integers satisfying $n \geq 2k + 4$, $a \neq 0, \infty$ and $b \neq \infty$ be complex numbers, and let $\mathcal{F}$ be a family of functions meromorphic in a domain $D$. If for each $f, g \in \mathcal{F}$, $f^n + af^{(k)}$ and $g^n + ag^{(k)}$ share $b$, and all zeros have multiplicity at least $k + 1$, then $\mathcal{F}$ is normal in $D$.

**Example**: Let $D = \{z : |z| < 1\}$ and $\mathcal{F} = \{f_n\}$ where

$$f_n(z) = \frac{1}{n\sqrt[4]{z}}, z \in D, n = 1, 2, 3, \ldots.$$

Clearly $f_n'' + f_n^9 = \frac{5n^8+5}{8n^9\sqrt[4]{z^9}}$. So for each pair $m, n$, $f_n'' + f_n^7$ and $f_m'' + f_m^7$ share 0 in $D$, but $\mathcal{F}$ is not normal at the point $z = 0$ since $f_n^\sharp(\frac{1}{n^4}) = \frac{n^4}{4(1+n)} \to \infty (n \to \infty)$. This example show that Theorem 1 is not valid if $f$ doesn't satisfy that all zeros have multiplicity at least $k + 1$.

## 2.Lemmas

In this section, we present some lemmas which will be needed in the sequel.

**Lemma 2.1**([8]). Let $\mathcal{F}$ be a family of functions meromorphic on the unit disc, all of whose zeros have multiplicity at least $k$, and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$. Then if $\mathcal{F}$ is not normal, there exist, for each $0 \leq \alpha \leq k$,

a) a number $0 < r < 1$;

b) points $z_n$, $|z_n| < r$;

c) functions $f_n \in F$; and

d) positive numbers $\rho_n \to 0$

such that $\rho_n^{-\alpha} f_n(z_n + \rho_n \xi) = g_n(\xi) \to g(\xi)$ locally uniformly with respect to the spherical

metric, where $g$ is a nonconstant meromorphic function on $\mathbb{C}$, all of whose zeros have multiplicity at least $k$, such that $g^\sharp(\xi) \leq g^\sharp(0) = kA + 1$. In particular, $g$ has order at most two; and, in case $g$ is an entire function, it is of exponential type.

**Lemma 2.2** [11] Let $f$ be a meromorphic function, then we have

$$T(r,f) < \overline{N}(r,f) + N(r,\frac{1}{f}) + N(r,\frac{1}{f-1}) - N(r,\frac{1}{f'}) + S(r,f), \tag{2.1}$$

where

$$S(r,f) = 2m(r,\frac{f'}{f}) + m(r,\frac{f'}{f-1}) + log|\frac{192f(0)(f(0)-1)}{f'(0)}|.$$

**Lemma 2.3** Let $f$ be a transcendental meromorphic function, $n$ and $k$ be two integers and $n \geq 2k+4$, and all zeros of $f$ are of multiplicity greater than $k+1$, then $f^n + af^{(k)}$ assumes every finite complex value $b$ infinitely often.

Proof: Let $\psi(z) = \frac{af^{(k)}(z)-b}{f^n(z)}$, and suppose that $\psi(z) = -1$ has only finite number of roots. Then by Lemma 2.2 we have

$$T(r,\psi) < \overline{N}(r,\psi) + \overline{N}(r,\frac{1}{\psi}) + \overline{N}(r,\frac{1}{\psi+1}) + S(r,\psi). \tag{2.2}$$

Now the poles of $\psi(z)$ occur only at zeros of $f(z)$, and those poles which are not simultaneously zeros of $af^{(k)} - b$ have multiplicity at least $n$. Zeros of $\psi(z)$ can occur only at zeros of $af^{(k)} - b$ which are not poles of $f(z)$. Thus

$$\begin{aligned} \overline{N}(r,\psi) + \overline{N}(r,\frac{1}{\psi}) &\leq \frac{1}{n}N(r,\psi) + \overline{N}(r,\frac{1}{af^{(k)}-b}) + \overline{N}(r,f) \\ &\leq \frac{1}{n}T(r,\psi) + T(r,f^{(k)}) + \overline{N}(r,f). \end{aligned} \tag{2.3}$$

By the first fundamental theorem, we have

$$\begin{aligned} \overline{N}(r,\psi) + \overline{N}(r,\frac{1}{\psi}) &\leq \frac{1}{n}T(r,\psi) + T(r,f) + (k+1)\overline{N}(r,f) + S(r,f) \\ &\leq \frac{1}{n}T(r,\psi) + (k+2)T(r,f) + S(r,f). \end{aligned} \tag{2.4}$$

Take (2.4) into (2.1), we have

$$(1-\frac{1}{n})T(r,\psi) < (k+2)T(r,f) + S(r,f). \tag{2.5}$$

On the other hand,

$$\begin{aligned} nT(r,f) &= T(r,f^n) = T(r,\frac{af^{(k)}-b}{\psi}) \\ &\leq T(r,f^{(k)}) + T(r,\psi) + O(1) \\ &\leq (k+1)T(r,f) + T(r,\psi) + S(r,f), \end{aligned}$$

that is

$$(n-k-1)T(r,f) \leq T(r,\psi) + S(r,f). \tag{2.6}$$

Combining (3.5) and (3.6) we obtain

$$(1-\frac{1}{n})T(r,\psi) < [\frac{k+2}{n-k-1} + O(1)]T(r,\psi) \tag{2.7}$$

which contradicts with the condition $n \geq 2k+4$.

Thus we proved Lemma 2.3.

**Lemma 2.4** Let $f$ be a nonconstant rational function, $n$ and $k$ be two integers and $n \geq 2k+4$, and all zeros of $f$ are of multiplicity greater than $k+1$, then $f^n + af^{(k)}$ has at least two distinct zeros.

Proof: Case 1. If $f^n + af^{(k)}$ has no zeros, it is easy to see that $f$ is not a polynomial, then $f$ is rational function but not a polynomial.

Let $f = \frac{P}{(z-z_1)^{m_1}(z-z_2)^{m_2}\cdots(z-z_t)^{m_t}} = \frac{P}{Q}$, $f^{(k)} = \frac{P_1}{Q_1}$. We denote $p = degP$, and $q = degQ$, then $degQ_1 = q+kt$, $degP_1 = p+k(t-1)$.

$$f^n + af^{(k)} = \frac{P^nQ_1 + aP_1Q^n}{Q^nQ_1}. \tag{2.8}$$

$deg(P^nQ_1) = np+q+kt$, $deg(P_1Q^n) = p+k(t-1)+nq$.

If $p \geq q$, then $np+q+kt-(p+k(t-1)+nq) = (n-1)(p-q)+k > 0$, that is $deg(P^nQ_1) > deg(P_1Q^n)$; If $p < q$, since $n \geq 2k+4$, then $np+q+kt-(p+k(t-1)+nq) = (n-1)(p-q)+k < 0$, that is $deg(P^nQ_1) < deg(P_1Q^n)$, thus $f^n + af^{(k)}$ have zeros, which is a contradiction.

Case 2. $f^n + af^{(k)}$ has only one distinct zero $z_0$.

If $f$ is a polynomial, then $f^n + af^{(k)} = A(z-z_0)^l$. From the condition that all zeros of $f$ are of multiplicity greater than $k+1$, we can deduce that $z_0$ is the only zero of $f$, so $f = b(z-z_0)^m, m \geq k+1$, where $b$ is a constant and $m$ is a positive integer.

$$f^{(k)} = bm(m-1)\cdots(m-k+1)(z-z_0)^{m-k} \tag{2.9}$$

and

$$f^n + af^{(k)} = A^n(z-z_0)^{nl} + c(z-z_0)^{m-k} = (z-z_0)^{m-k}[A(z-z_0)^{nl-m+k} + C], \tag{2.10}$$

thus $f^n + af^{(k)}$ has two distinct zeros, contradiction.

So $f$ is a non-polynomial rational function, then we can assume that

$$f(z) = \frac{A(z-z_0)^m}{(z-z_1)^{n_1}(z-z_2)^{n_2}\cdots(z-z_s)^{n_s}}. \tag{2.11}$$

where $A$ is a constant, and $s$ is a positive integer. By integration (2.4) we get

$$f^{(k)} = \frac{(z-z_0)^{m-k}g(z)}{(z-z_1)^{n_1+1}(z-z_2)^{n_2+1}\cdots(z-z_s)^{n_s+1}}, \tag{2.12}$$

From (2.11) and (2.12) we obtain

$$\begin{aligned} f^n + af^{(k)} &= \frac{A^n(z-z_0)^{nm} + a(z-z_0)^{m-k}g(z)(z-z_1)^{(n-1)n_1-k}\cdots(z-z_s)^{(n-1)n_s-k}}{(z-z_1)^{n_1}(z-z_2)^{n_2}\cdots(z-z_s)^{n_s}} \\ &= \frac{(z-z_0)^{m-k}[A^n(z-z_0)^{nm-m+k} + ag(z)(z-z_1)^{(n-1)n_1-k}\cdots(z-z_s)^{(n-1)n_s-k}]}{(z-z_1)^{n_1}(z-z_2)^{n_2}\cdots(z-z_s)^{n_s}}. \end{aligned} \tag{2.13}$$

On the other hand, since $f^n + af^{(k)}$ has only one zero, we have

$$f^n + af^{(k)} = \frac{C(z-z_0)^l}{(z-z_1)^{n_1}(z-z_2)^{n_2}\cdots(z-z_s)^{n_s}}. \tag{2.14}$$

Combining (2.13) and (2.14) we have

$$C(z-z_0)^l = (z-z_0)^{m-k}g_1(z),$$

where $g_1(z) = A^n(z-z_0)^{nm-m+k} + ag(z)(z-z_1)^{(n-1)n_1-k}\cdots(z-z_s)^{(n-1)n_s-k}$. If $l > m-k$, then $g_1(z)$ has a zero $z_0$, which is impossible. If $l = m-k$, $g_1(z) = C$, that is $A^n(z-z_0)^{nm-m+k} + ag(z)(z-z_1)^{(n-1)n_1-k}\cdots(z-z_s)^{(n-1)n_s-k} = C$, thus $nm-m+k = (n-1)N$, where $N = n_1 + n_2 + \cdots + n_s$, thus $(n-1)(N-m) = k$, which is impossible since $n \geq 2k+4$.

The proof of Lemma 2.4 is completed.

## 3. The Proof of Theorem 1

We may assume that $D = \Delta$, the unit disc. Suppose that $F$ is not normal on $\Delta$. Then by Lemma1, we can find $f_j \in F, z_j \in \Delta$, and $\rho_j \to 0^+$ such that $g_j(\xi) = \rho_n^{\frac{k}{n-1}} f_j(z_j + \rho_j\xi)$ converges locally uniformly with respect to the sphericity metric to a nonconstant meromorphic function $g$ on $\mathbb{C}$, all of whose zeros have multiplicity at least $k$, which satisfies $g^\sharp(\xi) \leq g^\sharp(0) = kA+1$, in particular, $g$ has order at most two.

On every compact subset of $\mathbb{C}$ we have

$$\rho_j^{\frac{nk}{n-1}}[f_j^n + af_j^{(k)} - b] = g_j^n(\xi) + ag_j^{(k)}(\xi) - \rho_j^{\frac{nk}{n-1}}b \to g^n(\xi) + ag^{(k)}(\xi). \tag{3.1}$$

If $g^n(\xi) + ag^{(k)}(\xi) \equiv 0$, then $g$ has no poles and $g$ is not a polynomial, thus $g$ is a transcendental entire function. From $g^n(\xi) + ag^{(k)}(\xi) \equiv 0$ we obtain $g^{n-1} = -a\frac{g^{(k)}}{g}$, by the first fundamental theorem we have

$$(n-1)T(r,g) = (n-1)m(r,g) = m(r,g^{n-1}) = m(r,-a\frac{g^{(k)}}{g}) = S(r,g),$$

since $n \geq 2k+4$, we obtain $T(r,g) = S(r,g)$, which is a contradiction, thus $g^n(\xi) + ag^{(k)}(\xi) \not\equiv 0$.

By Lemma 2.3 and Lemma 2.4 we obtain that $g^n + ag^{(k)}$ has at least two distinct zeros.

Next we prove that $g^n + ag^{(k)}$ has only one distinct zero.

Let $\xi_0$ and $\xi_0^*$ be two distinct zeros of $g^n + ag^{(k)}$. We choose a small $\delta > 0$ such that $D_1 \cap D_2 = \emptyset$, where $D_1 = \xi \in \mathbb{C} : |\xi - \xi_0| < \delta$ and $D_2 = \xi \in \mathbb{C} : |\xi - \xi_0^*| < \delta$.

From (3.1), Hurwitz's theorem implied that there exist points $\xi_j \in D_1$ and $\xi_j^* \in D_2$ such that for sufficiently large $j$

$$f_j^n(z_j + \rho_j\xi_j) + af_j^{(k)}(z_j + \rho_j\xi_j) = b,$$

$$f_j^n(z_j + \rho_j\xi_j^*) + af_j^{(k)}(z_j + \rho_j\xi_j^*) = b.$$

By the assumption of Theorem 1, we see that for each $f_m \in F$

$$f_m^n(z_j + \rho_j\xi_j) + af_m^{(k)}(z_j + \rho_j\xi_j) = b,$$

$$f_m^n(z_j + \rho_j\xi_j^*) + af_m^{(k)}(z_j + \rho_j\xi_j^*) = b.$$

Fix $m$ and let $j \to \infty$, we have $z_j + \rho_j\xi_j \to z_0$, and $z_j + \rho_j\xi_j^* \to z_0$, and

$$f_m^n(z_0) + af_m^{(k)}(z_0) = b.$$

Since the zeros of $f_m^n + af_m^{(k)} - b$ have no accumulation points, we deduce that $z_j + \rho_j\xi_j = z_0$ and $z_j + \rho_j\xi_j^* = z_0$, for sufficiently large $j$. Hence $\xi_j = \xi_j^* = (z_0 - z_j)/\rho_j$, which contradicts the fact that $\xi_j \in D_1$, $\xi_j^* \in D_2$ and $D_1 \cap D_2 = \emptyset$.

Thus we complete the proof of Theorem 1.

# References

[1] Hayman, W.K.; The spherical derivative of integral and meromorphic functions. Comment Math. Helv. 40. 117-148(1966).

[2] Schiff. J., Normal Families. Springer. Berlin (1993).

[3] Y. L.: Value distribution theory, Springer. Berlin (1993).

[4] Schwick. W.: Sharing values and normality. Arch. Math. 59, 50-54 (1992).

[5] D. C. Sun, On the normal criteriion of shared values, J. Wuhan Univ. Natur. Sci. Ed. 3(1994)9-12(in Chinese).

[6] P. Montel, Sur les familles de fonctions analytiques qui admettent des valeurs exceptionnelles dans un domaine, Ann. Sci. Ecole Norm. Sup.29(1912)487-535.

[7] M. Fang, L. Zalcman, A note on normality and shared values, J. Aust. Math. Soc. 32(2004)141-150.

[8] X. Pang, L. Zalcman, Normal families and shared values, Bull. London. Math. Soc. 32(2000)325-331.

[9] Q. Zhang, Normal criteria concerning sharing values, Kodai Math. J. 25(2002)8-14.

[10] Q. C. Zhang, Normal families of meromoephic functions concerning shared values, J. Math. Anal. Appl. 338(2008)545-551.

[11] W.K. Hayman, Picard values of meromorphic functions and their derivatives, Ann. of Math. 70(1959)9-42.

# A general iterative algorithm for solving a class of variational inequalities over the common fixed points set of a finite family of nonexpansive mappings in Banach spaces

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**Abstract** In this paper, we introduce a general iterative algorithm for finding a common element of the set of common fixed points of an finite family of nonexpansive mappings and the set of solutions of class of variational inequalities in uniformly convex and $q$-uniformly smooth Banach space. We prove that the sequence generated by the iterative algorithm converges strongly to the unique solution of the variational inequality under suitable conditions. Our result improves and extends the recent results announced by many others.

**Keywords** Strong convergence, Fixed point, nonexpansive mapping, Variational inequality, Banach space.

**Mathematics Subject Classification(2010)** 47H09; 47H10.

## 1. Introduction

Throughout this paper, we denote by $X$ and $X^*$ a real Banach space and the dual space of $X$, respectively. Let $C$ be a subset of $X$ and $T$ be a mapping. We denote the fixed points of $T$ by $F(T) = \{x \in C : Tx = x\}$ and denote $\to$ and $\rightharpoonup$ by strong and weak convergence, respectively. Let $q > 1$ be a real number.

A Banach space $X$ is said to be strictly convex, if whenever $x$ and $y$ are not colinear, then

$$\|x + y\| < \|x\| + \|y\|.$$

The modulus of convexity of $X$ is defined by

$$\delta_X(\epsilon) = \inf\left\{1 - \left\|\frac{1}{2}(x+y)\right\| : \|x\|, \|y\| \le 1, \|x-y\| \ge \epsilon\right\}$$

for all $\epsilon \in [0, 2]$. $X$ is said to be uniformly convex if $\delta_X(0) = 0$, and $\delta_X(\epsilon) > 0$ for all $0 < \epsilon \le 2$. It is known that every uniformly convex Banach space is strictly convex and reflexive(see [1]).

Let $S(X) = \{x \in X : \|x\| = 1\}$. Then the norm of $X$ is said to be Gâteaux differentiable if

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in S(X)$. In this case, $X$ is said to be smooth. Let $\rho_X : [0, \infty) \longrightarrow [0, \infty)$ be the modulus of smoothness of $X$ defined by

$$\rho_X(t) = \sup\left\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : x \in S(X), \|y\| \le t\right\}.$$

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A Banach space $X$ is said to be uniformly smooth if $\frac{\rho_X(t)}{t} \to 0$ as $t \to 0$. A Banach space $X$ is said to be $q$-uniformly smooth($q > 1$), if there exists a fixed constant $c > 0$ such that $\rho_X(t) \leq ct^q$. It is easy to see that $X$ is $q$-uniformly smooth, then $q \leq 2$ and $X$ is uniformly smooth. The(generalized)duality mapping $J_q : X \to 2^{X^*}$ is defined by

$$J_q(x) = \left\{x^* \in X^* : \langle x, x^* \rangle = \|x\|^q,\ \|x^*\| = \|x\|^{q-1}\right\},\ \forall x \in X,$$

where $\langle \cdot, \cdot \rangle$ denote the duality pairing between $X$ and $X^*$. In particular, $J = J_2$ is called the normalized duality mapping and $J_q(x) = \|x\|^{q-2} J_2(x)$ for $x \neq 0$. If $X$ is a Hilbert space, then $J = I$ where $I$ is the identity mapping. Further, we have the following properties of the generalized duality mapping $J_q$:

(1) $J_q(x) = \|x\|^{q-2} J_2(x)$ for all $x \in X$ with $x \neq 0$.

(2) $J_q(tx) = t^{q-1} J_q(x)$ for all $x \in X$ and $t \in [0, \infty)$.

(3) $J_q(-x) = -J_q(x)$ for all $x \in X$.

It is well-known that if $X$ is smooth, then $J_q$ is single-valued, which is denoted by $j_q$. The duality mapping $J_q$ from a smooth Banach space $X$ into $X^*$ is said to be weakly sequentially continuous generalized duality mapping if for all $\{x_n\} \subset X$ with $x_n \rightharpoonup x$ implies $J_q(x_n) \rightharpoonup J_q(x)$.

**Definition 1.1** A mapping $T : C \to X$ is said to be:

(1) $L$-Lipschitzian if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L \|x - y\|,\ \forall x, y \in C. \tag{1.1}$$

If $0 < L < 1$, then $T$ is a contraction and if $L = 1$, then $T$ is a nonexpansive mapping.

(2) $\alpha$-strongly accretive if there exists $j_q(x - y) \in J_q(x - y)$ and a constant $\alpha > 0$ such that

$$\langle Ax - Ay, j_q(x - y) \rangle \geq \alpha \|x - y\|^q,\ \forall x, y \in C. \tag{1.2}$$

**Remark 1.2** If $X := H$ is a real Hilbert space, accretive and strongly accretive mappings coincide with monotone and strongly monotone mappings, respectively.

Recall that the normal Manns iterative algorithm was introduced by Mann [1] in 1953. Since then, construction of fixed points for nonexpansive mappings and strict pseudo-contractions via the normal Manns iterative algorithm has extensively investigated by many authors (see, e.g. [2-8]).

variational inequality theory has emerged as an important tool in studying a wide class of obstacle, unilateral, free, moving, equilibrium problems arising in several branches of pure and applied sciences in a unified and general framework. Several numerical methods have been developed for solving variational inequalities and related optimization problems, see [9-14,25-28] and the references therein.

In 2010, Tian [11] introduced the general steepest-descent method

$$x_{n+1} = \alpha_n \rho f(x_n) + (I - \mu \alpha_n F) T x_n, \forall n \geq 0, \tag{1.3}$$

where $F$ is an $L$-Lipschitzian and $\eta$-strongly monotone operator. Under certain approximate conditions, the sequence $\{x_n\}$ generated by (1.5) converges strongly to a fixed point of $T$, which

solves the variational inequality:

$$\langle(\rho f-\mu F)x^*, x-x^*\rangle \leq 0, \forall x \in Fix(T).$$

Recently, Zhang et al. [12] proposed the following iterative algorithm:

$$x_{k+1}=\alpha_k\gamma V(x_k)+(I-\mu\alpha_k F)T_N^k\cdots T_1^k(x_k), \forall k\geq 0, \tag{1.4}$$

where $V$ is a $L$-Lipschitzian mapping, and $F$ is a Lipschitzian and strongly monotone mapping,

$$T_i^k := (1-\beta_k^i)I+\beta_k^i T_i.$$

$\{T_i\}_{i=1}^N$ is a finite family of nonexpansive mappings of $H$. They obtained that under some approximate assumptions on the operators and parameters, the sequence $\{x_k\}$ generated by (1.6) converges strongly to the unique solution of the variational inequality:

$$\langle(\mu F-\gamma V)x^*, (x-x^*)\rangle \geq 0, \forall x \in C=\bigcap_{i=1}^N Fix(T_i).$$

On the other hand, Ceng et al. [16] investigated implicit and explicit iterative schemes for finding the fixed points of a nonexpansive mapping $T$ on a nonempty, closed and convex subset $C$ in a real Hilbert space $H$ as follows:

$$x_t = P_C[t\gamma V x_t+(I-t\mu F)Tx_t] \tag{1.5}$$

and

$$x_{n+1}=P_C[\alpha_n\gamma V x_n+(I-\alpha_n\mu F)Tx_n], \forall n\geq 0, \tag{1.6}$$

where $V$ is a $L$-Lipschitzian mapping and $F$ is a $\kappa$-Lipschitzian and $\eta$-strongly monotone operator. Then they proved that under some approximate assumptions on the operators and parameters, the sequences generated by (1.7) and (1.8) converge strongly to the unique solution of the variational inequality

$$\langle(\mu F-\gamma V)x^*, x-x^*\rangle \geq 0, \forall x \in Fix(T). \tag{1.7}$$

Pongsakorn et al. [17] introduced the following iterative process:

$$x_{n+1}=\beta_n x_n+(1-\beta_n)P_C[\alpha_n\gamma S x_n+(I-\alpha_n F)Tx_n], \forall n\geq 1, \tag{1.8}$$

where $P_C$ is a metric projection from $H$ onto $C$, $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $(0,1)$, $S: C\to H$ is a Lipschitzian mapping, $F: C\to H$ is an invertible positive linear operator, and $T: C\to C$ is a nonexpansive mapping. Then they proved strong convergence theorems under different control conditions on $\{\alpha_n\}$ and $\{\beta_n\}$, the sequence $\{x_n\}$ generated by (1.10) converges to a fixed point of $T$, which is a unique solution of some variational inequalities.

The following questions naturally arise in connection with above results:

**Question 1.** Can theorem of Zhang et al. [12] be extended from a real Hilbert space to a general Banach space? such as $q$-uniformly smooth Banach space.

**Question 2.** Can we extend the iterative scheme of algorithm (1.6) to a more general iterative scheme?

The purpose of this paper is to give the affirmative answers to these questions mentioned above, motivated by Tian [11], Zhang et al. [12], Ceng et al. [16] and Pongsakorn et al. [17], we introduce a general iterative method. Under some suitable assumptions, we prove the strong convergence theorems of such iterative scheme in $q$-uniformly smooth Banach space which admits a weakly sequentially continuous generalized duality mapping. The results presented in this article extend and generalize the corresponding results announced by many others in the literature.

## 2. Preliminaries

Let $C$ and $D$ be nonempty subsets of a Banach space $X$ such that $C$ is nonempty closed convex and $D \subset C$, then a mapping $Q : C \to D$ is sunny provided $Q(x + t(x - Q(x))) = Q(x)$ for all $x \in C$ and $t \geq 0$, whenever $x + t(x - Q(x)) \in C$. A mapping $Q : C \to D$ is retraction if $Qx = x$ for all $x \in D$. Furthermore, $Q$ is a sunny nonexpansive retraction from $C$ onto $D$ if $Q$ is a retraction from $C$ onto $D$ which is also sunny and nonexpansive. It is well known that if $X := H$ is a real Hilbert space, then a sunny nonexpansive retraction $Q_C$ is coincident with the metric projection from $X$ onto $C$.

A subset $D$ of $C$ is called of a sunny nonexpansive retraction of $C$ if there exists a sunny nonexpansive retraction from $C$ onto $D$.

In order to prove our main results, we need the following lemmas.

**Lemma 2.1** ([15]). Let $1 < p < \infty$, $q \in (1, 2]$, $r > 0$ be given.
(i) If $X$ is uniformly convex, then there exists a continuous, strictly increasing and convex function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ such that

$$\|\lambda x + (1 - \lambda)y\|^p \leq \lambda\|x\|^p + (1 - \lambda)\|y\|^p - W_p(\lambda)\varphi(\|x - y\|), \quad x, y \in \mathfrak{B}_r, \quad 0 \leq \lambda \leq 1,$$

where $W_p(\lambda) = \lambda^p(1 - \lambda) + (1 - \lambda)^p\lambda$, $\mathfrak{B}_r = \{z \in X : \|z\| \leq r\}$.
(ii) Let $X$ be a real $q$-uniformly smooth Banach space, then there exists a constant $C_q > 0$ such that

$$\|x + y\|^q \leq \|x\|^q + q \langle y, j_q x\rangle + C_q \|y\|^q$$

for all $x, y \in X$.

**Lemma 2.2** ([21]). Assume $\{\alpha_n\}$ is a sequence of nonnegative real numbers satisfying the property

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\delta_n$ is a sequence such that
$(i)$ $\sum\limits_{n=0}^{\infty} \gamma_n = \infty$,
$(ii)$ $\limsup\limits_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ or $\sum\limits_{n=1}^{\infty} |\delta_n| < \infty$.
Then $\lim\limits_{n\to\infty} \alpha_n = 0$.

**Lemma 2.3** ([7]). Let $C$ be a nonempty convex subset of a real uniformly convex Banach space $X$ and $T : C \to X$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Then $I - T$ is demiclosed at zero.

**Lemma 2.4** ([20]). Let $C \subseteq X$ be a nonempty set. $T : C \to X$ is said to be $\alpha$-averaged mapping if there exists some number $\alpha \in (0,1)$ such that

$$T = (1-\alpha)I + \alpha S$$

for some nonexpansive mapping $S$.

$(i)$ The composite of finitely many averaged mappings is averaged. In particular, if $T_1$ is $\alpha_1$-averaged and $T_2$ is $\alpha_2$-averaged, where $\alpha_1, \alpha_2 \in (0,1)$, then both $T_1T_2$ and $T_2T_1$ are $\alpha$-averaged, where $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$.

$(ii)$ If the mapping $\{T_i\}_{i=1}^N$ are averaged and have a common fixed point, then

$$\bigcap_{i=1}^{N} F(T_i) = F(T_1 \cdots T_N).$$

In particular, if $N = 2$, we have $F(T_1) \bigcap F(T_2) = F(T_1T_2) = F(T_2T_1)$.

**Lemma 2.5** ([21], p.63). Let $q > 1$. Then the following inequality holds:

$$ab \leq \frac{1}{q}a^q + \frac{q-1}{q}b^{\frac{q}{q-1}}$$

for arbitrary positive real numbers $a, b$.

**Lemma 2.6** ([22]). Let $C$ be a nonempty closed convex subset of a real q-uniformly smooth Banach space $X$. Let $V : C \to X$ be a $k$-Lipschitz and $\eta$-strongly accretive operator with constants $k, \eta > 0$. Let $0 < \mu < (\frac{q\eta}{c_q k^q})^{\frac{1}{q-1}}$ and $\tau = \mu(\eta - \frac{C_q \mu^{q-1} k^q}{q})$. Then for $t \in \left(0, \min\left\{1, \frac{1}{\tau}\right\}\right)$, the mapping $S : C \to X$ define by $S := (I - t\mu V)$ is a contraction with a constant $1 - t\tau$.

**Lemma 2.7** ([23]). Let $C$ be a nonempty closed convex subset of a real smooth Banach space $X$. Let $\tilde{C}$ be a nonempty subset of $C$. Let $Q_C : C \to \tilde{C}$ be a retraction, and let $j$, $j_q$ be the normalized duality mapping and generalized duality mapping on $X$, respectively. Then the following are equivalent:
$(i)$ $Q_C$ is sunny and nonexpensive;
$(ii)$ $\|Q_Cx - Q_Cy\|^2 \leq \langle x - y, j(Q_Cx - Q_Cy)\rangle, \quad \forall x, y \in C$;
$(iii)$ $\langle x - Q_Cx, j(y - Q_Cx)\rangle \leq 0, \quad \forall x \in C, y \in \tilde{C}$;
$(iv)$ $\langle x - Q_Cx, j_q(y - Q_Cx)\rangle \leq 0, \quad \forall x \in C, y \in \tilde{C}$.

## 3. Main results

Let $C$ be a nonempty, closed and convex subset of a real $q$-uniformly smooth Banach space $X$. Let $Q_C$ be a sunny nonexpansive retraction from $X$ onto $C$. Let $F : C \to X$ be a $L$-Lipschitz and $\eta$-strongly accretive operator, $V : C \to X$ be $\alpha$-Lipschitzian mapping and $T$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Let $0 < \mu < (\frac{q\eta}{c_q L^q})^{\frac{1}{q-1}}$, $0 < \gamma\alpha < \tau$ with $\tau = \mu(\eta - \frac{C_q \mu^{q-1} L^q}{q})$. For each $t \in \left(0, \min\left\{1, \frac{1}{\tau}\right\}\right)$, we define the mapping $S_t : C \to C$ by

$$S_t x := Q_C[t\gamma V x + (I - t\mu F)Tx], \forall x \in C.$$

It is easy to see that $S_t$ is a contraction. Indeed, in terms of Lemma 2.6, we have

$$\begin{aligned}\|S_t x - S_t y\| &= \|Q_C[t\gamma V x + (I - t\mu F)Tx] - Q_C[t\gamma V y + (I - t\mu F)Ty]\| \\ &\leq \|[t\gamma V x + (I - t\mu F)Tx] - [t\gamma V y + (I - t\mu F)Ty]\| \\ &\leq t\gamma \|Vx - Vy\| + \|(I - t\mu F)Tx - (I - t\mu F)Ty\| \\ &\leq t\gamma\alpha \|x - y\| + (1 - t\tau) \|x - y\| \\ &= (1 - t(\tau - \gamma\alpha)) \|x - y\|.\end{aligned}$$

Hence, $S_t$ has a unique fixed point, denoted by $x_t$, which uniquely solve the fixed point equation

$$x_t = Q_C[t\gamma V x_t + (I - t\mu F)Tx_t]. \tag{3.1}$$

**Lemma 3.1** Let $C$ be a nonempty, closed and convex subset of a real uniformly convex and $q$-uniformly smooth Banach space $X$ which admits a weakly sequentially continuous generalized duality mapping $j_q$ from $x$ into $X^*$. Let $Q_C$ be a sunny nonexpansive retraction. Let $F : C \to X$ be a $L$-Lipschitz and $\eta$-strongly accretive operator, $V : C \to X$ be $\alpha$-Lipschitzian mapping and $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Let $0 < \mu < (\frac{q\eta}{c_q L^q})^{\frac{1}{q-1}}$ and $0 < \gamma\alpha < \tau$ with $\tau = \mu(\eta - \frac{C_q \mu^{q-1} L^q}{q})$. For each $t \in \left(0, \min\left\{1, \frac{1}{\tau}\right\}\right)$, let $\{x_t\}$ defined by (3.1), then $\{x_t\}$ converges strongly to $x^* \in F(T)$ as $t \to 0$, which $x^*$ is the unique solution of the variational inequality

$$\langle (\gamma V - \mu F)x^*, j_q(x - x^*)\rangle \leq 0, \quad \forall x \in F(T). \tag{3.2}$$

**Proof.** We observe that

$$\begin{aligned}\frac{C_q \mu^{q-1} L^q}{q} > 0 \Leftrightarrow \eta - \frac{C_q \mu^{q-1} L^q}{q} &< \eta \\ \Leftrightarrow \mu\left(\eta - \frac{C_q \mu^{q-1} L^q}{q}\right) &< \mu\eta \\ \Leftrightarrow \tau &< \mu\eta.\end{aligned} \tag{3.3}$$

It follows that

$$0 < \gamma\alpha < \tau < \mu\eta. \tag{3.4}$$

First, we show the uniqueness of solution of the variational inequality (3.2). Suppose that $\tilde{x}$ , $x^*$ are solutions of (3.2), then

$$\langle (\gamma V - \mu F)x^*, j_q(x - x^*)\rangle \leq 0, \quad \forall x \in F(T). \tag{3.5}$$

and

$$\langle (\gamma V - \mu F)\tilde{x}, j_q(x - \tilde{x})\rangle \leq 0, \quad \forall x \in F(T). \tag{3.6}$$

Adding up (3.5) and (3.6), we have

$$\begin{aligned}0 &\geq \langle(\mu F-\gamma V)\tilde{x}-(\mu F-\gamma V)x^*, j_q(\tilde{x}-x^*)\rangle\\ &= \mu\langle F\tilde{x}-Fx^*, j_q(\tilde{x}-x^*)\rangle-\gamma\langle V\tilde{x}-Vx^*, j_q(\tilde{x}-x^*)\rangle\\ &\geq \mu\eta\|\tilde{x}-x^*\|^q-\gamma\alpha\|\tilde{x}-x^*\|^q\\ &\geq (\mu\eta-\gamma\alpha)\|\tilde{x}-x^*\|^q.\end{aligned}$$

Therefore, $\tilde{x}=x^*$ and the uniqueness is proved. Below we use $x^*$ to denote the unique solution of (3.2).

We observe that $\{x_t\}$ is bounded. Indeed, taking $\bar{x}\in F(T)$, then, we have

$$\begin{aligned}\|x_t-\bar{x}\| &\leq \|t\gamma Vx_t+(I-t\mu F)Tx_t-\bar{x}\|\\ &= \|t(\gamma Vx_t-\mu F\bar{x})+(I-t\mu F)Tx_t-(I-t\mu F)\bar{x}\|\\ &\leq t\gamma\|Vx_t-V\bar{x}\|+t\|\gamma V\bar{x}-\mu F\bar{x}\|+(1-t\tau)\|Tx_t-\bar{x}\|\\ &\leq (1-t(\tau-\gamma\alpha))\|x_t-\bar{x}\|+t\|\gamma V\bar{x}-\mu F\bar{x}\|.\end{aligned}$$

therefore, $\|x_t-\bar{x}\|\leq\frac{\|\gamma V\bar{x}-\mu F\bar{x}\|}{\tau-\gamma\alpha}$, so are $\{Vx_t\}$ and $\{FTx_t\}$.

By the definition of $\{x_t\}$, we have

$$\begin{aligned}\|x_t-Tx_t\| &= \|Q_C[t\gamma Vx_t+(I-t\mu F)Tx_t-Q_CTx_t]\|\\ &\leq \|[t\gamma Vx_t+(I-t\mu F)Tx_t-Tx_t]\|\\ &= t\|\gamma Vx_t-\mu FTx_t\|\to 0\ as\ t\to 0.\end{aligned}\tag{3.7}$$

Next, we prove that $x_t\to\tilde{x}$ as $t\to 0$.

Setting $y_t=t\gamma Vx_t+(I-t\mu F)Tx_t$, we obtain $x_t=Q_Cy_t$. Assume that $\{t_n\}\subset(0,1)$ is a sequence such that $t_n\to 0$ as $n\to\infty$. Since $\{x_t\}$ is bounded and $X$ is reflexive, there exists a subsequence $\{x_{t_n}\}$ of $\{x_t\}$ such that $x_{t_n}\rightharpoonup\tilde{x}$.

we claim $\|x_{t_n}-\tilde{x}\|\to 0$.

$$\begin{aligned}\|x_{t_m}-\tilde{x}\|^q &= \langle x_{t_m}-\tilde{x}, j_q(x_{t_m}-\tilde{x})\rangle\\ &= \langle Q_Cy_{t_m}-y_{t_m}+y_{t_m}-\tilde{x}, j_q(x_{t_m}-\tilde{x})\rangle\\ &\leq \langle t_m\gamma Vx_{t_m}+(I-t_m\mu F)Tx_{t_m}-\tilde{x}, j_q(x_{t_m}-\tilde{x})\rangle\\ &= \langle t_m(\gamma Vx_{t_m}-\mu F\tilde{x})+(I-t_m\mu F)Tx_{t_m}-(I-t_m\mu F)\tilde{x}, j_q(x_{t_m}-\tilde{x})\rangle\\ &= t_m\langle\gamma Vx_{t_m}-\mu F\tilde{x}, j_q(x_{t_m}-\tilde{x})\rangle+\langle(I-t_m\mu F)Tx_{t_m}-(I-t_m\mu F)\tilde{x}, j_q(x_{t_m}-\tilde{x})\rangle\\ &\leq t_m\langle\gamma Vx_{t_m}-\mu F\tilde{x}, j_q(x_{t_m}-\tilde{x})\rangle+(1-t_m\tau)\|x_{t_m}-\tilde{x}\|^q.\end{aligned}$$

Thus, we have

$$\begin{aligned}\|x_{t_m}-\tilde{x}\|^q &\leq \frac{1}{\tau}\langle\gamma Vx_{t_m}-\mu F\tilde{x}, j_q(x_{t_m}-\tilde{x})\rangle\\ &= \frac{1}{\tau}\langle\gamma(Vx_{t_m}-V\tilde{x})+(\gamma V\tilde{x}-\mu F\tilde{x}), j_q(x_{t_m}-\tilde{x})\rangle\\ &\leq \frac{1}{\tau}\gamma\alpha\|x_{t_m}-\tilde{x}\|^q+\frac{1}{\tau}\langle\gamma V\tilde{x}-\mu F\tilde{x}, j_q(x_{t_m}-\tilde{x})\rangle\end{aligned}$$

which implies that

$$\|x_{t_n} - \tilde{x}\|^q \leq \frac{1}{\tau - \gamma\alpha} \langle \gamma V\tilde{x} - \mu F\tilde{x}, j_q(x_{t_n} - \tilde{x})\rangle . \tag{3.8}$$

By (3.8), we get that $\|x_{t_n} - \tilde{x}\| \to 0$. From (3.7), (3.8) and Lemma 2.3, we have $\bar{x} \in F(T)$. Next, we show that $x^*$ solves the variational inequality (3.2). Since

$$x_t = Q_C y_t = Q_C y_t - y_t + y_t,$$

we derive that

$$(\mu F - \gamma V)x_t = \frac{1}{t}(Q_C y_t - y_t) - \frac{1}{t}(I - T)x_t + \mu(Fx_t - FTx_t). \tag{3.9}$$

Note that $I - T$ is accretive(i.e.$\langle (I - T)x - (I - T)y, j_q(x - y)\rangle \geq 0$, for all $x, y \in C$). For all $z \in F(T)$, it follows from (3.9) and Lemma 2.7 that

$$\begin{aligned}
\langle (\mu F - \gamma V)x_t, j_q(x_t - z)\rangle &= \frac{1}{t}\langle Q_C y_t - y_t, j_q(Q_C y_t - z)\rangle - \frac{1}{t}\langle (I - T)x_t - (I - T)z, j_q(x_t - z)\rangle \\
&\quad + \mu\langle Fx_t - FTx_t, j_q(x_t - z)\rangle \\
&\leq \mu\langle Fx_t - FTx_t, j_q(x_t - z)\rangle \\
&\leq \mu\|Fx_t - FTx_t\|\,\|x_t - z\|^{q-1} \\
&\leq \|x_t - Tx_t\|\, M
\end{aligned} \tag{3.10}$$

where $M = \sup\left\{\mu L\|x_t - z\|^{q-1}\right\}$, where $t \in \left(0, \min\left\{1, \frac{1}{\tau}\right\}\right)$.

Now replacing $t$ in (3.10) with $t_n$ and taking the limit as $n \to \infty$, we noticing that $x_{t_n} - Tx_{t_n} \to \bar{x} - T\bar{x} = 0$ for $\bar{x} \in F(T)$, we obtain $\langle (\mu F - \gamma V)\bar{x} - (\mu F - \gamma V)\bar{x}, j_q(\bar{x} - z)\rangle \leq 0$. That is, $\bar{x} \in F(T)$ is the solution of variational inequality (3.2). Consequently, $\bar{x} = x^*$ by uniqueness. Therefore $x_t \to x^*$ as $t \to 0$. This completes the proof. □

**Theorem 3.2** Let $C$ be a nonempty, closed and convex subset of a real uniform convex and $q$-uniformly smooth Banach space $X$ which admits a weakly sequentially continuous generalized duality mapping $j_q$ from $X$ into $X^*$. Let $Q_C$ be a sunny nonexpansive retraction. Let $F : C \to X$ be a $L$-Lipschitz and $\eta$-strongly accretive operator, $V : C \to X$ be $\alpha$-Lipschitzian mapping and $\{S_i\}_{i=1}^N : C \to C$ be a finite family of nonexpansive mappings such that $\Omega = \bigcap_{i=1}^{N} F(S_i) \neq \emptyset$. Define $T_i^k := (1 - \beta_k^i)I + \beta_k^i S_i$, where $\beta_k^i \in (a, b) \subset (0, 1)$. For arbitrarily given $x_0 \in X$ and $0 < \mu < (\frac{q\eta}{c_q L^q})^{\frac{1}{q-1}}$, let $\{x_k\}$ be the sequence generated iteratively by :

$$x_{k+1} = \beta_k x_k + (1 - \beta_k)Q_C[\alpha_k\gamma V x_k + (I - \mu\alpha_k F)T_N^k \cdots T_1^k x_k], \quad \forall k \geq 0 \tag{3.11}$$

where $0 < \gamma\alpha < \tau$ with $\tau = \mu(\eta - \frac{C_q\mu^{q-1}L^q}{q})$. Assume that $\{\alpha_k\} \subset (0, 1)$ and $\{\beta_k\} \subset [0, 1)$ satisfying the following conditions:

$(i)$ $\lim\limits_{k\to\infty} \alpha_k = 0$ and $\sum\limits_{k=1}^{\infty} \alpha_k = \infty$;

$(ii)$ $\sum\limits_{k=1}^{\infty} |\beta_{k+1} - \beta_k| < \infty$, $\limsup\limits_{n\to\infty} \beta_k < 1$ and $\sum\limits_{k=1}^{\infty} |\alpha_{k+1} - \alpha_k| < \infty$;

$(iii)$ $\sum\limits_{k=1}^{\infty} \left|\beta_{k+1}^i - \beta_k^i\right| < \infty$.

Then the sequence $\{x_k\}$ converges strongly to the unique solution $x^*$ of the variational inequality:

$$\langle(\gamma V - \mu F)x^*, j_q(x - x^*)\rangle \leq 0, \forall x \in \Omega. \tag{3.12}$$

**Proof.** Set $y_k = Q_C[\alpha_n \gamma V x_k + (I - \mu\alpha_k F)T_N^k \cdots T_1^k x_k]$, then

$$x_{k+1} = \beta_k x_k + (1 - \beta_k)y_k. \tag{3.13}$$

Since our methods easily deduce the general case, we prove theorem 3.2 for $N = 2$. The proof is divided into five steps.

**Step 1.** First we show that $\{x_k\}$ is bounded. Indeed, taking some $p \in \Omega$, from (3.13) and Lemma 2.6, we have

$$\begin{aligned}
&\|y_k - p\| = \left\|Q_C[\alpha_k\gamma V x_k + (I - \mu\alpha_k F)T_2^k T_1^k x_k] - Q_C(p)\right\| \\
&\leq \left\|\alpha_k\gamma V x_k + (I - \mu\alpha_k F)T_2^k T_1^k x_k - p\right\| \\
&= \left\|(I - \mu\alpha_k F)T_2^k T_1^k x_k - (I - \mu\alpha_k F)p + \alpha_k(\gamma V x_k - \gamma V p) + \alpha_k(\gamma V p - \mu F p)\right\| \\
&\leq \left\|(I - \mu\alpha_k F)T_2^k T_1^k x_k - (I - \mu\alpha_k F)p\right\| + \alpha_k \left\|\gamma V x_k - \gamma V p\right\| + \alpha_k \left\|\gamma V p - \mu F p\right\| \\
&\leq (1 - \alpha_k\tau) \left\|x_k - p\right\| + \alpha_k\gamma\alpha \left\|x_k - p\right\| + \alpha_k \left\|\gamma V p - \mu F p\right\| \\
&\leq [1 - \alpha_k(\tau - \gamma\alpha)] \left\|x_k - p\right\| + \alpha_k \left\|\gamma V p - \mu F p\right\|.
\end{aligned}$$

Also, it follows that

$$\begin{aligned}
&\|x_{k+1} - p\| = \|\beta_k x_k + (1 - \beta_k)y_k - p\| \\
&= \|\beta_k(x_k - p) + (1 - \beta_k)(y_k - p)\| \\
&\leq \beta_k \|x_k - p\| + (1 - \beta_k) \|y_k - p\| \\
&\leq \beta_k \|x_k - p\| + (1 - \beta_k)[1 - \alpha_k(\tau - \gamma\alpha)] \|x_k - p\| + (1 - \beta_k)\alpha_k \|\gamma V p - \mu F p\| \\
&= [1 - \alpha_k(1 - \beta_k)(\tau - \gamma\alpha)] \|x_k - p\| + \alpha_k(1 - \beta_k) \|\gamma V p - \mu F p\| \\
&= [1 - \alpha_k(1 - \beta_k)(\tau - \gamma\alpha)] \|x_k - p\| + \alpha_k(1 - \beta_k)(\tau - \gamma\alpha)\frac{\|\gamma V p - \mu F p\|}{\tau - \gamma\alpha}.
\end{aligned}$$

By induction, we have

$$\|x_{k+1} - p\| \leq \max\left\{\|x_0 - p\|, \frac{\|\gamma V p - \mu F p\|}{\tau - \gamma\alpha}\right\}, \forall k \geq 0.$$

Hence $\{x_k\}$ is bounded. So are $\left\{T_2^k T_1^k(x_k)\right\}, \left\{F T_2^k T_1^k(x_k)\right\}$ and $\{V(x_k)\}$.

**Step 2.** We claim that

$$\lim_{k\to\infty} \|x_{k+1} - x_k\| = 0.$$

We note that

$$\begin{aligned}
&\|y_{k+1}-y_k\| \\
&= \left\|Q_C[\alpha_{k+1}\gamma V x_{k+1}+(I-\mu\alpha_{k+1}F)T_2^{k+1}T_1^{k+1}x_{k+1}]-Q_C[\alpha_k\gamma V x_k+(I-\mu\alpha_k F)T_2^kT_1^kx_k]\right\| \\
&\le \left\|[\alpha_{k+1}\gamma V x_{k+1}+(I-\mu\alpha_{k+1}F)T_2^{k+1}T_1^{k+1}x_{k+1}]-[\alpha_k\gamma V x_k+(I-\mu\alpha_k F)T_2^kT_1^kx_k]\right\| \\
&\le \|\alpha_{k+1}\gamma(Vx_{k+1}-Vx_k)\|+\left\|(I-\mu\alpha_{k+1}F)T_2^{k+1}T_1^{k+1}x_{k+1}-(I-\mu\alpha_{k+1}F)T_2^kT_1^kx_k\right\| \\
&\quad+\left\|(\alpha_{k+1}-\alpha_k)(\gamma Vx_k-\mu FT_2^kT_1^kx_k)\right\| \\
&\le \alpha_{k+1}\gamma\alpha\|x_{k+1}-x_k\|+(1-\alpha_{k+1}\tau)[\|x_{k+1}-x_k\|+\left\|T_2^{k+1}T_1^kx_k-T_2^kT_1^kx_k\right\| \\
&\quad+\left\|T_1^{k+1}x_k-T_1^kx_k\right\|]+|\alpha_{k+1}-\alpha_k|\cdot M_1 \\
&= [1-\alpha_{k+1}(\tau-\gamma\alpha)]\|x_{k+1}-x_k\|+[(1-\alpha_{k+1}\tau)[\left\|T_2^{k+1}T_1^kx_k-T_2^kT_1^kx_k\right\| \\
&\quad+\left\|T_1^{k+1}x_k-T_1^kx_k\right\|]+|\alpha_{k+1}-\alpha_k|\cdot M_1 \qquad (3.14)
\end{aligned}$$

where $M_1$ is a fixed constant satisfying

$$M_1 \ge \sup_{k\ge1}\{\left\|\gamma Vx_k-\mu FT_2^kT_1^kx_k\right\|\}.$$

It follows from (3.13) and (3.14) that

$$\begin{aligned}
&\|x_{k+2}-x_{k+1}\| = \|\beta_{k+1}x_{k+1}+(1-\beta_{k+1})y_{k+1}-\beta_kx_k-(1-\beta_k)y_k\| \\
&= \|\beta_{k+1}(x_{k+1}-x_k)+(1-\beta_{k+1})(y_{k+1}-y_k)+(\beta_{k+1}-\beta_k)(x_k-y_k)\| \\
&\le \beta_{k+1}\|x_{k+1}-x_k\|+(1-\beta_{k+1})\|y_{k+1}-y_k\|+|\beta_{k+1}-\beta_k|\,\|x_k-y_k\| \\
&\le \beta_{k+1}\|x_{k+1}-x_k\|+(1-\beta_{k+1})[1-\alpha_{k+1}(\tau-\gamma\alpha)]\|x_{k+1}-x_k\| \\
&\quad+(1-\alpha_{k+1}\tau)(1-\beta_{k+1})[\left\|T_2^{k+1}T_1^kx_k-T_2^kT_1^kx_k\right\|+\left\|T_1^{k+1}x_k-T_1^kx_k\right\|] \\
&\quad+(1-\beta_{k+1})|\alpha_k-\alpha_{k-1}|\cdot M_1+|\beta_{k+1}-\beta_k|\cdot M_2 \\
&\le [1-\alpha_{k+1}(1-\beta_{k+1})(\tau-\gamma\alpha)]\|x_{k+1}-x_k\|+(1-\beta_{k+1})(1-\alpha_{k+1}\tau) \\
&\quad[\left\|T_2^{k+1}T_1^kx_k-T_2^kT_1^kx_k\right\|+\left\|T_1^{k+1}x_k-T_1^kx_k\right\|] \\
&\quad+(1-\beta_{k+1})|\alpha_k-\alpha_{k-1}|\cdot M_1+|\beta_{k+1}-\beta_k|\cdot M_2 \qquad (3.15)
\end{aligned}$$

where $M_2 \ge \sup_{k\ge1}\{\|x_k-y_k\|\}$, then (3.5) reduces to the following:

$$\|x_{k+2}-x_{k+1}\| \le (1-\gamma_k)\|x_{k+1}-x_k\|+\delta_k.$$

where $\gamma_k = \alpha_{k+1}(1-\beta_{k+1})(\tau-\gamma\alpha)$, $\delta_k = (1-\beta_{k+1})(1-\alpha_{k+1}\tau)[\left\|T_2^{k+1}T_1^kx_k-T_2^kT_1^kx_k\right\|+\left\|T_1^{k+1}x_k-T_1^kx_k\right\|]+(1-\beta_{k+1})|\alpha_k-\alpha_{k-1}|\cdot M_1+|\beta_{k+1}-\beta_k|\cdot M_2$. It is easily seen that

$$\sum_{k=1}^{\infty}\gamma_k=\infty.$$

Note that

$$\begin{aligned}
\left\|T_1^{k+1}x_k-T_1^kx_k\right\| &= \left\|(1-\beta_{k+1}^1)x_k+\beta_{k+1}^1S_1x_k-(1-\beta_k^1)x_k+\beta_k^1S_1x_k\right\| \\
&= \left\|(\beta_{k+1}^1-\beta_k^1)(S_1x_k-x_k)\right\| \\
&\le \left|\beta_{k+1}^1-\beta_k^1\right|(\|S_1x_k\|+\|x_k\|). \qquad (3.16)
\end{aligned}$$

Similarly,

$$\left\|T_2^{k+1}T_1^k x_k - T_2^k T_1^k x_k\right\| \leq \left|\beta_{k+1}^2 - \beta_k^2\right| \left(\left\|S_2 T_1^k x_k\right\| + \left\|T_1^k x_k\right\|\right). \tag{3.17}$$

From condition (iii), (3.16) and (3.17), we get that

$$\sum_{k=1}^{\infty} \left\|T_1^{k+1}x_k - T_1^k x_k\right\| < \infty, \sum_{k=1}^{\infty} \left\|T_2^{k+1}T_1^k x_k - T_2^k T_1^k x_k\right\| < \infty. \tag{3.18}$$

Then we have that $\sum\limits_{n=1}^{\infty} |\delta_k| < \infty$. Therefore, from Lemma 2.2, it follows that

$$\lim_{k\to\infty} \|x_{k+1} - x_k\| = 0. \tag{3.19}$$

**Step 3.** We show that

$$\lim_{k\to\infty} \left\|x_k - T_2^k T_1^k x_k\right\| = 0.$$

Since

$$\begin{aligned} \|x_k - y_k\| &\leq \|x_k - x_{k+1}\| + \|x_{k+1} - y_k\| \\ &\leq \|x_k - x_{k+1}\| + \beta_k \|x_k - y_k\|, \end{aligned}$$

that is,

$$\|x_k - y_k\| \leq \frac{1}{1-\beta_k} \|x_k - x_{k+1}\|.$$

It follows from (ii) and (3.19) that

$$\lim_{k\to\infty} \|x_k - y_k\| = 0. \tag{3.20}$$

By (i), we have

$$\begin{aligned} \left\|y_k - T_2^k T_1^k x_k\right\| &= \left\|P_C[\alpha_k \gamma V x_k + (I - \mu\alpha_k F)T_2^k T_1^k x_k] - P_C T_2^k T_1^k x_k\right\| \\ &\leq \left\|\alpha_k \gamma V x_k + (I - \mu\alpha_k F)T_2^k T_1^k x_k - T_2^k T_1^k x_k\right\| \\ &\leq \alpha_k \left\|\gamma V x_k - \mu F T_2^k T_1^k x_k\right\| \to 0 \quad as \quad k \to \infty. \end{aligned} \tag{3.21}$$

Moreover, we observe that

$$\left\|x_k - T_2^k T_1^k x_k\right\| \leq \|x_k - y_k\| + \left\|y_k - T_2^k T_1^k x_k\right\|, \tag{3.22}$$

it follows from (3.20), (3.21) and (3.22) that

$$\lim_{k\to\infty} \left\|x_k - T_2^k T_1^k x_k\right\| = 0. \tag{3.23}$$

**Step 4.** We will show that

$$\limsup_{k\to\infty} \langle (\gamma V - \mu F)x^*, j_q(x_k - x^*)\rangle \leq 0, \tag{3.24}$$

where $x^*$ is the unique solution of (3.12). To show this, we take a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ such that

$$\limsup_{k\to\infty} \langle (\gamma V - \mu F)x^*, j_q(x_k - x^*)\rangle = \lim_{j\to\infty} \left\langle (\gamma V - \mu F)x^*, j_q(x_{k_j} - x^*)\right\rangle.$$

Without loss of generality, we may further assume that $x_{k_j} \rightharpoonup \hat{x}$ as $j \to \infty$ due to reflexivity of the Banach space $X$ and boundness of $\{x_k\}$. Since $\{\beta_k^i\}$ is bounded for $i = 1, 2$, we can assume that $\beta_{k_j}^i \to \beta_\infty^i$ as $j \to \infty$ where $0 < a \leq \beta_\infty^i \leq b < 1$ for $i = 1, 2$. Define $T_i^\infty = (1 - \beta_\infty^i)I + \beta_\infty^i S_i (i = 1, 2)$. From Lemma 2.4, we have

$$F(T_1^\infty T_2^\infty) = F(T_1^\infty) \cap F(T_2^\infty) = F(S_1) \cap F(S_2).$$

Note that

$$\left\| T_i^{k_j} x - T_i^\infty x \right\| \leq \left| \beta_{k_j}^i - \beta_\infty^i \right| (\|S_i x\| + \|x\|).$$

Hence, we deduce that

$$\lim_{j \to \infty} \sup_{x \in D} \left\| T_i^{k_j} x - T_i^\infty x \right\| = 0, \tag{3.25}$$

where $D$ is an arbitrary bounded subset of $X$. Combine (3.23) and (3.25), we obtain

$$\begin{aligned}
&\left\| x_{k_j} - T_2^\infty T_1^\infty x_{k_j} \right\| \\
&\leq \left\| x_{k_j} - T_2^{k_j} T_1^{k_j} x_{k_j} \right\| + \left\| T_2^{k_j} T_1^{k_j} x_{k_j} - T_2^\infty T_1^{k_j} x_{k_j} \right\| + \left\| T_2^\infty T_1^{k_j} x_{k_j} - T_2^\infty T_1^\infty x_{k_j} \right\| \\
&\leq \left\| x_{k_j} - T_2^{k_j} T_1^{k_j} x_{k_j} \right\| + \sup_{x \in D'} \left\| T_2^{k_j} x - T_2^\infty x \right\| + \sup_{x \in D'} \left\| T_1^{k_j} x - T_1^\infty x \right\|,
\end{aligned}$$

where $D'$ is a bounded subset including $\{T_1^{k_j} x_{k_j}\}$ and $D''$ is a bounded subset including $\{x_{k_j}\}$. Hence $\lim_{j \to \infty} \left\| x_{k_j} - T_2^\infty T_1^\infty x_{k_j} \right\| = 0$. From Lemma 2.3, we have $\hat{x} \in F(T_1^\infty T_2^\infty)$. Since Banach space $X$ has a weakly sequentially continuous generalized duality mapping $j_q : X \to X^*$, we obtain that

$$\begin{aligned}
\limsup_{k \to \infty} \langle (\gamma V - \mu F) x^*, j_q(y_k - x^*) \rangle &= \limsup_{k \to \infty} \langle (\gamma V - \mu F) x^*, j_q(x_k - x^*) \rangle \\
&= \lim_{j \to \infty} \left\langle (\gamma V - \mu F) x^*, j_q(x_{k_j} - x^*) \right\rangle \\
&= \langle (\gamma V - \mu F) x^*, j_q(\hat{x} - x^*) \rangle \leq 0.
\end{aligned}$$

**Step 5.** Finally we show that $x_k \to x^*$ as $k \to \infty$.

Set $y_k = Q_C u_k$ for all $k \geq 0$, where $u_k = \alpha_k \gamma V x_k + (I - \mu \alpha_k F) T_2^k T_1^k x_k$. From Lemma 2.7, we have

$$\begin{aligned}
&\|y_k - x^*\|^q = \langle Q_C u_k - x^*, j_q(Q_C u_k - x^*) \rangle \\
&= \langle Q_C u_k - u_k, j_q(Q_C u_k - x^*) \rangle + \langle u_k - x^*, j_q(Q_C u_k - x^*) \rangle \\
&\leq \langle u_k - x^*, j_q(y_k - x^*) \rangle \\
&= \left\langle \alpha_k \gamma V x_k + (I - \mu \alpha_k F) T_2^k T_1^k x_k - x^*, j_q(y_k - x^*) \right\rangle \\
&= \alpha_k \gamma \langle V x_k - V x^*, j_q(y_k - x^*) \rangle + \alpha_k \langle \gamma V x^* - \mu F x^*, j_q(y_k - x^*) \rangle \\
&\quad + \left\langle (I - \mu \alpha_k F) T_2^k T_1^k x_k - (I - \mu \alpha_k F) x^*, j_q(y_k - x^*) \right\rangle \\
&\leq \alpha_k \alpha \gamma \|x_k - x^*\| \|y_k - x^*\|^{q-1} + \alpha_k \langle \gamma V x^* - \mu F x^*, j_q(y_k - x^*) \rangle \\
&\quad + (1 - \alpha_k \tau) \|x_k - x^*\| \|y_k - x^*\|^{q-1} \\
&= [1 - \alpha_k(\tau - \alpha\gamma)] \|x_k - x^*\| \|y_k - x^*\|^{q-1} + \alpha_k \langle \gamma V(x^*) - \mu F(x^*), j_q(y_k - x^*) \rangle \\
&= [1 - \alpha_k(\tau - \alpha\gamma)] (\frac{1}{q} \|x_k - x^*\|^q + \frac{q-1}{q} \|y_k - x^*\|^q) + \alpha_k \langle \gamma V(x^*) - \mu F(x^*), j_q(y_k - x^*) \rangle,
\end{aligned}$$

which implies that

$$\|y_k - x^*\|^q \leq [1 - \alpha_k(\tau - \alpha\gamma)]\|x_k - x^*\|^q + q\alpha_k \langle \gamma V(x^*) - \mu F(x^*), j_q(y_k - x^*)\rangle . \quad (3.26)$$

Again from (3.26) and Lemma 2.1, we obtain

$$\begin{aligned}
&\|x_{k+1} - x^*\|^q = \|\beta_k x_k + (1-\beta_k)y_k - x^*\|^q \\
&= \|\beta_k(x_k - x^*) + (1-\beta_k)(y_k - x^*)\|^q \\
&\leq \beta_k\|x_k - x^*\|^q + (1-\beta_k)\|y_k - x^*\|^q \\
&\leq \beta_k\|x_k - x^*\|^q + (1-\beta_k)[1 - \alpha_k(\tau - \alpha\gamma)]\|x_k - x^*\|^q + q\alpha_k(1-\beta_k)\langle \gamma V(x^*) - \mu F(x^*), j_q(y_k - x^*)\rangle \\
&\leq [1 - \alpha_k(1-\beta_k)(\tau - \alpha\gamma)]\|x_k - x^*\|^q + q\alpha_k(1-\beta_k)\langle \gamma V(x^*) - \mu F(x^*), j_q(y_k - x^*)\rangle , \quad (3.27)
\end{aligned}$$

Put $a_k = \alpha_k(1-\beta_k)(\tau - \alpha\gamma)$ and $\delta_k = q\alpha_k(1-\beta_k)\langle \gamma V(x^*) - \mu F(x^*), j_q(y_k - x^*)\rangle$, thus (3.27) reduce to the following:

$$\|x_{k+1} - x^*\|^q \leq (1 - a_k)\|x_k - x^*\|^q + \delta_k. \quad (3.28)$$

Apply Lemma 2.2 to (3.28), we obtain $x_k \to x^*$ as $k \to \infty$. This completes the proof. □

**Remark 3.3** Theorem 3.2 extend and generalize Theorem 3.2 of Pongsakorn et al. [17] in the following aspects:

(i) From a real Hilbert space to a real $q$-uniformly smooth Banach space which admits a weakly sequentially continuous generalized duality mapping.

(ii) From an invertible positive linear operator $F : C \to H$ to the Lipschitzian and strongly accretive operator $F : C \to X$.

(iii) From nonexpansive mapping $T : C \to C$ to a finite family of nonexpansive mappings.

**Remark 3.4** Compared with the known results in the literature, our results are very different from Theorem 3.1 of Zhang et al. [12] in the following aspects:

(i) Theorem 3.2 extends and improves Theorem 3.1 of Zhang et al. [12] from real Hilbert space to a real $q$-uniformly smooth Banach space which admits a weakly sequentially continuous generalized duality mapping.

(ii) We generalize the iteration process in theorem 3.1 of Zhang et al. [12].

## References


[1] W. R. Mann, Mean Value Methods in Iteration, Proc. Amer. Math. Soc., (1953), **4(3)**:506-506.

[2] G. Marino, H. K. Xu, Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces, J. Math. Anal. Appl., (2007), **329**:336-349.

[3] H. Y. Zhou, Nonexpansive mappings and iterative methods in uniformly convex Banach spaces, Acta Mathematica Sinica(English Series), (2004), **20**:829-836.

[4] H. Y. Zhou, Convergence theorems for $\lambda$-strict pseudo-contractions in 2-uniformly smooth Banach spaces, Nolinear Anal., (2008), **69**:3160-3173.

[5] H. Y. Zhou, Convergence theorems for $\lambda$-strict pseudocontractions in $q$-uniformly smooth Banach spaces, Acta Mathematica Sinica(English Series), (2010), **26**:743-758.

[6] H. Zhang, Y. F. Su, Strong convergence theorems for strict pseudo-contractions in q-uniformly smooth Banach spaces, Nolinear Anal., (2009), **70**:3236-3242.

[7] Q. L. Dong, S. N. He, F. Su, Strong convergence of an iterative algorithm for an infinite family of strict pseudo-contractions in Banach spaces, Appl. Math. Comput., (2010), **216**:959-969.

[8] Y. S. Song, Strong convergence of viscosity approximation methods with strong pseudocontraction for Lipschitz pseudocontractive mappings, Positivity, (2009), **13**:643-655.
[9] Y. H. Yao, M. A. Noor, K. I. Noor, Y.-C. Liou, H. Yaqoob, Modified extragradient methods for a system of variational inequalities in Banach spaces, Acta Appl. Math., (2010), **110**:1211-1224.
[10] L. C. Ceng, C. Wang, J.C. Yao, Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities, Math. Methods Oper. Res., (2008), **67**:375-390.
[11] M. Tian, A general iterative algorithm for nonexpansive mappings in Hilbert spaces, Nolinear Anal., (2010), **73**:689-694.
[12] H. K. Xu, Viscosity approximation methods for nonexpansive mappings, J. Math. Anal. Appl., (2004), **298**:279-291.
[13] Y. H. Wang, W. Xu, strong convergence of a modified iterative algorithm for hierarchial fixed point problems and variational inequalities, Fixed Point Theory Appl., (2013), **2013**:121.
[14] C. J. Zhang, C.P. Yang, A new explicit iterative algorithm for solving a class of variational inequalities over the common fixed points set of a finite family of nonexpansive mappings, Fixed Point Theory Appl., (2014), **2014**:60.
[15] H. K. Xu, Inequalities in Banach spaces with applications, Nonlinear Anal-Theor., (1991), **16**:1127-1138.
[16] L. C. Ceng, Q. H. Anasri, J. C. Yao, some iterative methods for finding fixed points and for solving constrained convex minimization problems, Nolinear Anal., (2011), **74**:5289-5302.
[17] P. Sunthrayuth, Y. J. Cho, P. Kumam, general iterative algorithms approach to variational inequalities and minimum-norm fixed point for minimization and split feasibility problems, Opsearch, (2014), **51**:400-415.
[18] R. E. B. Jr, Properties of fixed point sets of nonexpansive mappings in Banach spaces, Trans. Amer. Math. Soc. (1973), **179**:251-262.
[19] A. Y. Matsushita, W. Takahashi, Strong convergence theorems for nonexpansive nonself-mappings without boundary conditions, Nolinear Anal., (2008), **68**:412-419.
[20] G. López, V. Martin, H. K. Xu, Iterative algorithm for the multiple-sets split feasibility problem, Biomedical Math., (2009), 243-279.
[21] D. S. Mitrinović, Analytic Inequalities, Springer-Verlag, New York, (1970).
[22] K. Aoyama, Y. Kimura, W. Takahashi, M. Toyoda, Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space, Nonlinear Anal., (2007), **67**:2350-2360.
[23] J. L. Guan, C. S. Hu, X. wang, a new general iterative algorithm with Meir-Keeler contractions for variational inequality problems in $q$-uniformly smooth Banach spaces, Fixed Point Theory Appl., (2013), **2013**:292.
[24] T. Suzuki, Strong convergence of Krasnoselskii and Mann'type sequences for one parameter nonex-pansive semigroups without Bochner integral, J. Math. Anal. Appl., (2005), **35**:227-239.
[25] M. A. Noor, K. I. Noor, E. Al-Said, Iterative methods for solving general quasi-vatiational inequalities, Optim Lett., (2010), **4**:513-530.
[26] M. A. Noor, On a system of general mixed variational inequalities, Optim Lett., (2009), **3**:437-451.
[27] M. A. Noor, Projection methods for nonconvex variational inequalities, Optim Lett., (2009), **3**:411-418.
[28] Y. H. Yao, N. Shahzad, Strong convergence of a proximal point algorithm with general errors, Optim Lett., (2012), **6**:621-628.

# FOURIER SERIES OF FUNCTIONS ASSOCIATED WITH POLY-BERNOULLI POLYNOMIALS

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Abstract. In this paper, we consider three types of functions associated with poly-Bernoulli functions and derive their Fourier series expansions. In addition, we will express each of them in terms of Bernoulli functions.

## 1. Introduction

As is well known, the *Bernoulli polynomials* $B_m(x)$ are given by the generating function

$$\frac{t}{e^t-1}e^{xt}=\sum_{m=0}^{\infty}B_m(x)\frac{t^m}{m!},\quad (\text{see [4,14,16,19]}). \tag{1.1}$$

For any integer $r$, the poly-Bernoulli polynomials $\mathbf{B}_m^{(r)}(x)$ of index $r$ are given by the generating function

$$\frac{Li_r(1-e^{-t})}{e^t-1}e^{xt}=\sum_{m=0}^{\infty}\mathbf{B}_m^{(r)}(x)\frac{t^m}{m!},\quad (\text{see [2,3,8,10,13,20]}), \tag{1.2}$$

where $Li_r(x)=\sum_{m=1}^{\infty}\frac{x^m}{m^r}$ is the $r$th polylogarithmic function for $r\geq 1$, and a rational function for $r\leq 0$.

We observe here that

$$\frac{d}{dx}(Li_{r+1}(x))=\frac{1}{x}Li_r(x). \tag{1.3}$$

We need to note the following as to the poly-Bernoulli polynomials:

$$\frac{d}{dx}\mathbf{B}_m^{(r)}(x)=m\mathbf{B}_{m-1}^{(r)}(x),(m\geq 1), \tag{1.4}$$

$$\begin{aligned}&\mathbf{B}_m^{(1)}(x)=B_m(x),\mathbf{B}_0^{(r)}(x)=1,\mathbf{B}_m^{(0)}(x)=x^m,\\&\mathbf{B}_m^{(0)}=\delta_{m,0},\mathbf{B}_m^{(r+1)}(1)-\mathbf{B}_m^{(r+1)}(0)=\mathbf{B}_{m-1}^{(r)},\ (m\geq 1).\end{aligned} \tag{1.5}$$

For any real number $x$, let

$$<x>=x-[x]\in[0,1) \tag{1.6}$$

denote the fractional part of $x$.

Fourier series expansion of higher-order Bernoulli functions were treated in the recent paper [15]. Here we will study three types of functions associated with poly-Bernoulli functions and derive their Fourier series expansions. In addition, we will express each of them in terms of Bernoulli functions.

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(1) $\alpha_m(< x >) = \sum_{k=0}^{m} \mathbb{B}_k^{(r+1)}(< x >) < x >^{m-k},\ (m \geq 1)$;
(2) $\beta_m(< x >) = \sum_{k=0}^{m} \frac{1}{k!(m-k)!}\mathbb{B}_k^{(r+1)}(< x >) < x >^{m-k},\ (m \geq 1)$;
(3) $\gamma_m(< x >) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)}\mathbb{B}_k^{(r+1)}(< x >) < x >^{m-k},\ (m \geq 2)$.

For elementary facts about Fourier analysis, the reader may refer to any book (for example, see [1,17,21]).

As to $\gamma_m(< x >)$, we note that the next polynomial identity follows immediately from Theorems 4.1 and 4.2, which is in turn derived from the Fourier series expansion of $\gamma_m(< x >)$:

$$\sum_{k=1}^{m-1} \frac{1}{k(m-k)}\mathbf{B}_k^{(r+1)}(x)x^{m-k},$$
$$= \frac{1}{m}\sum_{s=0}^{m}\binom{m}{s}\Big(\Lambda_{m-s+1} + \frac{H_{m-1}-H_{m-s}}{m-s+1}(\mathbf{B}_{m-s}^{(r)}+1)\Big)B_s(x)$$

where $H_l = \sum_{j=1}^{l}\frac{1}{j}$ are the harmonic numbers and

$$\Lambda_l = \sum_{k=1}^{l-1}\frac{\mathbf{B}_k^{(r+1)}}{k(l-k)} + \sum_{k=1}^{l-1}\frac{\mathbf{B}_{k-1}^{(r)}}{k(l-k)}.$$

The obvious polynomial identities can be derived also for $\alpha_m(< x >)$ and $\beta_m(< x >)$ from Theorems 2.1 and 2.2, and Theorems 3.1 and 3.2, respectively. It is remarkable that from the Fourier series expansion of the function $\sum_{k=1}^{m-1}\frac{1}{k(m-k)}B_k(< x >)B_{m-k}(< x >)$ we can derive the Faber-Pandharipande-Zagier identity (see [6,9,11,12]) and the Miki's identity (see [5,7,9,11,12,18]).

## 2. The function $\alpha_m(< x >)$

For integers $r, m$ with $m \geq 1$, we let

$$\alpha_m(x) = \sum_{k=0}^{m}\mathbf{B}_k^{(r+1)}(x)x^{m-k}.$$

Now, we consider the function

$$\alpha_m(< x >) = \sum_{k=0}^{m}\mathbf{B}_k^{(r+1)}(< x >) < x >^{m-k},\ (m \geq 1), \tag{2.1}$$

defined on $(-\infty, \infty)$, which is periodic with period 1.

The Fourier series of $\alpha_m(< x >)$ is

$$\sum_{n=-\infty}^{\infty} A_n^{(m)}e^{2\pi inx},$$

where

$$\begin{aligned} A_n^{(m)} &= \int_0^1 \alpha_m(< x >)e^{-2\pi inx}dx \\ &= \int_0^1 \alpha_m(x)e^{-2\pi inx}dx. \end{aligned} \tag{2.2}$$

Before proceeding any further, we need to observe the following.

$$\begin{aligned} \alpha_m'(x) &= \sum_{k=0}^{m}\left(k\mathbf{B}_{k-1}^{(r+1)}(x)x^{m-k} + (m-k)\mathbf{B}_k^{(r+1)}(x)x^{m-k-1}\right) \\ &= \sum_{k=1}^{m} k\mathbf{B}_{k-1}^{(r+1)}(x)x^{m-k} + \sum_{k=0}^{m-1}(m-k)\mathbf{B}_k^{(r+1)}(x)x^{m-k-1} \\ &= \sum_{k=0}^{m-1}(k+1)\mathbf{B}_k^{(r+1)}(x)x^{m-1-k} + \sum_{k=0}^{m-1}(m-k)\mathbf{B}_k^{(r+1)}(x)x^{m-1-k} \\ &= (m+1)\sum_{k=0}^{m-1}\mathbf{B}_k^{(r+1)}(x)x^{m-1-k} \\ &= (m+1)\alpha_{m-1}(x). \end{aligned} \tag{2.3}$$

From this, we see that

$$\left(\frac{\alpha_{m+1}(x)}{m+2}\right)' = \alpha_m(x), \tag{2.4}$$

$$\int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\left(\alpha_{m+1}(1) - \alpha_{m+1}(0)\right). \tag{2.5}$$

For $m \geq 1$, we put

$$\begin{aligned} \Delta_m &= \alpha_m(1) - \alpha_m(0) \\ &= \sum_{k=0}^{m}\left(\mathbf{B}_k^{(r+1)}(1) - \mathbf{B}_k^{(r+1)}\delta_{m,k}\right) \\ &= \mathbf{B}_0^{(r+1)}(1) + \sum_{k=1}^{m}\mathbf{B}_k^{(r+1)}(1) - \sum_{k=0}^{m}\mathbf{B}_k^{(r+1)}\delta_{m,k} \\ &= 1 + \sum_{k=1}^{m}\left(\mathbf{B}_k^{(r+1)} + \mathbf{B}_{k-1}^{(r)}\right) - \mathbf{B}_m^{(r+1)} \\ &= \sum_{k=0}^{m-1}\mathbf{B}_k^{(r+1)} + \sum_{k=0}^{m-1}\mathbf{B}_k^{(r)}. \end{aligned} \tag{2.6}$$

Then $\alpha_m(1) = \alpha_m(0) \Longleftrightarrow \Delta_m = 0$, and

$$\int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\Delta_{m+1}.$$

We are now ready to determine the Fourier coefficients $A_n^{(m)}$.
$Case1 : n \neq 0$.

$$\begin{aligned}
A_n^{(m)} &= \int_0^1 \alpha_m(x) e^{-2\pi inx} dx \\
&= -\frac{1}{2\pi in}\left[\alpha_m(x)e^{-2\pi inx}\right]_0^1 + \frac{1}{2\pi in}\int_0^1 \alpha_m'(x)e^{-2\pi inx}dx \\
&= -\frac{1}{2\pi in}\left(\alpha_m(1)-\alpha_m(0)\right) + \frac{m+1}{2\pi in}\int_0^1 \alpha_{m-1}(x)e^{-2\pi inx}dx \\
&= \frac{m+1}{2\pi in}A_n^{(m-1)} - \frac{1}{2\pi in}\Delta_m \\
&= \frac{m+1}{2\pi in}\left(\frac{m}{2\pi in}A_n^{(m-2)} - \frac{1}{2\pi in}\Delta_{m-1}\right) - \frac{1}{2\pi in}\Delta_m \\
&= \frac{(m+1)_2}{(2\pi in)^2}A_n^{(m-2)} - \sum_{j=1}^{2}\frac{(m+1)_{j-1}}{(2\pi in)^j}\Delta_{m-j+1} \\
&= \cdots \\
&= \frac{(m+1)!}{(2\pi in)^m}A_n^{(0)} - \sum_{j=1}^{m}\frac{(m+1)_{j-1}}{(2\pi in)^j}\Delta_{m-j+1} \\
&= -\frac{1}{m+2}\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1},
\end{aligned} \tag{2.7}$$

where $A_n^{(0)} = \int_0^1 e^{-2\pi inx}dx = 0$.

$Case2 : n = 0$.

$$A_0^{(m)} = \int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\Delta_{m+1}. \tag{2.8}$$

Here we recall the following facts about Bernoulli functions $B_m(< x >)$ :
(a) for $m \geq 2$,

$$B_m(< x >) = -m! \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^{2\pi inx}}{(2\pi in)^m}. \tag{2.9}$$

(b) for $m = 1$,

$$-\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^{2\pi inx}}{2\pi in} = \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases} \tag{2.10}$$

$\alpha_m(< x >), (m \geq 1)$ is piecewise $C^\infty$. Moreover, $\alpha_m(< x >)$ is continuous for those positive integers $m$ with $\Delta_m = 0$ and discontinuous with jump discontinuities at integers for those positive integers $m$ with $\Delta_m \neq 0$.

Assume first that $m$ is a positive integer with $\Delta_m = 0$. Then $\alpha_m(1) = \alpha_m(0)$. Hence $\alpha_m(< x >)$ is piecewise $C^\infty$, and continuous. Thus the Fourier series of $\alpha_m(< x >)$ converges uniformly to $\alpha_m(< x >)$, and

$$\begin{aligned}\alpha_m(< x >) &= \frac{1}{m+2}\Delta_{m+1} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}\right)e^{2\pi inx} \\ &= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2}\sum_{j=1}^{m}\binom{m+2}{j}\Delta_{m-j+1} \\ &\times\left(-j!\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^j}\right) \\ &= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2}\sum_{j=2}^{m}\binom{m+2}{j}\Delta_{m-j+1}B_j(< x >) \\ &+ \Delta_m \times \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}\end{aligned} \tag{2.11}$$

Now, we can state our first result.

**Theorem 2.1.** *For each positive integer $l$, let*

$$\Delta_l = \sum_{k=0}^{l-1}\boldsymbol{B}_k^{(r+1)} + \sum_{k=0}^{l-1}\boldsymbol{B}_k^{(r)}.$$

*Assume that $\Delta_m = 0$, for a positive integer $m$. Then we have the following.*

(a) $\sum_{k=0}^{m}\boldsymbol{B}_k^{(r+1)}(< x >) < x >^{m-k}$ *has the Fourier series expansion*

$$\begin{aligned}&\sum_{k=0}^{m}\boldsymbol{B}_k^{(r+1)}(< x >) < x >^{m-k} \\ &= \frac{1}{m+2}\Delta_{m+1} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}\right)e^{2\pi inx},\end{aligned}$$

*for all $x \in (-\infty, \infty)$, where the convergence is uniform.*

(b) $$\begin{aligned}&\sum_{k=0}^{m}\boldsymbol{B}_k^{(r+1)}(< x >) < x >^{m-k} \\ &= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2}\sum_{j=2}^{m}\binom{m+2}{j}\Delta_{m-j+1}B_j(< x >),\end{aligned}$$

*for all $x \in (-\infty, \infty)$, where $B_j(< x >)$ is the Bernoulli function.*

Assume next that $m$ is a positive integer with $\Delta_m \neq 0$. Then $\alpha_m(1) \neq \alpha_m(0)$. Hence $\alpha_m(< x >)$ is piecewise $C^\infty$, and discontinuous with jump discontinuities at

integers. The Fourier series of $\alpha_m(< x >)$ converges pointwise to $\alpha_m(< x >)$, for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}\left(\alpha_m(0) + \alpha_m(1)\right) = \alpha_m(0) + \frac{1}{2}\Delta_m,$$

for $x \in \mathbb{Z}$.

Next, we can state the second result.

**Theorem 2.2.** *For each positive integer $l$, let*

$$\Delta_l = \sum_{k=0}^{l-1} \boldsymbol{B}_k^{(r+1)} + \sum_{k=0}^{l-1} \boldsymbol{B}_k^{(r)}.$$

*Assume that $\Delta_m \neq 0$, for a positive integer $m$. Then we have the following.*

$$\begin{aligned}
&(\mathrm{a})\frac{1}{m+2}\Delta_{m+1} \\
&\quad + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left( -\frac{1}{m+2} \sum_{j=1}^{m} \frac{(m+2)_j}{(2\pi i n)^j} \Delta_{m-j+1} \right) e^{2\pi i n x} \\
&\quad = \begin{cases} \sum_{k=0}^{m} \boldsymbol{B}_k^{(r+1)}(< x >) < x >^{m-k}, & for\ x \notin \mathbb{Z}, \\ \boldsymbol{B}_m^{(r+1)} + \frac{1}{2}\Delta_m, & for\ x \in \mathbb{Z}. \end{cases}
\end{aligned}$$

$$\begin{aligned}
&(\mathrm{b})\frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2}\sum_{j=1}^{m} \binom{m+2}{j} \Delta_{m-j+1} B_j(< x >) \\
&= \sum_{k=0}^{m} \boldsymbol{B}_k^{(r+1)}(< x >) < x >^{m-k},\ for\ x \notin \mathbb{Z};
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2}\sum_{j=2}^{m} \binom{m+2}{j} \Delta_{m-j+1} B_j(< x >) \\
&= \boldsymbol{B}_m^{(r+1)} + \frac{1}{2}\Delta_m,\ for\ x \in \mathbb{Z}.
\end{aligned}$$

## 3. The fuction $\beta_m(< x >)$

Let $\beta_m(x) = \sum_{k=0}^{m} \frac{1}{k!(m-k)!} \mathbf{B}_k^{(r+1)}(x) x^{m-k},\ (m \geq 1)$.
Before proceeding further, we need to observe the following.

$$\begin{aligned}\beta'_m(x) &= \sum_{k=0}^{m}\left\{\frac{k}{k!(m-k)!}\mathbf{B}_{k-1}^{(r+1)}(x)x^{m-k}(x)\right.\\ &\left.+\frac{m-k}{k!(m-k)!}\mathbf{B}_{k}^{(r+1)}(x)x^{m-k-1}\right\}\\ &= \sum_{k=1}^{m}\frac{1}{(k-1)!(m-k)!}\mathbf{B}_{k-1}^{(r+1)}x^{m-k}\\ &+\sum_{k=0}^{m-1}\frac{1}{k!(m-1-k)!}\mathbf{B}_{k}^{(r+1)}(x)x^{m-k-1}\\ &= 2\sum_{k=0}^{m-1}\frac{1}{k!(m-1-k)!}\mathbf{B}_{k}^{(r+1)}(x)x^{m-1-k}\\ &= 2\beta_{m-1}(x).\end{aligned} \tag{3.1}$$

From this, we obtain $\left(\frac{\beta_{m+1}(x)}{2}\right)' = \beta_m(x)$,
and

$$\int_0^1 \beta_m(x)dx = \frac{1}{2}\Big(\beta_{m+1}(1)-\beta_{m+1}(0)\Big). \tag{3.2}$$

For $m \geq 1$, we have

$$\begin{aligned}\Omega_m &= \beta_m(1)-\beta_m(0)\\ &= \sum_{k=0}^{m}\frac{1}{k!(m-k)!}\mathbf{B}_{k}^{(r+1)}(1)-\frac{1}{m!}\mathbf{B}_{m}^{(r+1)}\\ &= \frac{1}{m!}+\sum_{k=1}^{m}\frac{1}{k!(m-k)!}\mathbf{B}_{k}^{(r+1)}(1)-\frac{1}{m!}\mathbf{B}_{m}^{(r+1)}\\ &= \frac{1}{m!}+\sum_{k=1}^{m}\frac{1}{k!(m-k)!}(\mathbf{B}_{k}^{(r+1)}+\mathbf{B}_{k-1}^{(r)})-\frac{1}{m!}\mathbf{B}_{m}^{(r+1)}\\ &= \sum_{k=0}^{m-1}\frac{1}{k!(m-k)!}\mathbf{B}_{k}^{(r+1)}+\sum_{k=1}^{m}\frac{1}{k!(m-k)!}\mathbf{B}_{k-1}^{(r)}\end{aligned} \tag{3.3}$$

Then $\beta_m(1)=\beta_m(0) \Longleftrightarrow \Omega_m = 0$.
Also,

$$\int_0^1 \beta_m(x)dx = \frac{1}{2}\Omega_{m+1}.$$

Now, we are going to consider the function

$$\beta_m(< x >) = \sum_{k=0}^{m}\frac{1}{k!(m-k)!}\mathbf{B}_{k}^{(r+1)}(< x >) < x >^{m-k},\ (m \geq 1),$$

defined on $(-\infty,\infty)$, which is periodic with period 1.

The Fourier series of $\beta_m(< x >)$ is

$$\sum_{n=-\infty}^{\infty} B_n^{(m)} e^{2\pi inx},$$

where

$$B_n^{(m)} = \int_0^1 \beta_m(< x >) e^{-2\pi inx} dx = \int_0^1 \beta_m(x) e^{-2\pi inx} dx.$$

Next, we want to determine the Fourier coefficients $B_n^{(m)}$.

Case 1:$n \neq 0$.

$$\begin{aligned}
B_n^{(m)} &= \int_0^1 \beta_m(x) e^{-2\pi inx} dx \\
&= -\frac{1}{2\pi in}\Big[\beta_m(x) e^{-2\pi inx}\Big]_0^1 + \frac{1}{2\pi in}\int_0^1 \beta_m'(x) e^{-2\pi inx} dx \\
&= -\frac{1}{2\pi in}\Big(\beta_m(1) - \beta_m(0)\Big) + \frac{1}{\pi in}\int_0^1 \beta_{m-1}(x) e^{-2\pi inx} dx \\
&= \frac{2}{2\pi in} B_n^{(m-1)} - \frac{1}{2\pi in}\Omega_m \\
&= \frac{2}{2\pi in}\Big(\frac{2}{2\pi in} B_n^{(m-2)} - \frac{1}{2\pi in}\Omega_{m-1}\Big) - \frac{1}{2\pi in}\Omega_m \\
&= \Big(\frac{2}{2\pi in}\Big)^2 B_n^{(m-2)} - \sum_{j=1}^{2} \frac{2^{j-1}}{(2\pi in)^j}\Omega_{m-j+1} \\
&= \cdots \\
&= \Big(\frac{2}{2\pi in}\Big)^m B_n^{(0)} - \sum_{j=1}^{m} \frac{2^{j-1}}{(2\pi in)^j}\Omega_{m-j+1} \\
&= -\sum_{j=1}^{m} \frac{2^{j-1}}{(2\pi in)^j}\Omega_{m-j+1},
\end{aligned} \tag{3.4}$$

where $B_n^{(0)} = \int_0^1 e^{-2\pi inx} dx = 0$.

Case 2: $n = 0$.

$$B_0^{(m)} = \int_0^1 \beta_m(x) dx = \frac{1}{2}\Omega_{m+1}. \tag{3.5}$$

$\beta_m(< x >)$, $(m \geq 1)$ is piecewise $C^\infty$. Moreover, $\beta_m(< x >)$ is continuous for those positive integers $m$ with $\Omega_m = 0$ and discontinuous with jump discontinuities at integers for those positive integers $m$ with $\Omega_m \neq 0$.

Assume first that $m$ is a positive integer with $\Omega_m = 0$. Then $\beta_m(1) = \beta_m(0)$. Hence $\beta_m(< x >)$ is piecewise $C^\infty$, and continuous. Thus the Fourier series of $\beta_m(< x >)$ converges uniformly to $\beta_m(< x >)$, and

$$\begin{aligned}
&\beta_m(< x >)\\
&= \frac{1}{2}\Omega_{m+1} + \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \Big(-\sum_{j=1}^{m} \frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1}\Big)e^{2\pi i n x}\\
&= \frac{1}{2}\Omega_{m+1} + \sum_{j=1}^{m} \frac{2^{j-1}}{j!}\Omega_{m-j+1} \times \Big(-j! \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^j}\Big)\\
&= \frac{1}{2}\Omega_{m+1} + \sum_{j=2}^{m} \frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(< x >)\\
&+ \Omega_m \times \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z},\\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}
\end{aligned} \tag{3.6}$$

Now, we can state our first result.

**Theorem 3.1.** *For each positive integer $l$, let*

$$\Omega_l = \sum_{k=0}^{l-1} \frac{1}{k!(l-k)!}\boldsymbol{B}_k^{(r+1)} + \sum_{k=1}^{l} \frac{1}{k!(l-k)!}\boldsymbol{B}_{k-1}^{(r)}.$$

*Assume that $\Omega_m = 0$, for a positive integer $m$. Then we have the following.*
(a) $\sum_{k=0}^{m} \frac{1}{k!(m-k)!}\boldsymbol{B}_k^{(r+1)}(< x >) < x >^{m-k}$ *has the Fourier series expansion*

$$\begin{aligned}
&\sum_{k=0}^{m} \frac{1}{k!(m-k)!}\boldsymbol{B}_k^{(r+1)}(< x >) < x >^{m-k}\\
&= \frac{1}{2}\Omega_{m+1} - \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \Big(-\sum_{j=1}^{m} \frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1}\Big)e^{2\pi i n x},
\end{aligned} \tag{3.7}$$

*for all $x \in (-\infty, \infty)$, where the convergence is uniform.*

(b) $\sum_{k=0}^{m} \frac{1}{k!(m-k)!}\boldsymbol{B}_k^{(r+1)}(< x >) < x >^{m-k} = \frac{1}{2}\Omega_{m+1} + \sum_{j=2}^{m} \frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(< x >)$,

*for all $x \in (-\infty, \infty)$, where $B_j(< x >)$ is the Bernoulli function.*

Assume next that $m$ is a positive integer with $\Omega_m \neq 0$. Then, $\beta_m(1) \neq \beta_m(0)$. Hence $\beta_m(< x >)$ is piecewise $C^\infty$ and discontinuous with jump discontinuities at integers. Thus the Fourier series of $\beta_m(< x >)$ converges pointwise to $\beta_m(< x >)$, for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}(\beta_m(0) + \beta_m(1)) = \beta_m(0) + \frac{1}{2}\Omega_m = \frac{1}{m!}\mathbf{B}_m^{(r+1)} + \frac{1}{2}\Omega_m, \tag{3.8}$$

for $x \in \mathbb{Z}$.

Now, we can state our second result.

**Theorem 3.2.** *For each positive integer $l$, let*

$$\Omega_l = \sum_{k=0}^{l-1} \frac{1}{k!(l-k)!}\boldsymbol{B}_k^{(r+1)} + \sum_{k=1}^{l} \frac{1}{k!(l-k)!}\boldsymbol{B}_{k-1}^{(r)}.$$

*Assume that $\Omega_m \neq 0$, for a positive integer $m$. Then we have the following.*

(a)
$$\frac{1}{2}\Omega_{m+1} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \Big(-\sum_{j=1}^{m} \frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1}\Big)e^{2\pi i n x}$$
$$= \begin{cases} \sum_{k=0}^{m} \frac{1}{k!(m-k)!}\boldsymbol{B}_k^{(r+1)}(< x >) < x >^{m-k}, & \text{for } x \notin \mathbb{Z}, \\ \frac{1}{m!}\boldsymbol{B}_m^{(r+1)} + \frac{1}{2}\Omega_m, & \text{for } x \in \mathbb{Z}. \end{cases}$$

(b)
$$\frac{1}{2}\Omega_{m+1} + \sum_{j=1}^{m} \frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(< x >)$$
$$= \sum_{k=0}^{m} \frac{1}{k!(m-k)!}\boldsymbol{B}_k^{(r+1)}(< x >) < x >^{m-k}, \quad \text{for } x \notin \mathbb{Z};$$
$$\frac{1}{2}\Omega_{m+1} + \sum_{j=2}^{m} \frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(< x >)$$
$$= \frac{1}{m!}\boldsymbol{B}_m^{(r+1)} + \frac{1}{2}\Omega_m, \qquad \text{for } x \in \mathbb{Z}.$$

## 4. The function $\gamma_m(< x >)$

Let $\gamma_m(x) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)}\mathbf{B}_k^{(r+1)}(x)x^{m-k}$, $(m \geq 2)$.

$$\gamma_m'(x) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)}\Big(k\mathbf{B}_{k-1}^{(r+1)}(x)x^{m-k} + (m-k)\mathbf{B}_k^{(r+1)}(x)x^{m-k-1}\Big)$$
$$= \sum_{k=0}^{m-2} \frac{1}{m-1-k}\mathbf{B}_k^{(r+1)}(x)x^{m-1-k} + \sum_{k=1}^{m-1} \frac{1}{k}\mathbf{B}_k^{(r+1)}(x)x^{m-1-k}$$
$$= \frac{1}{m-1}x^{m-1} + \sum_{k=1}^{m-2} \frac{1}{m-1-k}\mathbf{B}_k^{(r+1)}(x)x^{m-1-k} + \frac{1}{m-1}\mathbf{B}_{m-1}^{(r+1)}(x)$$
$$+ \sum_{k=1}^{m-2} \frac{1}{k}\mathbf{B}_k^{(r+1)}(x)x^{m-1-k}$$
$$= \frac{1}{m-1}\Big(x^{m-1} + \mathbf{B}_{m-1}^{(r+1)}(x)\Big) + (m-1)\sum_{k=1}^{m-2} \frac{1}{k(m-1-k)}\mathbf{B}_k^{(r+1)}(x)x^{m-1-k}$$
$$= \frac{1}{m-1}\Big(x^{m-1} + \mathbf{B}_{m-1}^{(r+1)}(x)\Big) + (m-1)\gamma_{m-1}(x). \tag{4.1}$$

Thus,
$$\gamma_m'(x) = \frac{1}{m-1}\Big(x^{m-1} + \mathbf{B}_{m-1}^{(r+1)}(x)\Big) + (m-1)\gamma_{m-1}(x).$$

From this, we obtain

$$\left(\frac{1}{m}\Big(\gamma_{m+1}(x)-\frac{1}{m(m+1)}\mathbf{B}_{m+1}^{(r+1)}(x)-\frac{1}{m(m+1)}x^{m+1}\Big)\right)'=\gamma_m(x).$$

$$\begin{aligned}
&\int_0^1\gamma_m(x)dx\\
&=\frac{1}{m}\Big[\gamma_{m+1}(x)-\frac{1}{m(m+1)}\mathbf{B}_{m+1}^{(r+1)}(x)-\frac{1}{m(m+1)}x^{m+1}\Big]_0^1\\
&=\frac{1}{m}\Big(\gamma_{m+1}(1)-\gamma_{m+1}(0)-\frac{1}{m(m+1)}\Big(\mathbf{B}_{m+1}^{(r+1)}(1)-\mathbf{B}_{m+1}^{(r+1)}(0)\Big)\\
&-\frac{1}{m(m+1)}\Big)\\
&=\frac{1}{m}\Big(\gamma_{m+1}(1)-\gamma_{m+1}(0)-\frac{1}{m(m+1)}\mathbf{B}_m^{(r)}-\frac{1}{m(m+1)}\Big).
\end{aligned}\tag{4.2}$$

For $m\geq 2$, we let

$$\begin{aligned}
\Lambda_m=\Lambda_m(r)&=\gamma_m(1)-\gamma_m(0)\\
&=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbf{B}_k^{(r+1)}(1)\\
&=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\Big(\mathbf{B}_k^{(r+1)}+\mathbf{B}_{k-1}^{(r)}\Big)\\
&=\sum_{k=1}^{m-1}\frac{\mathbf{B}_k^{(r+1)}}{k(m-k)}+\sum_{k=1}^{m-1}\frac{\mathbf{B}_{k-1}^{(r)}}{k(m-k)}.
\end{aligned}\tag{4.3}$$

Then,

$$\gamma_m(1)=\gamma_m(0)\ \Leftrightarrow\ \Lambda_m=0,\tag{4.4}$$

and

$$\int_0^1\gamma_m(x)dx=\frac{1}{m}\left(\Lambda_{m+1}-\frac{1}{m(m+1)}\mathbf{B}_m^{(r)}-\frac{1}{m(m+1)}\right).\tag{4.5}$$

We are going to consider

$$\gamma_m(<x>)=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbf{B}_k^{(r+1)}(<x>)<x>^{m-k},\tag{4.6}$$

defined on $(-\infty,\infty)$, which is periodic with period 1.

The Fourier series of $\gamma_m(<x>)$ is

$$\sum_{n=-\infty}^{\infty}C_n^{(m)}e^{2\pi inx},\tag{4.7}$$

where

$$C_n^{(m)}=\int_0^1\gamma_m(<x>)e^{-2\pi inx}dx=\int_0^1\gamma_m(x)e^{-2\pi inx}dx.\tag{4.8}$$

Now, we want to determine the Fourier coefficients $C_n^{(m)}$.
Case 1: $n \neq 0$. We can easily show that, for $l \geq 1$,

$$\begin{aligned} &\int_0^1 \mathbf{B}_l^{(r+1)}(x)e^{-2\pi inx}dx \\ &= \begin{cases} -\sum_{k=1}^{l} \frac{(l)_{k-1}}{(2\pi in)^k}\mathbf{B}_{l-k}^{(r)}(< x >), & \text{for } n \neq 0, \\ \frac{1}{l+1}\mathbf{B}_l^{(r)}, & \text{for } n = 0, \end{cases} \end{aligned} \tag{4.9}$$

$$\begin{aligned} &\int_0^1 x^l e^{-2\pi inx}dx \\ &= \begin{cases} -\sum_{k=1}^{l} \frac{(l)_{k-1}}{(2\pi in)^k}, & \text{for } n \neq 0, \\ \frac{1}{l+1}, & \text{for } n = 0. \end{cases} \end{aligned} \tag{4.10}$$

By using (4.9) and (4.10), we can obtain the following recursive relation.

$$\begin{aligned} C_n^{(m)} &= \int_0^1 \gamma_m(x)e^{-2\pi inx}dx \\ &= -\frac{1}{2\pi in}\Big[\gamma_m(x)e^{-2\pi inx}\Big]_0^1 + \frac{1}{2\pi in}\int_0^1 \gamma_m'(x)e^{-2\pi inx}dx \\ &= -\frac{1}{2\pi in}\Big(\gamma_m(1) - \gamma_m(0)\Big) \\ &+ \frac{1}{2\pi in}\int_0^1 \Big((m-1)\gamma_{m-1}(x) + \frac{1}{m-1}(x^{m-1} + \mathbf{B}_{m-1}^{(r+1)}(x))\Big)e^{-2\pi inx}dx \\ &= -\frac{1}{2\pi in}\Lambda_m + \frac{m-1}{2\pi in}C_n^{(m-1)} + \frac{1}{2\pi in(m-1)}\int_0^1 \mathbf{B}_{m-1}^{(r+1)}(x)e^{-2\pi inx}dx \\ &+ \frac{1}{2\pi in(m-1)}\int_0^1 x^{m-1}e^{-2\pi inx}dx \\ &= \frac{m-1}{2\pi in}C_n^{(m-1)} - \frac{1}{2\pi in}\Lambda_m - \frac{1}{2\pi in(m-1)}\Theta_m - \frac{1}{2\pi in(m-1)}\Phi_m, \end{aligned} \tag{4.11}$$

where, for $m \geq 2$,

$$\begin{aligned} \Lambda_m &= \gamma_m(1) - \gamma_m(0) = \sum_{k=1}^{m-1}\frac{\mathbf{B}_k^{(r+1)}}{k(m-k)} + \sum_{k=1}^{m-1}\frac{\mathbf{B}_{k-1}^{(r)}}{k(m-k)}, \\ \Theta_m &= \sum_{k=1}^{m-1}\frac{(m-1)_{k-1}}{(2\pi in)^k}\mathbf{B}_{m-k-1}^{(r)}, \\ \Phi_m &= \sum_{k=1}^{m-1}\frac{(m-1)_{k-1}}{(2\pi in)^k}. \end{aligned} \tag{4.12}$$

From the relation (4.11), we can get an expression for $C_n^{(m)}$.

$$
\begin{aligned}
C_n^{(m)} &= \frac{m-1}{2\pi in}C_n^{(m-1)} - \frac{1}{2\pi in}\Lambda_m - \frac{1}{2\pi in(m-1)}\Phi_m - \frac{1}{2\pi in(m-1)}\Theta_m \\
&= \frac{m-1}{2\pi in}\Big(\frac{m-2}{2\pi in}C_n^{(m-2)} - \frac{1}{2\pi in}\Lambda_{m-1} - \frac{1}{2\pi in(m-2)}\Phi_{m-1} - \frac{1}{2\pi in(m-2)}\Theta_{m-1}\Big) \\
&- \frac{1}{2\pi in}\Lambda_m - \frac{1}{2\pi in(m-1)}\Phi_m - \frac{1}{2\pi in(m-1)}\Theta_m \\
&= \frac{(m-1)_2}{(2\pi in)^2}C_n^{(m-2)} - \sum_{j=1}^{2}\frac{(m-1)_{j-1}}{(2\pi in)^j}\Lambda_{m-j+1} - \sum_{j=1}^{2}\frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Phi_{m-j+1} \\
&- \sum_{j=1}^{2}\frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Theta_{m-j+1} \\
&= \cdots \\
&= \frac{(m-1)!}{(2\pi in)^{m-1}}C_n^{(1)} - \sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j}\Lambda_{m-j+1} - \sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Phi_{m-j+1} \\
&- \sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Theta_{m-j+1},
\end{aligned}
\tag{4.13}
$$

where

$$
C_n^{(1)} = \int_0^1 \gamma_1(x)e^{-2\pi inx}dx = 0. \tag{4.14}
$$

Before proceeding further, we note the following:

$$
\begin{aligned}
&\sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Theta_{m-j+1} \\
=&\sum_{j=1}^{m-1}\frac{1}{m-j}\sum_{k=1}^{m-j}\frac{(m-1)_{j+k-2}}{(2\pi in)^{j+k}}\mathbf{B}_{m-j-k}^{(r)} \\
=&\sum_{j=1}^{m-1}\frac{1}{m-j}\sum_{s=j+1}^{m}\frac{(m-1)_{s-2}}{(2\pi in)^s}\mathbf{B}_{m-s}^{(r)} \\
=&\frac{1}{m}\sum_{s=1}^{m}\frac{(m)_s}{(2\pi in)^s}\frac{H_{m-1}-H_{m-s}}{m-s+1}\mathbf{B}_{m-s}^{(r)}.
\end{aligned}
\tag{4.15}
$$

Putting everything altogether, we obtain

$$
C_n^{(m)} = -\frac{1}{m}\sum_{s=1}^{m}\frac{(m)_s}{(2\pi in)^s}\Big(\Lambda_{m-s+1} \quad + \frac{H_{m-1}-H_{m-s}}{m-s+1}(\mathbf{B}_{m-s}^{(r)}+1)\Big). \tag{4.16}
$$

Case 2: $n = 0$.

$$
C_0^{(m)} = \int_0^1 \gamma_m(x)dx = \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)}\mathbf{B}_m^{(r)} - \frac{1}{m(m+1)}\right). \tag{4.17}
$$

$\gamma_m(< x >),\ (m \geq 2)$ is piecewise $C^\infty$. Moreover, $\gamma_m(< x >)$ is continuous for those positive integers $m \geq 2$ with $\Lambda_m = 0$ and discontinuous with jump discontinuities at integers for those positive integers $\geq 2$ with $\Lambda_m \neq 0$.

Assume first that $\Lambda_m = 0$. Then $\gamma_m(1) = \gamma_m(0)$. Hence $\gamma_m(< x >)$ is piecewise $C^\infty$, and continuous. Thus the Fourier series of $\gamma_m(< x >)$ converges uniformly to $\gamma_m(< x >)$, and

$$
\begin{aligned}
&\gamma_m(< x >) \\
&= \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)}\mathbf{B}_m^{(r)} - \frac{1}{m(m+1)}\right) \\
&- \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\Big(\frac{1}{m}\sum_{s=1}^{m}\frac{(m)_s}{(2\pi i n)^s}\Big(\Lambda_{m-s+1} + \frac{H_{m-1}-H_{m-s}}{m-s+1}(\mathbf{B}_{m-s}^{(r)}+1)\Big)\Big)e^{2\pi i n x} \\
&= \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)}\mathbf{B}_m^{(r)} - \frac{1}{m(m+1)}\right) \\
&+ \frac{1}{m}\sum_{s=1}^{m}\binom{m}{s}\Big(\Lambda_{m-s+1} + \frac{H_{m-1}-H_{m-s}}{m-s+1}(\mathbf{B}_{m-s}^{(r)}+1)\Big) \\
&\times \Big(-s!\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\frac{e^{2\pi i n x}}{(2\pi i n)^s}\Big) \\
&= \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)}\mathbf{B}_m^{(r)} - \frac{1}{m(m+1)}\right) \\
&+ \frac{1}{m}\sum_{s=2}^{m}\binom{m}{s}\Big(\Lambda_{m-s+1} + \frac{H_{m-1}-H_{m-s}}{m-s+1}(\mathbf{B}_{m-s}^{(r)}+1)\Big)B_s(< x >) \\
&+ \Lambda_m \times \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z} \end{cases} \\
&= \frac{1}{m}\sum_{\substack{s=0 \\ s\neq 1}}^{m}\binom{m}{s}\Big(\Lambda_{m-s+1} + \frac{H_{m-1}-H_{m-s}}{m-s+1}(\mathbf{B}_{m-s}^{(r)}+1)\Big)B_s(< x >) \\
&+ \Lambda_m \times \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}
\end{aligned}
\tag{4.18}
$$

Now, we are ready to state our first result.

**Theorem 4.1.** *For each integer $l \geq 2$, let*

$$\Lambda_l = \sum_{k=1}^{l-1}\frac{\boldsymbol{B}_k^{(r+1)}}{k(l-k)} + \sum_{k=1}^{l-1}\frac{\boldsymbol{B}_{k-1}^{(r)}}{k(l-k)},$$

*with $\Lambda_1 = 0$.*

*Assume that $\Lambda_m = 0$, for an integer $m \geq 2$. Then we have the following.*
(a) $\sum_{k=1}^{m-1} \frac{1}{k(m-k)} \boldsymbol{B}_k^{(r+1)}(< x >) < x >^{m-k}$ *has the Fourier series expansion*

$$\begin{aligned}
&\sum_{k=1}^{m-1} \frac{1}{k(m-k)} \boldsymbol{B}_k^{(r+1)}(< x >) < x >^{m-k} \\
&= \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)}\boldsymbol{B}_m^{(r)} - \frac{1}{m(m+1)}\right) \\
&- \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \Big(\frac{1}{m}\sum_{s=1}^{m} \frac{(m)_s}{(2\pi i n)^s}\Big(\Lambda_{m-s+1} + \frac{H_{m-1} - H_{m-s}}{m-s+1}(\boldsymbol{B}_{m-s}^{(r)} + 1)\Big)\Big) e^{2\pi i n x},
\end{aligned}$$

*for all $x \in (-\infty, \infty)$, where the convergence is uniform.*

(b)

$$\begin{aligned}
&\sum_{k=1}^{m-1} \frac{1}{k(m-k)} \boldsymbol{B}_k^{(r+1)}(< x >) < x >^{m-k} \\
&= \frac{1}{m}\sum_{\substack{s=0 \\ s\neq 1}}^{m} \binom{m}{s}\Big(\Lambda_{m-s+1} + \frac{H_{m-1} - H_{m-s}}{m-s+1}(\boldsymbol{B}_{m-s}^{(r)} + 1)\Big) B_s(< x >),
\end{aligned}$$

*for all $x \in (-\infty, \infty)$, where $B_k(< x >)$ is the Bernoulli function.*

Assume next that $m$ is a positive integer $\geq 2$, with $\Lambda_m \neq 0$. Then, $\gamma_m(1) \neq \gamma_m(0)$. Hence $\gamma_m(< x >)$ is piecewise $C^\infty$, and discontinuous with jump discontinuities at integers. Thus the Fourier series of $\gamma_m(< x >)$ convergence pointwise to $\gamma_m(< x >)$, for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}(\gamma_m(0) + \gamma_m(1)) = \gamma_m(0) + \frac{1}{2}\Lambda_m = \frac{1}{2}\Lambda_m,$$

for $x \in \mathbb{Z}$.

Next, we can state the second result.

**Theorem 4.2.** *For each integer $l \geq 2$, let*

$$\Lambda_l = \sum_{k=1}^{l-1} \frac{\boldsymbol{B}_k^{(r+1)}}{k(l-k)} + \sum_{k=1}^{l-1} \frac{\boldsymbol{B}_{k-1}^{(r)}}{k(l-k)},$$

*with $\Lambda_1 = 0$.*

*Assume that $\Lambda_m \neq 0$, for the an integer $m \geq 2$. Then we have the following.*
(a)

$$\begin{aligned}
&\frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)}\boldsymbol{B}_m^{(r)} - \frac{1}{m(m+1)}\right) \\
&- \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \Big(\frac{1}{m}\sum_{s=1}^{m} \frac{(m)_s}{(2\pi i n)^s}\Big(\Lambda_{m-s+1} + \frac{H_{m-1} - H_{m-s}}{m-s+1}(\boldsymbol{B}_{m-s}^{(r)} + 1)\Big)\Big) e^{2\pi i n x} \\
&= \begin{cases} \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \boldsymbol{B}_k^{(r+1)}(< x >) < x >^{m-k}, & \text{for } x \notin \mathbb{Z}, \\ \frac{1}{2}\Lambda_m, & \text{for } x \in \mathbb{Z}. \end{cases}
\end{aligned}$$

(b)

$$\frac{1}{m}\sum_{s=0}^{m}\binom{m}{s}\Big(\Lambda_{m-s+1}+\frac{H_{m-1}-H_{m-s}}{m-s+1}(\boldsymbol{B}_{m-s}^{(r)}+1)\Big)B_s(<x>)$$
$$=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\boldsymbol{B}_k^{(r+1)}(<x>)<x>^{m-k},\ \textit{for } x\notin\mathbb{Z};$$

$$\frac{1}{m}\sum_{\substack{s=0\\ s\neq 1}}^{m}\binom{m}{s}\Big(\Lambda_{m-s+1}+\frac{H_{m-1}-H_{m-s}}{m-s+1}(\boldsymbol{B}_{m-s}^{(r)}+1)\Big)B_s(<x>)$$
$$=\frac{1}{2}\Lambda_m,\ \textit{for } x\in\mathbb{Z}.$$

## References


1. M. Abramowitz, IA. Stegun, *Handbook of Mathematical Functions*, Dover, New York, **1970**.
2. T. Arakawa, M. Kaneko, *On poly-Bernoulli numbers,* Comment. Math. Univ. St. Paul. **48**(1999), no. 2, 159–167.
3. A. Bayad, Y. Hamahata, *Multiple polylogarithms and multi-poly-Bernoulli polynomials,* Funct. Approx. Comment. Math.**46**(2012), part 1, 45–61.
4. L. Carlitz , *A note on Bernoulli numbers and polynomials*, Elem. Math., **29**(1974), 90-92.
5. G. V. Dunne, C. Schubert, *Bernoulli number identities from quantum field theory and topological string theory,* Commun. Number Theory Phys. **7(2)**(2013), 225-249.
6. C. Faber, R. Pandharipande, *Hodge integrals and Gromov-Witten theory,* Invent. Math. **139(1)**(2000), 173-199.
7. I. M. Gessel, *On Miki's identities for Bernoulli numbers,* J. Number Theory **110(1)**(2005), 75-82.
8. M. Kaneko, *Poly-Bernoulli numbers,* J. Theor. Nombres Bordeaux **9**(1997), no. 1, 221–228.
9. D.S. Kim, D. V. Dolgy, T. Kim, S.-H. Rim, *Some Formulae for the Product of Two Bernoulli and Euler Polynomials,* Abstr. Appl. Anal. 2012, Art. ID 784307.
10. D. S. Kim, T. Kim, *A note on poly-Bernoulli and higher-order poly-Bernoulli polynomials,* Russ. J. Math. Phys., **22(1)** (2015), 26–33.
11. D.S. Kim, T. Kim, *Bernoulli basis and the product of several Bernoulli polynomials,* Int. J. Math. Math. Sci. 2012, Art. ID 463659.
12. D.S. Kim, T. Kim, *Some identities of higher order Euler polynomials arising from Euler basis,* Integral Transforms Spec. Funct., **24(9)** (2013), 734-738.
13. D. S. Kim, T. Kim, *Higher-order Bernoulli and poly-Bernoulli mixed type polynomials,* Georgian Math. J. **22**(2015), no. 2, 265–272.
14. T. Kim, *Some identities for the Bernoulli, the Euler and Genocchi numbers and polynomials*, Adv. Stud. Contemp. Math., **20**(2010), no.1, 23-28.
15. T. Kim, D.S. Kim, S.-H. Rim, D.-V. Dolgy, *Fourier series of higher-order Bernoulli functions and their applications,* J. Inequal. Appl. 2017, 2017:8, 7pp.
16. H. Liu, and W. Wang, *Some identities on the the Bernoulli, Euler and Genocchi poloynomials via power sums and alternate power sums*, Disc. Math., **309**(2009), 3346-3363.
17. J. E. Marsden, *Elementary classical analysis,* W. H. Freeman and Company, 1974.
18. K. Shiratani, S. Yokoyama, *An application of p-adic convolutions,* Mem. Fac. Sci. Kyushu Univ. Ser. A **36(1)**(1982), 7383.
19. H. M. Srivastava, *Some generalizations and basic extensions of the Bernoulli, Euler and Genocchi polynomials*, Appl. Math. and Inf. Sci., **5**(2011), no. 3, 390-414.
20. P. T. Young, *Bernoulli and poly-Bernoulli polynomial convolutions and identities of p-adic Arakawa-Kaneko zeta functions*, J. Number Theory, **12**, no. 5, (2016), 1295-1309.
21. D. G. Zill, M. R. Cullen, *Advanced Engineering Mathematics,* Jones and Bartlett Publishers 2006.

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# Fixed point results under constraint inequalities in Menger PM-spaces

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**Abstract**

In this paper, we discuss the existence of a fixed point of a self-mapping $T$ on a Menger PM-space under two constraint inequalities with respect to two partial orders under two kinds of contractive conditions and obtain some new fixed point results. We then present several useful consequences of our main results. We also give examples to show the validity of our main results.

**Keywords:** Menger PM-space; fixed point; constraint inequalities; partial order; implicit contraction

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## 1 Introduction and preliminaries

In 1942, Menger introduced the concept of a probabilistic metric space which give birth to a new branch called probabilistic analysis [1]-[2]. The theory of PM-spaces and its applications has attracted much attention since then and fixed point theory for nonlinear operators in the setting of PM-spaces was studied by many authors [3]-[11].

The research on the existence of fixed points for mappings in a metric space equipped with a partial order was initiated by Turinici [12]. Ran and Reurings [13] obtained fixed point results for continuous monotone operators in a partially ordered metric space and applied them to study the existence of positive solutions to certain classes of nonlinear matrix equations. The results in [13] were generalized by many authors from different aspects (see *e.g.* [14]-[17]). On the other hand, the fixed point problems for mappings in partially ordered Menger PM-spaces were also extensively studied (see *e.g.* [18]-[21]).

Let $(X, \mathscr{F}, \Delta)$ be a Menger PM-space and $X$ be endowed with two partial orders $\preceq_1$ and $\preceq_2$. Consider five self-mappings $T, A, B, C, D : X \to X$. In this paper, we are interested in the following problem: Find $x \in X$, such that

$$\begin{cases} x = Tx, \\ Ax \preceq_1 Bx, \\ Cx \preceq_2 Dx. \end{cases} \tag{1.1}$$

In [22], Jleli and Samet considered the existence of solutions to (1.1) in the framework of metric spaces. They introduced the concepts of $d$-regularity and $(A, B, C, D, \preceq_1, \preceq_2)$-stability, and obtained a fixed point result which guaranteed the existence of a fixed point of $T$ under two constraint inequalities. In [23], Ansari *et al.* argued that the result of Jleli and Samet holds by assuming that only $A$ and $B$ are continuous (or only $C$ and $D$ are continuous), and proved that (1.1) has a unique solution. Moreover, they considered the existence of solutions to (1.1) under a certain implicit contraction by introducing a more general class of functions.

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In this paper, we will discuss the existence of a fixed point of a self-mapping $T$ in the setting of Menger PM-spaces under two constraint inequalities with respect to two partial orders and obtain a new fixed point result, which extends the main results of [22] and [23] from metric spaces to Menger PM-spaces. As a consequence, we derive some corollaries of our main result. Also, some examples are given to show the validity of the new results.

We now recall some basic definitions in the theory of Menger PM-spaces.

A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is called a *distribution function* if it is nondecreasing left-continuous with $\sup_{t\in\mathbb{R}} F(t) = 1$ and $\inf_{t\in\mathbb{R}} F(t) = 0$.

We will denote by $\mathscr{D}$ the set of all distribution functions while $H$ will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, & t \le 0, \\ 1, & t > 0. \end{cases}$$

**Definition 1.1**[5] A mapping $\Delta : [0,1] \times [0,1] \to [0,1]$ is called a *triangular norm* (for short, a $t$-norm) if the following conditions are satisfied: $\Delta(a,1) = a$; $\Delta(a,b) = \Delta(b,a)$; $\Delta(a,c) \ge \Delta(b,d)$ for $a \ge b, c \ge d$;$\Delta(a,\Delta(b,c)) = \Delta(\Delta(a,b),c)$.

A typical example of a $t$-norm is $\Delta_{min}$ which is defined by $\Delta_{min}(a,b) = \min\{a,b\}$ for all $a,b \in [0,1]$.

**Definition 1.2**[5] A triplet $(X,\mathscr{F},\Delta)$ is called a *Menger probabilistic metric space* (for short, a *Menger PM-space*) if $X$ is a nonempty set, $\Delta$ is a $t$-norm and $\mathscr{F}$ is a mapping from $X \times X$ into $\mathscr{D}$ satisfying the following conditions (we denote $\mathscr{F}(x,y)$ by $F_{x,y}$):

(PM-1) $F_{x,y}(0) = 0$;

(PM-2) $F_{x,y}(t) = H(t)$ for all $t \in \mathbb{R}$ if and only if $x = y$;

(PM-3) $F_{x,y}(t) = F_{y,x}(t)$ for all $t \in \mathbb{R}$;

(PM-4) $F_{x,y}(t+s) \ge \Delta(F_{x,z}(t), F_{z,y}(s))$ for all $x,y,z \in X$ and $t,s \ge 0$.

**Remark 1.1**[5] If a Menger PM-space $(X,\mathscr{F},\Delta)$ satisfies the condition $\sup_{0<t<1} \Delta(t,t) = 1$, then $(X,\mathscr{F},\Delta)$ is a Hausdorff topological space in the $(\epsilon,\lambda)$-topology $\mathscr{T}$, *i.e.*, the family of sets $\{U_x(\epsilon,\lambda) : \epsilon > 0, \lambda \in (0,1]\}(x \in X)$ is a basis of neighborhoods of a point $x$ for $\mathscr{T}$, where $U_x(\epsilon,\lambda) = \{y \in X : F_{x,y}(\epsilon) > 1-\lambda)\}$ .

By virtue of the topology $\mathscr{T}$, a sequence $\{x_n\}$ is said to be $\mathscr{T}$-convergent to $x \in X$(we write $x_n \xrightarrow{\mathscr{T}} x(n \to \infty)$) if for any given $\epsilon > 0$ and $\lambda \in (0,1]$, there exists a positive integer $N = N(\epsilon,\lambda)$ such that $F_{x_n,x}(\epsilon) > 1-\lambda$ whenever $n \ge N$, which is equivalent to $\lim_{n\to\infty} F_{x_n,x}(t) = 1$ for all $t > 0$; $\{x_n\}$ is called a $\mathscr{T}$-Cauchy sequence in $(X,\mathscr{F},\Delta)$ if for any given $\epsilon > 0$ and $\lambda \in (0,1]$, there exists a positive integer $N = N(\epsilon,\lambda)$ such that $F_{x_n,x_m}(\epsilon) > 1-\lambda$ whenever $n,m \ge N$; $(X,\mathscr{F},\Delta)$ is said to be $\mathscr{T}$-complete if each $\mathscr{T}$-Cauchy sequence in $X$ is $\mathscr{T}$-convergent in $X$. It is worth noting that in a Menger PM-space, $\lim_{n\to\infty} x_n = x$ implies that $x_n \xrightarrow{\mathscr{T}} x(n \to \infty)$.

**Remark 1.2**[5] Let $(X,d)$ be a metric space and $\mathscr{F} : X \times X \to \mathscr{D}$ be defined by

$$\mathscr{F}(x,y)(t) = F_{x,y}(t) = H(t - d(x,y)), \forall x,y \in X \text{ and } t > 0. \tag{1.2}$$

Then $(X,\mathscr{F},\Delta_{min})$ is a $\mathscr{T}$-complete Menger PM-space induced by $(X,d)$.

The following definitions and notations will be needed in the sequel.

**Definition 1.3** Let $(X,\mathscr{F},\Delta)$ be a Menger PM-space and $\preceq$ be partial order on $X$. $\preceq$ is called *F-regular*, if for any sequences $\{a_n\},\{b_n\} \subset X$, we have

$$\lim_{n\to\infty} F_{a_n,a}(t) = \lim_{n\to\infty} F_{b_n,b}(t) = 1, \forall t > 0, \text{ and } a_n \preceq b_n \text{ for all } n \in \mathbb{N} \Longrightarrow a \preceq b,$$

where $(a,b) \in X \times X$.

**Definition 1.4** Let $X$ be a nonempty set endowed with two partial orders $\preceq_1$ and $\preceq_2$. Let $T, A, B, C, D : X \to X$ be five self-mappings. The mapping $T$ is called $(A, B, C, D, \preceq_1, \preceq_2)$-*stable*, if the following condition is satisfied:

$$x \in X, Ax \preceq_1 Bx \Longrightarrow CTx \preceq_2 DTx.$$

Denote by $\Psi$ the set of functions $\psi : [0,1] \to [0,1]$ satisfying the following conditions:

$(\psi_1)$ $\psi$ is lower semi-continuous;

$(\psi_2)$ $\psi(t) = 0$ if and only if $t = 1$.

Denote by $\Phi$ the set of functions $\phi : [0,1] \to [1,\infty)$ satisfying the following conditions:

$(\phi_1)$ $\phi$ is lower semi-continuous;

$(\phi_2)$ $\phi(t) = 1$ if and only if $t = 1$.

## 2 Main results

We are now ready to state and prove our main results.

**Theorem 2.1** Let $(X, \mathscr{F}, \Delta)$ be a $\mathscr{T}$-complete Menger PM-space endowed with two partial orders $\preceq_1$ and $\preceq_2$, and $T, A, B, C, D : X \to X$ be self-mappings. Suppose that the following conditions are satisfied:

(i) $\preceq_i$ is $F$-regular, $i = 1, 2$;

(ii) $A$ and $B$ are $\mathscr{T}$-continuous or $C$ and $D$ are $\mathscr{T}$-continuous;

(iii) there exists $x_0 \in X$, such that $Ax_0 \preceq_1 Bx_0$;

(iv) $T$ is $(A, B, C, D, \preceq_1, \preceq_2)$-stable;

(v) $T$ is $(C, D, A, B, \preceq_2, \preceq_1)$-stable;

(vi) there exists $\psi \in \Psi$ such that

$$Ax \preceq_1 Bx, Cy \preceq_2 Dy \Longrightarrow F_{Tx,Ty}(t) \geq F_{x,y}(t) + \psi(F_{x,y}(t)), \forall t > 0.$$

Then the sequence $\{T^n x_0\}$ converges to some $x^* \in X$, which is a solution to (1.1). Moreover, $x^*$ is the unique solution to (1.1).

**Proof.** By (iii), there exists $x_0 \in X$ such that $Ax_0 \preceq_1 Bx_0$. Construct a sequence $\{x_n\} \subset X$ by

$$x_n = T^n x_0, \quad n = 0, 1, 2, \cdots .$$

By (iv), we have $CTx_0 \preceq_2 DTx_0$, that is, $Cx_1 \preceq_2 Dx_1$. Thus, we get $Ax_0 \preceq_1 Bx_0$ and $Cx_1 \preceq_2 Dx_1$. By (v), we have $ATx_1 \preceq_1 BTx_1$, that is, $Ax_2 \preceq_1 Bx_2$. Again, by (iv), we obtain $CTx_2 \preceq_2 DTx_2$, that is, $Cx_3 \preceq_2 Dx_3$. Thus, we get $Ax_2 \preceq_1 Bx_2$ and $Cx_3 \preceq_2 Dx_3$. Continuing this process, we obtain

$$Ax_{2n} \preceq_1 Bx_{2n} \text{ and } Cx_{2n+1} \preceq_2 Dx_{2n+1}, n = 0, 1, 2, \cdots . \tag{2.1}$$

By (2.1) and (vi), we have

$$F_{x_{n+1},x_n}(t) = F_{Tx_n,Tx_{n-1}}(t) \geq F_{x_n,x_{n-1}}(t) + \psi(F_{x_n,x_{n-1}}(t)), \forall t > 0, n = 1, 2, \cdots , \tag{2.2}$$

which yields that

$$F_{x_{n+1},x_n}(t) \geq F_{x_n,x_{n-1}}(t), \forall t > 0, n = 1, 2, \cdots .$$

Thus, $\{F_{x_{n+1},x_n}(t)\}$ is an increasing sequence of positive numbers for each $t > 0$. Therefore, there exists some $r(t) \in [0,1]$, such that

$$\lim_{n\to\infty} F_{x_{n+1},x_n}(t) = r(t), \forall t > 0. \tag{2.3}$$

By (2.2), we have

$$\liminf_{n\to\infty} F_{x_{n+1},x_n}(t) \geq \liminf_{n\to\infty}(F_{x_n,x_{n-1}}(t) + \psi(F_{x_n,x_{n-1}}(t))), \forall t > 0.$$

Using (2.3) and $(\psi_1)$, we obtain

$$r(t) \geq r(t) + \psi(r(t)), \forall t > 0,$$

which implies that $\psi(r(t)) = 0, \forall t > 0$. By $(\psi_2)$, we get $r(t) = 1, \forall t > 0$, i.e.,

$$\lim_{n\to\infty} F_{x_{n+1},x_n}(t) = 1, \forall t > 0. \tag{2.4}$$

We now show that $\{x_n\}$ is a $\mathscr{T}$-Cauchy sequence in $(X, \mathscr{F}, \Delta)$. Suppose that this is not true. Then there exists $\epsilon_0 > 0$ and $\lambda_0 \in (0, 1]$, for which we can find two sequences of positive integers $\{m_k\}$ and $\{n_k\}$, such that for all positive integers $k$, we have

$$n(k) > m(k) > k, \;\; F_{x_{m(k)},x_{n(k)}}(\epsilon_0) \leq 1 - \lambda_0, \;\; F_{x_{m(k)},x_{n(k)-1}}(\epsilon_0) > 1 - \lambda_0. \tag{2.5}$$

For any $\delta \in (0, \epsilon_0)$, we have

$$F_{x_{m(k)},x_{n(k)}}(\epsilon_0) \geq \Delta(F_{x_{m(k)},x_{n(k)-1}}(\epsilon_0 - \delta), F_{x_{n(k)-1},x_{n(k)}}(\delta)).$$

Letting $k \to \infty$, by (2.4), we have

$$\liminf_{k\to\infty} F_{x_{m(k)},x_{n(k)}}(\epsilon_0) \geq \Delta(\liminf_{k\to\infty} F_{x_{m(k)},x_{n(k)-1}}(\epsilon_0 - \delta), 1) = \liminf_{k\to\infty} F_{x_{m(k)},x_{n(k)-1}}(\epsilon_0 - \delta).$$

Letting $\delta \to 0$, by the left-continuity of the distribution function and (2.5), we obtain

$$1 - \lambda_0 \geq \liminf_{k\to\infty} F_{x_{m(k)},x_{n(k)}}(\epsilon_0) \geq \liminf_{k\to\infty} F_{x_{m(k)},x_{n(k)-1}}(\epsilon_0) \geq 1 - \lambda_0,$$

which implies that

$$\liminf_{k\to\infty} F_{x_{m(k)},x_{n(k)}}(\epsilon_0) = 1 - \lambda_0. \tag{2.6}$$

On the other hand, for any $\delta \in (0, \epsilon_0)$, we have

$$F_{x_{m(k)},x_{n(k)+1}}(\epsilon_0) \geq \Delta(F_{x_{m(k)},x_{n(k)}}(\epsilon_0 - \delta), F_{x_{n(k)},x_{n(k)+1}}(\delta))$$

and

$$F_{x_{m(k)},x_{n(k)}}(\epsilon_0) \geq \Delta(F_{x_{m(k)},x_{n(k)+1}}(\epsilon_0 - \delta), F_{x_{n(k)+1},x_{n(k)}}(\delta)).$$

Letting $k \to \infty$, by (2.4), (2.6) and the left-continuity of the distribution function, we have

$$\liminf_{k\to\infty} F_{x_{m(k)},x_{n(k)+1}}(\epsilon_0) \geq 1 - \lambda_0 \;\text{ and }\; 1 - \lambda_0 \geq \liminf_{k\to\infty} F_{x_{m(k)},x_{n(k)+1}}(\epsilon_0),$$

which implies that

$$\liminf_{k\to\infty} F_{x_{m(k)},x_{n(k)+1}}(\epsilon_0) = 1 - \lambda_0. \tag{2.7}$$

Similarly, for any $\delta \in (0, \epsilon_0)$, we have

$$F_{x_{n(k)},x_{m(k)-1}}(\epsilon_0) \geq \Delta(F_{x_{m(k)},x_{n(k)}}(\epsilon_0 - \delta), F_{x_{m(k)-1},x_{m(k)}}(\delta))$$

and

$$F_{x_{m(k)},x_{n(k)}}(\epsilon_0) \geq \Delta(F_{x_{n(k)},x_{m(k)-1}}(\epsilon_0 - \delta), F_{x_{m(k)-1},x_{m(k)}}(\delta)).$$

Letting $k \to \infty$, by (2.4), (2.6) and the left-continuity of the distribution function, we have

$$\liminf_{k\to\infty} F_{x_{n(k)},x_{m(k)-1}}(\epsilon_0) \geq 1-\lambda_0 \text{ and } 1-\lambda_0 \geq \liminf_{k\to\infty} F_{x_{n(k)},x_{m(k)-1}}(\epsilon_0),$$

which implies that

$$\liminf_{k\to\infty} F_{x_{n(k)},x_{m(k)-1}}(\epsilon_0) = 1-\lambda_0. \tag{2.8}$$

Also, for any $\delta \in (0,\epsilon_0)$, we have

$$F_{x_{n(k)+1},x_{m(k)}}(\epsilon_0) \geq \Delta(F_{x_{n(k)+1},x_{m(k)+1}}(\epsilon_0-\delta), F_{x_{m(k)},x_{m(k)+1}}(\delta))$$

and

$$F_{x_{n(k)+1},x_{m(k)+1}}(\epsilon_0) \geq \Delta(F_{x_{n(k)+1},x_{m(k)}}(\epsilon_0-\delta), F_{x_{m(k)},x_{m(k)+1}}(\delta)).$$

Letting $k \to \infty$, we can similarly obtain

$$\liminf_{k\to\infty} F_{x_{n(k)+1},x_{m(k)+1}}(\epsilon_0) = 1-\lambda_0. \tag{2.9}$$

Note that for all $k$, there exists a positive integer $0 \leq i(k) \leq 1$ such that

$$n(k) - m(k) + i(k) \equiv 1(2).$$

By (2.1), for all $k > 1$, we have

$$Ax_{n(k)} \preceq_1 Bx_{n(k)} \text{ and } Cx_{m(k)-i(k)} \preceq_2 Dx_{m(k)-i(k)}$$

or

$$Ax_{m(k)-i(k)} \preceq_1 Bx_{m(k)-i(k)} \text{ and } Cx_{n(k)} \preceq_2 Dx_{n(k)}.$$

Then from (vi), we have

$$F_{Tx_{n(k)},Tx_{m(k)-i(k)}}(t) \geq F_{x_{n(k)},x_{m(k)-i(k)}}(t) + \psi(F_{x_{n(k)},x_{m(k)-i(k)}}(t)), \forall k \in \mathbb{Z}^+ \text{ and } t > 0,$$

that is,

$$F_{x_{n(k)+1},x_{m(k)-i(k)+1}}(t) \geq F_{x_{n(k)},x_{m(k)-i(k)}}(t) + \psi(F_{x_{n(k)},x_{m(k)-i(k)}}(t)), \forall k \in \mathbb{Z}^+ \text{ and } t > 0. \tag{2.10}$$

Set

$$\Gamma_1 := \{k > 1 | i(k) = 0\} \text{ and } \Gamma_2 := \{k > 1 | i(k) = 1\}.$$

Now consider the following two cases.

• Case 1. $\Gamma_1$ is a countably infinite set. By (2.10), we get

$$F_{x_{n(k)+1},x_{m(k)+1}}(\epsilon_0) \geq F_{x_{n(k)},x_{m(k)}}(\epsilon_0) + \psi(F_{x_{n(k)},x_{m(k)}}(\epsilon_0)), \forall k \in \Gamma_1,$$

which yields that

$$\liminf_{k\to\infty} F_{x_{n(k)+1},x_{m(k)+1}}(\epsilon_0) \geq \liminf_{k\to\infty} (F_{x_{n(k)},x_{m(k)}}(\epsilon_0) + \psi(F_{x_{n(k)},x_{m(k)}}(\epsilon_0))).$$

Combining (2.6), (2.9) and ($\psi_1$), we obtain

$$1-\lambda_0 \geq 1-\lambda_0 + \psi(1-\lambda_0),$$

which implies that $\psi(1-\lambda_0) = 0$. By ($\psi_2$), we get $\lambda_0 = 0$, which is in contradiction with $\lambda_0 > 0$.

• Case 2. $\Gamma_1$ is a finite set. In this case, $\Gamma_2$ must be a countably infinite set. By (2.10), we get

$$F_{x_{n(k)+1},x_{m(k)}}(\epsilon_0) \geq F_{x_{n(k)},x_{m(k)-1}}(\epsilon_0) + \psi(F_{x_{n(k)},x_{m(k)-1}}(\epsilon_0)), \forall k \in \Gamma_2,$$

which yields that

$$\liminf_{k\to\infty} F_{x_{n(k)+1},x_{m(k)}}(\epsilon_0) \geq \liminf_{k\to\infty}(F_{x_{n(k)},x_{m(k)-1}}(\epsilon_0) + \psi(F_{x_{n(k)},x_{m(k)-1}}(\epsilon_0))).$$

Combining (2.7), (2.8) and $(\psi_1)$, we obtain

$$1 - \lambda_0 \geq 1 - \lambda_0 + \psi(1 - \lambda_0),$$

which implies that $\psi(1 - \lambda_0) = 0$. By $(\psi_2)$, we get $\lambda_0 = 0$, which is in contradiction with $\lambda_0 > 0$.

Therefore, we deduce that $\{x_n\}$ is a $\mathscr{T}$-Cauchy sequence in $(X, \mathscr{F}, \Delta)$. Since $(X, \mathscr{F}, \Delta)$ is $\mathscr{T}$-complete, there exists $x^* \in X$, such that $x_n \xrightarrow{\mathscr{T}} x^* (n \to \infty)$, i.e.,

$$\lim_{n\to\infty} F_{x_n,x^*}(t) = 1, \forall t > 0. \tag{2.11}$$

On the other hand, by (2.1), we have

$$Ax_{2n} \preceq_1 Bx_{2n}, n = 0, 1, 2, \cdots.$$

Suppose first that $A$ and $B$ are $\mathscr{T}$-continuous, by (2.11), we have $Ax_{2n} \xrightarrow{\mathscr{T}} Ax^* (n \to \infty)$ and $Bx_{2n} \xrightarrow{\mathscr{T}} Bx^* (n \to \infty)$, i.e.,

$$\lim_{n\to\infty} F_{Ax_{2n},Ax^*}(t) = \lim_{n\to\infty} F_{Bx_{2n},Bx^*}(t) = 1, \forall t > 0.$$

Since $\preceq_1$ is $F$-regular, we get

$$Ax^* \preceq_1 Bx^*. \tag{2.12}$$

On the other hand, by (2.1), (2.12) and condition (vi), for all $t > 0$ and any $\delta \in (0, t)$, we obtain

$$\begin{aligned} F_{Tx^*,x^*}(t) &\geq \Delta(F_{Tx^*,Tx_{2n+1}}(t-\delta), F_{x_{2n+2},x^*}(\delta)) \\ &\geq \Delta(F_{x^*,x_{2n+1}}(t-\delta) + \psi(F_{x^*,x_{2n+1}}(t-\delta)), F_{x_{2n+2},x^*}(\delta)), n = 0, 1, 2, \cdots. \end{aligned}$$

Therefore, we get

$$F_{Tx^*,x^*}(t) \geq \Delta(\liminf_{n\to\infty} F_{x^*,x_{2n+1}}(t-\delta) + \psi(F_{x^*,x_{2n+1}}(t-\delta)), \liminf_{n\to\infty} F_{x_{2n+2},x^*}(\delta)), n = 0, 1, 2, \cdots.$$

Using $(\psi_1)$, $(\psi_2)$ and (2.11), we obtain

$$F_{Tx^*,x^*}(t) \geq \Delta(1 + \psi(1), 1) = \Delta(1, 1) = 1, \forall t > 0,$$

which implies that

$$Tx^* = x^*. \tag{2.13}$$

Now, since $T$ is $(A, B, C, D, \preceq_1, \preceq_2)$-stable, from (2.12), we get

$$CTx^* \preceq DTx^*,$$

which implies from (2.13) that

$$Cx^* \preceq Dx^*. \tag{2.14}$$

As a consequence, it follows from (2.12), (2.13) and (2.14) that $x^* \in X$ is a solution to (1.1).

Suppose now that $y^* \in X$ is another solution to (1.1), that is,

$$Ty^* = y^*, Ay^* \preceq_1 By^*, Cy^* \preceq_2 Dy^*, \text{ and } F_{x^*,y^*}(t_0) < 1 \text{ for some } t_0 > 0.$$

By condition (vi) and $(\psi_2)$, we obtain

$$F_{x^*,y^*}(t_0) = F_{Tx^*,Ty^*}(t_0) \geq F_{x^*,y^*}(t_0) + \psi(F_{x^*,y^*}(t_0)) > F_{x^*,y^*}(t_0),$$

which is a contradiction. Therefore, $x^*$ is the unique solution to (1.1).

If we require only the $\mathscr{T}$-continuity of $C$ and $D$ by condition (ii), we can also deduce the conclusions using similar arguments. This completes the proof.

**Example 2.1** Let $X = \mathbb{R}$ be the set of all real numbers equipped with the standard order $\leq$. Take $\preceq_1 = \preceq_2 = \leq$. Let $X$ be endowed with the standard metric $d(x, y) = |x - y|, (x, y) \in X \times X$ and $\mathscr{F} : X \times X \to \mathscr{D}$ is defined by (1.2). Then $(X, \mathscr{F}, \Delta_{min})$ is a $\mathscr{T}$-complete Menger PM-space induced by $(X, d)$. Let $T : X \to X$ be defined by

$$Tx = \begin{cases} -1, & \text{if } x < 0, \\ 2, & \text{if } x \geq 0, \end{cases}$$

and $A, B, C, D : X \to X$ be defined by

$$Ax = 2x^2, \ Bx = 6x, \ Cx = 2,$$

$$Dx = \begin{cases} 0, & \text{if } x < 2, \\ x, & \text{if } x \geq 2. \end{cases}$$

It is obvious that $\preceq_i$ is $F$-regular, $i = 1, 2$. Moreover, $A$ and $B$ are $\mathscr{T}$-continuous. Also, for $x_0 = 1$, we have $Ax_0 = 2 \leq 6 = Bx_0$. If for some $x \in X$, we have $Ax \leq Bx$, then $0 \leq x \leq 3$, which yields $CTx = 2 \leq 3 = D3 = DTx$. So $T$ is $(A, B, C, D, \preceq_1, \preceq_2)$-stable. If for some $x \in X$, we have $Cx \leq Dx$, then $x \geq 2$, which yields $ATx = A3 = 18 = B3 = BTx$. So $T$ is $(C, D, A, B, \preceq_1, \preceq_2)$-stable. Note that for any $(x, y) \in X \times X$, we have

$$Ax \leq Bx, Cy \leq Dy \Longrightarrow (x, y) \in [0, 3] \times [2, \infty) \Longrightarrow (Tx, Ty) = (3, 3).$$

Therefore,

$$Ax \preceq_1 Bx, Cy \preceq_2 Dy \Longrightarrow F_{Tx,Ty}(t) = 1 \geq F_{x,y}(t) + \psi(F_{x,y}(t)), \forall t > 0,$$

where $\psi(x) = 1 - x$ for $x \in [0, 1]$. By Theorem 2.1, (1.1) has a unique solution, which is $x^* = 3$. On the other hand, observe that $D$ is not $\mathscr{T}$-continuous. So we don't need $A, B, C, D$ are all $\mathscr{T}$-continuous to guarantee the existence of the solution.

**Remark 2.1** Note that although the conditions of Theorem 2.1 are sufficient to guarantee the existence of a unique solution to (1.1), $T$ need not have a unique fixed point, as the above example shows (in fact, the mapping $T$ has two fixed points $-2$ and 3).

Denote by $L$ the class of all functions $f : [0, 1] \times [1, \infty) \to \mathbb{R}$ satisfying

$(L_1)$ $f$ is $\mathscr{T}$-continuous;

$(L_2)$ $f(a, b) \geq a, \forall a \in [0, 1]$ and $b \in [1, \infty)$;

$(L_3)$ $a \in [0, 1], b \in [1, \infty), f(a, b) = a \Longrightarrow b = 1$;

$(L_4)$ $\forall a \in [0, 1], b_1 \geq b_2 \Longrightarrow f(a, b_1) \geq f(a, b_2)$.

We give the second main result of this paper by utilizing this new class of functions.

**Theorem 2.2** Let $(X, \mathscr{F}, \Delta)$ be a $\mathscr{T}$-complete Menger PM-space endowed with two partial orders $\preceq_1$ and $\preceq_2$, and $T, A, B, C, D : X \to X$ be self-mappings. Suppose that the following conditions are satisfied:

(i) $\preceq_i$ is $F$-regular, $i = 1, 2$;

(ii) $A$ and $B$ are $\mathscr{T}$-continuous or $C$ and $D$ are $\mathscr{T}$-continuous;

(iii) there exists $x_0 \in X$, such that $Ax_0 \preceq_1 Bx_0$;

(iv) $T$ is $(A, B, C, D, \preceq_1, \preceq_2)$-stable;

(v) $T$ is $(C, D, A, B, \preceq_2, \preceq_1)$-stable;

(vi) there exists $\phi \in \Phi$ and $f \in L$ such that

$$Ax \preceq_1 Bx, Cy \preceq_2 Dy \Longrightarrow F_{Tx,Ty}(t) \geq f(F_{x,y}(t), \phi(F_{x,y}(t))), \forall t > 0.$$

Then the sequence $\{T^n x_0\}$ converges to some $x^* \in X$, which is a solution to (1.1). Moreover, $x^*$ is the unique solution to (1.1).

**Proof.** By (iii), there exists $x_0 \in X$ such that $Ax_0 \preceq_1 Bx_0$. Construct a sequence $\{x_n\} \subset X$ by

$$x_n = T^n x_0, \quad n = 0, 1, 2, \cdots .$$

By similar arguments in the proof of Theorem 2.1, we obtain (2.1).

By (2.1) and (vi), we have

$$F_{x_{n+1},x_n}(t) = F_{Tx_n,Tx_{n-1}}(t) \geq f(F_{x_n,x_{n-1}}(t), \phi(F_{x_n,x_{n-1}}(t))), \forall t > 0, n = 1, 2, \cdots , \tag{2.15}$$

which by $(L_2)$ yields that

$$F_{x_{n+1},x_n}(t) \geq F_{x_n,x_{n-1}}(t), \forall t > 0, n = 1, 2, \cdots .$$

Thus, $\{F_{x_{n+1},x_n}(t)\}$ is an increasing sequence of positive numbers for each $t > 0$. Therefore, there exists some $p(t) \in [0, 1]$, such that

$$\lim_{n\to\infty} F_{x_{n+1},x_n}(t) = p(t), \forall t > 0. \tag{2.16}$$

By the properties of $f$ and $\phi$, we obtain

$$p(t) \geq f(p(t), \phi(p(t))) \geq p(t),$$

which yields that

$$f(p(t), \phi(p(t))) = p(t).$$

Hence, by $(L_1)$, we get $\phi(p(t)) = 1, \forall t > 0$, and thus $p(t) = 1, \forall t > 0$, that is,

$$\lim_{n\to\infty} F_{x_{n+1},x_n}(t) = 1, \forall t > 0. \tag{2.17}$$

We now show that $\{x_n\}$ is a $\mathscr{T}$-Cauchy sequence in $(X, \mathscr{F}, \Delta)$. Suppose that this is not true. Then there exists $\epsilon_0 > 0$ and $\lambda_0 \in (0, 1]$, for which we can find two sequences of positive integers $\{m_k\}$ and $\{n_k\}$, such that for all positive integers $k$, (2.5) holds. By similar arguments in the proof of Theorem 2.1, (2.6)−(2.9) hold.

Note that for all $k$, there exists a positive integer $0 \leq i(k) \leq 1$ such that

$$n(k) - m(k) + i(k) \equiv 1(2).$$

Similarly, by (2.1) and (vi), we have

$$F_{x_{n(k)+1},x_{m(k)-i(k)+1}}(t) \geq f(F_{x_{n(k)},x_{m(k)-i(k)}}(t) + \phi(F_{x_{n(k)},x_{m(k)-i(k)}}(t))), \forall k \in \mathbb{Z}^+ \text{ and } t > 0. \tag{2.18}$$

Set

$$\Gamma_1 := \{k > 1 | i(k) = 0\} \text{ and } \Gamma_2 := \{k > 1 | i(k) = 1\}.$$

Now consider the following two cases.

• Case 1. $\Gamma_1$ is a countably infinite set. By (2.18), we get

$$F_{x_{n(k)+1},x_{m(k)+1}}(\epsilon_0) \geq f(F_{x_{n(k)},x_{m(k)}}(\epsilon_0), \phi(F_{x_{n(k)},x_{m(k)}}(\epsilon_0))), \forall k \in \Gamma_1,$$

which yields that

$$\liminf_{k\to\infty} F_{x_{n(k)+1},x_{m(k)+1}}(\epsilon_0) \geq \liminf_{k\to\infty} f(F_{x_{n(k)},x_{m(k)}}(\epsilon_0), \phi(F_{x_{n(k)},x_{m(k)}}(\epsilon_0))).$$

Combining (2.6), (2.9) and the properties of $f$ and $\phi$, we obtain

$$1-\lambda_0 \geq f(1-\lambda_0, \phi(1-\lambda_0)) \geq 1-\lambda_0,$$

which implies that

$$f(1-\lambda_0, \phi(1-\lambda_0)) = 1-\lambda_0.$$

Again, by $(L_3)$ and $(\phi_2)$, we get $\lambda_0 = 0$, which is in contradiction with $\lambda_0 > 0$.

• Case 2. $\Gamma_1$ is a finite set. In this case, $\Gamma_2$ must be a countably infinite set. By (2.18), we get

$$F_{x_{n(k)+1},x_{m(k)}}(\epsilon_0) \geq f(F_{x_{n(k)},x_{m(k)-1}}(\epsilon_0), \phi(F_{x_{n(k)},x_{m(k)-1}}(\epsilon_0))), \forall k \in \Gamma_2,$$

which yields that

$$\liminf_{k\to\infty} F_{x_{n(k)+1},x_{m(k)}}(\epsilon_0) \geq \liminf_{k\to\infty} f(F_{x_{n(k)},x_{m(k)-1}}(\epsilon_0) + \phi(F_{x_{n(k)},x_{m(k)-1}}(\epsilon_0))).$$

Combining (2.7), (2.8) and $(\phi_1)$, we also obtain

$$1-\lambda_0 \geq f(1-\lambda_0, \phi(1-\lambda_0)) \geq 1-\lambda_0.$$

It follows from the properties of $f$ and $\phi$ that $\lambda_0 = 0$, which is a contradiction.

Therefore, we deduce that $\{x_n\}$ is a $\mathscr{T}$-Cauchy sequence in $(X, \mathscr{F}, \Delta)$. Since $(X, \mathscr{F}, \Delta)$ is $\mathscr{T}$-complete, there exists $x^* \in X$, such that $x_n \xrightarrow{\mathscr{T}} x^* (n \to \infty)$, i.e.,

$$\lim_{n\to\infty} F_{x_n,x^*}(t) = 1, \forall t > 0. \tag{2.19}$$

We only consider the case when $A$ and $B$ are $\mathscr{T}$-continuous (the proof is similar when $C$ and $D$ are $\mathscr{T}$-continuous). By (2.19), we have

$$\lim_{n\to\infty} F_{Ax_{2n},Ax^*}(t) = \lim_{n\to\infty} F_{Bx_{2n},Bx^*}(t) = 1, \forall t > 0.$$

Since $\preceq_1$ is $F$-regular, noting that (2.1) holds, we get

$$Ax^* \preceq_1 Bx^*. \tag{2.20}$$

On the other hand, by (2.1), (2.12) and condition (vi), for all $t > 0$ and any $\delta \in (0, t)$, we obtain

$$\begin{aligned} F_{Tx^*,x^*}(t) &\geq \Delta(F_{Tx^*,Tx_{2n+1}}(t-\delta), F_{x_{2n+2},x^*}(\delta)) \\ &\geq \Delta(f(F_{x^*,x_{2n+1}}(t-\delta), \phi(F_{x^*,x_{2n+1}}(t-\delta))), F_{x_{2n+2},x^*}(\delta)), n = 0, 1, 2, \cdots. \end{aligned}$$

By (2.19) and the properties of $f$ and $\phi$, we obtain

$$F_{Tx^*,x^*}(t) \geq \Delta(f(1, \phi(1)), 1) = f(1, \phi(1)) = f(1, 1) \geq 1, \forall t > 0,$$

which implies that

$$Tx^* = x^*. \tag{2.21}$$

Now, since $T$ is $(A, B, C, D, \preceq_1, \preceq_2)$-stable, from (2.20), we get

$$CTx^* \preceq DTx^*,$$

which implies from (2.21) that

$$Cx^* \preceq Dx^*. \tag{2.22}$$

As a consequence, it follows from (2.20), (2.21) and (2.22) that $x^* \in X$ is a solution to (1.1).

Suppose now that $y^* \in X$ is another solution to (1.1), that is,

$$Ty^* = y^*, Ay^* \preceq_1 By^*, Cy^* \preceq_2 Dy^*, \text{ and } F_{x^*,y^*}(t_0) < 1 \text{ for some } t_0 > 0.$$

By condition (vi) and $(\phi_2)$, we obtain

$$F_{x^*,y^*}(t_0) = F_{Tx^*,Ty^*}(t_0) \geq f(F_{x^*,y^*}(t_0), \phi(F_{x^*,y^*}(t_0))) \geq F_{x^*,y^*}(t_0),$$

which yields that

$$f(F_{x^*,y^*}(t_0), \phi(F_{x^*,y^*}(t_0))) = F_{x^*,y^*}(t_0).$$

By $(L_3)$ and $(\phi_2)$, we get $F_{x^*,y^*}(t_0) = 1$, which is a contradiction. Therefore, $x^*$ is the unique solution to (1.1). This completes the proof.

## 3 Some consequences

Setting $\preceq_1 = \preceq_2 = \preceq$, $C = B$ and $D = A$, problem (1.1) becomes the following one: Find $x \in X$, such that

$$\begin{cases} x = Tx, \\ Ax = Bx, \end{cases} \tag{3.1}$$

where $T, A, B : X \to X$ are given self-mappings and $(X, \mathscr{F}, \Delta)$ is a Menger PM-space endowed with a partial order $\preceq$.

Letting $\preceq_1 = \preceq_2 = \preceq$, $C = B$ and $D = A$, we can obtain the following corollary from Theorem 2.2.

**Corollary 3.1** Let $(X, \mathscr{F}, \Delta)$ be a $\mathscr{T}$-complete Menger PM-space endowed with a partial order $\preceq$, and $T, A, B : X \to X$ be self-mappings. Suppose that the following conditions are satisfied:

(i) $\preceq$ is $F$-regular;

(ii) $A$ and $B$ are $\mathscr{T}$-continuous;

(iii) there exists $x_0 \in X$, such that $Ax_0 \preceq_1 Bx_0$;

(iv) $x \in X, Ax \preceq Bx \Longrightarrow BTx \preceq ATx$;

(v) $x \in X, Bx \preceq Ax \Longrightarrow ATx \preceq BTx$;

(vi) there exists $\phi \in \Phi$ and $f \in L$ such that

$$Ax \preceq Bx, By \preceq_2 Ay \Longrightarrow F_{Tx,Ty}(t) \geq f(F_{x,y}(t), \phi(F_{x,y}(t))), \forall t > 0.$$

Then the sequence $\{T^n x_0\}$ converges to some $x^* \in X$, which is a solution to (3.1). Moreover, $x^*$ is the unique solution to (3.1).

Next, we provide an example to show the validity of Corollary 3.1.

**Example 3.1** Let $X = \{(6, 2), (2, 2), (8, 2), (5, 2), (5, 3)\} \subset \mathbb{R}^2$ and $\preceq$ be the partial order on $X$ defined by

$$(x_1, y_1), (x_2, y_2) \in X, \ (x_1, y_1) \preceq (x_2, y_2) \Longleftrightarrow x_1 \leq x_2, \ y_1 \leq y_2.$$

Let $X$ be endowed with the metric $d$ defined by

$$d((x_1,y_1),(x_2,y_2))=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2},\ \text{for } x=(x_1,y_1), y=(x_2,y_2)\in X.$$

Let $\mathscr{F}: X\times X\to\mathscr{D}$ be defined by (1.2). Then it follows from Remark 1.2 that $(X,\mathscr{F},\Delta_{min})$ is a $\mathscr{T}$-complete Menger PM-space.

Let $T, A, B: X\to X$ be the mappings defined by

$$\begin{cases} T(6,2)=(6,2),\\ T(2,2)=(8,2),\\ T(8,2)=(5,2),\\ T(5,2)=(6,2),\\ T(5,3)=(2,2), \end{cases} \quad \begin{cases} A(6,2)=(6,2),\\ A(2,2)=(4,2),\\ A(8,2)=(3,2),\\ A(5,2)=(7,2),\\ A(5,3)=(4,3), \end{cases} \quad \text{and} \quad \begin{cases} B(6,2)=(6,2),\\ B(2,2)=(5,2),\\ B(8,2)=(2,2),\\ B(5,2)=(8,2),\\ B(5,3)=(6,2). \end{cases}$$

It is obvious that

$$u\in X,\ u\preceq Bu\Longleftrightarrow u\in\{(6,2),(2,2),(2,2)\}$$

and

$$v\in X,\ Bv\preceq v\Longleftrightarrow v\in\{(6,2),(8,2)\}.$$

By the above definitions, it is easy to check that

$$u\in X, Au\preceq Bu\Longrightarrow BTu\preceq ATu$$

and

$$v\in X, Bv\preceq Av\Longrightarrow ATv\preceq BTv.$$

Now, let $(u,v)\in X\times X$ satisfies $Au\preceq Bu$ and $Bv\preceq Av$. Then we have

$$(u,v)\in\{((6,2),(6,2)),((6,2),(8,2)),((2,2),(6,2)),((2,2),(8,2)),((5,2),(6,2)),((5,2),(8,2))\}.$$

For $(u,v)=((6,2),(6,2))$, we have

$$F_{Tu,Tv}(t)=H(t-d(Tu,Tv))=H(t)=1\geq 1\cdot\phi(1)=F_{u,v}(t)\cdot\phi(F_{u,v}(t)),\forall t>0.$$

For $(u,v)=((6,2),(8,2))$, we have

$$F_{Tu,Tv}(t)=H(t-d(Tu,Tv))=H(t-1)\geq H(t-2)\cdot\phi(H(t-2))=F_{u,v}(t)\cdot\phi(F_{u,v}(t)),\forall t>0.$$

For $(u,v)=((2,2),(6,2))$, we have

$$F_{Tu,Tv}(t)=H(t-d(Tu,Tv))=H(t-2)\geq H(t-4)\cdot\phi(H(t-4))=F_{u,v}(t)\cdot\phi(F_{u,v}(t)),\forall t>0.$$

For $(u,v)=((2,2),(8,2))$, we have

$$F_{Tu,Tv}(t)=H(t-d(Tu,Tv))=H(t-3)\geq H(t-6))\cdot\phi(H(t-6))=F_{u,v}(t)\cdot\phi(F_{u,v}(t)),\forall t>0.$$

For $(u,v)=((5,2),(6,2))$, we have

$$F_{Tu,Tv}(t)=H(t-d(Tu,Tv))=H(t)=1\geq H(t-1)\cdot\phi(H(t-1))=F_{u,v}(t)\cdot\phi(F_{u,v}(t)),\forall t>0.$$

For $(u,v)=((5,2),(8,2))$, we have

$$F_{Tu,Tv}(t)=H(t-d(Tu,Tv))=H(t-1)\geq H(t-3)\cdot\phi(H(t-3))=F_{u,v}(t)\cdot\phi(F_{u,v}(t)),\forall t>0.$$

Then we have

$$Au \preceq Bu, Bv \preceq Av \Longrightarrow F_{Tu,Tv}(t) \geq f(F_{u,v}(t), \psi(F_{u,v}(t))), \forall t > 0,$$

where $f(a,b) = a \cdot b, \forall a \in [0,1]$ and $b \in [1,\infty)$. Therefore, all the conditions of Corollary 3.1 are satisfied and thus (3.1) has a unique solution. In fact, we observe that $x^* = (6,2)$ is the unique solution to (3.1). We would like to point out that there exists $t_0 = 3 > 0$, such that

$$F_{T(6,2),T(5,3)}(t_0) = H(t_0 - 4) = 0 < 1 = H(t_0 - \sqrt{2}) = F_{(6,2),(5,3)}(t_0),$$

which shows that $T$ is not a $SB$-contraction on $(X, \mathscr{F}, \Delta)$.

Furthermore, setting $A = I_X$ in (3.1), where $I_X$ denotes the identity mapping on $X$, we get another problem: Find $x \in X$, such that

$$\begin{cases} x = Tx, \\ x = Bx, \end{cases} \tag{3.2}$$

where $T, B : X \to X$ are given self-mappings and $(X, \mathscr{F}, \Delta)$ is a Menger PM-space endowed with a partial order $\preceq$. This problem is to find a common fixed point of two self-mappings on a partially ordered Menger PM-space.

Taking $A = I_X$ in Corollary 3.1, we obtain the following result immediately.

**Corollary 3.2** Let $(X, \mathscr{F}, \Delta)$ be a $\mathscr{T}$-complete Menger PM-space endowed with a partial order $\preceq$, and $T, B : X \to X$ be self-mappings. Suppose that the following conditions are satisfied:

(i) $\preceq$ is $F$-regular;

(ii) $B$ is $\mathscr{T}$-continuous;

(iii) there exists $x_0 \in X$, such that $x_0 \preceq_1 Bx_0$;

(iv) $x \in X, x \preceq Bx \Longrightarrow BTx \preceq Tx$;

(v) $x \in X, Bx \preceq x \Longrightarrow Tx \preceq BTx$;

(vi) there exists $\phi \in \Phi$ and $f \in L$ such that

$$x \preceq Bx, By \preceq y \Longrightarrow F_{Tx,Ty}(t) \geq f(F_{x,y}(t), \phi(F_{x,y}(t))), \forall t > 0.$$

Then the sequence $\{T^n x_0\}$ converges to some $x^* \in X$, which is a solution to (3.2). Moreover, $x^*$ is the unique solution to (3.2).

Furthermore, by taking $B = T$ in Corollary 3.2, we obtain a fixed point theorem of a self-mapping $T$ on a partially ordered Menger PM-space.

## References


[1] K. Menger, *Statistical metrics*, Proc. Natl. Acad. Sci. USA **28** (1942) 535-537.

[2] B. Schweizer, A. Sklar, *Statistical metric spaces*, Pacific J. Math. **10** (1960) 313-334.

[3] B. Schweizer, A. Sklar, *Probabilistic Metric Spaces*, North-Holland, Amsterdam, 1983.

[4] O. Hadžić, E. Pap, *Fixed Point Theory in Probabilistic Metric Spaces*, Kluwer Academic Publishers, Dordrecht, 2001.

[5] S. Chang, Y. J. Cho, S. M. Kang, *Nonlinear Operator Theory in Probabilistic Metric Spaces*, Nova Science Publishers Inc., New York, 2001.

[6] B. Schweizer, *Commentary on Probabilistic Geometry*, in Karl Menger, Selecta Mathematica, edited by B. Schweizer, A. Sklar, K. Sigmund, et al. Springer/ Wien/ New York, **2** (2003) 409-432.

[7] C. Zhu, *Several nonlinear operator problems in the Menger PN space*, Nonlinear Anal. **65** (2006) 1281-1284.

[8] C. Zhu, *Research on some problems for nonlinear operators*, Nonlinear Anal. **71** (2009) 4568-4571.

[9] Z. Wu, C. Zhu, J. Li, *Common fixed point theorems for two hybrid pairs of mappings satisfying the common property (E.A) in Menger PM-spaces*, Fixed Point Theory Appl. **2013** (2013) 25.

[10] Z. Wu, C. Zhu, J. Wang, *Common fixed point theorems for occasionally weakly biased mappings under contractive conditions in Menger PM-spaces*, J. Comput. Anal. Appl. **19(4)** (2015) 661-669.

[11] Z. Wu, C. Zhu, X. Zhang, *Some new fixed point theorems for single and set-valued admissible mappings in Menger PM-spaces*, Revista de la Real Academia de Ciencias Exactas, Fśicas y Naturales. Serie A. Matemáticas, **110** (2016) 755-769.

[12] M. Turinici, *Abstract comparison principles and multivariable Gronwall-Bellman inequalities*, J. Math. Anal. Appl. **117** (1986) 100-127.

[13] A. C. M. Ran, M. C. B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc. **132** (2004) 1435-1443.

[14] J. Harjani, K. Sadarangani, *Fixed point theorems for weakly contractive mappings in partially ordered sets*, Nonlinear Anal. **71** (2009) 3403-3410.

[15] E. Karapinar, *Couple fixed point theorems for nonlinear contractions in cone metric spaces*, Comput. Math. Appl. **59** (2010) 3656-3668.

[16] V. Lakshmikantham, L. *Ćirić*, *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Anal. **70** (2009) 4341-4349.

[17] B. Samet, C. Vetro, *Coupled fixed point theorems for multi-valued nonlinear contraction mappings in partially ordered metric spaces*, Nonlinear Anal. **74** (2011) 4260-4268.

[18] L. *Ćirić*, D. Mihet, R. Saadati, *Monotone generalized contractions in partially ordered probabilistic metric spaces*, Topol. Appl. **156** (2009) 2838-2844.

[19] J. Wu, *Some fixed point theorems for mixed monotone operators in partially ordered probabilistic metric spaces*, Fixed Point Theory Appl. **2014** (2014) 49.

[20] W. Xu, C. Zhu, Z. Wu, L. Zhu, *Fixed point theorems for two new types of cyclic weakly contractive mappings in partially ordered Menger PM-spaces*, J. Nonlinear Sci. Appl. **8** (2015) 412-422.

[21] Q. Tu, C. Zhu, Z. Wu, X. Mu, *Some new coupled fixed point theorems in partially ordered complete probabilistic metric spaces*, J. Nonlinear Sci. Appl. **9** (2016) 1116-1128.

[22] M. Jleli and B. Samet, *A fixed point problem under two constraint inequalities*, Fixed Point Theory Appl. **2016** (2016) 18.

[23] A. H. Ansari, P. Kumam and B. Samet, *A fixed point problem with constraint inequalities via an implict contraction*, J. Fixed Point Theory Appl. (2016) DOI 10.1007/s11784-016-0320-1.

# Fixed Point Theorems for several types of Meir-Keeler contraction mappings in $M_s-$metric spaces

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**Abstract:** In this paper, we first introduce several types of Meir-Keeler contractive mappings in the structure of $M_S-$metric spaces. Then we study some existence and uniqueness fixed point theorems for these types of MKC mappings in $M_s-$metric spaces via Gupta-Saxena type contraction and other fraction version type contractions. Also, we extend and improve very recent results in fixed point theory.

**MSC:** 47H10;54H25

**Keywords:** Fixed point; Meir-Keeler contraction mappings; $M_s-$metric space

## 1. Introduction and Preliminaries

In 2014, Nabil [1] established an extension of $S$-metric spaces to partial $S$-metric spaces and pointed out that every $S$-metric space is a partial $S$-metric space. Also, they obtained some fixed point results under certain contractive principle in partial $S-$metric spaces. Recently, Nabil et al.[2] have extended the concept of a partial $S-$metric space to the concept of an $M_s-$metric space. They gave a more general extension of almost any metric space with three dimensions and that is not by defining the self "distance" in a metric as in partial metric spaces, but they assumed that is not necessary that the self "distance" is less than the value of the metric among three distinct elements.

In 1969, Meir and Keeler [3] established a fixed point theorem in a metric space $(X, d)$ for mappings satisfying the condition that for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$\epsilon \leq d(x, y) < \epsilon + \delta \quad implies \quad d(Tx, Ty) < \epsilon, \tag{1}$$

$\forall x, y \in X$. This condition is called the Meri-Keeler contractive ($MKC$, for short) type condition. Since then, many authors extended and improved this condition and established fixed point results for new generalized MKC mappings, see [4]-[7].

In this paper, we establish some of the fixed point theorem for some types of MKC mappings in $M_s-$metric spaces. Also, we extend and improve very recent results in fixed point theory.

Next, we remind the reader of some definitions, notions, lemmas which are useful in the sequel.

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Now, we present the definition of an $M_s$-metric space, but first we introduce the following notations which are useful in the sequel:

(1) $m_{s_{x,y,z}} = \min\{m_s(x,x,x), m_s(y,y,y), m_s(z,z,z)\}$;
(2) $M_{s_{x,y,z}} = \max\{m_s(x,x,x), m_s(y,y,y), m_s(z,z,z)\}$.

**Definition 1.1.** [2] An $M_s$-metric on a nonempty set $X$ is a function $m_s : X^3 \mapsto \mathbb{R}^+$ that satisfies the following conditions: for $\forall x, y, z, t \in X$,

$(m_s1)$ $m_s(x,x,x) = m_s(y,y,y) = m_s(z,z,z) = m_s(x,y,z)$ if and only if $x = y = z$;
$(m_s2)$ $m_{s_{x,y,z}} \leq m_s(x,y,z)$;
$(m_s3)$ $m_s(x,x,y) = m_s(y,y,x)$;
$(m_s4)$ $(m_s(x,y,z) - m_{s_{x,y,z}}) \leq (m_s(x,x,t) - m_{s_{x,x,t}}) + (m_s(y,y,t) - m_{s_{y,y,t}}) + (m_s(z,z,t) - m_{s_{z,z,t}})$.

Then the pair $(X, m_s)$ is called an *$M_s$-metric space*.

Immediate examples of such $M_s-$metric space are:

(1) Let $X = [0, \infty)$ and $m_s : X^3 \mapsto \mathbb{R}^+$ be a mapping defined by

$$m_s(x,y,z) = \max\{x,y,z\} - \min\{x,y,z\},$$

for $\forall x, y, z \in X$. Then $m_s$ is an $M_s$-metric on $X$.

(2) Let $X$ be a nonempty set and $d$ be the ordinary metric on $X$. Define mapping $m_s : X^3 \mapsto [0, \infty)$ by

$$m_s(x,y,z) = d(x,y) + d(x,z) + d(y,z),$$

for $\forall x, y, z \in X$. Then $m_s$ is an $M_s$-metric on $X$.

(3) Let $X = \{1,2,3\}$ and define a mapping $m_s$ on $X$ by

$$\begin{aligned} &m_s(1,2,3) = 6, \;\; m_s(1,1,2) = m_s(2,2,1) = 10, \\ &m_s(1,1,3) = m_s(3,3,1) = m_s(2,2,3) = m_s(3,3,2) = 7, \\ &m_s(2,2,2) = 9, \;\; m_s(3,3,3) = 5, \;\; m_s(1,1,1) = 8. \end{aligned}$$

Then $m_s$ is an $M_s$-metric on $X$.

**Remark 1.1.** If $m_s$ is an $M_s$-metric on a nonempty set $X$, then two mappings $m_s^w, m_s^* : X^3 \mapsto \mathbb{R}^+$ defined by

$$m_s^w(x,y,z) = m_s(x,y,z) - 2m_{s_{x,y,z}} + M_{s_{x,y,z}}$$

and

$$m_s^*(x,y,z) = \begin{cases} m_s(x,y,z) - m_{s_{x,y,z}}, & x \neq y \neq z, \\ 0, & x = y = z = 0, \end{cases}$$

for all $x, y, z \in X$ are two ordinary $S$-metrics on $X$. In fact, if $m_s^w(x,y,z) = 0$, then we have

$$m_s(x,y,z) = 2m_{s_{x,y,z}} - M_{s_{x,y,z}}.$$

But, from the equation defined above and $(m_s2)$, it follows that

$$m_{s_{x,y,z}} = M_{s_{x,y,z}} = m_s(x,x,x) = m_s(y,y,y) = m_s(z,z,z).$$

So, by the equation above, we have that $m_s(x,y,z) = m_s(x,x,x) = m_s(y,y,y) = m_s(z,z,z)$ and so $x = y = z$. We can get the inequality property in the definition of an $S$-metric from Lemma 1.1 (7) and $(m_s4)$.

**Lemma 1.1.** Let $(X, m_s)$ be an $M_s$-metric space. Then, for all $x, y, z, t \in X$,

(1) $m_s(x,y,y) \leq m_s(x,x,y)$;
(2) $m_s(x,y,x) \leq 2m_s(x,x,y)$;
(3) $m_s(x,y,z) - m_{s_{x,y,z}} \leq (m_s(x,x,z) - m_{s_{x,x,z}}) + (m_s(y,y,z) - m_{s_{y,y,z}})$;
(4) $m_s(x,y,z) - m_{s_{x,y,z}} \leq (m_s(x,x,y) - m_{s_{x,x,y}}) + (m_s(z,z,y) - m_{s_{z,z,y}})$;
(5) $m_s(x,y,z) - m_{s_{x,y,z}} \leq (m_s(y,y,x) - m_{s_{y,y,x}}) + (m_s(z,z,x) - m_{s_{z,z,x}})$;
(6) $m_s(x,y,z) - m_{s_{x,y,z}} \leq \frac{2}{3}[(m_s(x,x,z) - m_{s_{x,x,z}}) + (m_s(z,z,y) - m_{s_{z,z,y}}) + (m_s(y,y,x) - m_{s_{y,y,x}})]$;
(7) $(M_{s_{x,y,z}} - m_{s_{x,y,z}}) \leq (M_{s_{x,x,t}} - m_{s_{x,x,t}}) + (M_{s_{y,y,t}} - m_{s_{y,y,t}}) + (M_{s_{z,z,t}} - m_{s_{z,z,t}})$.

*Proof.* (1)-(7) can be directly obtained from Definition 1.1. □

## 2. Topology for $M_s-$metric

It is clear that each $M_s$-metric $m_s$ on $X$ generates a topology $\tau_{m_s}$ on $X$. The set

$$\{B_{m_s}(x,\epsilon) : x \in X, \epsilon > 0\},$$

where

$$B_{m_s}(x,\epsilon) := \{y \in X : m_s(x,x,y) - m_{s_{x,x,y}} < \epsilon\},$$

for $\forall x \in X$ and $\epsilon > 0$, forms a base of $\tau_{m_s}$.

**Definition 2.1.** Let $(X, m_s)$ be an $M_s$-metric space. Then:

(1) A sequence $\{x_n\}$ in $X$ is said to be *convergent* to a point $x$ if

$$\lim_{n\to\infty}(m_s(x_n,x_n,x) - m_{s_{x_n,x_n,x}}) = 0.$$

(2) A sequence $\{x_n\}$ in $X$ is called an $M_s$-*Cauchy sequence* if

$$\lim_{n,m\to\infty}(m_s(x_n,x_n,x_m) - m_{s_{x_n,x_n,x_m}}), \lim_{n,m\to\infty}(M_{s_{x_n,x_n,x_m}} - m_{s_{x_n,x_n,x_m}})$$

exist and finite.

(3) An $M_s$-metric space is said to be *complete* if every $m_s$-Cauchy sequence $\{x_n\}$ in $X$ converges, with respect to $\tau_{m_s}$, to a point $x \in X$ such that

$$\lim_{n\to\infty}(m_s(x_n,x_n,x) - m_{s_{x_n,x_n,x}}) = 0, \quad \lim_{n\to\infty}(M_{s_{x_n,x_n,x}} - m_{s_{x_n,x_n,x}}) = 0.$$

**Lemma 2.1.** Let $(X, m_s)$ be an $M_s$-metric space. Then:

(1) $\{x_n\}$ is an $M_s$-Cauchy sequence in $(X, m_s)$ if and only if it is an $S$-Cauchy sequence in the $S$-metric space $(X, m_s^w)$.

(2) An $M_s$-metric space $(X, m_s)$ is complete if and only if the $S$-metric space $(X, m_s^w)$ is complete. Furthermore,

$$\begin{aligned}&\lim_{n\to\infty} m_s^w(x_n, x_n, x) = 0\\ \Longrightarrow\ &\lim_{n\to\infty}(m_s(x_n, x_n, x) - m_{s_{x_n,x_n,x}}) = 0,\ \lim_{n\to\infty}(M_{s_{x_n,x_n,x}} - m_{s_{x_n,x_n,x}}) = 0.\end{aligned}$$

*Proof.* It is obviously follows from the definitions of $M_s$-Cauchy sequence, $M_s$-completeness, $S$-Cauchy sequence and $S$-completeness. □

Meanwhile, the above assertions are true for $m_s^*$.

**Lemma 2.2.** Assume that $x_n \to x$ and $y_n \to y$ as $n \to \infty$ in an $M_s$-metric space $(X, m_s)$. Then:

$$\lim_{n\to\infty} m_s(x_n, x_n, y_n) - m_{s_{x_n,x_n,y_n}} = m_s(x, x, y) - m_{s_{x,x,y}}.$$

*Proof.* We have

$$\begin{aligned}&|(m_s(x_n, x_n, y_n) - m_{s_{x_n,x_n,y_n}}) - (m_s(x, x, y) - m_{s_{x,x,y}})|\\ \leq\ &2|m_s(x_n, x_n, x) - m_{s_{x_n,x_n,x}}| + 2|m_s(y_n, y_n, y) - m_{s_{y_n,y_n,y}}|.\end{aligned}$$

□

From Lemma 2.2, we can deduce the following lemma:

**Lemma 2.3.** Assume that $x_n \to x$ as $n \to \infty$ in an $M_s$-metric space $(X, m_s)$. Then:

$$\lim_{n\to\infty} m_s(x_n, x_n, y) - m_{s_{x_n,x_n,y}} = m_s(x, x, y) - m_{s_{x,x,y}},$$

$\forall y \in X$.

**Lemma 2.4.** Assume that $x_n \to x$ and $x_n \to y$ as $n \to \infty$ in an $M_s$-metric space $(X, m_s)$. Then: $m_s(x, x, y) = m_{s_{x,x,y}}$. Furthermore, if $m_s(x, x, x) = m_s(y, y, y)$, then $x = y$.

*Proof.* By Lemma 2.2, we have

$$0 = \lim_{n\to\infty} m_s(x_n, x_n, x_n) - m_{s_{x_n,x_n,x_n}} = m_s(x, x, y) - m_{s_{x,x,y}}.$$

□

**Lemma 2.5.** Let $\{x_n\}$ be a sequence in $M_s$-metric space $(X, m_s)$ such that there exists $r \in [0, 1)$ such that

$$m_s(x_{n+1}, x_{n+1}, x_n) \leq r m_s(x_n, x_n, x_{n-1}), \tag{2}$$

for $\forall n \in \mathbb{N}$. Then we have

(1) $\lim_{n\to\infty} m_s(x_n, x_n, x_{n-1}) = 0$.
(2) $\lim_{n\to\infty} m_s(x_n, x_n, x_n) = 0$.
(3) $\lim_{n,m\to\infty} m_{s_{x_n,x_n,x_m}} = 0$.
(4) $\{x_n\}$ is an $M_s-$Cauchy sequence.

*Proof.* From the equation (2), we have

$$m_s(x_n, x_n, x_{n-1}) \leq r m_s(x_{n-1}, x_{n-1}, x_{n-2}) \leq r^2 m_s(x_{n-2}, x_{n-2}, x_{n-3}) \leq \cdots \leq r^n m_s(x_1, x_1, x_0)$$

and so $\lim_{n\to\infty} m_s(x_n, x_n, x_{n-1}) = 0$, which implies that $(A)$ holds. From $(m_s 2)$ and (1), we have

$$\lim_{n\to\infty} m_{s_{x_n, x_n, x_{n-1}}} \leq \lim_{n\to\infty} m_s(x_n, x_n, x_{n-1}) = 0,$$

that is, (2) holds. Clearly, (3) and (4) hold. □

**Theorem 2.1.** The topology $\tau_{m_s}$ is not Hausdorff.

*Proof.* Let $x, y, z \in X$ be such that $a := m_s(x, x, x) < m_s(z, z, z) = \frac{a+b}{2} < b := m_s(y, y, y)$ with

$$\frac{b}{2} < \frac{k}{2} < m_s(y, y, y) < M_{s_{x,x,y}} = b, \;\; r = 2m_s(x, x, y) - a - b > 0,$$

$$\max\{m_s(x, x, z), m_s(z, z, y)\} \leq (2m_s(x, x, y) - k)\frac{\epsilon}{r}.$$

Without loss of generality, we assume that, for each $\epsilon > 0$, $\epsilon < r$. Now, we need to prove that the intersection of the following neighborhoods is not empty:

$$U_x = \{z \in X : m_s(x, x, z) - m_{s_{x,x,z}} < \epsilon\}, \;\; V_y = \{z \in X : m_s(y, y, z) - m_{s_{y,y,z}} < \epsilon\}.$$

To prove $z \in U_x$, we have

$$m_s(x, x, z) < (2m_s(x, x, y) - k)\frac{\epsilon}{r},$$

$$\begin{aligned} m_s(x, x, z) - m_{s_{x,x,z}} &< (2m_s(x, x, y) - k)\frac{\epsilon}{r} - a \\ &< (2m_s(x, x, y) - k - a)\frac{\epsilon}{r} \\ &< (2m_s(x, x, y) - a - b)\frac{\epsilon}{r} \\ &= \epsilon, \end{aligned}$$

and, for any $z \in V_y$, we also have

$$m_s(y, y, z) < \;\; (2m_s(x, x, y) - k)\tfrac{\epsilon}{r},$$

$$\begin{aligned} m_s(y, y, z) - m_{s_{y,y,z}} &< (2m_s(x, x, y) - k)\frac{\epsilon}{r} - \frac{a+b}{2} \\ &< (2m_s(x, x, y) - k)\frac{\epsilon}{r} - \frac{a+b}{2}\frac{\epsilon}{r} \\ &= (2m_s(x, x, y) - k - \frac{a+b}{2})\frac{\epsilon}{r} \\ &< (2m_s(x, x, y) - a - b)\frac{\epsilon}{r} \\ &= \epsilon. \end{aligned}$$

So, we can find $x, y \in X$ such that, for all nonempty neighborhoods $U_x$ of $x$ and $V_y$ of $y$, $U_x \cap V_y \neq \emptyset$. This completes the proof. □

## 3. Main Results

The following definition is new version of definition in [3] for an $M_s-$metric space.

**Definition 3.1.** A Meir-Keeler mapping is a mapping $T : X \mapsto X$ on an $M_s-$metric space $(X, m_s)$ such that

$$\forall \epsilon > 0 \quad \exists \delta(\epsilon) > 0 \quad such \quad that \quad \forall x, y \in X \epsilon \leq m_s(x, x, y) < \epsilon + \delta(\epsilon) \quad implies \quad m_s(Tx, Tx, Ty) < \epsilon. \quad (3)$$

**Theorem 3.1.** Let $(X, m_s)$ be a complete $M_s-$metric space and let $T$ be a mapping from $X$ into itself satisfying the following condition:

$$\forall \epsilon > 0 \quad \exists \delta(\epsilon) > 0 \quad such \quad that \quad \forall x, y \in X \epsilon \leq m_s(x, x, y) < \epsilon + \delta(\epsilon) \Rightarrow m_s(Tx, Tx, Ty) < \epsilon.$$

Then, $T$ has a unique fixed point $x^* \in X$. Moreover, for all $x \in X$ the sequence $\{T^n(x)\}$ converges to $x^*$.

*Proof.* It is easy to check that $T$ is a strict contractive mapping, i.e.,

$$x \neq y \Rightarrow m_s(Tx, Tx, Ty) < m_s(x, x, y). \quad (4)$$

Let $x_0 \in X$ and define the sequence $\{x_n\}$ by $x_n = T^n(x_0), n \in \mathbb{N}$. So we have

$$m_s(x_n, x_n, x_{n-1}) = m_s(Tx_{n-1}, Tx_{n-1}, Tx_{n-2}) < m_s(x_{n-1}, x_{n-1}, x_{n-2}), \quad \forall n \in \mathbb{N}. \quad (5)$$

So the sequence $\{m_s(x_n, x_n, x_{n-1})\}$ is bounded below and decreasing.
Then, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} m_s(x_n, x_n, x_{n-1}) = \lim_{n\to\infty} m_s(x_{n-1}, x_{n-1}, x_n) = r.$$

Next, we will show that $r = 0$. If $r > 0$, therefore, $m_s(x_n, x_n, x_{n-1}) \geq r$, for $\forall n \in \mathbb{N}$.
On the other hand, for $r > 0$, by the hypothesis that $T$ is MKC mapping, there exists $\delta(r) > 0$ such that

$$r \leq m_s(x_{n-1}, x_{n-1}, x_{n-2}) < r + \delta(r) \Rightarrow m_s(Tx_{n-1}, Tx_{n-1}, Tx_{n-2}) = m_s(x_n, x_n, x_{n-1}) < r,$$

which implies that it is a contradiction. Hence, $r = 0$. Then, we have that

$$\lim_{n\to\infty} m_s(x_n, x_n, x_{n-1}) = \lim_{n\to\infty} m_s(x_{n-1}, x_{n-1}, x_n) = 0. \quad (6)$$

and

$$\lim_{n\to\infty} \min\{m_s(x_{n-1}, x_{n-1}, x_{n-1}), m_s(x_n, x_n, x_n)\} = \lim_{n\to\infty} m_{s_{x_n, x_n, x_{n-1}}} < \lim_{n\to\infty} m_s(x_n, x_n, x_{n-1}) = 0. \quad (7)$$

And then, we also have that

$$\lim_{m,n\to\infty} m_{s_{x_m, x_m, x_n}} = 0 \quad and \quad \lim_{m,n\to\infty} M_{s_{x_m, x_m, x_n}} = 0. \quad (8)$$

Next, we claim that $\lim_{m,n\to\infty} m_s(x_m, x_m, x_n) = 0$, that is
for every $\epsilon > 0$, there exists $N \in \mathbb{N}$, such that

$$m_s(x_l, x_l, x_{l+k}) < \epsilon, \quad (9)$$

for all $l \geq N$ and $k \in \mathbb{N}$.
For $\forall \epsilon > 0$.
By (7), we can choose $N_1 \in \mathbb{N}$ such that for all $m, n > N_1$,

$$m_{s_{x_m, x_m, x_n}} < \frac{\epsilon}{4}. \tag{10}$$

Since $\{m_s(x_{n-1}, x_{n-1}, x_n\}$ converges to 0, as $n \to \infty$, for every $\delta > 0$, there exists $N_2 \in \mathbb{N}$ such that

$$m_s(x_{n-1}, x_{n-1}, x_n) < \frac{\delta}{4}, \quad \forall n > N_2 + 1. \tag{11}$$

Choose $\delta$ such as $\delta < \epsilon$. We will prove (8) by using mathematic induction on $k$.
For $k = 1$, (8) becomes that

$$m_s(x_l, x_l, x_{l+1}) < \frac{\delta}{4} < \frac{\epsilon}{4} < \epsilon,$$

and clearly holds for all $l > N_2 + 1$, due to (11) and the choice of $\delta$.
Assume that the inequality (9) holds for some $k = m$, that is

$$m_s(x_l, x_l, x_{l+m}) < \frac{\delta}{4} < \frac{\epsilon}{4} < \epsilon,$$

for $\forall l \geq N_2 + 1$.
For $k = m + 1$, we have to show that $m_s(x_l, x_l, x_{l+m}) < \epsilon$, for some $l \geq N$.
Take $N = \max\{N_1, N_2 + 1\}$. For all $l \geq N$, we have that

$$\begin{aligned}
& m_s(x_l, x_l, x_{l+m+1}) \\
\leq\ & (m_s(x_l, x_l, x_{l+m}) - m_{s_{x_l, x_l, x_{l+m}}}) + (m_s(x_l, x_l, x_{l+m}) - m_{s_{x_l, x_l, x_{l+m}}}) \\
+\ & (m_s(x_{l+m}, x_{l+m}, x_{l+m+1}) - m_{s_{x_{l+m}, x_{l+m}, x_{l+m+1}}}) + m_{s_{x_l, x_l, x_{l+m}}} \\
\leq\ & 2m_s(x_l, x_l, x_{l+m}) + m_s(x_{l+m}, x_{l+m}, x_{l+m+1}) + m_{s_{x_l, x_l, x_{l+m}}} \\
<\ & 2 \cdot \frac{\delta}{4} + \frac{\delta}{4} + \frac{\epsilon}{4} \\
<\ & 2 \cdot \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} \\
=\ & \epsilon.
\end{aligned}$$

Hence, (9) holds for $k = m + 1$.
Therefore, the claim is true.
So by (8) and $\lim_{m,n \to \infty} m_s(x_m, x_m, x_n) = 0$, we see that the sequence $\{x_n\}$ is a Cauchy sequence and by completeness of $X$, $x_n \to x^*$ in $m_s$ for some $x^* \in X$, that is

$$\lim_{n \to \infty} (m_s(x_n, x_n, x^*) - m_{s_{x_n, x_n, x^*}}) = 0. \tag{12}$$

But $m_{s_{x_n, x_n, x^*}} \to 0$, as $n \to \infty$, due to $m_s(x_n, x_n, x_n) \to 0$. So $m_s(x_n, x_n, x^*) \to 0$, as $n \to \infty$ .
Thus, by the hypothesis, we have that $m_s(Tx_n, Tx_n, Tx^*) < m_s(x_n, x_n, x^*) \to 0$, as $n \to \infty$. Hence, by $(m_s 2)$, we have that

$$m_{s_{Tx_n, Tx_n, Tx^*}} \leq m_s(Tx_n, Tx_n, Tx^*) \to 0.$$

Therefore, $Tx_n \to Tx^*$, as $n \to \infty$.
Equation (6) implies that $m_s(x_n, x_n, Tx_n) \to 0$, as $n \to \infty$. Since $m_{s_{x_n, x_n, Tx_n}} \to 0$, as $n \to \infty$, by

Lemma 2.2, we get $m_s(x^*, x^*, Tx^*) = m_{s_{x^*,x^*,Tx^*}}$.
On the other hand, by Lemma 2.2 and

$$Tx_{n-1} = x_n \to x^* \quad and \quad x_{n+1} = Tx_n \to Tx^*,$$

we have

$$\begin{aligned} 0 &= \lim_{n\to\infty}(m_s(x_n, x_n, Tx_n) - m_{s_{x_n,x_n,Tx_n}}) \\ &= \lim_{n\to\infty}(m_s(x_n, x_n, x_{n+1}) - m_{s_{x_n,x_n,Tx_n}}) \\ &= m_s(x^*, x^*, x^*) - m_{s_{x^*,x^*,Tx^*}} \\ &= m_s(Tx^*, Tx^*, Tx^*) - m_{s_{x^*,x^*,Tx^*}}. \end{aligned}$$

Thus, $m_s(x^*, x^*, x^*) = m_{s_{x^*,x^*,Tx^*}} = m_s(Tx^*, Tx^*, Tx^*)$.
And since

$$m_s(x^*, x^*, Tx^*) = m_s(Tx^*, Tx^*, x^*) = m_{s_{x^*,x^*,Tx^*}} = m_s(x^*, x^*, x^*) = m_s(Tx^*, Tx^*, Tx^*),$$

then, by Lemma 2.4, we have that $x^* = Tx^*$.
Uniqueness by the contraction (4) is clear. □

Next, we establish a fixed point theorem for a MKC mapping in $M_s-$metric space via Gupta-Saxena type contraction.

Put

$$C(x, x, y) = m_s(x, x, y) + \frac{(1 + m_s(x, x, Tx))m_s(y, y, Ty)}{1 + m_s(x, x, y)} + \frac{m_s(x, x, Tx)m_s(y, y, Ty)}{m_s(x, x, y)}.$$

**Theorem 3.2.** Let $(X, m_s)$ be a complete $M_s-$metric space and let $T$ be a continuous mapping from $X$ into itself satisfying the following condition:

$$\forall \epsilon > 0 \quad \exists \delta(\epsilon) > 0 \quad such \quad that \quad \forall x, y \in X \epsilon \le kC(x, x, y) < \epsilon + \delta(\epsilon) \Rightarrow m_s(Tx, Tx, Ty) < \epsilon, \quad (13)$$

for some $0 < k < \frac{1}{3}$.
Then, $T$ has a unique fixed point $x^* \in X$. Moreover, for all $x \in X$ the sequence $\{T^n(x)\}$ converges to $x^*$.

*Proof.* From definition of $C(x, x, y)$, it follows that $T$ is a strict contraction, i.e.,

$$x \neq y \Rightarrow m_s(Tx, Tx, Ty) < kC(x, x, y). \quad (14)$$

Let $x_0 \in X$ and define the sequence $\{x_n\}$ by $x_n = T^n(x_0), n \in \mathbb{N}$. So we have

$$\begin{aligned} C(x_{n-1}, x_{n-1}, x_n) &= m_s(x_{n-1}, x_{n-1}, x_n) + \frac{(1 + m_s(x_{n-1}, x_{n-1}, x_n))m_s(x_n, x_n, x_{n+1})}{1 + m_s(x_{n-1}, x_{n-1}, x_n)} \\ &+ \frac{m_s(x_{n-1}, x_{n-1}, x_n)m_s(x_n, x_n, x_{n+1})}{m_s(x_{n-1}, x_{n-1}, x_n)} \\ &= m_s(x_{n-1}, x_{n-1}, x_n) + 2m_s(x_n, x_n, x_{n+1}), \end{aligned}$$

and

$$\begin{aligned} m_s(x_n, x_n, x_{n+1}) &= m_s(Tx_{n-1}, Tx_{n-1}, Tx_n) \\ &< kC(x_{n-1}, x_{n-1}, x_n) \\ &= k(m_s(x_{n-1}, x_{n-1}, x_n) + 2m_s(x_n, x_n, x_{n+1})). \end{aligned}$$

Therefore,

$$m_s(x_n, x_n, x_{n+1}) < rm_s(x_{n-1}, x_{n-1}, x_n),$$

where $r = \frac{k}{1-2k} \in [0, 1)$ by the choice of $k$. Now, by Lemma 2.5, $\{x_n\}$ is a Cauchy sequence, and by completeness of $X$, $Tx_{n-1} = x_n \to x^*$ as $n \to \infty$ in $m_s$ for some $x^* \in X$. Since $T$ is a continuous mapping, so $x_n = Tx_{n-1} \to Tx^*$ as $n \to \infty$ in $m_s$. By Lemma 2.4, we find

$$m_s(x^*, x^*, Tx^*) = m_{s_{x^*, x^*, Tx^*}},$$

and

$$\begin{aligned} 0 &= \lim_{n\to\infty} (m_s(x_n, x_n, Tx_n) - m_{s_{x_n, x_n, Tx_n}}) \\ &= m_s(x^*, x^*, Tx^*) - m_{s_{x^*, x^*, Tx^*}} \\ &= m_s(Tx^*, Tx^*, Tx^*) - m_{s_{x^*, x^*, Tx^*}}. \end{aligned}$$

By Lemma 2.2, we have that

$$m_s(x^*, x^*, Tx^*) = m_{s_{x^*, x^*, Tx^*}} = m_s(x^*, x^*, x^*) = m_s(Tx^*, Tx^*, Tx^*).$$

Then, by Lemma 2.4, we have that $x^* = Tx^*$.
Uniqueness by the contraction (14) is clear. □

**Theorem 3.3.** Let $(X, m_s)$ be an $M_s-$metric space and let $T$ be a self-mapping defined on $X$. Assume that there exists a function $\varphi(t) : [0, \infty) \mapsto [0, \infty)$ satisfying the following conditions:

(1) $\varphi(0) = 0$ and $t > 0 \Rightarrow \varphi(t) > 0$;
(2) $\varphi$ is nondecreasing and right continuous;
(3) for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\epsilon \le \varphi(kC(x, x, y)) < \epsilon + \delta \Rightarrow \varphi(m_s(Tx, Tx, Ty)) < \epsilon, \tag{15}$$

for some $0 < k < \frac{1}{3}$ and for all $x, y \in X$ with $x \neq y$.
Then, (13) is satisfied.

*Proof.* Fix $\epsilon > 0$, so $\varphi(\epsilon) > 0$. By (15) there exists $\delta > 0$ such that

$$\forall x, y \in X, x \neq y, \varphi(\epsilon) < \varphi(kC(x, x, y)) < \varphi(\epsilon) + \delta \Rightarrow \varphi(m_s(Tx, Tx, Ty)) < \varphi(\epsilon).$$

In view of the fact that $\varphi$ is right continuous, then there exists $\delta' > 0$ such that

$$\varphi(\epsilon + \delta) < \varphi(\epsilon) + \delta'.$$

Now, for $x, y \in X$ with $x \neq y$ and fixed

$$\epsilon \leq kC(x, x, y) < \epsilon + \delta,$$

Since $\varphi$ is a nondecreasing mapping, we have

$$\varphi(\epsilon) \leq \varphi(kC(x, x, y)) < \varphi(\epsilon + \delta) < \varphi(\epsilon) + \delta'.$$

So we get

$$\varphi(m_s(Tx, Tx, Ty)) < \varphi(\epsilon),$$

which implies that $m_s(Tx, Tx, Ty) < \epsilon$, i.e., (13) is satisfied. □

**Corollary 3.1.** Let $(X, m_s)$ be an $M_s-$metric space and let $T$ be a self-mapping defined on $X$. Assume that there exists a function $h(t) : [0, \infty) \mapsto [0, \infty)$ is a locally integrable satisfying the following conditions:

(1) $\int_0^t h(s)ds > 0$ for all $t > 0$;
(2) for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\frac{1}{k}\epsilon \leq \int_0^{C(x,x,y)} h(s)ds < \frac{1}{k}\epsilon + \delta \Rightarrow \int_0^{\frac{1}{k}m_s(Tx,Tx,Ty)} h(s)ds < \frac{1}{k}\epsilon,$$

for some $0 < k < \frac{1}{3}$ and for all $x, u \in X$ with $x \neq y$.
Then (13) is satisfied.

*Proof.* Defining $\varphi(t)$ in Theorem 3.3 by $\varphi(t) = \int_0^t h(s)ds$, then we can draw the conclusion. □

Next, we establish a fixed point theorems for a MKC mapping in $M_s-$metric spaces via other fractional type contractions.

Put

$$M_A(x, y, z) = m_s(x, y, z) + \frac{m_s(x, Tx, Tx)m_s(y, Ty, Ty)m_s(z, Tz, Tz)}{m_s(x, y, z)m_s(Tx, Ty, Tz)} + \frac{m_s(x, Tx, Tx)m_s(y, Ty, Ty)m_s(z, Tz, Tz)}{m_s(x, y, z)^2}.$$

**Theorem 3.4.** Let $(X, m_s)$ be a complete $M_s-$metric space and let $T$ be a continuous mapping from $X$ into itself satisfying the following condition:

$$\forall \epsilon > 0 \quad \exists \delta(\epsilon) > 0 \quad such \quad that \quad \forall x, y, z \in X \epsilon \leq kM_A(x, y, z) < \epsilon + \delta(\epsilon) \Rightarrow m_s(Tx, Ty, Tz) < \epsilon. \quad (16)$$

for some $0 < k < \frac{1}{3}$.
Then, $T$ has a unique fixed point $x^* \in X$. Moreover, for all $x \in X$ the sequence $\{T^n(x)\}$ converges to $x^*$.

*Proof.* From definition of $M_A(x, y, z)$, it follows that $T$ is a strict contraction, i.e.,

$$x \neq y \neq z \Rightarrow m_s(Tx, Ty, Tz) < kM_A(x, y, z). \quad (17)$$

Let $x_0 \in X$ and define the sequence $\{x_n\}$ by $x_n = T^n(x_0), n \in \mathbb{N}$. So we have

$$\begin{aligned} M_A(x_{n-1}, x_{n-1}, x_n) &= m_s(x_{n-1}, x_{n-1}, x_n) + \frac{m_s(x_{n-1}, x_n, x_n)^2 m_s(x_n, x_{n+1}, x_{n+1})}{m_s(x_{n-1}, x_{n-1}, x_n) m_s(x_n, x_n, x_{n+1})} \\ &+ \frac{m_s(x_{n-1}, x_n, x_n)^2 m_s(x_n, x_{n+1}, x_{n+1})}{m_s(x_{n-1}, x_{n-1}, x_n)^2} \\ &\leq m_s(x_{n-1}, x_{n-1}, x_n) + \frac{m_s(x_{n-1}, x_{n-1}, x_n)^2 m_s(x_n, x_n, x_{n+1})}{m_s(x_{n-1}, x_{n-1}, x_n) m_s(x_n, x_n, x_{n+1})} \\ &+ \frac{m_s(x_{n-1}, x_n, x_n)^2 m_s(x_n, x_n, x_{n+1})}{m_s(x_{n-1}, x_{n-1}, x_n)^2} \quad (\text{" } \leq \text{"} by Lemma 1.1(1)) \\ &= m_s(x_{n-1}, x_{n-1}, x_n) + 2m_s(x_n, x_n, x_{n+1}), \end{aligned}$$

and

$$\begin{aligned} m_s(x_n, x_n, x_{n+1}) &= m_s(Tx_{n-1}, Tx_{n-1}, Tx_n) \\ &< k M_A(x_{n-1}, x_{n-1}, x_n) \\ &< k(m_s(x_{n-1}, x_{n-1}, x_n) + 2m_s(x_n, x_n, x_{n+1})). \end{aligned}$$

Therefore,

$$m_s(x_n, x_n, x_{n+1}) < r m_s(x_{n-1}, x_{n-1}, x_n),$$

where $r = \frac{k}{1-2k} \in [0, 1)$ by the choice of $k$. Now, by Lemma 2.5, $\{x_n\}$ is a Cauchy sequence, and by completeness of $X$, $Tx_{n-1} = x_n \to x^*$ as $n \to \infty$ in $m_s$ for some $x^* \in X$. Since $T$ is a continuous mapping, so $x_n = Tx_{n-1} \to Tx^*$ as $n \to \infty$ in $m_s$. By Lemma 2.4, we find

$$m_s(x^*, x^*, Tx^*) = m_{s_{x^*, x^*, Tx^*}},$$

and

$$\begin{aligned} 0 &= \lim_{n \to \infty} (m_s(x_n, x_n, Tx_n) - m_{s_{x_n, x_n, Tx_n}}) \\ &= m_s(x^*, x^*, Tx^*) - m_{s_{x^*, x^*, Tx^*}} \\ &= m_s(Tx^*, Tx^*, Tx^*) - m_{s_{x^*, x^*, Tx^*}}. \end{aligned}$$

By Lemma 2.2, we have that

$$m_s(x^*, x^*, Tx^*) = m_{s_{x^*, x^*, Tx^*}} = m_s(x^*, x^*, x^*) = m_s(Tx^*, Tx^*, Tx^*).$$

Then, by Lemma 2.4, we have that $x^* = Tx^*$.
Uniqueness by the contraction (17) is clear. □

Put

$$\begin{aligned} M_B(x, y, z) &= m_s(x, y, z) + \frac{(1 + m_s(x, x, Tx)) m_s(y, y, Ty)}{1 + m_s(x, y, z)} \\ &+ \frac{(1 + m_s(y, y, Ty)) m_s(z, z, Tz)}{1 + m_s(x, y, z)} + \frac{m_s(x, x, Tx) m_s(y, y, Ty) m_s(z, z, Tz)}{m_s(x, y, z)^2}. \end{aligned}$$

**Theorem 3.5.** Let $(X, m_s)$ be a complete $M_s-$metric space and let $T$ be a continuous mapping from $X$ into itself satisfying the following condition:

$$\forall \epsilon > 0 \quad \exists \delta(\epsilon) > 0 \quad such \quad that \quad \forall x, y, z \in X \epsilon \leq k M_B(x, y, z) < \epsilon + \delta(\epsilon) \Rightarrow m_s(Tx, Ty, Tz) < \epsilon. \quad (18)$$

for some $0 < k < \frac{1}{4}$.
Then, $T$ has a unique fixed point $x^* \in X$. Moreover, for all $x \in X$ the sequence $\{T^n(x)\}$ converges to $x^*$.

*Proof.* From definition of $M_B(x, y, z)$, it follows that $T$ is a strict contraction, i.e.,

$$x \neq y \neq z \Rightarrow m_s(Tx, Ty, Tz) < kM_B(x, y, z). \tag{19}$$

Let $x_0 \in X$ and define the sequence $\{x_n\}$ by $x_n = T^n(x_0), n \in \mathbb{N}$. So we have

$$\begin{aligned} M_B(x_{n-1}, x_{n-1}, x_n) &= m_s(x_{n-1}, x_{n-1}, x_n) + \frac{(1 + m_s(x_{n-1}, x_{n-1}, x_n))m_s(x_{n-1}, x_{n-1}, x_n)}{1 + m_s(x_{n-1}, x_{n-1}, x_n)} \\ &+ \frac{(1 + m_s(x_{n-1}, x_{n-1}, x_n))m_s(x_n, x_n, x_{n+1})}{1 + m_s(x_{n-1}, x_{n-1}, x_n)} + \frac{m_s(x_{n-1}, x_{n-1}, x_n)^2 m_s(x_n, x_n, x_{n+1})}{m_s(x_{n-1}, x_{n-1}, x_n)^2} \\ &= 2(m_s(x_{n-1}, x_{n-1}, x_n) + m_s(x_n, x_n, x_{n+1})), \end{aligned}$$

and

$$\begin{aligned} m_s(x_n, x_n, x_{n+1}) &= m_s(Tx_{n-1}, Tx_{n-1}, Tx_n) \\ &< kM_B(x_{n-1}, x_{n-1}, x_n) \\ &< 2k(m_s(x_{n-1}, x_{n-1}, x_n) + m_s(x_n, x_n, x_{n+1})). \end{aligned}$$

Therefore,

$$m_s(x_n, x_n, x_{n+1}) < rm_s(x_{n-1}, x_{n-1}, x_n),$$

where $r = \frac{2k}{1-2k} \in [0, 1)$ by the choice of $k$.
Then, the conclusion we can directly obtain by using the similar arguments to the proof of Theorem 3.4. □

Replacing $C(x, x, y)$ by $M_A(x, y, z)$ or $M_B(x, y, z)$ in Theorem 3.3 and Corollary 3.1, we can obtain the following theorem and corollary.

**Theorem 3.6.** Let $(X, m_s)$ be an $M_s-$metric space and let $T$ be a self-mapping defined on $X$. Assume that there exists a function $\varphi(t) : [0, \infty) \mapsto [0, \infty)$ satisfying the following conditions:

(1) $\varphi(0) = 0$ and $t > 0 \Rightarrow \varphi(t) > 0$;
(2) $\varphi$ is nondecreasing and right continuous;
(3) for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\epsilon \leq \varphi(kM_A(x, x, y)) < \epsilon + \delta \Rightarrow \varphi(m_s(Tx, Tx, Ty)) < \epsilon,$$

for some $0 < k < \frac{1}{3}$ and for all $x, y, z \in X$ with $x \neq y \neq z$.
Then, (16) is satisfied.

**Theorem 3.7.** Let $(X, m_s)$ be an $M_s-$metric space and let $T$ be a self-mapping defined on $X$. Assume that there exists a function $\varphi(t) : [0, \infty) \mapsto [0, \infty)$ satisfying the following conditions:

(1) $\varphi(0) = 0$ and $t > 0 \Rightarrow \varphi(t) > 0$;
(2) $\varphi$ is nondecreasing and right continuous;

(3) for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\epsilon \leq \varphi(kM_B(x,x,y)) < \epsilon + \delta \Rightarrow \varphi(m_s(Tx,Tx,Ty)) < \epsilon,$$

for some $0 < k < \frac{1}{4}$ and for all $x, y, z \in X$ with $x \neq y \neq z$.
Then, (18) is satisfied.

**Corollary 3.2.** Let $(X, m_s)$ be an $M_s-$metric space and let $T$ be a self-mapping defined on $X$. Assume that there exists a function $h(t) : [0,\infty) \mapsto [0,\infty)$ is a locally integrable satisfying the following conditions:

(1) $\int_0^t h(s)ds > 0$ for all $t > 0$;
(2) for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\frac{1}{k}\epsilon \leq \int_0^{M_A(x,y,z)} h(s)ds < \frac{1}{k}\epsilon + \delta \Rightarrow \int_0^{\frac{1}{k}m_s(Tx,Ty,Tz)} h(s)ds < \frac{1}{k}\epsilon,$$

for some $0 < k < \frac{1}{3}$ and for all $x, u \in X$ with $x \neq y \neq z$.
Then (16) is satisfied.

**Corollary 3.3.** Let $(X, m_s)$ be an $M_s-$metric space and let $T$ be a self-mapping defined on $X$. Assume that there exists a function $h(t) : [0,\infty) \mapsto [0,\infty)$ is a locally integrable satisfying the following conditions:

(1) $\int_0^t h(s)ds > 0$ for all $t > 0$;
(2) for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\frac{1}{k}\epsilon \leq \int_0^{M_B(x,y,z)} h(s)ds < \frac{1}{k}\epsilon + \delta \Rightarrow \int_0^{\frac{1}{k}m_s(Tx,Ty,Tz)} h(s)ds < \frac{1}{k}\epsilon,$$

for some $0 < k < \frac{1}{4}$ and for all $x, u \in X$ with $x \neq y \neq z$.
Then (18) is satisfied.

We are now in a position to define two new types of Meir-Keeler contractions on $M_s-$metric spaces, say type $A$ and type $B$.

**Definition 3.2.** Let $(X, m_s)$ ba an $M_s-$metric space. A self-mapping $T : X \mapsto X$ is said to be a Meir-Keeler contraction mapping of type $A$ if for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$\epsilon \leq M(x,y) < \epsilon + \delta(\epsilon) \quad implies \quad m_s(Tx,Tx,Ty) < \epsilon, \tag{20}$$

where $M(x,y) = \min\{m_s(x,x,y), m_s(x,x,Tx), m_s(y,y,Ty)\}$, for all $x, y \in X$.

**Definition 3.3.** Let $(X, m_s)$ ba an $M_s-$metric space. A self-mapping $T : X \mapsto X$ is said to be a Meir-Keeler contraction mapping of type $B$ if for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$\epsilon \leq N(x,y) < \epsilon + \delta(\epsilon) \quad implies \quad m_s(Tx,Tx,Ty) < \epsilon, \tag{21}$$

where $N(x,y) = \min\{m_s(x,x,y), \frac{1}{2}[m_s(x,x,Tx) + m_s(y,y,Ty)]\}$, for all $x, y \in X$.

**Remark 3.1.** (1) Suppose that $T : X \mapsto X$ is a Meir-Keeler contraction mapping of type $A$ (respectively, type $B$). Then

$$m_s(Tx, Tx, Ty) < M(x, y) \quad (respectively, N(x, y)),$$

for all $x, y \in X$ with $x \neq y$.

(2) It is readily verified that $M(x, y) \leq N(x, y)$ for all $x, y \in X$, where $M(x, y), N(x, y)$ are defined in Definition 3.2 and Definition 3.3.

**Theorem 3.8.** Let $(X, m_s)$ be a complete $M_s-$metric space and let $T$ be a mapping from $X$ into itself satisfying the following condition:

$$\forall \epsilon > 0 \quad \exists \delta(\epsilon) > 0 \quad such \quad that \quad \forall x, y \in X \epsilon \leq M(x, y) < \epsilon + \delta(\epsilon) \Rightarrow m_s(Tx, Tx, Ty) < \epsilon, \tag{22}$$

where $M(x, y) = \min\{m_s(x, x, y), m_s(x, x, Tx), m_s(y, y, Ty)\}$.
Then, $T$ has a unique fixed point $x^* \in X$. Moreover, for all $x \in X$ the sequence $\{T^n(x)\}$ converges to $x^*$.

*Proof.* It is readily verified that $T$ is a strict contractive mapping, i.e.,

$$x \neq y \Rightarrow m_s(Tx, Tx, Ty) < M(x, y), \tag{23}$$

where $M(x, y) = \min\{m_s(x, x, y), m_s(x, x, Tx), m_s(y, y, Ty)\}$.
Let $x_0 \in X$ and define the sequence $\{x_n\}$ by $x_n = T^n(x_0), n \in \mathbb{N}$. So we have

$$m_s(x_n, x_n, x_{n-1}) = m_s(Tx_{n-1}, Tx_{n-1}, Tx_{n-2}) < M(x_{n-1}, x_{n-2}).$$

where $M(x_{n-1}, x_{n-2}) = \min\{m_s(x_{n-1}, x_{n-1}, x_{n-2}), m_s(x_{n-1}, x_{n-1}, x_n), m_s(x_{n-2}, x_{n-2}, x_{n-1})\}$.
In what follows, we examine three cases.
Case1. Assume that $M(x_{n-1}, x_{n-2}) = m_s(x_{n-1}, x_{n-1}, x_{n-2})$. Then (22) becomes that

$$\epsilon \leq m_s(x_{n-1}, x_{n-1}, x_{n-2}) < \epsilon + \delta \Rightarrow m_s(x_n, x_n, x_{n-1}) < \epsilon.$$

Therefore, we deduce that

$$m_s(x_n, x_n, x_{n-1}) < \epsilon \leq m_s(x_{n-1}, x_{n-1}, x_{n-2}),$$

for all $n \in \mathbb{N}$. That is $\{m_s(x_n, x_n, x_{n-1})\}$ is a bounded below and decreasing, and it converges to some $r \geq 0$. To show that $r = 0$, we assume the contrary that $r > 0$. Then we must have

$$0 < r \leq m_s(x_n, x_n, x_{n-1}), \quad \forall n \in \mathbb{N}.$$

On the other hand, for $r > 0$, by the hypothesis that $T$ is MKC mapping of type $A$, there exists $\delta(r) > 0$ such that

$$r \leq m_s(x_{n-1}, x_{n-1}, x_{n-2}) < r + \delta(r) \Rightarrow m_s(Tx_{n-1}, Tx_{n-1}, Tx_{n-2}) = m_s(x_n, x_n, x_{n-1}) < r,$$

which is a contradiction. Hence, $r = 0$. Therefore, we get

$$\lim_{n\to\infty} m_s(x_n, x_n, x_{n-1}) = \lim_{n\to\infty} m_s(x_{n-1}, x_{n-1}, x_n) = 0. \tag{24}$$

Case2. Assume that $M(x_{n-1}, x_{n-2}) = m_s(x_{n-1}, x_{n-1}, x_n)$. Then (22) becomes that

$$\epsilon \leq m_s(x_{n-1}, x_{n-1}, x_n) < \epsilon + \delta \Rightarrow m_s(x_n, x_n, x_{n+1}) < \epsilon.$$

Therefore, we deduce that

$$m_s(x_n, x_n, x_{n-1}) < \epsilon \leq m_s(x_{n-1}, x_{n-1}, x_{n-2}),$$

for all $n \in \mathbb{N}$. That is $\{m_s(x_n, x_n, x_{n+1})\}$ is a bounded below and decreasing, and it converges to some $L \geq 0$. In fact, the limit $L$ of this sequence is 0, which can be shown by mimicking the proof of (24) done above.

Case3. Assume that $M(x_{n-1}, x_{n-2}) = m_s(x_{n-2}, x_{n-2}, x_{n-1})$. Then (22) becomes that

$$\epsilon \leq m_s(x_{n-2}, x_{n-2}, x_{n-1}) < \epsilon + \delta \Rightarrow m_s(x_{n-1}, x_{n-1}, x_n) < \epsilon.$$

Therefore, we deduce that

$$m_s(x_{n-1}, x_{n-1}, x_n) < \epsilon \leq m_s(x_{n-2}, x_{n-2}, x_{n-1}),$$

for all $n \in \mathbb{N}$. As in two cases above, the sequence $\{m_s(x_{n-1}, x_{n-1}, x_n)\}$ is a bounded below and decreasing, hence it converges to 0.

As a result, we see all three cases, the sequence $\{m_s(x_n, x_n, x_{n-1})\}$ converges to 0.

Form (21), we get that

$$\lim_{n\to\infty} \min\{m_s(x_{n-1}, x_{n-1}, x_{n-1}), m_s(x_n, x_n, x_n)\} = \lim_{n\to\infty} m_{s_{x_n, x_n, x_{n-1}}} < \lim_{n\to\infty} m_s(x_n, x_n, x_{n-1}) = 0. \quad (25)$$

and

$$\lim_{m,n\to\infty} m_{s_{x_m, x_m, x_n}} = 0 \quad \textit{and} \quad \lim_{m,n\to\infty} M_{s_{x_m, x_m, x_n}} = 0. \quad (26)$$

Using similar arguments as in proof of Theorem 3.1, it can be shown that

$$\lim_{m,n\to\infty} m_s(x_m, x_m, x_n) = 0.$$

So, by (26) and $\lim_{m,n\to\infty} m_s(x_m, x_m, x_n) = 0$, we get that the sequence $\{x_n\}$ is a Cauchy sequence and by completeness of $X$, $x_n \to x^*$ in $m_s$ for some $x^* \in X$, that is

$$\lim_{n\to\infty} (m_s(x_n, x_n, x^*) - m_{s_{x_n, x_n, x^*}}) = 0.$$

But $m_{s_{x_n, x_n, x^*}} \to 0$, as $n \to \infty$, due to $m_s(x_n, x_n, x_n) \to 0$. So $m_s(x_n, x_n, x^*) \to 0$, as $n \to \infty$ .

Thus, by the hypothesis, we have that $m_s(Tx_n, Tx_n, Tx^*) < M(x_n, x^*) \to 0$, as $n \to \infty$. Hence, by $(m_s 2)$, we have that

$$m_{s_{Tx_n, Tx_n, Tx^*}} \leq m_s(Tx_n, Tx_n, Tx^*) \to 0.$$

Therefore, $Tx_n \to Tx^*$, as $n \to \infty$.

Equation (24) implies that $m_s(x_n, x_n, Tx_n) \to 0$, as $n \to \infty$. Since $m_{s_{x_n, x_n, Tx_n}} \to 0$, as $n \to \infty$, by Lemma 2.2, we get $m_s(x^*, x^*, Tx^*) = m_{s_{x^*, x^*, Tx^*}}$.

On the other hand, by Lemma 2.2 and

$$Tx_{n-1} = x_n \to x^* \quad \textit{and} \quad x_{n+1} = Tx_n \to Tx^*,$$

we have

$$\begin{aligned} 0 &= \lim_{n\to\infty}(m_s(x_n,x_n,Tx_n) - m_{s_{x_n,x_n,Tx_n}}) \\ &= \lim_{n\to\infty}(m_s(x_n,x_n,x_{n+1}) - m_{s_{x_n,x_n,Tx_n}}) \\ &= m_s(x^*,x^*,x^*) - m_{s_{x^*,x^*,Tx^*}} \\ &= m_s(Tx^*,Tx^*,Tx^*) - m_{s_{x^*,x^*,Tx^*}}. \end{aligned}$$

Thus, $m_s(x^*,x^*,x^*) = m_{s_{x^*,x^*,Tx^*}} = m_s(Tx^*,Tx^*,Tx^*)$.
And since

$$m_s(x^*,x^*,Tx^*) = m_s(Tx^*,Tx^*,x^*) = m_{s_{x^*,x^*,Tx^*}} = m_s(x^*,x^*,x^*) = m_s(Tx^*,Tx^*,Tx^*),$$

then, by Lemma 2.4, we have that $x^* = Tx^*$.
Uniqueness by the contraction (23) is clear. □

In what follows, we present an existence and uniqueness theorem for fixed point of Meir-Keeler contraction of type $B$. Taking Remark 3.1 into account, we observe that the proof of this is similar to the proof of Theorem 3.8.

**Theorem 3.9.** Let $(X,m_s)$ be a complete $M_s-$metric space and let $T$ be a mapping from $X$ into itself satisfying the following condition:

$$\forall\epsilon>0 \quad \exists\delta(\epsilon)>0 \quad such \quad that \quad \forall x,y\in X \epsilon\le N(x,y)<\epsilon+\delta(\epsilon)\Rightarrow m_s(Tx,Tx,Ty)<\epsilon, \qquad (27)$$

where $N(x,y)=\min\{m_s(x,x,y),\frac{1}{2}[m_s(x,x,Tx)+m_s(y,y,Ty)]\}$.
Then, $T$ has a unique fixed point $x^*\in X$. Moreover, for all $x\in X$ the sequence $\{T^n(x)\}$ converges to $x^*$.

**Competing interests**

The authors declare that they have no competing interests.

**Authors' contributions**

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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# References

[1] M. Nabil, *A contraction principle in partial S-metric spaces*, Universal J. Math. and Mathmatical, **5**(2014), 109–119.

[2] M. Nabil, S. Nizar, A. Thabet, *A new extension of S-metric spaces*, J. Nonlinear Sci. Appl., in press.

[3] A. Meir, E. Keeler, *A theorem on contraction mappings*, J.Math.Anal.Appl., **28**(1969), 326–329.

[4] M. Maiti, T. K. Pal, *Generalizations of two fixed points theorems*, Bull. Calcutta Math. Soc., **70**(1978), 57–61.

[5] S. Park, B. E. Rhoades, *Meir-Keeler type contractive conditions*, Math. Japn., **26**(1)(1981), 13–20.

[6] C. Mongkolkeha, P. Kumam, *Best proximity points for asympototic proximal pointwise weaker Meir-Keeler-type $\psi-$contraction mappings*, J. Egypt. Math. Soc., **26**(1)(1981), 13–20.

[7] A. Latif, M. E. Gordji, E. Karapinar, W. Sintunavarat, *Fixed point results for generalized $(\alpha-\psi)-$Meir-Keeler contractive mappings and applications*, Journal of Inequalities and Applications, **68**(2014), 11 pages.

# Tender and naive weak closure operations on lower $BCK$-semilattices

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**Abstract.** The notion of tender and naive weak closure operation is introduced, and their relations and properties are investigated. Using a weak closure operation "$cl$" and an ideal $A$ of a lower $BCK$-semilattice $X$ with the greatest element 1, a new ideal $K$ of $X$ containing the ideal $A^{cl}$ of $X$ is established. Using this ideal $K$, a new function

$$cl_t : \mathcal{I}(X) \to \mathcal{I}(X),\ A \mapsto K$$

is given, and related properties are considered. We show that if "$cl$" is a tender (resp., naive) weak closure operation on $\mathcal{I}(X)$, then so are "$cl_t$" and "$cl_f$".

## 1. Introduction

In [2], Bordbar et al. introduced a weak closure operation, which is more general form than closure operation, on ideals of $BCK$-algebras, and investigated related properties. Regarding weak closure operation, they defined finite type and (strong) quasi-primeness, and investigated related properties. They also discussed positive implicative (resp., commutative and implicative) weak closure operations, and provided several examples to illustrate notions and properties.

In this paper, we introduce the notion of tender and naive weak closure operation, and investigate their relations and properties. Using a weak closure operation "$cl$" and an ideal $A$ of a lower $BCK$-semilattice $X$ with the greatest element 1, we construct a new ideal $K$ of $X$ containing the ideal $A^{cl}$ of $X$. Using this ideal $K$, we define a new function

$$cl_t : \mathcal{I}(X) \to \mathcal{I}(X),\ A \mapsto K$$

and investigate related properties. We show that if "$cl$" is a tender (resp., naive) weak closure operation on $\mathcal{I}(X)$, then so are "$cl_t$" and "$cl_f$".

## 2. Preliminaries

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A $BCK/BCI$-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a *$BCI$-algebra* if it satisfies the following conditions

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,
(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,
(III) $(\forall x \in X)\ (x * x = 0)$,
(IV) $(\forall x, y \in X)\ (x * y = 0,\ y * x = 0\ \Rightarrow\ x = y)$.

If a $BCI$-algebra $X$ satisfies the following identity

(V) $(\forall x \in X)\ (0 * x = 0)$,

then $X$ is called a *$BCK$-algebra*.

Any $BCK/BCI$-algebra $X$ satisfies the following conditions

(a1) $(\forall x \in X)\ (x * 0 = x)$,
(a2) $(\forall x, y, z \in X)\ (x \leq y\ \Rightarrow\ x * z \leq y * z,\ z * y \leq z * x)$,
(a3) $(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)$,
(a4) $(\forall x, y, z \in X)\ ((x * z) * (y * z) \leq x * y)$

where $x \leq y$ if and only if $x * y = 0$. A $BCK$-algebra $X$ is called a *lower $BCK$-semilattice* (see [6]) if $X$ is a lower semilattice with respect to the $BCK$-order.

A subset $A$ of a $BCK/BCI$-algebra $X$ is called an *ideal* of $X$ (see [6]) if it satisfies

$$0 \in A, \tag{2.1}$$

$$(\forall x \in X)\,(\forall y \in A)\,(x * y \in A\ \Rightarrow\ x \in A)\,. \tag{2.2}$$

Note that every ideal $A$ of a $BCK/BCI$-algebra $X$ satisfies the following implication (see [6]).

$$(\forall x, y \in X)\,(x \leq y,\ y \in A\ \Rightarrow\ x \in A)\,. \tag{2.3}$$

For any subset $A$ of $X$, the ideal generated by $A$ is defined to be the intersection of all ideals of $X$ containing $A$, and it is denoted by $\langle A \rangle$. If $A$ is finite, then we say that $\langle A \rangle$ is *finitely generated ideal* of $X$ (see [6]).

Let $\mathcal{I}(X)$ and $\mathcal{I}_f(X)$ be a set of all ideals of $X$ and a set of all finitely generated ideals of $X$, respectively.

We refer the reader to the books [5, 6] for further information regarding $BCK/BCI$-algebras.

## 3. **Tender and naive weak closure operations**

In what follows, let $X$ be a lower $BCK$-semilattice unless otherwise specified. For any $a, b \in X$, denote by $a \wedge b$ the greatest lower bound of $a$ and $b$.

**Definition 3.1** ([2])**.** An element $x$ of $X$ is called a *zeromeet element* of $X$ if the condition

$$(\exists\, y \in X \setminus \{0\})\,(x \wedge y = 0)$$

is valid. Otherwise, $x$ is called a *non-zeromeet element* of $X$.

Denote by $Z(X)$ the set of all zeromeet elements of $X$, that is,

$$Z(X) = \{x \in X \mid x \wedge y = 0 \text{ for some nonzero element } y \in X\}.$$

Obviously, $0 \in Z(X)$ and if $X$ has the greatest element 1, then $1 \in X \setminus Z(X)$.

**Lemma 3.2** ([2]). *For any $x, y \in X$, if $x, y \notin Z(X)$, then $x \wedge y \notin Z(X)$, that is, the set $X \setminus Z(X)$ is closed under the operation $\wedge$.*

**Definition 3.3** ([4]). For any nonempty subsets $A$ and $B$ of $X$, we denote

$$A \wedge B := \langle \{a \wedge b \mid a \in A, b \in B\} \rangle$$

which is called the *meet ideal* of $X$ generated by $A$ and $B$. In this case, we say that the operation "$\wedge$" is a *meet operation*. If $A = \{a\}$, then $\{a\} \wedge B$ is denoted by $a \wedge B$. Also, if $B = \{b\}$, then $A \wedge \{b\}$ is denoted by $A \wedge b$.

**Definition 3.4** ([3]). For any nonempty subsets $A$ and $B$ of $X$, we define a set

$$(A :_{\wedge} B) := \{x \in X \mid x \wedge B \subseteq A\}$$

whenever $x \wedge B$ exists for all $x \in X$, and it is called the *relative annihilator* of $B$ with respect to $A$.

**Lemma 3.5** ([3]). *If $A$ and $B$ are ideals of a lower $BCK$-semilattice $X$, then the relative annihilator $(A :_{\wedge} B)$ of $B$ with respect to $A$ is an ideal of $X$.*

**Definition 3.6** ([2]). A mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ is called a *weak closure operation* on $\mathcal{I}(X)$ if the following conditions are valid.

$$(\forall A \in \mathcal{I}(X))\,(A \subseteq cl(A))\,, \tag{3.1}$$

$$(\forall A, B \in \mathcal{I}(X))\,(A \subseteq B \;\Rightarrow\; cl(A) \subseteq cl(B))\,. \tag{3.2}$$

If a weak closure operation $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ satisfies the condition

$$(\forall A \in \mathcal{I}(X))\,(cl(cl(A)) = cl(A))\,, \tag{3.3}$$

then we say that "$cl$" is a closure operation on $\mathcal{I}(X)$ (see [1]). In what follows, we use $A^{cl}$ instead of $cl(A)$.

For non-zeromeet elements $a$ and $b$ of $X$ and $A \in \mathcal{I}(X)$, we consider two ideals

$$((a \wedge A)^{cl} :_{\wedge} \langle b \rangle) \text{ and } A^{cl},$$

and investigate their relations where "$cl$" is a weak closure operation on $\mathcal{I}(X)$. In the following example, we will check that there are following relations:

(1) $((a \wedge A)^{cl} :_{\wedge} \langle b \rangle) \subseteq A^{cl}$ for some $A \in \mathcal{I}(X)$ and some non-zeromeet elements $a$ and $b$ of $X$,

(2) $((a \wedge A)^{cl} :_{\wedge} \langle b \rangle) \supseteq A^{cl}$ for some $A \in \mathcal{I}(X)$ and some non-zeromeet elements $a$ and $b$ of $X$,

(3) They have no inclusion relation, that is,

$$((a \wedge A)^{cl} :_{\wedge} \langle b\rangle) \nsubseteq A^{cl} \text{ and } ((a \wedge A)^{cl} :_{\wedge} \langle b\rangle) \nsupseteq A^{cl}$$

for some $A \in \mathcal{I}(X)$ and some non-zeromeet elements $a$ and $b$ of $X$.

**Example 3.7.** Consider a lower $BCK$-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

| $*$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 | 0 |
| 2 | 2 | 2 | 0 | 0 | 0 |
| 3 | 3 | 3 | 3 | 0 | 3 |
| 4 | 4 | 4 | 4 | 4 | 0 |

We have 6 ideals of $X$, and they are $A_0 = \{0\}$, $A_1 = \{0, 1\}$, $A_2 = \{0, 1, 2\}$, $A_3 = \{0, 1, 2, 3\}$, $A_4 = \{0, 1, 2, 4\}$ and $A_5 = X$. Define a mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ by $A_0^{cl} = A_1$, $A_1^{cl} = A_2$, $A_2^{cl} = A_2$, $A_3^{cl} = A_5$, $A_4^{cl} = A_5$ and $A_5^{cl} = A_5$. Then "$cl$" is a weak closure operation on $\mathcal{I}(X)$. For non-zeromeet elements 1 and 3 of $X$, we have

$$((1 \wedge A_3)^{cl} :_{\wedge} \langle 3\rangle) = (A_2 :_{\wedge} \{0, 1, 2, 3\}) = \{0, 1, 2, 4\} = A_4 \subseteq A_5 = A_3^{cl}.$$

**Example 3.8.** Consider a lower $BCK$-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

| $*$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 | 0 |
| 2 | 2 | 1 | 0 | 0 | 0 |
| 3 | 3 | 3 | 3 | 0 | 3 |
| 4 | 4 | 4 | 4 | 4 | 0 |

We have 5 ideals of $X$, and they are $A_0 = \{0\}$, $A_1 = \{0, 1, 2\}$, $A_2 = \{0, 1, 2, 3\}$, $A_3 = \{0, 1, 2, 4\}$ and $A_4 = X$. Define a mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ by $A_0^{cl} = A_0$, $A_1^{cl} = A_2$, $A_2^{cl} = A_4$, $A_3^{cl} = A_4$ and $A_4^{cl} = A_4$. Then "$cl$" is a weak closure operation on $\mathcal{I}(X)$. For non-zeromeet elements 1 and 2 of $X$, we have

$$((2 \wedge A_1)^{cl} :_{\wedge} \langle 1\rangle) = (A_2 :_{\wedge} \{0, 1, 2\}) = \{0, 1, 2, 3, 4\} = X \supseteq A_2 = A_1^{cl}.$$

**Example 3.9.** Let $X = \{0, 1, 2, 3, 4\}$ be a lower $BCK$-semilattice which is given in Example 3.7. If we define a mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ by $A_0^{cl} = A_0$, $A_1^{cl} = A_2$, $A_2^{cl} = A_2$, $A_3^{cl} = A_5$, $A_4^{cl} = A_4$, $A_5^{cl} = A_5$. Then "$cl$" is a weak closure operation on $\mathcal{I}(X)$. For non-zeromeet elements 3 and 4, we have

$$((3 \wedge A_4)^{cl} :_{\wedge} \langle 4\rangle) = (A_2 :_{\wedge} \{0, 1, 2, 4\}) = \{0, 1, 2, 3\} = A_3$$

and $A_4^{cl} = A_4$. Therefore

$$((3 \wedge A_4)^{cl} :_{\wedge} \langle 4\rangle) \nsubseteq A_4^{cl} \text{ and } ((3 \wedge A_4)^{cl} :_{\wedge} \langle 4\rangle) \nsupseteq A_4^{cl}.$$

We consider the equality of $((a \wedge A)^{cl} :_{\wedge} \langle b\rangle)$ and $A^{cl}$, that is,

$$((a \wedge A)^{cl} :_{\wedge} \langle b\rangle) = A^{cl}. \tag{3.4}$$

**Proposition 3.10.** *If $X$ has the greatest element 1, then every weak closure operation "cl" on $\mathcal{I}(X)$ satisfies the equality (3.4) for some $A \in \mathcal{I}(X)$ and non-zeromeet elements $a$ and $b$ of $X$.*

*Proof.* The ideals $A = \{0\}$ and a non-zeromeet element $b = 1$ satisfy the equality (3.4) for all non-zeromeet element $a$ of $X$. □

The following example shows that the converse of Proposition 3.10 is not true in general.

**Example 3.11.** Consider a lower $BCK$-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

| $*$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 | 0 |
| 2 | 2 | 2 | 0 | 2 | 0 |
| 3 | 3 | 3 | 3 | 0 | 0 |
| 4 | 4 | 4 | 4 | 4 | 0 |

Note that 4 is the greatest element and there are 6 ideals of $X$, that is, $A_0 = \{0\}$, $A_1 = \{0, 1\}$, $A_2 = \{0, 1, 2\}$, $A_3 = \{0, 1, 3\}$, $A_4 = \{0, 1, 2, 3\}$ and $A_5 = X$. Define a mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ by $A_0^{cl} = A_0$, $A_1^{cl} = A_3$, $A_2^{cl} = A_2$, $A_3^{cl} = A_4$, $A_4^{cl} = A_5$ and $A_5^{cl} = A_5$. Note that 0 is the only zeromeet element of $X$. For non-zeromeet elements 2 and 3 of $X$, we have $(3 \wedge A_1)^{cl} = A_1^{cl} = A_3$. Hence

$$((3 \wedge A_1)^{cl} :_{\wedge} \langle 2 \rangle) = (A_3 :_{\wedge} A_2) = \{0, 1, 3\} = A_3 = A_1^{cl}.$$

Therefore "$cl$" satisfies the equality (3.4) for ideal $A_1$ and non-zeromeet elements 2 and 3 of $X$. But "$cl$" is not a weak closure operation because $A_1 \subseteq A_2$, but

$$A_1^{cl} = A_3 \nsubseteq A_2 = A_2^{cl}.$$

**Definition 3.12.** A weak closure operation "$cl$" on $\mathcal{I}(X)$ is said to be

- *tender* if for any $A \in \mathcal{I}(X)$ and any non-zeromeet elements $a$ and $b$ of $X$, the equality (3.4) is valid,
- *naive* if for any $A \in \mathcal{I}(X)$ there exist non-zeromeet elements $a$ and $b$ of $X$ such that the equality (3.4) is valid.

**Example 3.13.** Consider a lower $BCK$-semilattice $X = \{0, 1, 2, 3\}$ with the following Cayley table.

| $*$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 1 | 0 |
| 2 | 2 | 2 | 0 | 0 |
| 3 | 3 | 3 | 3 | 0 |

We have 5 ideals of $X$, and they are $A_0 = \{0\}$, $A_1 = \{0, 1\}$, $A_2 = \{0, 2\}$, $A_3 = \{0, 1, 2\}$, and $A_4 = X$. Note that 3 is only the non-zeromeet element of $X$. Define a mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$

by $A_0^{cl} = A_0$, $A_1^{cl} = A_3$, $A_2^{cl} = A_3$, $A_3^{cl} = A_3$ and $A_4^{cl} = A_4$. Then "$cl$" is a weak closure operation on $\mathcal{I}(X)$ and

$(3 \wedge A_0)^{cl} :_\wedge \langle 3\rangle) = (A_0^{cl} :_\wedge \langle 3\rangle) = (A_0 :_\wedge \langle 3\rangle) = A_0 = A_0^{cl}$,
$(3 \wedge A_1)^{cl} :_\wedge \langle 3\rangle) = (A_1^{cl} :_\wedge \langle 3\rangle) = (A_3 :_\wedge \langle 3\rangle) = A_3 = A_1^{cl}$,
$(3 \wedge A_2)^{cl} :_\wedge \langle 3\rangle) = (A_2^{cl} :_\wedge \langle 3\rangle) = (A_3 :_\wedge \langle 3\rangle) = A_3 = A_2^{cl}$,
$(3 \wedge A_3)^{cl} :_\wedge \langle 3\rangle) = (A_3^{cl} :_\wedge \langle 3\rangle) = (A_3 :_\wedge \langle 3\rangle) = A_3 = A_3^{cl}$,
$(3 \wedge A_4)^{cl} :_\wedge \langle 3\rangle) = (A_4^{cl} :_\wedge \langle 3\rangle) = (A_4 :_\wedge \langle 3\rangle) = A_4 = A_4^{cl}$.

Therefore "$cl$" is a tender weak closure operation on $\mathcal{I}(X)$. Also, it is a naive weak closure operation on $\mathcal{I}(X)$.

Obviously, every tender weak closure operation is a native weak closure operation. But the converse is not true in general as seen in the following example.

**Example 3.14.** Consider a lower $BCK$-semilattice $X = \{0, 1, 2, 3\}$ with the following Cayley table.

| $*$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 |
| 2 | 2 | 1 | 0 | 1 |
| 3 | 3 | 3 | 3 | 0 |

We have 3 ideals of $X$, and they are $A_0 = \{0\}$, $A_1 = \{0, 1, 2\}$ and $A_2 = X$. Define a mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ by $A_0^{cl} = A_1^{cl} = A_1$ and $A_2^{cl} = A_2$. We can easily check that "$cl$" is a naive weak closure operation on $\mathcal{I}(X)$. But, it is not a tender weak closure operation on $\mathcal{I}(X)$. In fact, we know that there are two non-zeromeet elements 2 and 3. Thus

$$((3 \wedge A_1)^{cl} :_\wedge \langle 2\rangle) = (A_1^{cl} :_\wedge \langle 2\rangle) = (A_1 :_\wedge A_1) = X \neq A_1 = A_1^{cl}.$$

**Definition 3.15** ([2])**.** Given a (weak) closure operation "$cl$" on $\mathcal{I}(X)$, we define a new operation $cl_f : \mathcal{I}(X) \to \mathcal{I}(X)$ by

$$(\forall A \in \mathcal{I}(X)) \left(A^{cl_f} = \cup\{B^{cl} \mid B \subseteq A,\ B \in \mathcal{I}_f(X)\}\right), \tag{3.5}$$

where $\mathcal{I}_f(X)$ is the set of all finitely generated ideals of $X$.

**Definition 3.16** ([2])**.** A (weak) closure operation "$cl$" on $\mathcal{I}(X)$ is said to be of *finite type* if the following assertion is valid.

$$(\forall A \in \mathcal{I}(X)) \left(A^{cl} = A^{cl_f}\right). \tag{3.6}$$

**Theorem 3.17.** *If "cl" is a tender weak closure operation on $\mathcal{I}(X)$, then*

$$((a \wedge A)^{cl_f} :_\wedge \langle b\rangle) \subseteq A^{cl_f}. \tag{3.7}$$

*for all $A \in \mathcal{I}(X)$ and every non-zeromeet elements $a$ and $b$ of $X$.*

*Proof.* Suppose that "$cl$" is a tender weak closure operation on $\mathcal{I}(X)$. Let $A \in \mathcal{I}(X)$ and consider non-zeromeet elements $a$ and $b$ of $X$. If $x \in ((a \wedge A)^{cl_f} :_\wedge \langle b\rangle)$, then $x \wedge \langle b\rangle \subseteq (a \wedge A)^{cl_f}$ and so $x \wedge z \in (a \wedge A)^{cl_f}$ for all $z \in \langle b\rangle$. Thus there exists a finitely generated ideal $B$ of $X$ such that $B \subseteq a \wedge A$ and $x \wedge z \in B^{cl}$. Since

$$B = \langle x_1, x_2, \cdots, x_n\rangle$$

for some $x_1, x_2, \cdots, x_n \in X$, it follows that $x_i \in a \wedge A$ and so that $x_i = a \wedge a_i$ for $a_i \in A$ and $i \in \{1, 2, \cdots, n\}$. Thus

$$B = \langle a \wedge a_1, a \wedge a_2, \cdots, a \wedge a_n\rangle.$$

Now put $C = \langle a_1, a_2, \cdots, a_n\rangle$. Then $C \subseteq A$. If $x \in B$, then

$$(\cdots((x * (a \wedge a_1)) * (a \wedge a_2)) * \cdots) * (a \wedge a_n) = 0. \tag{3.8}$$

Since $a_1, a_2, \cdots, a_n \in C$, we have

$$a \wedge a_i \in \{a \wedge c \mid c \in C\} \text{ for } i = 1, 2, \cdots, n.$$

Since $a \wedge C = \langle\{a \wedge c \mid c \in C\}\rangle$, it follows from (3.8) that $x \in a \wedge C$. Thus $B \subseteq a \wedge C$, and hence $x \wedge z \in B^{cl} \subseteq (a \wedge C)^{cl}$ which means that $x \in ((a \wedge C)^{cl} :_\wedge z)$. Since $z$ is an arbitrary element of $\langle b\rangle$, we have

$$x \in ((a \wedge C)^{cl} :_\wedge \langle b\rangle) = C^{cl}.$$

Since $C$ is a finitely generated ideal of $X$ which is contained in $A$, we have $x \in A^{cl_f}$. Therefore $((a \wedge A)^{cl_f} :_\wedge \langle b\rangle) \subseteq A^{cl_f}$. □

If the condition "tender" in Theorem 3.17 is omitted, then Theorem 3.17 is not true as seen in the following example.

**Example 3.18.** Consider the lower $BCK$-semilattice $X$ and the weak closure operation "$cl$" on $\mathcal{I}(X)$ as in Example 3.14. Then "$cl$" is a naive weak closure operation but it is not tender. In fact, for ideal $A_1$ and $2, 3 \in X \setminus Z(X)$, we have

$$((3 \wedge A_1)^{cl_f} = \cup\{B^{cl} \mid B \subseteq 3 \wedge A_1, B \in \mathcal{I}_f(X)\} = A_1.$$

Thus $((3 \wedge A_1)^{cl_f} :_\wedge \langle 2\rangle) = (A_1 :_\wedge A_1) = X$. But,

$$A_1^{cl_f} = \cup\{B^{cl} \mid B \subseteq A_1, B \in \mathcal{I}_f(X)\} = A_1.$$

Therefore $((3 \wedge A_1)^{cl_f} :_\wedge \langle 2\rangle) = X \nsubseteq A_1 = A_1^{cl_f}$, that is, the condition (3.7) is not true.

**Remark 3.19.** Example 3.18 also shows that the condition (3.7) does not hold whenever we use a naive weak closure operation instead of a tender weak closure operation on $\mathcal{I}(X)$.

**Lemma 3.20** ([3])**.** *If $A$ is an ideal of $X$, then $(A :_\wedge X) = A$ and $(A :_\wedge A) = X$.*

**Theorem 3.21.** *Suppose that $X$ has the greatest element 1. If "$cl$" is a naive weak closure operation on $\mathcal{I}(X)$, then so is "$cl_f$".*

*Proof.* Note that if "$cl$" is a weak closure operation on $\mathcal{I}(X)$, then so is "$cl_f$" (see [1, Lemma 4.1]). Suppose that $A$ is an ideal of $X$. Since $1 \wedge A = A$ and $\langle 1\rangle = X$, it follows from Lemma 3.20 that

$$A^{cl_f} = ((1 \wedge A)^{cl_f} :_{\wedge} \langle 1\rangle).$$

Therefore "$cl_f$" is naive weak closure operation on $\mathcal{I}(X)$. □

**Corollary 3.22.** *Suppose that $X$ has the greatest element 1. If "$cl$" is a tender weak closure operation on $\mathcal{I}(X)$, then "$cl_f$" is a naive weak closure operation on $\mathcal{I}(X)$.*

**Corollary 3.23.** *Suppose that $X$ has the greatest element 1. If "$cl$" is a naive weak closure operation on $\mathcal{I}(X)$, then*

$$(\forall A \in \mathcal{I}(X))(\exists a, b \in X \setminus Z(X)) \left(((a \wedge A)^{cl_f} :_{\wedge} \langle b\rangle) = A^{cl_f}\right).$$

**Lemma 3.24** ([5])**.** *Every commutative $BCK$-algebra $X$ satisfies the identity:*

$$(\forall x, y, z \in X)\left((x \wedge y) * (x \wedge z) = (x \wedge y) * z\right).$$

**Theorem 3.25.** *In a commutative $BCK$-algebra $X$ with the greatest element 1, if "$cl$" is a tender weak closure operation on $\mathcal{I}(X)$, then so is "$cl_f$".*

*Proof.* Suppose that "$cl$" is a tender weak closure operation on $\mathcal{I}(X)$. Note that if "$cl$" is a weak closure operation on $\mathcal{I}(X)$, then so is "$cl_f$" (see [1, Lemma 4.1]). For any $A \in \mathcal{I}(X)$, if $x \in A^{cl_f}$, then there exists $B \in \mathcal{I}_f(X)$ such that $B \subseteq A$ and $x \in B^{cl}$. Since "$cl$" is a tender weak closure operation, we have

$$B^{cl} = ((a \wedge B)^{cl} :_{\wedge} \langle b\rangle)$$

for every elements $a, b \in X \setminus Z(X)$. Thus $x \in ((a \wedge B)^{cl} :_{\wedge} \langle b\rangle)$ which means that

$$x \wedge \langle b\rangle \subseteq (a \wedge B)^{cl}.$$

Now we will show that $a \wedge B \subseteq C$ where $C = \langle a \wedge x_1, a \wedge x_2, \cdots, a \wedge x_n\rangle$. Let $p \in a \wedge B$. Then $p = a \wedge q$ for some $q \in B$. Since $B$ is finitely generated, we have

$$(\cdots((q * x_1) * x_2) * \cdots) * x_n = 0$$

for some $x_1, x_2, \cdots, x_n \in X$. It follows from Lemma 3.24 and (a3) that

$$\begin{aligned}((a \wedge q) * (a \wedge x_1)) * (a \wedge x_2) &= ((a \wedge q) * x_1) * (a \wedge x_2)\\ &= ((a \wedge q) * (a \wedge x_2)) * x_1\\ &= ((a \wedge q) * x_2) * x_1\\ &= ((a \wedge q) * x_1) * x_2\end{aligned}$$

and so from Lemma 3.24 and (a3) again that

$$\begin{aligned}&(((a\wedge q)*(a\wedge x_1))*(a\wedge x_2))*(a\wedge x_3)\\&=(((a\wedge q)*x_1)*x_2)*(a\wedge x_3)\\&=(((a\wedge q)*(a\wedge x_3))*x_1)*x_2\\&=(((a\wedge q)*x_3)*x_1)*x_2\\&=(((a\wedge q)*x_1)*x_2)*x_3.\end{aligned}$$

By the mathematical induction, we conclude that

$$\begin{aligned}&(\cdots(((a\wedge q)*(a\wedge x_1))*(a\wedge x_2))*\cdots)*(a\wedge x_n)\\&=(\cdots(((a\wedge q)*x_1)*x_2)*\cdots)*x_n.\end{aligned}\tag{3.9}$$

The inequality $a\wedge q\leq q$ implies from (a2) that

$$(\cdots(((a\wedge q)*x_1)*x_2)*\cdots)*x_n\leq(\cdots((q*x_1)*x_2)*\cdots)*x_n=0$$

which implies that

$$\begin{aligned}0&=(\cdots(((a\wedge q)*x_1)*x_2)*\cdots)*x_n\\&=(\cdots(((a\wedge q)*(a\wedge x_1))*(a\wedge x_2))*\cdots)*(a\wedge x_n).\end{aligned}$$

Hence $p=a\wedge q\in C$, and so $a\wedge B\subseteq C$. Thus

$$x\wedge\langle b\rangle\subseteq(a\wedge B)^{cl}\subseteq C^{cl}.$$

Since $C$ is a finitely generated ideal of $X$ which is contained in $a\wedge A$, it follows that $x\wedge\langle b\rangle\subseteq(a\wedge A)^{cl_f}$, that is, $x\in((a\wedge A)^{cl_f}:_\wedge\langle b\rangle)$. Thus $A^{cl_f}\subseteq((a\wedge A)^{cl_f}:_\wedge\langle b\rangle)$. Now let $x\in((a\wedge A)^{cl_f}:_\wedge\langle b\rangle)$. Then $x\wedge\langle b\rangle\subseteq(a\wedge A)^{cl_f}$, and so there exists $B\in\mathcal{I}_f(X)$ such that $B\subseteq a\wedge A$ and $x\wedge z\in B^{cl}$ for every $z\in\langle b\rangle$. Hence

$$x\in((1\wedge B)^{cl}:_\wedge\langle b\rangle).$$

Since "$cl$" is a tender weak closure operation on $\mathcal{I}(X)$, we have $((1\wedge B)^{cl}:_\wedge\langle b\rangle)=B^{cl}$ and $x\in B^{cl}$. Since $B\in\mathcal{I}_f(X)$ and $B\subseteq a\wedge A\subseteq A$, we get $x\in A^{cl_f}$. Therefore

$$A^{cl_f}=((a\wedge A)^{cl_f}:_\wedge\langle b\rangle).\tag{3.10}$$

Consequently we know that "$cl_f$" is a tender weak closure operation on $\mathcal{I}(X)$. □

**Lemma 3.26** ([3]). *For any nonempty subsets $A$, $B_1$ and $B_2$ of $X$, we have*

$$B_1\subseteq B_2\ \Rightarrow\ (A:_\wedge B_2)\subseteq(A:_\wedge B_1).\tag{3.11}$$

**Lemma 3.27** ([3]). *For any ideal $A$ of $X$, we have*

$$(\forall x,y\in X)\,(x\leq y\ \Rightarrow\ x\wedge A\subseteq y\wedge A)\,.\tag{3.12}$$

**Theorem 3.28.** *Assume that $X$ has the greatest element 1. For every weak closure operation "cl" on $\mathcal{I}(X)$ and every ideal $A$ of $X$, let*

$$K := \cup\{((a \wedge A)^{cl} :_{\wedge} \langle b\rangle) \mid a, b \in X \setminus Z(X)\}. \tag{3.13}$$

*Then $K$ is an ideal of $X$ containing $A^{cl}$.*

*Proof.* Suppose that $A$ is an ideal of $X$. Then $((a \wedge A)^{cl} :_{\wedge} \langle b\rangle)$ is an ideal of $X$ for every $a, b \in X \setminus Z(X)$ by Definition 3.3 and Lemma 3.5. Hence it is clear that $0 \in K$. Let $x, y \in X$ be such that $x \in K$ and $y * x \in K$. Then there exist $a_1, a_2, b_1, b_2 \in X \setminus Z(X)$ such that

$$x \in ((a_1 \wedge A)^{cl} :_{\wedge} \langle b_1\rangle) \text{ and } y * x \in ((a_2 \wedge A)^{cl} :_{\wedge} \langle b_2\rangle).$$

Thus $x \wedge \langle b_1\rangle \subseteq (a_1 \wedge A)^{cl}$ and $(y * x) \wedge \langle b_2\rangle \subseteq (a_2 \wedge A)^{cl}$. Since $a_i \leq 1$ for $i = 1, 2$, it follows from Lemma 3.27 that $a_i \wedge A \subseteq 1 \wedge A = A$ for $i = 1, 2$. Hence

$$(a_i \wedge A)^{cl} \subseteq (1 \wedge A)^{cl}$$

for $i = 1, 2$, which implies that $x \wedge \langle b_1\rangle \subseteq (1 \wedge A)^{cl}$ and $(y * x) \wedge \langle b_2\rangle \subseteq (1 \wedge A)^{cl}$. Therefore $x \in ((1 \wedge A)^{cl} :_{\wedge} \langle b_1\rangle)$ and $y * x \in ((1 \wedge A)^{cl} :_{\wedge} \langle b_2\rangle)$. Note that $\langle b_1 \wedge b_2\rangle \subseteq \langle b_1\rangle$ and $\langle b_1 \wedge b_2\rangle \subseteq \langle b_2\rangle$, and so

$$((1 \wedge A)^{cl} :_{\wedge} \langle b_1\rangle) \subseteq ((1 \wedge A)^{cl} :_{\wedge} \langle b_1 \wedge b_2\rangle)$$

and

$$((1 \wedge A)^{cl} :_{\wedge} \langle b_2\rangle) \subseteq ((1 \wedge A)^{cl} :_{\wedge} \langle b_1 \wedge b_2\rangle).$$

Therefore

$$x \in ((1 \wedge A)^{cl} :_{\wedge} \langle b_1 \wedge b_2\rangle)$$

and

$$y * x \in ((1 \wedge A)^{cl} :_{\wedge} \langle b_1 \wedge b_2\rangle).$$

It follows that $y \in ((1 \wedge A)^{cl} :_{\wedge} \langle b_1 \wedge b_2\rangle)$. Since $X \setminus Z(X)$ is closed under the operation $\wedge$ by Lemma 3.2, then $b_1 \wedge b_2 \in X \setminus Z(X)$. Therefore $y \in K$ and $K$ is an ideal of $X$. Obviously $A^{cl} \subseteq K$. □

Assume that $X$ has the greatest element 1 and define a new function

$$cl_t : \mathcal{I}(X) \to \mathcal{I}(X),\ A \mapsto K \tag{3.14}$$

where $K$ is the ideal in Theorem 3.28.

**Theorem 3.29.** *Assume that $X$ has the greatest element 1. If "cl" is a weak closure operation on $\mathcal{I}(X)$, then so is the function "$cl_t$" in* (3.14).

*Proof.* Let "$cl$" be a weak closure operation on $\mathcal{I}(X)$. For any ideal $A$ of $X$, we have

$$A \subseteq A^{cl} = ((1 \wedge A)^{cl} :_{\wedge} \langle 1\rangle) \subseteq A^{cl_t}.$$

Let $A$ and $B$ be ideals of $X$. Then

$$A \subseteq B \Rightarrow a \wedge A \subseteq a \wedge B \Rightarrow (a \wedge A)^{cl} \subseteq (a \wedge B)^{cl}$$
$$\Rightarrow ((a \wedge A)^{cl} :_{\wedge} \langle b\rangle) \subseteq ((a \wedge B)^{cl} :_{\wedge} \langle b\rangle),$$

and so $A^{cl_t} \subseteq B^{cl_t}$. Therefore "$cl_t$" is a weak closure operation on $\mathcal{I}(X)$. □

**Theorem 3.30.** *Assume that $X$ has the greatest element 1. If "$cl$" is a finite type weak closure operation on $\mathcal{I}(X)$, then so is the function "$cl_t$" in* (3.14).

*Proof.* Assume that "$cl$" is a finite type weak closure operation on $\mathcal{I}(X)$. Then "$cl_t$" is a weak closure operation on $\mathcal{I}(X)$ by Theorem 3.29. For any ideal $A$ of $X$, if $x \in A^{cl_t}$ then there exist non-zeromeet elements $a$ and $b$ of $X$ such that

$$x \in ((a \wedge A)^{cl} :_{\wedge} \langle b\rangle).$$

Thus $x \wedge \langle b\rangle \subseteq (a \wedge A)^{cl}$, and so $x \wedge z \in (a \wedge A)^{cl}$ for every element $z \in \langle b\rangle$. Since "$cl$" is of finite type, there exists a finitely generated ideal $B$ such that $B \subseteq a \wedge A$ and $x \wedge z \in B^{cl}$. On the other hand, we know that $B \subseteq a \wedge C$ for some finite generated ideal $C$ of $X$ since $B$ is finitely generated. Thus $x \wedge z \in B^{cl}$ implies that $x \wedge z \in (a \wedge C)^{cl}$, that is, $x \in ((a \wedge C)^{cl} :_{\wedge} z)$. Since $z \in \langle b\rangle$, it follows that

$$x \in ((a \wedge C)^{cl} :_{\wedge} \langle b\rangle)$$

and so that $x \in C^{cl_t}$. Therefore $A^{cl_t} \subseteq A^{(cl_t)_f}$. Obviously, $A^{(cl_t)_f} \subseteq A^{cl_t}$ and therefore $A^{cl_t}$ is a finite type weak closure operation on $\mathcal{I}(X)$. □

**Theorem 3.31.** *Assume that $X$ has the greatest element 1. If "$cl$" is a naive weak closure operation on $\mathcal{I}(X)$, then so is the function "$cl_t$" in* (3.14).

*Proof.* Suppose that "$cl$" is a naive weak closure operation on $\mathcal{I}(X)$. Then "$cl_t$" is a weak closure operation on $\mathcal{I}(X)$ by Theorem 3.29. For any $A \in \mathcal{I}(X)$, if $x \in A^{cl_t}$, then there exists $a, b \in X \setminus Z(X)$ such that $x \in ((a \wedge A)^{cl} :_{\wedge} \langle b\rangle)$. Thus

$$x \wedge \langle b\rangle \subseteq (a \wedge A)^{cl}.$$

Since $a \wedge A \in \mathcal{I}(X)$ and "$cl$" is naive, there exists $p, q \in X \setminus Z(X)$ such that

$$(a \wedge A)^{cl} = ((p \wedge (a \wedge A))^{cl} :_{\wedge} \langle q\rangle).$$

Hence $x \wedge \langle b\rangle \subseteq ((p \wedge (a \wedge A))^{cl} :_{\wedge} \langle q\rangle)$. Also

$$((p \wedge (a \wedge A))^{cl} :_{\wedge} \langle q\rangle) \subseteq (a \wedge A))^{cl_t}.$$

Therefore $x \in ((a \wedge A)^{cl_t} :_{\wedge} \langle b\rangle)$.

Conversely, suppose that $x \in ((a \wedge A)^{cl_t} :_\wedge \langle b\rangle)$. Then $x \wedge z \in (a \wedge A)^{cl_t}$ for every $z \in \langle b\rangle$, and so there exist $p, q \in X \setminus Z(X)$ such that

$$x \wedge z \in ((p \wedge (a \wedge A))^{cl} :_\wedge \langle q\rangle).$$

Hence $x \wedge \langle b\rangle \wedge \langle q\rangle \subseteq ((p \wedge a) \wedge A)^{cl}$. Also $\langle b \wedge q\rangle \subseteq \langle b\rangle \wedge \langle q\rangle$. Therefore

$$x \wedge \langle b \wedge q\rangle \subseteq ((p \wedge a) \wedge A)^{cl}$$

which means that

$$x \in (((p \wedge a) \wedge A)^{cl} :_\wedge \langle b \wedge q\rangle).$$

Since $X \setminus Z(X)$ is closed under the operation $\wedge$ by Lemma 3.2, we have $p \wedge a$, $b \wedge q \in X \setminus Z(X)$ and therefore $x \in A^{cl_t}$. Consequently, "$cl_t$" is a naive weak closure operation on $\mathcal{I}(X)$. □

**Theorem 3.32.** *Assume that $X$ has the greatest element 1. If "cl" is a tender weak closure operation on $\mathcal{I}(X)$, then so is the function "$cl_t$" in* (3.14).

*Proof.* It is similar to the proof of Theorem 3.31. □

## References


[1] H. Bordbar and M. M. Zahedi, A finite type of closure operations on $BCK$-algebra, Appl. Math. Inf. Sci. Lett. 4 (2016), no. 2, 1–9.

[2] H. Bordbar, M. M. Zahedi, S. S. Ahn and Y. B. Jun, Weak closure operations on ideals of $BCK$-algebras, J. Comput. Anal. Appl. (in press).

[3] H. Bordbar, M. M. Zahedi and Y. B. Jun, Relative annihilators in lower $BCK$-semilattices, Mathematical Sciences Letters (submitted).

[4] H. Bordbar, Sun Shin Ahn, M. M. Zahedi and Y. B. Jun, Semiring structures based on meet and plus ideals in lower $BCK$-semilattices, J. Comput. Anal. Appl. (in press).

[5] Y. Huang, $BCI$-algebra, Science Press, Beijing 2006.

[6] J. Meng and Y. B. Jun, $BCK$-algebras, Kyung Moon Sa Co., Seoul 1994.

# Some Functional Inequalities for Generalized Error Function

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**Abstract**

In the paper, the authors present some functional inequalities for the generalized error function. Concretely speaking, the authors present several inequalities of the generalized error function in terms of the arithmetic, logarithmic, and exponential means, find monotonicity, convexity, and concavity of the generalized error function, and, consequently, derive two Grünbaum type and Turán type inequalities.

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## 1 Introduction

It is well known that the error function $\operatorname{erf}(x)$ is defined by

$$\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,\mathrm{d}\,t,\quad x\in\mathbb{R}.$$

The function $\operatorname{erf}(x)$ has numerous applications in statistics, probability, partial differential equations, and so on. It is not difficult to directly verify that $\operatorname{erf}(x)$ is odd on $(-\infty,\infty)$, convex on $(-\infty,0)$, concave on $[0,\infty)$, strictly increasing on $\mathbb{R}$, and $\operatorname{erf}(0)=0$. Moreover, the limit $\lim_{x\to\infty}\operatorname{erf}(x)=1$ can be found in [1, 12, 17, 18, 23, 28] and the closely related references therein.

In 1955, Chu [13] established a double inequality

$$\sqrt{1-e^{-ax^2}}\le\operatorname{erf}(x)\le\sqrt{1-e^{-bx^2}},$$

where $0\le a\le1$ and $b\ge\frac{4}{\pi}$ for $x\ge0$. Later, Cao et al. [12, 17] obtained the double inequality

$$\frac{1}{\sqrt{1+(9\pi/16-1)/(n^2)}}\le\operatorname{erf}(n)<\frac{1}{\sqrt{1-3/(4n^2)}}$$

for $n \in \mathbb{N}$ and derived the probability integral $\int_0^\infty e^{-x^2}\,\mathrm{d}\,x = \frac{\sqrt{\pi}}{2}$. See also related texts in [18, p. 31]. In 1968, Mitrinović and Weinacht [21] obtained

$$\operatorname{erf}(x) + \operatorname{erf}(y) \le \operatorname{erf}(x+y) + \operatorname{erf}(x)\operatorname{erf}(y), \quad x, y \ge 0.$$

In 2003 and 2009, Alzer [2, 3] proved for all $x \ge y > 0$ the sharp double inequalities

$$\operatorname{erf}(1) < \frac{\operatorname{erf}(x+\operatorname{erf}(y))}{\operatorname{erf}(y+\operatorname{erf}(x))} < \frac{2}{\sqrt{\pi}} \quad \text{and} \quad 0 < \frac{\operatorname{erf}(x\operatorname{erf}(y))}{\operatorname{erf}(y\operatorname{erf}(x))} \le 1.$$

For $p > 0$ and $x \in (0, \infty)$, the generalized error function $\operatorname{erf}_p(x)$ is defined by

$$\operatorname{erf}_p(x) = \frac{p}{\Gamma(1/p)} \int_0^x e^{-t^p}\,\mathrm{d}\,t,$$

where $\Gamma(z)$ for $\Re(z) > 0$ stands for the classical Euler gamma function. It is easy to see that $\operatorname{erf}_2(x) = \operatorname{erf}(x)$. By an easy computation, we can see that $\operatorname{erf}_p(x)$ is odd on $(-\infty, \infty)$, convex on $(-\infty, 0)$, concave on $[0, \infty)$, strictly increasing on $\mathbb{R}$, $\operatorname{erf}_p(0) = 0$, and $\lim_{x\to\infty} \operatorname{erf}_p(x) = 1$.

In 1997, Alzer [5] established the double inequality

$$\Gamma\left(1+\frac{1}{p}\right)\left(1-e^{-bx^p}\right)^{1/p} < \int_0^x e^{-t^p}\,\mathrm{d}\,t < \Gamma\left(1+\frac{1}{p}\right)\left(1-e^{-ax^p}\right)^{1/p}, \quad x > 0,$$

where

$$\begin{cases} a = 1 \quad \text{and} \quad b = \left[\Gamma\left(1+\frac{1}{p}\right)\right]^{-p}, & 0 < p < 1; \\ a = \left[\Gamma\left(1+\frac{1}{p}\right)\right]^{-p} \quad \text{and} \quad b = 1, & p > 1 \end{cases}$$

are best possible constants. In 1999, Qi and Guo [26, Theorem 1] proved that

$$xe^{-(x/2)^p} < \int_0^x e^{-t^p}\,\mathrm{d}\,t < \frac{x}{2}\left(1+e^{-x^p}\right) \tag{1.1}$$

for $p \in (0, 1]$ and $x \in (0, \infty)$; if $p > 1$ and $0 < x < \left(1-\frac{1}{p}\right)^{1/p}$, the double inequality (1.1) reverses. In 1999, Qi et al [25, 27] constructed

$$\int_0^{\pi/2} e^{-t^2\sin^2 x}\sin x\,\mathrm{d}\,x \le \frac{1-e^{-t^2}}{t^2}, \quad \int_0^x e^{-t^2}\,\mathrm{d}\,t \ge \frac{1-e^{-x^2}}{x}, \quad \int_0^x e^{-t^\alpha}\,\mathrm{d}\,t \ge \frac{1-e^{-x^\alpha}}{x^{\alpha-1}},$$
$$\int_0^t e^{x^2}\,\mathrm{d}\,x < \frac{e^{t^2}-1}{t}, \quad \int_x^\infty e^{-t^\alpha}\,\mathrm{d}\,t \le \frac{e^{-x^\alpha}}{\alpha x^{\alpha-1}}, \quad \int_x^\infty e^{-t^3}\,\mathrm{d}\,t \ge \frac{e^{-x^3}}{3(x+1)^2}, \quad \int_0^x e^{t^\alpha}\,\mathrm{d}\,t \le \frac{e^{x^\alpha}-1}{x^{\alpha-1}},$$

where $\alpha \ge 1$. There are more results on $\operatorname{erf}_p$ in the papers [8, 16, 20, 22, 24, 25, 27] and the closely related references therein.

For two distinct positive numbers $x$ and $y$, the arithmetic, geometric, harmonic, logarithmic, exponential means and the power mean of order $t \in \mathbb{R}$ are respectively defined by $A(x,y) = \frac{x+y}{2}$, $G(x,y) = \sqrt{xy}$, $H(x,y) = \frac{1}{A(1/x,1/y)}$, and

$$L(x,y) = \frac{x-y}{\ln x - \ln y}, \qquad I(x,y) = \frac{1}{e}\left(\frac{x^x}{y^y}\right)^{1/(x-y)}, \qquad M_t(x,y) = \left(\frac{x^t+y^t}{2}\right)^{1/t}.$$

Recently, several mathematicians began to study inequalities for the error function $\operatorname{erf}(x)$ with respect to all kinds of means, including the arithmetic mean, harmonic mean, and the power mean. See the papers [4, 14, 15] and the closely related references therein.

Let $f : J \subseteq (0,\infty) \to (0,\infty)$ be continuous. Let $M$ and $N$ be means defined above. If

$$f(M(x,y)) \lessgtr N(f(x), f(y))$$

for all $x, y \in J$, we call $f$ $MN$-convex (or $MN$-concave, respectively).

In 2007, Anderson et al [6] studied the generalized convexity (or concavity, respectively) with respect to general means. In 2010, Baricz [7] presented that if $f$ is differentiable, then it is $(a,b)$-convex (or $(a,b)$-concave, respectively) on $J$ if and only if $\frac{x^{1-a}f'(x)}{f^{1-b}(x)}$ is increasing (or decreasing, respectively). We observe that the $(1,1)$-convexity means the $AA$-convexity, the $(1,0)$-convexity is the $AG$-convexity, and the $(0,0)$-convexity implies $GG$-convexity. It is easy to see that a function $f : J \subseteq (0,\infty) \to (0,\infty)$ is said to be geometrically convex if it is convex with respect to the geometric mean, that is, the inequality $f\big(x^\lambda y^{1-\lambda}\big) \le [f(x)]^\lambda [f(y)]^{1-\lambda}$ holds for all $x, y > 0$ and $\lambda \in (0,1)$; if the above inequality is reversed, the function $f$ is called geometrically concave. We note that a differentiable function $f$ is geometrically convex (or concave, respectively) if and only if $\frac{xf'(x)}{f(x)}$ is increasing (or decreasing, respectively).

The first aim of this paper is to present several inequalities of the generalized error function $\operatorname{erf}_p(x)$ in terms of the arithmetic, logarithmic, and exponential means $A(a,b)$, $L(a,b)$ and $I(a,b)$. The second aim is to find monotonicity, convexity, and concavity of the generalized error function $\operatorname{erf}_p(x)$ and, consequently, derive two Grünbaum type and Turán type inequalities.

Our main results can be stated as following theorems.

**Theorem 1.1.** *For all $p \in (0,\infty)$ and $x, y \in (0,\infty)$, we have*

$$L\big(\operatorname{erf}_p(x), \operatorname{erf}_p(y)\big) < \operatorname{erf}_p(L(x,y)) \quad \text{and} \quad I\big(\operatorname{erf}_p(x), \operatorname{erf}_p(y)\big) < \operatorname{erf}_p(A(x,y)).$$

**Theorem 1.2.** *For all $p \in [1,\infty)$ and $x, y \in (0,\infty)$, we have*

$$L\big(x^{p-1}\operatorname{erf}_p(x), y^{p-1}\operatorname{erf}_p(y)\big) < L^{p-1}(x,y)\operatorname{erf}_p(L(x,y))$$

*and*

$$I\big(x^{p-1}\operatorname{erf}_p(x), y^{p-1}\operatorname{erf}_p(y)\big) < A^{p-1}(x,y)\operatorname{erf}_p(A(x,y)).$$

**Theorem 1.3.** *Let $x, y, z \in (0,\infty)$ such that $z^2 = x^2 + y^2$. Then we have the Grünbaum type inequality*

$$1 + \operatorname{erf}_p\big(z^2\big) \ge \operatorname{erf}_p\big(x^2\big) + \operatorname{erf}_p\big(y^2\big).$$

**Theorem 1.4.** *For all $p \in (0,\infty)$, the function $x \mapsto \frac{\operatorname{erf}_p(x)}{\operatorname{arctan}_p x}$ is strictly decreasing in $x \in (0,\infty)$, where $\operatorname{arctan}_p(x)$ is defined by $\operatorname{arctan}_p x = \int_0^x \frac{1}{1+t^p}\,\mathrm{d}\,t$. Consequently, we have*

$$\frac{2}{\pi_p}\operatorname{arctan}_p x < \operatorname{erf}_p(x) < \frac{p}{\Gamma(1/p)}\operatorname{arctan}_p x.$$

*where $\frac{2}{\pi_p}$ and $\frac{p}{\Gamma(1/p)}$ are best possible constants.*

**Theorem 1.5.** *For all $x \in (0,\infty)$ fixed, the function $p \mapsto \mathrm{erf}_p(x)$ is strictly logarithmically concave in $p \in \left(0, \frac{1}{x_0}\right)$, where $x_0 = 1.461632\ldots$ is the only positive root of the digamma function $\psi(x)$. Consequently, the Turán type inequality*

$$\mathrm{erf}_p^2(x) > \mathrm{erf}_{p-1}(x)\,\mathrm{erf}_{p+1}(x)$$

*is valid for all $p \in \left(0, \frac{1}{x_0}\right)$ and $x \in (0,\infty)$.*

# 2 Lemmas

In order to prove our main results, we need the following lemmas.

**Lemma 2.1** ([10, Theorem 1, p. 138])**.** *Let $f : J \subseteq (0,\infty) \to (0,\infty)$.*

1. *If $f$ is increasing and logarithmically convex (or logarithmically concave, respectively), then it is LL-convex (or concave, respectively);*
2. *If $f$ is increasing and logarithmically convex (or logarithmically concave, respectively), then it is AL-convex (or concave, respectively).*

**Lemma 2.2** ([11, Theorem 1, p. 6])**.** *Let $f : J \subseteq (0,\infty) \to (0,\infty)$. If $f(x)$ is continuously differentiable, increasing, and logarithmically convex (or logarithmically concave, respectively), then*

$$I(f(x), f(y)) \geq f(I(x,y)) \quad \text{and} \quad I(f(x), f(y)) \leq f(A(x,y)).$$

**Lemma 2.3** ([29, Example 152, p. 124])**.** *If $f(x)$ and $g(x)$ are convex functions on $(0,\infty)$ such that $f_1(x) \geq 0$, $f_2(x) \geq 0$, and $f_1(0) = f_2(0) = 0$, then $\frac{f_1(x)f_2(x)}{x}$ is also convex on $(0,\infty)$.*

**Lemma 2.4** ([9, Lemma 3, p. 246])**.** *Let $f : (a,\infty) \to \mathbb{R}$ for $a \geq 0$ and $g(x) = \frac{f(x)-1}{x}$ be increasing on $(a,\infty)$. Then $h(x) = f\left(x^2\right)$ satisfies the Grünbaum type inequality*

$$1 + h(z) \geq h(x) + h(y) \tag{2.1}$$

*for $x, y \geq a$ and $z^2 = x^2 + y^2$. If $g(x)$ is decreasing, then the inequality (2.1) is reversed.*

**Lemma 2.5** ([19, Lemma 3.2, p. 523])**.** *Let $f(x), g(x)$ be continuous on $[a,b]$, differentiable on $(a,b)$, $g'(x) \neq 0$ on $(a,b)$, and $f(a) = g(a) = 0$ or $f(b) = g(b) = 0$. If $\frac{f'(x)}{g'(x)}$ is increasing (or decreasing, respectively) on $(a,b)$, then the ratio $\frac{f(x)}{g(x)}$ is also increasing (or decreasing, respectively) on $(a,b)$.*

# 3 Proofs of main results

Now we are in a position to prove our main results.

*Proof of Theorem 1.1.* A simple computation yields

$$\mathrm{erf}_p'(x) = \frac{p}{\Gamma(1/p)} e^{-x^p} > 0 \quad \text{and} \quad \mathrm{erf}_p''(x) = -\frac{p^2}{\Gamma(1/p)} x^{p-1} e^{-x^p} < 0.$$

Hence, the function $\mathrm{erf}_p(x)$ is strictly increasing and concave on $(0,\infty)$. Since the concavity implies the logarithmic concavity, considering Lemmas 2.1 and 2.2, Theorem 1.1 is thus proved. □

*Proof of Theorem 1.2.* Considering Lemmas 2.1 and 2.2, it suffices to prove that $x^{p-1}\operatorname{erf}_p(x)$ is strictly increasing and concave on $(0,\infty)$. This follows readily from a simple computation

$$\left[x^{p-1}\operatorname{erf}_p(x)\right]' = x^{p-2}\left[(p-1)\operatorname{erf}_p(x) + x\operatorname{erf}_p'(x)\right] > 0.$$

On the other hand, the function $f_1(x) = x^p$ and $f_2(x) = -\operatorname{erf}_p(x)$ are convex on $(0,\infty)$ and $f_1(0) = f_2(0) = 0$. Making use of Lemma 2.3 reveals that $-x^{p-1}\operatorname{erf}_p(x)$ is also convex on $(0,\infty)$. So, the function $x^{p-1}\operatorname{erf}_p(x)$ is strictly increasing and concave on $(0,\infty)$. The proof of Theorem 1.2 is complete. □

*Proof of Theorem 1.3.* By virtue of Lemma 2.4, we only need to prove that the function $\frac{\operatorname{erf}_p(x)-1}{x}$ is strictly increasing on $(0,\infty)$. A direct computation results in

$$\left[\frac{\operatorname{erf}_p(x)-1}{x}\right]' = \frac{1}{x^2}\left[\frac{p}{\Gamma(1/p)}e^{-x^p} - \operatorname{erf}_p(x) + 1\right] \triangleq \frac{g(x)}{x^2} \quad \text{and} \quad g'(x) = -\frac{p^2}{\Gamma(1/p)}x^p e^{-x^p} < 0.$$

Accordingly, the function $g(x)$ is strictly decreasing on $(0,\infty)$. Therefore, from the inequality $g(x) > \lim_{x\to\infty} g(x) = 0$, Theorem 1.3 follows immediately. □

*Proof of Theorem 1.4.* For proving the monotonicity of $\frac{\operatorname{erf}_p(x)}{\arctan_p x}$, we denote $g_1(x) = \operatorname{erf}_p(x)$ and $g_2(x) = \arctan_p x$. Then $g_1(0) = g_2(0) = 0$ and

$$\frac{g_1'(x)}{g_2'(x)} = \frac{p}{\Gamma(1/p)}(1+x^p)e^{-x^p} \triangleq \frac{p}{\Gamma(1/p)}q(x).$$

By a direct differentiation, it follows that $q'(x) = -px^{2p-1}e^{-x^p} < 0$ which implies that the function $q(x)$ is strictly decreasing on $(0,\infty)$. By Lemma 2.5, we complete the required proof. □

*Proof of Theorem 1.5.* Let $t \in (0,x)$ and $x \in (0,\infty)$ be fixed. Define the function

$$\alpha(p) = \ln\left[\frac{p}{\Gamma(1/p)}e^{-t^p}\right] = \ln p - \ln\left[\Gamma\left(\frac{1}{p}\right)\right] - t^p.$$

By a direct computation, we have

$$\alpha'(p) = \frac{1}{p} + \frac{1}{p^2}\psi\left(\frac{1}{p}\right) - t^p\ln t \quad \text{and} \quad \alpha''(p) = -\frac{1}{p^2} - \frac{2}{p^3}\psi\left(\frac{1}{p}\right) - \frac{1}{p^4}\psi'\left(\frac{1}{p}\right) - t^p\ln^2 t < 0.$$

Therefore, the function $\alpha(p)$ is strictly logarithmically concave. By the fact that integrating preserves the monotonicity and logarithmic concavity, it follows that the function $p \mapsto \operatorname{erf}_p(x)$ is strictly logarithmically concave in $p \in \left(0, \frac{1}{x_0}\right)$. □

# References

[1] M. Abramowitz and I. A. Stegun(Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series **55**, 10th printing, Dover Publications, New York and Washington, 1972.

[2] H. Alzer, *Functional inequalities for the error function*, Aequationes Math. **66** (2003), no. 1-2, 119–127; Available online at http://dx.doi.org/10.1007/s00010-003-2683-9.

[3] H. Alzer, *Functional inequalities for the error function, II*, Aequationes Math. **78** (2009), no. 1-2, 113–121; Available online at http://dx.doi.org/10.1007/s00010-009-2963-0.

[4] H. Alzer, *Error function inequalities*, Adv. Comput. Math. **33** (2010), no. 3, 349–379; Available online at http://dx.doi.org/10.1007/s10444-009-9139-2.

[5] H. Alzer, *On some inequalities for the incomplete gamma function*, Math. Comp. **66** (1997), no. 218, 771–778; Available online at http://dx.doi.org/10.1090/S0025-5718-97-00814-4.

[6] G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen, *Generalized convexity and inequalities*, J. Math. Anal. Appl. **335** (2007), no. 2, 1294–1308; Available online at http://dx.doi.org/10.1016/j.jmaa.2007.02.016.

[7] Á. Baricz, *Geometrically concave univariate distributions*, J. Math. Anal. Appl. **363** (2010), no. 1, 182–196; Available online at http://dx.doi.org/10.1016/j.jmaa.2009.08.029.

[8] Á. Baricz, B. A. Bhayo, and T. K. Pogány, *Functional inequalities for generalized inverse trigonometric and hyperbolic functions*, J. Math. Anal. Appl. **417** (2014), no. 1, 244–259; Available online at http://dx.doi.org/10.1016/j.jmaa.2014.03.039.

[9] Á. Baricz, B. A. Bhayo, and K. Riku, *Convexity properties of generalized trigonometric and hyperbolic functions*, Aequationes Math. **89** (2015), no. 3, 473–484; Available online at http://dx.doi.org/10.1007/s00010-013-0222-x.

[10] B. Bhayo and L. Yin, *Logarithmic mean inequality for generalized trigonometric and hyperbolic functions*, Acta Univ. Sapientiae Math. **6** (2014), no. 2, 135–145; Available online at http://dx.doi.org/10.1515/ausm-2015-0002.

[11] B. A. Bhayo and L. Yin, *On the generalized convexity and concavity*, Probl. Anal. Issues Anal. **4(22)** (2015), no. 1, 3–10; Available online at http://dx.doi.org/10.15393/j3.art.2015.2709.

[12] J. Cao, D.-W. Niu, and F. Qi, *A Wallis type inequality and a double inequality for probability integral*, Aust. J. Math. Anal. Appl. **5** (2007), no. 1, Art. 3; Available online at http://ajmaa.org/cgi-bin/paper.pl?string=v4n1/V4I1P3.tex.

[13] J. T. Chu, *On bounds for the normal integral*, Biometrika **42** (1955), 263–265.

[14] Y.-M. Chu, Y.-M. Li, W.-F. Xia, and X.-H. Zhang, *Best possible inequalities for the harmonic mean of error function*, J. Inequal. Appl. **2014**, 2014:525, 9 pages; Available online at http://dx.doi.org/10.1186/1029-242X-2014-525.

[15] Y.-M. Chu and T.-H. Zhao, *Concavity of the error function with respect to Hölder mean*, Math. Inequal. Appl. **19** (2016), no. 2, 589–595; Available online at http://dx.doi.org/10.7153/mia-19-43.

[16] Á. Elbert and A. Laforgia, *An inequality for the product of two integrals relating to the incomplete Gamma function*, J. Inequal. Appl. **5** (2000), no. 1, 39–51; Available online at http://dx.doi.org/10.1155/S1025583400000035.

[17] B.-N. Guo and F. Qi, *A class of completely monotonic functions involving divided differences of the psi and tri-gamma functions and some applications*, J. Korean Math. Soc. **48** (2011), no. 3, 655–667; Available online at http://dx.doi.org/10.4134/JKMS.2011.48.3.655.

[18] B.-N. Guo and F. Qi, *On the Wallis formula*, Internat. J. Anal. Appl. **8** (2015), no. 1, 30–38.

[19] R. Klén, M. Vuorinen and X.-H. Zhang, *Inequalities for the generalized trigonometric and hyperbolic functions*, J. Math. Anal. Appl. **409** (2014), no. 1, 521–529; Available online at http://dx.doi.org/10.1016/j.jmaa.2013.07.021.

[20] A. Laforgia and P. Natalini, *Supplements to known monotonicity results and inequalities for the gamma and incomplete gamma functions*, J. Inequal. Appl. **2006**, Article ID 48727, 8 pages; Available online at http://dx.doi.org/10.1155/JIA/2006/48727.

[21] D. S. Mitrinović and R. J. Weinacht, *Problems and Solutions: Solutions of Advanced Problems: 5555*, Amer. Math. Monthly **75** (1968), no. 10, 1129–1130; Available online at http://dx.doi.org/10.2307/2315774.

[22] P. Natalini and B. Palumbo, *Inequalities for the incomplete gamma function*, Math. Inequal. Appl. **3** (2000), no. 1, 69–77; Available online at http://dx.doi.org/10.7153/mia-03-08.

[23] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark(eds.), *NIST Handbook of Mathematical Functions*, Cambridge University Press, New York, 2010; Available online at http://dlmf.nist.gov/.

[24] F. Qi, *Monotonicity results and inequalities for the gamma and incomplete gamma functions*, Math. Inequal. Appl. **5** (2002), no. 1, 61–67; Available online at http://dx.doi.org/10.7153/mia-05-08.

[25] F. Qi, L.-H. Cui, and S.-L. Xu, *Some inequalities constructed by Tchebysheff's integral inequality*, Math. Inequal. Appl. **2** (1999), no. 4, 517–528; Available online at http://dx.doi.org/10.7153/mia-02-42.

[26] F. Qi and S.-L. Guo, *Inequalities for the incomplete gamma and related functions*, Math. Inequal. Appl. **2** (1999), no. 1, 47–53; Available online at http://dx.doi.org/10.7153/mia-02-05.

[27] F. Qi and J.-Q. Mei, *Some inequalities of the incomplete gamma and related functions*, Z. Anal. Anwendungen **18** (1999), no. 3, 793–799; Available online at http://dx.doi.org/10.4171/ZAA/914.

[28] K. Oldham, J. Myland, and J. Spanier, *An Atlas of Functions*, 2nd Ed., Springer, New York, 2009; Available online at http://dx.doi.org/10.1007/978-0-387-48807-3.

[29] C.-H. Xue and S.-L. Xu, *Mathematical Analysis: Selected Exercises and Their Solutions*, Vol. 1, Tsinghua University Press, Beijing, 2009. (Chinese)

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# Kantorovich Type Integral Inequalities for Tensor Products of Continuous Fields of Positive Operators

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**Abstract**

This paper establishes a number of Kantorovich type integral inequalities involving tensor products of continuous fields of positive operators parametrized by a locally compact Hausdorff space. Such integrals appear as Bochner integrals with respect to a finite Radon measure on that space. Kantorovich type inequalities in which the operator product are replaced by an operator mean are also investigated.

**Keywords:** continuous field of operators, Bochner integral, tensor product, operator mean, operator monotone function
**Mathematics Subject Classifications 2010:** 26D15, 46G10, 47A63, 47A64, 47A80.

## 1 Introduction

One of well-known analytic inequalities is the classical Kantorovich inequality [10], which states that for real numbers $a_i$ and $w_i$ such that $0 < a \leqslant a_i \leqslant b$ and $w_i \geqslant 0$ for all $1 \leqslant i \leqslant n$, we have

$$\left(\sum_{i=1}^n w_i a_i\right)\left(\sum_{i=1}^n \frac{w_i}{a_i}\right) \leqslant \frac{(a+b)^2}{4ab}\left(\sum_{i=1}^n w_i\right)^2. \tag{1.1}$$

This inequality can be regarded as a reverse version of weighted arithmetic-harmonic mean inequality. Applications of this inequality arise in convergence analysis for numerical methods and statistics. Various generalizations, variations, refinements and equivalences of this inequality in several settings have been investigated. Let us focus on an integral version of (1.1):

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**Theorem 1.1** (see *e.g.* [4]). *Let $J$ be a real interval equipped with a probability measure $\mu$. For any continuous function $f : J \to \mathbb{R}$ such that* $\mathrm{Range}(f) \subseteq [a, b]$ *for some $a, b > 0$, it holds that*

$$\int_J f^2 \, d\mu \leqslant \frac{(a+b)^2}{4ab} \left( \int_J f \, d\mu \right)^2. \tag{1.2}$$

Over the years, Kantorovich type inequalities were obtained in the contexts of matrices and operators, see *e.g.* [5, 7, 12, 14] and references therein. A matrix analogue of the inequality (1.1) involving Hadamard product (entrywise product, denoted by $\odot$) is given as follows.

**Theorem 1.2** ([13], Theorem 2.2). *For each $i = 1, 2, \ldots, n$, let $A_i$ and $W_i$ be positive definite matrices of the same size such that $0 < aI \leqslant A_i \leqslant bI$. Then*

$$\sum_{i=1}^n W_i^{\frac{1}{2}} A_i W_i^{\frac{1}{2}} \odot \sum_{i=1}^n W_i^{\frac{1}{2}} A_i^{-1} W_i^{\frac{1}{2}} \leqslant \frac{a^2+b^2}{2ab} \left( \sum_{i=1}^n W_i \odot \sum_{i=1}^n W_i \right). \tag{1.3}$$

Note that the constant bound $(a^2 + b^2)/(2ab)$ of the matrix case (1.3) is slightly different to that of scalar case $(a^2 + b^2)/(4ab)$ in (1.1) and (1.2). The inequality (1.3) can be viewed as a reverse of the Fiedler's inequality

$$A \circ A^{-1} \geqslant I$$

which holds for any positive definite matrix $A$ (see [6]). Kantorovich type inequality in which the operator product is replaced by an operator mean was considered in [15, 17].

In this paper, we establish certain integral inequalities of Kantorovich type for continuous fields of positive operators on a Hilbert space. The inequalities (1.1) and (1.2) are generalized in many ways in terms of Bochner integrals of operator-valued functions defined on a locally compact Hausdorff space equipped with a finite Radon measure. Instead of the Hadamard product in Theorem 1.2, we consider the (Hilbert) tensor product and Kubo-Ando operator mean. Our results include discrete inequalities as special cases.

This paper consists of four sections. Section 2 provides fundamental facts about continuous fields of operators and its integrability. Section 3 deals with Kantorovich type integral inequalities involving tensor products of continuous fields of operators. In Section 4, we recall Kubo-Ando theory of operator means and then derive Kantorovich type inequalities involving operator means.

# 2 Continuous field of operators and Bochner integrability

Throughout, let $\mathcal{H}$ be a complex Hilbert space. Denote by $\mathfrak{B}(\mathcal{H})$ and $\mathfrak{B}(\mathcal{H})^+$ the C$^*$-algebra of all bounded linear operators on $\mathcal{H}$ and its positive cone, respectively. Let $\mathbb{A}$ and $\mathbb{A}^+$ be a unital C$^*$-subalgebra of $\mathfrak{B}(\mathcal{H})$ and its positive

cone, respectively. Capital letters always denote operators on a Hilbert space. In particular, $I$ denotes the identity operator, where the underlying space is clear from the context. The spectrum of an operator $A$ is expressed as $\mathrm{Sp}(A)$. As usual, the operator norm of an operator $A$ is denoted by $\|A\|$. For selfadjoint elements $A, B \in \mathbb{A}$, the expression $A \leqslant B$ indicates that $B - A$ is a positive element, while $A > 0$ means that $A$ is positive and invertible.

Let us denote the supremum norm of a real-valued function $f$ defined on a set $E$ by $\|f\|_{\infty,E}$. The symbol $\|\cdot\|_1$ denotes the $L^1$-norm on a given set, which is clear from the context.

The next lemma asserts the continuity of the map $A \mapsto f(A)$. Here, $f(A)$ is the continuous functional calculus of $f$ on $\mathrm{Sp}(A)$.

**Lemma 2.1.** *Let $\Delta$ be a nonempty compact subset of $\mathbb{C}$ and let $f : \Delta \to \mathbb{C}$ be a continuous function. Let $\mathcal{A}$ be the subset of $\mathbb{A}$ consisting of all operators whose spectra are contained in $\Delta$. Then the map sending $A \in \mathcal{A}$ to $f(A) \in \mathbb{A}$ is continuous.*

*Proof.* Let $\epsilon > 0$. Weierstrass' approximation theorem guarantees the existence of a polynomial $p$ such that

$$\|f - p\|_{\infty,\Delta} \;<\; \frac{\epsilon}{3}.$$

Since the map $X \mapsto p(X)$ is continuous on $\mathcal{A}$, there is a positive constant $\delta$ such that $\|p(A) - p(B)\| < \frac{\epsilon}{3}$ whenever $\|A - B\| < \delta$. For any operators $A, B \in \mathcal{A}$ such that $\|A - B\| < \delta$, we have

$$\begin{aligned}
\|f(A) - f(B)\| &\leqslant \|f(A) - p(A)\| + \|p(A) - p(B)\| + \|p(B) - f(B)\| \\
&= \|f - p\|_{\infty,\sigma(A)} + \|p(A) - p(B)\| + \|f - p\|_{\infty,\sigma(B)} \\
&\leqslant \|f - p\|_{\infty,\Delta} + \|p(A) - p(B)\| + \|f - p\|_{\infty,\Delta} \\
&< \epsilon.
\end{aligned}$$

Note that the above equality holds since the Gelfand transform $f \mapsto f(A)$ is an isometry. Therefore the map $A \mapsto f(A)$ is continuous. □

From now on, let $\Omega$ be a locally compact Hausdorff space. Equip $\Omega$ with a Radon measure $\mu$, *i.e.*, $\mu$ is a Borel measure on $\Omega$ that is finite on all compact subsets, outer regular on all Borel subsets, and inner regular on all open subsets. A family $(A_t)_{t\in\Omega}$ of operators in $\mathbb{A}$ is said to be a *continuous field of operators* if the parametrization $t \mapsto A_t$ is continuous on $\Omega$. If we further assume the Lebesgue integrability of the function $t \mapsto \|A_t\|$, then the Bochner integral $\int_\Omega A_t \, d\mu(t)$ is well-defined as the element in $\mathbb{A}$ satisfying

$$\phi\left(\int_\Omega A_t \, d\mu(t)\right) \;=\; \int_\Omega \phi(A_t) \, d\mu(t)$$

for every $\phi$ in the norm dual of $\mathbb{A}$ (see *e.g.* [16]). Let $\mathcal{C}(\Omega; \mathbb{A}^+)$ be the set of all continuous fields$(A_t)_{t\in\Omega}$ such that $A_t \in \mathbb{A}^+$ for all $t \in \Omega$. If we want to

specify that $\mathrm{Sp}(A_t) \subseteq J$, for some subset $J \subseteq [0,\infty)$, we shall use the notation $\mathcal{C}(\Omega; \mathbb{A}^+, J)$.

The next lemma is useful for integrating any vector-valued function on a finite measure space.

**Lemma 2.2** (see *e.g.* [1], Theorem 11.44)**.** *Let $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ be a Banach space, and let $(\Gamma, \nu)$ be a finite measure space. Suppose that $f : \Gamma \to \mathbb{X}$ is a measurable function (here, $\mathbb{X}$ is equipped with the Borel $\sigma$-algebra). Then $f$ is Bochner integrable if and only if its norm function $\|f\|$ is Lebesgue integrable, i.e.,*

$$\int_\Gamma \|f\|\, d\nu \,<\, \infty.$$

*Here, $\|f\|$ is defined by $\|f\|(x) = \|f(x)\|_{\mathbb{X}}$ for any $x \in \mathbb{X}$.*

In what follows, suppose that $\mu$ is a finite Radon measure on $\Omega$. The integrability of a real-valued function is always in the sense of Lebesgue.

**Proposition 2.3.** *Let $\Delta$ be a nonempty compact subset of $\mathbb{C}$ and let $f : \Delta \to \mathbb{C}$ be a continuous function. Let $(A_t)_{t\in\Omega}$ be a continuous field of normal operators in $\mathbb{A}$ whose spectra are contained in $\Delta$. Let $(W_t)_{t\in\Omega}$ be a field in $\mathcal{C}(\Omega; \mathbb{A}^+)$. Suppose that the function $t \mapsto \|W_t\|$ is integrable on $\Omega$. Then we can form the Bochner integral*

$$\int_\Omega W_t^{\frac{1}{2}} f(A_t) W_t^{\frac{1}{2}}\, d\mu(t). \tag{2.1}$$

*In addition, if $f$ is nonnegative on $\Delta$, then the operator* (2.1) *is positive.*

*Proof.* By Lemma 2.2, it suffices to prove the integrability of the norm function $t \mapsto \|W_t^{\frac{1}{2}} f(A_t) W_t^{\frac{1}{2}}\|$. Since $t \mapsto A_t$ is continuous, the map $t \mapsto f(A_t)$ is continuous by Lemma 2.1, and hence so is the map $t \mapsto W_t^{\frac{1}{2}} f(A_t) W_t^{\frac{1}{2}}$. Thus

$$\begin{aligned}
\int_\Omega \|W_t^{\frac{1}{2}} f(A_t) W_t^{\frac{1}{2}}\|\, d\mu(t) \;&\leqslant\; \int_\Omega \|W_t^{\frac{1}{2}}\| \cdot \|f(A_t)\| \cdot \|W_t^{\frac{1}{2}}\|\, d\mu(t) \\
&=\; \int_\Omega \|W_t\|^{\frac{1}{2}} \cdot \|f\|_{\infty,\mathrm{Sp}(A_t)} \cdot \|W_t\|^{\frac{1}{2}}\, d\mu(t) \\
&\leqslant\; \int_\Omega \|f\|_{\infty,\Delta} \cdot \|W_t\|\, d\mu(t) \\
&=\; \|f\|_{\infty,\Delta} \int_\Omega \|W_t\|\, d\mu(t) \\
&<\; \infty.
\end{aligned}$$

Now, suppose $f(\Delta) \subseteq [0,\infty)$. The spectral mapping theorem implies that $f(A_t)$ is a positive element for all $t \in \Omega$. Therefore the resulting integral (2.1) is positive. □

**Remark 2.4.** For convenience, we may assume that $\Omega$ is a compact Hausdorff space. In this case, any Radon measure on $\Omega$ is always finite and hence every continuous field $(X_t)_{t\in\Omega}$ of operators is automatically Bochner integrable. Indeed, its norm function $t \mapsto \|X_t\|$ is bounded and, thus, integrable. It follows that the map $t \mapsto X_t$ is Bochner integrable by Lemma 2.2.

**Lemma 2.5** (see *e.g.* [1], Lemma 11.45)**.** *Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces and let $(\Gamma, \nu)$ be a measure space. Suppose that a function $f : \Gamma \to \mathbb{X}$ is Bochner integrable. If $T : \mathbb{X} \to \mathbb{Y}$ be a bounded linear operator, then the composition $T \circ f$ is also Bochner integrable and*

$$\int_\Gamma (T \circ f)\, d\nu \;=\; T\left(\int_\Gamma f\, d\nu\right).$$

The next proposition will be useful in later discussions.

**Proposition 2.6.** *Let $(A_t)_{t\in\Omega}$ be a bounded continuous field of operators in $\mathbb{A}$. For any $X \in \mathbb{A}$, we have*

$$\int_\Omega A_t\, d\mu(t) \otimes X \;=\; \int_\Omega (A_t \otimes X)\, d\mu(t), \tag{2.2}$$

$$X \otimes \int_\Omega A_t\, d\mu(t) \;=\; \int_\Omega (X \otimes A_t)\, d\mu(t). \tag{2.3}$$

*Proof.* By Lemma 2.2, the map $t \mapsto A_t$ is Bochner integrable on $\Omega$ since it is continuous and bounded. Note that the maps $T \mapsto T \otimes X$ and $T \mapsto X \otimes T$ are bounded linear operators from $\mathfrak{B}(\mathcal{H})$ to $\mathfrak{B}(\mathcal{H} \otimes \mathcal{H})$. It follows from Lemma 2.5 that the maps $t \mapsto A_t \otimes X$ and $t \mapsto X \otimes A_t$ are Bochner integrable on $\Omega$, and the properties (2.2) and (2.3) hold. □

# 3 Integral inequalities of Kantorovich type for tensor products of operators

In this section, we derive operators integral inequalities of Kantorovich type in which the operator product is given by the tensor product. From now on, let $a, b$ be constants such that $0 < a \leqslant b$. For each $A, B \in \mathfrak{B}(\mathcal{H})$, we denote

$$A \otimes_s B \;=\; \frac{1}{2}(A \otimes B + B \otimes A).$$

Recall that the tensor power $A^{\otimes 2}$ is defined to be $A \otimes A$.

**Lemma 3.1.** *The minimum constant $k$ for which the inequality*

$$A \otimes B^{-1} + A^{-1} \otimes B \;\leqslant\; kI. \tag{3.1}$$

*holds for all positive elements $A, B \in \mathbb{A}$ whose spectra are contained in $[a, b]$ is determined by $k = (a^2 + b^2)/(ab)$. Here, $I$ denotes the identity on $\mathcal{H} \otimes \mathcal{H}$.*

*Proof.* Since $\mathrm{Sp}(A), \mathrm{Sp}(B) \subseteq [a,b]$, we have

$$\begin{aligned}\mathrm{Sp}(A \otimes B^{-1}) &= \{xy \,:\, x \in \mathrm{Sp}(A), y \in \mathrm{Sp}(B^{-1})\} \\ &= \{xz^{-1} \,:\, x \in \mathrm{Sp}(A), z \in \mathrm{Sp}(B)\} \\ &\subseteq [a/b, b/a].\end{aligned}$$

Note also that $(A \otimes B^{-1})^{-1} = A^{-1} \otimes B$. Let us denote by $r(\cdot)$ the spectral radius of an operator. Spectral mapping theorem now implies that

$$\begin{aligned}\|A \otimes B^{-1} + A^{-1} \otimes B\| &= r(A \otimes B^{-1} + A^{-1} \otimes B) \\ &= \sup\{\lambda + \lambda^{-1} \,:\, \lambda \in \mathrm{Sp}(A \otimes B^{-1})\} \\ &\leqslant \sup\{\lambda + \lambda^{-1} \,:\, \lambda \in [a/b, b/a]\} \\ &= \frac{a^2+b^2}{ab}.\end{aligned}$$

Thus, we arrive at inequality (3.1). The constant $(a^2+b^2)/(ab)$ cannot be improved since the case $A = aI_{\mathcal{H}}$ and $B = bI_{\mathcal{H}}$ is reduced to the scalar case. □

The following theorem is an integral inequality of Kantorovich type.

**Theorem 3.2.** *Let $(A_t)_{t\in\Omega}$ be a field in $\mathcal{C}(\Omega; \mathbb{A}^+, [a,b])$. Let $(W_t)_{t\in\Omega}$ be a field in $\mathcal{C}(\Omega; \mathbb{A}^+)$ such that the function $t \mapsto \|W_t\|$ is integrable on $\Omega$. Then*

$$\int_\Omega W_t^{\frac{1}{2}} A_t W_t^{\frac{1}{2}}\, d\mu(t) \otimes_s \int_\Omega W_t^{\frac{1}{2}} A_t^{-1} W_t^{\frac{1}{2}}\, d\mu(t) \leqslant \frac{a^2+b^2}{2ab} \left(\int_\Omega W_t\, d\mu(t)\right)^{\otimes 2}. \quad (3.2)$$

*Moreover, the constant $(a^2+b^2)/(2ab)$ is best possible.*

*Proof.* For convenience, let us denote

$$X = \int_\Omega W_t{}^{\frac{1}{2}} A_t W_t{}^{\frac{1}{2}}\, d\mu(t) \text{ and } Y = \int_\Omega W_t{}^{\frac{1}{2}} A_t{}^{-1} W_t{}^{\frac{1}{2}}\, d\mu(t).$$

By Proposition 2.3 with $\Delta = [a,b]$, the operators $X, Y$ and $\int_\Omega W_t\, d\mu(t)$ are well-defined and positive via putting $f(x) = x$, $f(x) = 1/x$, and $f(x) = 1$, respectively. Using properties (2.2) and (2.3) in Proposition 2.6, we obtain

$$\begin{aligned}X \otimes Y &= \int_\Omega \left(W_t^{\frac{1}{2}} A_t^{-1} W_t^{\frac{1}{2}} \otimes \int_\Omega W_r^{\frac{1}{2}} A_r^{-1} W_r^{\frac{1}{2}}\, d\mu(r)\right) d\mu(t) \\ &= \iint_{\Omega^2} \left(W_t^{\frac{1}{2}} A_t W_t^{\frac{1}{2}} \otimes W_r^{\frac{1}{2}} A_r^{-1} W_r^{\frac{1}{2}}\right) d\mu(r)\, d\mu(t).\end{aligned}$$

Similarly, we have

$$Y \otimes X = \iint_{\Omega^2} \left(W_t^{\frac{1}{2}} A_t{}^{-1} W_t^{\frac{1}{2}} \otimes W_r^{\frac{1}{2}} A_r W_r^{\frac{1}{2}}\right) d\mu(r)\, d\mu(t).$$

It follows that

$$\begin{aligned} 2(X \otimes_s Y) &= \iint_{\Omega^2} \left( W_t^{\frac{1}{2}} A_t W_t^{\frac{1}{2}} \otimes W_r^{\frac{1}{2}} A_r^{-1} W_r^{\frac{1}{2}} + W_t^{\frac{1}{2}} A_t^{-1} W_t^{\frac{1}{2}} \otimes W_r^{\frac{1}{2}} A_r W_r^{\frac{1}{2}} \right) d\mu(r)\, d\mu(t) \\ &= \iint_{\Omega^2} (W_t \otimes W_r)^{\frac{1}{2}} \left( A_t \otimes A_r^{-1} + A_t^{-1} \otimes A_r \right) (W_t \otimes W_r)^{\frac{1}{2}}\, d\mu(r)\, d\mu(t). \end{aligned}$$

Lemma 3.1 together with Proposition 2.6 imply that

$$\begin{aligned} X \otimes_s Y &\leqslant \frac{1}{2} \int_\Omega \int_\Omega \frac{a^2+b^2}{ab} (W_t \otimes W_r)\, d\mu(r)\, d\mu(t) \\ &= \frac{a^2+b^2}{2ab} \int_\Omega \left( \int_\Omega W_r\, d\mu(r) \right) \otimes W_t\, d\mu(t) \\ &= \frac{a^2+b^2}{2ab} \int_\Omega W_t\, d\mu(t) \otimes \int_\Omega W_t\, d\mu(t). \end{aligned}$$

Therefore, we arrive at (3.2). The best possibility of the constant $(a^2+b^2)/(2ab)$ also comes from Lemma 3.1. □

The next result is an integral inequality of Kantorovich type in which the weights are scalars.

**Corollary 3.3.** *Let $(A_t)_{t\in\Omega}$ be a field in $\mathcal{C}(\Omega; \mathbb{A}^+, [a,b])$. Let $w : \Omega \to [0,\infty)$ be a continuous integrable function. Then*

$$\int_\Omega w(t) A_t\, d\mu(t) \otimes_s \int_\Omega w(t) A_t^{-1}\, d\mu(t) \leqslant \frac{a^2+b^2}{2ab} \|w\|_1^2 I. \tag{3.3}$$

*Proof.* From Theorem 3.2, put $W_t = w(t)I$ for each $t \in \Omega$. □

**Corollary 3.4.** *Let $(A_t)_{t\in\Omega}$ and $(B_t)_{t\in\Omega}$ be fields in $\mathcal{C}(\Omega; \mathbb{A}^+)$ such that*

*i)* $\mathrm{Sp}(A_t) \subseteq [a,b] \subseteq (0,\infty)$ *for each* $t \in \Omega$,

*ii) the function* $t \mapsto \|B_t\|$ *is integrable on* $\Omega$, *and*

*iii)* $A_t B_t = B_t A_t$ *for each* $t \in \Omega$.

*Then*

$$\int_\Omega A_t B_t\, d\mu(t) \otimes_s \int_\Omega A_t^{-1} B_t\, d\mu(t) \leqslant \frac{a^2+b^2}{2ab} \left( \int_\Omega B_t\, d\mu(t) \right)^2 I. \tag{3.4}$$

*Proof.* From Theorem 3.2, put $W_t = B_t$ for each $t \in \Omega$. □

**Corollary 3.5.** *Let $f, \phi : \Omega \to [0,\infty)$ be continuous functions. Assume that* $\mathrm{Range}(f) \subseteq [a,b] \subseteq (0,\infty)$ *and $\phi$ is integrable with $\int_\Omega \phi\, d\mu = 1$. Then*

$$\|\phi f\|_1 \leqslant \frac{a^2+b^2}{2ab} \frac{1}{\|\phi/f\|_1}.$$

*Proof.* From Corollary 3.4, put $\mathbb{A} = \mathbb{C}$. Note that $\int_\Omega (\phi/f)\, d\mu > 0$. □

Theorem 3.2 can be extended in the following way:

**Theorem 3.6.** *Let $(A_t)_{t\in\Omega}$ be a filed in $\mathcal{C}(\Omega; \mathbb{A}^+, [a,b])$. Let $(W_t)_{t\in\Omega}$ be a field in $\mathcal{C}(\Omega; \mathbb{A}^+)$ such that the function $t \mapsto \|W_t\|$ is integrable on $\Omega$. Let $f$ be a continuous real-valued function defined on $[a,b] \cup [1/b, 1/a]$ such that*

*(i) $f(x)f(1/x) \leqslant 1$ for all $x \in [a,b]$,*

*(ii) $f([a,b]) \subseteq [a,b]$ or $f([a,b]) \subseteq [1/b, 1/a]$.*

*Then*

$$\int_\Omega W_t^{\frac{1}{2}} f(A_t) W_t^{\frac{1}{2}}\, d\mu(t) \otimes_s \int_\Omega W_t^{\frac{1}{2}} f(A_t^{-1}) W_t^{\frac{1}{2}}\, d\mu(t) \leqslant \frac{a^2+b^2}{2ab} \left( \int_\Omega W_t\, d\mu(t) \right)^{\otimes 2}. \tag{3.5}$$

*Proof.* Since $\mathrm{Sp}(A_t^{-1}) \subseteq [1/b, 1/a]$ for each $t$, the function $t \mapsto W_t^{\frac{1}{2}} f(A_t^{-1}) W_t^{\frac{1}{2}}$ is Bochner integrable by Proposition 2.3. The assumption also implies that

$$f(A_t^{-1}) \;\leqslant\; f(A_t)^{-1}$$

for each $t \in \Omega$. The inequality (3.5) now follows from Theorem 3.2. Note that the constant $(a^2+b^2)/(2ab)$ is not affected. □

Theorem 3.6 is reduced to Theorem 3.2 by setting $f(x) = x$ or $f(x) = 1/x$.

**Corollary 3.7.** *Let $0 < a < b$. Consider three continuous functions $\phi : \Omega \to [a,b]$, $g : [a,b] \to (0,\infty)$ and $f : [a,b] \cup [1/b, 1/a] \to \mathbb{R}$. Suppose that*

*(i) $f(x)f(1/x) \leqslant 1$ for all $x \in [a,b]$,*

*(ii) $f([a,b]) \subseteq [a,b]$ or $f([a,b]) \subseteq [1/b, 1/a]$.*

*Then we have the bound*

$$\|(fg) \circ \phi\|_1 \;\leqslant\; \frac{a^2+b^2}{2ab} \frac{\|g \circ \phi\|_1^2}{\|(f \circ \frac{1}{\phi})(g \circ \phi)\|_1}.$$

*Proof.* It is a special case of Theorem 3.6 when $\mathbb{A} = \mathbb{C}$. □

# 4 Kantorovich type integral inequalities involving operator means

In this section, we establish integral analogues of Kantorovich inequality involving operator means. First of all, we recall some fundamental facts about operator means [11]; see also [9, Ch. 5].

**Definition 4.1.** A binary operation $\sigma : \mathfrak{B}(\mathcal{H})^+ \times \mathfrak{B}(\mathcal{H})^+ \to \mathfrak{B}(\mathcal{H})^+$ is called a *connection* if the following conditions hold for all $A, B, C, D \in \mathfrak{B}(\mathcal{H})^+$:

(i) (joint) monotonicity: $A \leqslant C, B \leqslant D \implies A \,\sigma\, B \leqslant C \,\sigma\, D$

(ii) transformer inequality: $C(A \,\sigma\, B)C \leqslant (CAC) \,\sigma\, (CBC)$

(iii) (joint) continuity from above: for any sequences $(A_n), (B_n)$ in $\mathfrak{B}(\mathcal{H})^+$, if $A_n \downarrow A$ and $B_n \downarrow B$, then $A_n \,\sigma\, B_n \downarrow A \,\sigma\, B$. Here, $X_n \downarrow X$ indicates that $(X_n)$ is a decreasing sequence converging strongly to $X$.

It follows that every connection $\sigma$ satisfies the following properties:

$$X(A \,\sigma\, B)X \;=\; (XAX) \,\sigma\, (XBX), \tag{4.1}$$

$$(A + B) \,\sigma\, (C + D) \;\geqslant\; (A \,\sigma\, C) + (B \,\sigma\, D) \tag{4.2}$$

for all $A, B, C, D \geqslant 0$ and $X > 0$. A *mean* is a connection $\sigma$ with idempotent property $A \,\sigma\, A = A$ for all $A \geqslant 0$.

Recall also that a continuous function $f : [0, \infty) \to \mathbb{R}$ is said to be *operator monotone* if the condition $0 \leqslant A \leqslant B$ implies $f(A) \leqslant f(B)$. Such $f$ is said to be *super-multiplicative* if $f(xy) \geqslant f(x)f(y)$ for all $x, y \geqslant 0$.

**Proposition 4.2** ([11])**.** *There is a one-to-one correspondence between operator connections and operator monotone functions from $[0, \infty)$ to itself such that*

$$f(A) \;=\; I \,\sigma\, A, \quad A \in \mathfrak{B}(\mathcal{H})^+. \tag{4.3}$$

*Moreover, $\sigma$ is an operator mean if and only if $f(1) = 1$.*

Such $f$ in this proposition is called the *representing function* of $\sigma$. Every operator connection $\sigma$ admits an integral representation (see *e.g.* [3])

$$A \,\sigma\, B \;=\; \int_0^1 A \,!_t\, B \, d\nu(t), \qquad A, B \in \mathfrak{B}(\mathcal{H})^+$$

for some finite Radon measure $\nu$ on the interval $[0, 1]$. Here, $!_t$ denotes the $t$-weighted harmonic mean. Hence if $A, B \in \mathbb{A}^+$, then $A \,\sigma\, B \in \mathbb{A}^+$ since the integral is a limit of finite sums.

**Lemma 4.3** ([2])**.** *For any operator connection $\sigma$ and $A, B \in \mathfrak{B}(\mathcal{H})^+$, we have*

$$\|A \,\sigma\, B\| \leqslant \|A\| \,\sigma\, \|B\|.$$

*Here, $\sigma$ on the right hand side is the induced connection on $[0, \infty)$ defined by $(a \,\sigma\, b)I = aI \,\sigma\, bI$ for any $a, b \in [0, \infty)$.*

**Lemma 4.4.** *Let $\sigma$ be an operator connection with associated super-multiplicative operator-monotone function. Then for all positive operators $A, B, C, D$, we have*

$$(A \,\sigma\, C) \otimes_s (B \,\sigma\, D) \leqslant (A \otimes_s B) \,\sigma\, (C \otimes_s D). \tag{4.4}$$

*Proof.* By a continuity argument using the monotonicity and the continuity from above of a connection, we may assume that $A, B > 0$. Putting $X = A^{-\frac{1}{2}} C A^{-\frac{1}{2}}$ and $Y = B^{-\frac{1}{2}} D B^{-\frac{1}{2}}$, we have from properties (4.1) and (4.3) that

$$\begin{aligned}(A \,\sigma\, C) \otimes (B \,\sigma\, D) &= (A \otimes B)^{\frac{1}{2}} [(I \,\sigma\, X) \otimes (I \,\sigma\, Y)](A \otimes B)^{\frac{1}{2}} \\ &= (A \otimes B)^{\frac{1}{2}} [f(X) \otimes f(Y)](A \otimes B)^{\frac{1}{2}} \\ &\leqslant (A \otimes B)^{\frac{1}{2}} [f(X \otimes Y)](A \otimes B)^{\frac{1}{2}} \\ &= (A \otimes B)^{\frac{1}{2}} [I \,\sigma\, (X \otimes Y)](A \otimes B)^{\frac{1}{2}} \\ &= (A \otimes B) \,\sigma\, (C \otimes D).\end{aligned}$$

Now, using property (4.2) yields

$$\begin{aligned}&(A \,\sigma\, C) \otimes (B \,\sigma\, D) + (B \,\sigma\, D) \otimes (A \,\sigma\, C) \\ &\quad\leqslant (A \otimes B) \,\sigma\, (C \otimes D) + (B \otimes A) \,\sigma\, (D \otimes C) \\ &\quad\leqslant [(A \otimes B) + (B \otimes A)] \,\sigma\, [(C \otimes D) + (D \otimes C)]\,. \qquad \square\end{aligned}$$

The following result can be regarded as a Kantorovich type integral inequality concerning an operator mean.

**Theorem 4.5.** *Let $(A_t)_{t\in\Omega}$ be a filed in $\mathcal{C}(\Omega; \mathbb{A}^+, [a, b])$. Let $(W_t)_{t\in\Omega}$ be a field in $\mathcal{C}(\Omega; \mathbb{A}^+)$ such that the function $t \mapsto \|W_t\|$ is integrable on $\Omega$. Let $\sigma$ be a mean associated with a super-multiplicative representing function. Then*

$$\begin{aligned}\int_\Omega W_t^{\frac{1}{2}} (A_t \,\sigma\, B_t) W_t^{\frac{1}{2}} \, d\mu(t) \;\otimes_s\; &\int_\Omega W_t^{\frac{1}{2}} (A_t^{-1} \,\sigma\, B_t^{-1}) W_t^{\frac{1}{2}} \, d\mu(t) \\ &\leqslant \frac{a^2 + b^2}{2ab} \left( \int_\Omega W_t \, d\mu(t) \right)^{\otimes 2}.\end{aligned} \tag{4.5}$$

*Proof.* The upper semicontinuity of $\sigma$, and the continuity of the maps $t \mapsto A_t$ and $t \mapsto B_t$ together imply the measurability of the map $t \mapsto A_t \sigma B_t$. Note that $\|A_t \sigma B_t\| \leqslant b$ by the monotonicity and the idempotency of $\sigma$, and the norm estimation in Lemma 4.3. It follows that the map $t \mapsto W_t^{\frac{1}{2}} (A_t \,\sigma\, B_t) W_t^{\frac{1}{2}}$ is Bochner integrable by Lemma 2.2. Similarly, the map $t \mapsto W_t^{\frac{1}{2}} (A_t^{-1} \,\sigma\, B_t^{-1}) W_t^{\frac{1}{2}}$

is Bochner integrable. Now, we have

$$\begin{aligned}
&\int_\Omega W_t^{\frac{1}{2}}(A_t\sigma B_t)W_t^{\frac{1}{2}}\,d\mu(t)\otimes_s\int_\Omega W_t^{\frac{1}{2}}(A_t^{-1}\,\sigma\,B_t^{-1})W_t^{\frac{1}{2}}\,d\mu(t)\\
&\leqslant\int_\Omega\left(W_t^{\frac{1}{2}}A_tW_t^{\frac{1}{2}}\,\sigma\,W_t^{\frac{1}{2}}B_tW_t^{\frac{1}{2}}\right)\,d\mu(t)\otimes_s\int_\Omega\left(W_t^{\frac{1}{2}}A_t^{-1}W_t^{\frac{1}{2}}\,\sigma\,W_t^{\frac{1}{2}}B_t^{-1}W_t^{\frac{1}{2}}\right)\,d\mu(t)\\
&\quad\text{(since }\sigma\text{ satisfies the transformer inequality)}\\
&\leqslant\left[\int_\Omega W_t{}^{\frac{1}{2}}A_tW_t{}^{\frac{1}{2}}\,d\mu(t)\,\sigma\,\int_\Omega W_t{}^{\frac{1}{2}}B_tW_t{}^{\frac{1}{2}}\,d\mu(t)\right]\\
&\quad\otimes_s\left[\int_\Omega W_t{}^{\frac{1}{2}}A_t^{-1}W_t{}^{\frac{1}{2}}\,d\mu(t)\,\sigma\,\int_\Omega W_t{}^{\frac{1}{2}}B_t^{-1}W_t{}^{\frac{1}{2}}\,d\mu(t)\right]\qquad\text{(by (4.2))}\\
&\leqslant\left[\int_\Omega W_t{}^{\frac{1}{2}}A_tW_t{}^{\frac{1}{2}}\,d\mu(t)\otimes_s\int_\Omega W_t{}^{\frac{1}{2}}A_t^{-1}W_t{}^{\frac{1}{2}}\,d\mu(t)\right]\\
&\quad\sigma\left[\int_\Omega W_t{}^{\frac{1}{2}}B_tW_t{}^{\frac{1}{2}}\,d\mu(t)\otimes_s\int_\Omega W_t{}^{\frac{1}{2}}B_t^{-1}W_t{}^{\frac{1}{2}}\,d\mu(t)\right]\qquad\text{(by Lemma 4.4)}\\
&\leqslant\frac{a^2+b^2}{2ab}\left(\int_\Omega W_t\,d\mu(t)\right)^{\otimes 2}\;\sigma\;\frac{a^2+b^2}{2ab}\left(\int_\Omega W_t\,d\mu(t)\right)^{\otimes 2}\qquad\text{(by Theorem 3.2)}\\
&=\frac{a^2+b^2}{2ab}\left(\int_\Omega W_t\,d\mu(t)\right)^{\otimes 2}.
\end{aligned}$$

□

Theorem 4.5 is reduced to Theorem 3.2 by putting $A_t = B_t$ for all $t \in \Omega$.

**Corollary 4.6.** *Let $(A_t)_{t\in\Omega}$ and $(B_t)_{t\in\Omega}$ be two fields in $\mathcal{C}(\Omega;\mathbb{A}^+,[a,b])$. Let $w:\Omega\to[0,\infty)$ be an integrable continuous function. Let $\sigma$ be an operator mean associated with a super-multiplicative representing function. Then*

$$\int_\Omega w(t)(A_t\,\sigma\,B_t)\,d\mu(t)\;\otimes_s\;\int_\Omega w(t)(A_t^{-1}\,\sigma\,B_t^{-1})\,d\mu(t)\;\leqslant\;\frac{a^2+b^2}{2ab}\|w\|_1^2 I. \quad (4.6)$$

*Proof.* From Theorem 4.5, put $W_t = w(t)I$ for all $t \in \Omega$. □

**Theorem 4.7.** *Let $0 < a \leqslant 1 \leqslant b$. Let $(A_t)_{t\in\Omega}$ be a field in $\mathcal{C}(\Omega;\mathbb{A}^+,[a,b])$. Let $(W_t)_{t\in\Omega}$ be a field in $\mathcal{C}(\Omega;\mathbb{A}^+)$ such that the function $t\mapsto\|W_t\|$ is integrable. For any super-multiplicative operator-monotone function $f:[0,\infty)\to[0,\infty)$ such that $f(1)=1$, we have*

$$\int_\Omega W_t^{\frac{1}{2}}f(A_t)W_t^{\frac{1}{2}}\,d\mu(t)\otimes_s\int_\Omega W_t^{\frac{1}{2}}f(A_t^{-1})W_t^{\frac{1}{2}}\,d\mu(t)\leqslant\frac{a^2+b^2}{2ab}\left(\int_\Omega W_t\,d\mu(t)\right)^{\otimes 2}. \quad (4.7)$$

*Proof.* Proposition 4.2 guarantees the existence of an operator mean $\sigma$ such that $f(A) = I\,\sigma\,A$ for all $A \geqslant 0$. The inequality (4.7) now follows from Theorem 4.5 by considering $I\,\sigma A_t$ instead of $A_t\,\sigma\,B_t$. □

**Corollary 4.8.** *Let $0 < a \leqslant 1 \leqslant b$ and $\alpha \in [-1, 1]$. Let $(A_t)_{t\in\Omega}$ be a field in $\mathcal{C}(\Omega, ; \mathbb{A}^+, [a, b])$, and let $(B_t)_{t\in\Omega}$ be a field in $\mathcal{C}(\Omega, ; \mathbb{A}^+)$ such that $A_tB_t = B_tA_t$ for each $t \in \Omega$. Then*

$$\int_\Omega A_t^\alpha B_t \, d\mu(t) \otimes_s \int_\Omega A_t^{-\alpha} B_t \, d\mu(t) \leqslant \frac{a^2 + b^2}{2ab} \left( \int_\Omega B_t \, d\mu(t) \right)^{\otimes 2}. \tag{4.8}$$

*Proof.* It suffices to assume that $\alpha \in [0, 1]$. The famous Löwner-Heinz states that the function $f(x) = x^\alpha$ is operator monotone (see *e.g.* [9, Ch.4]). Note that $f$ is also super-multiplicative and $f(1) = 1$. The inequality 4.8 now follows by replacing $W_t$ by $B_t$ in Theorem 4.7. □

The case $\mathbb{A} = \mathbb{C}$ in Corollary 4.8 reads as follows.

**Corollary 4.9.** *Let $0 < a \leqslant 1 \leqslant b$ and $\alpha \in [-1, 1]$. Let $\phi : \Omega \to [a, b]$ and $g : \Omega \to (0, \infty)$ be continuous functions. We have*

$$\|g\phi^\alpha\|_1 \leqslant \frac{a^2 + b^2}{2ab} \frac{\|g\|_1^2}{\|g\phi^{-\alpha}\|_1}.$$

The next result is a generalization of Theorem 1.2 in the context of operators in which the constant bound is given by $(a^2 + b^2)/(2ab)$.

**Corollary 4.10.** *Let $(A_t)_{t\in\Omega}$ be a field in $\mathcal{C}(\Omega; \mathbb{A}^+, [a, b])$. If $\mu(\Omega) = 1$, then*

$$\int_\Omega A_t^2 \, d\mu(t) \otimes_s I \leqslant \frac{a^2 + b^2}{2ab} \left( \int_\Omega A_t \, d\mu(t) \right)^{\otimes 2}. \tag{4.9}$$

*Proof.* From Corollary 4.8, put $\alpha = 1$ and $A_t = B_t$ for all $t \in \Omega$. □

**Remark 4.11.** Discrete versions for all results in this paper can be obtained by putting $\Omega$ to be a finite space endowed with the counting measure. For example, a discrete version of Theorem 4.5 is as follows: For each $i = 1, 2, \ldots, n$, let $A_i$ and $B_i$ be operators in $\mathbb{A}^+$ whose spectra are contained in $[a, b]$, and let $W_i \in \mathbb{A}^+$. Let $\sigma$ be an operator mean associated with a super-multiplicative operator-monotone function. Then

$$\sum_{i=1}^n W_i^{\frac{1}{2}} (A_i \, \sigma \, B_i) W_i^{\frac{1}{2}} \otimes_s \sum_{i=1}^n W_i^{\frac{1}{2}} (A_i^{-1} \, \sigma \, B_i^{-1}) W_i^{\frac{1}{2}} \leqslant \frac{a^2 + b^2}{2ab} \left( \sum_{i=1}^n W_i \right)^{\otimes 2}.$$

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# References


[1] C. D. Aliprantis, K. C. Border, *Infinite Dimensional Analysis*, Springer-Verlag, New York, 2006.

[2] Y. M. Arlinskii, Theory of operator means, *Ukrainian Math. J.,* 42(6), 723-730 (1990).

[3] P. Chansangiam, W. Lewkeeratiyutkul, Operator connections and Borel measures on the unit interval, *ScienceAsia*, 41, 273-279 (2015).

[4] S. S. Dragomir, A survey on Cauchy-Bunyakowsky-Schwarz type discrete inequalities, *J. Inequal. Pure and Appl. Math.,* 4(3), Art. 63 (2003).

[5] S. S. Dragomir, Operator Inequalities of the Jensen, Čebyšev and Grüss Type, Springer, New York, 2012.

[6] M. Fiedler, Uber eine ungleichung fur positiv definite matrizen, *Math. Nachrichten,* 23, 197-199 (1961)

[7] M. Fujii, H. Zuo, N. Cheng, Generalization on Kantorovich inequality, *J. Math. Inequal.,* 7, 517-522 (2013).

[8] F. Hansen, J. Pečarić, I. Perić, Jensen's operator inequality and its converses, *Math Scand.,* 100, 61-73 (2007).

[9] F. Hiai, D. Petz, *Introduction to Matrix Analysis and Applications*, Springer, New Delhi, 2014.

[10] L. V. Kantorovich, Functional analysis and applied mathematics, *Uspehi Mat. Nauk (N.S.),* 3, 89-185 (1948). (in Russian)

[11] F. Kubo, T. Ando, Means of positive linear operators *Math. Ann.,* 246, 205-224 (1980).

[12] S. Liu, H. Neudecker, Several matrix Kantorovich-type inequalities, *J. Math. Anal. Appl.,* 197, 23-26 (1996).

[13] J. S. Mathru, J. S. Aujla, Hadamard product versions of the Chebyshev and Kantorovich inequalities, *J. Ineq. Pure Applied Math,* 10, Article 51 (2009).

[14] M. S. Moslehian, Recent developments of the operator Kantorovich inequality, *Expositiones Mathematicae,* 30, 376-388 (2012).

[15] R. Nakamoto, M. Nakamura, Operator mean and Kantorovich inequality, *Math. Japon.,* 44, 495-498 (1996).

[16] G. K. Pedersen, *Analysis Now*, Springer-Verlag, New York, 1989.

[17] T. Yamazaki, An extension of Kantorovich inequality to $n$-operators via the geometric mean by Ando–Li–Mathias, *Linear Algebra Appl.,* 416, 688-695 (2006).

# Characteristic fuzzy sets and conditional fuzzy subalgebras

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**Abstract.** The notion of characteristic fuzzy sets is introduced. Using this notion, conditions for a subset of $BCK/BCI$-algebra to be a subalgebra are discussed. The notion of conditional fuzzy subalgebras is introduced, and several properties are investigated. Given a subalgebra of $BCK/BCI$-algebras, conditions for the characteristic fuzzy set to be a conditional fuzzy subalgebra of several types are provided.

## 1. Introduction

The notions of "membership" and "quasicoincidence" of fuzzy points and fuzzy sets were introduced by Pu and Liu in [13]. The idea of quasi-coincidence of a fuzzy point with a fuzzy set, played a vital role to generate some different types of fuzzy subgroups, called $(\alpha, \beta)$-fuzzy subgroups, introduced by Bhakat and Das [1]. In particular, $(\in, \in \vee q)$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. Recently, these notions are applied to several algebraic structures, for example, near rings (see [2]), hypernear-rings (see [3]), hemirings (see [4]), lattices (see [9]), pseudo-$BL$ algebras (see [15]), and $BL$-algebras (see [16]) etc. In $BCK/BCI$-algebras, many research articles have been published on $(\alpha, \beta)$-fuzzy subalgebras (see [6], [7], [8], [11], [12] and [14]) which is an important and useful generalization of the well-known concepts, called fuzzy subalgebras.

In this paper, we define characteristic fuzzy sets, as a generalization of crisp characteristic function, and conditional fuzzy subalgebra. Using this notion, we discuss conditions

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for a subset of $BCK/BCI$-algebra to be a subalgebra. Given a subalgebra of $BCK/BCI$-algebras, we provide conditions for the characteristic fuzzy set to be a conditional $(\in, q)$-fuzzy subalgebra, a conditional $(q, \in)$-fuzzy subalgebra, a conditional $(q, q)$-fuzzy subalgebra, a conditional $(\in, \in \wedge q)$-fuzzy subalgebra, and a conditional $(q, \in \wedge q)$-fuzzy subalgebra.

## 2. Preliminaries

By a *$BCI$-algebra* we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the axioms:

(a1) $((x * y) * (x * z)) * (z * y) = 0$,
(a2) $(x * (x * y)) * y = 0$,
(a3) $x * x = 0$,
(a4) $x * y = y * x = 0 \ \Rightarrow \ x = y$,

for all $x, y, z \in X$. We can define a partial ordering $\leq$ by $x \leq y$ if and only if $x * y = 0$. If a $BCI$-algebra $X$ satisfies the axiom

(a5) $0 * x = 0$ for all $x \in X$,

then we say that $X$ is a *$BCK$-algebra*. A nonempty subset $S$ of a $BCK/BCI$-algebra $X$ is called a *subalgebra* of $X$ if $x * y \in S$ for all $x, y \in S$. We refer the reader to the books [5] and [10] for further information regarding $BCK/BCI$-algebras.

A fuzzy set $\mu$ in a set $X$ of the form

$$\mu(y) := \begin{cases} t \in (0, 1] & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

is said to be a *fuzzy point* with support $x$ and value $t$ and is denoted by $x_t$.

For a fuzzy point $x_t$ and a fuzzy set $\mu$ in a set $X$, Pu and Liu [13] introduced the symbol $x_t \alpha \mu$, where $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$. To say that $x_t \in \mu$ (resp. $x_t \ q \ \mu$), we mean $\mu(x) \geq t$ (resp. $\mu(x) + t > 1$), and in this case, $x_t$ is said to *belong to* (resp. *be quasi-coincident with*) a fuzzy set $\mu$. To say that $x_t \in \vee q \ \mu$ (resp. $x_t \in \wedge q \ \mu$), we mean $x_t \in \mu$ or $x_t \, q \, \mu$ (resp. $x_t \in \mu$ and $x_t \, q \, \mu$). To say that $x_t \, \overline{\alpha} \, \mu$, we mean $x_t \alpha \mu$ does not hold, where $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$.

A fuzzy set $\mu$ in a $BCK/BCI$-algebra $X$ is called a *fuzzy subalgebra* of $X$ if it satisfies:

$$\mu(x * y) \geq \min\{\mu(x), \mu(y)\} \tag{2.1}$$

for all $x, y \in X$.

A fuzzy set $\mu$ in $X$ is said to be an $(\alpha, \beta)$-*fuzzy subalgebra* of $X$, where $\alpha, \beta \in \{\in,\ q\,,\ \in\vee q\,,\ \in\wedge q\,\}$ and $\alpha \neq \in\wedge q\,$, if it satisfies the following condition:

$$x_{t_1}\alpha\,\mu,\ y_{t_2}\alpha\,\mu \ \Rightarrow\ (x*y)_{\min\{t_1,t_2\}}\,\beta\,\mu. \tag{2.2}$$

for all $x, y \in X$ and $t_1, t_2 \in (0, 1]$.

**Lemma 2.1** ([7]). *A fuzzy set $\mu$ in $X$ is an $(\in, \in\vee q\,)$-fuzzy subalgebra of $X$ if and only if it satisfies:*

$$(\forall x, y \in X)\,(\mu(x*y) \geq \min\{\mu(x), \mu(y), 0.5\})\,. \tag{2.3}$$

## 3. Characteristic fuzzy sets

In what follows, let $X$ denote a $BCK/BCI$-algebra and $\varepsilon, \delta \in [0, 1]$ with $\varepsilon > \delta$ unless otherwise specified.

For a non-empty subset $S$ of $X$, define a *characteristic fuzzy set* $\mu_S^{(\varepsilon,\delta)}$ in $X$ as follows:

$$\mu_S^{(\varepsilon,\delta)}(x) := \begin{cases} \varepsilon & \text{if } x \in S, \\ \delta & \text{otherwise.} \end{cases}$$

In particular, the characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$ in $X$ with $\varepsilon = 1$ and $\delta = 0$ is the characteristic function $\chi_S$ of $S$ in $X$.

**Theorem 3.1.** *For any non-empty subset $S$ of $X$, the following are equivalent:*

(1) *$S$ is a subalgebra of $X$.*
(2) *The characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$ is a fuzzy subalgebra of $X$.*

*Proof.* Assume that $S$ is a subalgebra of $X$ and let $x, y \in X$. If $x, y \in S$, then $x * y \in S$ and so

$$\mu_S^{(\varepsilon,\delta)}(x*y) = \varepsilon = \min\left\{\mu_S^{(\varepsilon,\delta)}(x), \mu_S^{(\varepsilon,\delta)}(y)\right\}.$$

If $x \notin S$ or $y \notin S$, then $\mu_S^{(\varepsilon,\delta)}(x) = \delta$ or $\mu_S^{(\varepsilon,\delta)}(y) = \delta$. Hence

$$\mu_S^{(\varepsilon,\delta)}(x*y) \geq \delta = \min\left\{\mu_S^{(\varepsilon,\delta)}(x), \mu_S^{(\varepsilon,\delta)}(y)\right\}.$$

Therefore $\mu_S^{(\varepsilon,\delta)}$ is a fuzzy subalgebra of $X$.

Conversely, suppose that (2) is valid. Let $x, y \in S$. Then $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon$ and $\mu_S^{(\varepsilon,\delta)}(y) = \varepsilon$. It follows that $\mu_S^{(\varepsilon,\delta)}(x * y) \geq \min\left\{\mu_S^{(\varepsilon,\delta)}(x), \mu_S^{(\varepsilon,\delta)}(y)\right\} = \varepsilon$. Thus $x * y \in S$, and therefore $S$ is a subalgebra of $X$. □

**Theorem 3.2.** *If $S$ is a subalgebra of $X$, then the characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$ is an $(\in, \in \vee q)$-fuzzy subalgebra of $X$.*

*Proof.* Assume that $S$ is a subalgebra of $X$. For any $x, y \in X$, if $x, y \in S$, then $x * y \in S$ and so

$$\mu_S^{(\varepsilon,\delta)}(x * y) = \varepsilon \geq \min\left\{\mu_S^{(\varepsilon,\delta)}(x), \mu_S^{(\varepsilon,\delta)}(y), 0.5\right\}.$$

If $x \notin S$ or $y \notin S$, then $\mu_S^{(\varepsilon,\delta)}(x) = \delta$ or $\mu_S^{(\varepsilon,\delta)}(y) = \delta$. Hence

$$\mu_S^{(\varepsilon,\delta)}(x * y) \geq \delta \geq \min\left\{\mu_S^{(\varepsilon,\delta)}(x), \mu_S^{(\varepsilon,\delta)}(y), 0.5\right\}.$$

It follows from Lemma 2.1 that $\mu_S^{(\varepsilon,\delta)}$ is an $(\in, \in \vee q)$-fuzzy subalgebra of $X$. □

The converse of Theorem 3.2 is not true in general as seen in the following example.

**Example 3.3.** Let $X = \{0, a, b, c, d\}$ be a $BCK$-algebra with the following Cayley table:

| $*$ | $0$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $a$ | $a$ | $0$ | $0$ | $0$ | $0$ |
| $b$ | $b$ | $b$ | $0$ | $0$ | $0$ |
| $c$ | $c$ | $c$ | $b$ | $0$ | $0$ |
| $d$ | $d$ | $d$ | $c$ | $b$ | $0$ |

For a subset $S = \{0, c, d\}$ of $X$, consider a characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$ in $X$ with $\varepsilon = 0.7$ and $\delta = 0.5$. Then $\mu_S^{(\varepsilon,\delta)}$ is an $(\in, \in \vee q)$-fuzzy subalgebra of $X$, but $S$ is not a subalgebra of $X$ since $d * c = b \notin S$.

**Theorem 3.4.** *Assume that $\varepsilon \leq 0.5$. If the characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$ is an $(\in, \in \vee q)$-fuzzy subalgebra of $X$ then $S$ is a subalgebra of $X$.*

*Proof.* Let $x, y \in S$. Then $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_S^{(\varepsilon,\delta)}(y)$. Using Lemma 2.1, we have

$$\mu_S^{(\varepsilon,\delta)}(x * y) \geq \left\{\mu_S^{(\varepsilon,\delta)}(x), \mu_S^{(\varepsilon,\delta)}(y), 0.5\right\} = \{\varepsilon, 0.5\} = \varepsilon,$$

and so $x * y \in S$. Therefore $S$ is a subalgebra of $X$. □

**Corollary 3.5.** *A non-empty subset $S$ of $X$ is a subalgebra of $X$ if and only if the characteristic function $\chi_S$ of $S$ is an $(\in, \in \vee q)$-fuzzy subalgebra of $X$.*

*Proof.* Clearly, we can find the necessity by taking $\varepsilon = 1$ and $\delta = 0$ in Theorem3.2.

Conversely, suppose that the characteristic function $\chi_S$ of $S$ is an $(\in, \in \vee q)$-fuzzy subalgebra of $X$. Let $x, y \in S$. Then $\chi_S(x) = 1 = \chi_S(y)$, which implies from (2.3) that

$$\chi_S(x * y) \geq \min\{\chi_S(x), \chi_S(y), 0.5\} = \min\{1, 0.5\} = 0.5.$$

Hence $x * y \in S$, and therefore $S$ is a subalgebra of $X$. □

**Theorem 3.6.** *For a subset $S$ of $X$, let $\mu_S^{(\varepsilon,\delta)}$ is an $(\in, q)$-fuzzy subalgebra of $X$. If $\delta \leq 0.5$ or $\varepsilon + \delta \leq 1$, then $S$ is a subalgebra of $X$.*

*Proof.* Let $x, y \in S$ and assume that $\delta \leq 0.5$. Then $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon > \delta$ and $\mu_S^{(\varepsilon,\delta)}(y) = \varepsilon > \delta$, that is, $x_\delta \in \mu_S^{(\varepsilon,\delta)}$ and $y_\delta \in \mu_S^{(\varepsilon,\delta)}$. Hence $(x * y)_\delta = (x * y)_{\min\{\delta,\delta\}}\, q\, \mu_S^{(\varepsilon,\delta)}$, which implies that $\mu_S^{(\varepsilon,\delta)}(x * y) + \delta > 1$. Since $\delta \leq 0,5$, it follows that $\mu_S^{(\varepsilon,\delta)}(x * y) > 1 - \delta \geq \delta$. Thus $\mu_S^{(\varepsilon,\delta)}(x * y) = \varepsilon$ and $x * y \in S$. Therefore $S$ is a subalgebra of $X$.

Now, suppose that $\varepsilon + \delta \leq 1$. Then $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_S^{(\varepsilon,\delta)}(y)$, and so $x_\varepsilon \in \mu_S^{(\varepsilon,\delta)}$ and $y_\varepsilon \in \mu_S^{(\varepsilon,\delta)}$. Hence $(x * y)_\varepsilon = (x * y)_{\min\{\varepsilon,\varepsilon\}}\, q\, \mu_S^{(\varepsilon,\delta)}$, which implies that $\mu_S^{(\varepsilon,\delta)}(x * y) + \varepsilon > 1$. Therefore $\mu_S^{(\varepsilon,\delta)}(x*y) > 1-\varepsilon \geq \delta$, and thus $\mu_S^{(\varepsilon,\delta)}(x*y) = \varepsilon$, that is, $x*y \in S$. Consequently, $S$ is a subalgebra of $X$. □

**Theorem 3.7.** *Let $\varepsilon > 0.5$. If the characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$ is a $(q, \in)$-fuzzy subalgebra of $X$, then $S$ is a subalgebra of $X$.*

*Proof.* Let $x, y \in S$. Then $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_S^{(\varepsilon,\delta)}(y)$, which implies that

$$\mu_S^{(\varepsilon,\delta)}(x) + \varepsilon = \varepsilon + \varepsilon > 1 \text{ and } \mu_S^{(\varepsilon,\delta)}(y) + \varepsilon = \varepsilon + \varepsilon > 1,$$

that is, $x_\varepsilon\, q\, \mu_S^{(\varepsilon,\delta)}$ and $y_\varepsilon\, q\, \mu_S^{(\varepsilon,\delta)}$. Since $\mu_S^{(\varepsilon,\delta)}$ is a $(q, \in)$-fuzzy subalgebra of $X$, it follows that $(x * y)_\varepsilon = (x * y)_{\min\{\varepsilon,\varepsilon\}} \in \mu_S^{(\varepsilon,\delta)}$ and so that $\mu_S^{(\varepsilon,\delta)}(x * y) = \varepsilon$, that is, $x * y \in S$. Therefore $S$ is a subalgebra of $X$. □

**Theorem 3.8.** *Assume that $\varepsilon > 0.5$ and $\varepsilon + \delta \leq 1$. If the characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$ is a $(q, q)$-fuzzy subalgebra of $X$, then $S$ is a subalgebra of $X$.*

*Proof.* Let $x, y \in S$. Then $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_S^{(\varepsilon,\delta)}(y)$, which implies that

$$\mu_S^{(\varepsilon,\delta)}(x) + \varepsilon = \varepsilon + \varepsilon > 1 \text{ and } \mu_S^{(\varepsilon,\delta)}(y) + \varepsilon = \varepsilon + \varepsilon > 1,$$

that is, $x_\varepsilon \, q \, \mu_S^{(\varepsilon,\delta)}$ and $y_\varepsilon \, q \, \mu_S^{(\varepsilon,\delta)}$. Since $\mu_S^{(\varepsilon,\delta)}$ is a $(q,q)$-fuzzy subalgebra of $X$, it follows that $(x*y)_\varepsilon = (x*y)_{\min\{\varepsilon,\varepsilon\}} \, q \, \mu_S^{(\varepsilon,\delta)}$. Hence $\mu_S^{(\varepsilon,\delta)}(x*y) > 1-\varepsilon \geq \delta$, and therefore $\mu_S^{(\varepsilon,\delta)}(x*y) = \varepsilon$. This proves that $x * y \in S$, and $S$ is a subalgebra of $X$. □

**Theorem 3.9.** *Assume that $\varepsilon + \delta \leq 1$. If the characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$ is an $(\in, \in \wedge q)$-fuzzy subalgebra of $X$, then $S$ is a subalgebra of $X$.*

*Proof.* Assume that $\varepsilon + \delta \leq 1$ and the characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$ is an $(\in, \in \wedge q)$-fuzzy subalgebra of $X$. Let $x, y \in S$. Then $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_S^{(\varepsilon,\delta)}(y)$, and so $x_\varepsilon \in \mu_S^{(\varepsilon,\delta)}$ and $y_\varepsilon \in \mu_S^{(\varepsilon,\delta)}$. Hence $(x*y)_\varepsilon = (x*y)_{\min\{\varepsilon,\varepsilon\}} \in \wedge q \, \mu_S^{(\varepsilon,\delta)}$, that is, $(x*y)_\varepsilon = (x*y)_{\min\{\varepsilon,\varepsilon\}} \in \mu_S^{(\varepsilon,\delta)}$ and $(x * y)_\varepsilon = (x * y)_{\min\{\varepsilon,\varepsilon\}} \, q \, \mu_S^{(\varepsilon,\delta)}$. Hence $\mu_S^{(\varepsilon,\delta)}(x * y) \geq \varepsilon$ and $\mu_S^{(\varepsilon,\delta)}(x * y) + \varepsilon > 1$. If $\mu_S^{(\varepsilon,\delta)}(x * y) \geq \varepsilon$, then $\mu_S^{(\varepsilon,\delta)}(x * y) = \varepsilon$ and thus $x * y \in S$. If $\mu_S^{(\varepsilon,\delta)}(x * y) + \varepsilon > 1$, then $\mu_S^{(\varepsilon,\delta)}(x * y) > 1 - \varepsilon \geq \delta$ and so $\mu_S^{(\varepsilon,\delta)}(x * y) = \varepsilon$, which shows that $x * y \in S$. Therefore $S$ is a subalgebra of $X$. □

**Theorem 3.10.** *Assume that $\varepsilon > 0.5$ and $\varepsilon + \delta \leq 1$. If the characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$ is a $(q, \in \wedge q)$-fuzzy subalgebra or a $(q, \in \vee q)$-fuzzy subalgebra of $X$, then $S$ is a subalgebra of $X$.*

*Proof.* Let $x, y \in S$. Then $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_S^{(\varepsilon,\delta)}(y)$, which implies that

$$\mu_S^{(\varepsilon,\delta)}(x) + \varepsilon = \varepsilon + \varepsilon > 1 \text{ and } \mu_S^{(\varepsilon,\delta)}(y) + \varepsilon = \varepsilon + \varepsilon > 1,$$

that is, $x_\varepsilon \, q \, \mu_S^{(\varepsilon,\delta)}$ and $y_\varepsilon \, q \, \mu_S^{(\varepsilon,\delta)}$. If $\mu_S^{(\varepsilon,\delta)}$ is a $(q, \in \wedge q)$-fuzzy subalgebra of $X$, then

$$(x * y)_\varepsilon = (x * y)_{\min\{\varepsilon,\varepsilon\}} \in \wedge q \, \mu_S^{(\varepsilon,\delta)},$$

that is, $\mu_S^{(\varepsilon,\delta)}(x * y) \geq \varepsilon$ and $\mu_S^{(\varepsilon,\delta)}(x * y) + \varepsilon > 1$. If $\mu_S^{(\varepsilon,\delta)}(x * y) \geq \varepsilon$, then $x * y \in S$. If $\mu_S^{(\varepsilon,\delta)}(x * y) + \varepsilon > 1$, then $\mu_S^{(\varepsilon,\delta)}(x * y) > 1 - \varepsilon \geq \delta$ and so $\mu_S^{(\varepsilon,\delta)}(x * y) = \varepsilon$. Thus $x * y \in S$, and therefore $S$ is a subalgebra of $X$.

If $\mu_S^{(\varepsilon,\delta)}$ is a $(q, \in \vee q)$-fuzzy subalgebra of $X$, then $(x * y)_\varepsilon = (x * y)_{\min\{\varepsilon,\varepsilon\}} \in \vee q \, \mu_S^{(\varepsilon,\delta)}$, and so that $(x * y)_\varepsilon \in \mu_S^{(\varepsilon,\delta)}$ or $(x * y)_\varepsilon \, q \, \mu_S^{(\varepsilon,\delta)}$. If $(x * y)_\varepsilon \in \mu_S^{(\varepsilon,\delta)}$, then $\mu_S^{(\varepsilon,\delta)}(x * y) = \varepsilon$ and so $x * y \in S$. If $(x * y)_\varepsilon \, q \, \mu_S^{(\varepsilon,\delta)}$, then $\mu_S^{(\varepsilon,\delta)}(x * y) + \varepsilon > 1$. Since $\varepsilon + \delta \leq 1$, it follows that $\mu_S^{(\varepsilon,\delta)}(x * y) > 1 - \varepsilon \geq \delta$ and so that $\mu_S^{(\varepsilon,\delta)}(x * y) = \varepsilon$. Thus $x * y \in S$. Therefore $S$ is a subalgebra of $X$. □

**Lemma 3.11.** *We have the following relations among the types of* $(\in, \in\vee q)$, $(\in\vee q, \in)$, $(\in\vee q, q)$, $(\in\vee q, \in\wedge q)$, *and* $(\in\vee q, \in\vee q)$*:*

$$\begin{array}{ccccc} & & (\in, \in\vee q) & & \\ & & \Uparrow & & \\ (\in\vee q, \in) & \Longleftarrow & (\in\vee q, \in\vee q) & \Longrightarrow & (\in\vee q, q) \\ & \nwarrow\!\Uparrow\!\nearrow & & & \end{array} \tag{3.1}$$

Combining Lemma 3.11 and Theorem 3.4, we have the following corollary.

**Corollary 3.12.** *Assume that* $\varepsilon \le 0.5$*. If the characteristic fuzzy set* $\mu_S^{(\varepsilon,\delta)}$ *is any one of an* $(\alpha, \beta)$*-fuzzy subalgebra of* $X$ *with* $(\alpha, \beta) \in \{(\in\vee q, \in), (\in\vee q, \in\wedge q), (\in\vee q, \in\vee q)\}$*, then* $S$ *is a subalgebra of* $X$*.*

**Theorem 3.13.** *Assume that* $\varepsilon + \delta \le 1$*. If the characteristic fuzzy set* $\mu_S^{(\varepsilon,\delta)}$ *is a* $(\in\vee q, q)$*-fuzzy subalgebra of* $X$*, then* $S$ *is a subalgebra of* $X$*.*

*Proof.* If $S$ is not a subalgebra of $X$, then there exists $a, b \in S$ such that $a * b \notin S$. Thus $\mu_S^{(\varepsilon,\delta)}(a) = \varepsilon = \mu_S^{(\varepsilon,\delta)}(b)$ and $\mu_S^{(\varepsilon,\delta)}(a * b) = \delta$. Hence $a_\varepsilon \in \mu_S^{(\varepsilon,\delta)}$ and $b_\varepsilon \in \mu_S^{(\varepsilon,\delta)}$, which imply that $a_\varepsilon \in \vee q\, \mu_S^{(\varepsilon,\delta)}$ and $b_\varepsilon \in \vee q\, \mu_S^{(\varepsilon,\delta)}$. Since $\mu_S^{(\varepsilon,\delta)}(a * b) + \varepsilon = \delta + \varepsilon \le 1$, we have $(a * b)_\varepsilon\, \overline{q}\, \mu_S^{(\varepsilon,\delta)}$. This is a contradiction, and so $S$ is a subalgebra of $X$. □

## 4. Conditional $(\alpha, \beta)$-fuzzy subalgebras

We begin with a definition.

**Definition 4.1.** Let $\mathcal{R} := \{\rho \in (0, 1] \mid \rho \text{ has relations to } \varepsilon \text{ and/or } \delta\}$. A characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$ in $X$ is called an $\mathcal{R}$*-conditional* $(\alpha, \beta)$*-fuzzy subalgebra* of $X$, where $\alpha, \beta \in \{\in,\ q,\ \in\vee q,\ \in\wedge q\}$ and $\alpha \ne \in\wedge q$, if it satisfies the following condition:

$$(\forall x, y \in X)\,(\forall \rho_1, \rho_2 \in \mathcal{R})\left(x_{\rho_1} \alpha\, \mu_S^{(\varepsilon,\delta)},\ y_{\rho_2} \alpha\, \mu_S^{(\varepsilon,\delta)} \ \Rightarrow\ (x * y)_{\min\{\rho_1,\rho_2\}}\, \beta\, \mu_S^{(\varepsilon,\delta)}\right). \tag{4.1}$$

**Example 4.2.** (1) Let $X = \{0, 1, 2, 3, 4\}$ be a set with the following Cayley table:

| $*$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 1 | 0 | 1 |
| 2 | 2 | 2 | 0 | 2 | 0 |
| 3 | 3 | 1 | 3 | 0 | 3 |
| 4 | 4 | 4 | 2 | 4 | 0 |

Then $X$ is a $BCK$-algebra (see [10]). If we take $\mathcal{R} = \{\rho \in (0,1] \mid 0.3 < \rho \leq 0.7\}$, then $\mu_S^{(\varepsilon,\delta)}$ with $S := \{0,2,4\}$ is an $\mathcal{R}$-conditional $(\in, \in \wedge\, q)$-fuzzy subalgebra of $X$ where $\delta = 0.2$ and $\varepsilon = 0.7$.

(2) For a fixed element $a$ of a $BCI$-algebra $X$, let

$$S := \{x \in X \mid a * (a * x) = x\}.$$

For $\delta = 0.3$ and $\varepsilon = 0.6$, if we consider $\mathcal{R}_1 = \{\rho \in (0,1] \mid \rho > 0.4\}$ then $\mu_S^{(\varepsilon,\delta)}$ is an $\mathcal{R}_1$-conditional $(\in, q)$-fuzzy subalgebra of $X$. If we take $\mathcal{R}_2 = \{\rho \in (0,1] \mid \rho \leq 0.6\}$ then $\mu_S^{(\varepsilon,\delta)}$ is an $\mathcal{R}_2$-conditional $(q, \in)$-fuzzy subalgebra of $X$.

(3) Let $X = \{0,1,2,a,b\}$ be a set with the following Cayley table:

| $*$ | $0$ | $1$ | $2$ | $a$ | $b$ |
|---|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $a$ | $a$ |
| $1$ | $1$ | $0$ | $1$ | $a$ | $a$ |
| $2$ | $2$ | $2$ | $0$ | $a$ | $a$ |
| $a$ | $a$ | $a$ | $a$ | $0$ | $0$ |
| $b$ | $b$ | $a$ | $b$ | $1$ | $0$ |

Then $X$ is a $BCI$-algebra (see [5, 10]). Consider $\mathcal{R} = \{\rho \in (0,1] \mid 0.3 < \rho \leq 0.9\}$, Then $\mu_S^{(\varepsilon,\delta)}$ with $S := \{0,1,2\}$ is an $\mathcal{R}$-conditional $(q,q)$-fuzzy subalgebra of $X$ where $\delta = 0.1$ and $\varepsilon = 0.7$.

(4) Let $X$ be a $BCI$-algebra and let $S := \{x * a \mid x \in X\}$ for a fixed element $a \in X$. Consider $\mathcal{R} = \{\rho \in (0,1] \mid 0.3 < \rho \leq 0.7\}$. Then $\mu_S^{(\varepsilon,\delta)}$ is an $\mathcal{R}$-conditional $(q, \in \wedge\, q)$-fuzzy subalgebra of $X$ with $\delta = 0.1$ and $\varepsilon = 0.7$.

**Theorem 4.3.** *Let $\mathcal{R} := \{\rho \in (0,1] \mid \rho > \delta$ and $\varepsilon + \rho > 1\}$. If $S$ is a subalgebra of $X$, then $\mu_S^{(\varepsilon,\delta)}$ is an $\mathcal{R}$-conditional $(\in, q)$-fuzzy subalgebra of $X$.*

*Proof.* Let $x, y \in X$ and $\rho_1, \rho_2 \in \mathcal{R}$ be such that $x_{\rho_1} \alpha\, \mu_S^{(\varepsilon,\delta)}$ and $y_{\rho_2} \alpha\, \mu_S^{(\varepsilon,\delta)}$. Then $\mu_S^{(\varepsilon,\delta)}(x) \geq \rho_1 > \delta$ and $\mu_S^{(\varepsilon,\delta)}(y) \geq \rho_2 > \delta$, which imply that $x, y \in S$. Thus $x * y \in S$, and so $\mu_S^{(\varepsilon,\delta)}(x * y) = \varepsilon$. Hence

$$\mu_S^{(\varepsilon,\delta)}(x * y) + \min\{\rho_1, \rho_2\} = \varepsilon + \min\{\rho_1, \rho_2\} > 1,$$

that is, $(x * y)_{\min\{\rho_1,\rho_2\}}\, q\, \mu_S^{(\varepsilon,\delta)}$. Therefore $\mu_S^{(\varepsilon,\delta)}$ is an $\mathcal{R}$-conditional $(\in, q)$-fuzzy subalgebra of $X$. □

If we take $\varepsilon = 1$ and $\delta = 0$ in Theorems 3.6 and 4.3, then we have the following corollary.

**Corollary 4.4.** *A non-empty subset $S$ of $X$ is a subalgebra of $X$ if and only if the characteristic function $\chi_S$ of $S$ is an $(\in, q)$-fuzzy subalgebra of $X$.*

**Theorem 4.5.** *Let $\mathcal{R} := \{\rho \in (0,1] \mid \varepsilon \geq \rho \text{ and } \delta \leq 1-\rho\}$. If $S$ is a subalgebra of $X$, then $\mu_S^{(\varepsilon,\delta)}$ is an $\mathcal{R}$-conditional $(q, \in)$-fuzzy subalgebra of $X$.*

*Proof.* Let $x, y \in X$ and $\rho_1, \rho_2 \in \mathcal{R}$ be such that $x_{\rho_1}\, q\, \mu_S^{(\varepsilon,\delta)}$ and $y_{\rho_2}\, q\, \mu_S^{(\varepsilon,\delta)}$. Then $\mu_S^{(\varepsilon,\delta)}(x) + \rho_1 > 1$ and $\mu_S^{(\varepsilon,\delta)}(y) + \rho_2 > 1$, which imply that $\mu_S^{(\varepsilon,\delta)}(x) > 1 - \rho_1 \geq \delta$ and $\mu_S^{(\varepsilon,\delta)}(u) > 1 - \rho_2 \geq \delta$. Hence $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_S^{(\varepsilon,\delta)}(y)$, and so $x, y \in S$. Since $S$ is a subalgebra of $X$, we have $x * y \in S$. Thus $\mu_S^{(\varepsilon,\delta)}(x * y) = \varepsilon \geq \min\{\rho_1, \rho_2\}$, and hence $(x * y)_{\min\{\rho_1,\rho_2\}} \in \mu_S^{(\varepsilon,\delta)}$. Therefore $\mu_S^{(\varepsilon,\delta)}$ is an $\mathcal{R}$-conditional $(q, \in)$-fuzzy subalgebra of $X$. □

If we take $\varepsilon = 1$ and $\delta = 0$ in Theorems 3.7 and 4.5, then we have the following corollary.

**Corollary 4.6.** *A non-empty subset $S$ of $X$ is a subalgebra of $X$ if and only if the characteristic function $\chi_S$ of $S$ is a $(q, \in)$-fuzzy subalgebra of $X$.*

**Theorem 4.7.** *Let $\mathcal{R} := \{\rho \in (0,1] \mid \delta \leq 1-\rho < \varepsilon\}$. If $S$ is a subalgebra of $X$, then the characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$ is an $\mathcal{R}$-conditional $(q, q)$-fuzzy subalgebra of $X$.*

*Proof.* Let $x, y \in X$ and $\rho_1, \rho_2 \in \mathcal{R}$ be such that $x_{\rho_1}\, q\, \mu_S^{(\varepsilon,\delta)}$ and $y_{\rho_2}\, q\, \mu_S^{(\varepsilon,\delta)}$. Then $\mu_S^{(\varepsilon,\delta)}(x) + \rho_1 > 1$ and $\mu_S^{(\varepsilon,\delta)}(y) + \rho_2 > 1$, which imply that $\mu_S^{(\varepsilon,\delta)}(x) > 1 - \rho_1 \geq \delta$ and $\mu_S^{(\varepsilon,\delta)}(y) > 1 - \rho_2 \geq \delta$. It follows that $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_S^{(\varepsilon,\delta)}(y)$ and so that $x, y \in S$. Since $S$ is a subalgebra of $X$, we have $x * y \in S$ and so $\mu_S^{(\varepsilon,\delta)}(x * y) = \varepsilon$. Thus

$$\mu_S^{(\varepsilon,\delta)}(x * y) + \min\{\rho_1, \rho_2\} = \varepsilon + \min\{\rho_1, \rho_2\} > 1,$$

that is, $(x * y)_{\min\{\rho_1,\rho_2\}}\, q\, \mu_S^{(\varepsilon,\delta)}$. This shows that $\mu_S^{(\varepsilon,\delta)}$ is a $(q, q)$-fuzzy subalgebra of $X$. □

If we take $\varepsilon = 1$ and $\delta = 0$ in Theorems 3.8 and 4.7, then we have the following corollary.

**Corollary 4.8.** *A non-empty subset $S$ of $X$ is a subalgebra of $X$ if and only if the characteristic function $\chi_S$ of $S$ is a $(q, q)$-fuzzy subalgebra of $X$.*

Since the $(q, \in \vee q)$-fuzzy subalgebra is induced by a $(q, \in)$-fuzzy subalgebra or a $(q, q)$-fuzzy subalgebra, we have the following corollary by using Theorems 4.5 and 4.7.

**Corollary 4.9.** *Let $\mathcal{R}$ be any one of*

$$\{\rho \in (0,1] \mid \delta \leq 1-\rho < \varepsilon\} \text{ and } \{\rho \in (0,1] \mid \varepsilon \geq \rho \text{ and } \delta \leq 1-\rho\}.$$

*If $S$ is a subalgebra of $X$, then the characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$ is an $\mathcal{R}$-conditional $(q, \in \vee q)$-fuzzy subalgebra of $X$.*

**Theorem 4.10.** *Let* $\mathcal{R} := \{\rho \in (0,1] \mid \delta < \rho \leq \varepsilon \text{ and } 1-\rho < \varepsilon\}$. *If* $S$ *is a subalgebra of* $X$, *then the characteristic fuzzy set* $\mu_S^{(\varepsilon,\delta)}$ *is an* $\mathcal{R}$*-conditional* $(\in, \in \wedge\, q)$*-fuzzy subalgebra of* $X$.

*Proof.* Let $x, y \in X$ and $\rho_1, \rho_2 \in \mathcal{R}$ be such that $x_{\rho_1} \in \mu_S^{(\varepsilon,\delta)}$ and $x_{\rho_2} \in \mu_S^{(\varepsilon,\delta)}$. Then $\mu_S^{(\varepsilon,\delta)}(x) \geq \rho_1 > \delta$ and $\mu_S^{(\varepsilon,\delta)}(y) \geq \rho_2 > \delta$, which imply that $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_S^{(\varepsilon,\delta)}(y)$. Hence $x, y \in S$. Since $S$ is a subalgebra of $X$, we have $x * y \in S$. Hence $\mu_S^{(\varepsilon,\delta)}(x * y) = \varepsilon \geq \min\{\rho_1, \rho_2\}$, i.e., $(x * y)_{\min\{\rho_1,\rho_2\}} \in \mu_S^{(\varepsilon,\delta)}$. Now,

$$\mu_S^{(\varepsilon,\delta)}(x * y) + \min\{\rho_1, \rho_2\} = \varepsilon + \min\{\rho_1, \rho_2\} > 1$$

and so $(x * y)_{\min\{\rho_1,\rho_2\}}\, q\, \mu_S^{(\varepsilon,\delta)}$. Therefore $(x * y)_{\min\{\rho_1,\rho_2\}} \in \wedge\, q\, \mu_S^{(\varepsilon,\delta)}$, and consequently $\mu_S^{(\varepsilon,\delta)}$ is an $(\in, \in \wedge\, q)$-fuzzy subalgebra of $X$. □

If we take $\varepsilon = 1$ and $\delta = 0$ in Theorems 3.9 and 4.10, then we have the following corollary.

**Corollary 4.11.** *A non-empty subset* $S$ *of* $X$ *is a subalgebra of* $X$ *if and only if the characteristic function* $\chi_S$ *of* $S$ *is an* $(\in, \in \wedge\, q)$*-fuzzy subalgebra of* $X$.

**Theorem 4.12.** *Let* $\mathcal{R} := \{\rho \in (0,1] \mid \varepsilon \geq \rho \text{ and } \varepsilon + \rho > 1 \geq \delta + \rho\}$. *If* $S$ *is a subalgebra of* $X$, *then the characteristic fuzzy set* $\mu_S^{(\varepsilon,\delta)}$ *is an* $\mathcal{R}$*-conditional* $(q, \in \wedge\, q)$*-fuzzy subalgebra of* $X$.

*Proof.* Let $x, y \in X$ and $\rho_1, \rho_2 \in \mathcal{R}$ be such that $x_{\rho_1}\, q\, \mu_S^{(\varepsilon,\delta)}$ and $y_{\rho_2}\, q\, \mu_S^{(\varepsilon,\delta)}$. Then $\mu_S^{(\varepsilon,\delta)}(x) + \rho_1 > 1$ and $\mu_S^{(\varepsilon,\delta)}(y) + \rho_2 > 1$, which imply that $\mu_S^{(\varepsilon,\delta)}(x) > 1 - \rho_1 \geq \delta$ and $\mu_S^{(\varepsilon,\delta)}(y) > 1 - \rho_2 \geq \delta$. Hence $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_S^{(\varepsilon,\delta)}(y)$, and so $x, y \in S$. Since $S$ is a subalgebra of $X$, we have $x * y \in S$ and thus

$$\mu_S^{(\varepsilon,\delta)}(x * y) = \varepsilon \geq \min\{\rho_1, \rho_2\},$$

that is, $(x * y)_{\min\{\rho_1,\rho_2\}} \in \mu_S^{(\varepsilon,\delta)}$. Now, $\mu_S^{(\varepsilon,\delta)}(x * y) + \min\{\rho_1, \rho_2\} = \varepsilon + \min\{\rho_1, \rho_2\} > 1$, and so $(x * y)_{\min\{\rho_1,\rho_2\}}\, q\, \mu_S^{(\varepsilon,\delta)}$. Hence $(x * y)_{\min\{\rho_1,\rho_2\}} \in \wedge\, q\, \mu_S^{(\varepsilon,\delta)}$, and $\mu_S^{(\varepsilon,\delta)}$ is a $(q, \in \wedge\, q)$-fuzzy subalgebra of $X$. □

If we take $\varepsilon = 1$ and $\delta = 0$ in Theorems 3.10 and 4.12, then we have the following corollary.

**Corollary 4.13.** *A non-empty subset* $S$ *of* $X$ *is a subalgebra of* $X$ *if and only if the characteristic function* $\chi_S$ *of* $S$ *is an* $(q, \in \wedge\, q)$*-fuzzy subalgebra of* $X$.

Before ending our research, we pose an open question.

**Question.** *Given a subalgebra* $S$ *of* $X$, *when will the characteristic fuzzy set* $\mu_S^{(\varepsilon,\delta)}$ *in* $X$ *be a*

(1) *conditional* $(\in \vee q\,, \in)$*-fuzzy subalgebra of* $X$*?*
(2) *conditional* $(\in \vee q\,, q)$*-fuzzy subalgebra of* $X$*?*
(3) *conditional* $(\in \vee q\,, \in \vee q\,)$*-fuzzy subalgebra of* $X$*?*
(4) *conditional* $(\in \vee q\,, \in \wedge q\,)$*-fuzzy subalgebra of* $X$*?*

## 5. Conclusion

we have introduced the notions of characteristic fuzzy sets, as a generalization of crisp characteristic function, and conditional fuzzy subalgebra. Using this notion, we have discussed conditions for a subset of $BCK/BCI$-algebra to be a subalgebra. Given a subalgebra of $BCK/BCI$-algebras, we have provided conditions for the characteristic fuzzy set to be a conditional $(\in, q)$-fuzzy subalgebra, a conditional $(q, \in)$-fuzzy subalgebra, a conditional $(q, q)$-fuzzy subalgebra, a conditional $(\in, \in \wedge q\,)$-fuzzy subalgebra, and a conditional $(q, \in \wedge q\,)$-fuzzy subalgebra.

On the basis of these results, we will apply the notions of characteristic fuzzy sets and conditional fuzzy substructures to ideal and filter theory in several algebraic structures, for example, $BCK/BCI$-algebras, $MV$-algebras, $BL$-algebras, $MTL$-algebras, residuated lattices, $R_0$-algebras, lattice implication algebras, $EQ$-algebras etc.

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## References

[1] S. K. Bhakat and P. Das, $(\in, \in \vee \mathrm{q})$-fuzzy subgroup, Fuzzy Sets and Systems 80 (1996), 359–368.

[2] B. Davvaz, $(\in, \in \vee q\,)$-fuzzy subnear-rings and ideals, Soft Computing 10(3) (2006) 206–211.

[3] B. Davvaz and P. Corsini, Generalized fuzzy hyperideals of hypernear-rings and many valued implications, J. Intell. Fuzzy Systems 17 (2006) 241–251.

[4] W. A. Dudek, M. Shabir and M. Irfan Ali, $(\alpha, \beta)$-fuzzy ideals of hemirings, Comput. Math. Appl. 58 (2009) 310–321.

[5] Y. S. Huang, $BCI$-algebra, Science Press, Beijing, 2006.

[6] Y. B. Jun, On $(\alpha, \beta)$-fuzzy ideals of $BCK/BCI$-algebras, Sci. Math. Jpn. 60(3) (2004), 613–617.

[7] Y. B. Jun, On $(\alpha, \beta)$-fuzzy subalgebras of $BCK/BCI$-algebras, Bull. Korean Math. Soc. 42(4) (2005), 703–711.

[8] Y. B. Jun, Fuzzy subalgebras of type $(\alpha, \beta)$ in $BCK/BCI$-algebras, Kyungpook Math. J. 47 (2007), 403–410.

[9] O. Kazanci and B. Davvaz, More on fuzzy lattices, Comput. Math. Appl. 64 (2012) 2917–2925.

[10] J. Meng and Y. B. Jun, $BCK$-algebra, Kyungmoon Sa Co. Seoul, 1994.

[11] G. Muhiuddin and Abdullah M. Al-roqi, Classifications of (alpha, beta)-fuzzy ideals in BCK/BCI-algebras, Journal of Mathematical Analysis, 7 (6) (2016) 75–82.

[12] G. Muhiuddin and Abdullah M. Al-roqi, Subalgebras of BCK/BCI-algebras based on (alpha, beta)-type fuzzy sets, J. Comput. Anal. Appl., 18 (6) (2015) 1057–1064.

[13] P. M. Pu and Y. M. Liu, Fuzzy topology I, Neighborhood structure of a fuzzy point and Moore-Smith convergence, J. Math. Anal. Appl. 76 (1980), 571–599.

[14] J. Zhan, Y. B. Jun and B. Davvaz, On $(\in, \in \vee q)$-fuzzy ideals of $BCI$-algebras, Iran. J. Fuzzy Syst. 6(1) (2009), 81–94.

[15] J. Zhan, Y. B. Jun and W. A. Dudek, On $(\in, \in \vee q)$-fuzzy filters of pseudo-$BL$ algerbas, Bull. Malays. Math. Sci. Soc. 33(1) (2010) 57–67.

[16] J. Zhan and Y. Xu, Some types of generalized fuzzy filters of $BL$-algebras, Comput. Math. Appl. 56 (2008) 1604–1616.

# *Some common fixed point theorems for two pairs of self maps in dislocated metric spaces*

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**Abstract.** The purpose of this paper is to establish some common fixed point theorems for two pairs of self mappings in dislocated metric spaces which generalize, extend, and improve related results in the literature. Some applications of new results are provided.

**Keywords:** Dislocated metric space; fixed point; weakly compatible maps; common fixed point.

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# 1 Introduction and preliminaries

In 2000, Hitzler and Seda [4] presented the concept of dislocated metric space and generalized the well-known Banach contraction mapping principle in complete dislocated metric spaces. In recent years, the study of dislocated metric spaces has always attracted interest of researchers; see, for instance, [1–3, 5–8, 10–16] and the references cited therein. One of the main reasons for this lies in the fact that dislocated metric spaces play very important roles not only in topology but also in other branches of science involving mathematics especially in logic programming and electronics engineering.

In the present paper, we prove some common fixed point theorems in the setting of dislocated metric spaces for two pairs of weakly compatible self mappings which generalize, extend, and improve related results reported in the literature. We need the following auxiliary definitions and results.

**Definition 1.1** [4] *Let $\mathcal{X}$ be a nonempty set and let $d : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be a function satisfying the following conditions:*
*($c_1$) $d(\xi, \eta) = d(\eta, \xi)$ for all $\xi, \eta \in \mathcal{X}$;*
*($c_2$) $d(\xi, \eta) = d(\eta, \xi) = 0$ implies that $\xi = \eta$;*
*($c_3$) $d(\xi, \eta) \leq d(\xi, \zeta) + d(\zeta, \eta)$ for all $\xi, \eta, \zeta \in \mathcal{X}$.*
*Then $d$ is called dislocated metric (or d-metric) on $\mathcal{X}$. The nonempty set $\mathcal{X}$ together with d-metric, i.e., $(\mathcal{X}, d)$, is called a dislocated metric space.*

**Definition 1.2** [4] *A sequence $\{\xi_n\}$ in a d-metric space $(\mathcal{X}, d)$ is called Cauchy sequence if for given $\epsilon > 0$, there exists an $n_0 \in N$ such that $d(\xi_m, \xi_n) < \epsilon$ for all $m, n \geq n_0$.*

**Definition 1.3** [4] *A sequence $\{\xi_n\}$ in a d-metric space $(\mathcal{X}, d)$ converges with respect to $d$ (or in $d$) if there exists a $\xi \in \mathcal{X}$ such that*

$$\lim_{n\to\infty} d(\xi_n, \xi) = 0.$$

*In this case, $\xi$ is called a limit of sequence $\{\xi_n\}$ and we write $\xi_n \to \xi$ as $n \to \infty$.*

**Definition 1.4** [4] *A d-metric space $(\mathcal{X}, d)$ is called complete if every Cauchy sequence in it is convergent with respect to $d$.*

**Definition 1.5** [4] *Let $(\mathcal{X}, d)$ be a d-metric space. A mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is called contraction if there exists a $\lambda \in [0, 1)$ such that $d(\mathcal{T}\xi, \mathcal{T}\eta) \leq \lambda d(\xi, \eta)$ for all $\xi, \eta \in \mathcal{X}$.*

**Lemma 1.6** [11] *Let $(\mathcal{X}, d)$ be a d-metric space. If $g : \mathcal{X} \to \mathcal{X}$ is a contraction function, then $\{g^n(\xi_0)\}$ is a Cauchy sequence for each $\xi_0 \in \mathcal{X}$.*

**Lemma 1.7** [4] *Limits in a d-metric space are unique.*

**Definition 1.8** [9] *Let $A$ and $S$ be mappings from a metric space $(\mathcal{X}, d)$ into itself. Then, $A$ and $S$ are said to be weakly compatible if they commute at their coincident points; that is, $A\xi = S\xi$ for some $\xi \in \mathcal{X}$ yields $AS\xi = SA\xi$.*

**Theorem 1.9** [4] *Let $(\mathcal{X}, d)$ be a complete dislocated metric space and let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a contraction mapping. Then, $\mathcal{T}$ has a unique fixed point.*

**Remark 1.10** [3] *It is easy to verify that in a d-metric space, the following statements hold.*
*(i) A subsequence of a Cauchy sequence in d-metric space is a Cauchy sequence.*
*(ii) A Cauchy sequence in d-metric space with a convergent subsequence is also convergent.*
*(iii) Limits of a convergent sequence are unique.*
*(iv) A d-metric $d$ is continuous, i.e., $\xi_n \to \xi$ and $\eta_n \to \eta$ imply that $d(\xi_n, \eta_n) \to d(\xi, \eta)$ as $n \to \infty$.*

# 2 Main results

In this section, we prove some fixed point theorems in d-metric spaces.

**Theorem 2.1** *Let $(\mathcal{X}, d)$ be a complete dislocated metric space. Assume that $A, B, S, T : \mathcal{X} \to \mathcal{X}$ are continuous self mappings satisfying the conditions:*
*(i) $T(\mathcal{X}) \subset A(\mathcal{X})$ and $S(\mathcal{X}) \subset B(\mathcal{X})$;*
*(ii) the pairs $(S, A)$ and $(T, B)$ are weakly compatible;*
*(iii) $d(S\xi, T\eta) \leq a_1[d(A\xi, T\eta) + d(B\eta, S\xi)] + a_2[d(B\eta, T\eta) + d(A\xi, S\xi)] + a_3 d(A\xi, B\eta) + a_4[d(A\xi, S\xi) + d(A\xi, T\eta)] + a_5[d(A\xi, B\eta) + d(B\eta, T\eta)]$ for all $\xi, \eta \in \mathcal{X}$, where $a_1, a_2, a_3, a_4, a_5 \geq 0$ and $0 \leq 4a_1 + 2a_2 + a_3 + 3a_4 + 2a_5 < 1$.*
*Then $A$, $B$, $S$, and $T$ have a unique common fixed point.*

**Proof.** Define two sequences $\{\xi_n\}$ and $\{\eta_n\}$ by

$$\eta_{2n} := B\xi_{2n+1} = S\xi_{2n} \text{ and } \eta_{2n+1} := A\xi_{2n+2} = T\xi_{2n+1} \text{ for } n = 0, 1, 2, \ldots.$$

If $\eta_{2n} = \eta_{2n+1}$ for some $n$, then $B\xi_{2n+1} = T\xi_{2n+1}$. Therefore, $\xi_{2n+1}$ is a coincident point of $B$ and $T$. Also, if $\eta_{2n+1} = \eta_{2n+2}$ for some $n$, then $A\xi_{2n+2} = S\xi_{2n+2}$. Hence, $\xi_{2n+2}$ is a coincidence point of $A$ and $S$. Suppose now that $\eta_{2n} \neq \eta_{2n+1}$ for all $n$. Then, we conclude that

$$\begin{aligned}
d(\eta_{2n}, \eta_{2n+1}) &= d(S\xi_{2n}, T\xi_{2n+1}) \\
&\leq a_1[d(A\xi_{2n}, T\xi_{2n+1}) + d(B\xi_{2n+1}, S\xi_{2n})] \\
&\quad +a_2[d(B\xi_{2n+1}, T\xi_{2n+1}) + d(A\xi_{2n}, S\xi_{2n})] \\
&\quad +a_3 d(A\xi_{2n}, B\xi_{2n+1}) + a_4[d(A\xi_{2n}, S\xi_{2n}) + d(A\xi_{2n}, T\xi_{2n+1})] \\
&\quad +a_5[d(A\xi_{2n}, B\xi_{2n+1}) + d(B\xi_{2n+1}, T\xi_{2n+1})] \\
&\leq a_1[d(\eta_{2n-1}, \eta_{2n+1}) + d(\eta_{2n}, \eta_{2n})] \\
&\quad +a_2[d(\eta_{2n}, \eta_{2n+1}) + d(\eta_{2n-1}, \eta_{2n})] \\
&\quad +a_3 d(\eta_{2n-1}, \eta_{2n}) + a_4[d(\eta_{2n-1}, \eta_{2n}) + d(\eta_{2n-1}, \eta_{2n+1})] \\
&\quad +a_5[d(\eta_{2n-1}, \eta_{2n}) + d(\eta_{2n}, \eta_{2n+1})] \\
&\leq a_1[d(\eta_{2n-1}, \eta_{2n}) + d(\eta_{2n}, \eta_{2n+1}) \\
&\quad +d(\eta_{2n-1}, \eta_{2n}) + d(\eta_{2n}, \eta_{2n+1})] \\
&\quad +a_2[d(\eta_{2n}, \eta_{2n+1}) + d(\eta_{2n-1}, \eta_{2n})] + a_3 d(\eta_{2n-1}, \eta_{2n}) \\
&\quad +a_4[d(\eta_{2n-1}, \eta_{2n}) + d(\eta_{2n-1}, \eta_{2n}) + d(\eta_{2n}, \eta_{2n+1})] \\
&\quad +a_5[d(\eta_{2n-1}, \eta_{2n}) + d(\eta_{2n}, \eta_{2n+1})] \\
&= (2a_1 + a_2 + a_3 + 2a_4 + a_5)d(\eta_{2n-1}, \eta_{2n}) \\
&\quad +(2a_1 + a_2 + a_4 + a_5)d(\eta_{2n}, \eta_{2n+1}).
\end{aligned}$$

Therefore, we get

$$d(\eta_{2n}, \eta_{2n+1}) \leq \frac{2a_1 + a_2 + a_3 + 2a_4 + a_5}{1 - (2a_1 + a_2 + a_4 + a_5)} d(\eta_{2n-1}, \eta_{2n}).$$

Let

$$h = \frac{2a_1 + a_2 + a_3 + 2a_4 + a_5}{1 - (2a_1 + a_2 + a_4 + a_5)} < 1.$$

Then

$$d(\eta_n, \eta_{n+1}) \leq h d(\eta_{n-1}, \eta_n).$$

Similarly, we have

$$d(\eta_{n-1}, \eta_n) \leq hd(\eta_{n-2}, \eta_{n-1}).$$

Continuing this process, we obtain

$$d(\eta_n, \eta_{n+1}) \leq h^n d(\eta_0, \eta_1).$$

Now, for any $m, n$ satisfying $m > n$, using triangle inequality, we get

$$\begin{aligned} d(\eta_n, \eta_m) &\leq d(\eta_n, \eta_{n+1}) + d(\eta_{n+1}, \eta_{n+2}) + \cdots + d(\eta_{m-1}, \eta_m) \\ &\leq h^n d(\eta_0, \eta_1) + h^{n+1} d(\eta_0, \eta_1) + \cdots + h^{m-1} d(\eta_0, \eta_1) \\ &\leq (h^n + h^{n+1} + h^{n+2} + \cdots) d(\eta_0, \eta_1) \\ &= \frac{h^n}{1-h} d(\eta_0, \eta_1). \end{aligned}$$

Since $h \in [0, 1)$, $h^n \to 0$ as $n \to \infty$, which shows that $\{\eta_n\}$ is a Cauchy sequence in the complete dislocated metric space $(\mathcal{X}, d)$. Hence, there exists a point $p \in \mathcal{X}$ such that $\lim_{n\to\infty} \eta_n = p$ and

$$\lim_{n\to\infty} S\xi_{2n} = \lim_{n\to\infty} B\xi_{2n+1} = \lim_{n\to\infty} T\xi_{2n+1} = \lim_{n\to\infty} A\xi_{2n+2} = p.$$

Since $T(\mathcal{X}) \subset A(\mathcal{X})$, there exists a point $v \in \mathcal{X}$ such that $p = Av$. Therefore,

$$\begin{aligned} d(Sv, p) &= d(Sv, T\xi_{2n+1}) \\ &\leq a_1[d(Av, T\xi_{2n+1}) + d(B\xi_{2n+1}, Sv)] \\ &\quad + a_2[d(B\xi_{2n+1}, T\xi_{2n+1}) + d(Av, Sv)] \\ &\quad + a_3 d(Av, Sv) + a_4[d(Av, Sv) + d(Au, T\xi_{2n+1})] \\ &\quad + a_5[d(Av, B\xi_{2n+1}) + d(B\xi_{2n+1}, T\xi_{2n+1})]. \end{aligned}$$

Taking $n \to \infty$, we get

$$\begin{aligned} d(Sv, p) &\leq a_1[d(p, p) + d(p, Sv)] + a_2[d(p, p) + d(p, Sv)] \\ &\quad + a_3 d(p, Sv) + a_4[d(p, Sv) + d(p, p)] + a_5[d(p, p) + d(p, p)] \\ &\leq (2a_1 + 2a_2 + 2a_4 + 4a_5) d(p, Sv) + (a_1 + a_2 + a_3 + a_4) d(p, Sv) \\ &= (3a_1 + 3a_2 + a_3 + 3a_4 + 4a_5) d(p, Sv), \end{aligned}$$

which is a contradiction, and so $Sv = Av = p$. Again, since $S(\mathcal{X}) \subset B(\mathcal{X})$, there exists a point $u \in \mathcal{X}$ such that $p = Bu$. We claim now that $p = Tu$. If

$p \neq Tu$, then

$$\begin{aligned} d(p,Tu) &= d(Sv,Tu) \\ &\leq a_1[d(Av,Tu)+d(Bu,Sv)]+a_2[d(Bu,Tu)+d(Av,Sv)] \\ &\quad +a_3d(Av,Bu)+a_4[d(Av,Sv)+d(Av,Tu)] \\ &\quad +a_5[d(Av,Bu)+d(Bu,Tu)] \\ &= a_1[d(p,Tu)+d(p,p)]+a_2[d(p,Tu)+d(p,p)]+a_3d(p,p) \\ &\quad +a_4[d(p,p)+d(p,Tu)]+a_5[d(p,p)+d(p,Tu)] \\ &\leq (3a_1+3a_2+2a_3+3a_4+3a_5)d(p,Tu), \end{aligned}$$

which is a contradiction, and thus $p = Tu$. Hence, we have $Sv = Av = Tu = Bu = p$. Since $(S, A)$ are weakly compatible, $SAv = ASv$ implies that $Sp = Ap$. Next, we show that $p$ is the fixed point of $S$. If $Sp \neq p$, then

$$\begin{aligned} d(Sp,p) &= d(Sp,Tu) \\ &\leq a_1[d(Ap,Tu)+d(Bu,Sp)]+a_2[d(Bu,Tu)+d(Ap,Sp)] \\ &\quad +a_3d(Ap,Bu)+a_4[d(Ap,Sp)+d(Ap,Tu)] \\ &\quad +a_5[d(Ap,Bu)+d(Bu,Tu)] \\ &= a_1[d(Sp,p)+d(p,Sp)]+a_2[d(p,p)+d(Sp,Sp)]+a_3d(Sp,p) \\ &\quad +a_4[d(Sp,Sp)+d(Sp,p)]+a_5[d(Sp,p)+d(p,p)] \\ &\leq (2a_1+4a_2+a_3+3a_4+3a_5)d(Sp,p), \end{aligned}$$

which is a contradiction, and so $Sp = p$. This yields $Ap = Sp = p$. Again, $(T, B)$ are weakly compatible, and hence $TBu = BTu$ implies that $Tp = Bp$. Now, we show that $p$ is the fixed point of $T$. If $Tp \neq p$, then

$$\begin{aligned} d(p,Tp) &= d(Sp,Tp) \\ &\leq a_1[d(Ap,Tp)+d(Bp,Sp)]+a_2[d(Bp,Tp)+d(Ap,Sp)] \\ &\quad +a_3d(Ap,Sp)+a_4[d(Ap,Sp)+d(Ap,Tp)] \\ &\quad +a_5[d(Ap,Bp)+d(Bp,Tp)] \\ &= a_1[d(p,Tp)+d(Tp,p)]+a_2[d(Tp,Tp)+d(p,p)] \\ &\quad +a_3d(p,Tp)+a_4[d(p,Tp)+d(p,Tp)] \\ &\quad +a_5[d(p,Tp)+d(Tp,Tp)] \\ &\leq (2a_1+4a_2+a_3+2a_4+3a_5)d(p,Tp), \end{aligned}$$

which is a contradiction, and hence $p = Tp$. Therefore, we have $Ap = Bp = Sp = Tp = p$, which shows that $p$ is the common fixed point of the self mappings $A$, $B$, $S$, and $T$.

**Uniqueness**. Suppose that $v \neq u$ are two common fixed points of the mappings $A$, $B$, $S$, and $T$. Then, we have

$$\begin{aligned} d(v,u) &= d(Sv,Tu) \\ &\leq a_1[d(Av,Tu)+d(Bu,Sv)]+a_2[d(Bu,Tu)+d(Av,Sv)] \\ &\quad +a_3d(Av,Bu)+a_4[d(Av,Sv)+d(Av,Tu)] \\ &\quad +a_5[d(Av,Bu)+d(Bu,Tu)] \\ &= a_1[d(v,u)+d(u,v)]+a_2[d(u,u)+d(v,v)]+a_3d(v,u) \\ &\quad +a_4[d(v,v)+d(v,u)]+a_5[d(v,u)+d(u,u)] \\ &\leq (2a_1+4a_2+a_3+3a_4+3a_5)d(v,u), \end{aligned}$$

which is a contradiction, and therefore $v = u$. The proof is complete. □

Letting $A = B = I$ (an identity mapping), we can derive the following result from Theorem 2.1.

**Corollary 2.2** *Let $(\mathcal{X}, d)$ be a complete dislocated metric space. If $S, T : \mathcal{X} \to \mathcal{X}$ are continuous self mappings satisfying*

$$\begin{aligned} d(S\xi,T\eta) &\leq a_1[d(\xi,T\eta)+d(\eta,S\xi)]+a_2[d(\eta,T\eta)+d(\xi,S\xi)]+a_3d(\xi,\eta) \\ &\quad +a_4[d(\xi,S\xi)+d(\xi,T\eta)]+a_5[d(\xi,\eta)+d(\eta,T\eta)] \end{aligned}$$

*for all $\xi, \eta \in \mathcal{X}$, where $a_1, a_2, a_3, a_4, a_5 \geq 0$ and $0 \leq 4a_1+2a_2+a_3+3a_4+2a_5 < 1$, then $S$ and $T$ have a unique common fixed point.*

If $a_4 = a_5 = 0$ and $S = T$, then Corollary 2.2 reduces to the following result obtained by Isufati [6].

**Corollary 2.3** *Let $(\mathcal{X}, d)$ be a complete dislocated metric space. If $T : \mathcal{X} \to \mathcal{X}$ is a continuous self mapping satisfying*

$$d(T\xi,T\eta) \leq a_1[d(\xi,T\eta)+d(\eta,T\xi)]+a_2[d(\eta,T\eta)+d(\xi,T\xi)]+a_3d(\xi,\eta)$$

*for all $\xi, \eta \in \mathcal{X}$, where $a_1, a_2, a_3 \geq 0$ and $0 \leq 4a_1+2a_2+a_3 < 1$, then $T$ has a unique fixed point.*

Letting $a_4 = a_5 = 0$ in Theorem 2.1, we get the following result reported by Panthi and Jha [11].

**Corollary 2.4** *Let $(\mathcal{X}, d)$ be a complete dislocated metric space. If $A, B, S, T : \mathcal{X} \to \mathcal{X}$ are continuous self mappings satisfying the conditions:*
*(i) $T(\mathcal{X}) \subset A(\mathcal{X})$ and $S(\mathcal{X}) \subset B(\mathcal{X})$;*
*(ii) the pairs $(S, A)$ and $(T, B)$ are weakly compatible;*
*(iii) $d(S\xi, T\eta) \leq a_1[d(A\xi, T\eta) + d(B\eta, S\xi)] + a_2[d(B\eta, T\eta) + d(A\xi, S\xi)] + a_3 d(A\xi, B\eta)$ for all $\xi, \eta \in \mathcal{X}$, where $a_1, a_2, a_3 \geq 0$ and $0 \leq 4a_1 + 2a_2 + a_3 < 1$, then $A$, $B$, $S$, and $T$ have a unique common fixed point.*

**Remark 2.5** *Our results improve those obtained by Aage and Salunke [1, 2], Jha and Panthi [7], Jha et al. [8], Rao and Rangaswamy [12], and Shrivastava et al. [14].*

# 3 Further results without any continuity requirement

In this section, we prove some fixed point theorems without any continuity requirement in d-metric spaces.

**Theorem 3.1** *Let $(\mathcal{X}, d)$ be a complete dislocated metric space. Suppose that $A, B, S, T : \mathcal{X} \to \mathcal{X}$ are self mappings satisfying the conditions:*
*(i) $T(\mathcal{X}) \subset A(\mathcal{X})$ and $S(\mathcal{X}) \subset B(\mathcal{X})$;*
*(ii) the pairs $(S, A)$ and $(T, B)$ are weakly compatible;*
*(iii) $d(S\xi, T\eta) \leq a_1[d(A\xi, T\eta) + d(B\eta, S\xi)] + a_2[d(B\eta, T\eta) + d(A\xi, S\xi)] + a_3 d(A\xi, B\eta) + a_4[d(A\xi, S\xi) + d(A\xi, T\eta)] + a_5[d(A\xi, B\eta) + d(B\eta, T\eta)]$ for all $\xi, \eta \in \mathcal{X}$, where $a_1, a_2, a_3, a_4, a_5 \geq 0$ and $0 \leq 4a_1 + 2a_2 + a_3 + 3a_4 + 2a_5 < 1$, then $A$, $B$, $S$, and $T$ have a unique common fixed point.*

**Proof.** Let $\xi_0 \in \mathcal{X}$ be arbitrary. Choose $\xi_1 \in \mathcal{X}$ such that $B\xi_1 = S\xi_0$. Again, choose $\xi_2 \in \mathcal{X}$ such that $A\xi_2 = T\xi_1$. Continuing this process, choose $\xi_n \in \mathcal{X}$ such that $S\xi_{2n} = B\xi_{2n+1}$ and $T\xi_{2n+1} = A\xi_{2n+2}$ for $n = 0, 1, 2, \ldots$. To simplify, we consider the sequence $\{\eta_n\}$ which is defined by $\eta_{2n} := S\xi_{2n}$ and $\eta_{2n+1} := T\xi_{2n+1}$ for $n = 0, 1, 2, \ldots$. Next, we claim that $\{\eta_n\}$ is a Cauchy

sequence. Indeed, for $n \geq 1$, we have

$$\begin{aligned}
d(\eta_{2n}, \eta_{2n+1}) &= d(S\xi_{2n}, T\xi_{2n+1}) \\
&\leq a_1[d(A\xi_{2n}, T\xi_{2n+1}) + d(B\xi_{2n+1}, S\xi_{2n})] \\
&\quad +a_2[d(A\xi_{2n}, S\xi_{2n}) + d(B\xi_{2n+1}, T\xi_{2n+1})] \\
&\quad +a_3 d(A\xi_{2n}, B\xi_{2n+1}) + a_4[d(A\xi_{2n}, S\xi_{2n}) + d(A\xi_{2n}, T\xi_{2n+1})] \\
&\quad +a_5[d(A\xi_{2n}, B\xi_{2n+1}) + d(B\xi_{2n+1}, T\xi_{2n+1})] \\
&\leq a_1[d(\eta_{2n-1}, \eta_{2n+1}) + d(\eta_{2n}, \eta_{2n})] \\
&\quad +a_2[d(\eta_{2n-1}, \eta_{2n}) + d(\eta_{2n}, \eta_{2n+1})] \\
&\quad +a_3 d(\eta_{2n-1}, \eta_{2n}) + a_4[d(\eta_{2n-1}, \eta_{2n}) + d(\eta_{2n-1}, \eta_{2n+1})] \\
&\quad +a_5[d(\eta_{2n-1}, \eta_{2n}) + d(\eta_{2n}, \eta_{2n+1})] \\
&\leq a_1[d(\eta_{2n-1}, \eta_{2n}) + d(\eta_{2n}, \eta_{2n+1}) \\
&\quad +d(\eta_{2n}, \eta_{2n+1}) + d(\eta_{2n+1}, \eta_{2n})] \\
&\quad +a_2[d(\eta_{2n-1}, \eta_{2n}) + d(\eta_{2n}, \eta_{2n+1})] + a_3 d(\eta_{2n-1}, \eta_{2n}) \\
&\quad +a_4[d(\eta_{2n-1}, \eta_{2n}) + d(\eta_{2n-1}, \eta_{2n}) + d(\eta_{2n}, \eta_{2n+1})] \\
&\quad +a_5[d(\eta_{2n-1}, \eta_{2n}) + d(\eta_{2n}, \eta_{2n+1})] \\
&= (a_1 + a_2 + a_3 + 2a_4 + a_5) d(\eta_{2n-1}, \eta_{2n}) \\
&\quad +(3a_1 + a_2 + a_4 + a_5) d(\eta_{2n}, \eta_{2n+1}).
\end{aligned}$$

Hence, we conclude that

$$d(\eta_{2n}, \eta_{2n+1}) \leq h d(\eta_{2n-1}, \eta_{2n}),$$

where

$$h = \frac{a_1 + a_2 + a_3 + 2a_4 + a_5}{1 - (3a_1 + a_2 + a_4 + a_5)} \in [0, 1).$$

This implies that $\{\eta_n\}$ is a Cauchy sequence in $\mathcal{X}$. Then, by Remark 1.10, $\{S\xi_{2n}\}$, $\{B\xi_{2n+1}\}$, $\{T\xi_{2n+1}\}$, and $\{A\xi_{2n+2}\}$ are also Cauchy sequences. Assume that $S\xi$ is a complete subspace of $\mathcal{X}$, the sequence $\{S\xi_{2n}\}$ converges to some $Sa$ such that $a \in \mathcal{X}$. So, $\{\eta_n\}$, $\{B\xi_{2n+1}\}$, $\{T\xi_{2n+1}\}$, and $\{A\xi_{2n+2}\}$ also converge to $Sa$. Since $S\mathcal{X} \subset B\mathcal{X}$, there exists a $v \in \mathcal{X}$ such that $Sa = Bv$. We show that $Bv = Tv$. In fact, we have

$$\begin{aligned}
d(S\xi_{2n}, Tv) &\leq a_1[d(A\xi_{2n}, Tv) + d(Bv, S\xi_{2n})] \\
&\quad +a_2[d(A\xi_{2n}, S\xi_{2n}) + d(Bv, Tv)] \\
&\quad +a_3 d(A\xi_{2n}, Bv) + a_4[d(A\xi_{2n}, S\xi_{2n}) + d(A\xi_{2n}, Tv)] \\
&\quad +a_5[d(A\xi_{2n}, Bv) + d(Bv, Tv)].
\end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned} d(Bv,Tv) &\leq a_1[d(Bv,Tv)+d(Bv,Bv)]+a_2[d(Bv,Bv)+d(Bv,Tv)] \\ &\quad +a_3d(Bv,Bv)+a_4[d(Bv,Bv)+d(Bv,Tv)] \\ &\quad +a_5[d(Bv,Bv)+d(Bv,Tv)] \\ &\leq (a_1+a_2+a_3+a_4+a_5)d(Bv,Bv) \\ &\quad +(a_1+a_2+a_4+a_5)d(Bv,Tv) \\ &\leq (3a_1+3a_2+2a_3+3a_4+3a_5)d(Bv,Tv). \end{aligned}$$

Therefore, $d(Bv,Tv)=0$, which implies that $Tv=Bv$. Since $T\mathcal{X} \subset A\mathcal{X}$, there exists a $u \in \mathcal{X}$ such that $Tv=Au$. We show that $Su=Au$. Indeed, we have

$$\begin{aligned} d(Su,Au) &= d(Su,Tv) \\ &\leq a_1[d(Au,Tv)+d(Bv,Su)]+a_2[d(Au,Su)+d(Bv,Tv)] \\ &\quad +a_3d(Au,Bv)+a_4[d(Au,Su)+d(Au,Tv)] \\ &\quad +a_5[d(Au,Bv)+d(Bv,Tv)] \\ &\leq a_1[d(Au,Au)+d(Au,Su)]+a_2[d(Au,Su)+d(Au,Au)] \\ &\quad +a_3d(Au,Au)+a_4[d(Au,Su)+d(Au,Au)] \\ &\quad +a_5[d(Au,Au)+d(Au,Au)] \\ &\leq a_1[d(Au,Su)+d(Su,Au)+d(Au,Su)] \\ &\quad +a_2[d(Au,Su)+d(Au,Su)+d(Su,Au)] \\ &\quad +a_3[d(Au,Su)+d(Su,Au)] \\ &\quad +a_4[d(Au,Su)+d(Au,Su)+d(Su,Au)] \\ &\quad +a_5[d(Au,Su)+d(Su,Au)+d(Au,Su)+d(Su,Au)] \\ &= (3a_1+3a_2+2a_3+3a_4+4a_5)d(Au,Su). \end{aligned}$$

Hence, $d(Su,Au)=0$, which yields $Au=Su$, and so $Bv=Tv=Au=Su$. By virtue of the fact that $(S,A)$ are weakly compatible, we deduce that $ASu=SAu$, which yields $AAu=ASu=SAu=SSu$. The weak compatibility of $B$ and $T$ implies that $BTv=TBv$, from which it follows that $BBv=BTv=TBv=TTv$. Let us show that $Bv$ is a fixed point of $T$.

In fact, we have

$$\begin{aligned}
d(Bv,TBv) &= d(Su,TBv)\\
&\leq a_1[d(Au,TBv)+d(BBv,Su)]\\
&\quad +a_2[d(Au,Su)+d(BBv,TBv)]\\
&\quad +a_3d(Au,BBv)+a_4[d(Au,Su)+d(Au,TBv)]\\
&\quad +a_5[d(Au,BBv)+d(BBv,TBv)]\\
&\leq a_1[d(Bv,TBv)+d(TBv,Bv)]\\
&\quad +a_2[d(Bv,Bv)+d(TBv,TBv)]\\
&\quad +a_3d(Bv,TBv)+a_4[d(Bv,Bv)+d(Bv,TBv)]\\
&\quad +a_5[d(Bv,TBv)+d(TBv,TBv)]\\
&\leq 2a_1d(Bv,TBv)+a_2[d(Bv,TBv)+d(TBv,Bv)\\
&\quad +d(TBv,Bv)+d(Bv,TBv)]+a_3d(Bv,TBv)\\
&\quad +a_4[d(Bv,TBv)+d(TBv,Bv)+d(Bv,TBv)]\\
&\quad +a_5[d(Bv,TBv)+d(TBv,Bv)+d(Bv,TBv)]\\
&= (2a_1+4a_2+a_3+3a_4+3a_5)d(Bv,TBv),
\end{aligned}$$

which yields $d(Bv,TBv)=0$, and so $TBv=Bv$. Therefore, $Bv$ is a fixed point of $T$. It follows that $BBv=TBv=Bv$, which implies that $Bv$ is also a fixed point of $B$. On the other hand, we get

$$\begin{aligned}
d(SBv,Bv) &= d(SBv,TBv)\\
&\leq a_1[d(ABv,TBv)+d(BBv,SBv)]+a_2[d(ABv,SBv)\\
&\quad +d(BBv,TBv)]+a_3d(ABv,BBv)+a_4[d(ABv,SBv)\\
&\quad +d(ABv,TBv)]+a_5[d(ABv,BBv)+d(BBv,TBv)]\\
&\leq a_1[d(Bv,Bv)+d(Bv,SBv)]+a_2[d(SBv,Bv)\\
&\quad +d(Bv,SBv)]+a_3d(Bv,Bv)+a_4[d(Bv,Bv)\\
&\quad +d(Bv,Bv)]+a_5[d(Bv,Bv)+d(Bv,Bv)]\\
&\leq a_1[d(Bv,SBv)+d(SBv,Bv)+d(Bv,SBv)]\\
&\quad +a_2[d(SBv,Bv)+d(Bv,SBv)]+a_3[d(Bv,SBv)\\
&\quad +d(SBv,Bv)]+a_4[d(Bv,SBv)+d(SBv,Bv)\\
&\quad +d(Bv,SBv)+d(SBv,Bv)]+a_5[d(Bv,SBv)\\
&\quad +d(SBv,Bv)+d(Bv,SBv)+d(SBv,Bv)]\\
&= (3a_1+2a_2+2a_3+4a_4+4a_5)d(SBv,Bv),
\end{aligned}$$

which implies that $d(Bv, SBv) = 0$, and hence $SBv = Bv$. Therefore, $Bv$ is a fixed point of $S$. It follows that $ABv = SBv = Bv$, which shows that $Bv$ is also a fixed point of $A$. Then $Bv$ is a common fixed point of $A$, $B$, $S$, and $T$.

**Uniqueness**. Let $w, u \in \mathcal{X}$ be two fixed points such that $Aw = Bw = Sw = Tw$ and $Au = Bu = Su = Tu$. If $d(w, u) \neq 0$, then

$$\begin{aligned} d(w,u) &= d(Sw,Tu) \\ &\leq a_1[d(Aw,Tu)+d(Bu,Sw)]+a_2[d(Bu,Tu)+d(Aw,Sw)] \\ &\quad +a_3 d(Aw,Bu)+a_4[d(Aw,Sw)+d(Aw,Tu)] \\ &\quad +a_5[d(Aw,Bu)+d(Bu,Tu)] \\ &= a_1[d(w,u)+d(u,w)]+a_2[d(u,u)+d(w,w)]+a_3 d(w,u) \\ &\quad +a_4[d(w,w)+d(w,u)]+a_5[d(w,u)+d(u,u)] \\ &\leq (2a_1+4a_2+a_3+3a_4+3a_5)d(w,u), \end{aligned}$$

which is a contradiction. Hence, $d(v, u) = 0$, which implies that $v = u$. The proof is complete. □

**Remark 3.2** *One can derive from Theorem 3.1 a number of fixed point theorems for self mappings A, B, S, and T. For example, we have the following result by letting* $a_4 = a_5 = 0$.

**Corollary 3.3** *Let* $(\mathcal{X}, d)$ *be a complete dislocated metric space. If* $A, B, S, T : \mathcal{X} \to \mathcal{X}$ *are self mappings satisfying the conditions:*
*(i)* $T(\mathcal{X}) \subset A(\mathcal{X})$ *and* $S(\mathcal{X}) \subset B(\mathcal{X})$*;*
*(ii) the pairs* $(S, A)$ *and* $(T, B)$ *are weakly compatible;*
*(iii)* $d(S\xi, T\eta) \leq a_1[d(A\xi, T\eta) + d(B\eta, S\xi)] + a_2[d(B\eta, T\eta) + d(A\xi, S\xi)] + a_3 d(A\xi, B\eta)$ *for all* $\xi, \eta \in \mathcal{X}$*, where* $a_1, a_2, a_3 \geq 0$ *and* $0 \leq 4a_1 + 2a_2 + a_3 < 1$*, then A, B, S, and T have a unique common fixed point.*

**Remark 3.4** *Our results generalize, extend, and improve those obtained by Bennani et al. [3].*

# 4 Examples

The following examples illustrate theoretical results obtained in the previous sections.

**Example 4.1** *Assume that* $\mathcal{X} = [0,1]$*,* $d$ *is a usual metric, and define the mappings* $A$*,* $B$*,* $S$*, and* $T$ *by*

$$A\xi = \xi, \quad B\xi = \xi, \quad S\xi = 0, \quad \text{and} \quad T\xi = \frac{1}{12}\xi.$$

*Let*

$$a_1 = \frac{1}{20}, \quad a_2 = \frac{1}{24}, \quad a_3 = \frac{1}{28}, \quad a_4 = \frac{1}{30}, \quad \text{and} \quad a_5 = \frac{1}{34}.$$

*Then* $A$*,* $B$*,* $S$*, and* $T$ *satisfy all assumptions of Theorem 2.1. As a matter of fact,* $0 \in \mathcal{X}$ *is the unique common fixed point of the mappings* $A$*,* $B$*,* $S$*, and* $T$*.*

**Example 4.2** *Let* $\mathcal{X} = [0,1]$*,* $d(\xi,\eta) = |\xi| + |\eta|$*, and define the mappings* $A$*,* $B$*,* $S$*, and* $T$ *by*

$$A\xi = \xi, \quad B\xi = \xi, \quad S\xi = 0, \quad \text{and} \quad T\xi = \frac{\xi}{6}.$$

*Set*

$$a_1 = \frac{1}{25}, \quad a_2 = \frac{1}{28}, \quad a_3 = \frac{1}{32}, \quad a_4 = \frac{1}{36}, \quad \text{and} \quad a_5 = \frac{1}{40}.$$

*Then* $A$*,* $B$*,* $S$*, and* $T$ *satisfy all assumptions of Theorem 3.1. In fact,* $0 \in \mathcal{X}$ *is the unique common fixed point of the mappings* $A$*,* $B$*,* $S$*, and* $T$*.*

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# References

[1] C. T. Aage, J. N. Salunke, *Some results of fixed point theorem in dislocated quasi-metric spaces*, Bull. Marathwada Math. Soc., 9 (2008) 1–5.

[2] C. T. Aage, J. N. Salunke, *The results on fixed points in dislocated and dislocated quasi-metric space*, Appl. Math. Sci., 2 (2008) 2941–2948.

[3] S. Bennani, H. Bourijal, S. Mhanna, D. E. Moutawakil, *Some common fixed point theorems in dislocated metric spaces*, J. Nonlinear Sci. Appl., 8 (2015) 86–92.

[4] P. Hitzler, A. K. Seda, *Dislocated topologies*, J. Electr. Eng., 51 (2000) 3–7.

[5] P. Hitzler, *Generalized metrics and topology in logic programming semantics*, Ph.D. Thesis, National University of Ireland (University College, Cork), 2001.

[6] A. Isufati, *Fixed point theorems in dislocated quasi-metric space*, Appl. Math. Sci., 4 (2010) 217–223.

[7] K. Jha, D. Panthi, *A common fixed point theorem in dislocated metric space*, Appl. Math. Sci., 6 (2012) 4497–4503.

[8] K. Jha, K. P. R. Rao, D. Panthi, *A common fixed point theorem for four mappings in dislocated quasi-metric space*, Int. J. Math. Sci. Eng. Appl., 6 (2012) 417–424.

[9] G. Jungck, B. E. Rhoades, *Fixed points for set valued functions without continuity*, Indian J. Pure Appl. Math., 29 (1998) 227–238.

[10] S. Shukla, S. Balasubramanian, M. Pavlović, *A generalized Banach fixed point theorem*, Bull. Malays. Math. Sci. Soc., 39 (2016) 1529–1539.

[11] D. Panthi, K. Jha, *A common fixed point of weakly compatible mappings in dislocated metric space*, Kathmandu Univ. J. Sci. Eng. Tech., 8 (2012) 25–30.

[12] K. P. R. Rao, P. Rangaswamy, *Common fixed point theorem for four mappings in dislocated quasi-metric space*, Nepali Math. Sci. Rep., 30 (2010) 70–75.

[13] I. R. Sarma, P. S. Kumari, *On dislocated metric spaces*, Int. J. Math. Arch., 3 (2012) 72–77.

[14] R. Shrivastava, Z. K. Ansari, M. Sharma, *Some results on fixed points in dislocated and dislocated quasi-metric spaces*, J. Adv. Stud. Topol., 3 (2012) 25–31.

[15] R. K. Vats, *Weakly compatible maps in metric spaces*, J. Indian Math. Soc., 69 (2002) 139–143.

[16] F. M. Zeyada, G. H. Hassan, M. A. Ahmed, *A generalization of a fixed point theorem due to Hitzler and Seda in dislocated quasi-metric spaces*, Arab. J. Sci. Eng., 31 (2006) 111–114.

# QUADRATIC $\rho$-FUNCTIONAL INEQUALITIES IN NON-ARCHIMEDEAN BANACH SPACES

SUNGSIK YUN

Abstract. In this paper, we solve the quadratic $\rho$-functional inequalities

$$\begin{aligned} &\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \\ &\quad\le \left\|\rho\left(4f\left(\frac{x+y}{2}\right)+f(x-y)-2f(x)-2f(y)\right)\right\|, \end{aligned} \tag{0.1}$$

where $\rho$ is a fixed non-Archimedean number with $|\rho|<|2|$, and

$$\begin{aligned} &\left\|4f\left(\frac{x+y}{2}\right)+f(x-y)-2f(x)-2f(y)\right\| \\ &\quad\le \|\rho(f(x+y)+f(x-y)-2f(x)-2f(y))\|, \end{aligned} \tag{0.2}$$

where $\rho$ is a fixed non-Archimedean number with $|\rho|<1$.

Furthermore, we prove the Hyers-Ulam stability of the quadratic $\rho$-functional inequalities (0.1) and (0.2) in non-Archimedean Banach spaces.

## 1. Introduction and preliminaries

A *valuation* is a function $|\cdot|$ from a field $K$ into $[0,\infty)$ such that 0 is the unique element having the 0 valuation, $|rs|=|r|\cdot|s|$ and the triangle inequality holds, i.e.,

$$|r+s|\le |r|+|s|, \qquad \forall r,s\in K.$$

A field $K$ is called a *valued field* if $K$ carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations.

Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the triangle inequality is replaced by

$$|r+s|\le \max\{|r|,|s|\}, \qquad \forall r,s\in K,$$

then the function $|\cdot|$ is called a *non-Archimedean valuation*, and the field is called a *non-Archimedean field*. Clearly $|1|=|-1|=1$ and $|n|\le 1$ for all $n\in\mathbb{N}$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except for 0 into 1 and $|0|=0$.

Throughout this paper, we assume that the base field is a non-Archimedean field, hence call it simply a field.

**Definition 1.1.** ([8]) Let $X$ be a vector space over a field $K$ with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\|: X\to[0,\infty)$ is said to be a *non-Archimedean norm* if it satisfies the following conditions:

(i) $\|x\|=0$ if and only if $x=0$;

(ii) $\|rx\|=|r|\|x\| \qquad (r\in K, x\in X)$;

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(iii) the strong triangle inequality

$$\|x+y\| \leq \max\{\|x\|, \|y\|\}, \qquad \forall x, y \in X$$

holds. Then $(X, \|\cdot\|)$ is called a *non-Archimedean normed space*.

**Definition 1.2.** (i) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space $X$. Then the sequence $\{x_n\}$ is called *Cauchy* if for a given $\varepsilon > 0$ there is a positive integer $N$ such that

$$\|x_n - x_m\| \leq \varepsilon$$

for all $n, m \geq N$.

(ii) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space $X$. Then the sequence $\{x_n\}$ is called *convergent* if for a given $\varepsilon > 0$ there are a positive integer $N$ and an $x \in X$ such that

$$\|x_n - x\| \leq \varepsilon$$

for all $n \geq N$. Then we call $x \in X$ a limit of the sequence $\{x_n\}$, and denote by $\lim_{n\to\infty} x_n = x$.

(iii) If every Cauchy sequence in $X$ converges, then the non-Archimedean normed space $X$ is called a *non-Archimedean Banach space*.

The stability problem of functional equations originated from a question of Ulam [18] concerning the stability of group homomorphisms. The functional equation $f(x+y) = f(x)+f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [7] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [11] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [6] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. The functional equation $f\left(\frac{x+y}{2}\right) = \frac{1}{2}f(x) + \frac{1}{2}f(y)$ is called the *Jensen equation*.

The functional equation $f(x+y)+f(x-y) = 2f(x)+2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional equation was proved by Skof [17] for mappings $f : E_1 \to E_2$, where $E_1$ is a normed space and $E_2$ is a Banach space. Cholewa [5] noticed that the theorem of Skof is still true if the relevant domain $E_1$ is replaced by an Abelian group. The functional equation $2f\left(\frac{x+y}{2}\right) + 2\left(\frac{x-y}{2}\right) = f(x) + f(y)$ is called a *Jensen type quadratic equation*. The stability problems of various functional equations have been extensively investigated by a number of authors (see [1, 3, 4, 9, 10, 12, 13, 14, 15, 16, 19, 20]).

In Section 2, we solve the quadratic $\rho$-functional inequality (0.1) and prove the Hyers-Ulam stability of the quadratic $\rho$-functional inequality (0.1) in non-Archimedean Banach spaces.

In Section 3, we solve the quadratic $\rho$-functional inequality (0.2) and prove the Hyers-Ulam stability of the quadratic $\rho$-functional inequality (0.2) in non-Archimedean Banach spaces.

Throughout this paper, assume that $X$ is a non-Archimedean normed space and that $Y$ is a non-Archimedean Banach space. Let $|2| \neq 1$.

## 2. Quadratic $\rho$-functional inequality (0.1) in non-Archimedean normed spaces

Throughout this section, assume that $\rho$ is a fixed non-Archimedean number with $|\rho| < |2|$.

In this section, we solve the quadratic $\rho$-functional inequality (0.1) in non-Archimedean normed spaces.

**Lemma 2.1.** *If a mapping $f : G \to Y$ satisfies*

$$\begin{aligned}&\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \\ &\quad\le \left\|\rho\left(4f\left(\frac{x+y}{2}\right)+f(x-y)-2f(x)-2f(y)\right)\right\|\end{aligned} \tag{2.1}$$

*for all $x, y \in G$, then $f : G \to Y$ is quadratic.*

*Proof.* Assume that $f : G \to Y$ satisfies (2.1).

Letting $x = y = 0$ in (2.1), we get $\|2f(0)\| \le |\rho|\|f(0)\|$. So $f(0) = 0$.

Letting $y = x$ in (2.1), we get $\|f(2x) - 4f(x)\| \le 0$ and so $f(2x) = 4f(x)$ for all $x \in G$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{4}f(x) \tag{2.2}$$

for all $x \in G$.

It follows from (2.1) and (2.2) that

$$\begin{aligned}&\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \\ &\quad\le \left\|\rho\left(4f\left(\frac{x+y}{2}\right)+f(x-y)-2f(x)-2f(y)\right)\right\| \\ &\quad= |\rho|\|f(x+y)+f(x-y)-2f(x)-2f(y)\|\end{aligned}$$

and so

$$f(x+y)+f(x-y) = 2f(x)+2f(y)$$

for all $x, y \in G$. □

Now, we prove the Hyers-Ulam stability of the quadratic $\rho$-functional inequality (2.1) in non-Archimedean Banach spaces.

**Theorem 2.2.** *Let $r < 2$ and $\theta$ be nonnegative real numbers and let $f : X \to Y$ be a mapping satisfying*

$$\begin{aligned}\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \;&\le\; \left\|\rho\left(4f\left(\frac{x+y}{2}\right)+f(x-y)-2f(x)-2f(y)\right)\right\| \\ &+\; \theta(\|x\|^r+\|y\|^r)\end{aligned} \tag{2.3}$$

*for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x)-Q(x)\| \le \frac{2}{|2|^r}\theta\|x\|^r \tag{2.4}$$

*for all $x \in X$.*

*Proof.* Letting $x = y = 0$ in (2.3), we get $\|f(0)\| \le |\rho|\|2f(0)\|$. So $f(0) = 0$.

Letting $y = x$ in (2.3), we get

$$\|f(2x)-4f(x)\| \le 2\theta\|x\|^r \tag{2.5}$$

for all $x \in X$. So $\left\|f(x)-4f\left(\frac{x}{2}\right)\right\| \leq \frac{2}{|2|^r}\theta\|x\|^r$ for all $x \in X$. Hence

$$\begin{aligned}
&\left\|4^l f\left(\frac{x}{2^l}\right)-4^m f\left(\frac{x}{2^m}\right)\right\| \qquad (2.6)\\
&\leq \max\left\{\left\|4^l f\left(\frac{x}{2^l}\right)-4^{l+1} f\left(\frac{x}{2^{l+1}}\right)\right\|,\cdots,\left\|4^{m-1} f\left(\frac{x}{2^{m-1}}\right)-4^m f\left(\frac{x}{2^m}\right)\right\|\right\}\\
&= \max\left\{|4|^l\left\|f\left(\frac{x}{2^l}\right)-4f\left(\frac{x}{2^{l+1}}\right)\right\|,\cdots,|4|^{m-1}\left\|f\left(\frac{x}{2^{m-1}}\right)-4f\left(\frac{x}{2^m}\right)\right\|\right\}\\
&\leq \max\left\{\frac{|4|^l}{|2|^{rl}},\cdots,\frac{|4|^{m-1}}{|2|^{r(m-1)}}\right\}\frac{2}{|2|^r}\theta\|x\|^r = \frac{\theta}{|2|^{(r-2)l}}\frac{2}{|2|^r}\|x\|^r
\end{aligned}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (2.6) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.6), we get (2.4).

It follows from (2.3) that

$$\begin{aligned}
&\|Q(x+y)+Q(x-y)-2Q(x)-2Q(y)\|\\
&= \lim_{n\to\infty}|4|^n\left\|f\left(\frac{x+y}{2^n}\right)+f\left(\frac{x-y}{2^n}\right)-2f\left(\frac{x}{2^n}\right)-2f\left(\frac{y}{2^n}\right)\right\|\\
&\leq \lim_{n\to\infty}|4|^n|\rho|\left\|4f\left(\frac{x+y}{2^{n+1}}\right)+f\left(\frac{x-y}{2^n}\right)-2f\left(\frac{x}{2^n}\right)-2f\left(\frac{y}{2^n}\right)\right\|+\lim_{n\to\infty}\frac{|4|^n\theta}{|2|^{nr}}(\|x\|^r+\|y\|^r)\\
&= |\rho|\left\|4Q\left(\frac{x+y}{2}\right)+Q(x-y)-2Q(x)-2Q(y)\right\|
\end{aligned}$$

for all $x, y \in X$. So

$$\|Q(x+y)+Q(x-y)-2Q(x)-2Q(y)\| \leq \left\|\rho\left(4Q\left(\frac{x+y}{2}\right)+Q(x-y)-2Q(x)-2Q(y)\right)\right\|$$

for all $x, y \in X$. By Lemma 2.1, the mapping $h : X \to Y$ is quadratic.

Now, let $T : X \to Y$ be another quadratic mapping satisfying (2.4). Then we have

$$\begin{aligned}
\|Q(x)-T(x)\| &= \left\|4^q Q\left(\frac{x}{2^q}\right)-4^q T\left(\frac{x}{2^q}\right)\right\|\\
&\leq \max\left\{\left\|4^q Q\left(\frac{x}{2^q}\right)-4^q f\left(\frac{x}{2^q}\right)\right\|,\left\|4^q T\left(\frac{x}{2^q}\right)-4^q f\left(\frac{x}{2^q}\right)\right\|\right\} \leq \frac{2}{|2|^{(r-2)q+r}}\theta\|x\|^r,
\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $Q(x) = T(x)$ for all $x \in X$. This proves the uniqueness of $Q$. Thus the mapping $Q : X \to Y$ is a unique quadratic mapping satisfying (2.4). □

**Theorem 2.3.** *Let $r > 2$ and $\theta$ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (2.3)*. Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x)-Q(x)\| \leq \frac{2\theta}{|4|}\|x\|^r$$

*for all $x \in X$.*

*Proof.* It follows from (2.5) that

$$\left\|f(x)-\frac{1}{4}f(2x)\right\|\leq\frac{2\theta}{|4|}\|x\|^r$$

for all $x\in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

## 3. Quadratic $\rho$-functional inequality (0.2)

Throughout this section, assume that $\rho$ is a fixed non-Archimedean number with $|\rho|<1$.

In this section, we solve the quadratic $\rho$-functional inequality (0.2) in non-Archimedean normed spaces.

**Lemma 3.1.** *If a mapping $f:G\to Y$ satisfies*

$$\begin{aligned}\left\|4f\left(\frac{x+y}{2}\right)+f\left(x-y\right)-2f(x)-2f(y)\right\| \\ \leq\|\rho(f(x+y)+f(x-y)-2f(x)-2f(y))\|\end{aligned}\tag{3.1}$$

*for all $x,y\in G$, then $f:G\to Y$ is quadratic.*

*Proof.* Assume that $f:G\to Y$ satisfies (3.1).

Letting $x=y=0$ in (3.1), we get $\|f(0)\|\leq|\rho|\|2f(0)\|$. So $f(0)=0$.

Letting $y=0$ in (3.1), we get $\left\|4f\left(\frac{x}{2}\right)-f(x)\right\|\leq 0$ and so

$$4f\left(\frac{x}{2}\right)=f(x)\tag{3.2}$$

for all $x\in G$.

It follows from (3.1) and (3.2) that

$$\begin{aligned}&\|f(x+y)+f(x-y)-2f(x)-2f(y)\|\\&=\left\|4f\left(\frac{x+y}{2}\right)+f\left(x-y\right)-2f(x)-2f(y)\right\|\\&\leq|\rho|\|f(x+y)+f(x-y)-2f(x)-2f(y)\|\end{aligned}$$

and so

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

for all $x,y\in G$. □

Now, we prove the Hyers-Ulam stability of the quadratic $\rho$-functional inequality (3.1) in non-Archimedean Banach spaces.

**Theorem 3.2.** *Let $r<2$ and $\theta$ be nonnegative real numbers, and let $f:X\to Y$ be a mapping satisfying*

$$\begin{aligned}\left\|4f\left(\frac{x+y}{2}\right)+f\left(x-y\right)-2f(x)-2f(y)\right\| &\leq \|\rho(f(x+y)+f(x-y)-2f(x)-2f(y))\| \\ &+ \theta(\|x\|^r+\|y\|^r)\end{aligned}\tag{3.3}$$

*for all $x,y\in X$. Then there exists a unique quadratic mapping $Q:X\to Y$ such that*

$$\|f(x)-Q(x)\|\leq\theta\|x\|^r\tag{3.4}$$

*for all $x\in X$.*

*Proof.* Letting $x = y = 0$ in (3.3), we get $\|2f(0)\| \leq |\rho|\|f(0)\|$. So $f(0) = 0$.

Letting $y = 0$ in (3.3), we get

$$\left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \leq \theta\|x\|^r \tag{3.5}$$

for all $x \in X$. So

$$\begin{aligned}
&\left\|4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\| \qquad (3.6)\\
&\quad \leq \max\left\{\left\|4^l f\left(\frac{x}{2^l}\right) - 4^{l+1} f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, \left\|4^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\|\right\}\\
&\quad = \max\left\{|4|^l \left\|f\left(\frac{x}{2^l}\right) - 4f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, |4|^{m-1}\left\|f\left(\frac{x}{2^{m-1}}\right) - 4f\left(\frac{x}{2^m}\right)\right\|\right\}\\
&\quad \leq \max\left\{\frac{|4|^l}{|2|^{rl}}, \cdots, \frac{|4|^{m-1}}{|2|^{r(m-1)}}\right\}\theta\|x\|^r = \frac{\theta}{|2|^{(r-2)l}}\|x\|^r
\end{aligned}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (3.6) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.6), we get (3.4).

The rest of the proof is similar to the proof of Theorem 2.2. □

**Theorem 3.3.** *Let $r > 2$ and $\theta$ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (3.3)*. Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{|2|^r\theta}{|4|}\|x\|^r \tag{3.7}$$

*for all $x \in X$.*

*Proof.* It follows from (3.5) that

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \leq \frac{|2|^r\theta}{|4|}\|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned}
&\left\|\frac{1}{4^l}f(2^l x) - \frac{1}{4^m}f(2^m x)\right\| \qquad (3.8)\\
&\quad \leq \max\left\{\left\|\frac{1}{4^l}f\left(2^l x\right) - \frac{1}{4^{l+1}}f\left(2^{l+1}x\right)\right\|, \cdots, \left\|\frac{1}{4^{m-1}}f\left(2^{m-1}x\right) - \frac{1}{4^m}f\left(2^m x\right)\right\|\right\}\\
&\quad = \max\left\{\frac{1}{|4|^l}\left\|f\left(2^l x\right) - \frac{1}{4}f\left(2^{l+1}x\right)\right\|, \cdots, \frac{1}{|4|^{m-1}}\left\|f\left(2^{m-1}x\right) - \frac{1}{4}f\left(2^m x\right)\right\|\right\}\\
&\quad \leq \max\left\{\frac{|2|^{rl}}{|4|^{l+1}}, \cdots, \frac{|2|^{r(m-1)}}{|4|^{(m-1)+1}}\right\}|2|^r\theta\|x\|^r = \frac{|2|^r\theta}{|2|^{(2-r)l+2}}\|x\|^r
\end{aligned}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (3.8) that the sequence $\{\frac{1}{4^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence

$\{\frac{1}{4^n}f(2^nx)\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.8), we get (3.7).
The rest of the proof is similar to the proofs of Theorems 2.2 and 3.2. □

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## References

[1] M. Adam, *On the stability of some quadratic functional equation*, J. Nonlinear Sci. Appl. **4** (2011), 50–59.
[2] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.
[3] L. Cădariu, L. Găvruta and P. Găvruta, *On the stability of an affine functional equation,* J. Nonlinear Sci. Appl. **6** (2013), 60–67.
[4] A. Chahbi and N. Bounader, *On the generalized stability of d'Alembert functional equation*, J. Nonlinear Sci. Appl. **6** (2013), 198–204.
[5] P. W. Cholewa, *Remarks on the stability of functional equations,* Aequationes Math. **27** (1984), 76–86.
[6] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–43.
[7] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. U.S.A. **27** (1941), 222–224.
[8] M.S. Moslehian and Gh. Sadeghi, *A Mazur-Ulam theorem in non-Archimedean normed spaces*, Nonlinear Anal.–TMA **69** (2008), 3405–3408.
[9] C. Park, K. Ghasemi, S. G. Ghaleh and S. Jang, *Approximate n-Jordan $*$-homomorphisms in $C^*$-algebras,* J. Comput. Anal. Appl. **15** (2013), 365-368.
[10] C. Park, A. Najati and S. Jang, *Fixed points and fuzzy stability of an additive-quadratic functional equation,* J. Comput. Anal. Appl. **15** (2013), 452–462.
[11] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.
[12] S. Schin, D. Ki, J. Chang and M. Kim, *Random stability of quadratic functional equations: a fixed point approach*, J. Nonlinear Sci. Appl. **4** (2011), 37–49.
[13] S. Shagholi, M. Bavand Savadkouhi and M. Eshaghi Gordji, *Nearly ternary cubic homomorphism in ternary Fréchet algebras,* J. Comput. Anal. Appl. **13** (2011), 1106–1114.
[14] S. Shagholi, M. Eshaghi Gordji and M. Bavand Savadkouhi, *Stability of ternary quadratic derivation on ternary Banach algebras,* J. Comput. Anal. Appl. **13** (2011), 1097–1105.
[15] D. Shin, C. Park and Sh. Farhadabadi, *On the superstability of ternary Jordan $C^*$-homomorphisms,* J. Comput. Anal. Appl. **16** (2014), 964–973.
[16] D. Shin, C. Park and Sh. Farhadabadi, *Stability and superstability of $J^*$-homomorphisms and $J^*$-derivations for a generalized Cauchy-Jensen equation,* J. Comput. Anal. Appl. **17** (2014), 125–134.
[17] F. Skof, *Propriet locali e approssimazione di operatori*, Rend. Sem. Mat. Fis. Milano **53** (1983), 113–129.
[18] S. M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ. New York, 1960.
[19] C. Zaharia, *On the probabilistic stability of the monomial functional equation,* J. Nonlinear Sci. Appl. **6** (2013), 51–59.
[20] S. Zolfaghari, *Approximation of mixed type functional equations in p-Banach spaces*, J. Nonlinear Sci. Appl. **3** (2010), 110–122.

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# A Fast Inversion Free Iterative Algorithm for Solving $X + A^*X^{-1}A = I$

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**Abstract**

We introduce a new inversion free variant of the basic fixed point iteration method for obtaining a maximal positive definite solution of the nonlinear matrix equation $X + A^*X^{-1}A = I$ with $A$ normal. It has fewer operations and matrix-matrix multiplications than the existing algorithms. We derive convergence conditions for the iteration and some numerical results to illustrate the behavior of the new algorithm.

**AMS classification**: 15A24; 65F10; 65F35

**Key words:** Nonlinear matrix equation; Hermitian positive definite solution; Fixed point iteration method; Convergence rate.

## 1 Introduction

Consider the nonlinear matrix equation

$$X + A^*X^{-1}A = I, \tag{1.1}$$

where $A$ is an $n \times n$ complex normal matrix and $I$ the identity matrix. Here $A^*$ stands for the conjugate transpose of $A$.

Nonlinear matrix equation (1.1) has many applications. It often arises in control theory, dynamic programming, ladder networks, stochastic filtering, statistics, and etc.; see [1, 3–11, 14–23] and the references therein. It is well known that $X$ is a solution of (1.1) if and only if it solves

$$X = I - A^*(I - A^*X^{-1}A)^{-1}A.$$

Assuming that $A$ is invertible, we can write the above equation as

$$X = F^*(R + X^{-1})^{-1}F + I,$$

where $F = A^{-*}A$ and $R = -A^{-*}A^{-1}$. This is a special case of the discrete algebraic Riccati equation

$$X - F^*(R + X^{-1})^{-1}F - I = 0,$$

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where $I = I^*$ and $R = R^*$ is invertible. For more details about the discrete algebraic Riccati equation, we refer to [2, 13].

In [17], Zhan proposed the following inversion free iteration

$$M1 : \begin{cases} X_{n+1} = I - A^*Y_nA, \\ Y_{n+1} = Y_n(2I - X_nY_n), \end{cases}$$

starting from $X_0 = Y_0 = I$.

In [11], Guo and Lancaster proposed the following inversion free iteration

$$M2 : \begin{cases} Y_{n+1} = Y_n(2I - X_nY_n), \\ X_{n+1} = I - A^*Y_{n+1}A, \end{cases}$$

starting from $X_0 = Y_0 = I$.

When $A$ is a nonsingular matrix, Monsalve and Raydan proposed in [16] the following inversion free iteration

$$M3 : \begin{cases} X_0 = AA^*, \\ X_{n+1} = 2X_n - X_nA^{-*}(I - X_n)A^{-1}X_n, \quad n = 0, 1, \dots \end{cases}$$

to solve the minimal solution. The maximal solution of (1.1) can be obtained through $X_+ = I - Y_-$, where $Y_-$ is the minimal solution of the dual equation $Y + AY^{-1}A^* = I$. M3 generates a Hermitian sequence of $X_n$. The implementation of iteration M3 involves three matrix-matrix multiplications per iteration and the inverse operation of $A$ at the beginning only.

In [8], El-Sayed and Al-Dbiban proposed an algorithm that avoids the matrix inversion for every iteration, called an inversion free variant of the basic fixed point iteration.

$$M4 : \begin{cases} Y_{n+1} = (I - X_n)Y_n + I, \\ X_{n+1} = I - A^*Y_{n+1}A, \end{cases}$$

starting from $X_0 = Y_0 = I$.

It is important to notice that M1 and M2 generate a Hermitian sequence and require four matrix-matrix multiplications per iteration, while M4 requires three matrix-matrix multiplications per iteration but does not generate a Hermitian sequence. If $A$ is a normal matrix, we will prove that M4 generates a Hermitian sequence. And we propose the following algorithm.

$$M5 : \begin{cases} Y_0 = I, \\ Y_{n+1} = (AY_n)^*(AY_n) + I, \quad n = 0, 1, \dots. \end{cases}$$

The algorithm indicated by M5 is an inverse-free iterative method. Notice that it only requires to compute $X = I - A^*YA$ at the end of the process, and only needs two matrix-matrix multiplications per iteration. Therefore it is clearly inexpensive. By an inductive argument, it is also worth noticing that in M5, $Y_n$ is a Hermitian matrix for all $n$.

The following notations will be used throughout the paper. Let $\mathbb{C}^{n\times n}$ be the set of $n \times n$ complex matrices. The notation $B \geq 0 (B > 0)$ means that $B$ is a Hermitian positive semi-definite (definite) matrix. Moreover, $B \geq C (B > C)$ is used as a different notation for $B - C \geq 0 (B - C > 0)$. This induces a partial ordering on the Hermitian matrices. The symbols $\rho(A)$ and $\|A\|$ denote the spectral radius and the spectral norm of a square matrix $A$, respectively.

## 2 Conditions for the Existence of Solutions

The following lemmas are needed for our purpose.

**Lemma 2.1.** *For Algorithm M4, if $A$ is normal, then*

$$AX_n = X_nA, \quad AY_n = Y_nA,$$
$$A^*X_n = X_nA^*, \quad A^*Y_n = Y_nA^*$$

*for $n = 0, 1, \dots$.*

*Proof.* Since $Y_1 = Y_0 = X_0 = I$,

$$AY_0 = Y_0A, \quad AY_1 = Y_1A, \quad AX_0 = X_0A.$$

Because

$$X_1 = I - A^*Y_1A = I - A^*A,$$
$$Y_2 = (I - X_1)Y_1 + I = A^*A + I,$$

and $AA^* = A^*A$,

$$AY_2 = A(A^*A + I) = (A^*A + I)A = Y_2A.$$

That is, $AY_n = Y_nA$ is true for n=0, 1, 2. So, assume that $AY_n = Y_nA$ is true for $n = k$. Now we prove that $AY_n = Y_nA$ when $n = k + 1$. In fact

$$AY_{k+1} = A((I - X_k)Y_k + I) = AA^*Y_kAY_k + A = Y_{k+1}A.$$

This completes the induction for $n = k + 1$. Therefore,

$$AY_n = Y_nA$$

for $n = 0, 1, 2, \ldots$. We also have

$$AX_{n+1} = A(I - A^*Y_{n+1}A) = A - A^*Y_{n+1}AA + A = X_{n+1}A$$

for $n = 1, 2, \ldots$. The proof of $A^*X_n = X_nA^*$ and $A^*Y_n = Y_nA^*$ are similar to that of $AX_n = X_nA$ and $AY_n = Y_nA$, respectively. □

**Lemma 2.2.** *If A is normal, then Algorithm M4 generates a Hermitian sequence.*

*Proof.* Since $Y_1 = Y_0 = X_0 = I$, $Y_1$, $Y_0$, and $X_0$ are Hermitian. Assume that $Y_n$ is Hermitian for $n = k$. Then

$$Y_{k+1}^* = ((I - X_k)Y_k + I)^* = (A^*Y_kAY_k + I)^* = Y_kA^*Y_kA + I.$$

If $A$ is a normal matrix, then according to Lemma 2.1,

$$Y_{k+1}^* = A^*Y_kAY_k + I = Y_{k+1}.$$

This completes the induction for $n = k + 1$. Since $X_n = I - A^*Y_nA$,

$$X_n^* = I - A^*Y_n^*A = I - A^*Y_nA = X_n$$

for $n = 1, 2, \ldots$. This completes the proof. □

**Lemma 2.3.** *For Algorithm M4, if A is normal, then* $\{X_n, Y_n, X_{n+1}, Y_{n+1}\}$ *is a commuting family,* $n = 0, 1, 2, \ldots$.

*Proof.* Since $Y_1 = Y_0 = X_0 = I$, $X_1 = I - A^*A$, it is easy to check that $\{X_0, Y_0, X_1, Y_1\}$ is a commuting family. Now we prove that $\{X_n, Y_n, X_{n+1}, Y_{n+1}\}$ is a commuting family for $n = 1, 2, \ldots$. Since $X_n = I - A^*Y_nA$,

$$Y_{n+1} = (I - X_n)Y_n + I = A^*Y_nAY_n + I.$$

According to Lemma 2.1,

$$\begin{aligned} Y_{n+1}Y_n &= (A^*Y_nAY_n + I)Y_n \\ &= A^*Y_nAY_nY_n + Y_n \\ &= Y_nA^*Y_nAY_n + Y_n \\ &= Y_nY_{n+1}, \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} Y_{n+1}X_{n+1} &= Y_{n+1}(I - A^*Y_{n+1}A) \\ &= Y_{n+1} - Y_{n+1}A^*Y_{n+1}A \\ &= Y_{n+1} - A^*Y_{n+1}AY_{n+1} \\ &= X_{n+1}Y_{n+1}. \end{aligned}$$

This implies that $Y_nX_n = X_nY_n$ for $n = 1, 2, \ldots$. From (2.1) and Lemma 2.1,

$$\begin{aligned} Y_{n+1}X_n &= Y_{n+1}(I - A^*Y_nA) \\ &= Y_{n+1} - Y_{n+1}A^*Y_nA \\ &= Y_{n+1} - A^*Y_nAY_{n+1} \\ &= X_nY_{n+1}, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} X_{n+1}Y_n &= (I - A^*Y_{n+1}A)Y_n \\ &= Y_n - A^*Y_{n+1}AY_n \\ &= Y_n - Y_nA^*Y_{n+1}A \\ &= Y_nX_{n+1}. \end{aligned}$$

It follows from (2.2) and Lemma 2.1 that

$$\begin{aligned} X_{n+1}X_n &= (I - A^*Y_{n+1}A)X_n \\ &= X_n - A^*Y_{n+1}AX_n \\ &= X_n - X_nA^*Y_{n+1}A \\ &= X_nX_{n+1}. \end{aligned}$$

This completes the proof. □

**Lemma 2.4.** *If $0 < M \le N$, $0 < P \le Q$, and $\{M, N, P, Q\}$ is a commuting family, then*

$$MP \le NQ.$$

*Proof.* Since $M$, $N$, $P$, $Q$ are positive definite matrices, and $\{M, N, P, Q\}$ is a commuting family. By Theorem 2.5.5 in [12], there is a unitary $U$ such that

$$U^*MU = \Lambda, \quad U^*NU = \Omega, \quad U^*PU = \Sigma, \quad U^*MU = \Gamma,$$

where $\Lambda$, $\Omega$, $\Sigma$, and $\Gamma$ are diagonal. Since $M$, $N$, $P$, $Q$ are positive definite matrices, $\Lambda$, $\Omega$, $\Sigma$, and $\Gamma$ are positive diagonal matrices. Because $\{M, N, P, Q\}$ is a commuting family,

$$MP = U\Lambda\Sigma U = U\Lambda^{\frac{1}{2}}\Sigma\Lambda^{\frac{1}{2}}U = P^{\frac{1}{2}}MP^{\frac{1}{2}} \le P^{\frac{1}{2}}NP^{\frac{1}{2}} = N^{\frac{1}{2}}PN^{\frac{1}{2}}. \tag{2.3}$$

Since $0 < P \le Q$,

$$N^{\frac{1}{2}}PN^{\frac{1}{2}} \le N^{\frac{1}{2}}QN^{\frac{1}{2}} = NQ. \tag{2.4}$$

Combining (2.3) and (2.4), we have

$$MP \le NQ.$$

This completes the proof. □

Now, we prove that the sequence $\{X_n\}$ in Algorithm M4 is monotone decreasing and converges to the maximal solution $X_+$, and the sequence $\{Y_n\}$ in Algorithm M4 is monotone increasing and converges to $X_+^{-1}$.

**Theorem 2.5.** *Let $A$ be normal. If the nonlinear matrix equation (1.1) has a positive definite solution, and the two sequences $\{X_n\}$ and $\{Y_n\}$ are determined by Algorithm M4, then $\{X_n\}$ is monotone decreasing and converges to the maximal solution $X_+$, and $\{Y_n\}$ is monotone increasing and converges to $X_+^{-1}$.*

*Proof.* We will prove that

$$I = X_0 \geq X_1 \geq \cdots \geq X_n \geq X_+$$

and

$$I = Y_0 \leq Y_1 \leq \cdots \leq Y_n \leq X_+^{-1}.$$

Since $X_+$ is a solution of (1.1), i.e.,

$$X_+ = I - A^* X_+^{-1} A,$$

$X_0 = I \geq X_+$ and $I \leq X_+^{-1}$. Also

$$X_1 = I - A^* A \leq I = X_0$$

and

$$X_1 = I - A^* A \geq I - A^* X_+^{-1} A = X_+,$$

i.e., $I = X_0 \geq X_1 \geq X_+$.

For the sequence $\{Y_n\}$ we have $Y_0 = Y_1 = I$, and since $I \leq X_+^{-1}$, then $Y_0 = Y_1 \leq X_+^{-1}$. From Lemmas 2.3 and 2.4, on the one hand,

$$Y_2 = (I - X_1)Y_1 + I = A^* A + I \geq I = Y_1 = Y_0;$$

on the other hand

$$Y_2 = (I - X_1)Y_1 + I \leq (I - X_+)X_+^{-1} + I = X_+^{-1},$$

i.e., $Y_0 = Y_1 \leq Y_2 \leq X_+^{-1}$.

Assume that the above inequalities are true for $n = k$, i.e.,

$$I = X_0 \geq X_1 \geq \cdots \geq X_k \geq X_+ \tag{2.5}$$

and

$$I = Y_0 \leq Y_1 \leq \cdots \leq Y_k \leq X_+^{-1}. \tag{2.6}$$

Now we prove inequalities for $n = k + 1$. From (2.5) and (2.6), the sequences $\{X_n\}$ and $\{Y_n\}$ are Hermitian positive definite for $i = 0, 1, \ldots, k$. According to Algorithm M4, Lemmas 2.3 and 2.4, (2.5), and (2.6), we get

$$Y_{k+1} = (I - X_k)Y_k + I \geq (I - X_{k-1})Y_{k-1} + I = Y_k,$$

$$Y_{k+1} = (I - X_k)Y_k + I \leq (I - X_+)X_+^{-1} + I = X_+^{-1},$$

i.e., $Y_k \leq Y_{k+1} \leq X_+^{-1}$. Concerning the sequence $\{X_n\}$, we have

$$X_k - X_{k+1} = A^*(Y_{k+1} - Y_k)A$$

since $Y_{k+1} \geq Y_k$. Hence $X_k \geq X_{k+1}$. Therefore,

$$X_{k+1} = I - A^* Y_{k+1} A \geq I - A^* X_+^{-1} A = X_+,$$

i.e., $X_k \geq X_{k+1} \leq X_+$.

This completes the induction for $n = k + 1$. Thus,

$$I = X_0 \geq X_1 \geq \cdots \geq X_n \geq X_+$$

and

$$I = Y_0 \leq Y_1 \leq \cdots \leq Y_n \leq X_+^{-1}$$

are true for all $n$. Therefore, These are convergent sequences, i.e., $\lim_{n\to\infty} X_n$ and $\lim_{n\to\infty} Y_n$ exist. Taking limit in Algorithm M4 leads to $Y = X^{-1}$ and $X = I - A^* X^{-1} A$. Moreover, as each $X_n \geq X_+$ and $Y_n \leq X_+^{-1}$, then $X = X_+$ and $Y = X_+^{-1}$, respectively. This completes the proof. □

According to Theorem 2.5, we know that the sequence $\{Y_n\}$ determined by Algorithm M4 is monotone increasing and converges to $X_+^{-1}$. From Lemmas 2.1 and 2.3, if $A$ is normal, then $\{A, Y_n, Y_{n+1}\}$ is a commuting family, $n = 0, 1, 2, \ldots$. So we can amend Algorithm M4 and obtain the following algorithm.

$$M5 : \begin{cases} Y_0 = I, \\ Y_{n+1} = (AY_n)^*(AY_n) + I, \quad n = 0, 1, \ldots. \end{cases}$$

In this algorithm, M5 generates a Hermitian sequence, and requires two matrix-matrix multiplications per iteration. Notice that it only requires to compute $X = I - A^*YA$ at the end of the process. It is an inverse-free iterative method.

If $A$ is normal, from [9, Theorem 11], the nonlinear matrix equation (1.1) has a solution if and only if $\rho(A) \leq \frac{1}{2}$. Therefore, the nonlinear matrix equation (1.1) has a solution if and only if $\|A\| \leq \frac{1}{2}$.

**Lemma 2.6.** *Let $A$ be normal. Assume that nonlinear matrix equation (1.1) has a positive definite solution and the sequence $\{Y_n\}$ is determined by Algorithm M5. Then $\{Y_n\}$ satisfies $\|AY_n\| < 1$ for every $n = 0, 1, \ldots$.*

*Proof.* Since $A$ is normal and the nonlinear matrix equation (1.1) has a positive definite solution, $\|A\| \leq \frac{1}{2}$. Because $Y_0 = I$, $\|AY_0\| = \|A\| \leq \frac{1}{2} < 1$. For $Y_1$ we have $Y_1 = A^*A + I$, thus $\|AY_1\| = \|A(A^*A + I)\| \leq \|A\|^3 + \|A\| < 1$. That is, the inequality holds for $n = 0, 1$. So, assume that the inequality satisfies $n = k$, i.e., $\|AY_k\| < 1$. Now we prove the inequality when $n = k + 1$.

$$\begin{aligned} \|AY_{k+1}\| &= \|A\left((AY_k)^*(AY_k) + I\right)\| \\ &= \|A(AY_k)^*(AY_k) + A\| \\ &\leq \|A\|\|AY_k\|^2 + \|A\| \\ &< \|A\| + \|A\| \\ &\leq 1. \end{aligned}$$

This completes the induction for $n = k + 1$ and the lemma. □

**Lemma 2.7.** *Let $A$ be normal. The maximal solution $X_+$ of the nonlinear matrix equation (1.1) commutes with $A$.*

*Proof.* If $A$ is normal, by [18], we have

$$X_+ = \frac{1}{2}\left[I + (I - 4A^*A)^{1/2}\right].$$

So, $AX_+ = X_+A$. This completes the proof. □

**Theorem 2.8.** *Let $A$ be normal. If the nonlinear matrix equation (1.1) has a positive definite solution, then the sequence $\{Y_n\}$ determined by Algorithm M5 satisfies*

$$\|Y_{n+1} - X_+^{-1}\| \leq \|AX_+^{-1}\|\|Y_n - X_+^{-1}\|$$

*for all $n$ large enough.*

*Proof.* Since the nonlinear matrix equation (1.1) has a positive definite solution, $X_+ = \frac{1}{2}\left[I + (I - 4A^*A)^{1/2}\right]$ is the maximal solution. Then

$$X_+ + A^*X_+^{-1}A = I.$$

Multiplying by $X_+^{-1}$ on the right, we obtain

$$X_+^{-1} = (X_+^{-1}A)^*AX_+^{-1} + I.$$

By Lemma 2.7,

$$X_+^{-1} = (AX_+^{-1})^*(AX_+^{-1}) + I.$$

By Algorithm M5,

$$Y_{n+1} = (AY_n)^*(AY_n) + I. \tag{2.7}$$

Subtracting $X_+^{-1}$ from both sides of (2.7) we have that

$$\begin{aligned} Y_{n+1} - X_+^{-1} &= (AY_n)^*(AY_n) + I - X_+^{-1} \\ &= (AY_n)^*(AY_n) + I - \left((AX_+^{-1})^*(AX_+^{-1}) + I\right) \\ &= (AY_n)^*(AY_n) - (AX_+^{-1})^*(AX_+^{-1}) \\ &= (AY_n)^*(AY_n) - (AX_+^{-1})^*(AY_n) + (AX_+^{-1})^*(AY_n) - (AX_+^{-1})^*(AX_+^{-1}) \\ &= (Y_n - X_+^{-1})^*A^*(AY_n) + (AX_+^{-1})^*A(Y_n - X_+^{-1}). \end{aligned} \tag{2.8}$$

Taking norms in (2.8) and recalling that $\lim_{n\to\infty} Y_n = X_+^{-1}$ and $\|A\| \le \frac{1}{2}$, we have that

$$\begin{aligned} \|Y_{n+1} - X_+^{-1}\| &= \|(Y_n - X_+^{-1})^*A^*(AY_n) + (AX_+^{-1})^*A(Y_n - X_+^{-1})\| \\ &\le \|Y_n - X_+^{-1}\|\|A\|\left(\|AY_n\| + \|AX_+^{-1}\|\right) \\ &\le \|AX_+^{-1}\|\|Y_n - X_+^{-1}\|. \end{aligned}$$

This completes the proof. □

# 3 Numerical Experiments

To illustrate the performance of our method described in the previous section, in this section several interesting examples are given, which were carried out using MATLAB on a PC computer. We report the number of required iterations (denoted as IT), the norm of the residual (denoted as Res), the computing time in seconds (denoted as CPU), and the number of matrix-matrix (denoted as MM) multiplications required when the process is stopped. Assume $A$ is normal and (1.1) has a solution. Then

$$X_+ = \frac{1}{2}\left[I + (I - 4A^*A)^{1/2}\right] \tag{3.1}$$

is the maximal solution, and if $A$ is nonsingular,

$$X_- = \frac{1}{2}\left[I - (I - 4A^*A)^{1/2}\right]. \tag{3.2}$$

is the minimal solution [18, Theorem 4.1]. This allows us to test the local convergence behavior of the five methods by taking the computed $X_+$ by using (3.1) as the accurate maximal solution and $X_-$ by using (3.2) as the accurate minimal solution.

In our implementation,

$$\|Y_n - X_+^{-1}\| \le \|X_+^{-1}\| \times 10^{-9}$$

is used as the termination criterion for M1, M2, M4, and M5, and

$$\|X_n - X_-\| \le \|X_-\| \times 10^{-9}$$

is used as the termination criterion for M3. We compare our iteration M5, with the inverse-free methods M1, M2, M3 and M4 for solving (1.1). In all cases we describe also the initial guess for which convergence is guaranteed.

Since M3 converges to the minimal solution, we use M3 to find the minimal solution $Y_-$ of the dual equation $Y + AY^{-1}A^* = I$. We can obtain the maximal solution of (1.1) through $X_+ = I - Y_-$. The value of "Res" in our tables reports $\|F(X_n)\|_F = \|X_n + A^*X_n^{-1}A - I\|_F$ for M1, M2, M3, M4 and M5 when the process is stopped.

**Experiment 3.1.** *In this test, the matrix A is from [8] using Example 3.1*

$$A = \frac{1}{32}\begin{pmatrix} 0.2 & -0.1 & -0.5 & 0.1 \\ -0.1 & 0.6 & -0.5 & 0.7 \\ -0.5 & -0.5 & 0.1 & 0.8 \\ 0.1 & 0.7 & 0.8 & 0.5 \end{pmatrix}$$

Table 1: Performance of M1, M2, M3, M4 and M5 to solve (1) for Experiment 3.1.

| Scheme | IT | $\|F(X_n)\|_F$ | MM |
|---|---|---|---|
| M1 | 6 | 8.7390e-12 | 24 |
| M2 | 4 | 2.9322e-14 | 16 |
| M3 | 7 | 1.7493e-11 | 21 |
| M4 | 4 | 1.1655e-13 | 12 |
| M5 | 3 | 1.1655e-13 | 6 |

*Since $A$ is normal and $\|A\| = 0.0412 \leq 0.5$, from [9, Theorem 11], the nonlinear matrix equation (1.1) has a solution. In Table 1, we can see that M1 requires more matrix-matrix multiplications than the other methods to achieve convergence. M3 requires more iterations than the other four methods to satisfy the stopping criterion. M5 carries out fewer iterations and matrixCmatrix multiplications than all the other methods. For this experiment, we could say that M5 is the best option.*

**Experiment 3.2.** *In this test, let*

$$A = \frac{1}{14}\begin{pmatrix} 4 & -1 & & & \\ -1 & 4 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 4 & -1 \\ & & & -1 & 4 \end{pmatrix} \in \mathbb{C}^{m\times m}.$$

Table 2: Iterations and the number of matrix-matrix products for Experiment 3.2.

| m | $\|A\|$ | M1 | | M2 | | M3 | | M4 | | M5 | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | IT | MM | IT | MM | IT | MM | IT | MM | IT | MM |
| 4 | 0.4013 | 29 | 116 | 16 | 64 | 15 | 45 | 22 | 66 | 21 | 42 |
| 8 | 0.4200 | 33 | 132 | 18 | 72 | 17 | 51 | 26 | 78 | 25 | 50 |
| 16 | 0.4261 | 34 | 136 | 19 | 76 | 18 | 54 | 27 | 81 | 26 | 52 |
| 32 | 0.4279 | 35 | 140 | 19 | 76 | 18 | 54 | 28 | 84 | 27 | 54 |
| 64 | 0.4284 | 35 | 140 | 19 | 76 | 18 | 54 | 28 | 84 | 27 | 54 |
| 128 | 0.4285 | 35 | 140 | 19 | 76 | 18 | 54 | 28 | 84 | 27 | 54 |
| 256 | 0.4286 | 35 | 140 | 19 | 76 | 18 | 54 | 28 | 84 | 27 | 54 |
| 512 | 0.4286 | 35 | 140 | 19 | 76 | 18 | 54 | 28 | 84 | 27 | 54 |
| 1024 | 0.4286 | 35 | 140 | 19 | 76 | - | - | 28 | 84 | 27 | 54 |

Table 3: CPU time(s) for Experiment 3.2.

| m | M1 | M2 | M3 | M4 | M5 |
|---|---|---|---|---|---|
| 4 | 0.002191 | 0.000315 | 0.000634 | 0.000986 | 0.000246 |
| 8 | 0.002479 | 0.000821 | 0.000937 | 0.001263 | 0.000568 |
| 16 | 0.005854 | 0.001868 | 0.003523 | 0.002451 | 0.000956 |
| 32 | 0.012124 | 0.008553 | 0.006721 | 0.006701 | 0.004385 |
| 64 | 0.019125 | 0.014864 | 0.017950 | 0.022326 | 0.013876 |
| 128 | 0.075103 | 0.046237 | 0.075727 | 0.065126 | 0.049925 |
| 256 | 0.522263 | 0.300470 | 0.479174 | 0.383668 | 0.290019 |
| 512 | 3.557083 | 2.176928 | 20.980815 | 2.392791 | 1.886120 |
| 1024 | 46.442456 | 24.082342 | - | 27.282594 | 21.436297 |

*According to Table 2, we know that $\|A\| \leq 0.5$. Since $A$ is normal and $A$ is nonsingular, the nonlinear matrix equation $X + A^*X^{-1}A = I$ has a solution. In this test, the matrix sequence of $X_n$ in M3 is badly scaled*

Table 4: Errors for Experiment 3.2.

| m | M1 | M2 | M3 | M4 | M5 |
|---|---|---|---|---|---|
| 4 | 1.9737e-10 | 5.8016e-11 | 2.2961e-10 | 1.3549e-10 | 1.3549e-10 |
| 8 | 1.8449e-10 | 6.3914e-11 | 2.1623e-10 | 9.5667e-11 | 9.5667e-11 |
| 16 | 3.2512e-10 | 5.5185e-11 | 1.8543e-10 | 1.4121e-10 | 1.4121e-10 |
| 32 | 2.5631e-10 | 9.4206e-11 | 3.0952e-10 | 1.1751e-10 | 1.1751e-10 |
| 64 | 3.8878e-10 | 1.4330e-10 | 4.6222e-10 | 1.8003e-10 | 1.8003e-10 |
| 128 | 5.6750e-10 | 2.0941e-10 | 6.7030e-10 | 2.6386e-10 | 2.6386e-10 |
| 256 | 8.1479e-10 | 3.0082e-10 | 9.5949e-10 | 3.7953e-10 | 3.7953e-10 |
| 512 | 1.1608e-9 | 4.2868e-10 | 1.3654e-09 | 5.4118e-10 | 5.4118e-10 |
| 1024 | 1.647e-19 | 6.0853e-10 | - | 7.6847e-10 | 7.6847e-10 |

*when $m = 1024$. In Table 2, we can see that M1 requires more iterations and matrix-matrix multiplications than the other methods to satisfy the stopping criterion. We can also observe that M5 needs more iterations than M2 and M3 to reach convergence, but it carries out fewer matrix-matrix multiplications than M1, M2, M4. Table 2 shows that M5 and M3 require the same number of matrix-matrix multiplications, but M5 considerably outperforms M3 in CPU time from Table 3. From Table 3 we observe that M5 outperforms the other methods in CPU time. From our numerical results, we can see that M1 is the most expensive iteration out of the five methods. For this experiment, we could say that M5 is the best option.*

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# References

[1] W.N. Anderson, T.D. Morley, G.E. Trapp, Positive solutions to $X = A - BX^{-1}B^*$, Linear Algebra Appl., 134 (1990) 53-62.

[2] H. Dai, Z.-Z. Bai, On eigenvalue bounds and iteration methods for discrete algebraic Riccati equations, J. Comput. Math., 29 (2011) 341-366.

[3] X. Duan, A. Liao, On the existence of Hermitian positive definite solutions of the matrix equation $X^s + A^*X^{-t}A = Q$, Linear Algebra Appl., 429 (2008) 673-687.

[4] X. Duan, Q.-W. Wang, C.-M. Li, Positive definite solution of a class of nonlinear matrix equation. Linear Multilinear Algebra 62(2014) 839-852.

[5] X. Duan, Q.-W. Wang, C.-M. Li, On the matrix equation $X - \sum_{i=1}^{m} N_i^*X^{-1}N_i = I$ arising in an interpolation problem. Linear Multilinear Algebra 61(2013) 1192-1205.

[6] S.M. El-Sayed, An algorithm for computing positive definite solutions of the nonlinear matrix equation $X + A^*X^{-1}A = I$. Int. J. Comput. Math., 80 (12) (2003) 1527-1534.

[7] S.M. El-Sayed, A.M. Al-Dbiban, On positive definite solutions of the nonlinear matrix equation $X + A^*X^{-n}A = I$, Appl. Math. Comput., 151 (2004) 533-541.

[8] S.M. El-Sayed, A.M. Al-Dbiban, A new inversion free iteration for solving the equation $X + A^*X^{-1}A = Q$, J. Comput. Appl. Math. 181 (2005) 148-156.

[9] J.C. Engwerda, On the existence of a positive definite solution of the matrix equation $X + A^TX^{-1}A = I$, Linear Algebra Appl., 194 (1993) 91-108.

[10] X.-X. Guo, On Hermitian positive definite solution of nonlinear matrix equation $X + A^*X^{-2}A = Q$, J. Comput. Math., 23 (2005) 513-526.

[11] C.H. Guo, P. Lancaster, Iterative solution of two matrix equations, Math. Comp., 68 (228) (1999) 1589-1603.

[12] R.A. Horn, Ch.R. Johnson, Matrix Analysis (Second Edition), Cambridge University Press, Cambridge, 2013.

[13] B.C. Levy, R. Frezza, A.J. Krener, Modeling and estimation of discrete-time Gaussian reciprocal processes, IEEE Trans. Automat. Control 35 (1990) 1013-1023.

[14] A. Liao, G. Yao, X. Duan, Thompson metric method for solving a class of nonlinear matrix equation, Appl. Math. Comput., 216 (2010) 1831-1836.

[15] X. Liu, H. Gao, On the positive definite solutions of the matrix equations $X^s \pm A^T X^{-t} A = I$, Linear Algebra Appl., 368 (2003) 83-97.

[16] M. Monsalve, M. Raydan, A new inversion-free method for a rational matrix equation, Linear Algebra Appl., 433 (2010) 64-71.

[17] X. Zhan, Computing the extremal positive definite solutions of a matrix equation, SIAM J. Sci. Comput., 17 (5) (1996) 1167-1174.

[18] X. Zhan, J. Xie, On the matrix equation $X + A^T X^{-1} A = I$, Linear Algebra Appl., 247 (1996) 337-345.

[19] G. Zhang, W. Xie, J. Zhao, Positive definite solutions of the nonlinear matrix equation $X + A^* X^q A = Q(q > 0)$, Appl. Math. Comput., 217 (2011) 9182-9188.

[20] L. Zhao, Some inequalities for the nonlinear matrix equations, Math. Inequal. Appl., 16(3) (2013), 903-910.

[21] D. Zhou, G. Chen, G. Wu and X. Zhang, Some properties of the nonlinear matrix equation $X^s + A^* X^{-t} A = Q$, J. Math. Anal. Appl., 392 (2012) 75-82.

[22] D. Zhou, G. Chen, G. Wu and X. Zhang, On the Nonlinear Matrix Equation $X^s + A^* F(X) A = Q$ with $s \geq 1$, J. Comput. Math., 31 (2013) 209-220.

[23] D. Zhou, G. Chen and X. Zhang, Some inequalities for the nonlinear matrix equation $X^s + A^* X^{-t} A = Q$: trace, determinant and eigenvalue, Appl. Math. Comput., 224 (2013) 21-28.

# On the Ulam-Hyers Stability of Some Differential Equations involving Hadamard Fractional Derivatives☆

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**Abstract**

In this paper, we show the Ulam-Hyers stability and Ulam-Hyers-Rassias stability criterion for nonlinear Hadamard fractional relaxation differential equations on compact and unbounded time intervals. More explicit Ulam-Hyers stability and Ulam-Hyers-Rassias results are presented by virtue of estimation of Mittag-Leffler functions.

*Keywords:* Hadamard fractional derivative, Relaxation differential equations, Ulam-Hyers stability, Ulam-Hyers-Rassias stability, Mittag-Leffler functions.

## 1. Introduction

The widely application of fractional differential equations arise in various areas of physics and engineering (see [1, 2, 3, 4]). During the past decades, fractional differential equations has been more and more recognized as an alternative model to the classical differential equations. There are many interesting advance on the theory analysis for Caputo type and Riemann-Liouville type fractional differential equations as well as Hadamard type fractional differential equations (see, for example, [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]).

The well-known Ulam stability problem of functional equations originated are posed in 1940. Numerous monographs and special issues have appeared devoted to the theory of Ulam stability for functional equations and differential equations (see for example [16, 17, 18, 19, 20, 21, 22, 23]). Recently, Li and Wang [24] explore some fundamental properties of continuity, integrable estimation, asymptotic property on Mittag-Leffler functions for a Hadamard fractional differential equation with constant coefficient and present existence results for such equation by using fixed point theorems. However, to our knowledge, Ulam's stability results for nonlinear Hadamard fractional differential equation with constant coefficient have not been investigated extensively. Especially, there are few research on the Ulam's stability for this kind of equation on noncompact interval.

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In this paper, we investigate Ulam's type stability of Hadamard fractional differential equations with constant coefficient $\lambda \in \mathbb{R} \setminus \{0\}$ of the type:

$$ {}_HD^{\alpha}_{1^+} y(x) = \lambda y(x) + f(x, y(x)),\ 0 < \alpha < 1,\ x \in J = (1, e] \text{ or } (e, \infty), \tag{1} $$

where ${}_HD^{\alpha}_{1^+}$ denotes the left-sided Hadamard fractional derivative of order $\alpha$ with the low limit 1 (see Definition 2.2), and nonlinear term $f : J \times \mathbb{R} \to \mathbb{R}$ is a given function satisfying some certain conditions. Let $\varepsilon > 0$ and $\varphi : J \to \mathbb{R}^+$ be a continuous function.

Set $\bar{J} := [1, e]$ or $[e, \infty)$. Consider equation (1) and the following inequalities:

$$ |{}_HD^{\alpha}_{1^+} z(x) - \lambda z(x) - f(x, z(x))| \le \varepsilon,\ 0 < \alpha < 1,\ x \in J, \tag{2} $$

and

$$ |{}_HD^{\alpha}_{1^+} z(x) - \lambda z(x) - f(x, z(x))| \le \varepsilon\varphi(x),\ 0 < \alpha < 1,\ x \in J. \tag{3} $$

**Definition 1.1.** *Equation (1) is Ulam-Hyers stable if there exists a constant $c > 0$ such that for each $\varepsilon > 0$ and for each solution $z \in C_{\gamma,\ln}(\bar{J}, \mathbb{R})$ of inequality (2) there exists a solution $y \in C_{\gamma,\ln}(\bar{J}, \mathbb{R})$ of equation (1) with*

$$ |z(x) - y(x)| \le c\varepsilon,\ x \in J. $$

**Remark 1.2.** *A function $z \in C_{\gamma,\ln}(\bar{J}, \mathbb{R})$ is a solution of inequality (2) if and only if there exists a function $h \in C_{\gamma,\ln}(\bar{J}, \mathbb{R})$ such that (i) $|h(x)| \le \varepsilon, x \in J$, (ii) ${}_HD^{\alpha}_{1^+} z(x) = \lambda z(x) + f(x, z(x)) + h(x), x \in J$.*

**Definition 1.3.** *Equation (1) is Ulam-Hyers-Rassias stable stable if there exists a constant $c > 0$ such that for each $\varepsilon > 0$ and for each solution $z \in C_{\gamma,\ln}(\bar{J}, \mathbb{R})$ of inequality (3) there exists a solution $y \in C_{\gamma,\ln}(\bar{J}, \mathbb{R})$ of equation (1) with*

$$ |z(x) - y(x)| \le c\varepsilon\varphi(x),\ x \in J. $$

**Remark 1.4.** *A function $z \in C_{\gamma,\ln}(\bar{J}, \mathbb{R})$ is a solution of inequality (3) if and only if there exists a function $\tilde{h} \in C_{\gamma,\ln}(\bar{J}, \mathbb{R})$ such that (i) $|\tilde{h}(x)| \le \varepsilon\varphi(x), x \in J$, (ii) ${}_HD^{\alpha}_{1^+} z(x) = \lambda z(x) + f(x, z(x)) + \tilde{h}(x), x \in J$.*

The rest of this paper is organized as follows. In Section 2, some notations and preparation results are given. In Section 3, some useful remarks on bounded and unbounded time intervals are presented. Section 4 is devoted to to give Ulam-Hyers stability and Ulam-Hyers-Rassias stability criteria of the equation (1) on bounded and unbounded time intervals respectively. Finally, the reason on the equation (1) is not necessary Ulam-Hyers-Rassias stable is analysed.

## 2. Preliminaries

Let $Y$ be a Banach space endowed with the norm $\|\cdot\|_Y$. Set $J := (1, e]$ or $(e, \infty)$. Denote $C(J, Y)$ be the Banach space of all continuous functions from $J$ into $Y$ with the norm $\|y\|_C = \sup_{x\in J} \|y(x)\|_Y$. For $0 < \mu < 1$, we denote the set $C_{\mu,\ln}(\bar{J}, Y) := \{y(x) : y : J \to Y$ is continuous such that $\left(\ln \frac{x}{a}\right)^{\mu} y(x) \in C(\bar{J}, Y)\}$. Following [1, Theorem 3.29], $C_{\mu,\ln}(\bar{J}, Y)$ is a Banach space with the norm $\|y\|_{C_{\mu,\ln}} = \|(\ln x)^{\mu} y(x)\|_C = \sup_{x\in\bar{J}} \|(\ln x)^{\mu} y(x)\|_Y$.

The following definitions and lemmas will be used in this paper.

**Definition 2.1.** *(see [1, p.110, (2.7.1)]) The left-sided Hadamard fractional integral of order $\alpha \in \mathbb{R}^+$ of function $y(t)$ are defined by*

$$({}_H J^{\alpha}_{a^+} y)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} y(s) \frac{ds}{s}, \ (0 < a < t \le b),$$

*where $\Gamma(\cdot)$ is the Gamma function.*

**Definition 2.2.** *(see [1, p.111, (2.7.7)]) The left-sided Hadamard fractional derivative of order $\alpha \in [n-1, n)$, $n \in \mathbb{Z}^+$ of function $y(t)$ are defined by*

$$({}_H D^{\alpha}_{a^+} y)(t) = \frac{1}{\Gamma(n-\alpha)} \left(t\frac{d}{dt}\right)^n \int_a^t \left(\ln \frac{t}{s}\right)^{n-\alpha+1} y(s) \frac{ds}{s}, \ (0 < a < t \le b).$$

**Lemma 2.3.** *(see [4, Theorem 2.3]) Let $\alpha, \beta \in (0, 1]$ and $\beta < 1 + \alpha$ be arbitrary. Then the following statements hold:*

*(i) For all $z > 0$, we have*

$$\mathbb{E}_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} = \frac{1}{\alpha} z^{\frac{1-\beta}{\alpha}} \exp(z^{\frac{1}{\alpha}}) + \int_0^{\infty} S(v, z) dv,$$

*where*

$$S(v, z) = \frac{1}{\pi\alpha} v^{\frac{1-\beta}{\alpha}} \exp(-v^{\frac{1}{\alpha}}) \frac{v \sin(\pi(1-\beta)) - z \sin(\pi(1-\beta+\alpha))}{v^2 - 2vz\cos(\pi\alpha) + z^2}.$$

*(ii) For all $z < 0$, we have*

$$\mathbb{E}_{\alpha,\beta}(z) = \int_0^{\infty} S(v, z) dv,$$

*where*

$$S(v, z) = \frac{1}{\pi\alpha} v^{\frac{1-\beta}{\alpha}} \exp(-v^{\frac{1}{\alpha}}) \frac{v \sin(\pi(1-\beta)) - z \sin(\pi(1-\beta+\alpha))}{v^2 - 2vz\cos(\pi\alpha) + z^2}.$$

*We note that $\mathbb{E}_{\alpha}(z) = \mathbb{E}_{\alpha,1}(z)$.*

By virtue of Lemma 2.3, Li and Wang [24] derived the following useful results for two-parameter Mittag-Leffler function.

**Lemma 2.4.** *(see [24, Theorem 2.11]) Let $\lambda > 0$ be arbitrary, $\alpha, \beta \in (0, 1]$ and $\beta < 1 + \alpha$. Denote*

$$\omega(\alpha, \beta, \lambda) = \max\left\{ \frac{\alpha \sin(\beta\pi)\Gamma(2\alpha - \beta + 1)}{\lambda^2 \alpha\pi \sin^2(\pi\alpha)}, \frac{\alpha |\sin(\pi(\beta-\alpha))| \Gamma(\alpha - \beta + 1)}{\lambda\alpha\pi \sin^2(\pi\alpha)} \right\}.$$

*For all $x \in (1,\infty)$, we have*

$$\left|(\ln x)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^{\alpha}) - \frac{1}{\alpha}\lambda^{\frac{1-\alpha}{\alpha}}\exp(\lambda^{\frac{1}{\alpha}}\ln x)\right| \leq \frac{\omega(\alpha,\alpha,\lambda)}{(\ln x)^{\alpha+1}}.$$

*In particular, for all $x \in (1,\infty)$,*

$$\left|\mathbb{E}_{\alpha}(\lambda(\ln x)^{\alpha}) - \frac{1}{\alpha}\exp(\lambda^{\frac{1}{\alpha}}\ln x)\right| \leq \frac{\omega(\alpha,1,\lambda)}{(\ln x)^{\alpha}}.$$

Further, we give the following integral estimation.

**Lemma 2.5.** *Let $\lambda > 0$ be arbitrary, $\alpha \in (0,1]$, we have*

*(i) For all $x \in (1,e]$, we have*

$$\begin{aligned} J_1 &:= \left|\int_1^x \left((\ln x - \ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x - \ln t)^{\alpha}) - \lambda^{\frac{1-\alpha}{\alpha}}\mathbb{E}_{\alpha}(\lambda(\ln x)^{\alpha})\exp(-\lambda^{\frac{1}{\alpha}}\ln t)\right)\frac{dt}{t}\right| \\ &\leq \frac{\mathbb{E}_{\alpha,\alpha}(\lambda)}{\alpha} + \frac{\mathbb{E}_{\alpha}(\lambda)}{\lambda}. \end{aligned}$$

*(ii) For all $x \in (e,\infty)$, we have*

$$\begin{aligned} J_2 &:= \left|\int_1^x \left((\ln x - \ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x - \ln t)^{\alpha}) - (\ln x)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^{\alpha})\exp(-\lambda^{\frac{1}{\alpha}}\ln t)\right)\frac{dt}{t}\right| \\ &\leq \frac{\mathbb{E}_{\alpha,\alpha}(\lambda)}{\alpha} + \frac{\exp(\lambda^{\frac{1}{\alpha}})}{\alpha\lambda} + \omega(\alpha,\alpha,\lambda)\left(\frac{1}{\alpha} + \frac{2}{\lambda^{\frac{1}{\alpha}}}\right) := M(\alpha,\lambda). \end{aligned}$$

*Proof.* *(i)* For all $x \in (1,e]$, we obtain

$$\begin{aligned} J_1 &\leq \mathbb{E}_{\alpha,\alpha}(\lambda)\left|\int_1^x (\ln x - \ln t)^{\alpha-1}\frac{dt}{t}\right| + \lambda^{\frac{1-\alpha}{\alpha}}\mathbb{E}_{\alpha}(\lambda)\left|\int_1^x \exp(-\lambda^{\frac{1}{\alpha}}\ln t)\frac{dt}{t}\right| \\ &\leq \frac{\mathbb{E}_{\alpha,\alpha}(\lambda)}{\alpha} + \frac{\mathbb{E}_{\alpha}(\lambda)}{\lambda}, \end{aligned}$$

where we use the decreasing property of $\mathbb{E}_{\alpha,\alpha}(z)$ for $z > 0$.

*(ii)* For all $x > e$, by using Lemma 2.4, we have

$$\begin{aligned} I_1 &:= \left|\int_1^{\frac{x}{e}} \left((\ln x - \ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x - \ln t)^{\alpha}) - (\ln x)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^{\alpha})\exp(-\lambda^{\frac{1}{\alpha}}\ln t)\right)\frac{dt}{t}\right| \\ &\leq \left|\int_1^{\frac{x}{e}} \left((\ln x - \ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x - \ln t)^{\alpha}) - \frac{1}{\alpha}\lambda^{\frac{1-\alpha}{\alpha}}\exp(\lambda^{\frac{1}{\alpha}}(\ln x - \ln t))\right)\frac{dt}{t}\right| \\ &\quad + \left|\int_1^{\frac{x}{e}} \frac{1}{\alpha}\lambda^{\frac{1-\alpha}{\alpha}}\exp(\lambda^{\frac{1}{\alpha}}(\ln x - \ln t)) - (\ln x)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^{\alpha})\exp(-\lambda^{\frac{1}{\alpha}}\ln t)\right)\frac{dt}{t}\right| \\ &\leq \int_1^{\frac{x}{e}} \frac{\omega(\alpha,\alpha,\lambda)}{(\ln x - \ln t)^{\alpha+1}}\frac{dt}{t} + \int_1^{\frac{x}{e}} \frac{\omega(\alpha,\alpha,\lambda)}{(\ln x)^{\alpha+1}}\exp(-\lambda^{\frac{1}{\alpha}}\ln t)\frac{dt}{t} \\ &\leq \int_1^{\frac{x}{e}} \frac{\omega(\alpha,\alpha,\lambda)}{(\ln x - \ln t)^{\alpha+1}}\frac{dt}{t} + \omega(\alpha,\alpha,\lambda)\int_1^{\frac{x}{e}} \exp(-\lambda^{\frac{1}{\alpha}}\ln t)\frac{dt}{t} \\ &\leq \frac{\omega(\alpha,\alpha,\lambda)}{\alpha} + \frac{\omega(\alpha,\alpha,\lambda)}{\lambda^{\frac{1}{\alpha}}} \\ &= \omega(\alpha,\alpha,\lambda)\left(\frac{1}{\alpha} + \frac{1}{\lambda^{\frac{1}{\alpha}}}\right). \end{aligned}$$

According to Lemma 2.4 again, we have

$$(\ln x)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^{\alpha}) \leq \frac{\omega(\alpha,\alpha,\lambda)}{(\ln x)^{\alpha+1}} + \frac{1}{\alpha}\lambda^{\frac{1-\alpha}{\alpha}}\exp(\lambda^{\frac{1}{\alpha}}\ln x).$$

Thus, using the decreasing property of $\mathbb{E}_{\alpha,\alpha}(z)$ for $z>0$ again, one has

$$\begin{aligned}
I_2 &:= \left|\int_{\frac{x}{e}}^{x}\left((\ln x-\ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x-\ln t)^{\alpha}) - (\ln x)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^{\alpha})\exp(-\lambda^{\frac{1}{\alpha}}\ln t)\right)\frac{dt}{t}\right| \\
&\leq \left|\int_{\frac{x}{e}}^{x}(\ln x-\ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x-\ln t)^{\alpha})\frac{dt}{t}\right| + \left|\int_{\frac{x}{e}}^{x}(\ln x)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^{\alpha})\exp(-\lambda^{\frac{1}{\alpha}}\ln t)\frac{dt}{t}\right| \\
&\leq \mathbb{E}_{\alpha,\alpha}(\lambda)\int_{\frac{x}{e}}^{x}(\ln x-\ln t)^{\alpha-1}\frac{dt}{t} \\
&\quad + \left|\int_{\frac{x}{e}}^{x}\left(\frac{\omega(\alpha,\alpha,\lambda)}{(\ln x)^{\alpha+1}}\exp(-\lambda^{\frac{1}{\alpha}}\ln t) + \frac{\lambda^{\frac{1-\alpha}{\alpha}}}{\alpha}\exp(\lambda^{\frac{1}{\alpha}}(\ln x-\ln t))\right)\frac{dt}{t}\right| \\
&\leq \frac{\mathbb{E}_{\alpha,\alpha}(\lambda)}{\alpha} + \frac{\omega(\alpha,\alpha,\lambda)}{(\ln x)^{\alpha+1}}\int_{\frac{x}{e}}^{x}\exp(-\lambda^{\frac{1}{\alpha}}\ln t)\frac{dt}{t} + \frac{1}{\alpha}\lambda^{\frac{1-\alpha}{\alpha}}\int_{\frac{x}{e}}^{x}\exp(\lambda^{\frac{1}{\alpha}}(\ln x-\ln t))\frac{dt}{t} \\
&\leq \frac{\mathbb{E}_{\alpha,\alpha}(\lambda)}{\alpha} + \frac{\omega(\alpha,\alpha,\lambda)}{\lambda^{\frac{1}{\alpha}}} - \frac{\exp(-\lambda^{\frac{1}{\alpha}})}{\lambda^{\frac{1}{\alpha}}} + \frac{1}{\alpha}\lambda^{\frac{1-\alpha}{\alpha}}\lambda^{-\frac{1}{\alpha}}\exp(\lambda^{\frac{1}{\alpha}}) \\
&\leq \frac{\mathbb{E}_{\alpha,\alpha}(\lambda)}{\alpha} + \frac{\omega(\alpha,\alpha,\lambda)}{\lambda^{\frac{1}{\alpha}}} + \frac{\exp(\lambda^{\frac{1}{\alpha}})}{\alpha\lambda}.
\end{aligned}$$

From above, we obtain

$$J_2 \leq I_1 + I_2 \leq \frac{\mathbb{E}_{\alpha,\alpha}(\lambda)}{\alpha} + \frac{\exp(\lambda^{\frac{1}{\alpha}})}{\alpha\lambda} + \omega(\alpha,\alpha,\lambda)\left(\frac{1}{\alpha} + \frac{2}{\lambda^{\frac{1}{\alpha}}}\right).$$

The proof is completed □

To end this section, we recall the following inequality which will be used in the sequel.

**Lemma 2.6.** *(see [25, Lemma 23]) If $\lambda, v, w > 0$, then for any $t > a, a > 0$, we have*

$$\left(\ln\frac{t}{a}\right)^{1-v}\int_a^t\left(\ln\frac{t}{s}\right)^{v-1}\left(\ln\frac{s}{a}\right)^{\lambda-1}\left(\frac{s}{a}\right)^{-w}\frac{ds}{s} \leq Cw^{-\lambda},$$

*where $C$ is a positive constant independent of the time variable $t$.*

## 3. Some useful lemmas and remarks

Now we plan to give the following integral estimation.

**Lemma 3.1.** *Let $\lambda > 0$, $z \in C_{\gamma,\ln}([1,\infty),\mathbb{R})$ be a solution of inequality (2). Then $z$ is a solution of the following inequality:*

$$\begin{aligned}
&\left|z(x) - (\ln x)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^{\alpha})c_0 - \int_1^x(\ln x-\ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x-\ln t)^{\alpha})f(t,z(t))\frac{dt}{t}\right| \\
&\leq \int_1^x\frac{\omega(\alpha,\alpha,\lambda)\varepsilon}{(\ln x-\ln t)^{\alpha+1}}\frac{dt}{t} + \frac{\varepsilon x^{\lambda^{\frac{1}{\alpha}}}}{\alpha\lambda},
\end{aligned}$$

*where $c_0 = {}_H J_{1^+}^{1-\alpha} z(1^+)$.*

*Proof.* According to the Remark 1.2, we have

$$ {}_H D^{\alpha}_{1^+} z(x) = \lambda z(x) + f(x, z(x)) + h(x),\ x \in (1, \infty). $$

By [1, p.234,(4.1.89)-(4.1.95)] or [26, p.182, (7.2.60)-(7.2.64)], we obtain

$$\begin{aligned} z(x) &= (\ln x)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^{\alpha})c_0 + \int_1^x (\ln x - \ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x - \ln t)^{\alpha}) f(t, z(t))\frac{dt}{t} \\ &\quad + \int_1^x (\ln x - \ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x - \ln t)^{\alpha}) h(t)\frac{dt}{t},\ x \in (1, \infty). \end{aligned}$$

For all $1 < x < \infty$, using Lemma 2.4, one has

$$\begin{aligned} & \left| z(x) - (\ln x)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^{\alpha})c_0 - \int_1^x (\ln x - \ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x - \ln t)^{\alpha}) f(t, z(t))\frac{dt}{t} \right| \\ = & \left| \int_1^x (\ln x - \ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x - \ln t)^{\alpha}) h(t)\frac{dt}{t} \right| \\ \le & \left| \int_1^x \left( (\ln x - \ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x - \ln t)^{\alpha}) - \frac{1}{\alpha}\lambda^{\frac{1-\alpha}{\alpha}} \exp(\lambda^{\frac{1}{\alpha}}(\ln x - \ln t)) \right) h(t)\frac{dt}{t} \right| \\ & + \left| \int_1^x \frac{1}{\alpha}\lambda^{\frac{1-\alpha}{\alpha}} \exp(\lambda^{\frac{1}{\alpha}}(\ln x - \ln t)) h(t)\frac{dt}{t} \right| \\ \le & \int_1^x \frac{\omega(\alpha,\alpha,\lambda)\varepsilon}{(\ln x - \ln t)^{\alpha+1}}\frac{dt}{t} + \varepsilon \int_1^x \frac{1}{\alpha}\lambda^{\frac{1-\alpha}{\alpha}} \exp(\lambda^{\frac{1}{\alpha}}(\ln x - \ln t))\frac{dt}{t} \\ \le & \int_1^x \frac{\omega(\alpha,\alpha,\lambda)\varepsilon}{(\ln x - \ln t)^{\alpha+1}}\frac{dt}{t} + \frac{\varepsilon}{\alpha\lambda}\exp(\lambda^{\frac{1}{\alpha}}\ln x) - \frac{\varepsilon}{\lambda^{\frac{1}{\alpha}}} \\ \le & \int_1^x \frac{\omega(\alpha,\alpha,\lambda)\varepsilon}{(\ln x - \ln t)^{\alpha+1}}\frac{dt}{t} + \frac{\varepsilon x^{\lambda^{\frac{1}{\alpha}}}}{\alpha\lambda}. \end{aligned}$$

The proof is completed. □

**Remark 3.2.** *Note that for some fixed point $x_0$ and $x_0 > \delta > 1$, we have*

$$\int_1^{x_0} \frac{1}{(\ln x_0 - \ln t)^{\alpha+1}}\frac{dt}{t} = \lim_{\delta \to x_0} \int_1^{\delta} \frac{1}{(\ln x_0 - \ln t)^{\alpha+1}}\frac{dt}{t} = \lim_{\delta \to x_0} \frac{1}{\alpha}[(\ln x_0 - \ln \delta)^{-\alpha} - (\ln x_0)^{-\alpha}] = \infty,$$

*which yields that it is not possible to obtain some explicit estimation in this case.*

Next, we divide our time interval $(1, \infty)$ into two subintervals $(1, e]$ and $(e, \infty)$.

**Remark 3.3.** *Let $\lambda > 0$, $z \in C_{\gamma,\ln}([1, e], \mathbb{R})$ be a solution of inequality (2). Then $z$ is a solution of the following inequality:*

$$\begin{aligned} & \left| z(x) - (\ln x)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^{\alpha})c_0 - \int_1^x (\ln x - \ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x - \ln t)^{\alpha}) f(t, z(t))\frac{dt}{t} \right| \\ \le & \left| \int_1^x \left( (\ln x - \ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x - \ln t)^{\alpha}) - \lambda^{\frac{1-\alpha}{\alpha}}\mathbb{E}_{\alpha}(\lambda(\ln x)^{\alpha}) \exp(-\lambda^{\frac{1}{\alpha}}\ln t) \right) h(t)\frac{dt}{t} \right| \\ & + \left| \int_1^x \lambda^{\frac{1-\alpha}{\alpha}}\mathbb{E}_{\alpha}(\lambda(\ln x)^{\alpha}) \exp(-\lambda^{\frac{1}{\alpha}}\ln t) h(t)\frac{dt}{t} \right| \\ \le & \varepsilon\left( \frac{\mathbb{E}_{\alpha,\alpha}(\lambda)}{\alpha} + \frac{\mathbb{E}_{\alpha}(\lambda)}{\lambda} \right) + \frac{\varepsilon\mathbb{E}_{\alpha}(\lambda)}{\lambda} \\ \le & \varepsilon\left( \frac{\mathbb{E}_{\alpha,\alpha}(\lambda)}{\alpha} + \frac{2\mathbb{E}_{\alpha}(\lambda)}{\lambda} \right),\ x \in (1, e], \end{aligned}$$

*where we use Lemma 2.5(i), Remark 1.2 and $\mathbb{E}_{\alpha,\alpha}(z)$ is an increasing function for $z > 0$.*

**Remark 3.4.** *Let $\lambda > 0$, $z \in C_{\gamma,\ln}([e,\infty),\mathbb{R})$ be a solution of inequality (2). Then $z$ is a solution of the following inequality:*

$$\begin{aligned}
&\left|z(x) - (\ln x)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^{\alpha})c_0 - \int_1^x (\ln x - \ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x - \ln t)^{\alpha})f(t,z(t))\frac{dt}{t}\right| \\
\leq\ & \left|\int_1^x \left((\ln x - \ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x - \ln t)^{\alpha}) - (\ln x)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^{\alpha})\exp(-\lambda^{\frac{1}{\alpha}}\ln t)\right)h(t)\frac{dt}{t}\right| \\
&+ \left|\int_1^x (\ln x)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^{\alpha})\exp(-\lambda^{\frac{1}{\alpha}}\ln t)h(t)\frac{dt}{t}\right| \\
\leq\ & \varepsilon M(\alpha,\lambda) + \varepsilon\left|\int_1^x \left(\frac{\omega(\alpha,\alpha,\lambda)}{(\ln x)^{\alpha+1}}\exp(-\lambda^{\frac{1}{\alpha}}\ln t) + \frac{1}{\alpha}\lambda^{\frac{1-\alpha}{\alpha}}\exp(\lambda^{\frac{1}{\alpha}}(\ln x - \ln t))\right)\frac{dt}{t}\right| \\
\leq\ & \varepsilon M(\alpha,\lambda) + \frac{\varepsilon\omega(\alpha,\alpha,\lambda)}{\lambda^{\frac{1}{\alpha}}} + \frac{\varepsilon x^{\frac{1}{\alpha}}}{\lambda^{\frac{1}{\alpha}}} \\
=\ & \varepsilon\left(M(\alpha,\lambda) + \frac{\omega(\alpha,\alpha,\lambda)}{\lambda^{\frac{1}{\alpha}}} + \frac{x^{\frac{1}{\alpha}}}{\lambda^{\frac{1}{\alpha}}}\right),
\end{aligned}$$

*where we use Lemma 2.4, Lemma 2.5(ii) and Remark 1.2.*

**Remark 3.5.** *Let $\lambda < 0$, $z \in C_{\gamma,\ln}([1,\infty),\mathbb{R})$ be a solution of inequality (2). Then $z$ is a solution of the following inequality:*

$$\begin{aligned}
&\left|z(x) - (\ln x)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^{\alpha})c_0 - \int_1^x (\ln x - \ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x - \ln t)^{\alpha})f(t,z(t))\frac{dt}{t}\right| \\
\leq\ & \frac{\varepsilon(\ln x)^{\alpha}}{\Gamma(\alpha+1)},\ x \in (1,\infty),
\end{aligned}$$

*where we use the fact $\mathbb{E}_{\alpha,\alpha}(z) \leq \frac{1}{\Gamma(\alpha)}$ for $z < 0$.*

**Remark 3.6.** *Let $\lambda < 0$, $z \in C_{\gamma,\ln}([1,\infty),\mathbb{R})$ be a solution of inequality (3). Then $z$ is a solution of the following inequality:*

$$\begin{aligned}
&\left|z(x) - (\ln x)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^{\alpha})c_0 - \int_1^x (\ln x - \ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x - \ln t)^{\alpha})f(t,z(t))\frac{dt}{t}\right| \\
=\ & \left|\int_1^x (\ln x - \ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x - \ln t)^{\alpha})\tilde{h}(t)\frac{dt}{t}\right| \\
\leq\ & \frac{\varepsilon}{\Gamma(\alpha)}\int_1^x (\ln x - \ln t)^{\alpha-1}\varphi(t)\frac{dt}{t},\ x \in (1,\infty),
\end{aligned}$$

*where we use the fact $\mathbb{E}_{\alpha,\alpha}(z) \leq \frac{1}{\Gamma(\alpha)}$ for $z < 0$ again.*

## 4. Main results

### *4.1. Ulam-Hyers stability results*

We introduce the following assumptions:

$(A_1)$ $f : (1,e] \times \mathbb{R} \to \mathbb{R}$ be a function such that $f(\cdot, y(\cdot)) \in C_{\gamma\,\ln}[1,e]$, $1 - \alpha \leq \gamma < 1$.

$(A_2)$ There exists $L > 0$ such that

$$|f(x,y) - f(x,z)| \leq L|y - z| \ \text{ for each } x \in J \text{ and all } y, z \in \mathbb{R}.$$

$(A_3)$ $\omega = 1 - L\mathbb{E}_{\alpha,\alpha}(\lambda)B[1-\gamma,\alpha] > 0$.

**Theorem 4.1.** *Assume that $(A_1)$, $(A_2)$, and $(A_3)$ are satisfied. Then equation (1) with $\lambda > 0$ is Ulam-Hyers stable on $J = (1, e]$.*

*Proof.* Let $z \in C_{\gamma,\ln}([1, e], \mathbb{R})$ be a solution of inequality (2). By $(A_1)$, $(A_2)$, and $(A_3)$, one can apply Banach fixed point theorem to derive

$$\begin{cases} {}_H D^{\alpha}_{1^+} y(x) = \lambda y(x) + f(x, y(x)),\ 0 < \alpha < 1,\ x \in J, \\ {}_H J^{1-\alpha}_{1^+} z(1^+) = c_0, \end{cases}$$

has the unique solution

$$y(x) = (\ln x)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^{\alpha})c_0 + \int_1^x (\ln x - \ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x - \ln t)^{\alpha}) f(t, y(t))\frac{dt}{t}.$$

By using Lemma 2.5$(i)$ and Remark 3.3, we have

$$\begin{aligned} & |(z(x) - y(x))(\ln x)^{\gamma}| \\ = & \left|(\ln x)^{\gamma}\left(z(x) - (\ln x)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^{\alpha})c_0 - \int_1^x (\ln x - \ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x - \ln t)^{\alpha}) f(t, y(t))\frac{dt}{t}\right)\right| \\ \leq & \left|(\ln x)^{\gamma}\left(z(x) - (\ln x)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^{\alpha})c_0 - \int_1^x (\ln x - \ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x - \ln t)^{\alpha}) f(t, z(t))\frac{dt}{t}\right)\right| \\ + & \left|\int_1^x (\ln x)^{\gamma}(\ln x - \ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x - \ln t)^{\alpha})(f(t, z(t)) - f(t, y(t)))\frac{dt}{t}\right| \\ \leq & \varepsilon(\ln x)^{\gamma}\left(\frac{\mathbb{E}_{\alpha,\alpha}(\lambda)}{\alpha} + \frac{2\mathbb{E}_{\alpha}(\lambda)}{\lambda}\right) + L\mathbb{E}_{\alpha,\alpha}(\lambda)\int_1^x (\ln x)^{\gamma}(\ln x - \ln t)^{\alpha-1}(\ln t)^{-\gamma}\frac{dt}{t}\|z - y\|_{C_{\gamma,\ln}} \\ \leq & \varepsilon(\ln x)^{\gamma}\left(\frac{\mathbb{E}_{\alpha,\alpha}(\lambda)}{\alpha} + \frac{2\mathbb{E}_{\alpha}(\lambda)}{\lambda}\right) + L\mathbb{E}_{\alpha,\alpha}(\lambda)B[1-\gamma, \alpha]\|z - y\|_{C_{\gamma,\ln}}, \end{aligned}$$

which yields that

$$\|z - y\|_{C_{\gamma,\ln}} \leq \frac{\varepsilon}{\omega}\left(\frac{\mathbb{E}_{\alpha,\alpha}(\lambda)}{\alpha} + \frac{2\mathbb{E}_{\alpha}(\lambda)}{\lambda}\right)(\ln x)^{\gamma}.$$

Thus,

$$|z(x) - y(x)| \leq c\varepsilon, \quad c = \frac{1}{\omega}\left(\frac{\mathbb{E}_{\alpha,\alpha}(\lambda)}{\alpha} + \frac{2\mathbb{E}_{\alpha}(\lambda)}{\lambda}\right) > 0.$$

The proof is completed. □

**Remark 4.2.** *Let $\lambda < 0$. Assume that $(A_1)$ and $(A_2)$ are satisfied. One can use the above similar methods via Remark 3.5 to check that the equation (1) is Ulam-Hyers-Rassias stable on $J = (1, \infty)$ provided by $\rho = 1 - \frac{LB[1-\gamma,\alpha]}{\Gamma(\alpha)} > 0$. That is,*

$$|z(x) - y(x)| \leq c\varepsilon\varphi(x),\ x \in (1, \infty), \quad c = \frac{\varepsilon}{\rho\Gamma(\alpha+1)} > 0,\ \varphi(x) = (\ln x)^{\alpha}.$$

**Example 4.3.** *Let $\alpha = \frac{2}{3}$, $\lambda = \frac{1}{2}$ and $\gamma = \frac{1}{2}$. Consider the fractional order differential equation*

$${}_H D^{\frac{2}{3}}_{1^+} y(x) = \frac{1}{2}y(x) + \frac{1}{l}\sin^2 y(x),\ x \in (1, e], l > 0, \tag{4}$$

*and the inequality*

$$|{}_H D^{\frac{2}{3}}_{1^+} z(x) - \frac{1}{2}z(x) - \frac{1}{l}\sin^2 z(x)| \leq \varepsilon,\ x \in (1, e]. \tag{5}$$

*Let $z \in C_{\gamma,\ln}([1,e],\mathbb{R})$ be a solution of inequality (5). Then there exists a function $h(x) = \varepsilon \ln x \in C_{\gamma,\ln}([1,e],\mathbb{R})$ such that $|h(x)| \leq \varepsilon, x \in (1,e]$, and ${}_H D_{1^+}^{\frac{2}{3}} z(x) = \frac{1}{2} z(x) + \frac{1}{l}\sin^2 z(x) + h(x),\ x \in (1,e]$.*

*Define $f(x,y(x)) = \frac{1}{l}\sin^2(x),\ x \in (1,e]$ and $L = \frac{2}{l}$. Obviously, $(A_1)$ and $(A_2)$ hold. Moreover, we choose $l = 6\mathbb{E}_{\frac{2}{3},\frac{2}{3}}\left(\frac{1}{2}\right) B\left[\frac{1}{2},\frac{2}{3}\right]$, then*

$$\omega = 1 - \frac{2}{l}\mathbb{E}_{\frac{2}{3},\frac{2}{3}}\left(\frac{1}{2}\right) B\left[\frac{1}{2},\frac{2}{3}\right] = \frac{2}{3} > 0,$$

*which implies that $(A_3)$ holds. According to Theorem 4.2, we have*

$$|z(x) - y(x)| \leq \frac{3\varepsilon}{2}\left(1.5\mathbb{E}_{\frac{2}{3},\frac{2}{3}}\left(\frac{1}{2}\right) + 4\mathbb{E}_{\frac{2}{3}}\left(\frac{1}{2}\right)\right).$$

*Thus, equation (4) is Ulam-Hyers stable on $(1,e]$ with $c = \frac{3}{2}\left(1.5\mathbb{E}_{\frac{2}{3},\frac{2}{3}}\left(\frac{1}{2}\right) + 4\mathbb{E}_{\frac{2}{3}}\left(\frac{1}{2}\right)\right)$.*

*4.2. Ulam-Hyers-Rassias stability result*

Next, we introduce the following assumptions:

$(B_1)$ Let $\lambda < 0$ and $\gamma = 1 - \alpha$.

$(B_2)$ $f : J \times \mathbb{R} \to \mathbb{R}$ is jointly continuous and there exists $L(\cdot) \in C([1,\infty),\mathbb{R}^+)$ such that

$$|f(x,y) - f(x,z)| \leq L(x)|y - z| \ \text{ for each } x \in J \text{ and all } y, z \in \mathbb{R},$$

where $L(\cdot)$ satisfying

$$\int_1^x (\ln x)^\gamma (\ln x - \ln t)^{\alpha-1} (\ln t)^{-\gamma} L(t)\frac{dt}{t} \leq \widetilde{C}(\ln x)^{\alpha-1},\ \widetilde{C} > 0. \tag{6}$$

$(B_3)$ There exists a $\varphi(\cdot) \in C([1,\infty),\mathbb{R}^+)$ such that

$$\int_1^x (\ln x - \ln t)^{\alpha-1}\varphi(t)\frac{dt}{t} \leq \widehat{C}\varphi(t),\ \widehat{C} > 0. \tag{7}$$

$(B_4)$ $\omega' = 1 - \frac{\widetilde{C}}{\Gamma(\alpha)} > 0$.

**Theorem 4.4.** *Let $\lambda < 0$. Assume that $(B_1)$, $(B_2)$, $(B_3)$ and $(B_4)$ are satisfied. Then equation (1) is Ulam-Hyers-Rassias stable on $J = (1,\infty)$.*

*Proof.* Note that the fact $\mathbb{E}_{\alpha,\alpha}(z) \leq \frac{1}{\Gamma(\alpha)}$ for $z < 0$. By Remark 3.6, one can obtain

$$\begin{aligned}
& |(z(x) - y(x))(\ln x)^\gamma| \\
\leq\ & \left|(\ln x)^\gamma\left(z(x) - (\ln x)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^\alpha)c_0 - \int_1^x (\ln x - \ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x - \ln t)^\alpha) f(t,z(t))\frac{dt}{t}\right)\right| \\
& + \left|\int_1^x (\ln x)^\gamma (\ln x - \ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x - \ln t)^\alpha)(f(t,z(t)) - f(t,y(t)))\frac{dt}{t}\right| \\
\leq\ & \frac{1}{\Gamma(\alpha)}\left|\int_1^x (\ln x)^\gamma (\ln x - \ln t)^{\alpha-1}\varepsilon\varphi(t)\frac{dt}{t}\right| \\
& + \frac{1}{\Gamma(\alpha)}\int_1^x (\ln x)^\gamma (\ln x - \ln t)^{\alpha-1} L(t)|y - z|\frac{dt}{t} \\
\leq\ & \frac{\varepsilon(\ln x)^\gamma}{\Gamma(\alpha)}\widehat{C}\varphi(x) + \frac{1}{\Gamma(\alpha)}\int_1^x (\ln x)^\gamma (\ln x - \ln t)^{\alpha-1}(\ln t)^{-\gamma} L(t)\frac{dt}{t}\|y - z\|_{C_{\gamma,\ln}} \\
\leq\ & \frac{\varepsilon(\ln x)^\gamma}{\Gamma(\alpha)}\widehat{C}\varphi(x) + \frac{\widetilde{C}}{\Gamma(\alpha)}(\ln x)^{\gamma+\alpha-1}\|y - z\|_{C_{\gamma,\ln}}.
\end{aligned}$$

This yields that

$$\left(1-\frac{\widetilde{C}}{\Gamma(\alpha)}\right)\|y-z\|_{C_{\gamma,\ln}}\leq\frac{\varepsilon(\ln x)^{\gamma}}{\Gamma(\alpha)}\widehat{C}\varphi(x).$$

This implies that

$$|y(x)-z(x)|\leq\frac{\widehat{C}\varepsilon}{\omega'\Gamma(\alpha)}\varphi(x),\ x\in J.$$

The proof is completed. □

**Example 4.5.** *Let $\alpha=\frac{1}{2}$, $\lambda=-\frac{1}{2}$ and $\gamma=\frac{1}{2}$. Consider the fractional order differential equation*

$$_{H}D^{\frac{1}{2}}_{1^{+}}y(x)=-\frac{1}{2}y(x)+\frac{1}{lx^{3}}\sin^{2}y(x),\ x\in(1,\infty),l>0, \tag{8}$$

*and the inequality*

$$|_{H}D^{\frac{1}{2}}_{1^{+}}z(x)-\frac{1}{2}z(x)-\frac{1}{lx^{3}}\sin^{2}z(x)|\leq\varepsilon\varphi(x),\ x\in(1,\infty). \tag{9}$$

*Define $f(x,y(x))=\frac{1}{lx^{3}}\sin^{2}(x)$ and $L(x)=\frac{1}{lx^{3}}$, $x\in(1,\infty)$. Let $z\in C_{\gamma,\ln}([1,\infty),\mathbb{R})$ be a solution of inequality (9). There exists a function $h(x)=\frac{\varepsilon}{x}(\ln x)^{\alpha-1-\gamma}\in C_{\gamma,\ln}([1,\infty),\mathbb{R})$ such that $|h(x)|\leq\varepsilon(\ln x)^{\alpha-1-\gamma}:=\varphi(x)$, $x\in(1,\infty)$. Moreover, (5) via Lemma 2.6 reduces to*

$$\int_{1}^{x}(\ln x)^{\frac{1}{2}}(\ln x-\ln t)^{-\frac{1}{2}}(\ln t)^{-\frac{1}{2}}\frac{2}{lt^{3}}\frac{dt}{t}\leq\frac{2C}{l}\sqrt{3}(\ln x)^{-\frac{1}{2}},\ x\in(1,\infty),$$

*and (9) reduces to*

$$\int_{1}^{x}(\ln x)^{\gamma}(\ln x-\ln t)^{\alpha-1}\varepsilon(\ln x)^{\alpha-1-\gamma}\frac{dt}{t}\leq\hat{C}\varepsilon(\ln x)^{\alpha-1-\gamma}=\hat{C}\varepsilon(\ln x)^{-1},\ \hat{C}>0,\ x\in(1,\infty).$$

*From above, $(B_1)$, $(B_2)$ and $(B_3)$ hold. Now we choose $l=\frac{\Gamma(0.5)}{4\sqrt{3}C}$ and $\omega'=1-\frac{2C\sqrt{3}}{l}=\frac{1}{2}>0$, which implies that $(B_4)$ holds. According to Theorem 4.4, we have*

$$|z(x)-y(x)|\leq\frac{2\hat{C}\varepsilon}{\Gamma(0.5)}(\ln x)^{-1}.$$

*Thus, equation (8) is Ulam-Hyers-Rassias stable on $(1,\infty)$ with $c=\frac{2\hat{C}}{\Gamma(0.5)}$ and $\varphi(x)=\varepsilon(\ln x)^{-1}$, $x\in(1,\infty)$.*

*4.3. Final remarks*

Let $\lambda>0$. Assume that $(B_1)$ and $(B_2)$ are satisfied. It seems that we can not use the above approach to discuss Ulam-Hyers-Rassias stability of the equation (1) with $\lambda>0$ on $J=(e,\infty)$.

In fact, by Remark 3.4, one has

$$\begin{aligned}|(z(x)-y(x))(\ln x)^{\gamma}|&\leq\left(\varepsilon M(\alpha,\lambda)+\frac{\varepsilon\omega(\alpha,\alpha,\lambda)}{\lambda^{\frac{1}{\alpha}}}+\frac{\varepsilon x^{\frac{1}{\alpha}}}{\lambda^{\frac{1}{\alpha}}}\right)(\ln x)^{\gamma}\\&\quad+\left|\int_{1}^{x}(\ln x)^{\gamma}(\ln x-\ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x-\ln t)^{\alpha})L(t)|y(t)-z(t)|\frac{dt}{t}\right|\\&\leq\left(\varepsilon M(\alpha,\lambda)+\frac{\varepsilon\omega(\alpha,\alpha,\lambda)}{\lambda^{\frac{1}{\alpha}}}+\frac{\varepsilon x^{\frac{1}{\alpha}}}{\lambda^{\frac{1}{\alpha}}}\right)(\ln x)^{\gamma}+\Upsilon(x)\|y-z\|_{C_{\gamma,\ln}},\end{aligned} \tag{10}$$

where

$$\Upsilon(x) = (\ln x)^{\gamma}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^{\alpha})\left|\int_1^x (\ln x - \ln t)^{\alpha-1}L(t)(\ln t)^{-\gamma}\frac{dt}{t}\right|.$$

Set $L(t) = t^{-w},\ w > 0$. According to Lemma 2.6, we have

$$\begin{aligned}
\Upsilon(x) &= (\ln x)^{\gamma}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^{\alpha})\left|\int_1^x (\ln x - \ln t)^{\alpha-1}(\ln t)^{-\gamma}t^{-w}\frac{dt}{t}\right| \\
&= (\ln x)^{\gamma}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^{\alpha})\left|\int_1^x (\ln x - \ln t)^{\alpha-1}(\ln t)^{(1-\gamma)-1}t^{-w}\frac{dt}{t}\right| \\
&\leq Cw^{1-\gamma}\frac{\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^{\alpha})}{(\ln x)^{1-\alpha-\gamma}} \quad (C>0) \\
&\leq \frac{\lambda^{\frac{1-\alpha}{\alpha}}Cw^{1-\gamma}}{\alpha}x^{\lambda^{\frac{1}{\alpha}}+\gamma},
\end{aligned}$$

where we use the fact

$$\lim_{x\to\infty}\frac{\frac{1}{\alpha}\lambda^{\frac{1-\alpha}{\alpha}}\exp(\lambda^{\frac{1}{\alpha}}\ln x)}{(\ln x)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^{\alpha})} = 1.$$

Dividing $(\ln x)^{\gamma}$ by (10) we obtain

$$|(z(x)-y(x))| \leq \left(\varepsilon M(\alpha,\lambda) + \frac{\varepsilon\omega(\alpha,\alpha,\lambda)}{\lambda^{\frac{1}{\alpha}}} + \frac{\varepsilon x^{\frac{1}{\alpha}}}{\lambda^{\frac{1}{\alpha}}}\right) + \frac{\Upsilon(x)}{(\ln x)^{\gamma}}\|y-z\|_{C_{\gamma,\ln}}. \tag{11}$$

Obviously,

$$\lim_{x\to\infty}\frac{\Upsilon(x)}{(\ln x)^{\gamma}} = \frac{\lambda^{\frac{1-\alpha}{\alpha}}cw^{1-\gamma}}{\alpha}\lim_{x\to\infty}\frac{x^{\lambda^{\frac{1}{\alpha}}+\gamma}}{(\ln x)^{\gamma}} = \infty.$$

Thus, the term $\frac{\Upsilon(x)}{(\ln x)^{\gamma}}\|y-z\|_{C_{\gamma,\ln}}$ in (11) does not vanish. Therefore, the equation (1) with $\lambda > 0$ is not necessary Ulam-Hyers-Rassias stable on $J = (e,\infty)$.

## References


[1] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and applications of fractional differential equations, Elsevier Science B.V., 2006.

[2] D. Baleanu, J. A. T. Machado, A. C. J. Luo, Fractional dynamics and control, Springer, 2012.

[3] K. Diethelm, The analysis of fractional differential equations, Lecture Notes in Mathematics, 2010.

[4] R. Gorenflo, J. Loutchko, Y. Luchko, Computation of the Mittag-Leffler function $E_{\alpha,\beta}(z)$ and its derivative, Fract. Calc. Appl. Anal., 5(2002), 491-518. Correction: Fract. Calc. Appl. Anal., 6(2003), 111-112.

[5] I. Podlubny, Fractional differential equations, Academic Press, 1999.

[6] R. Hilfer, Application of fractional calculus in physics, World Scientific Publishing Company, Singapore, 2000.

[7] V. E. Tarasov, Fractional dynamics: Application of fractional calculus to dynamics of particles, fields and media, Springer, HEP, 2011.

[8] Y. Zhou, J. Wang, L. Zhang, Basic theory of fractional differential equations, World Scientific, Singapore, 2016.

[9] B. Ahmad, S. K. Ntouyas, A fully Hadamard type integral boundary value problem of a coupled system of fractional differential equations, Frac. Calc. Appl. Anal., 17(2014), 348-360.

[10] J. Wang, M. Fečkan, Y. Zhou, A survey on impulsive fractional differential equations, Fract. Calc. Appl. Anal., 19(2016), 806-831

[11] J. Wang, A. G. Ibrahim, M. Fečkan, Nonlocal impulsive fractional differential inclusions with fractional sectorial operators on banach spaces, Appl. Math. Comput., 257(2015), 103-118.

[12] J. Wang, Y. Zhang, On the concept and existence of solutions for fractional impulsive systems with Hadamard derivatives, Appl. Math. Lett., 39(2015), 85-90.

[13] J. Wang, X. Li, M. Fečkan, Y. Zhou, Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals via two kinds of convexity, Appl. Anal., 92(2013), 2241-2253.

[14] B. Ahmad, S. K. Ntouyas, Initial value problems of fractional order Hadamard-type functional differential equations, Electronic J. Differential Equations, 77(2015), 1-9.

[15] J. Wang, Y. Zhou, M. Medveď, Existence and stability of fractional differential equations with Hadamard derivative, Topol. Meth. Nonlinear Anal., 41(2013), 113-133.

[16] L. Cădariu, Stabilitatea Ulam-Hyers-Bourgin pentru ecuatii functionale, Ed. Univ. Vest Timişoara, Timişara, 2007.

[17] D. H. Hyers, G. Isac, Th. M. Rassias, Stability of functional equations in several variables, Birkhäuser, 1998.

[18] S.-M. Jung, Hyers-Ulam-Rassias stability of functional equations in mathematical analysis, Hadronic Press, Palm Harbor, 2001.

[19] J. Wang, Y. Zhou, M. Fečkan, Nonlinear impulsive problems for fractional differential equations and Ulam stability, Comp. Math. Appl., 64(2012) 3389-3405.

[20] J. Wang, M. Fečkan, A general class of impulsive evolution equations, Topol. Meth. Nonlinear Anal., 46(2015), 915-934.

[21] J. Wang, M. Fečkan, Y. Zhou, Ulam's type stability of impulsive ordinary differential equations, J. Math. Anal. Appl., 395(2012), 258-264.

[22] D. Popa, I. Raşa, Hyers-Ulam stability of some differential equations and differential operators, Handbook of Functional Equations, Springer, 2014, 301-322.

[23] J. Brzdęk, L. Cădariu, K. Ciepliński, Ulam's type stability and fixed points methods, J. Function Spaces, 2014(2014), Art. ID 829419, 16 pages.

[24] M. Li, J. Wang, Analysis of nonlinear Hadamard fractional differential equations via properties of Mittag-Leffler functions, J. Appl. Math. Comput., 51(2016), 487-508.

[25] M. D. Kassim, N. E. Tatar, Well-posedness and stability for a differential problem with Hilfer-Hadamard fractional derivative, Abst. Appl. Anal., 2013(2013), Art. ID 605029, 12 pages.

[26] R. Gorenflo, A. A. Kilbas, F. Mainardi, S. V. Rogosin, Mittag-Leffler functions, related topics and applications, Springer, 2014.

# LINEARLY STABLE PERIODIC SOLUTIONS FOR LAGRANGIAN EQUATION

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ABSTRACT. In this paper we study the existence and uniqueness of linearly stable periodic solutions for the Lagrangian equation. The proof is based on the eigenvalue theory combined with degree theory. Compared with those results in the literature, our conditions are weaker.

## 1. INTRODUCTION

This paper is devoted to the study of the existence and uniqueness of linear stablity of periodic solutions for the following nonlinear scalar Lagrangian equation

$$\ddot{x} + g(t, x) = 0, \tag{1.1}$$

where $g(t, x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a $T$-periodic function in $t$ and is semilinear in the following sense: There exist $T$-periodic functions $\phi, \Phi \in L^1(0, T)$ such that

$$\phi(t) \leq g_x(t, x) \leq \Phi(t),$$

uniformly in $t \in [0, T]$. We say that a $T$-periodic solution $\psi(t)$ of (1.1) is *linearly stable* if the linearized equation

$$\ddot{y} + (g_x(t, \psi(t))y = 0 \tag{1.2}$$

is stable. But it is not sufficient to guarantee that $\psi(t)$ is Lyapunov stable as (1.1) is a conservative system, Lyapunov stability of $\psi(t)$ cannot be determined by linearized equation (1.2) and involves higher order approximations of (1.1). Based on this idea, a practical method, now known as the third order approximation, has been developed by Ortega based on the Birkhoff normal forms and the Moser's twist theorem [17]. After there has been considerable progress on this topic. We refer the reader to [2, 3, 4, 5]. However, an "almost" necessary condition for $\psi(t)$ to be stable is that it is linearly stable. In this direction, it is worth mentioning the example found by Chu [3]. That is, the equilibrium $x(t) = 0$ of the motion of

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a pendulum with variable length and relativistic effects

$$\Big(\frac{x'}{\sqrt{1-x'^2}}\Big)' + l(t)\sin x = 0, \quad l(t) > 0, \quad l \in C(\mathbb{R}/T\mathbb{Z}),$$

is stable if its linearized equation

$$\ddot{x} + l(t)x = 0$$

is stable. In this paper it is shown how a topological invariant, the index of an oscillation, can be used to obtain linear stability results. Actually, it will be proved that the index characterizes linearly stable in certain case (see section 2). Our study is mainly motivated by [7], where a result is the following.

**Theorem 1.1.** *If there exists $a > 0$ and $\Phi \in L^1(0,T)$ such that*

$$a < g_x(t,x) \le \Phi(t), \tag{1.3}$$

*for all $x$ and a.e. $t \in [0,T]$, and $\|\Phi\|_p < K(2p^*)$ for some $p \in [1,+\infty]$, then (1.1) has a unique $T$-periodic solution which is linearly stable.*

It must be noticed that the strict positiveness assumption of $g_x(t,x)$ is crucial for this result since this implied the monotonicity of the nonlinearity and then a method of lower and upper coupled with the monotone iterative technique was used to get existence and linear stability. Unfortunately, in some interesting problems we find that the constant $g_x(t,x)$ changes sign and Theorem 1.1 cannot be applied. It should be pointed out that Ortega has presented the index characterized asymptotic stability in dissipative case. See the references [15, 16] and the surveys [18, 19]. In this paper, we try to obtain similar results for the conservative case following the ideas in [15, 16, 18, 19], see Theorem 2.3 below.

The purpose of the present paper is to extend Theorem 1.1 which hold when $g_x(t,x)$ changes sign, we prove the following theorems.

**Theorem 1.2.** *Suppose that $g(t,x) \in C^1(\mathbb{R}\times\mathbb{R})$ satisfies the following semilinearity condition: there exist $T$-periodic functions $\phi, \Phi \in L^1(0,T)$ such that*

$$\phi(t) \le g_x(t,x) \le \Phi(t), \tag{1.4}$$

*uniformly in $t$. Furthermore, assume*

$$\overline{\phi(t)} > 0 \text{ and } \underline{\lambda}_1(\Phi) > 0, \tag{1.5}$$

*here $\overline{\phi(t)}$ denotes the average of $\phi(t)$ over a period and $\underline{\lambda}_1$ is antiperiodic eigenvalue. Then (1.1) has a unique $T$-periodic solution which is linearly stable.*

Let us recall a lower bound for $\underline{\lambda}_1(\Phi)$ from [20]. Let us define the positive part of a function $\Phi$ as $\Phi^+ = \max\{\Phi, 0\}$. If the $L^p$ norm $\|\Phi^+\|_p$ satisfies

$$\|\Phi^+\|_p \le K(2p^*),\ p^* = p/p-1,$$

then (see (13) in [20])

$$\underline{\lambda}_1(\Phi) \ge \Big(\frac{\pi}{T}\Big)^2\Big(1 - \frac{\|\Phi^+\|_p}{K(2p^*)}\Big).$$

Here $K(q)$ is the best Sobolev constant in the following inequality:

$$C\|x\|_q^2 \le \|\dot{x}\|_2^2 \ \text{ for all } x \in H_0^1(0,T),$$

where $H_0^1(0,T)$ is a Sobolev space of all the $T$-periodic absolutely continuous functions $x$ such that $\int_0^T \dot{x}^2(t)dt < \infty$ with the norm

$$\|x\|_{1,T} = \Big( \int_0^T \dot{x}^2(t)dt + \int_0^T x^2(t)dt \Big)^{\frac{1}{2}}.$$

Explicitly,

$$K(q) = \begin{cases} \frac{2\pi}{qT^{1+2/q}} \Big( \frac{2}{2+q} \Big)^{1-2/q} \Big( \frac{\Gamma(\frac{1}{q})}{\Gamma(\frac{1}{2}+\frac{1}{q})} \Big)^2, & \text{if } 1 \le q < \infty, \\ \frac{4}{T}, & \text{if } q = \infty. \end{cases}$$

See Talenti [13]. Thus we have the following

**Corollary 1.3.** *Assume all conditions of Theorem 1.2 hold except (1.5) and assume*

$$\overline{\phi(t)} > 0 \text{ and } \|\Phi^+\|_p < K(2p^*), \quad 1 \le p \le +\infty.$$

*Then the same conclusion holds. In particular, when $p = +\infty$, we arrive at the usual criterion*

$$\|\Phi^+\|_\infty < K(2) = \frac{\pi^2}{T^2}.$$

## 2. Linear stability and index

2.1. **Hill's equation and eigenvalue theory.** To each function $a \in L^1(\mathbb{R}/T\mathbb{Z})$, we associate a linear equation

$$\ddot{x} + a(t)x = 0, \tag{2.1}$$

which is called Hill's equation and there are many studies about it. The book by Magnus and Winkler [11] is a classical reference.

Now we recall some standard notions in the theory of Hill's equations. Denote by $\Psi(t) = \phi_1(t) + i\phi_2(t)$ the complex-valued solution of (2.1) with the initial data: $\Psi(0) = 1$ and $\Psi'(0) = i$, where $\phi_1$ and $\phi_2$ are respectively the real and imaginary parts of $\Psi$. Let

$$M(t) = \begin{pmatrix} \phi_1(t) & \phi_2(t) \\ \dot{\phi}_1(t) & \dot{\phi}_2(t) \end{pmatrix}$$

be employed for the fundamental matrix solution of

$$\dot{X} = A(t)X, \ X(0) = I_2,$$

where the column vector function $X(t) = (x(t), x'(t))^T$, $I_2$ is the $2 \times 2$ identity matrix and $A(t)$ is the matrix function

$$A(t) = \begin{pmatrix} 0 & 1 \\ -a(t) & 0 \end{pmatrix}.$$

Liouville's theorem implies that the matrix solution $M(t)$ always satisfies

$$\det M(t) = 1.$$

This property motivates our interest in the symplectic group. The monodromy matrix associated with (2.1) is

$$M(T) = \begin{pmatrix} \phi_1(T) & \phi_2(T) \\ \dot{\phi}_1(T) & \dot{\phi}_2(T) \end{pmatrix}.$$

Then $M$ is symplectic, i.e., $\det M = 1$. The eigenvalues $\rho_i$, $i = 1, 2$, of $M$ are called the *Floquet multipliers* of (2.1). They satisfy $\rho_1 \cdot \rho_2 = 1$. We can classify (2.1)

into three types, according to the Floquet multipliers, as either *hyperbolic* when $|\rho_{1,2}| \neq 1$, or *elliptic* when $|\rho_{1,2}| = 1$ but $\rho_{1,2} \neq \pm 1$, or *parabolic* when $\rho_{1,2} = \pm 1$, respectively.

Next we introduce some notations on eigenvalues. Consider the eigenvalue problems

$$x'' + (\lambda + a(t))x = 0 \tag{2.2}$$

subject to the periodic boundary condition

$$x(0) - x(T) = x'(0) - x'(T) = 0, \tag{2.3}$$

or to the anti-periodic boundary condition

$$x(0) + x(T) = x'(0) + x'(T) = 0. \tag{2.4}$$

We use

$$\lambda_1^D(a) < \lambda_2^D(a) < \cdots < \lambda_n^D(a) \cdots$$

to denote all eigenvalues of (2.2) with the Dirichlet boundary condition $(D)$ :

$$x(0) = x(T) = 0. \tag{2.5}$$

The following are standard results for eigenvalue theory. See, e.g. Reference [11]. A partial generalization of these results to the one-dimensional $p$-Laplacian with periodic potentials is given in Reference [14].

**Theorem 2.1.** *There exist two sequences $\{\underline{\lambda}_n(a) : n \in \mathbb{N}\}$ and $\{\overline{\lambda}_n(a) : n \in \mathbb{Z}^+\}$ of the reals such that*

*$(P_1)$ they have the following order:*

$$-\infty < \overline{\lambda}_0(a) < \underline{\lambda}_1(a) \leq \overline{\lambda}_1(a) < \cdots < \underline{\lambda}_n(a) \leq \overline{\lambda}_n(a) < \cdots$$

*and $\underline{\lambda}_n(a) \to +\infty$, $\overline{\lambda}_n(a) \to +\infty$ as $n \to \infty$.*

*$(P_2)$ $\lambda$ is an eigenvalue of (2.2)-(2.3) if and only if $\lambda = \underline{\lambda}_n(a)$ or $\overline{\lambda}_n(a)$ for some even integer $n$; $\lambda$ is an eigenvalue of (2.2)-(2.4) if and only if $\lambda = \underline{\lambda}_n(a)$ or $\overline{\lambda}_n(a)$ for some odd integer $n$.*

*$(P_3)$ (Continuity) $\lambda_n^D(a), \underline{\lambda}_n(a)$, and $\overline{\lambda}_n(a)$ are continuous functions of $q$ with respect to the $L^1$-metric on $q$'s: $d(a_1, a_2) = \int_0^T |a_1(t) - a_2(t)|dt$.*

*$(P_4)$ the eigenvalues $\underline{\lambda}_n(a)$ and $\overline{\lambda}_n(a)$ can be recovered from the Dirichlet eigenvalues in the following way: for any $n \in \mathbb{N}$,*

$$\underline{\lambda}_n(a) = \min\{\lambda_n^D(a_{t_0}) : t_0 \in \mathbb{R}\},\ \overline{\lambda}_n(a) = \max\{\lambda_n^D(a_{t_0}) : t_0 \in \mathbb{R}\},$$

*here $a_{t_0}(t)$ denotes the translation of $a(t) : a_{t_0}(t) \equiv a(t + t_0)$.*

*$(P_5)$ (Comparison) the comparison results hold for all of these eigenvalues. If $a_1 \geq a_2$ then*

$$\underline{\lambda}_n(a_1) \leq \underline{\lambda}_n(a_2),\ \overline{\lambda}_n(a_1) \leq \overline{\lambda}_n(a_1),\ \lambda_n^D(q_1) \leq \lambda_n^D(q_2), \tag{2.6}$$

*for any $n \in \mathbb{N}$. If $a_1(t) \geq a_2(t)$ for all $t$, and $a_1(t) > a_2(t)$ for $t$ in a subset of positive measure, then all of the inequalities in (2.6) are strict.*

*$(P_6)$ (Nodal structure) The eigenfunction of $\overline{\lambda}_0(a)$ do not vanish everywhere. For $n \in \mathbb{N}$, the eigenfunctions of $\underline{\lambda}_n(a)$ or $\overline{\lambda}_n(a)$ have exactly $n-1$ zeros in the intervals of the form $(t_0, t_0 + T)$.*

2.2. **Definition of the index via the Poincaré map.** In this subsection we assume uniqueness for the initial value problem associated to (1.1). Given $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, let $x(t;\xi)$ be the solution of (1.1) satisfying

$$x(0) = \xi_1, \quad x'(0) = \xi_2.$$

The Poincaré map is defined as the mapping

$$P_T : D_T \subset \mathbb{R}^2 \to \mathbb{R}^2, \qquad P_T(\xi) = (x(T;\xi), \dot{x}(T;\xi)),$$

where $D_T = \{\xi \in \mathbb{R}^2 : x(t;\xi) \text{ is defined in } [0,T]\}$.

The standard theory of the Cauchy problem says that $D_T$ is open in $\mathbb{R}^2$ and $P_T$ is a homeomorphism between $D_T$ and $P_T(D_T)$. In addition, the fixed points of $P_T$ correspond to the initial conditions of the $T$-periodic solutions and the search of $T$-periodic solutions is reduced to the study of the equation in $\mathbb{R}^2$,

$$\xi = P_T(\xi).$$

Let $x$ be a $T$-periodic solution of 1.1 and $\xi_0 = (x(0), x'(0))$. The solution $x$ is said to be isolated (periodic $T$) if $\xi_0$ is an isolated fixed point of $P_T$. In such case the index of $x$ is defined in terms of the following formula

$$\mathrm{ind}_T(x) = i[P_T, \xi_0],$$

where $i$ refers to the definition of the local fixed point index in the plane employed in [1]. For more information on the index of periodic solutions see [8] and [12].

2.3. **Connection between linear stability and index.** In this subsection we proof a few lemmas that are crucial for the proofs of the main results. First we need the following.

**Definition 2.2.** Given a $T$-periodic solution $x(t)$ of (1.1). It will be said that $x(t)$ is *non-degenerate* of periodic $T$ if the variational equation is

$$\ddot{y} + g_x(t,x)y = 0 \tag{2.7}$$

has no periodic solutions different from zero of periodic $T$.

**Theorem 2.3.** *Assume that $x$ is a nondegenerate $T$-periodic solution of (1.1) such that the inequality below holds*

$$\underline{\lambda}_1(g_x(t,x)) \geq 0, \tag{2.8}$$

*for $t \in \mathbb{R}$. Then $x$ is linearly stable if and only if* $\mathrm{ind}_T(x)$*=1.*

**Remark 2.4.** Notice that a nondegenerate solution is always isolated and it is assumed that $x$ is a nondegenerate in order to employ linearization techniques. We do not know if Theorem 2.3 is still valid when nondegenerate is replaced by degenerate. Because the computation of the index in the degenerate case

$$\rho_1 = \rho_2 = 1$$

is more delicate the previous technique does not work and the index of $x$ depends not only on (2.7) but also on the nonlinear terms of the Taylor expansion of $g$. Some methods about computation in the degenerate case can be found in [**?**] or [9] for more details.

The crucial step in proof of Theorem 1.2 is the following observation on the Hill equation (2.1).

**Lemma 2.5.** *Assume that*

$$\underline{\lambda}_1(a) > 0. \tag{2.9}$$

*Then problem (2.1) does not admit any negative Floquet multipliers. In particular, (2.1) does not admit any nontrivial subharmonic periodic solution of order 2.*

**Proof.** Suppose that there is a nontrivial solution of (2.1) with a negative Floquet multiplier, i.e. $x(t+T) = \rho x(t),\ t \in \mathbb{R}$ for some $\rho < 0$. Hence, there exists $t_0 \in [0, T]$ with $x(t_0) = x(t_0 + T) = 0$. Thus the corresponding $x$ is a nontrivial solution of the following Dirichlet boundary value problem

$$\begin{cases} \ddot{x}(t) + a(t)x = 0, \\ x(t_0) = x(T + t_0) = 0. \end{cases}$$

That is, $x$ is an eigenfunction associated with eigenvalue $\lambda_k^D(a) = 0$ for some $k \geq 1$ of

$$\begin{cases} \ddot{x}(t) + (\lambda + a(t))x = 0, \\ x(t_0) = x(T + t_0) = 0, \end{cases}$$

and hence $\underline{\lambda}_1(a) \leq \lambda_1^D(a) \leq \lambda_k^D(a) = 0$, contradicting (2.9).

**Proof of Theorem 2.3.** When the inequality in (2.8) is not strict it is elementary to show that $x$ is linearly stable and $\mathrm{ind}_T(x) = 1$. Therefore it will be assumed that the inequality in (2.8) is strict, at least on a set of positive measure. Denote by $\rho_1, \rho_2$ ($|\rho_1| \geq |\rho_2|$) the Floquet multipliers of (2.1). By Lemma 2.5 the multipliers are either conjugate complex or real and positive. In the elliptic case,

$$\rho_1 = \overline{\rho_2}$$

if and only if

$$\mathrm{ind}_T(x) = \mathrm{sign}\{\det(I_2 - M(T))\} = \mathrm{sign}\{|1 - \rho_1|^2\} = 1.$$

In the hyperbolic case,

$$0 < \rho_1 < 1 < \rho_2$$

if and only if

$$\mathrm{ind}_T(x) = \mathrm{sign}\{\det(I_2 - M(T))\} = \mathrm{sign}\{(1 - \rho_1)(1 - \rho_2)\} = -1.$$

The parabolic case is excluded because $x$ is nondegenerate 1 cannot be a Floquet multipler.

The conclusion now follows from the well-known principle of stability for Hill equation (see Theorem 7.2 in [6]) that periodic system (2.1) is stable in the sense of Lyapunov if and only if (2.1) is elliptic, or is parabolic ($\rho_1 = \rho_2 = \pm 1$) with further property that all solutions of (2.1) satisfy $x(t + T) = x(t)$, the $T$-periodic solutions in case $\rho_1 = \rho_2 = 1$, or $x(t + T) = -x(t)$, the $T$-anti-periodic solutions in case $\rho_1 = \rho_2 = -1$.

## 3. Proof of Theorem 1.2

The proof of existence is based on the following two lemmas.

**Lemma 3.1.** *Assume that*

$$\overline{a(t)} > 0 \text{ and } \underline{\lambda}_1(a) > 0. \tag{3.1}$$

*Then Hill equation (2.1) has only the trivial $T$-periodic solution.*

**Proof.** Suppose on the contrary that (2.1) admits a nontrivial $T$-periodic solution $x(t)$. We claim that $x(t)$ vanishes at some $t_0 \in [0,T]$. If not, then $x(t) \neq 0$ for all $t$ in $\mathbb{R}$. By the periodic boundary conditions, we have $\dot{x}(T) = \dot{x}(0)$ and $\frac{\dot{x}(T)}{x(T)} = \frac{\dot{x}(0)}{x(0)}$. Dividing (2.1) by $x(t)$ and integrating by part gives that

$$\int_0^T \frac{\dot{x}(t)^2}{x(t)^2} + \int_0^T a(t)dt = 0,$$

which contradicts the hypothesis $\overline{a(t)} > 0$. So $x(t)$ has a zero in [0,T]. We may assume that $x(0) = 0$ so that $x(0) = x(T) = 0$. Thus the corresponding $x$ is a nontrivial solution of the Dirichlet boundary value problem (2.1)-(2.5) That is, $x$ is an eigenfunction associated with eigenvalue $\lambda_k^D(a) = 0$ for some $k \geq 1$ of (2.2)-(2.5), and hence $\underline{\lambda}_1(a) \leq \lambda_1^D(a) \leq \lambda_k^D(a) = 0$, contradicting (3.1).

**Lemma 3.2.** *Under the conditions of Lemma 3.1, the Hill equation (2.1) is stable.*

**Proof.** The proof will be completed using Theorem 2.3. To this end, we compute the local index and consider following parametric equation

$$L_\lambda = \ddot{x} + [\lambda a(t) + (1-\lambda)a_0]x = 0, \quad \lambda \in [0,1],$$

where $0 < a_0 < (\pi/T)^2$.

Let $a_\lambda = \lambda a(t) + (1-\lambda)a_0$. Since

$$\overline{a_\lambda} = \lambda \overline{a(t)} + (1-\lambda)a_0 > 0,$$

and

$$\underline{\lambda}_1(a_\lambda) \geq \lambda\ \underline{\lambda}_1(a) + (1-\lambda)\underline{\lambda}_1(a_0) > 0,$$

it follows from Lemma 3.1 that $L_\lambda x = 0$ does not admit a nontrivial $T$-periodic solution. Let $B_\epsilon$ be the $\epsilon$-ball of 0, then $L_\lambda x = 0$ has no $T$-periodic solution on $\partial B_\epsilon$ for $\lambda \in [0,1]$. By the homotopy invariance properties of the topological degree, we have that

$$\begin{aligned} \text{ind}(L_1, 0) &= \deg(L_1, B_\epsilon, 0) = \deg(L_\lambda, B_\epsilon, 0) \\ &= \deg(L_0, B_\epsilon, 0) = \text{sgn} \begin{vmatrix} 0 & 1 \\ -a_0 & 0 \end{vmatrix} = 1. \end{aligned}$$

The conclusion follows from Theorem 2.3.

**Proof of Theorem 1.2** In order to show that the conditions are sufficient, we divide the proof into two steps.

*Step 1: Uniqueness.* Suppose that $x_1(t)$ and $x_2(t)$ are two $T$-periodic solutions of (1.1). Then

$$[x_1(t) - x_2(t)]'' + [g(t, x_1(t)) - g(t, x_2(t))] = 0. \tag{3.2}$$

Setting $\widetilde{x}(t) = x_1(t) - x_2(t)$, we obtain, from (3.2), that

$$\widetilde{x}''(t) + \beta(t)\widetilde{x}(t) = 0, \tag{3.3}$$

where $\beta(t) = \frac{g(t,x_1) - g(t,x_2)}{x_1 - x_2}$. It follows from Lemma 3.1 that $\widetilde{x}(t) \equiv 0$, which implies that $x_1(t) \equiv x_2(t)$ for all $t \in \mathbb{R}$.

*Step 2: Existence and linearly stable.* Without loss of generality, we may assume that $g(t,0) = 0$, for otherwise we can reduce both sides of Eq. (1.1) by $g(t,0)$. A natural choice for the parametrized equation in applying homotopy invariance property is to take $H$ defined by

$$H_\lambda(x) = \ddot{x}(t) + \lambda g(t,x) + (1-\lambda)\Phi(t)x = 0, \tag{3.4}$$

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in which $\Phi(t)$ is as in Theorem 1.2.

We claim that there is $R > 0$ such that equation (3.4) has no solution on $\partial B_R$ in $L^\infty[0,T]$ for all $\lambda \in [0,1]$. Suppose the assertion is not true. Let $x_n = x_n(t)$ be a sequence of $T$-periodic solutions such that $\|x_n\| \to \infty$ and $\lambda_n \in [0,1]$ be the corresponding sequence. Let $z_n(t) = \frac{x_n(t)}{\|x_n\|}$. First, dividing (3.4) by $\|x_n\|$, then multiplying by $\varphi(t) \in C_T^2$ and finally integrating by parts we have that

$$\int_0^T \Big\{ z_n \ddot{\varphi} + \Big[ \lambda_n g(t, x_n) + (1-\lambda_n)\Phi(t)x_n \Big] \cdot \varphi / \|x_n\| \Big\} dt = 0.$$

The conditions of Theorem 1.2 imply that $\{[\lambda_n g(t,x_n) + (1-\lambda_n)\Phi(t)x_n]/\|x_n\|\}$ is bounded and hence is pre-compact in weak star topology in $L^1[0,T]$. Thus there is a subsequence such that $g(t,x_n)/x_n \rightharpoonup \alpha(t)$ and $\lambda_n \to \lambda$. Taking the limit as $n \to \infty$, one obtains that $z_n \to z$,

$$\int_0^T (z\ddot{\varphi} + z\omega(t)\varphi) dt = 0,$$

where $\omega(t) = \lambda\alpha(t) + (1-\lambda)\Phi(t)$ satisfying the conditions of Lemma 3.1. It follows from Lemma 3.1 that $z(t) \equiv 0$, which contradicts $\|z(t)\| = 1$. This shows the boundedness of the periodic solutions of (3.4). By the standard argument one can verify that the $C^1$-norm is bounded independently of $\lambda$. Next, by applying the homotopy invariance property, we have that

$$\deg(H_1, B_R, 0) = \deg(H_0, B_R, 0).$$

From the same reasonings of Lemma 3.2 one proves that $\deg(H_0, B_R, 0) = 1$. This completes the existence of $T$-periodic solution $x$, and linear stability can be obtained by Lemma 3.2.

## 4. Final remark

Theorem 2.3 can be applied to other kinds of problems. In particular, to the piecewise linear equation

$$\ddot{x} + \mu(t)x^+ - \nu(t)x^- = 0, \tag{4.1}$$

where $x^+ = \max\{x, 0\}, x^- = \max\{-x, 0\}, \mu, \nu \in L^1(0,T)$. The equation (4.1) is very popular since a series of works of Lazer and McKenna [10] as a simple mathematical model for vertical oscillations of a long-span suspension bridge. The following result can be proved in a way similar to Theorem 1.2

*Equation (4.1) has a unique $T$-periodic solution which is linearly stable if $\overline{\mu(t)} > 0, \overline{\nu(t)} > 0$ and*

$$\| \max_{t\in\mathbb{R}} \{\mu(t), \nu(t)\} \|_p < K(2p^*), \quad 1 \le p \le +\infty.$$

## References

1. S. Chow, J.K. Hale, Methods of Bifurcation Theory, Spring-Verlag, 1982.
2. J. Chu, M. Zhang, Rotation numbers and Lyapunov stability of elliptic periodic solutions, Discrete Contin. Dyn. Syst. 21 (2008) 1071-1094.
3. J. Chu, The stability of the equilibrium of a nonlinear planar system and application to the relativistic oscillator, J. Differential Equations 247 (2009) 530-542.
4. J. Chu, M. Li, Twist periodic solutions of second order singular differential equations, J. Math. Anal. Appl. 355 (2009) 830-838.

5. J. Chu, J. Lei, M. Zhang, Lyapunov stability for conservative systems with lower degree of freedom, Discrete Contin. Dyn. Syst. 16 (2011) 423-443.
6. J.K. Hale, Ordinary Differential Equations, Second ed., Krieger, New York, 1980.
7. P.J. Torres, M. Zhang, A monotone iterative scheme for a nonlinear second order equation based on a generalized anti-maximum principle, Math. Nachr. 251 (2003), 101-107.
8. M.A. Krasnosel'skii, The operator of translation along the trayectories of differential equations, Trans. Math. Monograph 19. American Mathematical Society, Providence, 1968.
9. M.A. Krasnosel'skii, A.I. Perov, A.I. Povolotskiy, and P.P. Zabreiko, Plane Vector Fields, Academic Press, 1966.
10. A.C. Lazer, P.J. McKenna, Large-amplitude periodic oscillations in suspension bridges: Some new connection with nonlinear analysis, SIAM Rev. 32 (1990) 537-578.
11. W. Magnus, S. Winkler, Hill's Equations, Wiley, New York, 1966.
12. F. Nakajima, Index theorems and bifurcations in Duffing's equations, In "Recent Topics in Nonlinear PDE II." edited by Masuda and Mimura, 133-162. North Holland, 1985.
13. G. Talenti, Best constant in Sobolev inequality, Ann. Mat. Pura Appl. 110 (1976) 353-372.
14. M. Zhang, The rotation number approach to eigenvalues of the one-dimensional $p$-Laplacian with periodic potentials, J. London Math. Soc. 64 (2001) 125-143.
15. R. Ortega, Stability and index of periodic solutions of an equation of Duffing equation, Boll. Un. Mat. Ital. 3-B (1989) 533-546.
16. R. Ortega, Topological degree and stability of periodic solutions for certain differential equations, J. London Math. Soc. 42 (1990) 505-516.
17. R. Ortega, The twist coefficient of periodic solutions of a time-dependent Newton's equation, J. Dynam. Differential Equations, 4 (1992) 651-665.
18. R. Ortega, A criterion for asymptotic stability based on topological degree, in Proceedings of the First World Congress of Nonlinear Analysis, Tampa, 1992.
19. R. Ortega, Some applications of the topological degree to stability theory, in Topological Methods in Differential Equations and Inclusions (Montreal, PQ, 1994), A. Granas and M. Frigon (Eds.), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. Dordrecht: Kluwer, 472 (1995) 377-409.
20. M. Zhang, W. Li, A Lyapunov-type stability criterion using $L^\alpha$ norms, Proc. Amer. Math. Soc. 130 (2002) 3325-3333.

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# On a system of three max-type nonlinear difference equations

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**Abstract:** The primary focus of this paper is to investigate the boundedness and asymptotic behavior of the following symmetric system of max-type difference equations

$$x_{n+1}=\max\{c,\frac{y_n^p}{z_{n-1}^q}\}, y_{n+1}=\max\{c,\frac{z_n^p}{x_{n-1}^q}\}, z_{n+1}=\max\{c,\frac{x_n^p}{y_{n-1}^q}\}, \quad n\in\mathbb{N}_0,$$

where the parameters $c, p, q \in (0,\infty)$ and the initial conditions $x_{-1}, x_0, y_{-1}, y_0, z_{-1}, z_0$ are arbitrary positive real numbers. Our main results considerably improve results appearing in the literature (see, Stević, (2014) [29]).

**Keywords:** max-type system, difference equations, boundedness, global attractivity.

## 1. Introduction

In last few decades there has been a great interest in studying nonlinear difference equations and systems for developing some new techniques which can be used in investigating the models describing real life situations in biology, control theory, economics, etc. (see, e.g., [1-15] and the references therein). Recently, the so-called max-type difference equation has attracted more and more attention. However, the maxima operator is not a smooth function in n-dimensional real vector space so that the techniques which use

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derivatives could be of almost no use, so the study of max-type systems of difference equations become more difficult. Some studies of these difference equations have been presented in [16-25].

Paper [26] is one of the first such papers on max-type difference equations. It studies positive solutions of the difference equation

$$x_{n+1}=\max\{a,\frac{x_n^p}{x_{n-1}^p}\},\quad n\in\mathbb{N}_0, \tag{1.1}$$

where initial values $x_{-1}$, $x_0$, and parameters $a$ and $p$ are positive numbers.

In [27], Stevo Stević studied the boundedness character of positive solutions to the following max-type difference equation,

$$x_n=\max\{A,\frac{x_{n-1}^p}{x_{n-k}^r}\},\quad n\in\mathbb{N}_0, \tag{1.2}$$

where $k\in N\setminus\{1\}$, the parameters $A$ and $r$ are positive and $p$ is a nonnegative real number.

As an extension of (1.2), Stevo Stević studied the boundedness character and global attractivity of positive solutions of the following symmetric system of max-type difference equation

$$x_{n+1}=\max\{c,\frac{y_n^p}{x_{n-1}^p}\},\quad y_{n+1}=\max\{c,\frac{x_n^p}{y_{n-1}^p}\},\quad n\in\mathbb{N}_0, \tag{1.3}$$

where $c,p\in(0,+\infty)$ (see [28]).

Above results motivated Stevo Stević to continuously investigate the behavior of positive solutions of the following max-type system of differences

$$x_{n+1}=\max\{c,\frac{y_n^p}{z_{n-1}^p}\},y_{n+1}=\max\{c,\frac{z_n^p}{x_{n-1}^p}\},z_{n+1}=\max\{c,\frac{x_n^p}{y_{n-1}^p}\},\quad n\in\mathbb{N}_0, \tag{1.4}$$

where the parameters $c$ and $p$ are positive real numbers. It is proved that system (1.4) is permanent when $p\in(0,4)$ and so forth (see [29]).

Motivated by works [26-29], the primary focus of this paper is to investigate the boundedness character and global attractivity of the following max-type difference equations

$$x_{n+1}=\max\{c,\frac{y_n^p}{z_{n-1}^q}\},y_{n+1}=\max\{c,\frac{z_n^p}{x_{n-1}^q}\},z_{n+1}=\max\{c,\frac{x_n^p}{y_{n-1}^q}\},\quad n\in\mathbb{N}_0, \tag{1.5}$$

where $c, p, q \in (0,+\infty)$ . It is obvious that the paper can be considered as a continuation of studying special cases of the next systems of difference equations

$$x_{n+1} = \max\{A_n, \frac{y_{n-m}^p}{z_{n-k}^q}\}, y_{n+1} = \max\{A_n, \frac{z_{n-m}^p}{x_{n-k}^q}\}, z_{n+1} = \max\{A_n, \frac{x_{n-m}^p}{y_{n-k}^q}\}, \quad n \in \mathbb{N}_0,$$

where $m, k \in N$ , $p, q \in (0,+\infty)$ , and $\left(A_n\right)_{n\in N_0}$ is a sequence of positive numbers. For more related papers in this research area, see, for example, [30-33] and the references therein.

The rest of the paper is organized as follows. In Section 2, we will focus our attention on the buondedness character of solutions of system (1.5) by developing new iterative method and inequality technique. In Section 3, we will investigate the asymptotic behavior of solutions of system (1.5). Then we show an example and carry out numerical simulations in Section 4, from which it can be seen that all simulations agree with the theoretical results. We finally conclude our paper in Section 5.

## 2. Boundedness character of solutions

This section is devoted to analyzing the boundedness of the positive solutions to the max-type difference systems (1.5).

**Theorem 2.1.** Assume that $f(\lambda) = \lambda^2 - p\lambda + q$ and (a) there is $\lambda_1 > 1$ such that $f(\lambda_1) = 0$, or (b) there is $\lambda_1 = \lambda_2 = 1$ such that $f(\lambda_1) = f(\lambda_2) = 0$ . Then the system (1.5) has positive unbounded solutions.

**Proof.** Obviously, from (1.5), we can easily see that

$$x_{n+1} \geq \frac{y_n^p}{z_{n-1}^q}, \quad y_{n+1} \geq \frac{z_n^p}{x_{n-1}^q}, \quad z_{n+1} \geq \frac{x_n^p}{y_{n-1}^q}. \tag{2.1}$$

By taking logarithm in (2.1), for any $n \in \mathbb{N}_0$ , we obtain

$$\ln x_{n+1} \geq p \ln y_n - q \ln z_{n-1}, \quad \ln y_{n+1} \geq p \ln z_n - q \ln x_{n-1}, \quad \ln z_{n+1} \geq p \ln x_n - q \ln y_{n-1}. \tag{2.2}$$

Moreover, it follows that

$$\ln x_{n+1} y_{n+1} z_{n+1} \geq p \ln x_n y_n z_n - q \ln x_{n-1} y_{n-1} z_{n-1}. \tag{2.3}$$

Let $v_n = \ln x_n y_n z_n$ , where $n \geq -1$, then inequality (2.3) becomes

$$v_{n+1} \geq p v_n - q v_{n-1}, \quad n \in \mathbb{N}_0. \tag{2.4}$$

By hypothesis (a), we have that $f(\lambda_1) = 0$ and $\lambda_1 > 1$.

Let

$$f_1(\lambda)=\frac{f(\lambda)}{\lambda-\lambda_1}=\lambda+a\,, \tag{2.5}$$

then it follows that

$$f(\lambda)=(\lambda+a)(\lambda-\lambda_1)\,. \tag{2.6}$$

Thus, we can obtain $p=\lambda_1-a$ and $\mathrm{q}=-a\lambda_1$.

Set

$$u_n=v_n+av_{n\text{-}1},\quad n\in\mathbb{N}_0\,. \tag{2.7}$$

Then inequation (2.4) can be written in the following form

$$\begin{aligned} v_{n+1}-pv_n+qv_{n-1}&=v_{n+1}-(\lambda_1-a)v_n-a\lambda_1v_{n-1}\\ &=v_{n+1}+av_n-\lambda_1(v_n+av_{n-1})\\ &=u_{n+1}-\lambda_1u_n\\ &\geq 0. \end{aligned} \tag{2.8}$$

That is

$$u_{n+1}\geq\lambda_1u_n\,. \tag{2.9}$$

Let $v_{-1}, v_0$ be chosen such that

$$v_0\geq|\,a\,||\,v_{-1}\,|\,. \tag{2.10}$$

This, along with (2.9), yields to

$$u_{n+1}\geq\lambda_1^nu_0,\ \text{and}\ \ u_0>0\,. \tag{2.11}$$

Letting $n\to\infty$ in (2.11), from assumption (a) $\lambda_1>1$ and $u_0>0$, it follows that

$$u_n=v_n+av_{n\text{-}1}\to+\infty\ \ \text{as}\ \ n\to+\infty\,. \tag{2.12}$$

Hence $\{v_n\}_{n\geq-1}$ is unbounded. As $v_n=\ln x_ny_nz_n$, it follows that

$$x_ny_nz_n\to\infty\ \ \text{as}\ \ n\to\infty\,, \tag{2.13}$$

which along with $\sqrt{x_n^2+y_n^2+z_n^2}\geq\sqrt{3}\sqrt[3]{x_ny_nz_n}$ implies

$$\sqrt{x_n^2+y_n^2+z_n^2}\to+\infty\,, \tag{2.14}$$

from which it follows that the sequence $\{(x_n,y_n,z_n)\}_{n\geq-1}$ is unbounded.

By hypothesis (b), we have $p=2, q=1$. Then from (2.1) we get

$$\frac{x_{n+1}}{y_n} \geq \frac{y_n}{z_{n-1}}, \quad \frac{y_{n+1}}{z_n} \geq \frac{z_n}{x_{n-1}}, \quad \frac{z_{n+1}}{x_n} \geq \frac{x_n}{y_{n-1}}. \tag{2.15}$$

Moreover, one has

$$\frac{x_{n+1}y_{n+1}y_{n+1}}{x_n y_n z_n} \geq \frac{x_n y_n z_n}{x_{n-1}y_{n-1}z_{n-1}} \geq \cdots \geq \frac{x_0 y_0 z_0}{x_{-1}y_{-1}z_{-1}}, \ n \in N_0. \tag{2.16}$$

and consequently

$$x_n y_n z_n \geq (\frac{y_0 x_0 z_0}{x_{-1}y_{-1}z_{-1}})^n x_0 y_0 z_0, \ n \in N_0. \tag{2.17}$$

If we choose the initial conditions $x_{-1}, y_{-1}, z_{-1}, x_0, y_0, z_0$ such that $x_0 y_0 z_0 > x_{-1}y_{-1}z_{-1} > 0$ then we obtain (2.13) and consequently (2.14), which implies that the sequence $\{(x_n, y_n, z_n)\}_{n\geq -1}$ is unbounded, and then the proof of Theorem 2.1 is completed.

Next, we study the different cases concerning with the boundedness of positive solutions to the systems (1.5).

**Theorem 2.2.** If $c>0$, $p>0$ and $p^2<4q$, then the solutions to system (1.5) are bounded.

**Proof.** Assume that $(x_n, y_n, z_n)_{n\geq -1}$ is a positive solution to systems (1.5). Then the following estimate obviously holds

$$\min\{x_n, y_n, z_n\} \geq c, \quad n \in \mathbb{N}_0. \tag{2.18}$$

Due to the symmetry among $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$, as long as we prove the boundedness of $\{x_n\}$, other sequences $\{y_n\}$ and $\{z_n\}$ can be proved as well.

From systems (1.5), it follows that

$$x_{n+1} = \max\{c, \frac{y_n^p}{z_{n-1}^q}\} = \max\{c, \frac{c^p}{z_{n-1}^q}, \frac{z_{n-1}^{p^2-q}}{x_{n-2}^{pq}}\}, n \in \mathbb{N}_0. \tag{2.19}$$

**Case1.** When $p^2 \leq q$, we get

$$x_{n+1} \leq \max\{c, \frac{1}{c^{q-p}}, \frac{1}{c^{pq-p^2+q}}\}. \tag{2.20}$$

Thus, the sequence $\{x_n\}_{n\geq -1}$ is bounded.

**Case2.** When $p^2 > q$, let sequence $\{a_l\}_{l\geq 0}$ be defined as follows

$$a_{l+1} = q/(p-a_l), \quad a_0 = 0, \quad l \in N_0 \cdot \tag{2.21}$$

From (1.5) and (2.21), we have

$$\begin{aligned}
x_{n+1} &= \max\{c, \frac{y_n^p}{z_{n-1}^q}\} = \max\{c, \frac{c^p}{z_{n-1}^q}, \frac{z_{n-1}^{p^2-q}}{x_{n-2}^{pq}}\} \\
&= \max\{c, (\frac{c}{z_{n-1}^{q/p}})^p, (\frac{z_{n-1}}{x_{n-2}^{q/(p-q/p)}})^{(p-q/p)p}\} \\
&= \max\{c, (\frac{c}{z_{n-1}^{q/p}})^p, (\frac{c}{x_{n-2}^{q/(p-q/p)}})^{(p-q/p)p}, (\frac{x_{n-2}^{p-q/(p-q/p)}}{y_{n-3}^q})^{(p-q/p)p}\} \\
&= \cdots\cdots \\
&= \max\{c, (\frac{c}{z_{n-1}^{q/p}}, (\frac{c}{x_{n-2}^{q/(p-q/p)}}, \cdots, (\frac{x_{n-(3k-1)}^{p-a_{3k-1}}}{y_{n-3k}^q})^{p-a_{3k-2}}, \cdots)^{(p-q/p)})^p\} \\
&= \max\{c, (\frac{c}{z_{n-1}^{q/p}}, (\frac{c}{x_{n-2}^{q/(p-q/p)}}, \cdots, (\frac{c^{p-a_{3k-1}}}{y_{n-3k}^q}, \frac{y_{n-3k}^{p(p-a_{3k-1})-q}}{z_{n-(3k+1)}^q})^{p-a_{3k-2}}, \cdots)^{(p-q/p)})^p\} \\
&= \max\{c, (\frac{c}{z_{n-1}^{q/p}}, (\frac{c}{x_{n-2}^{q/(p-q/p)}}, \cdots, (\frac{y_{n-3k}^{p-a_{3k}}}{z_{n-(3k+1)}^q})^{p-a_{3k-1}}, \cdots)^{(p-q/p)})^p\} \\
&= \max\{c, (\frac{c}{z_{n-1}^{q/p}}, (\frac{c}{x_{n-2}^{q/(p-q/p)}}, \cdots, (\frac{z_{n-(3k+1)}^{p-a_{3k+1}}}{x_{n-(3k+2)}^q})^{p-a_{3k}}, \cdots)^{(p-q/p)})^p\}.
\end{aligned} \tag{2.22}$$

From the monotonicity of $g(x) = q/(p-x)$ on the interval $(0, p)$ along with the fact $0 = a_0 < a_1 = q/p$, it follows that the sequence $\{a_l\}$ is increasing as far as $a_l \le p$ for every $l \in \mathrm{N}_0$. Hence, we have $\lim_{l\to+\infty} a_l = x^*$, $x^* \in (0, p]$ and $x^*$ is the solution of the following equation

$$f(x) = x(p-x) - q = 0\,. \tag{2.23}$$

However the equation (2.23) has no real roots existing in $(0, p]$ when $p^2 < 4q$, which is contradiction. Hence there is $l_0 \in \mathbb{N}$ such that $a_{l_0-1} < p$ and $a_{l_0} \ge p$.

If $l_0 = 3k$, then by using (2.18) in (2.22) it follows that

$$\begin{aligned}
x_{n+1} &= \max\{c, (\frac{c}{z_{n-1}^{q/p}}, (\frac{c}{x_{n-2}^{q/(p-q/p)}}, \cdots, (\frac{y_{n-3k}^{p-a_{3k}}}{z_{n-(3k+1)}^q})^{p-a_{3k-1}}, \cdots)^{(p-q/p)})^p\} \\
&\le \max\{c, (\frac{c}{c^{a_1}}, (\frac{c}{c^{a_2}}, \cdots, (\frac{1}{c^{q-p+a_{3k}}})^{p-a_{3k-1}}, \cdots)^{(p-a_1)})^p\}
\end{aligned} \tag{2.24}$$

for $n > 3k$, from which the boundedness of $\{x_n\}_{n\ge -1}$ follows in this case.

If $l_0 = 3k+1$, then it follows that

$$\begin{aligned} x_{n+1} &= \max\{c, (\frac{c}{z_{n-1}^{q/p}}, (\frac{c}{x_{n-2}^{q/(p-q/p)}}, \cdots, (\frac{z_{n-(3k+1)}^{p-a_{3k+1}}}{x_{n-(3k+2)}^{q}})^{p-a_{3k}}, \cdots)^{(p-q/p)})^{p}\} \\ &\le \max\{c, (\frac{c}{c^{a_1}}, (\frac{c}{c^{a_2}}, \cdots, (\frac{1}{c^{q-p+a_{3k+1}}})^{p-a_{3k}}, \cdots)^{(p-a_1)})^{p}\} \end{aligned} \tag{2.25}$$

for $n \ge 3k+1$, from which the boundedness of $\{x_n\}_{n\ge -1}$ follows in this case.

If $l_0 = 3k+2$, then it follows that

$$\begin{aligned} x_{n+1} &= \max\{c, (\frac{c}{z_{n-1}^{q/p}}, (\frac{c}{x_{n-2}^{q/(p-q/p)}}, \cdots, (\frac{x_{n-(3k+2)}^{p-a_{3k+2}}}{y_{n-3(k+1)}^{q}})^{p-a_{3k+1}}, \cdots)^{(p-q/p)})^{p}\} \\ &\le \max\{c, (\frac{c}{c^{a_1}}, (\frac{c}{c^{a_2}}, \cdots, (\frac{1}{c^{q-p+a_{3k+2}}})^{p-a_{3k+1}}, \cdots)^{(p-a_1)})^{p}\} \end{aligned} \tag{2.26}$$

for $n \ge 3k+1$, from which the boundedness of $\{x_n\}_{n\ge -1}$ follows in this case.

Combined the case 1 $p^2 \le q$ and the case 2 $q < p^2 < 4q$, we can obtain that the sequence $\{x_n\}_{n\ge -1}$ is bounded when $p^2 < 4q$. In the same way, we can prove that the sequence $\{y_n\}$ and $\{z_n\}$ are bounded as well. Hence, every solution to systems (1.5) is bounded when $p^2 < 4q$.

**Theorem 2.3.** Assume that $c > 0$, $q > 0$ and $p = 1$. Then the solutions to systems (1.5) are bounded.

**Proof**. Assume that $\{(x_n, y_n, z_n)\}$ is any positive solution to systems (1.5) in particular $p = 1$. We can easily know that $x_n \ge c, y_n \ge c, z_n \ge c$. Therefore, we have

$$x_{n+1} \le \max\{c, \frac{y_n}{c^q}\}, \quad y_{n+1} \le \max\{c, \frac{z_n}{c^q}\}, \quad z_{n+1} \le \max\{c, \frac{x_n}{c^q}\}, \quad n \in N_0. \tag{2.27}$$

From the above (2.27), it follows that

$$x_{n+1} \le \max\{c, \frac{y_n}{c^q}\} \le \max\{c, \frac{c}{c^q}, \frac{z_{n-1}}{c^{2q}}\} \le \max\{c, \frac{c}{c^q}, \frac{c}{c^{2q}}, \frac{x_{n-2}}{c^{3q}}\}. \tag{2.28}$$

Set

$$v_{n+1} = \max\{c, \frac{c}{c^q}, \frac{c}{c^{2q}}, \frac{v_{n-2}}{c^{3q}}\}, n = 1, 2, \cdots \text{ and } v_1 = x_1, v_0 = x_0, v_{-1} = x_{-1}. \tag{2.29}$$

Assume that $\{v_n\}$ is the solution to (2.29). Then $v_n$ is greater than $x_n$ for any $n > 2$.

**Case 1**. $c > 1$.

**(a).** If $v_{-1} \le c^{3q+1}$, $v_0 \le c^{3q+1}$ and $v_1 \le c^{3q+1}$, from (2.29) we can obtain that $c^q > 1$ and $\frac{v_{n-2}}{c^{3q}} < c$, so $v_2 = c, v_5 = c, v_8 = c, \cdots$, which implies that $v_{3n-1} = c$. Moreover, $v_4 = c$, $v_7 = c$, $v_{10} = c, \cdots$, which implies that $v_{3n+1} = c$ and similarly $v_{3n} = c$. Hence, the boundedness of $\{v_n\}_{n \ge -1}$ follows in this case.

**(b).** If $v_{-1} > c^{3q+1}, v_0 > c^{3q+1}$ and $v_1 > c^{3q+1}$, from (2.29) we can obtain that $c < z_5 < z_2 < z_{-1}$. Through iteration, we can get that $\{v_{3n-1}\}$ is monotonically decreasing. Additionally, $v_n \ge c$ for any $n = 3k-1, k \in N$, we can obtain that $\{v_{3n-1}\}$ is bounded. Similarly, $\{v_{3n}\}$ and $\{v_{3n+1}\}$ are bounded as well. Hence, the boundedness of $\{v_n\}_{n \ge -1}$ follows in this case.

**(c).** If $v_{-1} \le c^{3q+1}$, $v_0 \le c^{3q+1}$ and $v_1 > c^{3q+1}$, from above proof we can obtain that $v_{3n-1} = c$, $v_{3n} = c$ and $\{v_{3n+1}\}$ is monotonically decreasing. Additionally $v_n \ge c$, we can obtain the boundedness of $\{v_n\}_{n \ge -1}$ follows in this case.

**(d).** If $v_{-1} \le c^{3q+1}$, $v_0 > c^{3q+1}$ and $v_1 \le c^{3q+1}$, from above proof we can obtain that $v_{3n-1} = c$, $v_{3n+1} = c$ and $\{v_{3n}\}$ is monotonically decreasing. Additionally $v_n \ge c$, we can obtain the boundedness of $\{v_n\}_{n \ge -1}$ follows in this case.

**(e).** If $v_{-1} > c^{3q+1}, v_0 \le c^{3q+1}$ and $v_1 \le c^{3q+1}$, from above proof we can obtain that $v_{3n} = c$, $v_{3n+1} = c$ and $\{v_{3n-1}\}$ is monotonically decreasing. Additionally $v_n \ge c$, we can obtain the boundedness of $\{v_n\}_{n \ge -1}$ follows in this case.

**(f).** If $v_{-1} \le c^{3q+1}$, $v_0 > c^{3q+1}$ and $v_1 > c^{3q+1}$, from above proof we can obtain that $v_{3n-1} = c$, $\{v_{3n}\}$ and $\{v_{3n+1}\}$ are monotonically decreasing. Additionally $v_n \ge c$, we can obtain the boundedness of $\{v_n\}_{n \ge -1}$ follows in this case.

**(g).** If $v_{-1} > c^{3q+1}$, $v_0 \le c^{3q+1}$ and $v_1 > c^{3q+1}$, from above proof we can obtain that $v_{3n} = c$, $\{v_{3n-1}\}$ and $\{v_{3n+1}\}$ are monotonically decreasing. Additionally $v_n \ge c$, we can obtain the boundedness of $\{v_n\}_{n \ge -1}$ follows in this case.

**(h).** If $v_{-1} > c^{3q+1}$, $v_0 > c^{3q+1}$ and $v_1 \le c^{3q+1}$, from above proof we can obtain that $v_{3n+1} = c$, $\{v_{3n-1}\}$ and $\{v_{3n}\}$ are monotonically decreasing. Additionally $v_n \ge c$, we can obtain the boundedness of $\{v_n\}_{n \ge -1}$ follows in this case.

Due to the bounededness of $\{v_n\}_{n\geq -1}$ and $x_n \leq v_n$, we can obtain the boundedness of $\{x_n\}$. Similarly $\{y_n\}$ and $\{z_n\}$ are bounded as well. Hence, every positive solution to systems (1.5) is bounded.

**Case 2**. $0 < c \leq 1$.

**(a)** If $q \geq 1$, in fact $x_n \geq c$, from (1.5) it follows that

$$\begin{aligned} x_{n+1} &= \max\{c, \frac{y_n}{z_{n-1}^q}\} = \max\{c, \frac{c}{z_{n-1}^q}, \frac{z_{n-1}}{z_{n-1}^q x_{n-2}^q}\} \\ &= \max\{c, \frac{c}{z_{n-1}^q}, \frac{1}{z_{n-1}^{q-1} x_{n-2}^q}\} \leq \max\{c, \frac{1}{c^{q-1}}, \frac{1}{c^{2q-1}}\} \end{aligned} \tag{2.30}$$

for $n \in N$, which means that $\{x_n\}$ is bounded.

**(b)**. If $0 < q < 1$, let sequence $\{a_l\}_{l\geq 0}$ be defined as follows

$$a_{l+1} = a_l - b_l, \quad b_{l+1} = qa_l, \quad a_1 = 1-q, b_1 = q, \quad l \in \mathbb{N}. \tag{2.31}$$

Thus, from (1.5) we have

$$\begin{aligned} x_{n+1} &= \max\{c, \frac{c}{z_{n-1}^q}, \frac{z_{n-1}^{1-q}}{x_{n-2}^q}\} = \max\{c, \frac{c}{z_{n-1}^q}, \frac{z_{n-1}^{a_1}}{x_{n-2}^{b_1}}\} \\ &= \max\{c, \frac{c}{z_{n-1}^q}, \frac{c^{1-q}}{x_{n-2}^q}, \frac{x_{n-2}^{(1-q)-q}}{y_{n-3}^{q(1-q)}}\} = \max\{c, \frac{c}{z_{n-1}^q}, \frac{c^{a_1}}{x_{n-2}^{b_1}}, \frac{x_{n-2}^{a_1-b_1}}{y_{n-3}^{qa_1}}\} \\ &= \max\{c, \frac{c}{z_{n-1}^q}, \frac{c^{1-q}}{x_{n-2}^q}, \frac{c^{(1-q)-q}}{y_{n-3}^{q(1-q)}}, \frac{y_{n-3}^{[(1-q)-q]-q(1-q)}}{z_{n-4}^{q[(1-q)-q]}}\} = \max\{c, \frac{c}{z_{n-1}^q}, \frac{c^{a_1}}{x_{n-2}^{b_1}}, \frac{c^{a_1-b_1}}{y_{n-3}^{qa_1}}, \frac{y_{n-3}^{a_2-b_2}}{z_{n-4}^{qa_2}}\} \\ &= \cdots\cdots \\ &= \max\{c, \frac{c}{z_{n-1}^q}, \frac{c^{a_1}}{x_{n-2}^{b_1}}, \frac{c^{a_1-b_1}}{y_{n-3}^{b_2}}, \frac{c^{a_2-b_2}}{z_{n-4}^{b_3}} \cdots, \frac{x_{n-(3k-2)}^{a_{3k-2}-b_{3k-2}}}{y_{n-3k}^{b_{3k-1}}}\} = \max\{c, \frac{c}{z_{n-1}^q}, \frac{c^{a_1}}{x_{n-2}^{b_1}}, \frac{c^{a_2}}{y_{n-3}^{b_2}}, \frac{c^{a_3}}{z_{n-4}^{b_3}} \cdots, \frac{x_{n-(3k-1)}^{a_{3k-1}}}{y_{n-3k}^{b_{3k-1}}}\} \\ &= \max\{c, \frac{c}{z_{n-1}^q}, \frac{c^{a_1}}{x_{n-2}^{b_1}}, \frac{c^{a_1-b_1}}{y_{n-3}^{b_2}}, \frac{c^{a_2-b_2}}{z_{n-4}^{b_3}} \cdots, \frac{y_{n-3k}^{a_{3k-1}-b_{3k-1}}}{z_{n-(3k+1)}^{b_{3k}}}\} = \max\{c, \frac{c}{z_{n-1}^q}, \frac{c^{a_1}}{x_{n-2}^{b_1}}, \frac{c^{a_2}}{y_{n-3}^{b_2}}, \frac{c^{a_3}}{z_{n-4}^{b_3}} \cdots, \frac{y_{n-3k}^{a_{3k}}}{z_{n-(3k+1)}^{b_{3k}}}\} \\ &= \max\{c, \frac{c}{z_{n-1}^q}, \frac{c^{a_1}}{x_{n-2}^{b_1}}, \frac{c^{a_1-b_1}}{y_{n-3}^{b_2}}, \frac{c^{a_2-b_2}}{z_{n-4}^{b_3}} \cdots, \frac{z_{n-(3k+1)}^{a_{3k}-b_{3k}}}{x_{n-(3k+2)}^{b_{3k+1}}}\} = \max\{c, \frac{c}{z_{n-1}^q}, \frac{c^{a_1}}{x_{n-2}^{b_1}}, \frac{c^{a_2}}{y_{n-3}^{b_2}}, \frac{c^{a_3}}{z_{n-4}^{b_3}} \cdots, \frac{z_{n-3k+1}^{a_{3k+1}}}{x_{n-(3k+2)}^{b_{3k+1}}}\} \end{aligned} \tag{2.32}$$

for every $k \in N$.

From (2.31), we can deduce

$$a_{l+1} - a_l + qa_{l-1} = 0, l \in N. \tag{2.33}$$

It is easy to see that the general solution of difference equation (2.33) is

$$a_l = c_1\lambda_1^l + c_2\lambda_2^l, \quad c_1, c_2 \in R, \tag{2.34}$$

where $\lambda_{1,2} = (1 \pm \sqrt{1-4q})/2$. The fact $|\lambda_{1,2}| < 1$ along with (2.34) implies that the sequence $\lim_{l\to+\infty} a_l = 0$. From this and (2.31) we get $\lim_{l\to+\infty} b_l = 0$.

Now note that from (2.32) it follows that

$$x_{n+1} \le \max\{c, \frac{c}{z_{n-1}^q}, \frac{c^{a_1}}{x_{n-2}^{b_1}}, \frac{c^{a_2}}{y_{n-3}^{b_2}}, \frac{c^{a_3}}{z_{n-4}^{b_3}} \cdots, \frac{x_{n-(3k-1)}^{a_{3k-1}}}{y_{n-3k}^{b_{3k-1}}}\}, \tag{2.35}$$

or

$$\mathrm{x}_{n+1} \le \max\{c, \frac{c}{z_{n-1}^q}, \frac{c^{a_1}}{x_{n-2}^{b_1}}, \frac{c^{a_2}}{y_{n-3}^{b_2}}, \frac{c^{a_3}}{z_{n-4}^{b_3}} \cdots, \frac{y_{n-3k}^{a_{3k}}}{z_{n-(3k+1)}^{b_{3k}}}\}, \tag{2.36}$$

or

$$x_{n+1} \le \max\{c, \frac{c}{z_{n-1}^q}, \frac{c^{a_1}}{x_{n-2}^{b_1}}, \frac{c^{a_2}}{y_{n-3}^{b_2}}, \frac{c^{a_3}}{z_{n-4}^{b_3}} \cdots, \frac{z_{n-(3k+1)}^{a_{3k+1}}}{x_{n-(3k+2)}^{b_{3k+1}}}\}. \tag{2.37}$$

The convergence of $\{a_l\}_{l\ge-1}$ and $\{b_l\}_{l\ge-1}$ along with (2.35)-(2.37) implies the boundedness of $\{x_n\}_{n\ge-1}$. Since systems (1.5) is symmetric, the boundedness of $\{x_n\}_{n\ge-1}$ implies the boundedness of $\{y_n\}_{n\ge-1}$ and $\{z_n\}_{n\ge-1}$, as claimed.

**Theorem 2.4.** Assume that $c > 0$, $p \in (0,1)$, then the solutions to systems (1.5) are bounded.

**Proof.** Assume that $(x_n, y_n, z_n)_{n\ge-1}$ is a positive solution to systems (1.5). Then the following estimate obviously holds

$$\min\{x_n, y_n, z_n\} \ge c, \quad n \in \mathbb{N}_0. \tag{2.38}$$

Hence

$$\begin{aligned} x_{n+1} &\le \max\{c, \frac{y_n^p}{c^q}\} \le \max\{c, \frac{c^p}{c^q}, \frac{z_{n-1}^{p^2}}{c^{pq+q}}\} \\ &\le \max\{c, \frac{c^p}{c^q}, \frac{c^{p^2}}{c^{pq+q}}, \frac{x_{n-2}^{p^3}}{c^{p^2q+pq+q}}\}. \end{aligned} \tag{2.39}$$

Let $\{v_n\}$ be the solutions of the following difference equation (2.40) and $v_{-1} = x_{-1}$, $v_0 = x_0, v_1 = x_1$.

$$v_{n+1}=\max\{c,\frac{1}{c^{q-p}},\frac{1}{c^{pq-p^2+q}},\frac{v_{n-2}^{p^3}}{c^{p^2q+pq+q}}\},\quad n=1,2,3,\cdots. \tag{2.40}$$

Since $p\in(0,1)$, the following function

$$f(x)=\max\{c,\frac{1}{c^{q-p}},\frac{1}{c^{pq-p^2+q}},\frac{x^{p^3}}{c^{p^2q+pq+q}}\} \tag{2.41}$$

is a concave function for sufficiently large $x$. Thus, it follows that there is a fixed point $x^*$, such that $f(x)<x$ for $x>x^*$. It is easy to see that if $v_2\in(0,x^*]$, the sequence $\{v_n\}_{n\geq 2}$ is bounded above by $x^*$ and if $v_2>x^*$, it is non-increasing and bounded below by $x^*$. Hence the sequence $\{v_n\}$ is bounded and consequently the sequence $\{x_n\}$ is bounded too. In the same way, we can prove that the sequence $\{y_n\}$ and $\{z_n\}$ are bounded as well. Hence, every solution to systems (1.5) is bounded as claimed.

## 3. Asymptotic behavior of solutions

This section is devoted to analyzing the asymptotic behavior of solutions to system (1.5) for $c>1$ and $c\in(0,1]$.

**Theorem 3.1.** Assuming $0<p\leq 1$, when $c\in(0,1]$ then $\{(x_n,y_n,z_n)\}$ converges to $(x^*,y^*,z^*)=(1,1,1)$, while $c>1$, $\{(x_n,y_n,z_n)\}$ converges to $(x^*,y^*,z^*)=(c,c,c)$.

**Proof. Case 1.** $c\in(0,1]$.

Due to the positivity of the solution to $\{(x_n,y_n,z_n)\}$ with initial data $x_0,x_{-1}>0$, $y_0,y_{-1}>0$ and $z_0,z_{-1}>0$, the systems (1.5) can be transformed to the following systems (3.1) by the change of $x_n=e^{s_n}$, $y_n=e^{t_n}$, $z_n=e^{r_n}$, where $s_n\geq 0,\ t_n\geq 0,\ r_n\geq 0,\ n>k$ and $\ln c<0$.

$$\begin{aligned} s_{n+1}&=\max\{\ln c,\ pt_n-qr_{n-1}\}\leq\max\{\ln c,\ pt_n\}\leq pt_n,\\ t_{n+1}&=\max\{\ln c,\ pr_n-qs_{n-1}\}\leq\max\{\ln c,\ pr_n\}\leq pr_n,\\ r_{n+1}&=\max\{\ln c,\ ps_n-qt_{n-1}\}\leq\max\{\ln c,\ ps_n\}\leq ps_n. \end{aligned} \tag{3.1}$$

Obviously, from systems (3.1), inequality (3.2) follows

$$s_{n+3}\leq p^3s_n,\quad t_{n+3}\leq p^3t_n,\quad r_{n+3}\leq p^3r_n. \tag{3.2}$$

For $p\leq 1$, $s_{n+3}<s_n$ can be obtained from inequality (3.2). The sequences $\{s_{3n}\}$, $\{s_{3n+1}\}$ and $\{s_{3n+2}\}$ are monotone decreasing. In addition, as $s_n>0$, we can get that $\lim_{n\to\infty}s_n=0$ and $\lim_{n\to\infty}x_n=\lim_{n\to\infty}e^{s_n}=1$. Then $\{x_n\}$ converges to 1. $\{y_n\}$ and $\{z_n\}$ can be proved

in the same way. Therefore, when $c \in (0,1]$, if $p \le 1$, then $\{(x_n, y_n, z_n)\}$ converges to $(x^*, y^*, z^*) = (1,1,1)$.

**Case 2. c>1.**

Correspondingly, systems (1.5) can be transformed to systems (3.3) by the change of $x_n = c^{s_n}, y_n = c^{t_n}, z_n = c^{r_n}$, where $s_n \ge 1, t_n \ge 1, r_n \ge 1$ due to $x_n \ge c, y_n \ge c, z_n \ge c$.

$$\begin{aligned} s_{n+1} &= \max\{1, pt_n - qr_{n-1}\}, \\ t_{n+1} &= \max\{1, pr_n - qs_{n-1}\}, \\ r_{n+1} &= \max\{1, ps_n - qt_{n-1}\}. \end{aligned} \tag{3.3}$$

Furthermore, systems (3.3) can be written as systems (3.4).

$$\begin{aligned} s_{n+1} - 1 &= \max\{0, pt_n - qr_{n-1} - 1\}, \\ t_{n+1} - 1 &= \max\{0, pr_n - qs_{n-1} - 1\}, \\ r_{n+1} - 1 &= \max\{0, ps_n - qt_{n-1} - 1\}. \end{aligned} \tag{3.4}$$

Similar proof has been fully given in case $c \in (0,1]$ and here the rest proof is omitted. Then the result is much alike with the one in above case:

$$\lim_{n\to\infty} s_n = 1, \ \lim_{n\to\infty} t_n = 1, \ \lim_{n\to\infty} r_n = 1, \tag{3.5}$$

and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} c^{s_n} = c, \quad \lim_{n\to\infty} y_n = \lim_{n\to\infty} c^{t_n} = c, \ \lim_{n\to\infty} z_n = \lim_{n\to\infty} c^{r_n} = c. \tag{3.6}$$

Therefore, assume $0 < p \le 1$ and $c > 1$, then $(x_n, y_n, z_n)$ converges to $(x^*, y^*, z^*) = (c,c,c)$. Hence Theorem 3.1 is proved completely.

## 4. Simulation experiment

In this section, some numerical simulations are given to support our theoretical analysis. As examples, we consider the following difference equations

$$x_{n+1} = \max\{0.5, \frac{y_n^{0.5}}{z_{n-1}^2}\}, y_{n+1} = \max\{0.5, \frac{z_n^{0.5}}{x_{n-1}^2}\}, y_{n+1} = \max\{0.5, \frac{z_n^{0.5}}{x_{n-1}^2}\}, n \in \mathbb{N}_0, \tag{4.1}$$

$$x_{n+1} = \max\{1, \frac{y_n}{z_{n-1}^2}\}, \quad y_{n+1} = \max\{1, \frac{z_n}{x_{n-1}^2}\}, \quad z_{n+1} = \max\{1, \frac{x_n}{y_{n-1}^2}\}, \quad n \in \mathbb{N}_0, \tag{4.2}$$

and

$$x_{n+1} = \max\{1.5, \frac{y_n^{0.5}}{z_{n-1}^2}\}, y_{n+1} = \max\{1.5, \frac{z_n^{0.5}}{x_{n-1}^2}\}, z_{n+1} = \max\{1.5, \frac{x_n^{0.5}}{y_{n-1}^2}\}, \quad n \in \mathbb{N}_0. \tag{4.3}$$

By employing Matlab R2013b, we solve the numerical solutions of the above equations, which are shown respectively in the following Figures.

More precisely, the initial conditions of (4.1) are that $x_{-1}=1.5$, $x_0=1.2$, $y_{-1}=0.5$, $y_0=0.8$, $z_{-1}=1$, and $z_0=1.5$. It is easy to show that the equations (4.1) satisfy the conditions of Theorem 2.2. Fig.4.1 shows that the solutions of the equations (4.1) are bounded. The initial conditions of (4.2) are that $x_{-1}=1.5$, $x_0=1.2$, $y_{-1}=0.5$, $y_0=0.8$, $z_{-1}=2.5$, and $z_0=3$. It is easy to show that the equations (4.2) satisfy the conditions of Theorem 2.3 and Theorem 3.1. Figure 4.2 shows the solutions to equations (4.2) are bounded and globally attractive. The initial conditions of equations (4.3) are that $x_{-1}=1.5$, $x_0=1.2$, $y_{-1}=0.5$, $y_0=0.8$, $z_{-1}=1$, and $z_0=1.5$. It is easy to show that the equations (4.3) satisfy the conditions of Theorem 2.4 and Theorem 3.1. Figure 4.3 shows the solutions to the equations (4.3) are bounded and globally attractive.

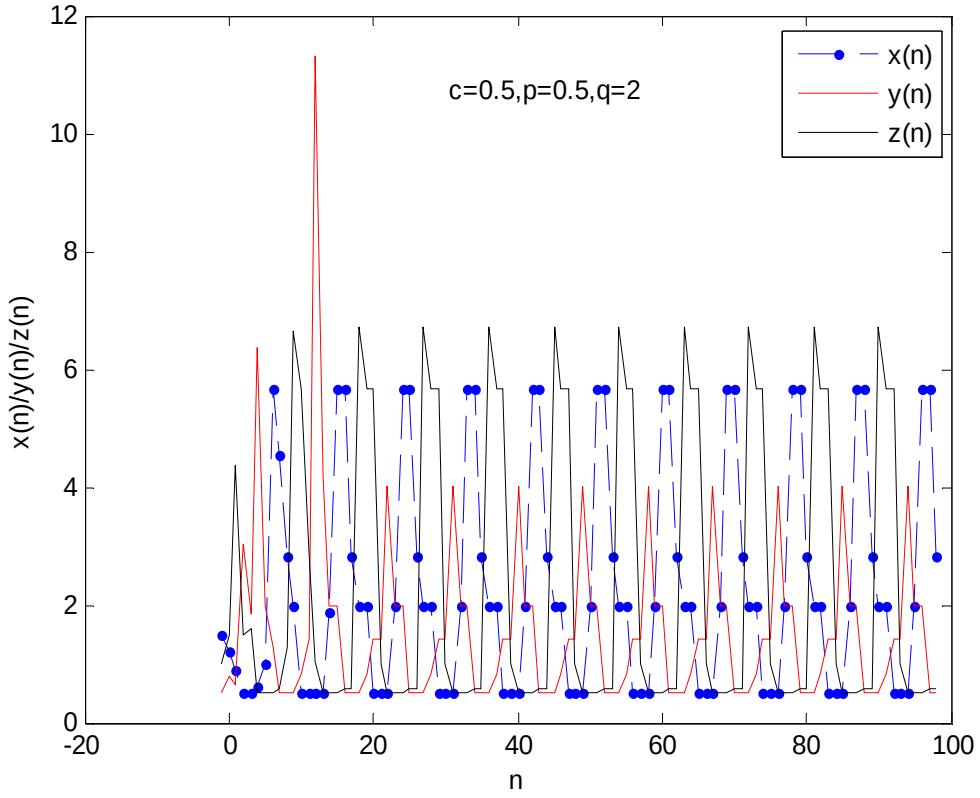



Figure 4.1. the solutions to equation (4.1)

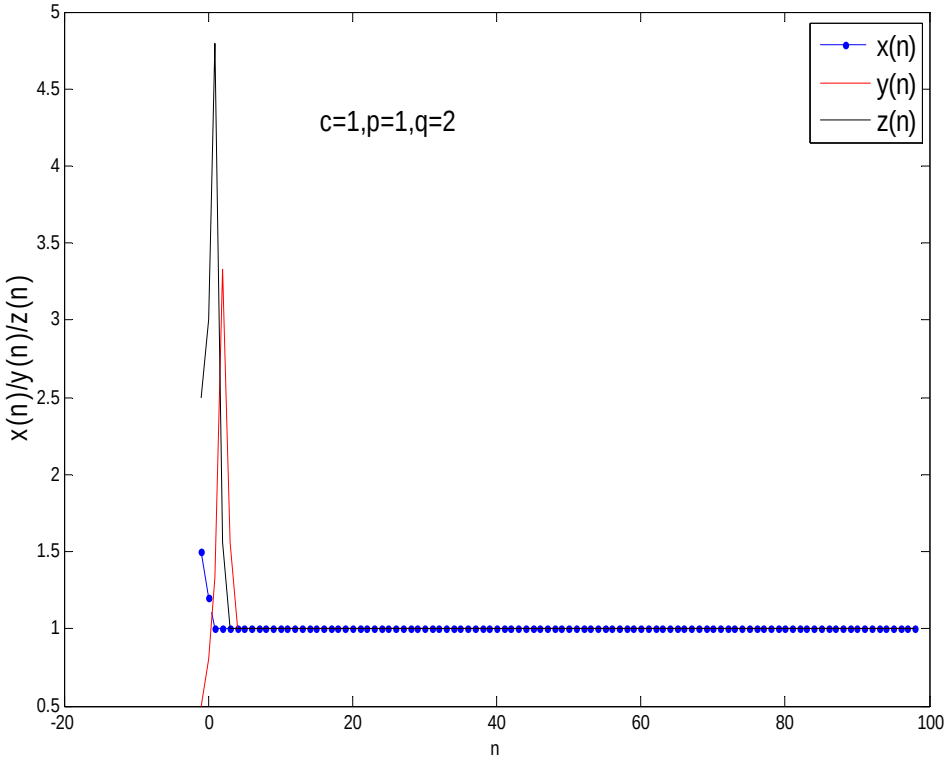



Figure4.2. the solutions to equation (4.2)

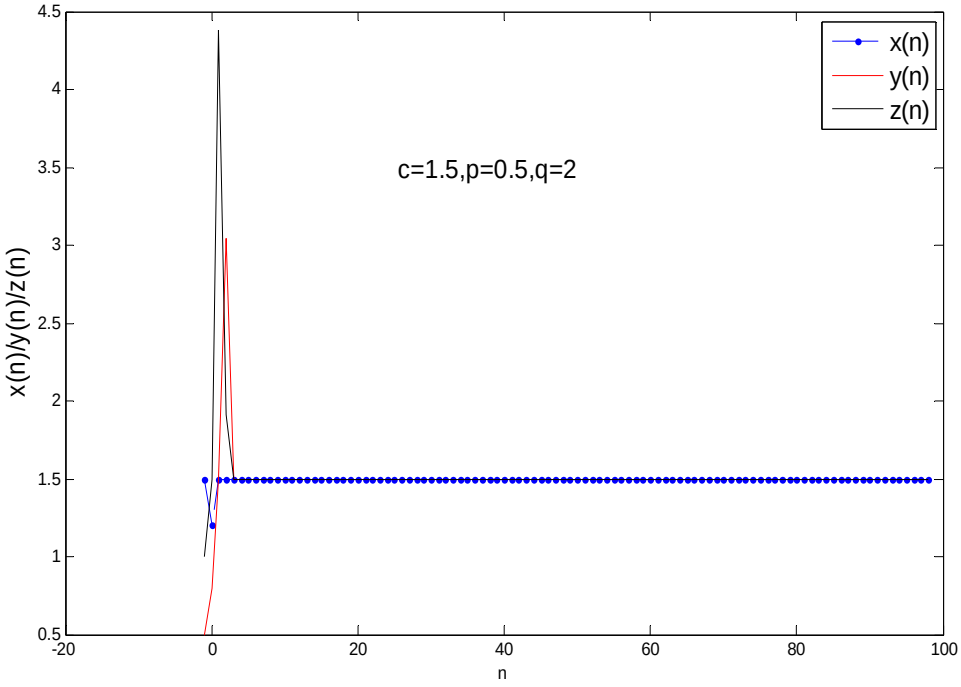



Figure 4.3. the solutions to equation (4.3)

## 5. Conclusion

It is obvious that the system of three max-type difference equations (1.5) is the extension of the models in [26-29]. In this paper, we have dealt with the problem of boundedness character and global attractivity for a class of max-type difference system. And we have obtained some sufficient conditions which ensure the boundedness character and global attractivity of the max-type system. Especially, the sufficient conditions that we obtained are very simple, which provide flexibility for the application and analysis of max-type difference system. These results generalize and improve some previous works. In addition, we present the use of a new iteration method for symmetric systems of max-type difference equations. This technique is a powerful tool for solving various difference equations and it can be applied to other nonlinear differential equations in mathematical physics. Computations are performed using the software package Matlab R2013b. Finally, some numerical examples are given to show the validity of the obtained theoretic results

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## References

[1] E. C. Pielou, Population and Community Ecology, Gordon and Breach, London, 1975.

[2] E. P. Popov, Automatic Regulation and Control, Nauka, Moscow, 1966 (in Russian).

[3] S. Stević, Behaviour of the positive solutions of the generalized Beddington-Holt equation, Panamer. Math. J. 10 (4) (2000) 77-85.

[4] E. M. Elsayed, On the solutions and periodic nature of some systems of difference equations, International Journal of Biomathematics, 7 (6) (2014), Article ID: 1450067.

[5] G. Papaschinopoulos, M. Radin, and C. J. Schinas, Study of the asymptotic behavior of the solutions of three systems of difference equations of exponential form, Applied Mathematics and Computation, 218 (9) (2012) 5310-5318.

[6] Y. Muroya, E. Ishiwata, and N. Guglielmi, Global stability for nonlinear difference equations with variable coefficients, Journal of mathematical analysis and applications, 334 (1) (2007) 232-247.

[7] E. A. Grove, G. Ladas, Periodicities in Nonlinear Difference Equations, Chapman & Hall, CRC Press, 2005.

[8] V. L. Kocic, G. Ladas, Global Behaviour of Nonlinear Difference Equations of Higher Order with Applications, Kluwer Academic Publishers, Dordrecht, 1993.

[9] C. Y. Wang, X. J. Fang, R. Li, On the solution for a system of two rational difference equations, Journal of Computational Analysis and Applications, 20 (1) (2016) 175-186.

[10] C. Y. Wang, X. J. Fang, R. Li, On the dynamics of a certain four-order fractional difference equations, Journal of Computational Analysis and Applications, 22 (5) (2017) 968-976.

[11] C. Y. Wang, S. Wang, W. Wang, Global asymptotic stability of equilibrium point for a family of rational difference equations, Applied Mathematics Letters, 24 (5) (2011) 714-718.

[12] A. Khaliq, F. Alzahranib, E. M. Elsayed, Global attractivity of a rational difference equation of order ten, Journal of Nonlinear Sciences and Applications, 9 (6) (2016) 4465-4477.

[13] M. M. El-Dessoky, E. M. Elsayed, E. O. Alzahrani, The form of solutions and periodic nature for some rational difference equations systems, Journal of Nonlinear Sciences and Applications, 9 (10) (2016) 5629-5647.

[14] W. T. Patula and H. D. Voulov, On a max type recurrence relation with periodic coefficients, Journal of Difference Equations and Applications, 10 (3) (2004) 329-338.

[15] Q. Xiao, Q. H. Shi, Eventually periodic solutions of a max-type equation, Mathematical and Computer Modelling, 57(3-4)(2013) 992-996.

[16] E. M. Elsayed, B. Irićanin, S. Stević, On the max-type equation $x_{n+1} = \max\{a_n / x_n, x_{n-1}\}$, Ars Combinatoria, 95 (2010) 187-192.

[17] M. M. El-Dessoky, On the periodicity of solutions of max-type difference equation, Mathematical Methods in the Applied Sciences, 38 (15) (2015) 3295-3307.

[18] S. Stević, M. A. Alghamdi, A. Alotaibi, and N Shahzad, Eventual periodicity of some systems of max-type difference equations, Applied Mathematics and Computation, 236 (2014) 635-641.

[19] E. M. Elsayed, Solution for systems of difference equations of rational form of order two, Computational and Applied Mathematics, 33 (3) (2014) 751-765.

[20] E. M. Elsayed, Behavior and Expression of the Solutions of Some Rational Difference Equations, Journal of Computational Analysis and Applications, 15(1) (2013) 73-81.

[21] S. Stević, Global stability of a max-type difference equation, Applied Mathematics and Computation, 216 (1) (2010) 354-356.

[22] S. Stević, Periodicity of a class of nonautonomous max-type difference equations, Applied Mathematics and Computation, 217 (23) (2011) 9562-9566.

[23] E. M. Elsayed, Solutions of Rational Difference System of Order Two, Mathematical and Computer Modelling, 55 (2012) 378–384.

[24] E. M. Elsayed and B. D. Iricanin, on a max-type and a min-type difference equation, Applied Mathematics and Computation, 215 (2) (2009) 608-614.

[25] S. Stević, On a generalized max-type difference equation from automatic control theory, Nonlinear Analysis: Theory, Methods & Applications, 72 (3) (2010) 1841-1849.

[26] S. Stević,, On the recursive sequence $x_{n+1} = \max\{c, x_n^p / x_{n-1}^p\}$ , Applied Mathematics Letters, 21 (8) (2008) 791–796.

[27] S. Stević, On a nonlinear generalized max-type difference equation, Journal of Mathematical Analysis and Applications, 376 (2011), no. 1, 317-328.

[28] S. Stević, On a symmetric system of max-type difference equations, *A*pplied Mathematics and Computation, 219 (15) (2013) 8407-8412.

[29] S. Stevic, On positive solutions of a system of max-type difference equations, Journal of Computational Analysis and Applications, 16 (5) (2014) 906-915.

[30] Tarek F. Ibrahim, Closed form expression of tractable semi-max-type two-dimensional system of difference equations with variable coefficients, Journal of the Egyptian Mathematical Society, 24 (4) (2016) 538-544.

[31] S. Stević, On periodic solutions of a class of k-dimensional systems of max-type difference equations, Advances in Difference Equations, 2016 (2016), Article ID: 251.

[32] T. X. Sun, H. J. Xi, Dynamics of a max-type system of difference equations, Analysis and Mathematical Physics, 6 (4) (2016) 393-402.

[33] T. X. Sun, H. J. Xi, On the solutions of a system of difference equations with maximum, Applied Mathematics and Computation 290 (2016) 292-297.

# The convexity of n-dimensional fuzzy mappings and the saddle point conditions of the fuzzy optimization problems

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**Abstract** The purpose of this work is to consider the optimization problem of n-dimensional fuzzy number valued functions. Firstly, the differentiability and convexity of n-dimensional fuzzy number valued function are discussed by means of the support function and a new order relation, which is built in the aid of the support function and the order of vector. Secondly, the fuzzy Lagrange function of fuzzy nonlinear programming is presented and weak duality theorems are obtained. At last, the saddle point of fuzzy lagrangian function is defined, the sufficient and necessity conditions of saddle point are given.
**Keywords** Fuzzy numbers; fuzzy programming; saddle point; duality theorem

## 1. Introduction

Since the concept and operations of fuzzy set were introduced by Zadeh, many studies have focused on the theoretical aspects and applications of fuzzy sets, one of the main stream is the fuzzy optimization in operation research. In 1970, Bellman and Zadeh[1] inspired the development of fuzzy optimization by providing the aggregation operators, which combined the fuzzy goals and fuzzy decision space. After this motivation and inspiration, there came out a lot of articles dealing with the fuzzy optimization problems and the insightful survey can be seen in [3, 7, 11].

The duality of fuzzy linear programming was first studied by Rodder and Zimmermann[9] who considered the economic interpretation of the dual variables. Zhong and Shi[20] presented a parametric approach for duality in fuzzy multi criteria and multi constrainted linear programming which extended fuzzy linear programming approaches. Wu[14]formulate the fuzzy primal and dual linear programming problems with fuzzy coefficients by using fuzzy scalar product, prove the weak and strong duality theorems. Wu[15] discuss the saddle-point optimality conditions in fuzzy optimization problems by introducing the fuzzy scalar product. In Wu[16], under a general setting partial ordering, the duality theorems and saddle point optimality of fuzzy nonlinear programming problems are derived. Zhang[21]discuss the saddle-points and minimax theorems under fuzzy environment, obtain the KT conditions for fuzzy programming and consider the "perturbed" convex fuzzy programming. Gong[5] propose the fuzzy Lagrangian function of a fuzzy optimization problem by considering a total ordering on the set of fuzzy numbers, and the saddle point of fuzzy Lagrangian function with its optimality condition were dicussed.Howere,the fuzzy number in these research above is on the real line,which is one dimensional.There are few studies on n-dimensional fuzzy numbers, maybe the ranking of n-dimensional fuzzy numbers has been a bottleneck for researchers.

The differentiability of fuzzy mappings from an open subset of a normed space into the n-dimension fuzzy number space $E^n$ was developed by Puri and Ralescu[8], which generalized and extended the concept of Hukuhara differentiability for set mappings. Wang and Wu[12] proposed the directional derivative,differential and sub-differential of fuzzy mappings from $R^n$ into $E$ using Hukuhara difference. Hai[6] characterize the generalized difference of n-dimensional fuzzy number valued functions by means of support functions and give the order relation $\preceq_s$by the aid of support function. Yan[18] give the order relation on $E$ considering the left and right endpoints and weights, which is a total order. Based on this, we give the order relation of n- dimensional fuzzy numbers by means of the support function and the order of vector, this order is partial and practical.

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The purpose of this work is to consider the optimization problem of n-dimensional fuzzy number valued function. First, we present the terminology used in the present paper, and give the order relation of n-dimensional fuzzy-number-valued function. In section 3, the differentiability and convexity are introduced and its relations is studied. For nonlinear fuzzy programming problem, the weak duality theorem the saddle point of fuzzy lagrangian function is is presented, further, the sufficient and necessity condition of saddle point are obtained.

## 2. Definitions and preliminaries

In this section,basic definitions and operations for fuzzy numbers are presented.

**Definition 2.1**[17] $X = R^n, n \geq 1$ is the real n-dimensional Euclidean space, A fuzzy number is a mapping $\widetilde{u} : R^n \to [0,1]$ with the following properties:

(1) $\widetilde{u}$ is a normal fuzzy set, i.e.there exists $x_0 \in R^n$ such that $\widetilde{u}(x_0) = 1$,

(2) $\widetilde{u}$ is a convex fuzzy set, i.e. $\widetilde{u}(\lambda x + (1-\lambda)y) \geq \min\{\widetilde{u}(x), \widetilde{u}(y)\}$ for any $x, y \in R^n$ and $\lambda \in [0,1]$.

(3) $\widetilde{u}$ is upper semi-continuous.

(4) $[\widetilde{u}]^0 = cl(\text{supp}\widetilde{u}) = \overline{\{x \in R^n : \widetilde{u}(x) > 0\}}$ is compact.

we will denote $E^n$ the set of fuzzy numbers.It is clear that any $\widetilde{u} \in E^n, r \in [0,1], [\widetilde{u}]^r = \{x \in R^n : \widetilde{u}(x) \geq r\}$ denoted as $r-$level cut is a compact convex set. Further, we give the representation theorem of these compact convex sets.

**Theorem 2.2**[17] Let $\widetilde{u} \in E^n$, then

(1)$[\widetilde{u}]^r$ is a nonempty compact convex subset of $R^n$ for any $r \in [0,1]$,

(2)$[\widetilde{u}]^{r_1} \subseteq [\widetilde{u}]^{r_2}$, for $0 \leq r_2 \leq r_1 \leq 1$,

(3)If $r_k > 0$ and $r_k$ is a nondecreasing sequence converging to $r \in (0,1]$, then $\bigcap_{k=1}^{\infty}[\widetilde{u}]^{r_k} = [\widetilde{u}]^r$,

Conversely, if $\{[A]^r \subseteq R^n : r \in [0,1]\}$ satisfies the conditions (1)-(3),then there exists a unique $\widetilde{u} \in E^n$ such that $[\widetilde{u}]^r = [A]^r$ for each $r \in (0,1]$ and $[\widetilde{u}]^0 = cl(\bigcup_{r\in(0,1]}[\widetilde{u}]^r) \subseteq [A]^0$.

Let $\widetilde{u}, \widetilde{v} \in E^n$ and $k \in R$, the addition $\widetilde{u}+\widetilde{v}$ and scalar multiplication $k\widetilde{u}$ is defined as: for any $x \in R^n$,

$$(\widetilde{u} + \widetilde{v})(x) = \sup_{s+t=x} \min\{\widetilde{u}(s), \widetilde{v}(t)\},$$

$$(k\widetilde{u})(x) = \widetilde{u}(\frac{x}{k}), k \neq 0, 0\widetilde{u} = \widetilde{0},$$

where $\widetilde{0}(x) = 1$ when $x = 0, \widetilde{0}(x) = 0$ when $x \neq 0$.

It is easy to get that the addition $\widetilde{u} + \widetilde{v}$ and scalar multiplication $k\widetilde{u}$ have the level cut:

$$[\widetilde{u} + \widetilde{v}]^r = [\widetilde{u}]^r + [\widetilde{v}]^r = \{x + y : x \in [\widetilde{u}]^r, y \in [\widetilde{u}]^r\},$$

$$[k\widetilde{u}]^r = k[\widetilde{u}]^r = \{kx : x \in [\widetilde{u}]^r\}.$$

The Hausdorff distance $D : E^n \times E^n \to [0, +\infty)$ is defined by

$$D(\widetilde{u}, \widetilde{v}) = \sup_{r\in[0,1]} d([\widetilde{u}]^r, \widetilde{v}]^r),$$

where $d$ is Harsdorff metric given by $d(A,B) = \inf\{\varepsilon : N(A,\varepsilon) \supset B, N(B,\varepsilon) \supset A\}$, and $N(A,\varepsilon) = \{x \in R^n : d(x,A) = \inf_{y\in A} d(x,y) \leq \varepsilon\}$is the $\varepsilon-$ neighborhood of $A$. Then $(E^n, D)$ is a complete metric space, and satisfies $D(\widetilde{u} + \widetilde{w}, \widetilde{v} + \widetilde{w}) = D(\widetilde{u}, \widetilde{v}), D(k\widetilde{u}, k\widetilde{v}) = |k|D(\widetilde{u}, \widetilde{v})$ for any $\widetilde{u}, \widetilde{v}, \widetilde{w} \in E^n$ and $k \in R$.

**Definition 2.3**[17] Let $\widetilde{u} \in E^n$, the support function of $\widetilde{u}$ is defined by

$$\widetilde{u}^*(r,p) = \sup_{a\in[\widetilde{u}]^r} \langle a, p\rangle, (r,p) \in I \times S^{n-1},$$

where $I = [0,1], S^{n-1} = \{x \in R^n : \|x\| = 1\}$ be the unit sphere of $R^n$ and $\langle\cdot,\cdot\rangle$be the inner product in $R^n$, that is $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$, where $x = (x_1, x_2, \cdots, x_n) \in R^n, y = (y_1, y_2, \cdots, y_n) \in R^n$. Also, assume that $\widetilde{u} = (\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_n)$, then $\langle x, \widetilde{u}\rangle = \sum_{i=1}^n x_i\widetilde{u}_i, \widetilde{u}_i \in E^n$.

**Theorem 2.4**[6,17]Let $\widetilde{u} \in E^n$, then support function$\widetilde{u}^*$ satisfy:

(1)$\widetilde{u}^*(r, p+q) \leq \widetilde{u}^*(r,p) + \widetilde{u}^*(r,q)$ for $p, q \in S^{n-1}$.

(2)$\widetilde{u}^*(r, kp) = k\widetilde{u}^*(r,p), k \geq 0$.

(3)$\widetilde{u}^*$ is uniformly bounded on $I \times S^{n-1}$, and $|\widetilde{u}^*(r,p)| \leq \sup_{a\in\widetilde{u}]^0} \|a\|$,

(4)$\widetilde{u}^*(r,p)$ is nonincreasing and left continuous on $r \in [0,1]$, right continuous at $r = 0$ for each fixed $p \in S^{n-1}$.

(5)$\widetilde{u}^*(r,\cdot)$ is uniformly Lipschitz continuous for $r \in [0,1]$, that is

$$|\widetilde{u}^*(r,p) - \widetilde{v}^*(r,q)| \leq (\sup_{a\in\widetilde{u}]^0} \|a\|)\|x-y\|,$$

(6)

$$d([\widetilde{u}]^r, [\widetilde{v}]^r) = \sup_{p\in S^{n-1}} |\widetilde{u}^*(r,p) - \widetilde{v}^*(r,p)|,$$

for any $r \in [0,1], \widetilde{u}, \widetilde{u} \in E^n$.

(7)$(-\widetilde{u})^*(r,p) = \widetilde{u}^*(r,-p)$.

**Theorem 2.5** Let $\widetilde{u}, \widetilde{v} \in E^n$, then

(1)$(s\widetilde{u} + t\widetilde{v})^*(r,p) = s\widetilde{u}^*(r,p) + t\widetilde{v}^*(r,p), s,t \geq 0$,

(2)$D(\widetilde{u},\widetilde{v}) = \sup_{r\in[0,1]} \|\widetilde{u}^*(r,p) - \widetilde{v}^*(r,p)\| = \sup_{r\in[0,1]} \sup_{p\in S^{n-1}} |\widetilde{u}^*(r,p) - \widetilde{v}^*(r,p)|$.

**Proof** (1)We prove that $(\widetilde{u}+\widetilde{v})^*(r,p) = \widetilde{u}^*(r,p)+\widetilde{v}^*(r,p)$ firstly. From the definition of support function,

$$(\widetilde{u}+\widetilde{v})^*(r,p) = \sup_{a\in[\widetilde{u}+\widetilde{v}]^r} \langle a,p\rangle = \sup_{a\in[\widetilde{u}]^r+[\widetilde{v}]^r} \langle a,p\rangle = \sup_{b\in[\widetilde{u}]^r, c\in[\widetilde{v}]^r} \langle b+c,p\rangle$$

$$= \sup_{b\in[\widetilde{u}]^r, c\in[\widetilde{v}]^r} (\langle b,p\rangle + \langle c,p\rangle) = \widetilde{u}^*(r,p) + \widetilde{v}^*(r,p).$$

in addition,for any $k \geq 0$,

$$(k\widetilde{u})^*(r,p) = \sup_{a\in[k\widetilde{u}]^r} \langle a,p\rangle = \sup_{a\in k[\widetilde{u}]^r} \langle a,p\rangle = \sup_{\frac{a}{k}\in[\widetilde{u}]^r} k\langle \frac{a}{k},p\rangle = k\widetilde{u}^*(r,p),$$

therefore,we get (1).

(2)

$$D(\widetilde{u},\widetilde{v}) = \sup_{r\in[0,1]} d([\widetilde{u}]^r,[\widetilde{u}]^r) = \sup_{r\in[0,1]} \sup_{p\in S^{n-1}} |\widetilde{u}^*(r,p) - \widetilde{v}^*(r,p)| = \sup_{r\in[0,1]} \|\widetilde{u}^*(r,p) - \widetilde{v}^*(r,p)\|.$$

we denote by $\mathcal{K}^n$ and $\mathcal{K}^n_{\mathfrak{C}}$ the spaces of (nonempty) compact convex sets of $R^n$ respectively.The generalized Hukuhara difference of two set $A, B \in \mathcal{K}^n_{\mathfrak{C}}$(gH-difference for short)is defined in[4,6,10] as follows:

$$A \ominus_{gH} B = C \Leftrightarrow \begin{cases} (a)A = B + C, \\ \text{or(b)B=A+(-1)C} \end{cases}$$

where $A + B = \{x + y : x \in A, y \in B\}, kA = \{kx : x \in A\}, k \in R$. Stefanini[10] extent the generalized Hukuhara difference to the fuzzy case. For any $\widetilde{u}, \widetilde{v} \in E^n$, the generalized Hukuhara difference(gH-difference for short)is the fuzzy number $\widetilde{w}$, if it exist,then

$$\widetilde{u} \ominus_{gH} \widetilde{v} = \widetilde{w} \Leftrightarrow \begin{cases} (a)\widetilde{u} = \widetilde{v} + \widetilde{w}, \\ \text{or}(b)\widetilde{v} = \widetilde{u} + (-1)\widetilde{w}. \end{cases}$$

From the theorem 2.4,it is easy to have the follows:

**Theorem 2.6** Let $\widetilde{u}, \widetilde{v} \in E^n, \widetilde{u} \ominus_{gH} \widetilde{v} = \widetilde{w}$, Then $\widetilde{w}^*(r,p) = \widetilde{u}^*(r,p) - \widetilde{v}^*(r,p), r \in [0,1], p \in S^{n-1}$

**Proof** Since $\widetilde{u} \ominus_{gH} \widetilde{v} = \widetilde{w}$, then either $(a)\widetilde{u} = \widetilde{v} + \widetilde{w}$, or $(b)\widetilde{v} = \widetilde{u} + (-1)\widetilde{w}$. For $(a)$, from 2.5(1),we have

$$\widetilde{u}^*(r,p) = \widetilde{v}^*(r,p) + \widetilde{w}^*(r,p).$$

For (b),

$$\widetilde{v}^*(r,p) = \widetilde{u}^*(r,p) + (-1)\widetilde{w}^*(r,p) = \widetilde{u}^*(r,p) + \widetilde{w}^*(r,-p),$$

then

$$\widetilde{w}^*(r,p) = -\widetilde{w}^*(r,-p) = \widetilde{u}^*(r,p) - \widetilde{v}^*(r,p)$$

for any $r \in [0,1], p \in S^{n-1}$.

**Definition 2.7**[18]Let $\widetilde{u}, \widetilde{v} \in E, \widetilde{u} \preceq \widetilde{v}$ denoted as

$$\int_0^1 r(\widetilde{u}^-(r) + \widetilde{u}^+(r))dr \leq \int_0^1 r(\widetilde{v}^-(r) + \widetilde{v}^+(r))dr,$$

where $[\widetilde{u}]^r = [\widetilde{u}^-(r), \widetilde{u}^+(r)], [\widetilde{v}]^r = [\widetilde{v}^-(r), \widetilde{v}^+(r)], r \in [0,1]$.

For any $x = (x_1, x_2, \cdots, x_n), y = (y_1, y_2, \cdots, y_n) \in R^n$, we define: $x \leq y$ if and only if $x_i \leq y_i$for any $i(i = 1, 2, \cdots, n)$, and $x < y$ means $x \leq y$ and there exist $m$, such that $x_m < y_m (m = 1, 2, \cdots, n)$.

**Definition 2.8** For any $\widetilde{u}, \widetilde{v} \in E^n$, we say that $\widetilde{u} \preceq \widetilde{v}$ if

$$(\tau(\widetilde{u}_1), \tau(\widetilde{u}_2), \cdots, \tau(\widetilde{u}_n) \leq (\tau(\widetilde{v}_1), \tau(\widetilde{v}_2), \cdots, \tau(\widetilde{v}_n),$$

where $\tau(\widetilde{u}_i) = \int_0^1 r(\widetilde{u}^*(r, e_i^+) - \widetilde{u}^*(r, e_i^-))dr, e_i^+ = (x_1, x_2, \cdots, x_n) \in S^{n-1}, e_i^- = (y_1, y_2, \cdots, y_n) \in S^{n-1}$ and$x_j = 1, y_j = -1$ when $j = i$, $x_j = y_j = 0$ when $j \neq i (i, j = 1, 2, \cdots, n)$. we say that $\widetilde{u} \prec \widetilde{v}$ if $\widetilde{u} \preceq \widetilde{v}$ and there exist $i(i = 1, 2, \cdots, n)$, such that $\tau(\widetilde{u}_i) < \tau(\widetilde{v}_i)$. Particularly, Def 2.8 is just as Def 2.7 when $n = 1$, it means the Def 2.8 is the extension of the Def 2.7. $\widetilde{u} \preceq \widetilde{v}$ also denoted as $\widetilde{v} \succeq \widetilde{u}$.

$\min(\widetilde{u}, \widetilde{v}) = \widetilde{w}$ if and only if $\tau(\widetilde{w}_i) = \min(\tau(\widetilde{u}_i), \tau(\widetilde{v}_i))(i = 1, 2, \cdots, n)$. In the follows, we denote $(\tau(\widetilde{u}_1), \tau(\widetilde{u}_2), \cdots, \tau(\widetilde{u}_n)) = H[\widetilde{u}]$,

## 3. Differentiability and convexity

Wang and Wu[12] present the directional derivative of the fuzzy mapping $F : R^n \to E$, that is characterized by the directional derivative of the real function. Below we give the differentiability of $F : R^n \to E^n$, and transformed it into the differentiability of functional in Banach space, then defined gradient and convexity and studied its relations.

**Definition 3.1** Let $F : M(\subset R^n) \to E^n$ be a fuzzy-number-valued function, $x^0 = (x_1^0, x_2^0, \cdots, x_n^0) \in$ intM. If there exist $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_n \in E^n$, such that

$$\lim_{x \to x^0} \frac{D(F(x), F(x^0) + \sum_{j=1}^n (x_j - x_j^0)\widetilde{u}_j)}{d(x, x^0)} = 0,$$

then we call $F$ is differentiable at $x^0$, and denote$\bigtriangledown F(x^0) = (\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_n)$ the gradient of $F$ at $x^0$, where $x = (x_1, x_2, \cdots, x_n)$.

**Theorem 3.2** Let $F : M(\subset R^n) \to E^n$ be a fuzzy-number-valued function, $x^0 = (x_1^0, x_2^0, \cdots, x_n^0) \in$ intM. $F$ is differentiable at $x^0$ if and only if

$$F(x)^*(r,p) = F(x^0)^*(r,p) + \sum_{j=1}^n (x_j - x_j^0)\widetilde{u}_j^*(r,p) + o(d(x, x^0)) \tag{3.1}$$

for any $r \in [0,1]$ and $p \in S^{n-1}$.

**Proof** $F$ is differentiable at $x^0$, if and only if

$$\lim_{x \to x^0} \frac{D(F(x), F(x^0) + \sum_{j=1}^n (x_j - x_j^0)\widetilde{u}_j)}{d(x, x^0)} = 0,$$

if and only if

$$\lim_{x \to x^0} \frac{\sup_{r \in [0,1]} \sup_{p \in S^{n-1}} |F(x)^*(r,p) - F(x^0)^*(r,p) - \sum_{j=1}^n (x_j - x_j^0)\widetilde{u}_j^*(r,p)|}{d(x, x^0)} = 0,$$

if and only if

$$\lim_{x \to x^0} \frac{|F(x)^*(r,p) - F(x^0)^*(r,p) - \sum_{j=1}^n (x_j - x_j^0)\widetilde{u}_j^*(r,p)|}{d(x, x^0)} = 0$$

for any $r \in [0,1]$ and $p \in S^{n-1}$, if and only if

$$F(x)^*(r,p) = F(x^0)^*(r,p) + \sum_{j=1}^n (x_j - x_j^0)\widetilde{u}_j^*(r,p) + o(d(x, x^0))$$

for any $r \in [0,1]$ and $p \in S^{n-1}$.

**Theorem 3.3** Let $F : M(\subset R^n) \to E^n$ be a fuzzy-number-valued function, $F$ is differentiable at $\overline{x}$, if $\overline{x}$ is a local minimum solution, then $\nabla F(\overline{x}) = (\widetilde{0}, \widetilde{0}, \cdots, \widetilde{0})$.

**Proof** Since $\overline{x}$ is a local minimum solution, then there exist a $\delta > 0$, such that $F(x) \succeq F(\overline{x})$ for any $x \in \bigcup(\overline{x}, \delta) \cap M$. that is

$$\tau(F(x)_i) = \int_0^1 r[F(x)^*(r, e_i^+) - F(x)^*(r, e_i^-)]dr \geq \int_0^1 r[F(\overline{x})^*(r, e_i^+) - F(\overline{x})^*(r, e_i^-)]dr = \tau(F(\overline{x})_i)$$

for $1 \leq i \leq n$. From the theorem 3.2 and arbitrariness of $x$,

$$\tau((\widetilde{u}_j)_i) = \int_0^1 r[\widetilde{u}_j^*(r, e_i^+) - \widetilde{u}_j^*(r, e_i^-)]dr = 0 (i, j = 1, 2, \cdots, n),$$

thus $\widetilde{u}_j = \widetilde{0}$, and $\nabla F(\overline{x}) = (\widetilde{0}, \widetilde{0}, \cdots, \widetilde{0})$.

**Definition 3.4** Let $F : M(\subset R^n) \to E^n$ be a fuzzy-number-valued function,$M$ be a convex set. Then $F(x)$ is said to be a convex fuzzy-number-valued function on $M$ if for any $x, y \in M, \lambda \in [0,1]$, such that $\lambda x + (1-\lambda)y \in M$, we have

$$F(\lambda x + (1-\lambda)y) \preceq \lambda F(x) + (1-\lambda)F(y). \tag{3.2}$$

we call $F(x)$ is a strictly convex fuzzy-number-valued function on $M$, if for any $x, y \in M, x \neq y, \lambda \in [0,1]$,such that $\lambda x + (1-\lambda)y \in M$, we have

$$F(\lambda x + (1-\lambda)y) \prec \lambda F(x) + (1-\lambda)F(y).$$

**Theorem 3.5** Let $M$ be an open convex set, $F : M \to E^n$, and $F$ is differentiable, then $F$ is convex if and only if

$$F(x) \succeq F(y) + \langle \nabla F(y), x - y \rangle \tag{3.3}$$

for any $x, y \in M$.

**Proof** Assume that $F$ is convex, then for any $x, y \in M, \lambda \in (0,1)$, we have

$$F(\lambda x + (1-\lambda)y) \preceq \lambda F(x) + (1-\lambda)F(y),$$

that is

$$H[F(\lambda x + (1-\lambda)y)] \leq \lambda H[F(x)] + (1-\lambda)H[F(y)].$$

thus

$$\tau(F(\lambda x + (1-\lambda)y)_i) \leq \lambda \tau(F(x)_i) + (1-\lambda)\tau(F(y)_i)$$

for any $1 \leq i \leq n$, that is

$$\int_0^1 r[F(\lambda x + (1-\lambda)y)^*(r, e_i^+) - F(\lambda x + (1-\lambda)y)^*(r, e_i^-)]dr$$

$$\leq \lambda \int_0^1 r[F(x)^*(r, e_i^+) - F(x)^*(r, e_i^-)]dr + (1-\lambda)\int_0^1 r[F(y)^*(r, e_i^+) - F(y)^*(r, e_i^-)]dr.$$

Since $F$ is differentiable, then

$$F(\lambda x + (1-\lambda)y)^*(r,p) - F(y)^*(r,p) = \sum_{j=1}^n \lambda(x_j - y_j)\widetilde{v}_j^*(r,p) + \lambda o\|x - y\|,$$

for any $r \in [0,1]$ and $p \in S^{n-1}$, where $\nabla F(y) = (\widetilde{v}_1, \widetilde{v}_2, \cdots, \widetilde{v}_n)$, therefore,

$$\int_0^1 r[F(\lambda x + (1-\lambda)y)^*(r, e_i^+) - F(\lambda x + (1-\lambda)y)^*(r, e_i^-)]dr - \int_0^1 r[F(y)^*(r, e_i^+) - F(y)^*(r, e_i^-)]dr$$

$$\leq \lambda(\int_0^1 r[F(x)^*(r,e_i^+) - F(x)^*(r,e_i^-)]dr - \int_0^1 r[F(y)^*(r,e_i^+) - F(y)^*(r,e_i^-)]dr),$$

thus

$$\int_0^1 r[\sum_{j=1}^n \lambda(x_j - y_j)\widetilde{v}_j^*(r,e_i^+) - \sum_{j=1}^n \lambda(x_j - y_j)\widetilde{v}_j^*(r,e_i^-)]dr$$

$$\leq \lambda(\int_0^1 r[F(x)^*(r,e_i^+) - F(x)^*(r,e_i^-)]dr - \int_0^1 r[F(y)^*(r,e_i^+) - F(y)^*(r,e_i^-)]dr),$$

that is

$$\tau(\sum_{j=1}^n (x_j - y_j)(\widetilde{v}_j)_i) \leq \tau(F(x)_i) - \tau(F(y)_i),$$

We have

$$\sum_{j=1}^n (x_j - y_j)\widetilde{v}_j \preceq F(x) - F(y),$$

therefore

$$F(x) \succeq F(y) + \langle \nabla F(y), x - y\rangle.$$

Conversely, assume that for any $x^{(1)}, x^{(2)} \in M$, we have

$$F(x^{(2)}) \succeq F(x^{(1)}) + \langle \nabla F(x^{(1)}), x^{(2)} - x^{(1)}\rangle.$$

Let $y$ be a point between the $x^{(1)}$ and $x^{(2)}$, then $y = \lambda x^{(1)} + (1-\lambda)x^{(2)}$ for some $\lambda \in (0,1)$, and $y \in M$ since $M$ is a convex set. Based on the assumption, we have

$$F(x^{(1)}) \succeq F(y) + \langle \nabla F(y), x^{(1)} - y\rangle,$$

$$F(x^{(2)}) \succeq F(y) + \langle \nabla F(y), x^{(2)} - y\rangle,$$

that is

$$H[F(x^{(1)})] \geq H[F(y)] + H[\langle \nabla F(y), x^{(1)} - y\rangle], \tag{3.4}$$

$$H[F(x^{(2)})] \geq H[F(y)] + H[\langle \nabla F(y), x^{(2)} - y\rangle]. \tag{3.5}$$

From (3.4)and (3.5),

$$\tau(F(x^{(1)})_i) \geq \tau(F(y)_i) + \tau(\langle \nabla F(y), x^{(1)} - y\rangle)_i), \tag{3.6}$$

$$\tau(F(x^{(2)})_i) \geq \tau(F(y)_i) + \tau(\langle \nabla F(y), x^{(2)} - y\rangle)_i) \tag{3.7}$$

for $1 \leq i < n$. Multiple(3.6),(3.7)by $\lambda, (1-\lambda)$ respectively, and then add the result, we have

$$\lambda\tau(F(x^{(1)})_i) + (1-\lambda)\tau(F(x^{(2)})_i) \geq \tau(F(y)_i),$$

that is

$$F(\lambda x^{(1)} + (1-\lambda)x^{(2)}) \preceq \lambda F(x^{(1)}) + (1-\lambda)F(x^{(2)}),$$

thus $F$ is a convex fuzzy-number-valued function.

## 4. The duality and the saddle point

Duality plays an important role in the development of optimization theory and algorithm. In this section, the duality theory of fuzzy nonlinear programming is introduced, and the weak duality theorems are obtained. At the same time, the Lagrange function of fuzzy nonlinear programming and saddle point are defined, and then discusses the relation between the saddle point of Lagrange function and the optimal solution of prime problem and dual problem and given saddle point optimality conditions.

Let $X \subset R^n$ be an open set, $F(x), G_i(x)(i = 1, 2, \cdots, m)$ be fuzzy-valued functions on $X$, now we consider the following primal fuzzy optimization:

$$(FP) \quad \begin{cases} \min F(x), \\ G_j(x) \preceq \widetilde{0}(j = 1, 2, \cdots, m), \end{cases} \tag{4.1}$$

where $S = \{x \in X | G_j(x) \preceq \widetilde{0}(j = 1, 2, \cdots, m)\}$ is the set of feasible solutions for problem $(FP)$, and denote $x \in S$ the feasible solution for problem $(FP)$.

We define the fuzzy-valued Lagrangian function for the primal problem as follow:

$$L(x, u) = F(x) + \sum_{j=1}^{m} u_j G_j(x)$$

for all $x \in S$ and all $(u_1, u_2, \cdots, u_m) \in R^m, u_j \geq 0(j = 1, 2, \cdots, m)$.

Now we define the dual fuzzy optimization problem as follow:

$$(FD) \quad \begin{cases} \max L(u) \\ u_j \geq 0(j = 1, 2, \cdots, m). \end{cases} \tag{4.2}$$

where $L(u) = \min_x L(x, u), u = (u_1, u_2, \cdots, u_m) \in R^m$.

**Theorem 4.1** (Weak Duality Theorem) Let $x \in X(\subset R^n), u \in Y(\subset R^m)$ be the feasible solution of problems (FP) and (FD) respectively. then

$$F(x) \succeq L(u).$$

**Proof** From the definition of $L(u)$, we have

$$L(u) = \min_x L(x, u) = \min_x (F(x) + \sum_{j=1}^{m} u_j G_j(x)) \preceq F(x) + \sum_{j=1}^{m} u_j G_j(x). \tag{4.3}$$

that is

$$\tau(L(u)_i) \leq \tau(F(x)_i) + \sum_{j=1}^{m} u_j \tau(G_j(x)_i),$$

for $1 \leq i < n$. Since $x$ and $u$ is the feasible solution of problems (FP) and (FD) respectively, that is $u_j \geq 0$ and $G_j(x) \preceq \widetilde{0}(j = 1, 2, \cdots, m)$, thus $\tau(G_j(x)_i) \leq 0(i = 1, 2, \cdots, n)$, then we have $\tau(L(u)_i) \leq \tau(F(x)_i)$for $1 \leq i < n$, then $L(u) \preceq F(x)$.

From above it follows easily:

**Proposition 4.2** For the problems (FP) and (FD), we have

$$\min\{F(x) | G_j(x) \preceq \widetilde{0}, x \in X, j = 1, 2, \cdots, m\} \succeq \max\{L(u) | u \geq 0\}.$$

**Proposition 4.3** Assume that

$$F(\overline{x}) \preceq L(\overline{u}),$$

where $\overline{x} \in \{x | G_j(x) \preceq \widetilde{0}, x \in X, j = 1, 2, \cdots, m\}$, $\overline{u} \geq 0$, then $\overline{x}$ and $\overline{u}$ are the optimal solutions of problems (FP) and (FD) respectively.

**Definition 4.4** Let $\overline{x} \in X(\subset R^n), \overline{u} \in Y(\subset R^m)$, then $(\overline{x}, \overline{u})$ is called a saddle point of the fuzzy-valued Lagrangian function $L : X \times Y \to E^n$ if and only if

$$L(\overline{x}, u) \preceq L(\overline{x}, \overline{u}) \preceq L(x, \overline{u}) \tag{4.4}$$

holds for every $(x, u) \in X \times Y$.

**Theorem 4.5** Let $(\overline{x}, \overline{u})$ be a saddle point of the fuzzy-valued Lagrangian function $L(x, u)$, then $\overline{x}$ and $\overline{u}$ are the optimal solutions of problems (FP) and (FD) respectively.

**Proof** Assume that $(\overline{x},\overline{u})$ be a saddle point, we are going to prove $\overline{x}\in S$ firstly. It follows easily from the definition of saddle point, $L(\overline{x},u)\preceq L(\overline{x},\overline{u})$ holds for all $u\in R^m$, that is

$$F(\overline{x})+\sum_{j=1}^{m}u_jG_j(\overline{x})\preceq F(\overline{x})+\sum_{j=1}^{m}\overline{u}_jG_j(\overline{x}),$$

so

$$\tau(F(\overline{x})_i)+\sum_{j=1}^{m}u_j\tau(G_j(\overline{x})_i)\leq\tau(F(\overline{x})_i)+\sum_{j=1}^{m}\overline{u}_j\tau(G_j(\overline{x})_i),$$

$$\sum_{j=1}^{m}(u_j-\overline{u}_j)\tau(G_j(\overline{x})_i)\leq 0. \tag{4.5}$$

holds for any $1\leq i<n$, Let $u_k=\overline{u}_k+1$ and $u_j=\overline{u}_j, j\neq k$, from (5.5)we have $\tau(G_k(\overline{x})_i)\leq 0(k=1,2,\cdots,m)$.It says that $\overline{x}$ is a feasible solution of (FP). Now we'll prove $\overline{x}$ and $\overline{u}$ are the optimal solutions of problems (FP) and (FD) respectively. Let $u_j(j=1,2,\cdots,m)$ in (4.5) be taken as 0, then $\sum_{j=1}^{m}(-\overline{u}_j)\tau(G_j(\overline{x})_i)\leq 0$. since $\overline{u}_j\geq 0,\tau(G_j(\overline{x})_i)\leq 0$, then we have

$$\sum_{j=1}^{m}(-\overline{u}_j)\tau(G_j(\overline{x})_i)=0. \tag{4.6}$$

From the right inequality of (4.4), $\tau(F(\overline{x})_i)\leq\tau(F(x)_i)+\sum_{j=1}^{m}(\overline{u}_j\tau(G_j(x)_i)$ holds for all $x\in S$. So $F(\overline{x})\succeq L(\overline{u})$. That is $\overline{x}$ and $\overline{u}$ are the optimal solutions of problems (FP) and (FD) from the proposition 4.3.

**Lemma 4.6**[16] Let $X$ be a nonempty convex set in a real vector space $R^n$, $F:X\to R, G=(G_1,G_2,\cdots,G_n), G_i:R^n\to R(i=1,2,\cdots,n)$ be convex functions. We consider the following conditions.

Condition a: $F(x)<0$ and $G(x)\leq 0$ for some $x\in X$;

Condition b: $u_0F(x)+\langle u,G(x)\rangle\geq 0$ for all $x\in X,(u_0,u)\geq 0$ and $(u_0,u)\neq 0$.

If $\overline{x}$ does not satisfy Condition a, then Condition b has a solution $(u_0,u)$ when $\overline{x}$ substitute $x$.

**Theorem 4.7** Let $X\subset R^n$ be a nonempty convex sets, $F:X\to E^n, G_j:X\to E^n(j=1,2,\cdots,m)$ be convex fuzzy-valued functions, $\overline{x}$ be an optimal solution of problem $(FP)$, assume that there exist $x$ , such that $G_j(x)\preceq\widetilde{0}$, then there exists $\overline{u}\geq 0$,such that

$$L(x,\overline{u})\succeq F(\overline{x})$$

holds for every $x\in X$.

**Proof** $\overline{x}$ be an optimal solution of $(FP)$, then $F(x)\succeq F(\overline{x})$ for any $x\in X$, that is

$$\tau(F(x)_i)=\int_0^1 r[F(x)^*(r,e_i^+)-F(x)^*(r,e_i^-)]dr\geq\int_0^1 r[F(\overline{x})^*(r,e_i^+)-F(\overline{x})^*(r,e_i^-)]dr=\tau(F(\overline{x})_i)$$

for $1\leq i<n$, Since $F:X\to E^n, G_j:X\to E^n(j=1,2,\cdots,m)$ be convex fuzzy-valued functions, then $\tau(F(x)_i),\tau(G_j(x)_i)(j=1,2,\cdots,m,i=1,2,\cdots,n)$ are convex real-valued functions. Therefore we consider the following systems:

$$\int_0^1 r[F(x)^*(r,e_i^+)-F(x)^*(r,e_i^-)]dr-\int_0^1 r[F(\overline{x})^*(r,e_i^+)-F(\overline{x})^*(r,e_i^-)]dr<0,$$

$$\int_0^1 r[G_j^*(r,e_i^+)-G_j^*(r,e_i^-)]dr\leq 0(j=1,2,\cdots,m),$$

the system has no solution on $X$, then from lemma 4.6, there exists $(u_0,u)\geq 0$ and $(u_0,u)\neq 0$ such that

$$u_0(\tau(F(x)_i)-\tau(F(\overline{x})_i))+\sum_{j=1}^{m}u_j\tau(G_j(x)_i)\geq 0$$

for every $x \in X$. assume that $u_0 = 0$, then $\sum_{j=1}^{m} u_j \tau(G_j(x)_i) \geq 0$ holds for every $x \in X$, since there exists $x$, such that $\tau(G_j(x)_i) \leq 0$ for $1 \leq i \leq n$, then $u_j = 0(j = 1, 2, \cdots, m)$, it contracts $(u_0, u) \neq 0$, thus $u_0 > 0$. dividing the inequality by $u_0$, we have

$$\tau(F(x)_i) - \tau(F(\overline{x})_i) + \sum_{j=1}^{m} u'_j \tau(G_j(x)_i) \geq 0,$$

where $u'_j = \frac{u_j}{u_0}(j = 1, 2, \cdots, m)$. then

$$F(x) + \sum_{j=1}^{m} u'_j G_j(x) \succeq F(\overline{x}).$$

denote $\overline{u}$ by $u'$, and then there exists $\overline{u} \geq 0$, such that $L(x, \overline{u}) \succeq F(\overline{x})$.

**Theorem 4.8** Let $X \subset R^n$ be a nonempty convex set, $F : X \to E^n, G_j : X \to E^n(j = 1, 2, \cdots, m)$ be convex fuzzy-valued functions, $\overline{x}$ be a optimal solution of problem $(FP)$, assume there exists $x$, such that $G_j(x) \prec \widetilde{0}$, then there exists $\overline{u} \geq 0$, such that $(\overline{x}, \overline{u})$ be a saddle point of the fuzzy-valued Lagrangian function $L(x, u)$.

**Proof** Let $\overline{x}$ be an optimal solution of problem $(FP)$, from the theorem 4.7, there exists $\overline{u} \geq 0$ such that

$$L(x, \overline{u}) \succeq F(\overline{x}) \tag{4.7}$$

holds for every $x \in X$. then

$$L(\overline{x}, \overline{u}) = F(\overline{x}) + \sum_{j=1}^{m} \overline{u}_j G_j(\overline{x}) \succeq F(\overline{x}),$$

and $\overline{u}_j \geq 0, G_j(\overline{x}) \prec \widetilde{0}(j = 1, 2, \cdots, m)$, so

$$\sum_{j=1}^{m} \overline{u}_j G_j(\overline{x}) = \widetilde{0}. \tag{4.8}$$

That is

$$L(\overline{x}, \overline{u}) = F(\overline{x}) + \sum_{j=1}^{m} \overline{u}_j G_j(\overline{x}) = F(\overline{x}),$$

From (4.7), we have

$$L(\overline{x}, \overline{u}) \prec L(x, \overline{u}). \tag{4.9}$$

From the definition of $L(x, u)$, we have

$$L(\overline{x}, u) = F(\overline{x}) + \sum_{j=1}^{m} u_j G_j(\overline{x}),$$

and $G_j(\overline{x}) \prec \widetilde{0}, u_j \geq 0(j = 1, 2, \cdots, m)$, then

$$L(\overline{x}, u) \prec F(\overline{x}) = L(\overline{x}, \overline{u}). \tag{4.10}$$

the (4.9),(4.10) indicate that $(\overline{x}, \overline{u})$ be a saddle point of the fuzzy-valued Lagrangian function $L(x, u)$.

## 5. Conclusion

In this article, we introduced the convexity and differentiability of n-dimensional fuzzy-number-valued function by means of a new order relation, which Pave a way for n-dimensional fuzzy optimization problem, so the saddle point optimal condition can be implemented.The n-dimensional fuzzy number valued function has been embedded into a complete Banach space, which is expressed by its support function, but the order of its support function is improper for fuzzy number, and establish this new order relationship more grasp the location information of fuzzy numbers. In the future work, we will discuss the application of n-dimensional fuzzy optimization in practice.

## References

[1] R.E. Bellman, L.A. Zadeh, Decision making in a fuzzy environment, Management Science 17 (1970) 141-164.

[2] B. Bede, L. Stefanini, Generalized differentiablity of fuzzy-valued functions, Fuzzy Sets and Systems 230 (2013) 119-141.

[3] M. Delgado, J. Kacporzyk, J.L. Verdegay, M.A. Vila, Fuzzy optimization: Recent Advances, New Advances, New York: Physica-Verlag, 1994.

[4] Z.T. Gong, H.Yang, Ill-posed fuzzy initial-boundary value problems based on generalized differentiability and regularization, Fuzzy Sets Syst. 295 (2016) 99-113.

[5] Z.T. Gong, H.X. Li, Saddle point optimality conditions in fuzzy optimization problems, Advances in Intelligent and Soft Computing 54 (2009) 7-14.

[6] S.X. Hai, Z.T. Gong, H.X. Li, Generalized differentiability for n-dimensional fuzzy-number-valued functions and fuzzy optimization, Information Sciences 374 (2016) 151-163.

[7] Y.J. Lai, C.L. Hwang, Fuzzy multiple objective decision making: methods and applications, Lecture Notes in Economics and Mathematical Systems, New York: Springer-Verlag, 1992.

[8] M.L. Puri, D.A. Ralescu, Differentials of fuzzy functions, J. Math. Anal. Appl. 91 (1983) 552-558.

[9] W. Rodder, H.J. Zimmermann, Duality in fuzzy linear programming, In Extremal Methods and System Analysis, (Edited by A.V.Fiacco and K.O.Kortanek) pp.415-429, Berlin, 1980.

[10] L. Stefanini, A generalization of Hukuhara difference and division for interval and fuzzy arithmetic, Fuzzy Sets Syst. 161 (2010) 1564-1584.

[11] R. Sloweński, J. Teghem, Stochastic versus fuzzy approaches to multiobjective mathematical programming under uncertainty, Boston: Kluwer Academic Publishers, 1990.

[12] G.X. Wang, C.X. Wu, Directional derivatives and subdifferential of convex fuzzy mappings and application in convex fuzzy programming, Fuzzy Sets Syst. 138 (2003) 559-591.

[13] C.X. Wu, M. Ma, J.X. Fang, Structure theory of fuzzy analysis, Guiyang: Guizhou Scientific Publication, 1994.(in Chinese).

[14] H.C. Wu, Duality theorems in fuzzy linear programming problems with fuzzy coefficients, Fuzzy Optimization and Decision Making 2 (2003) 61-73.

[15] H.C. Wu, Saddle point optimality conditions in fuzzy optimization problems, Fuzzy Sets and Systems 14 (2003) 131-141.

[16] H.C. Wu, Duality theorems in fuzzy mathematical programming problems based on the concept of necessity, Fuzzy Sets and Systems 139 (2003) 363-377.

[17] C.X. Wu, M. Ma, The Basic of fuzzy analysis, BeiJing: National Defence Industrial Press, 1991.

[18] H. Yan, J.P. Xu, A class of convex fuzzy mappings, Fuzzy Sets and Systems 129 (2002) 47-56.

[19] L.A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 331-353.

[20] Y. Zhong, Y. Shi, Duality in fuzzy multi-criteria and multi-constraint level linear programming: A parametric approach, Fuzzy Sets and Systems 132 (2002) 335-346.

[21] C. Zhang, Duality theory in fuzzy mathematical programming problems with fuzzy coefficients, Computers and Mathematics with Application 49 (2005) 1709-1730.

# Complex harmonic poles in the evolution of macromolecules depolymerization

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**Abstract**

The full comprehension and handling of the phenomenon of shattering, sometime happening during the process of polymer chain degradation [29, 32], remains unsolved when using the traditional evolution equations describing the degradation. This traditional model has been proved to be very hard to handle as it involves evolution of two intertwined quantities. moreover, the explicit form of its solution is, in general, impossible to obtain. In this article, we explore the possibility of generalizing evolution equation modeling the polymer chain degradation and analyze the model with $\beta$ derivative. We consider the general case where the breakup rate depends on the size of the chain breaking up. In the process, the alternative version of Sumudu integral transform is used to provide an explicit form of the general solution representing the evolution of polymer sizes distribution. In particular, we show that this evolution exhibits existence of complex periodic properties due to the presence of cosine and sine functions governing the solutions. Numerical simulations are performed for some particular cases and proves that such a system describing the polymer chain degradation contains complex and simple harmonic poles whose effects are given by these functions or a combination of them. This result may be crucial in the ongoing research to better handle and explain the phenomenon of shattering.

**Keywords:** $\beta$- derivative; depolymerization; replicated fractional poles; simple and complex harmonic motion; shattering

## 1 Introduction, motivation and Justification

Depolymerization is the process where polymers or biopolymers are converted into monomers or mixtures of monomers. Polymers range from familiar synthetic plastics such as polystyrene (also called styrofoam) to natural biopolymers such as DNA and proteins that are fundamental to biological structure and function. Historically, products arising from the linkage of repeating units by covalent chemical bonds have been the primary focus of polymer science; emerging important areas of the science now focus on non-covalent links. Polyisoprene of latex rubber and the polystyrene of styrofoam are examples of polymeric natural/biological and synthetic polymers, respectively. In biological contexts, essentially all biological macromolecules, i.e. proteins (polyamides), nucleic

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acids (polynucleotides), and polysaccharides are purely polymeric, composed in large part of polymeric components, for instance, isoprenylated/lipid-modified glycoproteins, where small lipidic molecule and oligosaccharide modifications occur on the polyamide backbone of the protein.

Today, it is widely known that the Newtonian concept of derivative can no longer satisfy all the complexity of the natural occurrences. A couple of complex phenomena and features happening in some areas of sciences or engineering are still (partially) unexplained by the traditional existing methods and remain open problems. Usually in mathematical modeling of a natural phenomenon that changes, the evolution is described by a family of time-parameter operators, that map an initial given state of the system to all subsequent states that takes the system during the evolution. A widely devotion has been predominantly offered to way of looking at that evolution in which time's change is described as transitions from one state to another. Hence, this is how the theory of semigroups was developed [16, 25], providing the mathematicians with very interesting tools to investigate and analyze resulting mathematical models. However, most of the phenomena scientists try to analyze and describe mathematically are complex and very hard to handle. Some of them like depolymerization, the rock fractures and fragmentation processes are difficult to analyze [11, 33] and often involve evolution of two intertwined quantities: the number of particles and the distribution of mass among the particles in the ensemble [15, 20, 28]. Then, though linear, they display non-linear features such as phase transition (called "shattering") causing the appearance of a "dust" of "zero-size" particles with nonzero mass. The phenomena of "shattering" remain (partially) unexplained by traditional models.

Another example is the groundwater flowing within a leaky aquifer. Recall that an aquifer is an underground layer of water-bearing permeable rock or unconsolidated materials (gravel, sand, or silt) from which groundwater can be extracted using a water well. Then, how do we explain accurately the observed movement of water within the leaky aquifer? As an attempt to answer this question, Hantush [17, 18] proposed an equation with the same name and his model has since been used by many hydro-geologists around the world. However, it is necessary to note that the model does not take into account all the non-usual details surrounding the movement of water through a leaky geological formation. Indeed, due to the deformation of some aquifers, the Hantush equation is not able to account for the effect of the changes in the mathematical formulation. Hence, all those non-usual features are beyond the usual models' resolutions and need other techniques and methods of modeling with more parameters involved.

Furthermore, time's evolution and changes occurring in some systems do not happen on the same manner after a fixed or constant interval of time and do not follow the same routine as one would expect. For instance, a huge variation can occur in a fraction of second causing a major change that may affect the whole system's state forever. Indeed, it has turned out recently that many phenomena in different fields, including sciences, en-

gineering and technology can be described very successfully by the models using fractional order differential equations [4, 6, 9, 10, 13, 14, 19, 22, 27]. Hence, differential equations with fractional derivative have become a useful tool for describing nonlinear phenomena that are involved in many branches of chemistry, engineering, biology, ecology and numerous domains of applied sciences. Many mathematical models, including those in acoustic dissipation, mathematical epidemiology, continuous time random walk, biomedical engineering, fractional signal and image processing, control theory, Levy statistics, fractional phase-locked loops, fractional Brownian, porous media, fractional filters motion and nonlocal phenomena have proved to provide a better description of the phenomenon under investigation than models with the conventional integer-order derivative [6, 22, 26].

One of the attempts to enhance mathematical models was to introduce the concept of derivative with fractional order. There exist in the literature number of definitions of fractional derivatives, including Riemann–Liouville and Caputo derivatives respectively defined as

$$D_x^{\alpha}\left(f(x)\right)=\frac{1}{\Gamma\left(n-\alpha\right)}\left(\frac{d}{dx}\right)^n\int_0^x\left(x-t\right)^{n-\alpha-1}f\left(t\right)dt, \tag{1}$$

$n-1<\alpha\leq n$ and

$$D_x^{\alpha}\left(f(x)\right)=\frac{1}{\Gamma\left(n-\alpha\right)}\int_0^x\left(x-t\right)^{n-\alpha-1}\left(\frac{d}{dt}\right)^n f\left(t\right)dt, \tag{2}$$

$n-1<\alpha\leq n$. A new fractional derivative with no singular kernel was recently proposed by Caputo et al. in [7]. However, Caputo fractional derivative [8], for instance, is the one mostly used for modelling real world problems in the field [4, 6, 13–15, 20, 28]. However, this derivative exhibits some limitations like not obeying the traditional chain rule; which chain rule represents one of the key elements of the match asymptotic method [20, 28]. Recall that the match asymptotic method has never been used to solve any kind of fractional differential equations because of the nature and properties of fractional derivatives. Hence, the conformable fractional derivative was proposed [2, 21]. This fractional derivative is theoretically very easier to handle and obeys the chain rule. But it also exhibits a huge failure that is expressed by the fact that the fractional derivative of any differentiable function at the point zero is zero. This does not make any sense in a physical point of view and then, a modified new version, the $\beta$–derivative was proposed in order to skirt the noticed weakness. The main aim of this new derivative was, first of all, to extend the well-known match asymptotic method to the scope of the fractional differential equation and later to describe the boundary layers problems within the folder of fractional calculus. The $\beta$–derivative was defined as [1, 15, 20]:

$${}_0^A D_t^{\beta} g(t)=\begin{cases}\lim\limits_{\varepsilon\to 0}\frac{g\left(t+\varepsilon\left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right)-g(t)}{\varepsilon} & \text{for all } t\geq 0,\quad 0<\beta\leq 1\\ g(t) & \text{for all } t\geq 0,\beta=0,\end{cases} \tag{3}$$

where $g$ is a function such that $g:[0,\ \infty)\to\mathbb{R}$ and $\Gamma$ the gamma-function

$$\Gamma(\zeta)=\int_0^\infty t^{\zeta-1}e^{-t}dt.$$

If the above limit of exists then $g$ is said to be $\beta-$differentiable.

Note that for $\beta=1$, we have ${}_0^AD_t^\beta g(t)=\frac{d}{dt}g(t)$. Moreover, unlike other derivatives with fractional parameters, the $\beta$–derivative of a function can be locally defined at a certain point, the same way like the first order derivative. For a general order, let us say $m\beta$, the $m\beta$–derivative of $g$ is defined as

$${}_0^AD_t^{m\beta}g(t)={}_0^A\,D_t^\beta\left({}_0^AD_t^{(m-1)\beta}g(t)\right)\quad\text{for all } t\geq 0, m\in\mathbb{N},\ 0<\beta\leq 1 \tag{4}$$

Notice that the $m\beta$–derivative of a given function provides information about the previous $n-1$–derivatives of the same function. For instance we have

$$\begin{aligned}{}_0^AD_t^{2\beta}g(t)&={}_0^A\,D_t^\beta\left({}_0^AD_t^\beta g(t)\right)\\&=\left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\left[(1-\beta)\left(t+\frac{1}{\Gamma(\beta)}\right)^{-\beta}g'+\left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}g''\right].\end{aligned} \tag{5}$$

This gives the $\beta$–derivative a unique property of memory, that is not provided by any other derivative. It is also easy to verify that for $\beta=1$, we recover the second derivative of $g$. For more properties and details on this new derivative, the readers can consult the reference [1, 15, 20, 28].

### 1.1 The kinetic equation

The evolution of the sizes distribution occurring during polymer chain degradation is well known [12, 15, 32] to be described by the following integrodifferential equation

$$\frac{\partial}{\partial t}g(x,t)=-g(x,t)\int_0^x H(y,x-y)dy+2\int_x^\infty g(y,t)H(x,y-x)dy,\qquad x,\,t>0. \tag{6}$$

Expressing the solution of equation (6) in its explicit form is very hard since fragmentation (or polymer chain degradation) processes, as explained in the previous section, are difficult to analyse as they involve evolution of two intertwined quantities: the distribution of mass among the particles in the ensemble and the number of particles in it. That is why, though linear, they display non-linear features such as "shattering" phenomena which they cannot fully explain [11, 15, 33]. Then, in order to have a broader idea about the evolution of polymer chain degradation and maybe trying to understand the phenomenon of shattering as described here above, we explore the possibility of extending the analysis by considering the $\beta$–derivative defined in the previous section. This yields the following integrodifferential equation:

$$ {}_0^A D_t^\beta g(x,t) = -g(x,t)\int_0^x H_\beta(y,x-y)dy + 2\int_x^\infty g(y,t)H_\beta(x,y-x)dy, \qquad x,\, t>0. \tag{7} $$

subject to the initial condition

$$ g(x,0) = g_0(x), \qquad x>0 \tag{8} $$

where $g(x,t)$ represents the density of $x$-groups (i.e. groups of size $x$) at time $t$ and $H_\beta(x,y)$ gives the average fragmentation rate, that is, the average number at which clusters of size $x+y$ undergo splitting to form an $x$-group and a $y$-group.

## 2 Some useful properties in the $\beta-$differentiation

Recall that there is a growing problem about the choice of the type of fractional derivative to use among the large number of its existing versions. We already mentioned the incapacity of most of them to explicitly provide the variation of the functions. Moreover, many models using fractional derivatives are not easy to handle analytically. The $\beta$–derivative allows us to palliate some insufficiencies of other fractional derivatives and then, we were able to successfully extend the well-known match asymptotic method [20, 28] to the scope of the fractional differential equation and also describe the boundary layers problems within the scope of fractional calculus. Next we recall some properties of the $\beta$–derivative all proved in [15, 20, 28].

**Theorem** 2.1. *Assuming that, a given function, say $g:[a,\infty)\to\mathbb{R}$ is $\beta-$differentiable at a given point, say $t_0\geq a$, $\beta\in(0,\,1]$, then $g$ is also continuous at $t_0$.*

**Theorem** 2.2. *Assuming that $f$ is $\beta-$differentiable on an open interval $(a,\,b)$ then*

1. *If ${}_0^A D_t^\beta f(t)<0$ for all $t\in(a,\,b)$ then $f$ is decreasing on $(a,b)$;*
2. *If ${}_0^A D_t^\beta f(t)>0$ for all $t\in(a,\,b)$ then $f$ is increasing on $(a,b)$;*
3. *If ${}_0^A D_t^\beta f(t)=0$ for all $t\in(a,\,b)$ then $f$ is constant on $(a,b)$.*

**Theorem** 2.3. *Assuming that, $g\neq 0$ and $f$ are two $\beta-$differentiable functions with $\beta\in(0,1]$ then the following relations are satisfied*

1. ${}_0^A D_t^\beta\left(af(t)+bg(t)\right)=a{}_0^A D_t^\beta\left(f(t)\right)+b{}_0^A D_t^\beta\left(g(t)\right)$ *for all real numbers $a$ and $b$;*
2. ${}_0^A D_t^\beta(c)=0$ *for any given constant $c$;*
3. ${}_0^A D_t^\beta\left(f(t)g(t)\right)=g(t)\,{}_0^A D_t^\beta\left(f(t)\right)+f(t)\,{}_0^A D_t^\beta\left(g(t)\right)$;
4. ${}_0^A D_t^\beta\left(\frac{f(t)}{g(t)}\right)=\frac{g(t){}_0^A D_t^\beta(f(t))-f(t){}_0^A D_t^\beta(g(t))}{g^2(t)}$.

**Theorem** 2.4. *Let $f:[a,\infty)\to\mathbb{R}$ be a function such that $f$ is differentiable and also $\beta-$differentiable. Let $g$ be a function defined in the range of $f$ and also differentiable, then we have the following rule*

$$ {}_0^A D_t^\beta (gof(t)) = \left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta} f'(t)g'(f(t)) \tag{9}$$

**Definition 2.1.** Let $f:[a,\infty)\to\mathbb{R}$ be a given function, then we propose that the $\beta-$ integral of $f$ is

$$ {}_a^A I_t^\beta (f(t)) = \int_a^t \left(\xi+\frac{1}{\Gamma(\beta)}\right)^{\beta-1} f(\xi)d\xi \tag{10}$$

The above operator is the inverse operator of the proposed fractional derivative. We shall present to underpin this statement by the following theorem.

**Theorem** 2.5. ${}_0^A D_t^\beta \left[{}_0^A I_t^\beta f(t)\right] = f(t)$ for all $t \geq 0$ with $f$ a given continuous and differentiable function.

*Proof.* [1, Theorem 7] □

**Theorem** 2.6.

$$ {}_a^A I_t^\beta \left[D_t^\beta f(t)\right] = f(t)-f(a) \tag{11}$$

for all $t \geq a$ with $f$ a given continuous and differentiable function.

*Proof.* [1, Theorem 8] □

## 3 Solutions to the model

Note that these above models (6) and (7) are well applicable in many branches of natural sciences, including physics, chemistry, engineering, biology, ecology, just to name a few, and in numerous domains of applied sciences, such as the rock fractures and break of droplets. Various types of fragmentation equations have been comprehensively analyzed in numerous works (see, e.g., [12, 30, 33]). In the domain of polymer science, the fragmentation dynamics has also been of considerable interest, since degradation of bonds or depolymerisation results in fragmentation, see [5, 23, 32]. In [23], the authors used statistical arguments to find and analyze the size distribution of the model. The authors in [5] analysed the model in combination with the inverse process, that is, the coagulation process, and provided a similar result for the size distribution. However, the classical fragmentation model (6) has been proved to be unable to fully describe some bizarre phenomena observed in such a degradation process, like for instance shattering as described above and also in [11, 23, 32, 33]. Recall that shattering is a phenomenon seen as an explosive or dishonest Markov process, see e.g. [3, 24] and has been associated with an infinite cascade of breakup events creating a 'dust' of particles of zero size which, however, carry non-zero mass. Hence, to have explicit solutions to the model, we consider

the case where the breakup rate depends on the size of the chain breaking and takes the form

$$H_\beta(x,y)=(x+y)^\nu, \quad \nu\in\mathbb{R} \tag{12}$$

Substituting in equation (7) yields

$$D_t^\beta(g(x,t))=-x^{\nu+1}g(x,t)+2\int_x^\infty y^\nu g(y,t)dy, \quad 0\le\beta\le 1 \tag{13}$$

Taking the the modified Sumudu transform $S_\beta$ (see the Appendix below) of both sides of equation (13) yields

$$S_\beta\left(D_t^\beta g(x,t),r\right)=-x^{\nu+1}G_s^\beta(x,r)+2\int_x^\infty y^\nu G_s^\beta(y,r)dy,$$

where $G_s^\beta(x,r)$ represents the the modified Sumudu transform $S_\beta(g(x,t),r)$ of $g(x,t)$. Using the relation (23) of Appendix, we obtain

$$r^{-2}(G_s^\beta(x,r)-g_0(x))=-x^{\nu+1}G_s^\beta(x,r)+2\int_x^\infty y^\nu G_s^\beta(y,r)dy,$$

rearranged to have

$$\left(1+x^{\nu+1}r^2\right)G_s^\beta(x,r)-2r^2\int_x^\infty y^\nu G_s^\beta(y,r)dy=g_0(x). \tag{14}$$

Next, it is important to mention that considering the differential equation (13), it is implicitly required that the function $\xi\longrightarrow g(\xi,t)$ is integrable, in the sense of Lebesgue, on any interval $[\epsilon,\infty)$ for $\epsilon>0$ and almost every $\xi>0$. Obviously, the same assertion applies to the functions $\xi\longrightarrow g_0(\xi)$ and $\xi\longrightarrow G_s^\beta(\xi,r), \quad 0\le\beta\le 1$.

This allows us to put

$$Z(x,r)=-2r^2\int_x^\infty y^\nu G_s^\beta(y,r)dy \tag{15}$$

knowing that the integrand will be integrable over any interval $[\epsilon,\infty)$ and the integral will be absolutely continuous at each $x>0$. The substitution of $Z(x,r)$ into (14) yields the partial differential equation

$$\left(\frac{1+x^{\nu+1}r^2}{1+r^2x^\nu}\right)\partial_x Z(x,r)+Z(x,r)=g_0(x). \tag{16}$$

Choosing the constant in the general solution so as to have solutions converging to zero at $\infty$, we obtain its solution given as

$$Z(x,r)=2r^2e^{-\sigma_{r,\nu}(x)}\int_x^\infty\frac{\xi^\nu g_0(\xi)}{1+r^2\xi^{\nu+1}}e^{\sigma_{r,\nu}(\xi)}d\xi$$

where

$$\sigma_{r,\nu}(x)=\int_0^x \frac{2r^2\xi^\nu}{1+r^2\xi^{\nu+1}}d\xi=\ln\left(1+r^2x^{\nu+1}\right)^{\frac{2}{\nu+1}}. \tag{17}$$

Thus, substituting $Z(x,r)$ into (15) yields the solution of (14) given as

$$\begin{aligned} G_s^\beta(x,r)&=\frac{-1}{x^\nu}\left(\frac{2r^2x^\nu}{1+r^2x^{\nu+1}}e^{-\sigma_{r,\nu}(x)}\right)\int_\infty^x \frac{\xi^\nu g_0(\xi)}{1+r^2\xi^{\nu+1}}e^{\sigma_{r,\nu}(\xi)}d\xi+\frac{g_0(x)}{1+r^2x^{\nu+1}}\\ &=\frac{g_0(x)}{1+r^2x^{\nu+1}}-\frac{2r^2}{\left(1+r^2x^{\nu+1}\right)^{\frac{2}{\nu+1}+1}}\int_\infty^x \xi^\nu\left(1+r^2\xi^{\nu+1}\right)^{\frac{2}{\nu+1}-1}g_0(\xi)d\xi \end{aligned} \tag{18}$$

Applying the inverse of the modified Sumudu transform, which coincides with the inverse Sumudu transform, we are finally lead to the solution of the model (13), given by

$$\begin{aligned} g(x,t)&=S_\beta^{-1}(G_s^\beta(x,r),t)\\ &=g_0(x)S_\beta^{-1}\left(\frac{1}{1+r^2x^{\nu+1}},t\right)-2\int_\infty^x \xi^\nu g_0(\xi)S_\beta^{-1}\left(\frac{r^2\left(1+r^2\xi^{\nu+1}\right)^{\frac{2}{\nu+1}-1}}{\left(1+r^2x^{\nu+1}\right)^{\frac{2}{\nu+1}+1}},t\right)d\xi\\ &=g_0(x)\cos(t\sqrt{x^{\nu+1}})-2\int_\infty^x \xi^\nu g_0(\xi)S_\beta^{-1}\left(\frac{r^2\left(1+r^2\xi^{\nu+1}\right)^{\frac{2}{\nu+1}-1}}{\left(1+r^2x^{\nu+1}\right)^{\frac{2}{\nu+1}+1}},t\right)d\xi \end{aligned} \tag{19}$$

***Remark*** *3.1.* The expression $g(x,t)$ in (19) is well-defined only if the integral

$$\int_\infty^x \xi^\nu g_0(\xi)S_\beta^{-1}\left(\frac{r^2\left(1+r^2\xi^{\nu+1}\right)^{\frac{2}{\nu+1}-1}}{\left(1+r^2x^{\nu+1}\right)^{\frac{2}{\nu+1}+1}},t\right)d\xi$$

converges.

We are now capable of taking some specific values of $\nu$ to see the exact expression of the solution.

- For $\nu=1$, expression (19) becomes

$$\begin{aligned} g(x,t)&=g_0(x)S_\beta^{-1}\left(\frac{1}{1+r^2x^2},t\right)-2\int_\infty^x \xi g_0(\xi)S_\beta^{-1}\left(\frac{r^2}{\left(1+r^2x^2\right)^2},t\right)d\xi\\ &=g_0(x)\cos xt-\frac{t\sin xt}{x}\int_\infty^x \xi g_0(\xi)d\xi \end{aligned} \tag{20}$$

- For $\nu=-3$, expression (19) becomes

$$\begin{aligned} g(x,t) &= g_0(x)S_\beta^{-1}\left(\frac{1}{1+r^2x^{-2}},t\right)-2\int_\infty^x \xi g_0(\xi)S_\beta^{-1}\left(r^2\left(1+r^2\xi^{-2}\right)^{-2},t\right)d\xi \\ &= g_0(x)\cos\frac{t}{x}-2\int_\infty^x \xi g_0(\xi)\frac{\xi t\sin\frac{t}{\xi}}{2}d\xi \qquad (21) \\ &= g_0(x)\cos\frac{t}{x}-\int_\infty^x t\xi^2 g_0(\xi)\sin\frac{t}{\xi}d\xi \end{aligned}$$

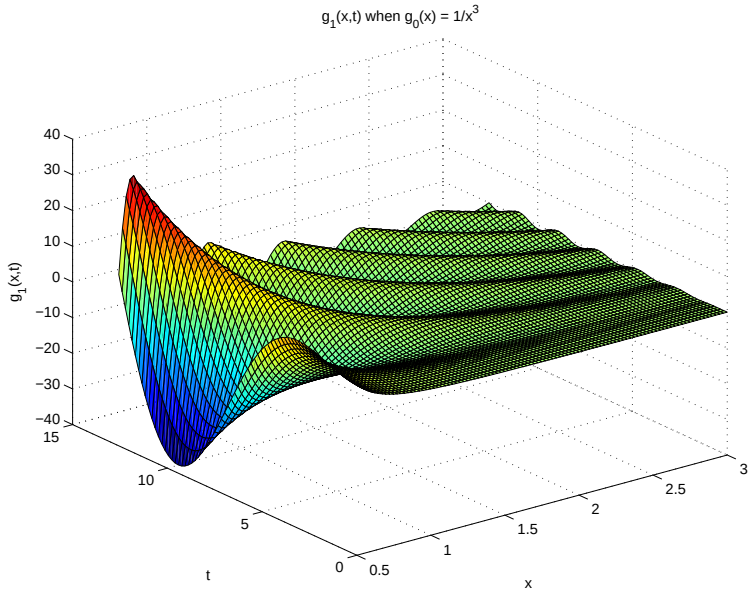



**Fig.** 1. $g(x,t)$ when $\nu=1$ and $g_0(x)=1/x^3$

## 4 Concluding remarks

We have explored the possibility of using new and alternative methods to generalize evolution equation modeling the polymer chain degradation. In the process, a modified version of the Sumudu transform is exploited to perform analysis of the system endowed the $\beta-$derivative and where the breakup rate depends on the size of the chain breaking up. Explicit forms of the solutions in some particular cases showed that the dynamics of this evolution exhibits complex periodic properties due to the presence of cosine and sine functions, as shown in Figs. 1 to 6, plotted for a positive value ($\nu=1$) and a negative value ($\nu=-3$) of $\nu$. Figs. 1 to 3 represent the solution for $\nu=1$ with initial condition $g_0(x)=1/x^3$ : Fig.1 is the 2−D surface plot while Fig. 2 and 3 are respectively its cross

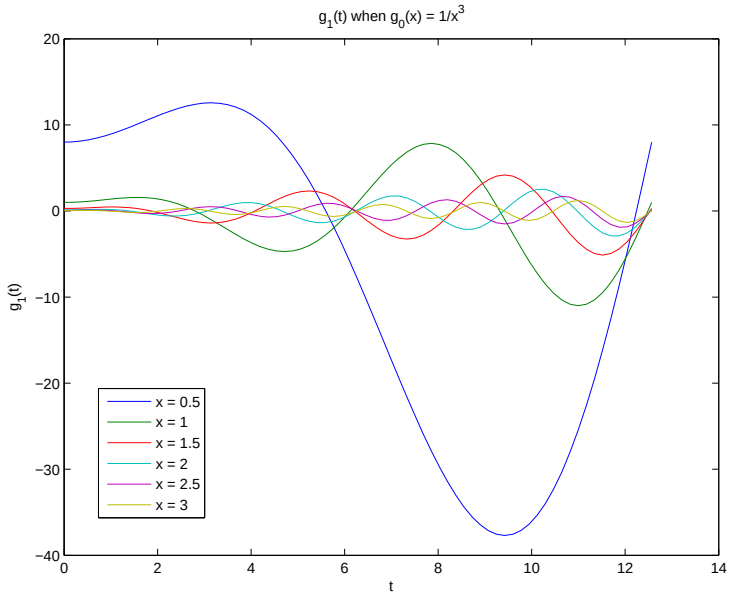



**Fig.** 2. $g(x,t)$ as a function of $t$ when $\nu=1$ and $g_0(x)=1/x^3$, for a few values of $x$

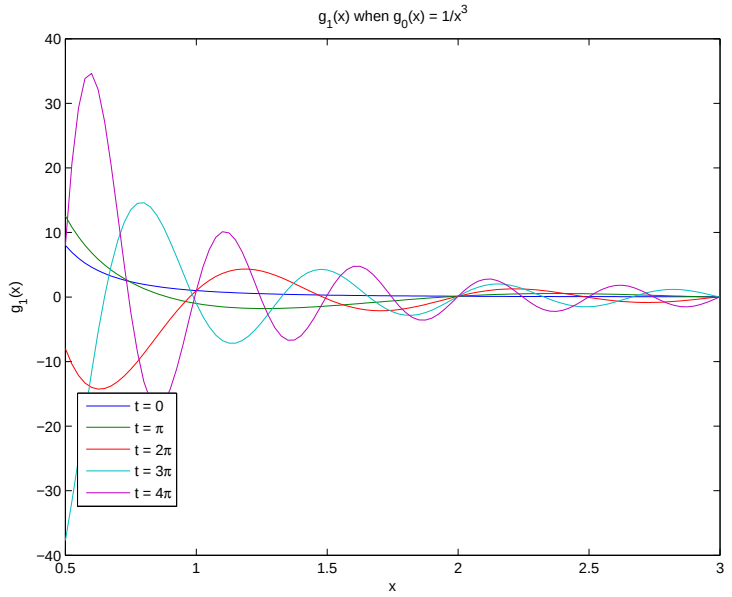



**Fig.** 3. $g(x,t)$ as a function of $x$ when $\nu=1$ and $g_0(x)=1/x^3$, for a few values of $t:\ 0,\pi,2\pi,3\pi,4\pi$

section and longitudinal section drawn for some specific values of the size $x$ and time $t$. A similar reasoning applies to Figs. 4 to 6, but this time with $\nu=-3$. This infers existence of complex and simple harmonic poles in the dynamics of polymer chain degradation whose effects are characterized by these functions or a combination of them. This work improved the preceding one with the inclusion of a more general expression of the breakup rate derivative and $\beta-$derivative. This work might be a breakthrough that may lead to

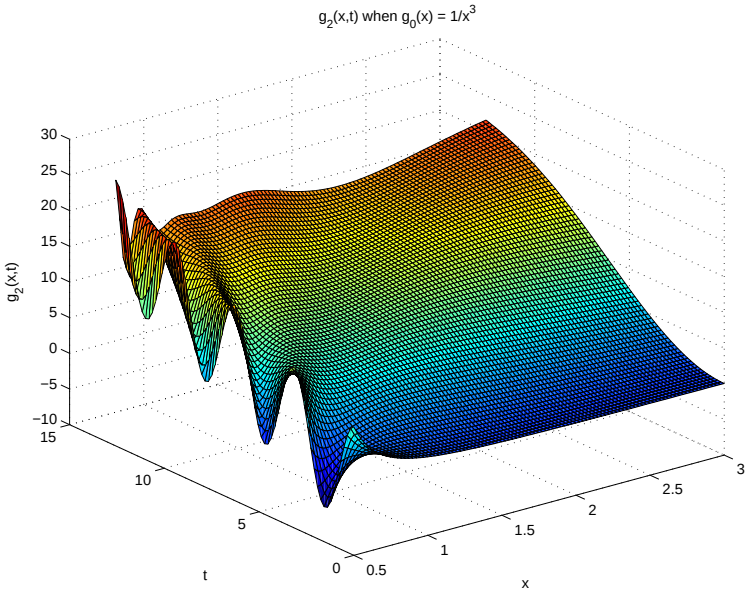



**Fig.** 4. $g(x,t)$ when $\nu=-3$ and $g_0(x)=1/x^3$

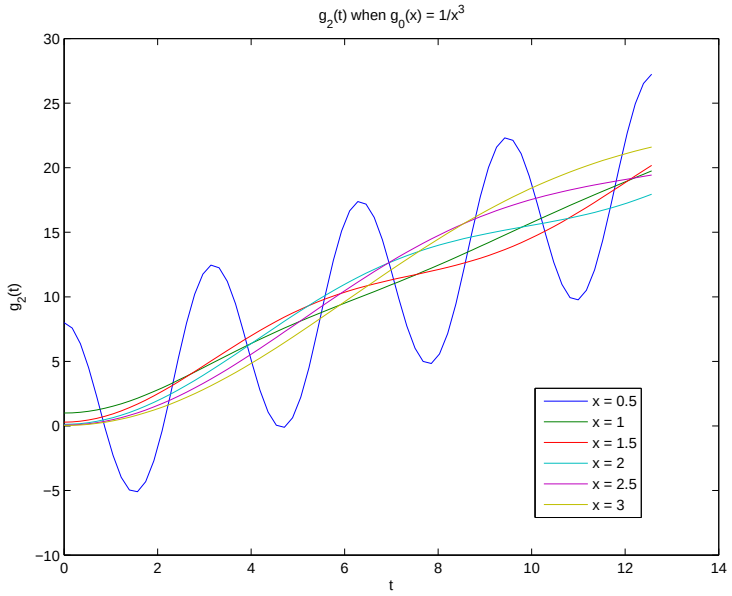



**Fig.** 5. $g(x,t)$ as a function of $t$ when $\nu=-3$ and $g_0(x)=1/x^3$, for a few values of $x$

a better understanding of bizarre phenomena happening in some dynamics such as the phenomenon of shattering.

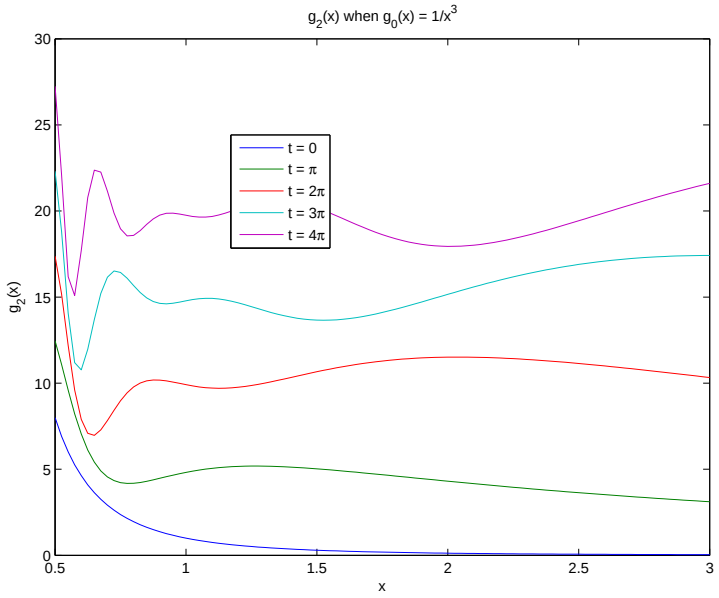



**Fig.** 6. $g(x,t)$ as a function of $x$ when $\nu=-3$ and $g_0(x)=1/x^3$, for a few values of $t$: $0,\pi,2\pi,3\pi,4\pi$

## Appendix: The new Sumudu integral transform

**Definition:** Let $g$ be a function defined in $(0,\infty)$, then, we define the modified Sumudu transform of $g$ as

$$S_\beta(g(t),u)=\int_0^\infty \left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta-\lceil\beta\rceil}\frac{1}{u}e^{-\frac{t}{u}}g(t)dt, \tag{22}$$

where $\lceil\beta\rceil$ is the smallest integer greater or equal to $\beta$. Since $\beta\in(0,1]$ in this article then, $\beta-\lceil\beta\rceil=\beta-1$.

## An important property of the modified Sumudu transform:

If $S(g(t),u)$ is the well known Sumudu transform of $g$ defined in [31] as

$$S(g(t),u)=\int_0^\infty \frac{1}{u}exp\left[-\frac{t}{u}\right]g(t)dt,$$

then, we have the following relation:

$$S_\beta({}_0^A D_t^\beta g^{n-1}(t),u)=\frac{1}{u^n}S(g(t),u)-\sum_{k=0}^{n-1}\frac{1}{u^{n-k}}g^{(k)}(0) \tag{23}$$

*Proof.* By definition we have

$$S_\beta({}_0^A D_t^\beta g^{n-1}(t),u) = \int_0^\infty \left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}$$
$$\frac{1}{u}exp\left[-\frac{t}{u}\right]\left(\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}\lim_{\varepsilon\to 0}\frac{g^{n-1}\left(t+\varepsilon\left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right)-g^{n-1}(t)}{\varepsilon}\right)dt \qquad (24)$$
$$= \int_0^\infty \left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}\frac{1}{u}exp\left[-\frac{t}{u}\right]\left(\left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\lim_{\eta\to 0}\frac{g^{n-1}(t+\eta)-g^{n-1}(t)}{\eta}\right)dt$$

where we have put $\eta = \varepsilon\left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta} \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. Hence, making use of the well known property of Sumudu transform $S(g(t),u)$ [31], we obtain

$$S_\beta({}_0^A D_t^\beta g^{n-1}(t),u) = S(g^n(t),u) = \frac{1}{u^n}S(g(t),u) - \sum_{k=0}^{n-1}\frac{1}{u^{n-k}}g^{(k)}(0),$$

which concludes the proof. □

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## References

[1] A. Atangana and E.F. Doungmo Goufo, Extension of Match Asymptotic Method to Fractional Boundary Layers Problems, Mathematical Problems in Engineering, Volume 2014, Article ID 107535, (2014).
http://dx.doi.org/10.1155/2014/107535

[2] M. Abu Hammad and R. Khalil, Conformable fractional Heat differential equation, International Journal of Pure and Applied Mathematics, vol. 94, no. 2, pp. 215–221, 2014.

[3] W.J. Anderson, Continuous-Time Markov Chains. An Applications-Oriented Approach, Springer Verlag, New York, 1991.

[4] E.G. Bazhlekova, Subordination principle for fractional evolution equations, Fractional Calculus & Applied Analysis, vol. 3, Number 3, pp. 213 – 230, 2000.

[5] R. Blatz, J.N. Tobobsky, Statistical investigation of fragments of polymer molecules, J. Phys. Chem., V.49. P.77., 1945.

[6] D. Brockmann, L. Hufnagel, Front propagation in reaction-superdiffusion dynamics: Taming Lévy flights with fluctuations, Phys. Review Lett. 98, No 17, Article ID 178301, 2007; DOI:10.1103/PhysRevLett.98.178301.

[7] M. Caputo and M. Fabrizio, A New Definition of Fractional Derivative without Singular Kernel, Progr. Fract. Differ. Appl. 1:2, pp. 1–13, 2015.

[8] M. Caputo, Linear models of dissipation whose Q is almost frequency independent II, Geophys. J. R. Ast. Soc. 13, No 5, 529539, 1967; Reprinted in: Fract. Calc. Appl. Anal. 11, No 1, pp. 3-14, 2008.

[9] E. Demirci, A. Unal and N. Özalp , A fractional order seir model with density dependent death rate, Hacettepe Journal of Mathematics and Statistics, 40, No2, 287–295, 2011
[10] K. Diethelm, The Analysis of Fractional Differential Equations, Springer, Berlin, 2010.
[11] E. F. Doungmo Goufo, Non-local and Non-autonomous Fragmentation-Coagulation Processes in Moving Media, PhD thesis, North-West University, South Africa, 2014.
[12] E.F. Doungmo Goufo, S.C. Oukouomi Noutchie, Honesty in discrete, nonlocal and randomly position structured fragmentation model with unbounded rates. Comptes Rendus Mathematique, C.R Acad. Sci Paris, Ser, I, (2013); http://dx.doi.org/10.1016/j.crma.2013.09.023.
[13] Doungmo Goufo EF, Chaotic processes using the two-parameter derivative with non-singular and non-local kernel: Basic theory and applications, Chaos, Vol. 26, No 8, 2016 http://dx.doi.org/10.1063/1.4958921
[14] E. F. Doungmo Goufo, A biomathematical view on the fractional dynamics of cellulose degradation, Fractional Calculus and Applied Analysis, 2015 (in press)
[15] E.F. Doungmo Goufo, Evolution equations with a parameter and application to transport-convection differential equations, Turkish Journal Of Mathematics, 2016, DOI: 10.3906/mat-1603-107, Available online: 27.06.2016 at http://journals.tubitak.gov.tr/math/accepted.htm
[16] K-J. Engel and R. Nagel, One-Parameter Semigroups for Linear Evolution Equations, Graduate Texts in Mathematics (Book 194), Springer, 2000
[17] M. S. Hantush, Analysis of data from pumping tests in leaky aquifers, Transactions, American Geophysical Union, vol. 37,no. 6, pp. 702-714, 1956.
[18] M. S. Hantush, C.E. Jacob, Non-steady radial flow in an infinite leaky aquifer, Transactions, American Geophysical Union, vol. 36, no. 1, pp. 95-100, 1955.
[19] R. Hilfer, Application of Fractional Calculus in Physics, World Scientific, Singapore, 1999.
[20] J. Kestin, L.N. Persen, The transfer of heat across a turbulent boundary layer at very high prandtl numbers. Int. J. Heat Mass Transfer 5, 1962, pp.355–371.
[21] R. Khalil, M. Al Horani, A. Yousef, M. Sababheh, A new definition of fractional derivative, Journal of Computational and Applied Mathematics, vol. 264, pp. 65–70, 2014.
[22] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations. Elsevier Sci. B.V., Amsterdam, 2006.
[23] H. Mark, R Simha, Degradation of long chain molecules, Trans Faraday 35, 611–618, 1940
[24] J.R. Norris, Markov Chains, Cambridge University Press, Cambridge, 1998.
[25] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Applied Mathematical Sciences, Vol. 44, Springer-Verlag, 1983.
[26] S. Pooseh, H.S Rodrigues, Torres D.F.M., Fractional derivatives in dengue epidemics. In: Simos, T.E., Psihoyios, G., Tsitouras, C., Anastassi, Z. (eds.) Numerical Analysis and Applied Mathematics, ICNAAM, American Institute of Physics, Melville, 2011, pp. 739–742.
[27] Prüss J., Evolutionary Integral Equations and Applications, Birkhäuser, Basel–Boston–Berlin, 1993.
[28] H. Schlichting, Boundary-Layer Theory (7 ed.) New York (USA): McGraw-Hill, 1979.
[29] G. T., Tsao, Structures of Cellulosic Materials and their Hydrolysis by Enzymes, Perspectives in Biotechnology and Applied Microbiology, 205–212, 1986
[30] W. Wagner, Explosion phenomena in stochastic coagulation-fragmentation models, Ann. Appl. Probab. 15 (3) 2081–2112, 2005.
[31] G. K. Watugala, Sumudu transform: a new integral transform to solve differential equations and control engineering problems, International Journal of Mathematical Education in Science and Technology 24, 35–43, 1993.
[32] R.M. Ziff, E.D. McGrady, The kinetics of cluster fragmentation and depolymerization, J. Phys. A 18 3027–3037, 1985.
[33] R.M. Ziff, E.D. McGrady, Shattering transition in fragmentation, Phys. Rev. Lett. 58 (9) 892–895, 1987.

# A fixed point convergence theorem with applications in left multivariate fractional calculus

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**Abstract**

A fixed point theorem is given under general conditions on the operators involved in a Banach space setting. The results find applications in left multivariate fractional calculus.

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**Key Words and Phrases:** Fixed point, Banach space, semi-local convergence, left multivariate fractional calculus.

## 1 Introductions

Numerous problems can be formulated as an equation like

$$R(x) = 0, \tag{1.1}$$

where $R$ is a continuous operator defined on a subset $\Omega$ of a Banach space $B_1$ with values in a Banach space $B_2$ using Mathematical Modelling [1], [7], [11], [12], [16], [18]. The solutions denoted by $x^*$ can be found in explicit form only in special cases. That is why most solution methods for these equations are usually iterative. Let $\mathcal{L}(B_1, B_2)$ denote the space of bounded linear operators from $B_1$ into $B_2$. Let also $A(\cdot) : \Omega \to \mathcal{L}(B_1, B_1)$ be a continuous operator. Set

$$F = LR, \tag{1.2}$$

where $L \in \mathcal{L}(B_2, B_1)$. We shall approximate $x^*$ using a sequence $\{x_n\}$ generated by the fixed point scheme:

$$\begin{array}{l} x_{n+1} := x_n + z_n, \ \ A(x_n) z_n + F(x_n) = 0 \\ \Leftrightarrow z_n = Q(z_n) := (I - A(x_n)) z_n - F(x_n), \end{array} \tag{1.3}$$

where $x_0 \in \Omega$. The sequence $\{x_n\}$ defined by

$$x_{n+1} = Q(x_n) = Q^{(n+1)}(x_0) \tag{1.4}$$

exists. In case of convergence we write:

$$Q^{\infty}(x_0) := \lim_{n \to \infty} (Q^n(x_0)) = \lim_{n \to \infty} x_n. \tag{1.5}$$

Many methods in the literature can be considered special cases of method (1.3). We can choose $A$ to be: $A(x) = F'(x)$ (Newton's method), $A(x) = F'(x_0)$ (Modified Newton's method), $A(x) = [x, g(x); F]$, $g : \Omega \to B_1$ (Steffensen's method). Many other choices for $A$ can be found in [1-20] and the references there in. Therefore, it is important to study the convergence of method (1.3) under generalized conditions. In particular, we present the semi-local convergence of method (1.3) using only continuity assumptions on operator $F$ and for a so general operator $A$ as to allow applications to left multivariate fractional calculus and other areas.

The rest of the paper is organized as follows: Section 2 contains the semi-local convergence of method (1.3). In the concluding Section 3, we suggest some applications to left multivariate fractional calculus.

## 2 Convergence

Let $B(w, \xi)$, $\overline{B}(w, \xi)$ stand, respectively for the open and closed balls in $B_1$ with center $w \in B_1$ and of radius $\xi > 0$.

We present the semi-local convergence of method (1.3) in this section.

**Theorem 2.1** *Let $F : \Omega \subset B_1 \to B_2$, $A(\cdot) : \Omega \to \mathcal{L}(B_1, B_1)$ and $x_0 \in \Omega$ be as defined in the Introduction. Suppose: there exist $\delta_0 \in (0,1)$, $\delta_1 \in (0,1)$, $\eta \geq 0$ such that for each $x, y \in \Omega$*

$$\delta := \delta_0 + \delta_1 < 1, \tag{2.1}$$

$$\|F(x_0)\| \leq \eta, \tag{2.2}$$

$$\|I - A(x)\| \leq \delta_0, \tag{2.3}$$

$$\|F(y) - F(x) - A(x)(y - x)\| \leq \delta_1 \|y - x\| \tag{2.4}$$

*and*

$$\overline{B}(x_0,\delta)\subseteq\Omega, \tag{2.5}$$

*where*

$$\rho=\frac{\eta}{1-\delta}. \tag{2.6}$$

*Then, sequence* $\{x_n\}$ *generated for* $x_0\in\Omega$ *by*

$$x_{n+1}=x_n+Q^{\infty}(0),\quad Q_n(z):=(I-A(x_n))z-F(x_n) \tag{2.7}$$

*is well defined in* $\overline{B}(x_0,\rho)$*, remains in* $\overline{B}(x_0,\rho)$ *for each* $n=0,1,2,...$ *and converges to* $x^*$ *which is the only solution of equation* $F(x)=0$ *in* $\overline{B}(x_0,\rho)$*. Moreover, an apriori error estimate is given by the sequence* $\{\rho_n\}$ *defined by*

$$\rho_0:=\rho,\ \ \rho_n=T_n^{\infty}(0),\ \ T_n(t)=\delta_0+\delta_1\rho_{n-1} \tag{2.8}$$

*for each* $n=1,2,...$ *and satisfying*

$$\lim_{n\to\infty}\rho_n=0. \tag{2.9}$$

*Furthermore, an aposteriori error estimate is given by the sequence* $\{\sigma_n\}$ *derfined by*

$$\sigma_n:=H_n^{\infty}(0),\ H_n(t)=\delta t+\delta_1 p_{n-1}, \tag{2.10}$$

$$q_n:=\|x_n-x_0\|\leq\rho-\rho_n\leq\rho, \tag{2.11}$$

*where*

$$p_{n-1}:=\|x_n-x_{n-1}\|\quad \text{for each}\ \ n=1,2,... \tag{2.12}$$

**Proof.** We shall show using mathematical induction the following assertion is true:

$(A_n)$ $x_n\in X$ and $\rho_n\geq 0$ are well defined and such that

$$\rho_n+p_{n-1}\leq\rho_{n-1}. \tag{2.13}$$

By the definition of $\rho$, (2.3)-(2.6) we have that there exists $r\leq\rho$ (Lemma 1.4 [7, pp. 3]) such that

$$\delta_0\tau+\|F(x_0)\|=r$$

and

$$\delta_0^k r\leq\delta_0^k\rho\to 0\text{ as }k\to\infty.$$

That is (Lemma 1.5 [7, pp. 4]) $x_1$ is well defined and $p_0\leq r$.

We need the estimate:

$$T_1(\rho-r)=\delta_0(\rho-r)+\delta_1\rho_0=$$

$$\delta_0\rho-\delta_0 r+\delta_1\rho=G_0(\rho)-r=\rho-r.$$

That is (Lemma 1.4 [7, pp. 3]) $\rho_1$ exists and satisfies

$$\rho_1 + p_0 \leq \rho - r + r = \rho = \rho_0.$$

Hence $(I_0)$ is true. Suppose that for each $k = 1, 2, ..., n$, assertion $(I_k)$ is true. We must show: $x_{k+1}$ exists and find a bound $r$ for $p_k$. Indeed, we have in turn that

$$\delta_0\rho_k + \delta_1\left(\rho_{k-1} - \rho_k\right) = \delta_0\rho_k + \delta_1\rho_{k-1} - \delta_1\rho_k$$
$$= T_k\left(\rho_k\right) - \delta_1\rho_k \leq \rho_k.$$

That is there exists $r \leq \rho_k$ such that

$$r = \delta_0 r + \delta_1\left(\rho_{k-1} - \rho_k\right) \quad \text{and} \quad \left(\delta_0 + \delta_1\right)^i r \to 0 \tag{2.14}$$

as $i \to \infty$.

The induction hypothesis gives that

$$q_k \leq \sum_{m=0}^{k-1} p_m \leq \sum_{m=0}^{k-1}\left(\rho_m - \rho_{m+1}\right) = \rho - \rho_k \leq \rho,$$

so $x_k \in \overline{B}\left(x_0, \rho\right) \subseteq \Omega$ and $x_1$ satisfies $\left\|I - A\left(x_1\right)\right\| \leq \delta_0$ (by (2.3)).

Using the induction hypothesis, (1.3) and (2.4), we get

$$\left\|F\left(x_k\right)\right\| = \left\|F\left(x_k\right) - F\left(x_{k-1}\right) - A\left(x_{k-1}\right)\left(x_k - x_{k-1}\right)\right\| \tag{2.15}$$
$$\leq \delta_1 p_{k-1} \leq \delta_1\left(\rho_{k-1} - \rho_k\right)$$

leading together with (2.14) to:

$$\delta_0 r + \left\|F\left(x_k\right)\right\| \leq r,$$

which implies $x_{k+1}$ exists and $p_k \leq r \leq \rho_k$. It follows from the definition of $\rho_{k+1}$ that

$$T_{k+1}\left(\rho_k - r\right) = T_k\left(\rho_k\right) - r = \rho_k - r,$$

so $\rho_{k+1}$ exists and satisfies

$$\rho_{k+1} + p_k \leq \rho_k - r + r = \rho_k$$

so the induction for $(I_n)$ is completed.

Let $j \geq k$. Then, we obtain in turn that

$$\left\|x_{j+k} - x_k\right\| \leq \sum_{i=k}^{j} p_i \leq \sum_{i=k}^{j}\left(\rho_j - \rho_{j+1}\right) = \rho_k - \rho_{j+k} \leq \rho_k. \tag{2.16}$$

We also have using induction that

$$\rho_{k+1} = T_{k+1}\left(\rho_{k+1}\right) \leq T_{k+1}\left(\rho_k\right) \leq \delta\rho_k \leq ... \leq \delta^{k+1}\rho. \tag{2.17}$$

Hence, by (2.1) and (2.17) $\lim_{k\to\infty}\rho_k = 0$, so $\{x_k\}$ is a complete sequence in a Banach space $X$ and as such it converges to some $x^*$. By letting $j \to \infty$ in (2.16), we conclude that $x^* \in \overline{B}(x_k, \rho_k)$. Moreover, by letting $k \to \infty$ in (2.15) and using the continuity of $F$ we get that $F(x^*) = 0$. Notice that

$$H_k(\rho_k) \leq T_k(\rho_k) \leq \rho_k,$$

so the apriori bound exists. That is $\sigma_k$ is smaller in general than $\rho_k$. Clearly, the conditions of the theorem are satisfied for $x_k$ replacing $x_0$ (by (2.16)). Hence, by (2.8) $x^* \in \overline{B}(x_n, \sigma_n)$, which completes the proof for the aposteriori bound. ■

**Remark 2.2** *(a) It follows from the proof of Theorem 2.1 that the conclusions hold, if $A(\cdot)$ is replaced by a more general continuous operator $A : \Omega \to B_1$.*

*(b) In the next section some applications are suggested for special choices of the "A" operators with $\gamma_0 := \delta_0$ and $\gamma_1 := \delta_1$.*

# 3 Applications to left multivariate fractional calculus

Our presented earlier semi-local convergence results, see Theorem 2.1, apply in the next two multivariate fractional settings given that the following inequalities are fulfilled:

$$\|1 - A(x)\|_\infty \leq \gamma_0 \in (0,1), \tag{3.1}$$

and

$$\left\| (F(y) - F(x))\overrightarrow{i} - A(x)(y-x) \right\| \leq \gamma_1 \|y - x\|, \tag{3.2}$$

where $\gamma_0, \gamma_1 \in (0,1)$, furthermore

$$\gamma = \gamma_0 + \gamma_1 \in (0,1), \tag{3.3}$$

for all $x, y \in \prod_{i=1}^{k} [a_i^*, b_i^*]$, where $a_i < a_i^* < b_i^* < b_i$, $i = 1, ..., k$.

Above $\overrightarrow{i}$ is the unit vector in $\mathbb{R}^k$, $k \in \mathbb{N}$, $\left\|\overrightarrow{i}\right\| = 1$, and $\|\cdot\|$ is a norm in $\mathbb{R}^k$.

The specific functions $A(x)$, $F(x)$ will be described next.

I) Consider the left multidimensional Riemann-Liouville fractional integral of order $\alpha = (\alpha_1, ..., \alpha_k)$ $(\alpha_i > 0,\ i = 1, ..., k)$:

$$\left(I_{a+}^{\alpha} f\right)(x) = \frac{1}{\prod_{i=1}^{k} \Gamma(\alpha_i)} \int_{a_1}^{x_1} ... \int_{a_k}^{x_k} \prod_{i=1}^{k} (x_i - t_i)^{\alpha_i - 1} f(t_1, ..., t_k)\, dt_1 ... dt_k, \tag{3.4}$$

where $\Gamma$ is the gamma function, $f \in L_\infty\left(\prod_{i=1}^{k}[a_i,b_i]\right)$, $a=(a_1,...,a_k)$, and $x=(x_1,...,x_k)\in\prod_{i=1}^{k}[a_i,b_i]$.

By [6], we get that $\left(I_{a+}^{\alpha}f\right)$ is a continuous function on $\prod_{i=1}^{k}[a_i,b_i]$. Furthermore by [6] we get that $I_{a+}^{\alpha}$ is a bounded linear operator, which is a positive operator, plus that $\left(I_{a+}^{\alpha}f\right)(a)=0$.

In particular, $\left(I_{a+}^{\alpha}f\right)$ is continuous on $\prod_{i=1}^{k}[a_i^*,b_i^*]$.

Thus there exist $x_1,x_2\in\prod_{i=1}^{k}[a_i^*,b_i^*]$ such that

$$\begin{array}{l}\left(I_{a+}^{\alpha}f\right)(x_1)=\min\left(I_{a+}^{\alpha}f\right)(x),\\ \left(I_{a+}^{\alpha}f\right)(x_2)=\max\left(I_{a+}^{\alpha}f\right)(x),\end{array}\tag{3.5}$$

over all $x\in\prod_{i=1}^{k}[a_i^*,b_i^*]$.

We assume that

$$\left(I_{a+}^{\alpha}f\right)(x_1)>0.\tag{3.6}$$

Hence

$$\left\|I_{a+}^{\alpha}f\right\|_{\infty,\prod_{i=1}^{k}[a_i^*,b_i^*]}=\left(I_{a+}^{\alpha}f\right)(x_2)>0.\tag{3.7}$$

Here, we define

$$Jf(x)=mf(x),\ \ 0<m<\frac{1}{2},\tag{3.8}$$

for any $x\in\prod_{i=1}^{k}[a_i^*,b_i^*]$.

Therefore the equation

$$Jf(x)=0,\ \ x\in\prod_{i=1}^{k}[a_i^*,b_i^*],\tag{3.9}$$

has the same solutions as the equation

$$F(x):=\frac{Jf(x)}{2\left(I_{a+}^{\alpha}f\right)(x_2)}=0,\quad x\in\prod_{i=1}^{k}[a_i^*,b_i^*].\tag{3.10}$$

Notice that

$$I_{a+}^{\alpha}\left(\frac{f}{2\left(I_{a+}^{\alpha}f\right)(x_2)}\right)(x)=\frac{\left(I_{a+}^{\alpha}f\right)(x)}{2\left(I_{a+}^{\alpha}f\right)(x_2)}\leq\frac{1}{2}<1,\ \ x\in\prod_{i=1}^{k}[a_i^*,b_i^*].\tag{3.11}$$

Call

$$A(x) := \frac{\left(I_{a+}^{\alpha} f\right)(x)}{2\left(I_{a+}^{\alpha} f\right)(x_2)}, \ \forall\ x \in \prod_{i=1}^{k} [a_i^*, b_i^*]. \tag{3.12}$$

We notice that

$$0 < \frac{\left(I_{a+}^{\alpha} f\right)(x_1)}{2\left(I_{a+}^{\alpha} f\right)(x_2)} \leq A(x) \leq \frac{1}{2}, \ \ \forall\ x \in \prod_{i=1}^{k} [a_i^*, b_i^*]. \tag{3.13}$$

Hence, the first condition (3.1) is fulfilled by

$$|1 - A(x)| = 1 - A(x) \leq 1 - \frac{\left(I_{a+}^{\alpha} f\right)(x_1)}{2\left(I_{a+}^{\alpha} f\right)(x_2)} =: \gamma_0, \ \ \forall\ x \in \prod_{i=1}^{k} [a_i^*, b_i^*]. \tag{3.14}$$

Hence, $\|1 - A(x)\|_{\infty} \leq \gamma_0$, where $\|\cdot\|_{\infty}$ is over $\prod_{i=1}^{k} [a_i^*, b_i^*]$. Clearly $\gamma_0 \in (0,1)$.

Next, we assume that $\frac{f(x)}{2\left(I_{a+}^{\alpha} f\right)(x_2)}$ is a contraction, that is

$$\left| \frac{f(x)}{2\left(I_{a+}^{\alpha} f\right)(x_2)} - \frac{f(y)}{2\left(I_{a+}^{\alpha} f\right)(x_2)} \right| \leq \theta \|x - y\|, \ \ \text{all } x, y \in \prod_{i=1}^{k} [a_i^*, b_i^*], \ \ 0 < \theta < 1. \tag{3.15}$$

Hence

$$\left| \frac{mf(x)}{2\left(I_{a+}^{\alpha} f\right)(x_2)} - \frac{mf(y)}{2\left(I_{a+}^{\alpha} f\right)(x_2)} \right| \leq m\theta \|x - y\| \leq \frac{\theta}{2} \|x - y\|, \ \ \text{all } x, y \in \prod_{i=1}^{k} [a_i^*, b_i^*]. \tag{3.16}$$

Set $\lambda = \frac{\theta}{2}$, it is $0 < \lambda < \frac{1}{2}$. We have that

$$|F(x) - F(y)| \leq \lambda \|x - y\|, \tag{3.17}$$

all $x, y \in \prod_{i=1}^{k} [a_i^*, b_i^*]$.

Equivalently we have

$$|Jf(x) - Jf(y)| \leq 2\lambda \left(I_{a+}^{\alpha} f\right)(x_2) \|x - y\|, \ \ \text{all } x, y \in \prod_{i=1}^{k} [a_i^*, b_i^*]. \tag{3.18}$$

We observe that

$$\left\| (F(y) - F(x)) \overrightarrow{i} - A(x)(y - x) \right\| \leq$$

$$|F(y) - F(x)| + |A(x)| \|y - x\| \leq \tag{3.19}$$

$\lambda \|y - x\| + |A(x)| \|y - x\| = (\lambda + |A(x)|) \|y - x\| =: (\psi_1), \ \ \forall\ x, y \in \prod_{i=1}^{k} [a_i^*, b_i^*]$.

By [6], we have that

$$\left|\left(I_{a+}^{\alpha}f\right)(x)\right| \leq \left(\prod_{i=1}^{k}\frac{\left(b_i-a_i\right)^{\alpha_i}}{\Gamma\left(\alpha_i+1\right)}\right)\|f\|_{\infty}, \tag{3.20}$$

$\forall\ x \in \prod_{i=1}^{k}[a_i^*, b_i^*]$, where $\|\cdot\|_{\infty}$ now is over $\prod_{i=1}^{k}[a_i, b_i]$.

Hence

$$|A(x)| = \frac{\left|\left(I_{a+}^{\alpha}f\right)(x)\right|}{2\left(I_{a+}^{\alpha}f\right)(x_2)} \leq \frac{1}{2\left(I_{a+}^{\alpha}f\right)(x_2)}\left(\prod_{i=1}^{k}\frac{\left(b_i-a_i\right)^{\alpha_i}}{\Gamma\left(\alpha_i+1\right)}\right)\|f\|_{\infty} < \infty, \tag{3.21}$$

$\forall\ x \in \prod_{i=1}^{k}[a_i^*, b_i^*]$.

Therefore we get

$$(\psi_1) \leq \left(\lambda + \frac{1}{2\left(I_{a+}^{\alpha}f\right)(x_2)}\left(\prod_{i=1}^{k}\frac{\left(b_i-a_i\right)^{\alpha_i}}{\Gamma\left(\alpha_i+1\right)}\right)\|f\|_{\infty}\right)\|y-x\|, \tag{3.22}$$

$\forall\ x, y \in \prod_{i=1}^{k}[a_i^*, b_i^*]$.

Call

$$0 < \gamma_1 := \lambda + \frac{1}{2\left(I_{a+}^{\alpha}f\right)(x_2)}\left(\prod_{i=1}^{k}\frac{\left(b_i-a_i\right)^{\alpha_i}}{\Gamma\left(\alpha_i+1\right)}\right)\|f\|_{\infty}, \tag{3.23}$$

and by choosing $(b_i - a_i)$ small enough, $i = 1, ..., k$, we can make $\gamma_1 \in (0, 1)$, fulfilling (3.2).

Next, we call and we need that

$$0 < \gamma := \gamma_0 + \gamma_1 = \left(1 - \frac{\left(I_{a+}^{\alpha}f\right)(x_1)}{2\left(I_{a+}^{\alpha}f\right)(x_2)}\right) +$$

$$\left(\lambda + \frac{1}{2\left(I_{a+}^{\alpha}f\right)(x_2)}\left(\prod_{i=1}^{k}\frac{\left(b_i-a_i\right)^{\alpha_i}}{\Gamma\left(\alpha_i+1\right)}\right)\|f\|_{\infty}\right) < 1, \tag{3.24}$$

equivalently,

$$\lambda + \frac{1}{2\left(I_{a+}^{\alpha}f\right)(x_2)}\left(\prod_{i=1}^{k}\frac{\left(b_i-a_i\right)^{\alpha_i}}{\Gamma\left(\alpha_i+1\right)}\right)\|f\|_{\infty} < \frac{\left(I_{a+}^{\alpha}f\right)(x_1)}{2\left(I_{a+}^{\alpha}f\right)(x_2)}, \tag{3.25}$$

equivalently,

$$2\lambda\left(I_{a+}^{\alpha}f\right)(x_2) + \left(\prod_{i=1}^{k}\frac{\left(b_i-a_i\right)^{\alpha_i}}{\Gamma\left(\alpha_i+1\right)}\right)\|f\|_{\infty} < \left(I_{a+}^{\alpha}f\right)(x_1), \tag{3.26}$$

which is possible for small $\lambda$ and small $(b_i - a_i)$, all $i = 1, ..., k$. That is $\gamma \in (0, 1)$, fulfilling (3.3). So our numerical method converges and solves (3.9).

II) Let $\alpha = (\alpha_1, ..., \alpha_k)$, $\alpha_i > 0$, $m_i = \lceil \alpha_i \rceil$ ($\lceil \cdot \rceil$ ceiling function), $\alpha_i \notin \mathbb{N}$, $i = 1, ..., k \in \mathbb{N}$, and $G \in C^{\sum_{i=1}^{k} m_i - 1}\left(\prod_{i=1}^{k} [a_i, b_i]\right)$, such that

$$0 \neq \frac{\partial^{\sum_{i=1}^{k} m_i} G}{\partial x_1^{m_1} ... \partial x_k^{m_k}} \in L_\infty \left(\prod_{i=1}^{k} [a_i, b_i]\right).$$

Here we consider the multivariate left Caputo type fractional mixed partial derivative of order $\alpha$:

$$D_{*a}^{\alpha} G(x) = \frac{1}{\prod_{i=1}^{k} \Gamma(m_i - \alpha_i)} \int_{a_1}^{x_1} ... \int_{a_k}^{x_k} \prod_{i=1}^{k} (x_i - t_i)^{m_i - \alpha_i - 1} \cdot \tag{3.27}$$

$$\frac{\partial^{\sum_{i=1}^{k} m_i} G(t_1, ..., t_k)}{\partial t_1^{m_1} ... \partial t_k^{m_k}} dt_1 ... dt_k,$$

where again $\Gamma$ is the gamma function, $a = (a_1, ..., a_k)$, $\forall\ x = (x_1, ..., x_k) \in \prod_{i=1}^{k} [a_i, b_i]$. Notice here that $m_i - \alpha_i > 0$, $i = 1, ..., k$.

By [6], we get that $D_{*a}^{\alpha} G$ is a continuous function on $\prod_{i=1}^{k} [a_i, b_i]$, and it holds that $D_{*a}^{\alpha} G(a) = 0$.

In particular $D_{*a}^{\alpha} G$ is continuous on $\prod_{i=1}^{k} [a_i^*, b_i^*]$, where $a_i < a_i^* < b_i^* < b_i$, $i = 1, ..., k$.

Therefore there exist $x_1, x_2 \in \prod_{i=1}^{k} [a_i^*, b_i^*]$ such that

$$\begin{array}{l} (D_{*a}^{\alpha} G)(x_1) = \min (D_{*a}^{\alpha} G)(x), \\ (D_{*a}^{\alpha} G)(x_2) = \max (D_{*a}^{\alpha} G)(x), \end{array} \tag{3.28}$$

over all $x \in \prod_{i=1}^{k} [a_i^*, b_i^*]$.

We assume that

$$(D_{*a}^{\alpha} G)(x_1) > 0. \tag{3.29}$$

Hence

$$\left\| D_{*a}^{\alpha} G \right\|_{\infty, \prod_{i=1}^{k} [a_i^*, b_i^*]} = (D_{*a}^{\alpha} G)(x_2) > 0. \tag{3.30}$$

Here we define

$$JG(x) = mG(x), \ \ 0 < m < \frac{1}{2}, \tag{3.31}$$

for any $x \in \prod_{i=1}^{k} [a_i^*, b_i^*]$.

Therefore the equation

$$JG(x) = 0, \ \ x \in \prod_{i=1}^{k} [a_i^*, b_i^*], \tag{3.32}$$

has the same solutions as the equation

$$F(x) := \frac{JG(x)}{2D_{*a}^{\alpha} G(x_2)} = 0, \quad x \in \prod_{i=1}^{k} [a_i^*, b_i^*]. \tag{3.33}$$

Notice that

$$D_{*a}^{\alpha}\left(\frac{G(x)}{2D_{*a}^{\alpha} G(x_2)}\right) = \frac{D_{*a}^{\alpha} G(x)}{2D_{*a}^{\alpha} G(x_2)} \leq \frac{1}{2} < 1, \quad x \in \prod_{i=1}^{k} [a_i^*, b_i^*]. \tag{3.34}$$

We call

$$A(x) := \frac{D_{*a}^{\alpha} G(x)}{2D_{*a}^{\alpha} G(x_2)}, \ \ \forall \ x \in \prod_{i=1}^{k} [a_i^*, b_i^*]. \tag{3.35}$$

We notice that

$$0 < \frac{D_{*a}^{\alpha} G(x_1)}{2D_{*a}^{\alpha} G(x_2)} \leq A(x) \leq \frac{1}{2}. \tag{3.36}$$

Hence, the first condition (3.1) is fulfilled by

$$|1 - A(x)| = 1 - A(x) \leq 1 - \frac{D_{*a}^{\alpha} G(x_1)}{2D_{*a}^{\alpha} G(x_2)} =: \gamma_0, \ \ \forall \ x \in \prod_{i=1}^{k} [a_i^*, b_i^*]. \tag{3.37}$$

Hence

$$\|1 - A(x)\|_{\infty} \leq \gamma_0, \tag{3.38}$$

where $\|\cdot\|_{\infty}$ is over $\prod_{i=1}^{k} [a_i^*, b_i^*]$.

Clearly $\gamma_0 \in (0, 1)$.

Next we assume that $\frac{G(x)}{2(D_{*a}^{\alpha} G)(x_2)}$ is a contraction, that is

$$\left| \frac{G(x)}{2\left(D_{*a}^{\alpha} G\right)(x_2)} - \frac{G(y)}{2D_{*a}^{\alpha} G(x_2)} \right| \leq \theta \|x - y\|, \ \ \text{all } x, y \in \prod_{i=1}^{k} [a_i^*, b_i^*], \tag{3.39}$$

with $0 < \theta < 1$.

Hence

$$\left| \frac{mG(x)}{2\left(D_{*a}^{\alpha} G\right)(x_2)} - \frac{mG(y)}{2\left(D_{*a}^{\alpha} G\right)(x_2)} \right| \leq m\theta \|x - y\| \leq \frac{\theta}{2} \|x - y\|, \tag{3.40}$$

all $x,y\in\prod_{i=1}^{k}[a_i^*,b_i^*]$.

Set $\lambda=\frac{\theta}{2}$, it is $0<\lambda<\frac{1}{2}$. We have that

$$|F(x)-F(y)|\leq\lambda\|x-y\|, \tag{3.41}$$

all $x,y\in\prod_{i=1}^{k}[a_i^*,b_i^*]$.

Equivalently we have

$$|JG(x)-JG(y)|\leq 2\lambda\left(D_{*a}^{\alpha}G\right)(x_2)\|x-y\|, \quad \text{all } x,y\in\prod_{i=1}^{k}[a_i^*,b_i^*]. \tag{3.42}$$

We observe that

$$\left\|(F(y)-F(x))\overrightarrow{i}-A(x)(y-x)\right\|\leq$$

$$|F(y)-F(x)|+|A(x)|\|y-x\|\leq$$

$$\lambda\|y-x\|+|A(x)|\|y-x\|=(\lambda+|A(x)|)\|y-x\|=:(\psi_2), \tag{3.43}$$

$\forall\ x,y\in\prod_{i=1}^{k}[a_i^*,b_i^*]$.

By (3.27), we notice that

$$|D_{*a}^{\alpha}G(x)|\leq\frac{1}{\prod_{i=1}^{k}\Gamma(m_i-\alpha_i)}\cdot$$

$$\left(\int_{a_1}^{x_1}...\int_{a_k}^{x_k}\prod_{i=1}^{k}(x_i-t_i)^{m_i-\alpha_i-1}dt_1...dt_k\right)\left\|\frac{\partial^{\sum_{i=1}^{k}m_i}G}{\partial x_1^{m_1}...\partial x_k^{m_k}}\right\|_{\infty}$$

$$=\frac{1}{\prod_{i=1}^{k}\Gamma(m_i-\alpha_i)}\left(\prod_{i=1}^{k}\frac{(x_i-a_i)^{m_i-\alpha_i}}{m_i-\alpha_i}\right)\left\|\frac{\partial^{\sum_{i=1}^{k}m_i}G}{\partial x_1^{m_1}...\partial x_k^{m_k}}\right\|_{\infty}$$

$$=\left(\prod_{i=1}^{k}\frac{(x_i-a_i)^{m_i-\alpha_i}}{\Gamma(m_i-\alpha_i+1)}\right)\left\|\frac{\partial^{\sum_{i=1}^{k}m_i}G}{\partial x_1^{m_1}...\partial x_k^{m_k}}\right\|_{\infty}. \tag{3.44}$$

We have proved that

$$|D_{*a}^{\alpha}G(x)|\leq\left(\prod_{i=1}^{k}\frac{(b_i-a_i)^{m_i-\alpha_i}}{\Gamma(m_i-\alpha_i+1)}\right)\left\|\frac{\partial^{\sum_{i=1}^{k}m_i}G}{\partial x_1^{m_1}...\partial x_k^{m_k}}\right\|_{\infty}, \tag{3.45}$$

$\forall\ x \in \prod_{i=1}^{k} [a_i^*, b_i^*]$, where $\|\cdot\|_\infty$ now is over $\prod_{i=1}^{k} [a_i, b_i]$.

Hence we get

$$|A(x)| \leq \frac{1}{2D_{*a}^{\alpha} G(x_2)} \left( \prod_{i=1}^{k} \frac{(b_i - a_i)^{m_i - \alpha_i}}{\Gamma(m_i - \alpha_i + 1)} \right) \left\| \frac{\partial^{\sum_{i=1}^{k} m_i} G}{\partial x_1^{m_1} ... \partial x_k^{m_k}} \right\|_\infty < \infty, \quad (3.46)$$

$\forall\ x \in \prod_{i=1}^{k} [a_i^*, b_i^*]$.

Therefore we obtain

$$(\psi_2) \leq \left( \lambda + \frac{1}{2D_{*a}^{\alpha} G(x_2)} \left( \prod_{i=1}^{k} \frac{(b_i - a_i)^{m_i - \alpha_i}}{\Gamma(m_i - \alpha_i + 1)} \right) \left\| \frac{\partial^{\sum_{i=1}^{k} m_i} G}{\partial x_1^{m_1} ... \partial x_k^{m_k}} \right\|_\infty \right) \|y - x\|, \quad (3.47)$$

$\forall\ x, y \in \prod_{i=1}^{k} [a_i^*, b_i^*]$.

Call

$$0 < \gamma_1 := \lambda + \frac{1}{2D_{*a}^{\alpha} G(x_2)} \left( \prod_{i=1}^{k} \frac{(b_i - a_i)^{m_i - \alpha_i}}{\Gamma(m_i - \alpha_i + 1)} \right) \left\| \frac{\partial^{\sum_{i=1}^{k} m_i} G}{\partial x_1^{m_1} ... \partial x_k^{m_k}} \right\|_\infty, \quad (3.48)$$

and by choosing $(b_i - a_i)$ small enough, $i = 1, ..., k$, we can make $\gamma_1 \in (0, 1)$, fulfilling (3.2).

Next we call and we need that

$$0 < \gamma := \gamma_0 + \gamma_1 = \left( 1 - \frac{D_{*a}^{\alpha} G(x_1)}{2D_{*a}^{\alpha} G(x_2)} \right) +$$

$$\left\{ \lambda + \frac{1}{2D_{*a}^{\alpha} G(x_2)} \left( \prod_{i=1}^{k} \frac{(b_i - a_i)^{m_i - \alpha_i}}{\Gamma(m_i - \alpha_i + 1)} \right) \left\| \frac{\partial^{\sum_{i=1}^{k} m_i} G}{\partial x_1^{m_1} ... \partial x_k^{m_k}} \right\|_\infty \right\} < 1, \quad (3.49)$$

equivalently,

$$\lambda + \frac{1}{2D_{*a}^{\alpha} G(x_2)} \left( \prod_{i=1}^{k} \frac{(b_i - a_i)^{m_i - \alpha_i}}{\Gamma(m_i - \alpha_i + 1)} \right) \left\| \frac{\partial^{\sum_{i=1}^{k} m_i} G}{\partial x_1^{m_1} ... \partial x_k^{m_k}} \right\|_\infty < \frac{D_{*a}^{\alpha} G(x_1)}{2D_{*a}^{\alpha} G(x_2)}, \quad (3.50)$$

equivalently,

$$2\lambda D_{*a}^{\alpha} G(x_2) + \left( \prod_{i=1}^{k} \frac{(b_i - a_i)^{m_i - \alpha_i}}{\Gamma(m_i - \alpha_i + 1)} \right) \left\| \frac{\partial^{\sum_{i=1}^{k} m_i} G}{\partial x_1^{m_1} ... \partial x_k^{m_k}} \right\|_{\infty} < D_{*a}^{\alpha} G(x_1), \quad (3.51)$$

which is possible for small $\lambda$ and small $(b_i - a_i)$, all $i = 1, ..., k$. That is $\gamma \in (0, 1)$, fulfilling (3.3). So our numerical method converges and solves (3.32).

# References


[1] S. Amat, S. Busquier, S. Plaza, *Chaotic dynamics of a third-order Newton-type method*, J. Math. Anal. Applic. **366**, 1, (2010), 164-174.

[2] G. Anastassiou, *Fractional Differentiation Inequalities*, Springer, New York, 2009.

[3] G. Anastassiou, *Fractional Representation Formulae and Right Fractional Inequalities*, Mathematical and Computer Modelling, **54**, (10-12), (2011), 3098-3115.

[4] G. Anastassiou, *Intelligent Mathematics: Computational Analysis*, Springer, Heidelberg, 2011.

[5] G. Anastassiou, *Advanced Inequalities*, World Scientific Publ. Corp., Singapore, New York, 2011.

[6] G. Anastassiou, *On left multidimensional Riemann-Liouville fractional integral*, J. of Computational Analysis and Applications, accepted, 2015.

[7] G. Anastassiou, I.K. Argyros, *Studies in Computational Intelligence*, 624, Intelligent Numerical Methods: Applications to Fractional Calculus, Springer, Heidelberg, 2016.

[8] I.K. Argyros, *A unifying local-semilocal convergence analysis and applications for two-point Newton-like methods in Banach space*, J. Math. Anal. Appl. **298**, (2004), 374-397.

[9] I.K. Argyros, *Convergence and Applications of Newton-type iterations*, Springer-Verlag Publ., New York, 2008.

[10] I.K. Argyros, *On a class of Newton-like methods for solving nonlinear equations*, J. Comput. Appl. Math. **228**, (2009), 115-122.

[11] I.K. Argyros, *A semilocal convergence analysis for directional Newton methods*, Math. Comp., AMS **80**, (2011), 327-343.

[12] I.K. Argyros, Y.J. Cho, S. Hilout, *Numerical methods for equations and its applications*, CRC Press/Taylor and Fracncis Group, New York, 2012.

[13] I.K. Argyros, S. Hilout, *Weaker conditions for the convergence of Newton's method*, J. Complexity **28**, (2012), 364-387.

[14] J.A. Ezquérro, J.M. Gutiérrez, M.A. Hernández, N. Romero, M.J. Rubio, *The Newton method: From Newton to Kantorovich* (Spanish), Gac. R. Soc. Mat. Esp. **13**, (2010), 53-76.

[15] J.A. Ezquérro, M.A. Hernández, *Newton-type methods of high order and domains of semilocal and global convergence*, Appl. Math. Comput. **214**, 1, (2009), 142-154.

[16] L.V. Kantorovich, G.P. Akilov, *Functional Analysis in Normed Spaces*, Pergamon Press, New York, 1964.

[17] A.A. Magréñán, *Different anomalies in a Jarratt family of iterative root finding methods*, Appl. Math. Comput. **233**, (2014), 29-38.

[18] A.A. Magréñán, *A new tool to study real dynamics: The convergence plane*, Appl. Math. Comput. **248**, (2014), 215-224.

[19] F.A. Potra, V. Ptak, *Nondiscrete induction and iterative processes*, Pitman Publ., London, 1984.

[20] P.D. Proinov, *New general convergence theory for iterative processes and its applications to Newton-Kantorovich type theorems*, J. Complexity, **26**, (2010), 3-42.

# Stability of delay-distributed virus dynamics model with cell-to-cell transmission and CTL immune response

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**Abstract**

In this paper, we study the stability analysis of a virus dynamics model with CTL immune response and with both cell-to-cell and virus-to-cell transmissions. The model contains three types of distributed time delays. The existence and global stability of all steady states of the model are determined by two parameters, the basic reproduction number ($R_0$) and the CTL immune response activation number ($R_1$). By using suitable Lyapunov functionals, we show that if $R_0 \leq 1$, then the infection-free steady state $E_0$ is globally asymptotically stable; if $R_1 \leq 1 < R_0$, then the CTL-inactivated infection steady state $E_1$ is globally asymptotically stable; if $R_1 > 1$, then the CTL-activated infection steady state $E_2$ is globally asymptotically stable. Numerical simulations are conducted to support the theoretical results.

**Keywords:** Virus dynamics; CTL immune response; Global stability; time delay; cell-to-cell transmission.

## 1 Introduction

During the past decades, several mathematical models have been proposed to describe the dynamical behavior of many human viruses such as HIV, HBV, HCV and HTLV-I (see e.g. [1]-[27]). Studying the global stability of the model's equilibria has become one of the most important features which help us to better understanding of the virus dynamics. Thus, several researchers have devoted extensive efforts to study the global stability of virus dynamics models (see e.g. [2]-[13]). All the above mentioned works focus on cell-free viral spread in a compartment such as the bloodstream. Recently, some viral infection models have been proposed to model both virus-to-cell and cell-to-cell transmissions (see [28]-[29]). The viral infection model with cell-to-cell transmission and distributed time delay has been proposed in [29] as:

$$\dot{T}(t) = \lambda - dT(t) - \beta_1 T(t)V(t) - \beta_2 T(t)T^*(t), \tag{1}$$

$$\dot{T}^*(t) = \int_0^\infty \left[\beta_1 T(t-s)V(t-s)ds + \beta_2 T(t-s)T^*(t-s)\right] f(s)e^{-\mu_1 s}ds - \mu_1 T^*(t), \tag{2}$$

$$\dot{V}(t) = bT^*(t-s)ds - cV(t), \tag{3}$$

where, $T(t)$, $T^*(t)$ and$V(t)$ are the concentrations of the uninfected cells which are susceptible to infection, infected cells that produces viruses, and free virus particles at time $t$, respectively; $\beta_1$ is the virus-to-cell infection rate constant; $\beta_2$ is the cell-to-cell infection rate constant; $\mu_1$ and $c$ are death rate constants of the infected cells and viruses, respectively; $b$ is the average number of viruses that bud out from an infected cell. $e^{-\mu_1 s}$ is the survival rate of infected cells during the time delay $s$, where $s$ is assumed to be distributed according to a probability distribution $f(s)$.

It is observed that, all the viral infection models with cell-to-cell transmission did not consider the effect of immune response. The immune response is universal and necessary to eliminate or control the disease after viral infection. The Cytotoxic T Lymphocyte (CTL) cells are responsible to attack and kill the infected cells. Several viral infection models have been introduced in the literature to model the CTL immune response to several diseases [23]-[27]. However, in [23]-[27], only virus-to-cell transmission has been considered. Therefore, our aim in this paper is to propose and analyze a delay-distributed virus dynamics model with virus-to-cell and cell-to-cell transmissions and takes into account the CTL immune response.

# 2 The model

In this section, we propose a virus dynamics model with cell-to-cell transmission and CTL immune response.

$$\dot{T}(t) = \lambda - dT(t) - \beta_1 T(t)V(t) - \beta_2 T(t)T^*(t), \tag{4}$$

$$\dot{T}^*(t) = \int_0^\infty f_1(s)e^{-\mu_1 s}\left[\beta_1 T(t-s)V(t-s) + \beta_2 T(t-s)T^*(t-s)\right]ds - \mu_1 T^*(t) - pT^*(t)Z(t), \tag{5}$$

$$\dot{V}(t) = b\int_0^\infty e^{-\mu_2 s} f_2(s)T^*(t-s)ds - cV(t), \tag{6}$$

$$\dot{Z}(t) = kT^*(t)Z(t) - qZ(t), \tag{7}$$

where, $Z(t)$ is the concentration of CTL immune cells at time $t$. The infected cells are killed by the CTL immune response with rate $pT^*(t)Z(t)$, where $p$ is constant. The CTLs are proliferated at a rate $kT^*(t)Z(t)$ and die at a rate $qZ(t)$. All the other variables and parameters of the model have the same meanings as given in (1)-(3).

Let us assume that the probability distribution function $f_i(s)$ satisfy $f_i(s) > 0$, $i = 1, 2$ and

$$\int_0^\infty f_i(s)ds = 1, \quad \int_0^\infty f_i(u)e^{\ell u}du < \infty, \quad i = 1, 2,$$

where $\ell > 0$. Denote

$$\eta_i = \int_0^\infty f_i(s)e^{-\mu_i s}ds, \quad i = 1, 2.$$

Thus $0 < \eta_i \leq 1$. Define the Banach space of fading memory type

$$C_\alpha = \{\phi \in C((-\infty, 0], \mathbb{R}) : \phi(\theta)e^{\alpha\theta} \text{ is uniformly continuous for } \theta \in (-\infty, 0] \text{ and } \|\phi\| < \infty\}$$

where $\alpha$ is a positive constant and $\|\phi\| = \sup_{\theta \leq 0} |\phi(\theta)|\, e^{\alpha\theta}$. Let

$$C_\alpha^+ = \{\phi \in C_\alpha : \phi(\theta) \geq 0 \ for \ \theta \in (-\infty, 0]\}.$$

The initial conditions for system (4)-(7) are given as:

$$\begin{aligned} &T(\theta) = \varphi_1(\theta),\ T^*(\theta) = \varphi_2(\theta), V(\theta) = \varphi_3(\theta),\ Z(\theta) = \varphi_4(\theta), \quad \text{for } \theta \in (-\infty, 0], \\ &\varphi_i \in C_\alpha^+,\ \ i = 1, ..., 4. \end{aligned} \tag{8}$$

By the fundamental theory of functional differential equations [33], system (4)-(7) with initial conditions (8) has a unique solution.

## 2.1 Non-negativity and boundedness of solutions

We show the non-negativity and boundedness of the solutions of model (4)-(7).

**Lemma 1**. The solutions $(T(t), T^*(t), V(t), Z(t))$ of model (4)-(7) with initial conditions (8) are non-negative and ultimately bounded.

**Proof**: First we prove $T(t) > 0$ for all $t \geq 0$. Assume the contrary and let $t_1 > 0$ such that $T(t_1) = 0$. Then from Eq. (4), we have $\dot{T}(t_1) = \lambda > 0$. Therefore $T(t) < 0$ for $t \in (t_1 - \varepsilon, t_1)$ and $\varepsilon > 0$ is sufficiently small. This contradicts with the fact of $T(t) > 0$ for $t \in [0, t_1)$. It follows that $T(t) > 0$ for $t \geq 0$. From Eqs. (5)-(7), we have

$$\begin{aligned}
T^*(t) &= \varphi_2(0)e^{-\int_0^t(\mu_1+pZ(\zeta))d\zeta} \\
&\quad + \int_0^t e^{-\int_\eta^t(\mu_1+pZ(\zeta))d\zeta} \int_0^\infty f_1(s)e^{-\mu_1 s}\left[\beta_1 T(\eta-s)V(\eta-s) + \beta_2 T(\eta-s)T^*(\eta-s)\right] ds d\eta, \\
V(t) &= \varphi_3(0)e^{-ct} + b\int_0^t e^{-c(t-\zeta)} \int_0^\infty f_2(s)e^{-\mu_2 s}T^*(\zeta-s) ds d\zeta, \\
Z(t) &= \varphi_4(0)e^{-\int_0^t(q-kT^*(\zeta))d\zeta},
\end{aligned}$$

which yield that $T^*(t) \geq 0$, $V(t) \geq 0$ and $Z(t) \geq 0$for all $t \geq 0$.

Next we show the boundedness of the solutions. From Eq. (7) we have $\limsup_{t\to\infty} T(t) \leq \dfrac{\lambda}{d}$. Let

$$F(t) = \int_0^\infty f_1(s)e^{-\mu_1 s}T(t-s)ds + T^*(t) + \frac{p}{k}Z(t).$$

Then

$$\begin{aligned}
\dot{F}(t) &= \int_0^\infty f_1(s)e^{-\mu_1 s}\left[\lambda - dT(t-s) - \beta_1 T(t-s)V(t-s) - \beta_2 T(t-s)T^*(t-s)\right] ds - \\
&\quad + \int_0^\infty \beta_1 T(t-s)V(t-s)f_1(s)e^{-\mu_1 s}ds + \int_0^\infty \beta_2 T(t-s)T^*(t-s)f_1(s)e^{-\mu_1 s}ds - \mu_1 T^*(t) - \frac{pq}{k}Z(t) \\
&= \lambda\eta_1 - d\int_0^\infty f_1(s)e^{-\mu_1 s}T(t-s)ds - \mu_1 T^*(t) - \frac{pq}{k}Z(t) \\
&\leq \lambda - \sigma\left(\int_0^\infty f_1(s)e^{-\mu_1 s}T(t-s)ds + T^*(t) + \frac{p}{k}Z(t)\right) = \lambda - \sigma F(t),
\end{aligned}$$

where, $\sigma = \min\{d, \mu_1, q\}$. Hence, $\limsup_{t\to\infty} F(t) \leq \dfrac{\lambda}{\sigma}$. Since $\int_0^\infty f_1(s)e^{-\mu_1 s}T(t-s)ds > 0$ , $T^* \geq 0$ and $Z \geq 0$, then $\limsup_{t\to\infty} T^*(t) \leq L_1$ and $\limsup_{t\to\infty} Z(t) \leq L_2$, where $L_1 = \dfrac{\lambda}{\sigma}$ and $L_2 = \frac{k}{p}L_1$. From Eq. (6) we have

$$\dot{V} = b\int_0^\infty e^{-\mu_2 s}f_2(s)T^*(t-s)ds - cV(t) \leq b\eta_2 L_1 - cV(t) \leq bL_1 - cV(t).$$

Thus $\limsup_{t\to\infty} V(t) \leq L_3$, where $L_3 = \dfrac{bL_1}{c}$. Therefore, $T(t), T^*(t), V(t)$ and $Z(t)$ are ultimately bounded. □

## 2.2 Steady States

**Lemma 1.**

(i) If $R_0 \leq 1$, then there exists only positive steady state $E_0$,

(i) if $R_1 \leq 1 < R_0$, then there exist only two positive steady states $E_0$ and $E_1$,

(ii) if $R_1 > 1$, then there exist three positive steady states $E_0$, $E_1$ and $E_2$.

**The proof.** Let the R.H.S of system (4)-(7) be equal zero

$$0 = \lambda - dT - \beta_1 TV - \beta_2 TT^*, \tag{9}$$

$$0 = \eta_1\left(\beta_1 TV + \beta_2 TT^*\right) - \mu_1 T^* - pT^*Z, \tag{10}$$

$$0 = \eta_2 bT^* - cV, \tag{11}$$

$$0 = kT^*Z - qZ. \tag{12}$$

Solving Eqs. (9)-(12) we find that the system has three steady states, infection-free steady state $E_0 = (T_0, 0, 0, 0, 0)$, where $T_0 = \frac{\lambda}{d}$, CTL-inactivated infection steady state $E_1(T_1, T_1^*, V_1, 0)$ and CTL-activated infection steady state $E_2(T_2, T_2^*, V_2, Z_2)$, where

$$T_1 = \frac{T_0}{R_0}, \qquad T_1^* = \frac{dc}{(b\beta_1\eta_2 + c\beta_2)}(R_0 - 1), \qquad V_1 = \frac{b\eta_2 T_1^*}{c},$$
$$T_2 = \frac{k\lambda c}{kdc + bq\beta_1\eta_2 + q\beta_2 c}, \quad T_2^* = \frac{q}{k}, \quad V_2 = \frac{b\eta_2}{c}T_2^*, \quad Z_2 = \frac{\mu_1}{p}(R_1 - 1),$$

and

$$R_0 = \frac{T_0\eta_1}{\mu_1 c}(b\beta_1\eta_2 + \beta_2 c), \qquad R_1 = \frac{kdc}{q(b\beta_1\eta_2 + \beta_2 c) + kdc}R_0,$$

where $R_0$ represents the basic infection reproduction number which describes the average number of newly infected cells generated from one infected cell at the beginning of the infectious process and $R_1$ represents the immune response activation number which expresses the CTL load during the lifespan of a CTL cell. Clearly $R_0 > R_1$. □

## 2.3 Global stability analysis

In this section, we study the global stability of all the steady states of system (4)-(7) employing the method of Lyapunov function. We will use the follwing function $g(x) = x - 1 - \ln x$ and the notation $(T, T^*, V, Z) = (T(t), T^*(t), V(t), Z(t))$.

**Theorem 1.** If $R_0 \leq 1$, then $E_0$ is GAS.

**Proof.** Define a Lyapunov functional $L$ as follows:

$$\begin{aligned} L(T, T^*, V, Z) = {} & T_0 g\left(\frac{T}{T_0}\right) + \frac{1}{\eta_1}T^* + \frac{\beta_1 T_0}{c}V + \frac{p}{\eta_1 k}Z \\ & + \frac{1}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1 s}\int_0^s [\beta_1 T(t-\theta)V(t-\theta) + \beta_2 T(t-\theta)T^*(t-\theta)]d\theta ds \\ & + \frac{b\beta_1 T_0}{c}\int_0^\infty f_2(s)e^{-\mu_2 s}\int_0^s T^*(t-\theta)d\theta ds. \end{aligned}$$

Calculating the derivative of $L$ along the solutions of the system (4)-(7), we obtain

$$\begin{aligned} \frac{dL}{dt} = {} & \left(1 - \frac{T_0}{T}\right)(\lambda - dT - \beta_1 TV - \beta_2 TT^*) \\ & + \frac{1}{\eta_1}\left[\int_0^\infty f_1(s)e^{-\mu_1 s}\left[\beta_1 T(t-s)V(t-s) + \beta_2 T(t-s)T^*(t-s)\right]ds - \mu_1 T^* - pT^*Z\right] \\ & + \frac{\beta_1 T_0}{c}\left[b\int_0^\infty e^{-\mu_2 s}f_2(s)T^*(t-s)ds - cV\right] + \frac{p}{\eta_1 k}\left[kT^*Z - qZ\right] \\ & + \frac{1}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1 s}\left[\beta_1 TV + \beta_2 TT^* - \beta_1 T(t-s)V(t-s) - \beta_2 T(t-s)T^*(t-s)\right]ds \\ & + \frac{b\beta_1 T_0}{c}\int_0^\infty f_2(s)e^{-\mu_2 s}\left[T^* - T^*(t-s)\right]ds \\ = {} & \left(1 - \frac{T_0}{T}\right)(\lambda - dT) + \left(\beta_2 T_0 + \frac{b\beta_1 T_0\eta_2}{c} - \frac{\mu_1}{\eta_1}\right)T^* - \frac{pq}{\eta_1 k}Z \\ = {} & -d\frac{(T - T_0)^2}{T} + \frac{\mu_1}{\eta_1}(R_0 - 1)T^* - \frac{pq}{\eta_1 k}Z. \end{aligned} \tag{13}$$

If $R_0 \leq 1$, then $\frac{dL}{dt} \leq 0$ for all $T, T^*, Z > 0$. Thus the solutions of system (4)-(7) limit to $M$, the largest invariant subset of $\left\{(T, T^*, V, Z) : \frac{dL}{dt} = 0\right\}$. Clearly, it follows from Eq. (13) that $\frac{dL}{dt} = 0$ if and only if $T = T_0$,

$T^* = 0$ and $Z = 0$. Noting that $M$ is invariant, for each element of $M$ we have $T^* = 0$ and $Z = 0$, then $\dot{T}^* = 0$. From Eq. (5) we drive that

$$0 = \dot{T}^* = \int_0^\infty f_1(s)e^{-\mu_1 s}\beta_1 T_0 V(t-s)ds.$$

It follows that $V = 0$. Hence $\frac{dL}{dt} = 0$ if and only if $T = T_0$, $T^* = 0$, $V = 0$ and $Z = 0$. LaSalle's invariance principle implies that $E_0$ is GAS when $R_0 \leq 1$. □

**Theorem 2.** If $R_1 \leq 1 < R_0$ , then $E_1$ is GAS.

**Proof.** Define the following Lyapunov functional

$$\begin{aligned}U(T,T^*,V,Z) = {} & T_1 g\left(\frac{T}{T_1}\right) + \frac{1}{\eta_1}T_1^* g\left(\frac{T^*}{T_1^*}\right) + \frac{\beta_1 T_1 V_1}{b\eta_2 T_1^*}V_1 g\left(\frac{V}{V_1}\right) + \frac{p}{\eta_1 k}Z \\ & + \frac{\beta_1 T_1 V_1}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1 s}\int_0^s g\left(\frac{T(t-\theta)V(t-\theta)}{T_1 V_1}\right)d\theta ds \\ & + \frac{\beta_2 T_1 T_1^*}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1 s}\int_0^s g\left(\frac{T(t-\theta)T^*(t-\theta)}{T_1 T_1^*}\right)d\theta ds \\ & + \frac{\beta_1 T_1 V_1}{\eta_2}\int_0^\infty f_2(s)e^{-\mu_2 s}\int_0^s g\left(\frac{T^*(t-\theta)}{T_1^*}\right)d\theta ds.\end{aligned}$$

The time derivative of $U$ along the trajectories of (4)-(7) is given by

$$\begin{aligned}\frac{dU}{dt} = {} & \left(1-\frac{T_1}{T}\right)(\lambda - dT - \beta_1 TV - \beta_2 TT^*) \\ & + \frac{1}{\eta_1}\left(1-\frac{T_1^*}{T^*}\right)\left(\int_0^\infty f_1(s)e^{-\mu_1 s}\left[\beta_1 T(t-s)V(t-s) + \beta_2 T(t-s)T^*(t-s)\right]ds - \mu_1 T^* - pT^*Z\right) \\ & + \frac{\beta_1 T_1 V_1}{b\eta_2 T_1^*}\left(1-\frac{V_1}{V}\right)\left(b\int_0^\infty f_2(s)e^{-\mu_2 s}T^*(t-s)ds - cV\right) + \frac{p}{\eta_1 k}\left(kT^*Z - qZ\right) \\ & + \frac{\beta_1 T_1 V_1}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1 s}\left(\frac{TV}{T_1 V_1} - \frac{T(t-s)V(t-s)}{T_1 V_1} + \ln\left(\frac{T(t-s)V(t-s)}{TV}\right)\right)ds \\ & + \frac{\beta_2 T_1 T_1^*}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1 s}\left(\frac{TT^*}{T_1 T_1^*} - \frac{T(t-s)T^*(t-s)}{T_1 T_1^*} + \ln\left(\frac{T(t-s)T^*(t-s)}{TT^*}\right)\right)ds \\ & + \frac{\beta_1 T_1 V_1}{\eta_2}\int_0^\infty f_2(s)e^{-\mu_2 s}\left(\frac{T^*}{T_1^*} - \frac{T^*(t-s)}{T_1^*} + \ln\left(\frac{T^*(t-s)}{T^*}\right)\right)ds. \qquad (14)\end{aligned}$$

Collecting terms of Eq. (14) and applying the steady state condtions for $E_1$:

$$\lambda - dT_1 = \beta_1 T_1 V_1 + \beta_2 T_1 T_1^* = \frac{\mu_1}{\eta_1}T_1^* = \frac{c\mu_1}{b\eta_1\eta_2}V_1,$$

we get

$$\begin{aligned}\frac{dU}{dt} = {} & -\frac{d}{T}(T-T_1)^2 + (\beta_1 T_1 V_1 + \beta_2 T_1 T_1^*)\left(1-\frac{T_1}{T}\right) \\ & - \frac{\beta_1 T_1 V_1}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1 s}\frac{T(t-s)V(t-s)T_1^*}{T_1 V_1 T^*}ds - \frac{\beta_2 T_1 T_1^*}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1 s}\frac{T(t-s)T^*(t-s)}{T_1 T^*}ds \\ & - \frac{\beta_1 T_1 V_1}{\eta_2}\int_0^\infty f_2(s)e^{-\mu_2 s}\frac{V_1 T^*(t-s)}{VT_1^*}ds + \frac{\beta_1 T_1 V_1}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1 s}\ln\left(\frac{T(t-s)V(t-s)}{TV}\right)ds \\ & + \frac{\beta_2 T_1 T_1^*}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1 s}\ln\left(\frac{T(t-s)T^*(t-s)}{TT^*}\right)ds \\ & + \frac{\beta_1 T_1 V_1}{\eta_2}\int_0^\infty f_2(s)e^{-\mu_2 s}\ln\left(\frac{T^*(t-s)}{T^*}\right)ds + \frac{p}{\eta_1}\left(T_1^* - \frac{q}{k}\right)Z + 2\beta_1 T_1 V_1 + \beta_2 T_1 T_1^*.\end{aligned}$$

Consider the following equalities

$$\ln\left(\frac{T(t-s)V(t-s)}{TV}\right)=\ln\left(\frac{T(t-s)V(t-s)T_i^*}{T_iV_iT^*}\right)+\ln\left(\frac{T_i}{T}\right)+\ln\left(\frac{V_iT^*}{VT_i^*}\right),$$
$$\ln\left(\frac{T(t-s)T^*(t-s)}{TT^*}\right)=\ln\left(\frac{T(t-s)T^*(t-s)}{T_iT^*}\right)+\ln\left(\frac{T_i}{T}\right),$$
$$\ln\left(\frac{T^*(t-s)}{T^*}\right)=\ln\left(\frac{V_iT^*(t-s)}{VT_i^*}\right)+\ln\left(\frac{VT_i^*}{V_iT^*}\right),\qquad i=1,2. \tag{15}$$

Using Eq. (15) with $i=1$ we get

$$\begin{aligned}
\frac{dU}{dt}&=-\frac{d}{T}\left(T-T_1\right)^2-\left(\beta_1T_1V_1+\beta_2T_1T_1^*\right)\left(\frac{T_1}{T}-1-\ln\left(\frac{T_1}{T}\right)\right)\\
&-\frac{\beta_1T_1V_1}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1s}\left[\frac{T(t-s)V(t-s)T_1^*}{T_1V_1T^*}-1-\ln\left(\frac{T(t-s)V(t-s)T_1^*}{T_1V_1T^*}\right)\right]ds\\
&-\frac{\beta_2T_1T_1^*}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1s}\left[\frac{T(t-s)T^*(t-s)}{T_1T^*}-1-\ln\left(\frac{T(t-s)T^*(t-s)}{T_1T^*}\right)\right]ds\\
&-\frac{\beta_1T_1V_1}{\eta_2}\int_0^\infty f_2(s)e^{-\mu_2s}\left[\frac{V_1T^*(t-s)}{VT_1^*}-1-\ln\left(\frac{V_1T^*(t-s)}{VT_1^*}\right)\right]ds+\frac{p}{\eta_1}\left(T_1^*-\frac{q}{k}\right)Z\\
&=-\frac{d}{T}\left(T-T_1\right)^2-\left(\beta_1T_1V_1+\beta_2T_1T_1^*\right)g\left(\frac{T_1}{T}\right)-\frac{\beta_1T_1V_1}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1s}g\left(\frac{T(t-s)V(t-s)T_1^*}{T_1V_1T^*}\right)ds\\
&-\frac{\beta_2T_1T_1^*}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1s}g\left(\frac{T(t-s)T^*(t-s)}{T_1T^*}\right)ds-\frac{\beta_1T_1V_1}{\eta_2}\int_0^\infty f_2(s)e^{-\mu_2s}g\left(\frac{V_1T^*(t-s)}{VT_1^*}\right)ds\\
&+\frac{p}{\eta_1}\frac{\beta_1bq\eta_2+\beta_2qc+dck}{(\beta_1bq\eta_2+\beta_2qc)k}\left(R_1-1\right)Z.
\end{aligned}$$

Hence, if $R_1\le 1$, then we obtain that $\frac{dU}{dt}\le 0$ and then solutions of system (4)-(7) limit to $M$, the largest invariant subset of $\left\{(T,T^*,V,Z):\frac{dU}{dt}=0\right\}$. It can be seen that, $\frac{dU}{dt}=0$ if and only if

$$\frac{T_1}{T}=\frac{T(t-s)V(t-s)T_1^*}{T_1V_1T^*}=\frac{T(t-s)T^*(t-s)}{T_1T^*}=\frac{V_1T^*(t-s)}{VT_1^*}=1.$$

LaSalle's invariance principle implies the global stability of $E_1$. □

**Theorem 3.** If $R_1>1$, then $E_2$ is GAS.

Define the following Lyapunov functional

$$\begin{aligned}
W(T,T^*,V,Z)&=T_2g\left(\frac{T}{T_2}\right)+\frac{1}{\eta_1}T_2^*g\left(\frac{T^*}{T_2^*}\right)+\frac{\beta_1T_2V_2}{b\eta_2T_2^*}V_2g\left(\frac{V}{V_2}\right)+\frac{p}{\eta_1k}Z_2g\left(\frac{Z}{Z_2}\right)\\
&+\frac{\beta_1T_2V_2}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1s}\int_0^s g\left(\frac{T(t-\theta)V(t-\theta)}{T_2V_2}\right)d\theta ds\\
&+\frac{\beta_2T_2T_2^*}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1s}\int_0^s g\left(\frac{T(t-\theta)T^*(t-\theta)}{T_2T_2^*}\right)d\theta ds\\
&+\frac{\beta_1T_2V_2}{\eta_2}\int_0^\infty f_2(s)e^{-\mu_2s}\int_0^s g\left(\frac{T^*(t-\theta)}{T_2^*}\right)d\theta ds.
\end{aligned}$$

The time derivative of $W$ along the trajectories of (4)-(7) is given by

$$\begin{aligned}\frac{dW}{dt} &= \left(1-\frac{T_2}{T}\right)(\lambda - dT - \beta_1 TV - \beta_2 TT^*)\\ &+ \frac{1}{\eta_1}\left(1-\frac{T_2^*}{T^*}\right)\left(\int_0^\infty f_1(s)e^{-\mu_1 s}\left(\beta_1 T(t-s)V(t-s)+\beta_2 T(t-s)T^*(t-s)\right)ds - \mu_1 T^* - pT^*Z\right)\\ &+ \frac{\beta_1 T_2 V_2}{b\eta_2 T_2^*}\left(1-\frac{V_2}{V}\right)\left(b\int_0^\infty f_2(s)e^{-\mu_2 s}T^*(t-s)ds - cV\right) + \frac{p}{\eta_1 k}\left(1-\frac{Z_2}{Z}\right)(kT^*Z - qZ)\\ &+ \frac{\beta_1 T_2 V_2}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1 s}\left(\frac{TV}{T_2V_2} - \frac{T(t-s)V(t-s)}{T_2V_2} + \ln\left(\frac{T(t-s)V(t-s)}{TV}\right)\right)ds\\ &+ \frac{\beta_2 T_2 T_2^*}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1 s}\left(\frac{TT^*}{T_2T_2^*} - \frac{T(t-s)T^*(t-s)}{T_2T_2^*} + \ln\left(\frac{T(t-s)T^*(t-s)}{TT^*}\right)\right)ds\\ &+ \frac{\beta_1 T_2 V_2}{\eta_2}\int_0^\infty f_2(s)e^{-\mu_2 s}\left(\frac{T^*}{T_2^*} - \frac{T^*(t-s)}{T_2^*} + \ln\left(\frac{T^*(t-s)}{T^*}\right)\right)ds.\end{aligned}$$

Using the following steady state conditions for $E_2$

$$\lambda - dT_2 = \beta_1 T_2 V_2 + \beta_2 T_2 T_2^* = \frac{p}{\eta_1}T_2^* Z_2 + \frac{\mu_1}{\eta_1}T_2^*, \quad T_2^* = \frac{q}{k}, \quad V_2 = \frac{bq\eta_2}{ck},$$

we get

$$\begin{aligned}\frac{dW}{dt} &= -\frac{d}{T}(T-T_2)^2 + (\beta_1 T_2 V_2 + \beta_2 T_2 T_2^*)\left(1-\frac{T_2}{T}\right)\\ &- \frac{\beta_1 T_2 V_2}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1 s}\frac{T(t-s)V(t-s)T_2^*}{T_2V_2T^*}ds - \frac{\beta_2 T_2 T_2^*}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1 s}\frac{T(t-s)T^*(t-s)}{T_2T^*}ds\\ &- \frac{\beta_1 T_2 V_2}{\eta_2}\int_0^\infty f_2(s)e^{-\mu_2 s}\frac{V_2T^*(t-s)}{VT_2}ds + \frac{\beta_1 T_2 V_2}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1 s}\ln\left(\frac{T(t-s)V(t-s)}{TV}\right)ds\\ &+ \frac{\beta_2 T_2 T_2^*}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1 s}\ln\left(\frac{T(t-s)T^*(t-s)}{TT^*}\right)ds\\ &+ \frac{\beta_1 T_2 V_2}{\eta_2}\int_0^\infty f_2(s)e^{-\mu_2 s}\ln\left(\frac{T^*(t-s)}{T^*}\right)ds + 2\beta_1 T_2 V_2 + \beta_2 T_2 T_2^*.\end{aligned}$$

Using Eq. (15) with $i=2$ we get

$$\begin{aligned}\frac{dW}{dt} &= -\frac{d}{T}(T-T_2)^2 - (\beta_1 T_2 V_2 + \beta_2 T_2 T_2^*)\,g\left(\frac{T_2}{T}\right) - \frac{\beta_1 T_2 V_2}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1 s}g\left(\frac{T(t-s)V(t-s)T_2^*}{T_2V_2T^*}\right)ds\\ &- \frac{\beta_2 T_2 T_2^*}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1 s}g\left(\frac{T(t-s)T^*(t-s)}{T_2T^*}\right)ds - \frac{\beta_1 T_2 V_2}{\eta_2}\int_0^\infty f_2(s)e^{-\mu_2 s}g\left(\frac{V_2T^*(t-s)}{VT_2^*}\right)ds.\end{aligned}$$

Noting that $T, T^*, V, Z > 0$, we have that $\frac{dW}{dt} \le 0$. The solutions of model (4)-(7) converge to $M$, the largest invariant subset of $\left\{(T,T^*,V,Z) : \frac{dW}{dt} = 0\right\}$. We have $\frac{dW}{dt} = 0$ if and only if $T = T_2$ and $g = 0$ i.e.,

$$\frac{T_2}{T} = \frac{T(t-s)V(t-s)T_2^*}{T_2V_2T^*} = \frac{T(t-s)T^*(t-s)}{T_2T^*} = \frac{V_2T^*(t-s)}{VT_2^*} = 1. \tag{16}$$

If $T = T_2$, then from Eq. (16) we get $T^* = T_2^*$ and $V = V_2$. The set $M$ is invariant and for any element belongs to $M$ satisfies $T^* = T_2^*$ and

$$\dot{T}^* = 0 = \eta_1(\beta_1 T_2 V_2 + \beta_2 T_2 T_2^*) - \mu_1 T_2^* - pT_2^* Z,$$

which gives $Z = Z_2$. Therefore, $\frac{dW}{dt} = 0$ if and only if $T = T_2, T^* = T_2^*, V = V_2$ and $Z = Z_2$. The global asymptotic stability of $E_2$ follows from LaSalle's invariance principle. □

# 3 Numerical simulations

In this section, we perform numerical simulations for the model (4)-(7) with particular distribution functions $f_1(s)$ and $f_2(s)$ as:

$$f_1(s) = \delta(s - s_1),\ f_2(s) = \delta(s - s_2),$$

where $\delta(.)$ is the dirac delta function, $s_1$ and $s_2$ are positive constants. Then, we can see that,

$$\int_0^\infty f_i(s)ds = 1,\ \ \eta_i = \int_0^\infty \delta(s - s_i)e^{-\mu_i s}ds = e^{-\mu_i s_i},\ i = 1, 2,$$

$$\int_0^\infty \delta(s - s_1)e^{-\mu s}\phi(t - s)ds = e^{-\mu s_1}\phi(t - s_1),$$

for any function $\phi$. With such choice, model (4)-(7) leads to:

$$\dot{T}(t) = \lambda - dT(t) - \beta_1 T(t)V(t) - \beta_2 T(t)T^*(t), \tag{17}$$

$$\dot{T}^*(t) = \left[\beta_1 T(t - s_1)V(t - s_1) + \beta_2 T(t - s_1)T^*(t - s_1)\right] e^{-\mu_1 s_1} - \mu_1 T^*(t) - pT^*(t)Z(t),$$

$$\dot{V}(t) = be^{-\mu_2 s_2}T^*(t - s_2) - cV(t), \tag{18}$$

$$\dot{Z}(t) = kT^*(t)Z(t) - qZ(t). \tag{19}$$

The parameters $R_0$ and $R_1$ become $R_0 = \dfrac{e^{-\mu_1 s_1}\lambda(b\beta_1 e^{-\mu_2 s_2} + \beta_2 c)}{c\mu_1 d}$, $R_1 = \dfrac{kdc}{q(b\beta e^{-\mu_2 s_2} + \beta_2 c) + kdc}R_0$.

Now we perform some numerical simulations for model (17)-(19) with parameters values given in Table 1.

Table 1: The values of the parameters of model (17 )-(19).

| Parameter | Value | Parameter | Value |
|---|---|---|---|
| $\lambda$ | 10 | $c$ | 3 |
| $d$ | 0.01 | $q$ | 0.1 |
| $p$ | 0.1 | $\beta_2$ | 0.0001 |
| $b$ | 10 | $\mu_1$ | 0.9 |
| $s_1$ | Varied | $\mu_2$ | 0.1 |
| $s_2$ | Varied | $\beta_1, k$ | Varied |

## 3.1 Effect of the parameters $\beta_1$ and $k$ on the stability of steady states

To show the global stablity of the steady states we consider three different initial conditions:

IC1: $\varphi_1(\theta) = 600$, $\varphi_2(\theta) = 1, \varphi_3(\theta) = 1, \varphi_4(\theta) = 10$,

IC2: $\varphi_1(\theta) = 200$, $\varphi_2(\theta) = 0.5, \varphi_3(\theta) = 3, \varphi_4(\theta) = 5$,

IC3: $\varphi_1(\theta) = 700$, $\varphi_2(\theta) = 5, \varphi_3(\theta) = 9, \varphi_4(\theta) = 12$,

where, $\theta \in [-\max\{s_1, s_2\}, 0]$.

In this case we choose $s_1 = 0.5$, $s_2 = 0.9$ and study the following subcases:

**(i)**: $R_0 < 1$. We choose, $\beta_1 = 0.0001$ and $k = 0.008$, then we compute $R_0 = 0.295489$ and $R_1 = 0.194228$. From Lemma 2 we have that the system has one steady state $E_0$. From Figures 1-4 we can see that, the concentration of uninfected cells is increasing and tends its normal value $\lambda/d = 1000$, while the concentrations of infected cells, free viruses and CTls are decaying and approaching zero. It means that, $E_0$ is GAS and the virus will be removed. This result support the result of Theorem 1.

**(ii)**: $R_1 \leq 1 < R_0$. We choose $\beta_1 = 0.001$ and $k = 0.008$, and then, $R_0 = 2.317257$ and $R_1 = 0.455395$. Lemma 2 state that the system has two steady states $E_0$ and $E_1$. Figures 5-8 show that the numerical results are consistent with Theorem 2. We can see that, the solution of the system converges to the steady $E_1(431.54, 4.03, 12.77, 0)$ for the initial conditions IC1-IC3.

**(iii)**: $R_1 > 1$: In this case, we choose $\beta_1 = 0.001$ and $k = 0.03$ and then $R_1 = 1.108600 > 1$. According to Lemma 2, the system has three steady states $E_0, E_1$ and $E_2$. From Figures 9-12 we can see that, the solutions of the system approach the steady state $E_2(478.41, 3.33, 10.57, 0.98)$ for large $t$ and for the initial conditions IC1-IC3. This support the result of Theorems 3.

## 3.2 Effect of the time delays on the stability of steady states

In this case, we consider the initial condition IC2. We take the values $\beta_1 = 0.001$ and $k = 0.03$. Without loss of generality we let $S = s_1 = s_2$. In Table 2, we present the values of $R_0$, $R_1$ and the steady states of system (17 )-(19) with different values of $S$.

From Table 2 we can see that, the values of $R_0$ and $R_1$ are decreased as $S$ is increased. Using the values of the parameters given in Table 1, we obtain that the following:

(i) if $0 \leq S < 0.8447$, then $E_2$ exists and it is GAS,

(ii) if $0.8447 \leq S < 0.8868$, then $E_1$ exists and it is GAS,

(iii) if $S \geq 0.8868$, then $E_0$ is GAS.

Figures 13-16 show that the numerical results are also compatible with the results of Theorems 1-3. From a biological point of view, the intracellular delay plays a similar role as an antiviral treatment in eliminating the virus. We observe that, sufficiently large delay suppresses viral replication and clears the virus. This gives us some suggestions on new drugs to prolong the increase the intracellular delay period.

Table 2: The values of steady states, $R_0$ and $R_1$ for model (17)-(19) with different values of the delay parameter $S$.

| Delay parameter | Steady states | $R_0$ | $R_1$ |
|---|---|---|---|
| $S = 0.0$ | $E_2(578.98, 1.11, 1.23, 27.89)$ | 6.54 | 3.79 |
| $S = 0.2$ | $E_2(626.03, 1.11, 1.01, 17.56)$ | 4.40 | 2.76 |
| $S = 0.4$ | $E_2(670.65, 1.11, 0.83, 9.87)$ | 2.96 | 1.99 |
| $S = 0.7$ | $E_2(731.69, 1.11, 0.61, 1.99)$ | 1.64 | 1.2 |
| $S = 0.80$ | $E_2(751.30, 1.11, 0.55, 0)$ | 1.33 | 1 |
| $S = 0.9$ | $E_1(904.23, 0.39, 0.18, 0)$ | 1.11 | 0.85 |
| $S = 0.95$ | $E_0(1000, 0, 0, 0)$ | 1 | 0.78 |
| $S = 1$ | $E_0(1000, 0, 0, 0)$ | 0.91 | 0.71 |
| $S = 1.5$ | $E_0(1000, 0, 0, 0)$ | 0.34 | 0.29 |
| $S = 2$ | $E_0(1000, 0, 0, 0)$ | 0.13 | 0.12 |

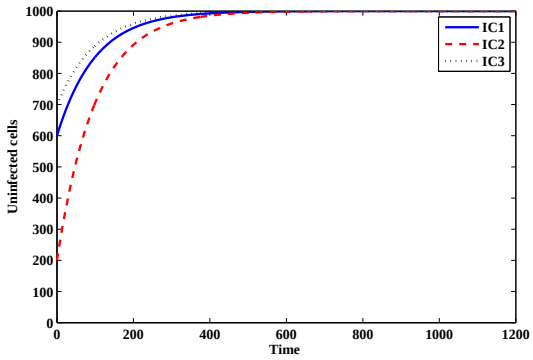



Figure 1: The evolution of uninfected cells with initial IC1-IC3 in case of $R_0 \leq 1$.

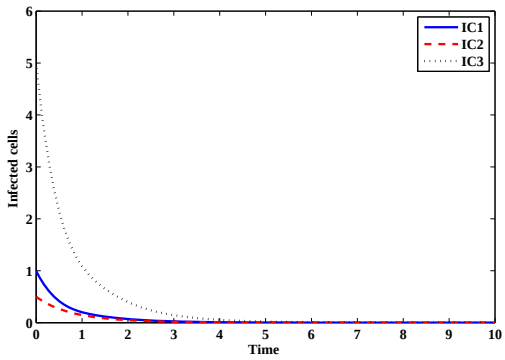



Figure 2: The evolution of infected cells with initial IC1-IC3 in case of $R_0 \leq 1$.

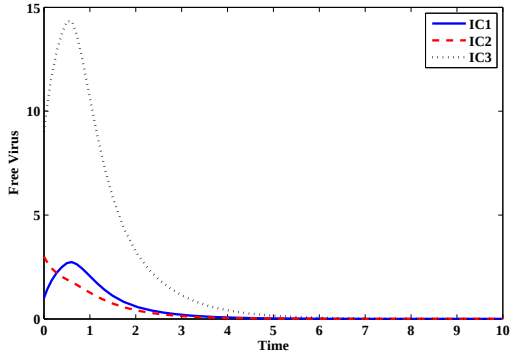



Figure 3: The evolution of free viruses with initial IC1-IC3 in case of $R_0 \leq 1$.

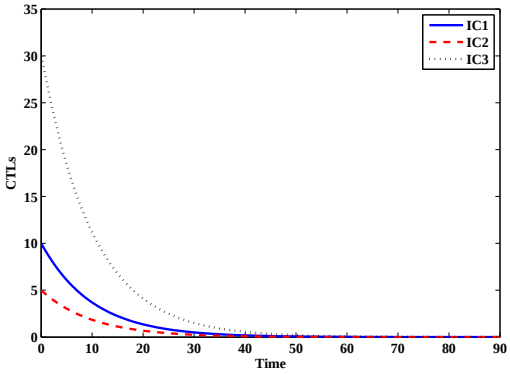



Figure 4: The evolution of CTLs with initial IC1-IC3 in case of $R_0 \leq 1$.

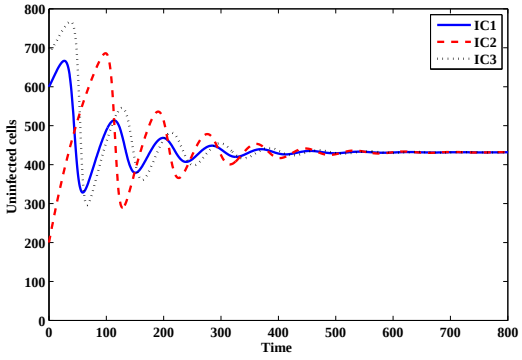



Figure 5: The evolution of uninfected cells with initial IC1-IC3 in case of $R_1 \leq 1 < R_0$.

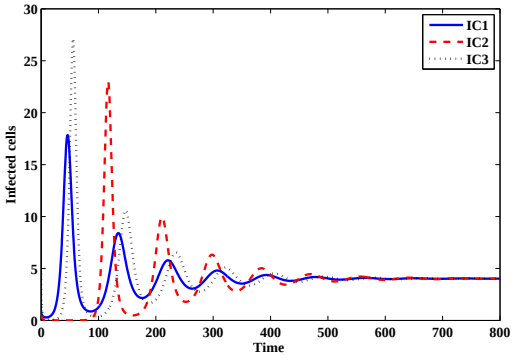



Figure 6: The evolution of infected cells with initial IC1-IC3 in case of $R_1 \leq 1 < R_0$.

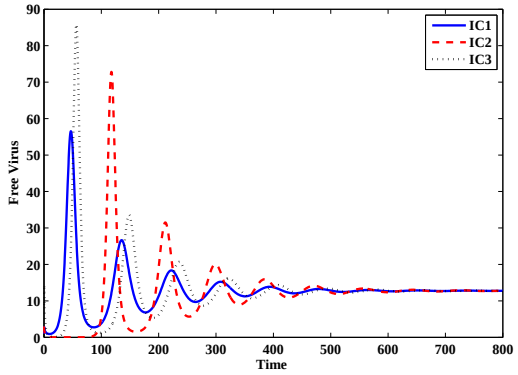



Figure 7: The evolution of free viruses with initial IC1-IC3 in case of $R_1 \leq 1 < R_0$.

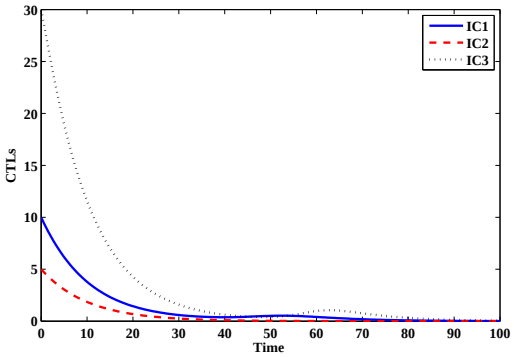



Figure 8: The evolution of CTLs with initial IC1-IC3 in case of $R_1 \leq 1 < R_0$.

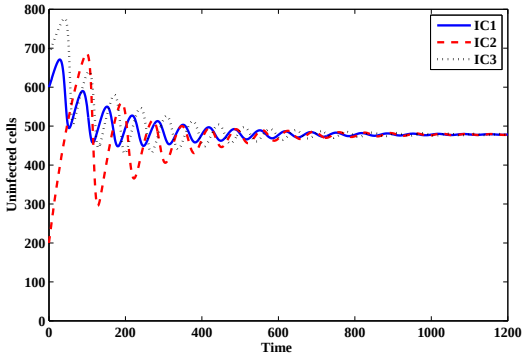



Figure 9: The evolution of uninfected cells with initial IC1-IC3 in case of $R_1 > 1$.

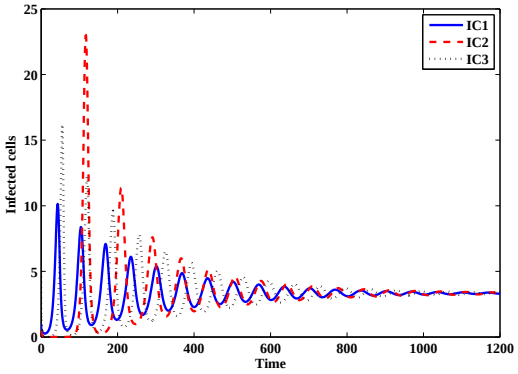



Figure 10: The evolution of infected cells with initial IC1-IC3 in case of $R_1 > 1$.

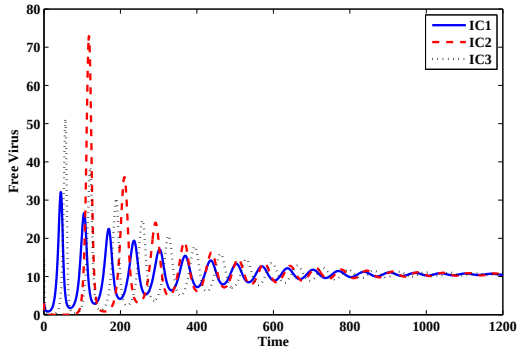



Figure 11: The evolution of free viruses with initial IC1-IC3 in case of $R_1 > 1$.

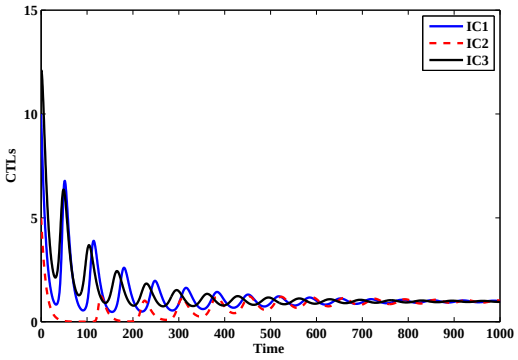



Figure 12: The evolution of CTLs with initial IC1-IC3 in case of $R_1 > 1$.

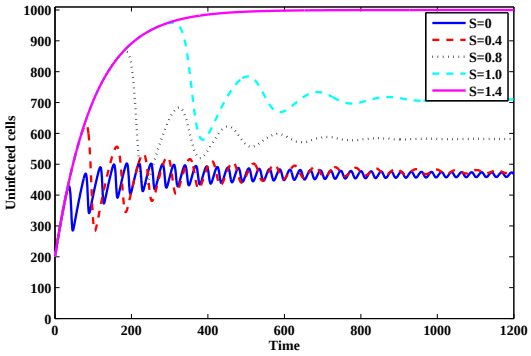



Figure 13: The evolution of uninfected cells with different delay parameter $S$.

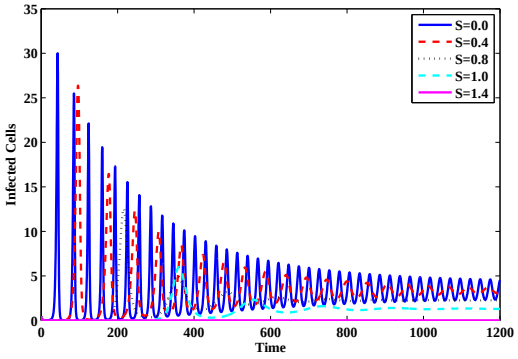



Figure 14: The evolution of infected cells with different delay parameter $S$.

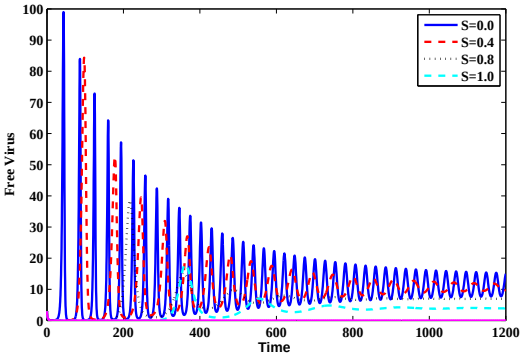



Figure 15: The evolution of free viruses with different delay parameter $S$.

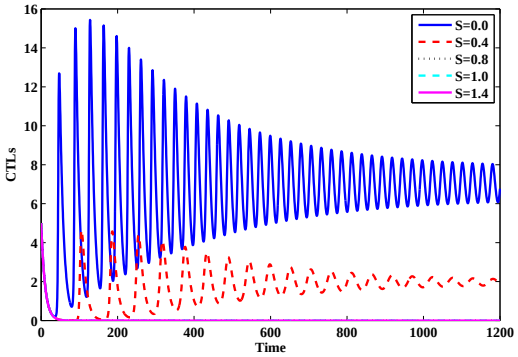



Figure 16: The evolution of CTLs with different delay parameter $S$.

# 4 Acknowledgment

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# References

[1] M. A. Nowak and C. R. M. Bangham, *Population dynamics of immune responses to persistent viruses*, Science, **272** (1996) 74-79.

[2] M. A. Obaid, *Dynamical behaviors of a nonlinear virus infection model with latently infected cells and immune response*, Journal of Computational Analysis and Applications, **21**(1) (2016), 182-193.

[3] A. M. Elaiw and N. H. AlShamrani, *Mathematical analysis of a cell mediated immunity in a virus dynamics model with nonlinear infection rate and removal*, Journal of Computational Analysis and Applications, **21**(3) (2016), 578-586.

[4] C. Connell McCluskey, Y. Yang, *Global stability of a diffusive virus dynamics model with general incidence function and time delay*, Nonlinear Analysis: Real World Applications, **25** (2015), 64-78.

[5] S. Liu, and L. Wang, *Global stability of an HIV-1 model with distributed intracellular delays and a combination therapy*, Mathematical Biosciences and Engineering, **7**(3) (2010), 675-685.

[6] A. M. Elaiw, I. A. Hassanien, and S. A. Azoz, *Global stability of HIV infection models with intracellular delays*, Journal of the Korean Mathematical Society, **49**(4) (2012), 779-794.

[7] A.M. Elaiw and S.A. Azoz, *Global properties of a class of HIV infection models with Beddington-DeAngelis functional response,* Mathematical Methods in the Applied Sciences, **36** (2013), 383-394.

[8] A.M. Elaiw, *Global properties of a class of HIV models*, Nonlinear Analysis: Real World Applications, **11** (2010), 2253-2263.

[9] A. M. Elaiw, and N. A. Almuallem, *Global properties of delayed-HIV dynamics models with differential drug efficacy in co-circulating target cells*, Applied Mathematics and Computation, **265** (2015), 1067-1089.

[10] A. M. Elaiw, and N. A. Almuallem, *Global dynamics of delay-distributed HIV infection models with differential drug efficacy in cocirculating target cells*, Mathematical Methods in the Applied Sciences, **39** (2016), 4-31.

[11] A. M. Elaiw and N. H. AlShamrani, *Global stability of humoral immunity virus dynamics models with nonlinear infection rate and removal*, Nonlinear Analysis: Real World Applications, **26**, pp. 161-190, 2015.

[12] B. Buonomo, and C. Vargas-De-Le, *Global stability for an HIV-1 infection model in cluding an eclipse stage of infected cells*, Journal of Mathematical Analysis and Applications, **385** (2012), 709-720.

[13] K. Wang, A. Fan, and A. Torres, *Global properties of an improved hepatitis B virus model*, Nonlinear Analysis: Real World Applications, **11** (2010), 3131-3138.

[14] P. K. Roy, A. N. Chatterjee, D. Greenhalgh, and Q. J.A. Khan, *Long term dynamics in a mathematical model of HIV-1 infection with delay in different variants of the basic drug therapy model*, Nonlinear Analysis: Real World Applications, **14** (2013), 1621-1633.

[15] M. Y. Li and L. Wang, *Backward bifurcation in a mathematical model for HIV infection in vivo with anti-retroviral treatment*, Nonlinear Analysis: Real World Applications, **17** (2014), 147-160.

[16] B. Li, Y. Chen, X. Lu and S. Liu, *A delayed HIV-1 model with virus waning term*, Mathematical Biosciences and Engineering, **13** (2016), 135-157.

[17] D. Huang, X. Zhang, Y. Guo, and H. Wang, *Analysis of an HIV infection model with treatments and delayed immune response*, Applied Mathematical Modelling, **40**(4) (2016), 3081-3089.

[18] Y. Zhao, D. T. Dimitrov, H. Liu, and Y. Kuang, *Mathematical insights in evaluating state dependent effectiveness of HIV prevention interventions*, Bulletin of Mathematical Biology, **75**(4), pp. 649-675, 2013.

[19] C. Monica and M. Pitchaimani, *Analysis of stability and Hopf bifurcation for HIV-1 dynamics with PI and three intracellular delays*, Nonlinear Analysis: Real World Applications, **27** (2016), 55-69.

[20] A. U. Neumann, N. P. Lam, H. Dahari, D. R. Gretch, T. E. Wiley, T. J, Layden, and A. S. Perelson, *Hepatitis C viral dynamics in vivo and the antiviral efficacy of interferon-alpha therapy*, Science, **282** (1998), 103-107.

[21] L. Wang, M. Y. Li, and D. Kirschner, *Mathematical analysis of the global dynamics of a model for HTLV-I infection and ATL progression*, Mathematical Biosciences, **179** (2002) 207-217.

[22] X. Shi, X. Zhou, and X. Son, *Dynamical behavior of a delay virus dynamics model with CTL immune response*, Nonlinear Analysis: Real World Applications, **11** (2010), 1795-1809.

[23] A. M. Elaiw, R. M. Abukwaik and E. O. Alzahrani, *Global properties of a cell mediated immunity in HIV infection model with two classes of target cells and distributed delays*, International Journal of Biomathematics, **7**(5) (2014) 1450055, 25 pages.

[24] C. Lv, L. Huang, and Z. Yuan, *Global stability for an HIV-1 infection model with Beddington-DeAngelis incidence rate and CTL immune response*, Communications in Nonlinear Science and Numerical Simulation, **19** (2014), 121-127.

[25] H. Shu, L. Wang and J. Watmough, *Global stability of a nonlinear viral infection model with infinitely distributed intracellular delays and CTL imune responses*, SIAM Journal of Applied Mathematics, **73**(3) (2013), 1280-1302.

[26] Y. Nakata, Global dynamics of a cell mediated immunity in viral infection models with distributed delays, Journal of Mathematical Analysis and Applications, **375** (2011) 14-27.

[27] X. Wang, A. M. Elaiw and X. Song, *Global properties of a delayed HIV infection model with CTL immune response*, Applied Mathematics and Computation, **218**(18) (2012), 9405-9414.

[28] F. Li and J. Wang, *Analysis of an HIV infection model with logistic target cell growth and cell-to-cell transmission*, Chaos, Solitons and Fractals, **81**(2015), 136-145.

[29] X. Lai and X. Zou , *Modeling cell-to-cell spread of HIV-1 with logistic target cell growth*, Journal of Mathematical Analysis and Applications, **426** (2015), 563–584.

[30] X. Lai, X. Zou, Modelling HIV-1 virus dynamics with both virus-to-cell infection and cell-to-cell transmission, SIAM Journal of Applied Mathematics, **74** (2014), 898–917.

[31] Y. Yanga, L. Zou and S. Ruanc, *Global dynamics of a delayed within-host viral infection model with both virus-to-cell and cell-to-cell transmissions*, Mathematical Biosciences, **270** (2015), 183-191.

[32] H. Pourbashash, S. S. Pilyugin and P. De Leenheer, *Global analysis of within host virus models with cell-to-cell viral transmission*, J. AIMS, **19**(10) (2014), 3341-3357.

[33] J. K. Hale, and S. M. V. Lunel, *Introduction to functional differential equations*, Springer-Verlag, New York, (1993).

# Dynamical behavior of HIV-1 infection with saturated virus-target and infected-target incidences and delays

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**Abstract**

This paper study the dynamical behavior of HIV-1 infection model with saturated virus-target and infected-target incidences. The model is incorporated by two types of intracellular distributed time delays. The model generalizes all the existing HIV-1 infection models with cell-to-cell transmission presented in the literature by considering saturated incidence rate. The nonnegativity and boundedness of the solutions of the model as well as global stability of the steady states are studied. The global stability are established using Lyapunov method. Using MATLAB we conduct some numerical simulations to confirm our results. The effect of the saturated incidence of the HIV-1 dynamics is shown.

**Keywords:** HIV-1 dynamics; Global stability; time delay; cell-to-cell transfer.

## 1 Introduction

It is known that human immunodeficiency virus type 1 (HIV-1) infects the CD4$^+$ T cells which play the central role in the immune system of the human body. Mathematical models that describe the dynamics of HIV-1 are helpful in understanding the virus dynamics and improving diagnosis and treatment strategies. The basic HIV-1 infection model has been given in [1] as:

$$\dot{T} = \rho - dT - \beta TV \tag{1}$$

$$\dot{T}^* = \beta TV - \mu T^*, \tag{2}$$

$$\dot{V} = bT^* - cV, \tag{3}$$

where, $T$, $T^*$ and $V$ are the concentrations of the uninfected CD4$^+$ T cells, infected cells, and free HIV-1 particles, respectively. The CD4$^+$ T cells are replenished at rate $\rho$, die at rate $dT$ and become infected at rate $\beta TV$, where $\beta$ is the virus-target incidence rate constant. The infected cells are die at rate $\mu$. The HIV-1 particles are produced from infected cells at rate $bT^*$ and cleared at rate $cV$. Parameters $\rho, d, \beta, \mu, b$ and $c$ are all positive.

In model (1)-(3), the infection rate is given by bilinear incidence $\beta TV$. In case when the concentration of the viruses is high, this bilinear incidence may not describe the HIV-1 dynamics accurately. Therefore, the model has been modified to incorporate the saturated incidence rate [2]:

$$\dot{T} = \rho - dT - \beta T\left(\frac{V}{1+\alpha V}\right) \tag{4}$$

$$\dot{T}^* = \beta T\left(\frac{V}{1+\alpha V}\right) - \mu T^*, \tag{5}$$

$$\dot{V} = bT^* - cV, \tag{6}$$

where, $\alpha$ is the saturation constant. Moreover, several works have been done to modify the basic model (1)-(3) by considering different effects such as: CTL immune response [3]-[5], humoral immune response [6]-[8], nonlinear incidence rate [9]-[11], intracellular time delay [10], [12], [13], [15], antiviral treatments [15]-[17], latently infected cells [18]-[19] and two types of target cells [20]-[22]. All the these works assume that the uninfected $CD4^+$ T cells becomes infected due to HIV-1 contacts. Recently, it has been reported that the uninfected $CD4^+$ T cells can also become infected due to direct contact with infected cells (see [23]-[26]). However, in [23]-[26], the rates of virus-target and infected-target infection are based on the mass action principle.

The aim of this paper is to study the dynamical behavior of HIV-1 infection model with saturated virus-target and infected-target incidences. Both discrete and distributed time delays are incorporated. We study the global stability analysis of the model using Lyapunov method.

# 2 HIV-1 model with discrete delays

We formulate an HIV-1 infection model with saturated virus-target and infected-target incidences and two types of discrete time delays as:

$$\dot{T}(t) = \rho - dT(t) - \frac{\beta_1 T(t)V(t)}{1+\alpha_1 V(t)} - \frac{\beta_2 T(t)T^*(t)}{1+\alpha_2 T^*(t)}, \tag{7}$$

$$\dot{T}^*(t) = e^{-\delta_1\tau_1}\left[\frac{\beta_1 T(t-\tau_1)V(t-\tau_1)}{1+\alpha_1 V(t-\tau_1)} + \frac{\beta_2 T(t-\tau_1)T^*(t-\tau_1)}{1+\alpha_2 T^*(t-\tau_1)}\right] - \mu T^*(t), \tag{8}$$

$$\dot{V}(t) = be^{-\delta_2\tau_2}T^*(t-\tau_2) - cV(t). \tag{9}$$

Parameter $\tau_1$ represents for the time between the virus or the infected cell contacts with an uninfected $CD4^+$ T cell, until it becomes infected but not yet producer cell. The parameter $\tau_2$ represents the time needed for new HIV-1 to be mature. The factor $e^{-\delta_1\tau_1}$ is the loss of $CD4^+$ T cells during the interval $[t-\tau_1, t]$ while, $e^{-\delta_2\tau_2}$ represents the loss of infected cells during the interval $[t-\tau_2, t]$, where $\delta_1$ and $\delta_2$ are positive constants.

The initial conditions for system (7)-(9) are given as:

$$\begin{aligned} &T(\eta) = \varphi_1(\eta),\ T^*(\eta) = \varphi_2(\eta), V(\eta) = \varphi_3(\eta), \\ &\varphi_j(\eta) \geq 0, \quad \eta \in [-\tau, 0], \qquad j = 1,2,3 \end{aligned} \tag{10}$$

where $\tau = max\{\tau_1, \tau_2\}$ and $(\varphi_1(\eta), \varphi_2(\eta), \varphi_3(\eta)) \in C([-\tau : 0), \mathbb{R}^3_+)$, where $C$ is the Banach space of continuous functions mapping the interval $[-\tau, 0)$ into $\mathbb{R}^3_+$. System (7)-(9) with initial conditions (10) has a unique solution [27].

## 2.1 Basic properties

The non-negativity and boundedness of the solutions of system (7)-(9) is established in the following lemma:

**Lemma 1**. All solutions $(T(t), T^*(t), V(t))$ of model (7)-(9) with initial conditions (10) are non-negative and ultimately bounded.

**Proof**: From Eq. (7), we have $\dot{T}\ |_{T=0} = \rho > 0$, therefore $T(t) > 0$ for $t \in (0, \varpi_1)$ where $(0, \varpi_1)$ is the maximal interval of existence of solution of system (7)-(9) with (10). Moreover, from Eqs. (8)-(9), we have

$$T^*(t) = e^{-\mu t}\varphi_2(0) + e^{-\delta_1\tau_1}\int_0^t e^{-\mu(t-\eta)}\left[\frac{\beta_1 T(\eta-\tau_1)V(\eta-\tau_1)}{1+\alpha_1 V(\eta-\tau_1)} + \frac{\beta_2 T(\eta-\tau_1)T^*(\eta-\tau_1)}{1+\alpha_2 T^*(\eta-\tau_1)}\right]d\eta \geq 0,$$

$$V(t) = e^{-ct}\varphi_3(0) + be^{-\delta_2\tau_2}\int_0^t e^{-c(t-\eta)}T^*(\eta-\tau_2)d\eta \geq 0,$$

for $t \in [0, \tau]$. By recursive argument we obtain $T^*(t), V(t) \geq 0$ for all $t \geq 0$.

From Eq. (7) we know $\lim\limits_{t\to\infty}\sup T(t)\leq\dfrac{\rho}{d}$. Let $F_1(t)=e^{-\delta_1\tau_1}T(t-\tau_1)+T^*(t)$. Then

$$\begin{aligned}\dot{F}_1(t)&=e^{-\delta_1\tau_1}\left[\rho-dT(t-\tau_1)-\frac{\beta_1T(t-\tau_1)V(t-\tau_1)}{1+\alpha_1V(t-\tau_1)}-\frac{\beta_2T(t-\tau_1)T^*(t-\tau_1)}{1+\alpha_2T^*(t-\tau_1)}\right]\\&+e^{-\delta_1\tau_1}\left[\frac{\beta_1T(t-\tau_1)V(t-\tau_1)}{1+\alpha_1V(t-\tau_1)}+\frac{\beta_2T(t-\tau_1)T^*(t-\tau_1)}{1+\alpha_2T^*(t-\tau_1)}\right]-\mu T^*(t)\\&=\rho e^{-\delta_1\tau_1}-de^{-\delta_1\tau_1}T(t-\tau_1)-\mu T^*(t)\\&\leq\rho-\sigma\left(e^{-\delta_1\tau_1}T(t-\tau_1)+T^*(t)\right)=\rho-\sigma F_1(t),\end{aligned}$$

where, $\sigma=\min\{d,\mu\}$. Hence, $\limsup_{t\to\infty}F_1(t)\leq\dfrac{\rho}{\sigma}$ and then $\limsup_{t\to\infty}T^*(t)\leq\dfrac{\rho}{\sigma}$ . From Eq. (9) we have

$$\dot{V}(t)=be^{-\delta_2\tau_2}T^*(t-\tau_2)-cV(t)\leq be^{-\delta_2\tau_2}\frac{\rho}{\sigma}-cV(t)<b\frac{\rho}{\sigma}-cV(t).$$

Thus $\limsup\limits_{t\to\infty}V(t)\leq\dfrac{b\rho}{c\sigma}$. Therefore, $T(t),T^*(t)$ and $V(t)$ are all ultimately bounded.□

Now we prove the existence of the steady state of the model (7)-(9).

**Lemma 2.**

(i) If $\mathcal{R}_0\leq1$, then there exists only positive steady state $S_0$,

(ii) If $1<\mathcal{R}_0$, then there exist two positive steady states $S_0$ and $S_1$.

**The proof.** Let the R.H.S of system (7)-(9) equal to zero

$$0=\rho-dT-\frac{\beta_1TV}{1+\alpha_1V}-\frac{\beta_2TT^*}{1+\alpha_2T^*},\tag{11}$$

$$0=e^{-\delta_1\tau_1}\left(\frac{\beta_1TV}{1+\alpha_1V}+\frac{\beta_2TT^*}{1+\alpha_2T^*}\right)-\mu T^*,\tag{12}$$

$$0=e^{-\delta_2\tau_2}bT^*-cV,\tag{13}$$

Solving Eqs. (11)-(13) we find that the system has two steady states, disease-free steady state $S_0=(T_0,0,0)$, where $T_0=\dfrac{\rho}{d}$ and endemic steady state $S_1(T_1,T_1^*,V_1)$, where

$$T_1=\frac{\mu c\left(1+\alpha_1V_1\right)\left(be^{-\delta_2\tau_2}+\alpha_2cV_1\right)}{be^{-(\delta_1\tau_1+\delta_2\tau_2)}\left[\beta_1\left(be^{-\delta_2\tau_2}+\alpha_2cV_1\right)+\beta_2c\left(1+\alpha_1V_1\right)\right]},\qquad T_1^*=\frac{-B+\sqrt{B^2-4AC}}{2A},$$

$$V_1=\frac{be^{-\delta_2\tau_2}T_1^*}{c},$$

where

$$\begin{aligned}A&=\mu be^{-\delta_2\tau_2}\left(d\alpha_1\alpha_2+\beta_1\alpha_2+\beta_2\alpha_1\right),\\B&=\beta_2(\mu c-\rho\alpha_1be^{-(\delta_1\tau_1+\delta_2\tau_2)})+\beta_1be^{-\delta_2\tau_2}(\mu-\rho\alpha_2e^{-\delta_1\tau_1})+d\mu(c\alpha_2+\alpha_1be^{-\delta_2\tau_2}),\\C&=d\mu c\left(1-\mathcal{R}_0\right).\end{aligned}\tag{14}$$

and

$$\mathcal{R}_0=\frac{T_0e^{-\delta_1\tau_1}\left(b\beta_1e^{-\delta_2\tau_2}+\beta_2c\right)}{\mu c},$$

where $\mathcal{R}_0$ represents the basic infection reproduction number. □

## 2.2 Global properties

In the following we established the global stability of the two steady states by of system (7)-(9) by constructing suitable Lyapunov functionals. Through the paper we will use the following function $g(x)=x-1-\ln x$ and the notation $(T,T^*,V)=(T(t),T^*(t),V(t))$.

**Theorem 1.** If $\mathcal{R}_0 \leq 1$, then $S_0$ is globally asymptotically stable.

**Proof.** Define a Lyapunov functional

$$L_1(T,T^*,V) = T_0 g\left(\frac{T}{T_0}\right) + \frac{1}{e^{-\delta_1\tau_1}}T^* + \frac{\beta_1 T_0}{c}V + \int_0^{\tau_1}\left[\frac{\beta_1 T(t-\eta)V(t-\eta)}{1+\alpha_1 V(t-\eta)} + \frac{\beta_2 T(t-\eta)T^*(t-\eta)}{1+\alpha_2 T^*(t-\eta)}\right]d\eta$$
$$+ \frac{b\beta_1 T_0}{c}e^{-\delta_2\tau_2}\int_0^{\tau_2} T^*(t-\eta)d\eta.$$

We evaluate $\frac{dL_1}{dt}$ along the solutions of the system (7)-(9),

$$\begin{aligned}
\frac{dL_1}{dt} &= \left(1-\frac{T_0}{T}\right)\left(\rho - dT - \frac{\beta_1 TV}{1+\alpha_1 V} - \frac{\beta_2 TT^*}{1+\alpha_2 T^*}\right)\\
&+ \frac{1}{e^{-\delta_1\tau_1}}\left[e^{-\delta_1\tau_1}\left(\frac{\beta_1 T(t-\tau_1)V(t-\tau_1)}{1+\alpha_1 V(t-\tau_1)} + \frac{\beta_2 T(t-\tau_1)T^*(t-\tau_1)}{1+\alpha_2 T^*(t-\tau_1)}\right) - \mu T^*\right]\\
&+ \frac{\beta_1 T_0}{c}\left[be^{-\delta_2\tau_2}T^*(t-\tau_2) - cV\right] + \frac{\beta_1 TV}{1+\alpha_1 V} + \frac{\beta_2 TT^*}{1+\alpha_2 T^*}\\
&- \frac{\beta_1 T(t-\tau_1)V(t-\tau_1)}{1+\alpha_1 V(t-\tau_1)} - \frac{\beta_2 T(t-\tau)T^*(t-\tau_1)}{1+\alpha_2 T^*(t-\tau_1)} + \frac{b\beta_1 T_0}{c}e^{-\delta_2\tau_2}\left[T^* - T^*(t-\tau_2)\right]\\
&= \left(1-\frac{T_0}{T}\right)(\rho - dT) - \alpha_1\beta_1 T_0\frac{V^2}{1+\alpha_1 V} - \alpha_2\beta_2 T_0\frac{T^{*2}}{1+\alpha_2 T^*}\\
&+ \frac{\mu}{e^{-\delta_1\tau_1}}\left(\frac{T_0 b\beta_1 e^{-(\delta_1\tau_1+\delta_2\tau_2)}}{\mu c} + \frac{T_0\beta_2 e^{-\delta_1\tau_1}}{\mu} - 1\right)T^*\\
&= -d\frac{(T-T_0)^2}{T} - \alpha_1\beta_1 T_0\frac{V^2}{1+\alpha_1 V} - \alpha_2\beta_2 T_0\frac{T^{*2}}{1+\alpha_2 T^*} + \frac{\mu}{e^{-\delta_1\tau_1}}(\mathcal{R}_0 - 1)T^*. \qquad (15)
\end{aligned}$$

If $\mathcal{R}_0 \leq 1$, then $\frac{dL_1}{dt} \leq 0$ for all $T, T^*, V > 0$ and $\frac{dL_1}{dt} = 0$ if and only if $T = T_0$ , $T^* = 0$ and $V = 0$. Let $D_0 = \left\{(T,T^*,V) : \frac{dL_1}{dt} = 0\right\}$. It is easy to show that $S_0$ is the largest invariant subset of $D_0$. LaSalle's invariance principle implies that $S_0$ is globally asymptotically stable when $\mathcal{R}_0 \leq 1$.□

**Theorem 2.** If $1 < \mathcal{R}_0$, then $S_1$ is globally asymptotically stable.

**Proof.** Define

$$\begin{aligned}
U(T,T^*,V) &= T_1 g\left(\frac{T}{T_1}\right) + \frac{1}{e^{-\delta_1\tau_1}}T_1^* g\left(\frac{T^*}{T_1^*}\right) + \frac{\beta_1 T_1 V_1}{be^{-\delta_2\tau_2}T_1^*(1+\alpha_1 V_1)}V_1 g\left(\frac{V}{V_1}\right)\\
&+ \frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1}\int_0^{\tau_1} g\left(\frac{T(t-\eta)V(t-\eta)(1+\alpha_1 V_1)}{T_1 V_1(1+\alpha_1 V(t-\eta))}\right)d\eta\\
&+ \frac{\beta_2 T_1 T_1^*}{1+\alpha_2 T_1^*}\int_0^{\tau_1} g\left(\frac{T(t-\eta)T^*(t-\eta)(1+\alpha_2 T_1^*)}{T_1 T_1^*(1+\alpha_2 T^*(t-\eta))}\right)d\eta + \frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1}\int_0^{\tau_2} g\left(\frac{T^*(t-\eta)}{T_1^*}\right)d\eta.
\end{aligned}$$

Evaluating $\frac{dU_1}{dt}$ along the trajectories of (7)-(9) as:

$$\begin{aligned}\frac{dU_1}{dt} &= \left(1-\frac{T_1}{T}\right)\left(\rho - dT - \frac{\beta_1 TV}{1+\alpha_1 V} - \frac{\beta_2 TT^*}{1+\alpha_2 T^*}\right)\\ &+ \frac{1}{e^{-\delta_1\tau_1}}\left(1-\frac{T_1^*}{T^*}\right)\left(e^{-\delta_1\tau_1}\left[\frac{\beta_1 T(t-\tau_1)V(t-\tau_1)}{1+\alpha_1 V(t-\tau_1)} + \frac{\beta_2 T(t-\tau_1)T^*(t-\tau_1)}{1+\alpha_2 T^*(t-\tau_1)}\right] - \mu T^*\right)\\ &+ \frac{\beta_1 T_1 V_1}{be^{-\delta_2\tau_2}T_1^*\left(1+\alpha_1 V_1\right)}\left(1-\frac{V_1}{V}\right)\left(be^{-\delta_2\tau_2}T^*(t-\tau_2) - cV\right)\\ &+ \frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1}\left(\frac{TV\left(1+\alpha_1 V_1\right)}{T_1 V_1\left(1+\alpha_1 V\right)} - \frac{T(t-\tau_1)V(t-\tau_1)\left(1+\alpha_1 V_1\right)}{T_1 V_1\left(1+\alpha_1 V(t-\tau_1)\right)}\right)\\ &+ \frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1}\ln\left(\frac{T(t-\tau_1)V(t-\tau_1)\left(1+\alpha_1 V\right)}{TV\left(1+\alpha_1 V(t-\tau_1)\right)}\right)\\ &+ \frac{\beta_2 T_1 T_1^*}{1+\alpha_2 T_1^*}\left(\frac{TT^*\left(1+\alpha_2 T_1^*\right)}{T_1 T_1^*\left(1+\alpha_2 T^*\right)} - \frac{T(t-\tau_1)T^*(t-\tau_1)\left(1+\alpha_2 T_1^*\right)}{T_1 T_1^*\left(1+\alpha_2 T^*(t-\tau_1)\right)}\right)\\ &+ \frac{\beta_2 T_1 T_1^*}{1+\alpha_2 T_1^*}\ln\left(\frac{T(t-\tau_1)T^*(t-\tau_1)\left(1+\alpha_2 T^*\right)}{TT^*\left(1+\alpha_2 T^*(t-\tau_1)\right)}\right)\\ &+ \frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1}\left(\frac{T^*}{T_1^*} - \frac{T^*(t-\tau_2)}{T_1^*} + \ln\left(\frac{T^*(t-\tau_2)}{T^*}\right)\right). \end{aligned} \tag{16}$$

Collecting terms of Eq. (16) and applying the steady state conditions for $S_1$:

$$\rho - dT_1 = \frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1} + \frac{\beta_2 T_1 T_1^*}{1+\alpha_2 T_1^*} = \frac{\mu}{e^{-\delta_1\tau_1}}T_1^* = \frac{c\mu}{be^{-(\delta_1\tau_1+\delta_2\tau_2)}}V_1,$$

we get

$$\begin{aligned}\frac{dU_1}{dt} &= -\frac{d}{T}\left(T-T_1\right)^2 + \left(1-\frac{T_1}{T}\right)\left(\frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1} + \frac{\beta_2 T_1 T_1^*}{1+\alpha_2 T_1^*}\right) + \frac{\beta_1 T_1 V}{1+\alpha_1 V} + \frac{\beta_2 T_1 T^*}{1+\alpha_2 T^*}\\ &- \frac{\beta_1 T_1^* T(t-\tau_1)V(t-\tau_1)}{T^*\left(1+\alpha_1 V(t-\tau_1)\right)} - \frac{\beta_2 T_1^* T(t-\tau_1)T^*(t-\tau_1)}{T^*\left(1+\alpha_2 T^*(t-\tau_1)\right)} - \frac{\beta_2 T_1 T^*}{1+\alpha_2 T_1^*} + \frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1}\\ &+ \frac{\beta_2 T_1 T_1^*}{1+\alpha_2 T_1^*} - \frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1}\frac{V_1 T^*(t-\tau_2)}{VT_1^*} - \frac{\beta_1 T_1 V}{1+\alpha_1 V_1} + \frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1}\\ &+ \frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1}\ln\left(\frac{T(t-\tau_1)V(t-\tau_1)\left(1+\alpha_1 V\right)}{TV\left(1+\alpha_1 V(t-\tau_1)\right)}\right) + \frac{\beta_2 T_1 T_1^*}{1+\alpha_2 T_1^*}\ln\left(\frac{T(t-\tau_1)T^*(t-\tau_1)\left(1+\alpha_2 T^*\right)}{TT^*\left(1+\alpha_2 T^*(t-\tau_1)\right)}\right)\\ &+ \frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1}\ln\left(\frac{T^*(t-\tau_2)}{T^*}\right). \end{aligned}$$

Consider the following equalities:

$$\begin{aligned}\ln\left(\frac{T(t-\tau_1)V(t-\tau_1)\left(1+\alpha_1 V\right)}{TV\left(1+\alpha_1 V(t-\tau_1)\right)}\right) &= \ln\left(\frac{T(t-\tau_1)V(t-\tau_1)\left(1+\alpha_1 V_1\right)T_1^*}{T_1 V_1\left(1+\alpha_1 V(t-\tau_1)\right)T^*}\right) + \ln\left(\frac{T_1}{T}\right)\\ &+ \ln\left(\frac{1+\alpha_1 V}{1+\alpha_1 V_1}\right) + \ln\left(\frac{V_1 T^*}{VT_1^*}\right),\\ \ln\left(\frac{T(t-\tau_1)T^*(t-\tau_1)\left(1+\alpha_2 T^*\right)}{TT^*\left(1+\alpha_2 T^*(t-\tau_1)\right)}\right) &= \ln\left(\frac{T(t-\tau_1)T^*(t-\tau_1)\left(1+\alpha_2 T_1^*\right)}{T_1 T^*\left(1+\alpha_2 T^*(t-\tau_1)\right)}\right) + \ln\left(\frac{T_1}{T}\right)\\ &+ \ln\left(\frac{1+\alpha_2 T^*}{1+\alpha_2 T_1^*}\right),\\ \ln\left(\frac{T^*(t-\tau_2)}{T^*}\right) &= \ln\left(\frac{V_1 T^*(t-\tau_2)}{VT_1^*}\right) + \ln\left(\frac{VT_1^*}{V_1 T^*}\right). \end{aligned} \tag{17}$$

Using Eqs. (17) we get

$$\begin{aligned}\frac{dU_1}{dt} &= -\frac{d}{T}(T-T_1)^2 + \frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1}\left[\frac{(1+\alpha_1 V_1)V}{(1+\alpha_1 V)V_1} - \frac{V}{V_1} - 1 + \frac{1+\alpha_1 V}{1+\alpha_1 V_1}\right] \\ &+ \frac{\beta_2 T_1 T_1^*}{1+\alpha_2 T_1^*}\left[\frac{(1+\alpha_2 T_1^*)T^*}{(1+\alpha_2 T^*)T_1^*} - \frac{T^*}{T_1^*} - 1 + \frac{1+\alpha_2 T^*}{1+\alpha_2 T_1^*}\right] - \frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1}\left[\frac{T_1}{T} - 1 - \ln\left(\frac{T_1}{T}\right)\right] \\ &- \frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1}\left[\frac{V_1 T^*(t-\tau_2)}{V T_1^*} - 1 - \ln\left(\frac{V_1 T^*(t-\tau_2)}{V T_1^*}\right) + \frac{1+\alpha_1 V}{1+\alpha_1 V_1} - 1 - \ln\left(\frac{1+\alpha_1 V}{1+\alpha_1 V_1}\right)\right] \\ &- \frac{\beta_2 T_1 T_1^*}{1+\alpha_2 T_1^*}\left[\frac{T_1}{T} - 1 - \ln\left(\frac{T_1}{T}\right)\right] - \frac{\beta_2 T_1 T_1^*}{1+\alpha_2 T_1^*}\left[\frac{1+\alpha_2 T^*}{1+\alpha_2 T_1^*} - 1 - \ln\left(\frac{1+\alpha_2 T^*}{1+\alpha_2 T_1^*}\right)\right] \\ &- \frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1}\left[\frac{T(t-\tau_1)V(t-\tau_1)(1+\alpha_1 V_1)T_1^*}{T_1 V_1 (1+\alpha_1 V(t-\tau_1))T^*} - 1 - \ln\left(\frac{T(t-\tau_1)V(t-\tau_1)(1+\alpha_1 V_1)T_1^*}{T_1 V_1 (1+\alpha_1 V(t-\tau_1))T^*}\right)\right] \\ &- \frac{\beta_2 T_1 T_1^*}{1+\alpha_2 T_1^*}\left[\frac{T(t-\tau_1)T^*(t-\tau_1)(1+\alpha_2 T_1^*)}{T_1 T^*(1+\alpha_2 T^*(t-\tau_1))} - 1 - \ln\left(\frac{T(t-\tau_1)T^*(t-\tau_1)(1+\alpha_2 T_1^*)}{T_1 T^*(1+\alpha_2 T^*(t-\tau_1))}\right)\right]\end{aligned}$$

Then

$$\begin{aligned}\frac{dU_1}{dt} &= -\frac{d}{T}(T-T_1)^2 - \frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1}\left[\frac{\alpha_1 (V-V_1)^2}{(1+\alpha_1 V)(1+\alpha_1 V_1)V_1}\right] - \frac{\beta_2 T_1 T_1^*}{1+\alpha_2 T_1^*}\left[\frac{\alpha_2 (T^*-T_1^*)^2}{(1+\alpha_2 T^*)(1+\alpha_2 T_1^*)T_1^*}\right] \\ &- \frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1}\left[g\left(\frac{T_1}{T}\right) + g\left(\frac{T(t-\tau_1)V(t-\tau_1)(1+\alpha_1 V_1)T_1^*}{T_1 V_1 (1+\alpha_1 V(t-\tau_1))T^*}\right) + g\left(\frac{1+\alpha_1 V}{1+\alpha_1 V_1}\right) + g\left(\frac{V_1 T^*(t-\tau_2)}{V T_1^*}\right)\right] \\ &- \frac{\beta_2 T_1 T_1^*}{1+\alpha_2 T_1^*}\left[g\left(\frac{T_1}{T}\right) + g\left(\frac{T(t-\tau_1)T^*(t-\tau_1)(1+\alpha_2 T_1^*)}{T_1 T^*(1+\alpha_2 T^*(t-\tau_1))}\right) + g\left(\frac{1+\alpha_2 T^*}{1+\alpha_2 T_1^*}\right)\right]. \qquad (18)\end{aligned}$$

Since $\mathcal{R}_0 > 1$, then $T, T^*, V > 0$. From Eq. (18) we have $\frac{dU_1}{dt} \leq 0$ and $\frac{dU_1}{dt} = 0$ ocurs at $S_1$. Let $D_1 = \left\{(T, T^*, V) : \frac{dU1}{dt} = 0\right\}$. It is clear that $S_1$ is the largest invariant subset of $D_1$. Using LaSalle's invariance principle we obtain that $S_1$ is globally asymptotically stable when $\mathcal{R}_0 > 1$. □

# 3 HIV-1 model with distributed delays

In this section, we formulate an HIV-1 infection model with saturated virus-target and infected-target incidences and two types of distributed time delays:

$$\dot{T}(t) = \rho - dT(t) - \frac{\beta_1 T(t)V(t)}{1+\alpha_1 V(t)} - \frac{\beta_2 T(t)T^*(t)}{1+\alpha_2 T^*(t)}, \qquad (19)$$

$$\dot{T}^*(t) = \int_0^\infty f_1(s)e^{-\delta_1 s}\left[\frac{\beta_1 T(t-s)V(t-s)}{1+\alpha_1 V(t-s)} + \frac{\beta_2 T(t-s)T^*(t-s)}{1+\alpha_2 T^*(t-s)}\right]ds - \mu T^*(t), \qquad (20)$$

$$\dot{V}(t) = b\int_0^\infty f_2(s)e^{-\delta_2 s}T^*(t-s)ds - cV(t). \qquad (21)$$

Let us assume that the probability distribution functions $f_i(s)$ satisfy $f_i(s) > 0$, $i = 1, 2$ and

$$\int_0^\infty f_i(s)ds = 1, \quad \int_0^\infty f_i(u)e^{\ell u}du < \infty, \quad i = 1, 2,$$

where $\ell > 0$. Denote $\eta_i = \int_0^\infty f_i(s)e^{-\delta_i s}ds$, $\quad i = 1, 2$, thus, $0 < \eta_i \leq 1$. Define the Banach space of fading memory type

$$C_\gamma = \{\phi \in C((-\infty, 0], \mathbb{R}) : e^{\alpha\eta}\phi(\eta) \text{ is uniformly continuous for } \eta \in (-\infty, 0] \text{ and } \|\phi\| < \infty\}$$

where $\gamma$ is a positive constant and $\|\phi\| = \sup_{\eta \leq 0} |\phi(\eta)| e^{\gamma\eta}$. Let

$$C_\gamma^+ = \{\phi \in C_\gamma : \phi(\eta) \geq 0 \ for \ \ \eta \in (-\infty, 0]\}.$$

The initial conditions for system (19)-(21) are given as:

$$T(\eta) = \varphi_1(\eta),\ T^*(\eta) = \varphi_2(\eta), V(\eta) = \varphi_3(\eta),\quad \text{for } \eta \in (-\infty, 0],$$
$$\varphi_i \in C^+_\gamma,\ \ i = 1, 2, 3. \tag{22}$$

System (7)-(9) with initial conditions (22) has a unique solution [27].

## 3.1 Basic properties

The non-negativity and boundedness of the solutions of model (19)-(21) will be established in the next lemma.

**Lemma 3**. The solutions $(T(t), T^*(t), V(t))$ of model (19)-(21) with initial conditions (22) are non-negative and ultimately bounded.

**Proof**: Similar to the proof of Lemma 1, one can show $T(t) > 0$ for all $T(t) > 0$ for $t \in (0, \varpi_2)$, where $(0, \varpi_2)$ is the maximal interval of existence of solution of system (19)-(21) with (22). From Eqs. (20)-(21), we have

$$T^*(t) = e^{-\mu t}\varphi_2(0) + \int_0^t e^{-\mu(t-\eta)} \int_0^\infty f_1(s)e^{-\delta_1 s}\left[\frac{\beta_1 T(\eta - s)V(\eta - s)}{1 + \alpha_1 V(\eta - s)} + \frac{\beta_2 T(\eta - s)T^*(\eta - s)}{1 + \alpha_2 T^*(\eta - s)}\right] ds d\eta \geq 0,$$
$$V(t) = e^{-ct}\varphi_3(0) + b\int_0^t e^{-c(t-\eta)} \int_0^\infty f_2(s)e^{-\delta_2 s}T^*(\eta - s)ds d\eta \geq 0.$$

From Eq. (19) we have $\lim\limits_{t\to\infty} \sup T(t) \leq \dfrac{\rho}{d}$. Let $F(t) = \int_0^\infty f_1(s)e^{-\delta_1 s}T(t - s)ds + T^*(t)$. Then

$$\begin{aligned}
\dot{F}_2(t) &= \int_0^\infty f_1(s)e^{-\delta_1 s}\left[\rho - dT(t - s) - \frac{\beta_1 T(t - s)V(t - s)}{1 + \alpha_1 V(t - s)} - \frac{\beta_2 T(t - s)T^*(t - s)}{1 + \alpha_2 T^*(t - s)}\right] ds \\
&+ \int_0^\infty f_1(s)e^{-\delta_1 s}\left[\frac{\beta_1 T(t - s)V(t - s)}{1 + \alpha_1 V(t - s)} + \frac{\beta_2 T(t - s)T^*(t - s)}{1 + \alpha_2 T^*(t - s)}\right] ds - \mu T^*(t) \\
&= \rho\eta_1 - d\int_0^\infty f_1(s)e^{-\delta_1 s}T(t - s)ds - \mu T^*(t) \\
&\leq \rho - \sigma\left(\int_0^\infty f_1(s)e^{-\delta_1 s}T(t - s)ds + T^*(t)\right) = \rho - \sigma F_2(t),
\end{aligned}$$

where, $\sigma = \min\{d, \mu\}$. Hence, $\limsup_{t\to\infty} F_2(t) \leq \dfrac{\rho}{\sigma}$. Since $\int_0^\infty f_1(s)e^{-\delta_1 s}T(t - s)ds > 0$ and $T^* \geq 0$ , then $\limsup_{t\to\infty} T^*(t) \leq \dfrac{\rho}{\sigma}$ . From Eq. (21) we have

$$\dot{V}(t) = b\int_0^\infty f_2(s)e^{-\delta_2 s}T^*(t - s)ds - cV(t) \leq b\eta_2\frac{\rho}{\sigma} - cV(t) \leq b\frac{\rho}{\sigma} - cV(t).$$

Thus $\limsup\limits_{t\to\infty} V(t) \leq \dfrac{b\rho}{c\sigma}$. Therefore, $T(t), T^*(t)$ and $V(t)$ are ultimately bounded.□

The existence of the steady state of the model (19)-(21) will be shown in the next lemma.

**Lemma 4.**

(i) If $\mathcal{R}_0 \leq 1$, then there exists only positive steady state $S_0$,

(ii) if $1 < \mathcal{R}_0$, then there exist only two positive steady states $S_0$ and $S_1$.

**The proof.** Let the R.H.S of system (19)-(21) be equal zero

$$0 = \rho - dT - \frac{\beta_1 TV}{1 + \alpha_1 V} - \frac{\beta_2 TT^*}{1 + \alpha_2 T^*}, \tag{23}$$
$$0 = \eta_1\left(\frac{\beta_1 TV}{1 + \alpha_1 V} + \frac{\beta_2 TT^*}{1 + \alpha_2 T^*}\right) - \mu T^*, \tag{24}$$
$$0 = \eta_2 bT^* - cV, \tag{25}$$

Solving Eqs. (23)-(25) we find that the system has two steady states, disease-free steady state $S_0 = (T_0, 0, 0, 0)$, where $T_0 = \frac{\rho}{d}$, and endemic steady state $S_1(T_1, T_1^*, V_1)$, where

$$T_1 = \frac{\mu c\,(1+\alpha_1 V_1)\,(b\eta_2 + \alpha_2 c V_1)}{b\eta_1\eta_2\,[\beta_1\,(b\eta_2+\alpha_2 c V_1) + \beta_2 c\,(1+\alpha_1 V_1)]}, \qquad T_1^* = \frac{-B+\sqrt{B^2-4AC}}{2A}, \qquad V_1 = \frac{b\eta_2 T_1^*}{c},$$

where

$$\begin{aligned} A &= \mu b\eta_2\,(d\alpha_1\alpha_2 + \beta_1\alpha_2 + \beta_2\alpha_1)\,, \\ B &= \beta_2(\mu c - \rho\alpha_1 b\eta_1\eta_2) + \beta_1 b\eta_2(\mu - \rho\alpha_2\eta_1) + d\mu(c\alpha_2 + \alpha_1 b\eta_2), \\ C &= d\mu c\,(1-\mathcal{R}_0)\,, \end{aligned} \tag{26}$$

and

$$\mathcal{R}_0 = \frac{T_0\eta_1}{\mu c}\,(b\beta_1\eta_2 + \beta_2 c)\,,$$

where $\mathcal{R}_0$ represents the basic infection reproduction number. □

## 3.2 Global properties

In this section, we study the global stability of all the steady states of system (19)-(21) employing the method of Lyapunov function.

**Theorem 3.** If $\mathcal{R}_0 \le 1$, then $S_0$ is globally asymptotically stable.

**Proof.** Define

$$\begin{aligned} L_2(T,T^*,V) = T_0 g\left(\frac{T}{T_0}\right) &+ \frac{1}{\eta_1}T^* + \frac{\beta_1 T_0}{c}V \\ &+ \frac{1}{\eta_1}\int_0^\infty f_1(s)e^{-\delta_1 s}\int_0^s \left[\frac{\beta_1 T(t-\eta)V(t-\eta)}{1+\alpha_1 V(t-\eta)} + \frac{\beta_2 T(t-\eta)T^*(t-\eta)}{1+\alpha_2 T^*(t-\eta)}\right] d\eta ds \\ &+ \frac{b\beta_1 T_0}{c}\int_0^\infty f_2(s)e^{-\delta_2 s}\int_0^s T^*(t-\eta)\,d\eta ds. \end{aligned}$$

Calculating $\frac{dL_2}{dt}$ along the solutions of the system (19)-(21), we obtain

$$\begin{aligned} \frac{dL_2}{dt} &= \left(1-\frac{T_0}{T}\right)\left(\rho - dT - \frac{\beta_1 TV}{1+\alpha_1 V} - \frac{\beta_2 TT^*}{1+\alpha_2 T^*}\right) \\ &+ \frac{1}{\eta_1}\left[\int_0^\infty f_1(s)e^{-\delta_1 s}\left[\frac{\beta_1 T(t-s)V(t-s)}{1+\alpha_1 V(t-s)} + \frac{\beta_2 T(t-s)T^*(t-s)}{1+\alpha_2 T^*(t-s)}\right] ds - \mu T^*\right] \\ &+ \frac{\beta_1 T_0}{c}\left[b\int_0^\infty f_2(s)e^{-\delta_2 s}T^*(t-s)ds - cV\right] \\ &+ \frac{1}{\eta_1}\int_0^\infty f_1(s)e^{-\delta_1 s}\left[\frac{\beta_1 TV}{1+\alpha_1 V} + \frac{\beta_2 TT^*}{1+\alpha_2 T^*} - \frac{\beta_1 T(t-s)V(t-s)}{1+\alpha_1 V(t-s)} - \frac{\beta_2 T(t-s)T^*(t-s)}{1+\alpha_2 T^*(t-s)}\right] ds \\ &+ \frac{b\beta_1 T_0}{c}\int_0^\infty f_2(s)e^{-\delta_2 s}\left[T^* - T^*(t-s)\right] ds \\ &= \left(1-\frac{T_0}{T}\right)(\rho - dT) - \alpha_1\beta_1 T_0\frac{V^2}{1+\alpha_1 V} - \alpha_2\beta_2 T_0\frac{T^{*2}}{1+\alpha_2 T^*} + \frac{\mu}{\eta_1}\left(\frac{T_0 b\beta_1\eta_1\eta_2}{\mu c} + \frac{T_0\beta_2\eta_1}{\mu} - 1\right)T^* \\ &= -d\frac{(T-T_0)^2}{T} - \alpha_1\beta_1 T_0\frac{V^2}{1+\alpha_1 V} - \alpha_2\beta_2 T_0\frac{T^{*2}}{1+\alpha_2 T^*} + \frac{\mu}{\eta_1}(\mathcal{R}_0 - 1)T^*. \end{aligned} \tag{27}$$

If $\mathcal{R}_0 \le 1$, then $\frac{dL_2}{dt} \le 0$ for all $T, T^*, V > 0$. Similar to the proof of Theorem 1 one can easily show that $S_0$ is globally asymptotically stable when $\mathcal{R}_0 \le 1$.□

**Theorem 4.** If $1 < \mathcal{R}_0$ , then $S_1$ is globally asymptotically stable.

**Proof.** We construct the following Lyapunov functional

$$\begin{aligned}U_2(T,T^*,V,Z) &= T_1 g\left(\frac{T}{T_1}\right)+\frac{1}{\eta_1}T_1^* g\left(\frac{T^*}{T_1^*}\right)+\frac{\beta_1 T_1 V_1}{b\eta_2 T_1^*\left(1+\alpha_1 V_1\right)}V_1 g\left(\frac{V}{V_1}\right)\\ &+\frac{\beta_1 T_1 V_1}{\eta_1\left(1+\alpha_1 V_1\right)}\int_0^\infty f_1(s)e^{-\delta_1 s}\int_0^s g\left(\frac{T(t-\eta)V(t-\eta)\left(1+\alpha_1 V_1\right)}{T_1 V_1\left(1+\alpha_1 V(t-\eta)\right)}\right)d\eta ds\\ &+\frac{\beta_2 T_1 T_1^*}{\eta_1\left(1+\alpha_2 T_1^*\right)}\int_0^\infty f_1(s)e^{-\delta_1 s}\int_0^s g\left(\frac{T(t-\eta)T^*(t-\eta)\left(1+\alpha_2 T_1^*\right)}{T_1 T_1^*\left(1+\alpha_2 T^*(t-\eta)\right)}\right)d\eta ds\\ &+\frac{\beta_1 T_1 V_1}{\eta_2\left(1+\alpha_1 V_1\right)}\int_0^\infty f_2(s)e^{-\delta_2 s}\int_0^s g\left(\frac{T^*(t-\eta)}{T_1^*}\right)d\eta ds.\end{aligned}$$

We evaluate $\frac{dU_2}{dt}$ along the trajectories of (19)-(21) is given by

$$\begin{aligned}\frac{dU_2}{dt} &= \left(1-\frac{T_1}{T}\right)\left(\rho-dT-\frac{\beta_1 TV}{1+\alpha_1 V}-\frac{\beta_2 TT^*}{1+\alpha_2 T^*}\right)\\ &+\frac{1}{\eta_1}\left(1-\frac{T_1^*}{T^*}\right)\left(\int_0^\infty f_1(s)e^{-\delta_1 s}\left[\frac{\beta_1 T(t-s)V(t-s)}{1+\alpha_1 V(t-s)}+\frac{\beta_2 T(t-s)T^*(t-s)}{1+\alpha_2 T^*(t-s)}\right]ds-\mu T^*\right)\\ &+\frac{\beta_1 T_1 V_1}{b\eta_2 T_1^*\left(1+\alpha_1 V_1\right)}\left(1-\frac{V_1}{V}\right)\left(b\int_0^\infty f_2(s)e^{-\delta_2 s}T^*(t-s)ds-cV\right)\\ &+\frac{\beta_1 T_1 V_1}{\eta_1\left(1+\alpha_1 V_1\right)}\int_0^\infty f_1(s)e^{-\delta_1 s}\left(\frac{TV\left(1+\alpha_1 V_1\right)}{T_1 V_1\left(1+\alpha_1 V\right)}-\frac{T(t-s)V(t-s)\left(1+\alpha_1 V_1\right)}{T_1 V_1\left(1+\alpha_1 V(t-s)\right)}\right)ds\\ &+\frac{\beta_1 T_1 V_1}{\eta_1\left(1+\alpha_1 V_1\right)}\int_0^\infty f_1(s)e^{-\delta_1 s}\ln\left(\frac{T(t-s)V(t-s)\left(1+\alpha_1 V\right)}{TV\left(1+\alpha_1 V(t-s)\right)}\right)ds\\ &+\frac{\beta_2 T_1 T_1^*}{\eta_1\left(1+\alpha_2 T_1^*\right)}\int_0^\infty f_1(s)e^{-\delta_1 s}\left(\frac{TT^*\left(1+\alpha_2 T_1^*\right)}{T_1 T_1^*\left(1+\alpha_2 T^*\right)}-\frac{T(t-s)T^*(t-s)\left(1+\alpha_2 T_1^*\right)}{T_1 T_1^*\left(1+\alpha_2 T^*(t-s)\right)}\right)ds\\ &+\frac{\beta_2 T_1 T_1^*}{\eta_1\left(1+\alpha_2 T_1^*\right)}\int_0^\infty f_1(s)e^{-\delta_1 s}\ln\left(\frac{T(t-s)T^*(t-s)\left(1+\alpha_2 T^*\right)}{TT^*\left(1+\alpha_2 T^*(t-s)\right)}\right)ds\\ &+\frac{\beta_1 T_1 V_1}{\eta_2\left(1+\alpha_1 V_1\right)}\int_0^\infty f_2(s)e^{-\delta_2 s}\left(\frac{T^*}{T_1^*}-\frac{T^*(t-s)}{T_1^*}+\ln\left(\frac{T^*(t-s)}{T^*}\right)\right)ds.\end{aligned}\tag{28}$$

Collecting terms of Eq. (28) and applying the steady state conditions for $S_1$:

$$\rho-dT_1=\frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1}+\frac{\beta_2 T_1 T_1^*}{1+\alpha_2 T_1^*}=\frac{\mu}{\eta_1}T_1^*=\frac{c\mu}{b\eta_1\eta_2}V_1,$$

we get

$$\begin{aligned}\frac{dU_2}{dt} &= -\frac{d}{T}\left(T-T_1\right)^2+\left(1-\frac{T_1}{T}\right)\left(\frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1}+\frac{\beta_2 T_1 T_1^*}{1+\alpha_2 T_1^*}\right)+\frac{\beta_1 T_1 V}{1+\alpha_1 V}\\ &+\frac{\beta_2 T_1 T^*}{1+\alpha_2 T^*}-\frac{T_1^*}{\eta_1 T^*}\int_0^\infty f_1(s)e^{-\delta_1 s}\frac{\beta_1 T(t-s)V(t-s)}{1+\alpha_1 V(t-s)}ds\\ &-\frac{T_1^*}{\eta_1 T^*}\int_0^\infty f_1(s)e^{-\delta_1 s}\frac{\beta_2 T(t-s)T^*(t-s)}{1+\alpha_2 T^*(t-s)}ds-\frac{\beta_2 T_1 T^*}{1+\alpha_2 T_1^*}+\frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1}\\ &+\frac{\beta_2 T_1 T_1^*}{1+\alpha_2 T_1^*}-\frac{\beta_1 T_1 V_1}{\eta_2\left(1+\alpha_1 V_1\right)}\int_0^\infty f_2(s)e^{-\delta_2 s}\frac{V_1 T^*(t-s)}{VT_1^*}ds-\frac{\beta_1 T_1 V}{1+\alpha_1 V_1}\\ &+\frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1}+\frac{\beta_1 T_1 V_1}{\eta_1\left(1+\alpha_1 V_1\right)}\int_0^\infty f_1(s)e^{-\delta_1 s}\ln\left(\frac{T(t-s)V(t-s)\left(1+\alpha_1 V\right)}{TV\left(1+\alpha_1 V(t-s)\right)}\right)ds\\ &+\frac{\beta_2 T_1 T_1^*}{\eta_1\left(1+\alpha_2 T_1^*\right)}\int_0^\infty f_1(s)e^{-\delta_1 s}\ln\left(\frac{T(t-s)T^*(t-s)\left(1+\alpha_2 T^*\right)}{TT^*\left(1+\alpha_2 T^*(t-s)\right)}\right)ds\\ &+\frac{\beta_1 T_1 V_1}{\eta_2\left(1+\alpha_1 V_1\right)}\int_0^\infty f_2(s)e^{-\delta_2 s}\ln\left(\frac{T^*(t-s)}{T^*}\right)ds.\end{aligned}$$

Using Eq. (17) we get

$$\begin{aligned}
\frac{dU_2}{dt} &= -\frac{d}{T}\left(T-T_1\right)^2+\frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1}\left[\frac{\left(1+\alpha_1 V_1\right)V}{\left(1+\alpha_1 V\right)V_1}-\frac{V}{V_1}-1+\frac{1+\alpha_1 V}{1+\alpha_1 V_1}\right]\\
&+\frac{\beta_2 T_1 T_1^*}{1+\alpha_2 T_1^*}\left[\frac{\left(1+\alpha_2 T_1^*\right)T^*}{\left(1+\alpha_2 T^*\right)T_1^*}-\frac{T^*}{T_1^*}-1+\frac{1+\alpha_2 T^*}{1+\alpha_2 T_1^*}\right]\\
&-\frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1}\left[\frac{T_1}{T}-1-\ln\left(\frac{T_1}{T}\right)\right]-\frac{\beta_2 T_1 T_1^*}{1+\alpha_2 T_1^*}\left[\frac{T_1}{T}-1-\ln\left(\frac{T_1}{T}\right)\right]\\
&-\frac{\beta_1 T_1 V_1}{\eta_1\left(1+\alpha_1 V_1\right)}\int_0^\infty f_1(s)e^{-\delta_1 s}\left[\frac{T(t-s)V(t-s)\left(1+\alpha_1 V_1\right)T_1^*}{T_1 V_1\left(1+\alpha_1 V(t-s)\right)T^*}-1\right]ds\\
&+\frac{\beta_1 T_1 V_1}{\eta_1\left(1+\alpha_1 V_1\right)}\int_0^\infty f_1(s)e^{-\delta_1 s}\ln\left(\frac{T(t-s)V(t-s)\left(1+\alpha_1 V_1\right)T_1^*}{T_1 V_1\left(1+\alpha_1 V(t-s)\right)T^*}\right)ds\\
&-\frac{\beta_2 T_1 T_1^*}{\eta_1\left(1+\alpha_2 T_1^*\right)}\int_0^\infty f_1(s)e^{-\delta_1 s}\left[\frac{T(t-s)T^*(t-s)\left(1+\alpha_2 T_1^*\right)}{T_1 T^*\left(1+\alpha_2 T^*(t-s)\right)}-1\right]ds\\
&+\frac{\beta_2 T_1 T_1^*}{\eta_1\left(1+\alpha_2 T_1^*\right)}\int_0^\infty f_1(s)e^{-\delta_1 s}\ln\left(\frac{T(t-s)T^*(t-s)\left(1+\alpha_2 T_1^*\right)}{T_1 T^*\left(1+\alpha_2 T^*(t-s)\right)}\right)ds\\
&-\frac{\beta_1 T_1 V_1}{\eta_2\left(1+\alpha_1 V_1\right)}\int_0^\infty f_2(s)e^{-\delta_2 s}\left[\frac{V_1 T^*(t-s)}{V T_1^*}-1-\ln\left(\frac{V_1 T^*(t-s)}{V T_1^*}\right)\right]ds\\
&-\frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1}\left[\frac{1+\alpha_1 V}{1+\alpha_1 V_1}-1-\ln\left(\frac{1+\alpha_1 V}{1+\alpha_1 V_1}\right)\right]-\frac{\beta_2 T_1 T_1^*}{1+\alpha_2 T_1^*}\left[\frac{1+\alpha_2 T^*}{1+\alpha_2 T_1^*}-1-\ln\left(\frac{1+\alpha_2 T^*}{1+\alpha_2 T_1^*}\right)\right]\\
&=-\frac{d}{T}\left(T-T_1\right)^2-\frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1}\left[\frac{\alpha_1\left(V-V_1\right)^2}{\left(1+\alpha_1 V\right)\left(1+\alpha_1 V_1\right)V_1}\right]-\frac{\beta_2 T_1 T_1^*}{1+\alpha_2 T_1^*}\left[\frac{\alpha_2\left(T^*-T_1^*\right)^2}{\left(1+\alpha_2 T^*\right)\left(1+\alpha_2 T_1^*\right)T_1^*}\right]\\
&-\frac{\beta_1 T_1 V_1}{\eta_1\left(1+\alpha_1 V_1\right)}\int_0^\infty f_1(s)e^{-\delta_1 s}\left[g\left(\frac{T_1}{T}\right)+g\left(\frac{T(t-s)V(t-s)\left(1+\alpha_1 V_1\right)T_1^*}{T_1 V_1\left(1+\alpha_1 V(t-s)\right)T^*}\right)+g\left(\frac{1+\alpha_1 V}{1+\alpha_1 V_1}\right)\right]ds\\
&-\frac{\beta_2 T_1 T_1^*}{\eta_1\left(1+\alpha_2 T_1^*\right)}\int_0^\infty f_1(s)e^{-\delta_1 s}\left[g\left(\frac{T_1}{T}\right)+g\left(\frac{T(t-s)T^*(t-s)\left(1+\alpha_2 T_1^*\right)}{T_1 T^*\left(1+\alpha_2 T^*(t-s)\right)}\right)+g\left(\frac{1+\alpha_2 T^*}{1+\alpha_2 T_1^*}\right)\right]ds\\
&-\frac{\beta_1 T_1 V_1}{\eta_2\left(1+\alpha_1 V_1\right)}\int_0^\infty f_2(s)e^{-\delta_2 s}g\left(\frac{V_1 T^*(t-s)}{V T_1^*}\right)ds.
\end{aligned}$$

Similar to the proof of Theorem 2, one can easily show that $S_1$ is globally asymptotically stable. □

# 4 Numerical simulations

In order to illustrate our theoretical results, we will perform numerical simulations for system (7)-(9). We use the data given in Table 1.

Table 1: The data of system (7)-(9).

| Parameter | Value | Parameter | Parameter |
|---|---|---|---|
| $\lambda$ | 10 cells mm$^{-3}$day$^{-1}$ | $\tau_1$ | Varied |
| $d$ | 0.01 day$^{-1}$ | $\tau_2$ | Varied |
| $\beta_1$ | Varied | $\delta_1$ | 0.9 day$^{-1}$ |
| $\beta_2$ | 0.0001 cells$^{-1}$ mm$^3$day$^{-1}$ | $\delta_2$ | 0.1 day$^{-1}$ |
| $\alpha_1$ | Varied | $b$ | 10 virus cells$^{-1}$day$^{-1}$ |
| $\alpha_2$ | Varied | $c$ | 3 day$^{-1}$ |
| $\mu$ | 0.9 day$^{-1}$ | | |

### 4.1 Effect of the parameter $\beta_1$ on the stability of steady states

To show the global stability of the steady states we consider three different initial conditions:

IC1: $\varphi_1(\eta) = 600$, $\varphi_2(\eta) = 1, \varphi_3(\eta) = 1$,

IC2: $\varphi_1(\eta) = 200$, $\varphi_2(\eta) = 0.5, \varphi_3(\eta) = 3$,

IC3: $\varphi_1(\eta) = 700$, $\varphi_2(\eta) = 5, \varphi_3(\eta) = 9$,

where, $\eta \in [-\max\{\tau_1, \tau_2\}, 0]$.

In this case we choose $\tau_1 = 0.5$ day, $\tau_2 = 0.9$ day, $\alpha_1 = 0.009$ virus $^{-1}$ mm$^3$, $\alpha_2 = 0.005$ cells $^{-1}$ mm$^3$ and study the following subcases for the initial conditions IC1-IC3.:

**(i)** $\mathcal{R}_0 \leq 1$. We choose, $\beta_1 = 0.0001$ virus $^{-1}$ mm$^3$day$^{-1}$, then we compute $\mathcal{R}_0 = 0.2867 < 1$. From Lemma 2 we have that the system has one steady state $S_0$. From Figures 1-3 we can see that, the concentration of uninfected CD4$^+$ T cells is increasing and tends its normal value $\rho/d = 1000$, while the concentrations of infected cells and free HIV-1 are decaying and approaching zero for all the three initial conditions IC1-IC3. It means that, $S_0$ is globally asymptotically stable and the virus will be removed. This result support the result of Theorem 1.

**(ii)** $\mathcal{R}_0 > 1$. We take $\beta_1 = 0.001$ virus $^{-1}$ mm$^3$day$^{-1}$, and then, $\mathcal{R}_0 = 2.2292 > 1$. Lemma 2 state that the system has two positive steady states $S_0$ and $S_1$. It is clear from Figures 4-6 that, both the numerical results and the theoretical results given in Theorem 2 are consistent. It is seen that, the solutions of the system converges to the steady $S_1(491.6543, 3.6015, 10.9717)$, for all the three initial conditions IC1-IC3.

### 4.2 Effect of the saturation infection on the HIV-1 dynamics

In this case, we consider the initial condition IC2. We take the values $\tau_1 = 0.5$ day, $\tau_2 = 0.9$ day and$\beta_1 = 0.001$ virus$^{-1}$ mm$^3$day$^{-1}$. Figures 7-9 show the effect of saturation infection. We observe that, as $\alpha_1$ and $\alpha_2$ are increased, both the virus-target and infected-target infection rates are decreased, and then the concentration of the CD4$^+$ T cells are increased, while the concentrations of the infected cells and free HIV-1 particles are decreased.

### 4.3 Effect of the time delays on the stability of steady states

In this case, we consider the initial condition IC2. We take the values $\beta_1 = 0.001$ virus $^{-1}$ mm$^3$day$^{-1}$, $\alpha_1 = 0.009$ virus$^{-1}$ mm$^3$ and $\alpha_2 = 0.005$ cells $^{-1}$ mm$^3$. Let us consider the case $\tau = \tau_1 = \tau_2$. The values of $\mathcal{R}_0$ and the steady states of system (7)-(9) with different values of $\tau$ are presented in Table 2.

Table 2: The values of steady states, $R_0$ for model (7)-(9) with different values of the delay parameter $\tau$.

| Delay parameter | Steady states | $R_0$ |
|---|---|---|
| $\tau = 0.0$ | $E_1 = (319.8688, 7.5570, 25.1900)$ | 3.8148 |
| $\tau = 0.2$ | $E_1 = (373.1935, 5.8172, 19.0069)$ | 3.1251 |
| $\tau = 0.6$ | $E_1 = (517.7076, 3.1228, 9.8032)$ | 2.0973 |
| $\tau = 0.9$ | $E_1 = (670.8935, 1.6267, 4.9557)$ | 1.5552 |
| $\tau = 1.$ | $E_1 = (733.0161, 1.2061, 3.6377)$ | 1.4077 |
| $\tau =$1.3431 | $E_0 = (1000, 0, 0, 0)$ | 1 |
| $\tau = 1.5$ | $E_0 = (1000, 0, 0, 0)$ | 0.8552 |
| $\tau = 2$ | $E_0 = (1000, 0, 0, 0)$ | 0.5196 |
| $\tau = 2.5$ | $E_0 = (1000, 0, 0, 0)$ | 0.3157 |

From Table 2 we can see that, $\mathcal{R}_0$ is decreased as $\tau$ is increased. Using the data given in Table 1, we get:

(i) if $0 \leq \tau < 1.343070098$, then $S_1$ exists and it is globally asymptotically stable,

(ii) if $\tau \geq 1.343070098$, then $S_0$ is globally asymptotically stable.

Figures 10-12 show that the numerical results are also compatible with the results of Theorems 1 and 2. It can be seen that when the time delay is increased, the system can be stabilized around the disease-free steady state $S_0$. This means that the delay plays a similar job as the antiviral treatment in clearing the HIV-1 from the plasma.

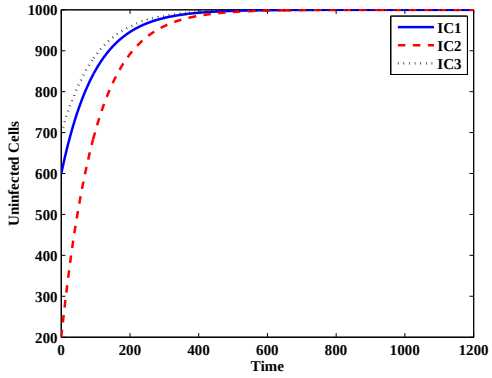



Figure 1: The evolution of uninfected CD4$^+$ T cells with initial IC1-IC3 in case of $R_0 \leq 1$.

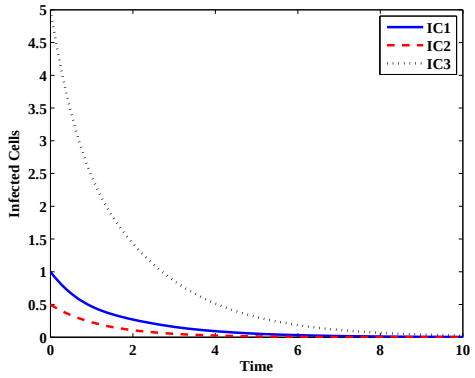



Figure 2: The evolution of infected cells with initial IC1-IC3 in case of $R_0 \leq 1$.

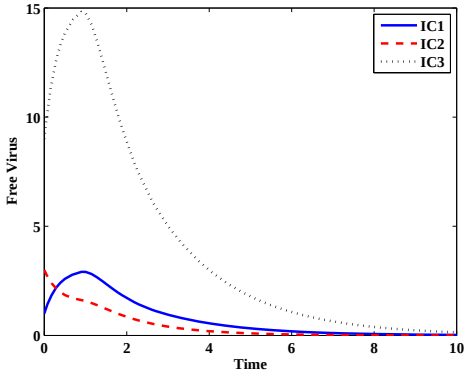



Figure 3: The evolution of free HIV-1 with initial IC1-IC3 in case of $R_0 \leq 1$.

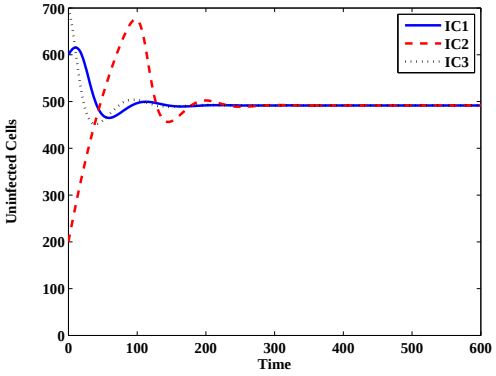



Figure 4: The evolution of uninfected CD4$^+$ T cells with initial IC1-IC3 in case of $R_0 > 1$.

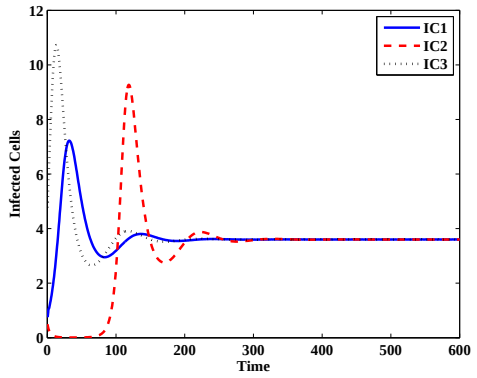



Figure 5: The evolution of infected cells with initial IC1-IC3 in case of $R_0 > 1$.

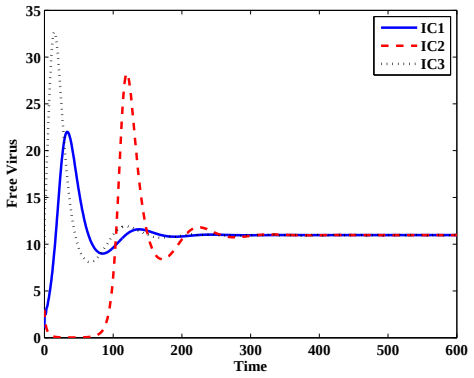



Figure 6: The evolution of free HIV-1 with initial IC1-IC3 in case of $R_0 > 1$.

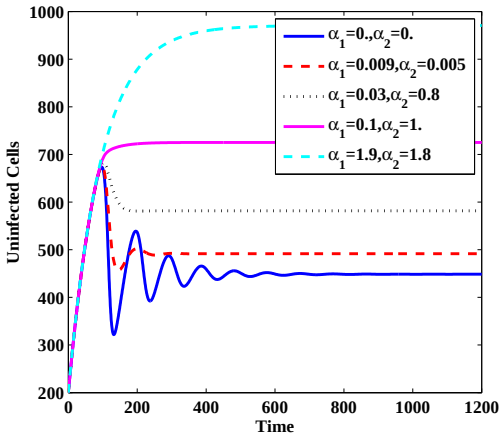



Figure 7: The evolution of uninfected CD4$^+$ T cells with different saturation parameters $\alpha_1$,$\alpha_2$.

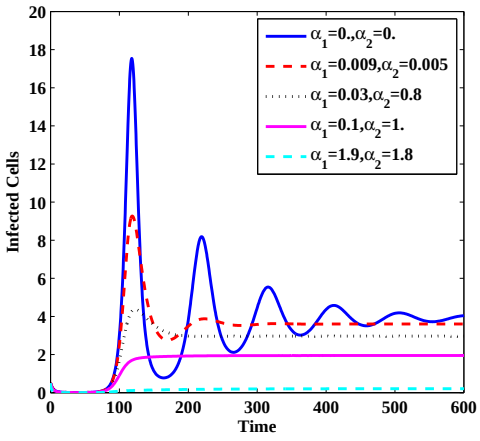



Figure 8: The evolution of infected cells with different saturation parameters $\alpha_1$,$\alpha_2$.

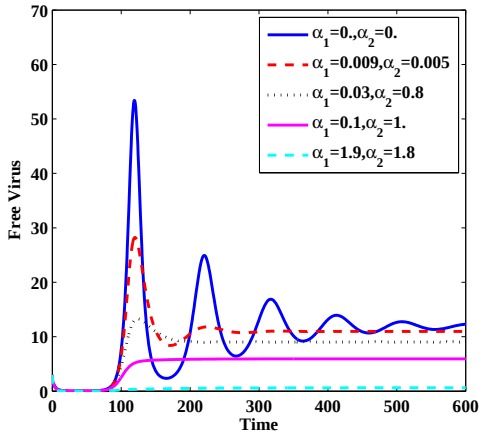



Figure 9: The evolution of free HIV-1 with different saturation parameters $\alpha_1$,$\alpha_2$.

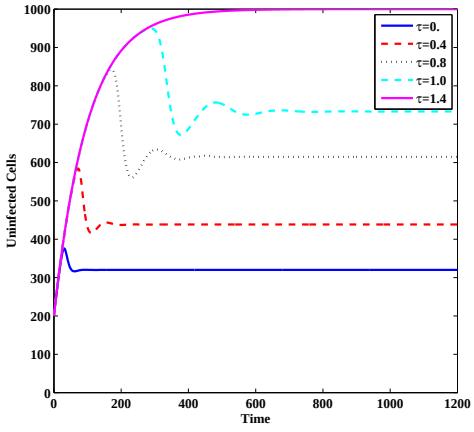



Figure 10: The evolution of uninfected CD4$^+$ T cells with different delay parameter $\tau$.

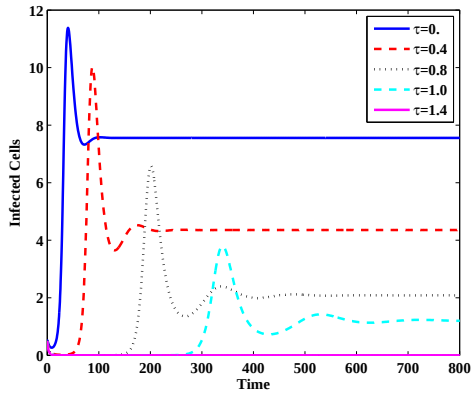



Figure 11: The evolution of infected cells with different delay parameter $\tau$.

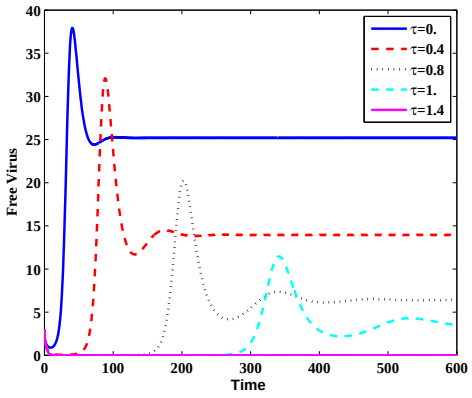



Figure 12: The evolution of free HIV-1 with different delay parameter $\tau$.

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## References

[1] M. A. Nowak and C. R. M. Bangham, *Population dynamics of immune responses to persistent viruses*, Science, **272** (1996) 74-79.

[2] X. Song, A.U. Neumann, *Global stability and periodic solution of the viral dynamics*, J. Math. Anal. Appl., 329 (2007), 281-297.

[3] X. Shi, X. Zhou, and X. Son, *Dynamical behavior of a delay virus dynamics model with CTL immune response*, Nonlinear Analysis: Real World Applications, **11** (2010), 1795-1809.

[4] H. Shu, L. Wang and J. Watmough, *Global stability of a nonlinear viral infection model with infinitely distributed intracellular delays and CTL imune responses*, SIAM Journal of Applied Mathematics, **73**(3) (2013), 1280-1302.

[5] A. M. Elaiw and N. H. AlShamrani, *Mathematical analysis of a cell mediated immunity in a virus dynamics model with nonlinear infection rate and removal*, Journal of Computational Analysis and Applications, **21**(3) (2016), 578-586.

[6] T. Wang, Z. Hu and F. Liao, *Stability and Hopf bifurcation for a virus infection model with delayed humoral immunity response*, J. Math. Anal. Appl., **411** (2014) 63-74.

[7] A. M. Elaiw and N. H. AlShamrani, *Global stability of humoral immunity virus dynamics models with nonlinear infection rate and removal*, Nonlinear Analysis: Real World Applications, **26**, pp. 161-190, 2015.

[8] M. A. Obaid, *Dynamical behaviors of a nonlinear virus infection model with latently infected cells and immune response*, Journal of Computational Analysis and Applications, **21**(1) (2016), 182-193.

[9] R. Xu, *Global stability of an HIV-1 infection model with saturation infection and intracellular delay*, J. Math. Anal. Appl., **375** (2011), 75-81.

[10] G. Huang, Y. Takeuchi and W. Ma, *Lyapunov functionals for delay differential equations model of viral infections*, SIAM J. Appl. Math., **70**(7) (2010), 2693-2708.

[11] S. Xu, *Global stability of the virus dynamics model with Crowley-Martin functional response*, Electronic Journal of Qualitative Theory of Differential Equations, **2012**, (2012), 1-9.

[12] N.M. Dixit and A.S. Perelson, *Complex patterns of viral load decay under antiretroviral therapy: Influence of pharmacokinetics and intracellular delay*, J. Theoret. Biol., **226** (2004), 95-109.

[13] M. Y. Li and H. Shu, *Impact of intracellular delays and target cell dynamics on in vivo viral infections*, SIAM J. Appl. Math., **70** (7) (2010), 2434-2448.

[14] D. Huang, X. Zhang, Y. Guo, and H. Wang, *Analysis of an HIV infection model with treatments and delayed immune response*, Applied Mathematical Modelling, **40**(4) (2016), 3081-3089.

[15] C. Monica and M. Pitchaimani, *Analysis of stability and Hopf bifurcation for HIV-1 dynamics with PI and three intracellular delays*, Nonlinear Analysis: Real World Applications, **27** (2016), 55-69.

[16] M. Y. Li and L. Wang, *Backward bifurcation in a mathematical model for HIV infection in vivo with anti-retroviral treatment*, Nonlinear Analysis: Real World Applications, **17** (2014), 147-160.

[17] A. M. Elaiw, and N. A. Almuallem, *Global dynamics of delay-distributed HIV infection models with differential drug efficacy in cocirculating target cells*, Mathematical Methods in the Applied Sciences, **39** (2016), 4-31.

[18] B. Buonomo, and C. Varglobally asymptotically stable-De-Le, *Global stability for an HIV-1 infection model including an eclipse stage of infected cells*, Journal of Mathematical Analysis and Applications, **385** (2012), 709-720.

[19] A. Korobeinikov, *Global properties of basic virus dynamics models*, Bull. Math. Biol. 66, (2004), 879-883.

[20] A. M. Elaiw, I. A. Hassanien, and S. A. Azoz, *Global stability of HIV infection models with intracellular delays*, Journal of the Korean Mathematical Society, **49**(4) (2012), 779-794.

[21] A.M. Elaiw and S.A. Azoz, *Global properties of a class of HIV infection models with Beddington-DeAngelis functional response,* Mathematical Methods in the Applied Sciences, **36** (2013), 383-394.

[22] A. M. Elaiw, and N. A. Almuallem, *Global properties of delayed-HIV dynamics models with differential drug efficacy in co-circulating target cells*, Applied Mathematics and Computation, **265** (2015), 1067-1089.

[23] F. Li and J. Wang, *Analysis of an HIV infection model with logistic target cell growth and cell-to-cell transmission*, Chaos, Solitons and Fractals, **81**(2015), 136-145.

[24] X. Lai and X. Zou , *Modeling cell-to-cell spread of HIV-1 with logistic target cell growth*, Journal of Mathematical Analysis and Applications, **426** (2015), 563–584.

[25] X. Lai, X. Zou, Modelling HIV-1 virus dynamics with both virus-to-cell infection and cell-to-cell transmission, SIAM Journal of Applied Mathematics, **74** (2014), 898–917.

[26] Y. Yanga, L. Zou and S. Ruanc, *Global dynamics of a delayed within-host viral infection model with both virus-to-cell and cell-to-cell transmissions*, Mathematical Biosciences, **270** (2015), 183-191.

[27] J. K. Hale, and S. M. V. Lunel, *Introduction to functional differential equations*, Springer-Verlag, New York, (1993).

# Equations on Banach space valued functions of abstract $g$-fractional calculus

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**Abstract**

The aim of this paper is utilize proper iterative methods for solving equations on Banach spaces. The differentiability of the operator involved is not assumed neither the convexity of its domain. Applications of the semi-local convergence are suggested including Banach space valued functions of fractional calculus, where all integrals are of Bochner-type.

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**Key Words and Phrases:** iterative method, Banach space, semi-local convergence, Fractional Calculus, Bochner-type integral.

## 1 Introduction

Let $B_1, B_2$ stand for Banach space and let $\Omega$ stand for an open subset of $B_1$. Let also $U(z,\rho) := \{u \in B_1 : \|u - z\| < \rho\}$ and let $\overline{U}(z,\rho)$ stand for the closure of $U(z,\rho)$.

Many problems in Computational Sciences, Engineering, Mathematical Chemistry, Mathematical Physics, Mathematical Economics and other disciplines can written as

$$F(x) = 0 \tag{1.1}$$

using Mathematical Modeling [1]-[17], where $F : \Omega \to B_2$ is a continuous operator. The solution $x^*$ of equation (1.1) is sought in closed form, but this is

attainable only in special cases. That explains why most solution methods for such equations are usually iterative. There is a plethora of iterative methods for solving equation (1.1), more the [2, 6, 7, 9 - 13, 15, 16].

**Newton's method** [6, 7, 11, 15, 16]:

$$x_{n+1} = x_n - F'(x_n)^{-1} F(x_n). \tag{1.2}$$

Secant method:

$$x_{n+1} = x_n - [x_{n-1}, x_n; F]^{-1} F(x_n), \tag{1.3}$$

where $[\cdot, \cdot; F]$ denotes a divided difference of order one on $\Omega \times \Omega$ [7, 15, 16].

Newton-like method:

$$x_{n+1} = x_n - E_n^{-1} F(x_n), \tag{1.4}$$

where $E_n = E(F)(x_n)$ and $E : \Omega \to \mathcal{L}(B_1, B_2)$ the space of bounded linear operators from $B_1$ into $B_2$. Other methods can be found in [7], [11], [15], [16] and the references therein.

In the present study we consider the new method defined for each $n = 0, 1, 2, ...$ by

$$x_{n+1} = G(x_n)$$
$$G(x_{n+1}) = G(x_n) - A_n^{-1} F(x_n), \tag{1.5}$$

where $x_0 \in \Omega$ is an initial point, $G : B_3 \to \Omega$ ($B_3$ a Banach space), $A_n = A(F)(x_{n+1}, x_n) = A(x_{n+1}, x_n)$ and $A : \Omega \times \Omega \to \mathcal{L}(B_1, B_2)$. Method (1.5) generates a sequence which we shall show converges to $x^*$ under some Lipschitz-type conditions (to be precised in Section 2). Although method (1.5) (and Section 2) is of independent interest, it is nevertheless designed especially to be used in $g$-Abstract Fractional Calculus (to be precised in Section 3). As far as we know such iterative methods have not yet appeared in connection to solve equations in Abstract Fractional Calculus.

In this paper we present the semi-local convergence of method (1.5) in Section 2. Some applications to Abstract $g$-Fractional Calculus are suggested in Section 3 on a certain Banach space valued functions, where all the integrals are of Bochner-type [8], [14].

## 2 Semi-local Convergence analysis

We present the semi-local convergence analysis of method (1.5) using conditions $(M)$:

$(m_1)$ $F : \Omega \subset B_1 \to B_2$ is continuous, $G : B_3 \to \Omega$ is continuous and $A(x, y) \in \mathcal{L}(B_1, B_2)$ for each $(x, y) \in \Omega \times \Omega$.

$(m_2)$ There exist $\beta > 0$ and $\Omega_0 \subset B_1$ such that $A(x,y)^{-1} \in \mathcal{L}(B_2, B_1)$ for each $(x,y) \in \Omega_0 \times \Omega_0$ and

$$\left\| A(x,y)^{-1} \right\| \leq \beta^{-1}.$$

Set $\Omega_1 = \Omega \cap \Omega_0$.

$(m_3)$ There exists a continuous and nondecreasing function $\psi : [0,+\infty)^3 \to [0,+\infty)$ such that for each $x, y \in \Omega_1$

$$\|F(x) - F(y) - A(x,y)(G(x) - G(y))\| \leq$$

$$\beta\psi(\|x-y\|, \|x-x_0\|, \|y-x_0\|) \|G(x) - G(y)\|.$$

$(m_4)$ There exists a continuous and nondecreasing function $\psi_0 : [0,+\infty) \to [0,+\infty)$ such that for each $x \in \Omega_1$

$$\|G(x) - G(x_0)\| \leq \psi_0(\|x - x_0\|) \|x - x_0\|.$$

$(m_5)$ For $x_0 \in \Omega_0$ and $x_1 = G(x_0) \in \Omega_0$ there exists $\eta \geq 0$ such that

$$\left\| A(x_1, x_0)^{-1} F(x_0) \right\| \leq \eta.$$

$(m_6)$ There exists $s > 0$ such that

$$\psi(\eta, s, s) < 1,$$

$$\psi_0(s) < 1$$

and

$$\|G(x_0) - x_0\| \leq s \leq \frac{\eta}{1 - q_0},$$

where $q_0 = \psi(\eta, s, s)$.

$(m_7)$ $\overline{U}(x_0, s) \subset \Omega$.

Next, we present the semi-local convergence analysis for method (1.5) using the conditions $(M)$ and the preceding notation.

**Theorem 2.1** *Assume that the conditions $(M)$ hold. Then, sequence $\{x_n\}$ generated by method (1.5) starting at $x_0 \in \Omega$ is well defined in $U(x_0, s)$, remains in $U(x_0, s)$ for each $n = 0, 1, 2, ...$ and converges to a solution $x^* \in \overline{U}(x_0, s)$ of equation $F(x) = 0$. The limit point $x^*$ is the unique solution of equation $F(x) = 0$ in $\overline{U}(x_0, s)$.*

**Proof.** By the definition of $s$ and $(m_5)$, we have $x_1 \in U(x_0, s)$. The proof is based on mathematical induction on $k$. Suppose that $\|x_k - x_{k-1}\| \leq q_0^{k-1}\eta$ and $\|x_k - x_0\| \leq s$.

We get by (1.5), $(m_2)-(m_5)$ in turn that

$$\|G(x_{k+1})-G(x_k)\| = \left\|A_k^{-1}F(x_k)\right\| =$$

$$\left\|A_k^{-1}\left(F(x_k)-F(x_{k-1})-A_{k-1}\left(G(x_k)-G(x_{k-1})\right)\right)\right\|$$

$$\leq \left\|A_k^{-1}\right\| \|F(x_k)-F(x_{k-1})-A_{k-1}\left(G(x_k)-G(x_{k-1})\right)\| \leq$$

$$\beta^{-1}\beta\psi\left(\|x_k-x_{k-1}\|, \|x_{k-1}-x_0\|, \|y_k-x_0\|\right)\|G(x_k)-G(x_{k-1})\| \leq$$

$$\psi(\eta,s,s)\|G(x_k)-G(x_{k-1})\| = q_0\|G(x_k)-G(x_{k-1})\| \leq q_0^k\|x_1-x_0\| \leq q_0^k\eta \tag{2.1}$$

and by $(m_6)$

$$\|x_{k+1}-x_0\| = \|G(x_k)-x_0\| \leq \|G(x_k)-G(x_0)\| + \|G(x_0)-x_0\|$$

$$\leq \psi_0\left(\|x_k-x_0\|\right)\|x_k-x_0\| + \|G(x_0)-x_0\|$$

$$\leq \psi_0(s)\,s + \|G(x_0)-x_0\| \leq s.$$

The induction is completed. Moreover, we have by (2.1) that for $m=0,1,2,...$

$$\|x_{k+m}-x_k\| \leq \frac{1-q_0^m}{1-q_0}q_0^k\eta.$$

It follows from the preceding inequation that sequence $\{G(x_k)\}$ is complete in a Banach space $B_1$ and as such it converges to some $x^* \in \overline{U}(x_0,s)$ (since $\overline{U}(x_0,s)$ is a closed ball). By letting $k\to+\infty$ in (2.1) we get $F(x^*)=0$. We also get by (1.5) that $G(x^*)=x^*$. To show the uniqueness part, let $x^{**}\in U(x_0,s)$ be a solution of equation $F(x)=0$ and $G(x^{**})=x^{**}$. By using (1.5), we obtain in turn that

$$\|x^{**}-G(x_{k+1})\| = \left\|x^{**}-G(x_k)+A_k^{-1}F(x_k)-A_k^{-1}F(x^{**})\right\| \leq$$

$$\left\|A_k^{-1}\right\|\|F(x^{**})-F(x_k)-A_k\left(G(x^{**})-G(x_k)\right)\| \leq$$

$$\beta^{-1}\beta\psi_0\left(\|x^{**}-x_k\|, \|x_{k+1}-x_0\|, \|x_k-x_0\|\right)\|G(x^{**})-G(x_k)\| \leq$$

$$q_0\|G(x^{**})-G(x_k)\| \leq q_0^{k+1}\|x^{**}-x_0\|,$$

so $\lim_{k\to+\infty} x_k = x^{**}$. We have shown that $\lim_{k\to+\infty} x_k = x^*$, so $x^*=x^{**}$. ■

**Remark 2.2** *(1) Condition $(m_2)$ can become part of condition $(m_3)$ by considering*

$(m_3)'$ *There exists a continuous and nondecreasing function $\varphi:[0,+\infty)^3\to[0,+\infty)$ such that for each $x,y\in\Omega_1$*

$$\left\|A(x,y)^{-1}\left[F(x)-F(y)-A(x,y)\left(G(x)-G(y)\right)\right]\right\| \leq$$

$$\varphi\left(\left\|x-y\right\|,\left\|x-x_0\right\|,\left\|y-x_0\right\|\right)\left\|G\left(x\right)-G\left(y\right)\right\|.$$

*Notice that*

$$\varphi\left(u_1,u_2,u_3\right)\leq\psi\left(u_1,u_2,u_3\right)$$

*for each* $u_1\geq 0$, $u_2\geq 0$ *and* $u_3\geq 0$. *Similarly, a function* $\varphi_1$ *can replace* $\psi_1$ *for the uniqueness of the solution part. These replacements are of Mysovskii-type [6], [11], [15] and influence the weaking of the convergence criterion in* $(m_6)$, *error bounds and the precision of* $s$.

*(2) Suppose that there exist* $\beta>0$, $\beta_1>0$ *and* $L\in\mathcal{L}\left(B_1,B_2\right)$ *with* $L^{-1}\in\mathcal{L}\left(B_2,B_1\right)$ *such that*

$$\left\|L^{-1}\right\|\leq\beta^{-1}$$

$$\left\|A\left(x,y\right)-L\right\|\leq\beta_1$$

*and*

$$\beta_2:=\beta^{-1}\beta_1<1.$$

*Then, it follows from the Banach lemma on invertible operators [11], and*

$$\left\|L^{-1}\right\|\left\|A\left(x,y\right)-L\right\|\leq\beta^{-1}\beta_1=\beta_2<1$$

*that* $A\left(x,y\right)^{-1}\in\mathcal{L}\left(B_2,B_1\right)$. *Let* $\beta=\frac{\beta^{-1}}{1-\beta_2}$. *Then, under these replacements, condition* $(m_2)$ *is implied, therefore it can be dropped from the conditions* $(M)$.

**Remark 2.3** *Section 2 has an interest independent of Section 3. It is worth noticing that the results especially of Theorem 2.1 can apply in Abstract g-Fractional Calculus as illustrated in Section 3. By specializing function* $\psi$, *we can apply the results of say Theorem 2.1 in the examples suggested in Section 3. In particular for (3.21), we choose for* $u_1\geq 0$, $u_2\geq 0$, $u_3\geq 0$

$$\psi\left(u_1,u_2,u_3\right)=\frac{\lambda\mu_1^{\nu}}{\beta\Gamma\left(\nu\right)\left(\nu+1\right)},$$

*if* $\left|g\left(x\right)-g\left(y\right)\right|\leq\mu_1$ *for each* $x,y\in\left[a,b\right]$;

$$\psi\left(u_1,u_2,u_3\right)=\frac{\lambda\mu_2^{\nu}}{\beta\Gamma\left(\nu\right)\left(\nu+1\right)},$$

*if* $\left|g\left(x\right)-g\left(y\right)\right|\leq\xi_2\left\|x-y\right\|$ *for each* $x,y\in\left[a,b\right]$ *and* $\mu_2=\xi_2\left|b-a\right|$;

$$\psi\left(u_1,u_2,u_3\right)=\frac{\lambda\mu_3^{\nu}}{\beta\Gamma\left(\nu\right)\left(\nu+1\right)},$$

*if* $\left|g\left(x\right)\right|\leq\xi_3$ *for each* $x,y\in\left[a,b\right]$ *and* $\mu_3=2\xi_3$, *where* $\lambda,\nu$ *and* $F$ *are defined in Section 3. Other choices of function* $\psi$ *are also possible.*

*Notice that with these choices of function* $\psi$ *and* $f=F$ *and* $g=G$, *crucial condition* $(m_3)$ *is satisfied, which justifies our definition of method (1.5). We can provide similar choices for the other examples of Section 3.*

# 3 Applications to $X$-valued $g$-Fractional Calculus of Canavati type

Here we deal with Banach space $(X, \|\cdot\|)$ valued functions $f$ of real domain $[a,b]$. All integrals here are of Bochner-type, see [14]. The derivatives of $f$ are defined similarly to numerical ones, see [17], pp. 83-86 and p. 93.

Here both needed backgrounds come from [5].

Let $\nu > 1$, $\nu \notin \mathbb{N}$, with integral part $[\nu] = n \in \mathbb{N}$. Let $g : [a,b] \to \mathbb{R}$ be a strictly increasing function, such that $g \in C^1([a,b])$, $g^{-1} \in C^n([g(a), g(b)])$, and let $f \in C^n([a,b], X)$. It clear then we obtain that $\left(f \circ g^{-1}\right) \in C^n([g(a), g(b)], X)$. Let $\alpha := \nu - [\nu] = \nu - n$ $(0 < \alpha < 1)$.

(I) See [5]. Let $h \in C([g(a), g(b)], X)$, we define the $X$-valued left Riemann-Liouville fractional integral as

$$\left(J_{\nu}^{z_0} h\right)(z) := \frac{1}{\Gamma(\nu)} \int_{z_0}^{z} (z-t)^{\nu-1} h(t)\, dt, \tag{3.1}$$

for $g(a) \le z_0 \le z \le g(b)$, where $\Gamma$ is the gamma function.

We define the subspace $C_{g(x)}^{\nu}([g(a), g(b)], X)$ of $C^n([g(a), g(b)], X)$, where $x \in [a,b]$ :

$$C_{g(x)}^{\nu}([g(a), g(b)], X) := \left\{ h \in C^n([g(a), g(b)], X) : J_{1-\alpha}^{g(x)} h^{(n)} \in C^1([g(x), g(b)], X) \right\}. \tag{3.2}$$

So let $h \in C_{g(x)}^{\nu}([g(a), g(b)], X)$; we define the $X$-valued left $g$-generalized fractional derivative of $h$ of order $\nu$, of Canavati type, over $[g(x), g(b)]$ as

$$D_{g(x)}^{\nu} h := \left(J_{1-\alpha}^{g(x)} h^{(n)}\right)'. \tag{3.3}$$

Clearly, for $h \in C_{g(x)}^{\nu}([g(a), g(b)], X)$, there exists

$$\left(D_{g(x)}^{\nu} h\right)(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_{g(x)}^{z} (z-t)^{-\alpha} h^{(n)}(t)\, dt, \tag{3.4}$$

for all $g(x) \le z \le g(b)$.

In particular, when $f \circ g^{-1} \in C_{g(x)}^{\nu}([g(a), g(b)], X)$ we have that

$$\left(D_{g(x)}^{\nu}\left(f \circ g^{-1}\right)\right)(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_{g(x)}^{z} (z-t)^{-\alpha} \left(f \circ g^{-1}\right)^{(n)}(t)\, dt, \tag{3.5}$$

for all $z : g(x) \le z \le g(b)$.

We have that $D_{g(x)}^{n}\left(f \circ g^{-1}\right) = \left(f \circ g^{-1}\right)^{(n)}$ and $D_{g(x)}^{0}\left(f \circ g^{-1}\right) = f \circ g^{-1}$.

From [5] we have for $\left(f \circ g^{-1}\right) \in C_{g(x)}^{\nu}([g(a), g(b)], X)$, where $x \in [a,b]$, ($X$-valued left fractional Taylor's formula) that

$$f(y) - f(x) = \sum_{k=1}^{n-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(x))}{k!} (g(y) - g(x))^k + \tag{3.6}$$

$$\frac{1}{\Gamma(\nu)}\int_{g(x)}^{g(y)}(g(y)-t)^{\nu-1}\left(D_{g(x)}^{\nu}\left(f\circ g^{-1}\right)\right)(t)\,dt,\quad \text{for all } y\in[a,b]: y\geq x.$$

Alternatively, for $\left(f\circ g^{-1}\right)\in C_{g(y)}^{\nu}([g(a),g(b)],X)$, where $y\in[a,b]$, we can write (again $X$-valued left fractional Taylor's formula) that:

$$f(x)-f(y)=\sum_{k=1}^{n-1}\frac{\left(f\circ g^{-1}\right)^{(k)}(g(y))}{k!}(g(x)-g(y))^{k}+ \tag{3.7}$$

$$\frac{1}{\Gamma(\nu)}\int_{g(y)}^{g(x)}(g(x)-t)^{\nu-1}\left(D_{g(y)}^{\nu}\left(f\circ g^{-1}\right)\right)(t)\,dt,\quad \text{for all } x\in[a,b]: x\geq y.$$

Here we consider $f\in C^{n}([a,b],X)$, such that $\left(f\circ g^{-1}\right)\in C_{g(x)}^{\nu}([g(a),g(b)],X)$, for every $x\in[a,b]$; which is the same as $\left(f\circ g^{-1}\right)\in C_{g(y)}^{\nu}([g(a),g(b)],X)$, for every $y\in[a,b]$ (i.e. exchange roles of $x$ and $y$); we write that as $\left(f\circ g^{-1}\right)\in C_{g+}^{\nu}([g(a),g(b)],X)$.

We have that

$$\left(D_{g(y)}^{\nu}\left(f\circ g^{-1}\right)\right)(z)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dz}\int_{g(y)}^{z}(z-t)^{-\alpha}\left(f\circ g^{-1}\right)^{(n)}(t)\,dt, \tag{3.8}$$

for all $z: g(y)\leq z\leq g(b)$.

So here we work with $f\in C^{n}([a,b],X)$, such that $\left(f\circ g^{-1}\right)\in C_{g+}^{\nu}([g(a),g(b)],X)$.

We define the $X$-valued left linear fractional operator

$$(A_1(f))(x,y):=\begin{cases}\sum_{k=1}^{n-1}\frac{\left(f\circ g^{-1}\right)^{(k)}(g(x))}{k!}(g(y)-g(x))^{k-1}+\\ \left(D_{g(x)}^{\nu}\left(f\circ g^{-1}\right)\right)(g(y))\frac{(g(y)-g(x))^{\nu-1}}{\Gamma(\nu+1)},\quad y>x,\\ \\ \sum_{k=1}^{n-1}\frac{\left(f\circ g^{-1}\right)^{(k)}(g(y))}{k!}(g(x)-g(y))^{k-1}+\\ \left(D_{g(y)}^{\nu}\left(f\circ g^{-1}\right)\right)(g(x))\frac{(g(x)-g(y))^{\nu-1}}{\Gamma(\nu+1)},\quad x>y,\\ \\ f^{(n)}(x),\quad x=y.\end{cases} \tag{3.9}$$

We may assume that (see [12], p. 3)

$$\|(A_1(f))(x,x)-(A_1(f))(y,y)\|=\left\|f^{(n)}(x)-f^{(n)}(y)\right\|=$$

$$\left\|\left(f^{(n)}\circ g^{-1}\right)(g(x))-\left(f^{(n)}\circ g^{-1}\right)(g(y))\right\|\leq\Phi\left|g(x)-g(y)\right|, \tag{3.10}$$

where $\Phi>0$; for any $x,y\in[a,b]$.

We make the following estimations:

(i) case of $y>x$ : We have that

$$\|f(y)-f(x)-(A_1(f))(x,y)(g(y)-g(x))\|=$$

$$\left\| \frac{1}{\Gamma(\nu)} \int_{g(x)}^{g(y)} (g(y)-t)^{\nu-1} \left( D^{\nu}_{g(x)} \left( f \circ g^{-1} \right) \right)(t)\, dt - \right.$$

$$\left. \left( D^{\nu}_{g(x)} \left( f \circ g^{-1} \right) \right)(g(y)) \frac{(g(y)-g(x))^{\nu}}{\Gamma(\nu+1)} \right\|$$

(by [1], p. 426, Theorem 11.43)

$$= \frac{1}{\Gamma(\nu)} \left\| \int_{g(x)}^{g(y)} (g(y)-t)^{\nu-1} \left( \left( D^{\nu}_{g(x)} \left( f \circ g^{-1} \right) \right)(t) - \left( D^{\nu}_{g(x)} \left( f \circ g^{-1} \right) \right)(g(y)) \right) dt \right\| \tag{3.11}$$

(by [8])

$$\leq \frac{1}{\Gamma(\nu)} \int_{g(x)}^{g(y)} (g(y)-t)^{\nu-1} \left\| \left( D^{\nu}_{g(x)} \left( f \circ g^{-1} \right) \right)(t) - \left( D^{\nu}_{g(x)} \left( f \circ g^{-1} \right) \right)(g(y)) \right\| dt$$

(we assume here that

$$\left\| \left( D^{\nu}_{g(x)} \left( f \circ g^{-1} \right) \right)(t) - \left( D^{\nu}_{g(x)} \left( f \circ g^{-1} \right) \right)(g(y)) \right\| \leq \lambda_1 |t-g(y)|, \tag{3.12}$$

for every $t, g(y), g(x) \in [g(a), g(b)]$ such that $g(y) \geq t \geq g(x)$; $\lambda_1 > 0$)

$$\leq \frac{\lambda_1}{\Gamma(\nu)} \int_{g(x)}^{g(y)} (g(y)-t)^{\nu-1} (g(y)-t)\, dt = \tag{3.13}$$

$$\frac{\lambda_1}{\Gamma(\nu)} \int_{g(x)}^{g(y)} (g(y)-t)^{\nu} dt = \frac{\lambda_1}{\Gamma(\nu)} \frac{(g(y)-g(x))^{\nu+1}}{(\nu+1)}. \tag{3.14}$$

We have proved that

$$\| f(y) - f(x) - (A_1(f))(x,y)(g(y)-g(x)) \| \leq \frac{\lambda_1}{\Gamma(\nu)} \frac{(g(y)-g(x))^{\nu+1}}{(\nu+1)}, \tag{3.15}$$

for all $x, y \in [a,b] : y > x$.

(ii) Case of $x > y$ : We observe that

$$\| f(y) - f(x) - (A_1(f))(x,y)(g(y)-g(x)) \| =$$

$$\| f(x) - f(y) - (A_1(f))(x,y)(g(x)-g(y)) \| =$$

$$\left\| \frac{1}{\Gamma(\nu)} \int_{g(y)}^{g(x)} (g(x)-t)^{\nu-1} \left( D^{\nu}_{g(y)} \left( f \circ g^{-1} \right) \right)(t)\, dt - \right.$$

$$\left. \left( D^{\nu}_{g(y)} \left( f \circ g^{-1} \right) \right)(g(x)) \frac{(g(x)-g(y))^{\nu}}{\Gamma(\nu+1)} \right\| = \tag{3.16}$$

$$\frac{1}{\Gamma(\nu)} \left\| \int_{g(y)}^{g(x)} (g(x)-t)^{\nu-1} \left( \left( D^{\nu}_{g(y)} \left( f \circ g^{-1} \right) \right)(t) - \left( D^{\nu}_{g(y)} \left( f \circ g^{-1} \right) \right)(g(x)) \right) dt \right\| \leq$$

$$\frac{1}{\Gamma(\nu)}\int_{g(y)}^{g(x)}(g(x)-t)^{\nu-1}\left\|\left(D_{g(y)}^{\nu}\left(f\circ g^{-1}\right)\right)(t)-\left(D_{g(y)}^{\nu}\left(f\circ g^{-1}\right)\right)(g(x))\right\|dt \tag{3.17}$$

(we assume that

$$\left\|\left(D_{g(y)}^{\nu}\left(f\circ g^{-1}\right)\right)(t)-\left(D_{g(y)}^{\nu}\left(f\circ g^{-1}\right)\right)(g(x))\right\|\leq\lambda_2\left|t-g(x)\right|, \tag{3.18}$$

for all $t, g(x), g(y)\in[g(a), g(b)]$ such that $g(x)\geq t\geq g(y)$; $\lambda_2>0$)

$$\leq\frac{\lambda_2}{\Gamma(\nu)}\int_{g(y)}^{g(x)}(g(x)-t)^{\nu-1}(g(x)-t)\,dt= \tag{3.19}$$

$$\frac{\lambda_2}{\Gamma(\nu)}\int_{g(y)}^{g(x)}(g(x)-t)^{\nu}\,dt=\frac{\lambda_2}{\Gamma(\nu)}\frac{(g(x)-g(y))^{\nu+1}}{(\nu+1)}.$$

We have proved that

$$\left\|f(y)-f(x)-(A_1(f))(x,y)(g(y)-g(x))\right\|\leq\frac{\lambda_2}{\Gamma(\nu)}\frac{(g(x)-g(y))^{\nu+1}}{(\nu+1)}, \tag{3.20}$$

for any $x, y\in[a,b]: x>y$.

**Conclusion 3.1** *Set $\lambda:=\max(\lambda_1,\lambda_2)$. Then*

$$\left\|f(y)-f(x)-(A_1(f))(x,y)(g(y)-g(x))\right\|\leq\frac{\lambda}{\Gamma(\nu)}\frac{|g(y)-g(x)|^{\nu+1}}{(\nu+1)}, \tag{3.21}$$

$\forall\ x, y\in[a,b]$ *(the case of $x=y$ is trivially true).*

We may choose that $\frac{\lambda}{\Gamma(\nu)}<1$.

Also we notice here that $\nu+1>2$.

(II) See [5] again. Let $h\in C([g(a), g(b)], X)$, we define the $X$-valued right Riemann-Liouville fractional integral as

$$\left(J_{z_0-}^{\nu}h\right)(z):=\frac{1}{\Gamma(\nu)}\int_{z}^{z_0}(t-z)^{\nu-1}h(t)\,dt, \tag{3.22}$$

for $g(a)\leq z\leq z_0\leq g(b)$.

We define the subspace $C_{g(x)-}^{\nu}([g(a), g(b)], X)$ of $C^n([g(a), g(b)], X)$, where $x\in[a,b]$:

$$C_{g(x)-}^{\nu}([g(a), g(b)], X):=\left\{h\in C^n([g(a), g(b)], X): J_{g(x)-}^{1-\alpha}h^{(n)}\in C^1([g(a), g(x)], X)\right\}. \tag{3.23}$$

So let $h\in C_{g(x)-}^{\nu}([g(a), g(b)], X)$; we define the $X$-valued right $g$-generalized fractional derivative of $h$ of order $\nu$, of Canavati type, over $[g(a), g(x)]$ as

$$D_{g(x)-}^{\nu}h:=(-1)^{n-1}\left(J_{g(x)-}^{1-\alpha}h^{(n)}\right)'. \tag{3.24}$$

Clearly, for $h \in C^{\nu}_{g(x)-}([g(a),g(b)],X)$, there exists

$$\left(D^{\nu}_{g(x)-}h\right)(z) = \frac{(-1)^{n-1}}{\Gamma(1-\alpha)}\frac{d}{dz}\int_z^{g(x)} (t-z)^{-\alpha} h^{(n)}(t)\,dt, \tag{3.25}$$

for all $g(a) \le z \le g(x) \le g(b)$.

In particular, when $f \circ g^{-1} \in C^{\nu}_{g(x)-}([g(a),g(b)],X)$ we have that

$$\left(D^{\nu}_{g(x)-}\left(f\circ g^{-1}\right)\right)(z) = \frac{(-1)^{n-1}}{\Gamma(1-\alpha)}\frac{d}{dz}\int_z^{g(x)} (t-z)^{-\alpha}\left(f\circ g^{-1}\right)^{(n)}(t)\,dt, \tag{3.26}$$

for all $g(a) \le z \le g(x) \le g(b)$.

We get that

$$\left(D^{n}_{g(x)-}\left(f\circ g^{-1}\right)\right)(z) = (-1)^n\left(f\circ g^{-1}\right)^{(n)}(z), \tag{3.27}$$

and

$$\left(D^{0}_{g(x)-}\left(f\circ g^{-1}\right)\right)(z) = \left(f\circ g^{-1}\right)(z), \tag{3.28}$$

for all $z \in [g(a),g(x)]$, see [5].

From [5] we have, for $\left(f\circ g^{-1}\right) \in C^{\nu}_{g(x)-}([g(a),g(b)],X)$, where $x \in [a,b]$, $\nu \ge 1$ ($X$-valued right fractional Taylor's formula) that:

$$f(y) - f(x) = \sum_{k=1}^{n-1} \frac{\left(f\circ g^{-1}\right)^{(k)}(g(x))}{k!}(g(y)-g(x))^k +$$

$$\frac{1}{\Gamma(\nu)}\int_{g(y)}^{g(x)} (t-g(y))^{\nu-1}\left(D^{\nu}_{g(x)-}\left(f\circ g^{-1}\right)\right)(t)\,dt, \quad \text{all } a \le y \le x. \tag{3.29}$$

Alternatively, for $\left(f\circ g^{-1}\right) \in C^{\nu}_{g(y)-}([g(a),g(b)],X)$, where $y \in [a,b]$, $\nu \ge 1$ (again $X$-valued right fractional Taylor's formula) that:

$$f(x) - f(y) = \sum_{k=1}^{n-1} \frac{\left(f\circ g^{-1}\right)^{(k)}(g(y))}{k!}(g(x)-g(y))^k +$$

$$\frac{1}{\Gamma(\nu)}\int_{g(x)}^{g(y)} (t-g(x))^{\nu-1}\left(D^{\nu}_{g(y)-}\left(f\circ g^{-1}\right)\right)(t)\,dt, \quad \text{all } a \le x \le y. \tag{3.30}$$

Here we consider $f \in C^n([a,b],X)$, such that $\left(f\circ g^{-1}\right) \in C^{\nu}_{g(x)-}([g(a),g(b)],X)$, for every $x \in [a,b]$; which is the same as $\left(f\circ g^{-1}\right) \in C^{\nu}_{g(y)-}([g(a),g(b)],X)$, for every $y \in [a,b]$; (i.e. exchange roles of $x$ and $y$) we write that as $\left(f\circ g^{-1}\right) \in C^{\nu}_{g-}([g(a),g(b)],X)$.

We have that

$$\left(D^{\nu}_{g(y)-}\left(f\circ g^{-1}\right)\right)(z) = \frac{(-1)^{n-1}}{\Gamma(1-\alpha)}\frac{d}{dz}\int_z^{g(y)} (t-z)^{-\alpha}\left(f\circ g^{-1}\right)^{(n)}(t)\,dt, \tag{3.31}$$

for all $g(a) \leq z \leq g(y) \leq g(b)$.

So here we work with $f \in C^n([a,b],X)$, such that $\left(f \circ g^{-1}\right) \in C^{\nu}_{g^-}([g(a),g(b)],X)$.

We define the $X$-valued right linear fractional operator

$$(A_2(f))(x,y) := \begin{cases} \sum_{k=1}^{n-1} \frac{\left(f\circ g^{-1}\right)^{(k)}(g(x))}{k!} (g(y)-g(x))^{k-1} - \\ \left(D^{\nu}_{g(x)-}\left(f\circ g^{-1}\right)\right)(g(y)) \frac{(g(x)-g(y))^{\nu-1}}{\Gamma(\nu+1)}, \quad x>y, \\ \\ \sum_{k=1}^{n-1} \frac{\left(f\circ g^{-1}\right)^{(k)}(g(y))}{k!} (g(x)-g(y))^{k-1} - \\ \left(D^{\nu}_{g(y)-}\left(f\circ g^{-1}\right)\right)(g(x)) \frac{(g(y)-g(x))^{\nu-1}}{\Gamma(\nu+1)}, \quad y>x, \\ \\ f^{(n)}(x), \quad x=y. \end{cases} \tag{3.32}$$

We may assume that ([12], p. 3)

$$\|(A_2(f))(x,x) - (A_2(f))(y,y)\| = \left\|f^{(n)}(x) - f^{(n)}(y)\right\| \leq \Phi^* |g(x)-g(y)|, \tag{3.33}$$

where $\Phi^* > 0$; for any $x,y \in [a,b]$.

We make the following estimations:

(i) case of $x > y$ : We have that

$$\begin{aligned} &\|f(x) - f(y) - (A_2(f))(x,y)(g(x)-g(y))\| = \\ &\|f(y) - f(x) - (A_2(f))(x,y)(g(y)-g(x))\| = \\ &\|f(y) - f(x) + (A_2(f))(x,y)(g(x)-g(y))\| = \end{aligned} \tag{3.34}$$

$$\left\| \frac{1}{\Gamma(\nu)} \int_{g(y)}^{g(x)} (t-g(y))^{\nu-1} \left(D^{\nu}_{g(x)-}\left(f\circ g^{-1}\right)\right)(t)\,dt - \left(D^{\nu}_{g(x)-}\left(f\circ g^{-1}\right)\right)(g(y)) \frac{(g(x)-g(y))^{\nu}}{\Gamma(\nu+1)} \right\| \tag{3.35}$$

(by [1], p. 426, Theorem 11.43)

$$= \frac{1}{\Gamma(\nu)} \left\| \int_{g(y)}^{g(x)} (t-g(y))^{\nu-1} \left( \left(D^{\nu}_{g(x)-}\left(f\circ g^{-1}\right)\right)(t) - \left(D^{\nu}_{g(x)-}\left(f\circ g^{-1}\right)\right)(g(y)) \right) dt \right\|$$

(by [8])

$$\leq \frac{1}{\Gamma(\nu)} \int_{g(y)}^{g(x)} (t-g(y))^{\nu-1} \left\| \left(D^{\nu}_{g(x)-}\left(f\circ g^{-1}\right)\right)(t) - \left(D^{\nu}_{g(x)-}\left(f\circ g^{-1}\right)\right)(g(y)) \right\| dt \tag{3.36}$$

(we assume here that

$$\left\| \left(D^{\nu}_{g(x)-}\left(f\circ g^{-1}\right)\right)(t) - \left(D^{\nu}_{g(x)-}\left(f\circ g^{-1}\right)\right)(g(y)) \right\| \leq \rho_1 |t-g(y)|, \tag{3.37}$$

for every $t, g(y), g(x) \in [g(a), g(b)]$ such that $g(x) \geq t \geq g(y)$; $\rho_1 > 0$)

$$\leq \frac{\rho_1}{\Gamma(\nu)} \int_{g(y)}^{g(x)} (t - g(y))^{\nu-1} (t - g(y))\, dt =$$

$$\frac{\rho_1}{\Gamma(\nu)} \int_{g(y)}^{g(x)} (t - g(y))^{\nu}\, dt = \frac{\rho_1}{\Gamma(\nu)} \frac{(g(x) - g(y))^{\nu+1}}{(\nu+1)}. \tag{3.38}$$

We have proved that

$$\|f(x) - f(y) - (A_2(f))(x,y)(g(x) - g(y))\| \leq \frac{\rho_1}{\Gamma(\nu)} \frac{(g(x) - g(y))^{\nu+1}}{(\nu+1)}, \tag{3.39}$$

$\forall\ x, y \in [a,b] : x > y$.

(ii) Case of $x < y$ : We have that

$$\|f(x) - f(y) - (A_2(f))(x,y)(g(x) - g(y))\| =$$

$$\|f(x) - f(y) + (A_2(f))(x,y)(g(y) - g(x))\| = \tag{3.40}$$

$$\left\| \frac{1}{\Gamma(\nu)} \int_{g(x)}^{g(y)} (t - g(x))^{\nu-1} \left( D^{\nu}_{g(y)-} \left( f \circ g^{-1} \right) \right)(t)\, dt - \right.$$

$$\left. \left( D^{\nu}_{g(y)-} \left( f \circ g^{-1} \right) \right)(g(x)) \frac{(g(y) - g(x))^{\nu}}{\Gamma(\nu+1)} \right\| =$$

$$\frac{1}{\Gamma(\nu)} \left\| \int_{g(x)}^{g(y)} (t - g(x))^{\nu-1} \left( \left( D^{\nu}_{g(y)-} \left( f \circ g^{-1} \right) \right)(t) - \left( D^{\nu}_{g(y)-} \left( f \circ g^{-1} \right) \right)(g(x)) \right) dt \right\| \leq$$

$$\frac{1}{\Gamma(\nu)} \int_{g(x)}^{g(y)} (t - g(x))^{\nu-1} \left\| \left( D^{\nu}_{g(y)-} \left( f \circ g^{-1} \right) \right)(t) - \left( D^{\nu}_{g(y)-} \left( f \circ g^{-1} \right) \right)(g(x)) \right\| dt \tag{3.41}$$

(we assume that

$$\left\| \left( D^{\nu}_{g(y)-} \left( f \circ g^{-1} \right) \right)(t) - \left( D^{\nu}_{g(y)-} \left( f \circ g^{-1} \right) \right)(g(x)) \right\| \leq \rho_2 |t - g(x)|, \tag{3.42}$$

for any $t, g(x), g(y) \in [g(a), g(b)] : g(y) \geq t \geq g(x)$; $\rho_2 > 0$)

$$\leq \frac{\rho_2}{\Gamma(\nu)} \int_{g(x)}^{g(y)} (t - g(x))^{\nu-1} (t - g(x))\, dt =$$

$$\frac{\rho_2}{\Gamma(\nu)} \int_{g(x)}^{g(y)} (t - g(x))^{\nu}\, dt = \tag{3.43}$$

$$\frac{\rho_2}{\Gamma(\nu)} \frac{(g(y) - g(x))^{\nu+1}}{(\nu+1)}. \tag{3.44}$$

We have proved that

$$\left\| f(x) - f(y) - \left(A_2(f)\right)(x,y)\left(g(x) - g(y)\right)\right\| \leq \frac{\rho_2}{\Gamma(\nu)} \frac{\left(g(y) - g(x)\right)^{\nu+1}}{(\nu+1)}, \tag{3.45}$$

$\forall\ x, y \in [a,b] : x < y$.

**Conclusion 3.2** *Set* $\rho := \max(\rho_1, \rho_2)$. *Then*

$$\left\| f(x) - f(y) - \left(A_2(f)\right)(x,y)\left(g(x) - g(y)\right)\right\| \leq \frac{\rho}{\Gamma(\nu)} \frac{\left|g(x) - g(y)\right|^{\nu+1}}{(\nu+1)}, \tag{3.46}$$

$\forall\ x, y \in [a,b]$ *((3.46) is trivially true when* $x = y$*).*

One may choose $\frac{\rho}{\Gamma(\nu)} < 1$.
Here again $\nu + 1 > 2$.

**Conclusion 3.3** *Based on (3.10) and (3.21) of (I), and based on (3.33) and (3.46) of (II), using our numerical results presented earlier, we can solve numerically* $f(x) = 0$.

Some examples for $g$ follow:

$g(x) = e^x,\ \ x \in [a,b] \subset \mathbb{R}$,
$g(x) = \sin x$,
$g(x) = \tan x$,
where $x \in \left[-\frac{\pi}{2} + \varepsilon, \frac{\pi}{2} - \varepsilon\right]$, with $\varepsilon > 0$ small.

# References


[1] C.D. Aliprantis and K.C. Border, *Infinite Dimensional Analysis*, Springer, New York, 2006.

[2] S. Amat, S. Busquier, S. Plaza, *Chaotic dynamics of a third-order Newton-type method*, J. Math. Anal. Applic. 366, 1, (2010), 164-174.

[3] G.A. Anastassiou, *Strong Right Fractional Calculus for Banach space valued functions*, Revista Proyecciones, Vol. 36, No. 1 (2017), 149-186.

[4] G.A. Anastassiou, *A strong Fractional Calculus Theory for Banach space valued functions*, Nonlinear Functional Analysis and Applications (Korea), accepted for publication, 2017.

[5] G.A. Anastassiou, *Principles of general fractional analysis for Banach space valued functions*, submitted for publication (2017).

[6] I.K. Argyros, *A unifying local-semilocal convergence analysis and applications for two-point Newton-like methods in Banach space*, J. Math. Anal. Appl. 298, (2004), 374-397.

[7] I.K. Argyros, A. Magrenan, *Iterative methods and their dynamics with applications*, CRC Press, New York, 2017.

[8] Bochner integral. *Encyclopedia of Mathematics*. URL: http://www.ency clopedia ofmath.org/index.php?title=Bochner_integral& oldid=38659.

[9] M. Edelstein, *On fixed and periodic points under contractive mappings*, J. London Math. Soc. 37, (1962), 74-79.

[10] J.A. Ezquerro, J.M. Gutierrez, M.A. Hernandez, N. Romero, M.J. Rubio, *The Newton method: From Newton to Kantorovich* (Spanish), Gac. R. Soc. Mat. Esp. 13, (2010), 53-76.

[11] L.V. Kantorovich, G.P. Akilov, *Functional Analysis in Normed Spaces*, Pergamon Press, New York, 1982.

[12] G.E. Ladas and V. Lakshmikantham, *Differential equations in abstract spaces,* Academic Press, New York, London, 1972.

[13] A. Magrenan, *A new tool to study real dynamics: The convergence plane*, Appl. Math. Comput. 248, (2014), 215-224.

[14] J. Mikusinski, *The Bochner integral*, Academic Press, New York, 1978.

[15] F.A. Potra, V. Ptak, *Nondiscrete induction and iterative processes*, Pitman Publ., London, 1984.

[16] P.D. Proinov, *New general convergence theory for iterative processes and its applications to Newton-Kantorovich type theorems*, J. Complexity, 26, (2010), 3-42.

[17] G.E. Shilov, *Elementary Functional Analysis*, Dover Publications, Inc., New York, 1996.

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